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EXTERNAL VISCOUS DAMPING.

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# FLEXURAL VIBRATION OF BEAMS
# WITH EXTERNAL VISCOUS DAMPING

by

Robert Otto Meitz

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I hereby recommend that this dissertation prepared under my direction by Robert Otto Meitz

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be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Roger A. Anderson
Dissertation Director

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After inspection of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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SIGNED: Robert Otto Meitz

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ABSTRACT

The primary goal of this investigation was to determine the effects of concentrated external viscous damping proportional to transverse velocity on the free and forced vibration of beams. The model used to represent this problem was the elementary Bernoulli-Euler differential equation for flexural vibration, which was modified by adding terms representing the type of damping under consideration.

Starting from this modified equation, general solutions were derived in terms of both exact and approximate representations of the normal modes of free damped vibration. Galerkin's method was used to construct the approximate solution, which may be applied to a wide variety of beam configurations for which an exact solution is either impossible or prohibitively difficult. Both general solutions are presented in a form applicable to any beam configuration with homogeneous boundary conditions, either distributed or concentrated damping, and arbitrary forcing input and initial conditions.

Numerical results for a specific example were calculated from both solutions. The example chosen was a uniform simply supported beam with a dashpot attached at

the center of its span. Normal mode functions and characteristic frequencies were calculated for a range of dashpot strength by both methods; comparison showed very good agreement between exact and approximate values.

Forced vibration due to sinusoidal motion of the beam supports was considered, and the steady state strain energy and power dissipated by damping were computed using the approximate solution.

Results for the example showed that the influence of concentrated damping may be interpreted in terms of its effects on the normal modes of vibration. These effects were found to differ both qualitatively and quantitatively for the various modes.

The relationship between the normal modes of damped vibration and their more familiar undamped counterparts was examined, and a physically meaningful description of the nature of these modes was offered. Comments on the utility of the normal mode solutions and suggestions for further investigation are included.

# I. INTRODUCTION

The vibratory behavior of mechanical systems represented by linear models may be examined in a number of different ways. Perhaps the most frequently used approach is the normal mode method in which solutions are constructed by summing responses associated with independent modes of motion. This method has several advantages which make it particularly attractive for both discrete and continuous models. First, it leads directly to a set of uncoupled equations of motion which may be solved by elementary methods. Second, the normal mode solution facilitates physical interpretation of the behavior of the system in response to forcing inputs. Finally, the response of the normal coordinates may be found by evaluating convolution integrals (by numerical methods if necessary); consequently, the solution for arbitrary inputs is relatively simple.

When viscous damping appears in the model of the system, the existence of classical normal modes depends on the distribution of damping in relation to other parameters of the system. Conditions governing the existence of such modes are discussed in papers by Caughey[3] and Caughey and O'Kelly.[4] Because a normal mode solution in the usual sense is not often available, the analysis of damped vibrating systems can present formidable difficulty. Under appropriate conditions, the problem may be circumvented by using the normal mode functions of the system without damping to transform the equations of motion into a set coupled by damping terms alone. If the coupling terms can be assumed to be sufficiently small, they are neglected, and the resulting uncoupled equations may be solved to produce an approximate normal mode solution. The normal mode approach may be abandoned entirely in favor of an alternate technique such as the method of mobilities used by Plunkett.[11,12] Neither of these approaches is entirely satisfactory. The approximate normal mode solution will not be adequate for some problems; while other methods sacrifice the advantages of the normal mode solution.

Although systems with viscous damping may not yield to a straightforward application of usual normal mode techniques, they do possess normal modes in a general sense. Foss[5] has described a modal solution for both free and forced vibration of damped discrete systems. In addition, he derived the orthogonality conditions associated with the integral equation of motion for flexural vibration of damped beams. Foss defined a set of velocity coordinates as the first time derivatives of the existing generalized coordinates and introduced these velocity coordinates into the second order time derivatives appearing in the equations of motion. Then, the equations which define the velocity and the equations of motion form a new set of reduced equations, which contains only first order time derivatives and possesses an orthogonal mode solution. Although this approach has the fundamental advantages cited for normal mode methods, two drawbacks are apparent. First, reducing the order of the equations doubles the number of coordinates and increases the complexity of the solution. The solution could become unmanageable for complicated systems which require a large number of coordinates to describe their motion. Second, the modes of a damped system are described by complex vectors for discrete models and complex functions for continuous models. Consequently, physical interpretation of the modal solution is not as convenient as in the case of undamped systems. Perhaps for these reasons, the method proposed by Foss has been neglected. Although his paper was published nearly ten years ago, a diligent search of the subsequent literature did not turn up a single attempt to use this approach. Despite the difficulties cited above, the advantages of a modal solution suggested its use for the problem considered in this dissertation.

The flexural vibration of damped beams is also a relatively neglected topic. Cases where classical normal modes exist offer no unusual difficulties and have been explored; a paper by Stanek[13] provides a good example of this type of problem. Beams with concentrated damping are interesting from a practical standpoint and have been considered by McBride,[9] Plunkett,[11] and Young.[14] None of these authors developed a complete solution, but McBride's results illustrate some significant effects of concentrated damping. He investigated a uniform cantilever beam with a dashpot attached to the free end. In this case, damping enters the problem through boundary conditions, and McBride was able to develop a closed form solution. A portion of the development of the example to be considered below follows his approach and yields similar but more complete results.

The basic objective of the study described in this dissertation is to determine the effects of adding external damping to a beam as a means of controlling its vibration response. Particular emphasis is placed on concentrated damping and periodic input. To achieve this objective, a solution of the differential equation of motion was needed; the normal mode approach was selected to provide a solution of general applicability and hopefully to promote physical insight.

Exact closed form expressions for the characteristic frequencies and mode functions of vibrating beams are rarely attainable, even for relatively simple configurations. Effective approximate methods of generating the mode shapes are essential for normal mode solutions of practical problems. This observation suggests that a general approximate mode solution for the damped case should be derived. Several techniques are frequently used in undamped beam problems; one of these, Galerkin's method, is adaptable to damped systems and quite efficient in terms of the number of coordinates which must be considered to obtain good results.

Every investigation needs a starting point. In this case, it is the familiar Bernoulli-Euler partial differential equation modified by adding a term to account for external viscous damping. This equation is introduced in the next chapter and further modified to derive the relations which characterize the complex normal modes. Galerkin's method is also introduced at this point. The following chapters contain parallel developments of closed form and Galerkin's method solutions, first in very general terms and, subsequently, for a simple example where damping is concentrated at a point. This example permits an explicit frequency equation which holds for all levels of damping however large. The frequency equation and its associated representation of mode functions are evaluated numerically to show some of the effects of viscous damping.

Corresponding results obtained by Galerkin's method are compared to the closed form. Finally, the effects of damping on the forced vibration response of the beam with sinusoidal loading are investigated using the Galerkin's method solution.

## II. THEORY

### The Equation of Motion

For slender beams, the well-known Bernoulli-Euler equation is based on the oldest and most often used model for flexural vibration. The fundamental assumptions on which this equation rests are as follows:

a. Initially plane cross sections of the beam remain plane during deformation.

b. The beam is initially straight, and its slope remains small everywhere along the span during deformation.

c. The mean axial deflection of any cross section is zero.

d. Deformation due to shear is neglected.

e. The effect of rotary inertia is neglected.

f. Dissipative effects are not considered.

The derivation of the equation is available in vibration texts, e.g., Anderson,[1] and will not be repeated here.

To account for external viscous damping, the Bernoulli-Euler equation must be modified. Let us assume that, in general, the motion of the beam is resisted by a distributed force with intensity proportional to the transverse velocity. This assumption leads to an additional term, consisting of the product of the transverse velocity and a distributed parameter which defines the force intensity (force per unit length) generated by the damping mechanism with a unit velocity applied. Point damping of the beam by one or more isolated dashpots may be viewed as a special case in which the distributed damping parameter becomes the sum of a number of terms containing delta functions.

Figure 2.1 illustrates the general configuration to be considered. Equation (2.1) is the modified Bernoulli-Euler equation for this configuration.

$$\frac{\partial 2}{\partial \bar{x}^2}\left(EI\,\frac{\partial^2 \bar{y}}{\partial \bar{x}^2}\right) + D\,\frac{\partial \bar{y}}{\partial \bar{t}} + \mu\,\frac{\partial^2 \bar{y}}{\partial \bar{t}^2} = \bar{f}\,(\bar{x},\,\bar{t}) \qquad (2.1)$$

where:

| | | |
|---|---|---|
| EI | = | flexural rigidity of the beam |
| $\bar{f}(\bar{x},\bar{t})$ | = | distributed transverse force applied |
| L | = | length of the beam |
| $\bar{t}$ | = | time |
| $\bar{x}$ | = | axial beam coordinate |
| $\bar{y}$ | = | transverse deflection of the beam |
| $\mu$ | = | mass density (per unit length) of the beam |
| D | = | distributed viscous damping parameter expressing the ratio of force per unit |

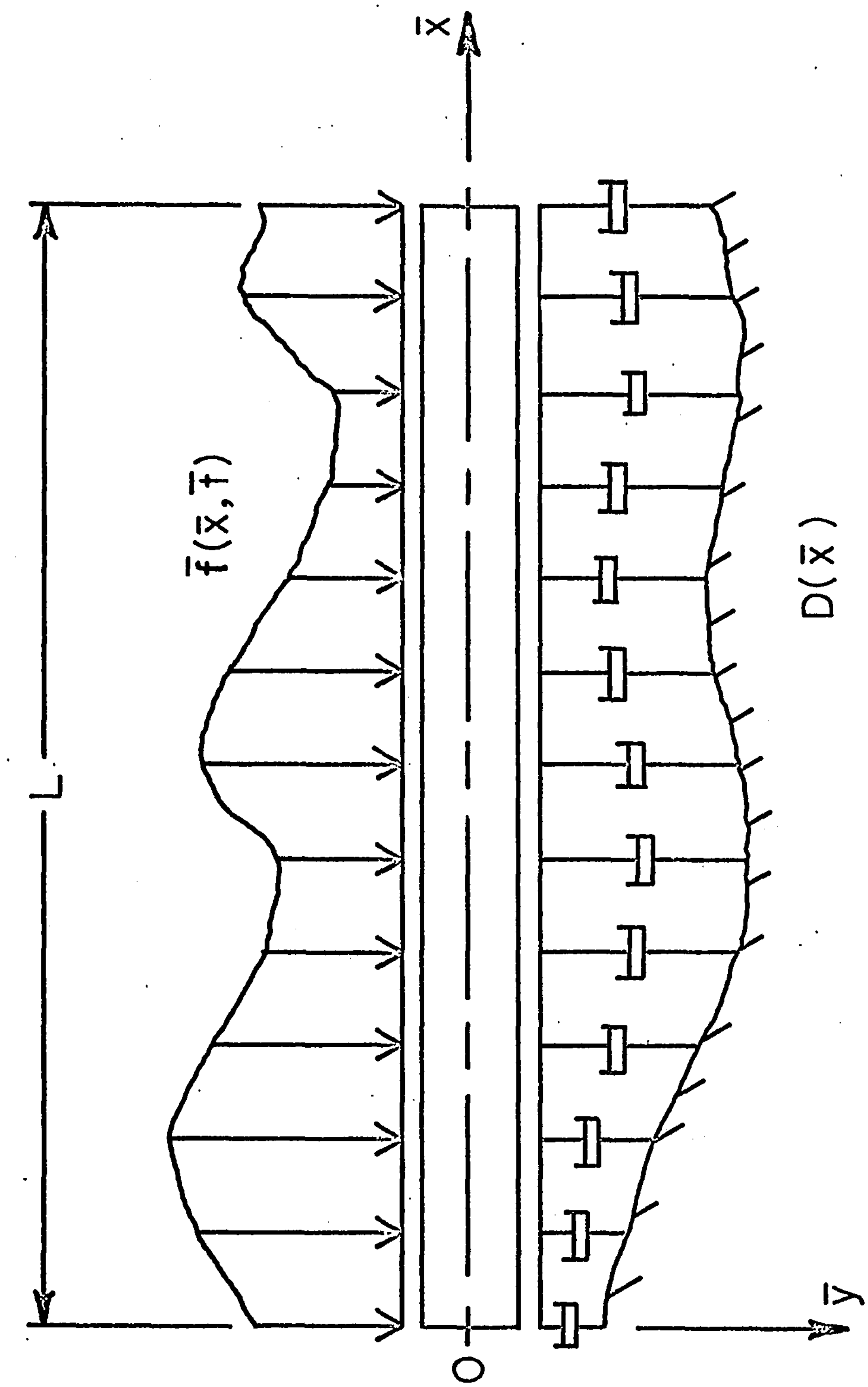



Fig. 2.1 General Configuration

length to unit transverse velocity applied to the damping mechanism.

The flexural rigidity, mass density, and damping parameter vary along the beam. We assume that the beam may be constrained only at its ends and that the possible end conditions are limited to the following cases:

Fixed end -- transverse deflection and slope are zero.

$$\bar{y} = 0, \ \frac{\partial \bar{y}}{\partial \bar{x}} = 0 \tag{2.2a}$$

Simply supported end -- transverse deflection and bending moment are zero.

$$\bar{y} = 0, \ EI \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} = 0 \tag{2.2b}$$

Free end -- bending moment and shear force are zero.

$$EI \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} = 0, \ \frac{\partial}{\partial \bar{x}} \left(EI \frac{\partial^2 \bar{y}}{\partial \bar{x}^2}\right) = 0 \tag{2.2c}$$

The results to be developed in subsequent sections are most conveniently expressed in dimensionless quantities. To avoid having two complete sets of notation, Eqs. (2.1) and (2.2) will be written in non-dimensional form at the beginning, resulting in

$$\frac{\partial^2}{\partial x^2} \left(k \frac{\partial^2 y}{\partial x^2}\right) + c \frac{\partial y}{\partial t} + m \frac{\partial^2 y}{\partial t^2} = f(x,t) \tag{2.3}$$

and end conditions

$$y = 0, \ \frac{\partial y}{\partial x} = 0 \tag{2.4a}$$

$$y = 0, \ k \frac{\partial^2 y}{dx^2} = 0 \tag{2.4b}$$

$$k \frac{\partial^2 y}{\partial x^2} = 0, \ \frac{\partial}{\partial y} \left(k \frac{\partial^2 y}{\partial x^2}\right) = 0 \tag{2.4c}$$

The dimensionless quantities in Eqs. (2.3) and (2.4) are defined as follows.

$$
\begin{aligned}
c &= \frac{L^2 \ D}{\sqrt{E_o I_o \mu_o}} \\
f &= \frac{L^3 \bar{f}}{E_o I_o} \\
k &= \frac{E \ I}{E_o I_o} \\
m &= \frac{\mu}{\mu_o} \\
t &= \frac{\sqrt{E_o I_o} \ \bar{t}}{\sqrt{\mu_o} \ \ L^2} \\
x &= \frac{\bar{x}}{L} \\
y &= \frac{\bar{y}}{L}
\end{aligned}
\tag{2.5}
$$

$E_o I_o$ and $\mu_o$ are reference values of flexural rigidity and mass density taken at a selected location along the beam.

For a uniform beam

$$k(x) = m(x) \equiv 1, \quad 0 \leq x \leq 1$$

Dashpots applied at one or more discrete points along the beam may be represented by writing $D(\overline{x})$ in terms of delta functions. For M dashpots

$$D(\overline{x}) = \sum_{r=1}^{M} D_r \, \delta(\overline{x} - \overline{x}_r)$$

In non-dimensional form

$$c(x) = \sum_{r=1}^{M} c_r \, \delta(x - x_r)$$

$$c_r = \frac{L \, D_r}{\sqrt{E_o I_o \mu_o}} \tag{2.6}$$

Normal mode solutions of Eq. (2.3) may be developed in a straightforward manner if we first transform this equation into two equations of the first order in time.

$$m \frac{\partial y}{\partial t} - mv = 0$$

$$\frac{\partial^2}{\partial x^2}\left(k \frac{\partial^2 y}{\partial x^2}\right) + c \frac{\partial y}{\partial t} + m \frac{\partial v}{\partial t} = f(x,t) \tag{2.7}$$

The first equation is an identity used to define the velocity, v; the second is the equation of motion reduced to the first order in time.

## Orthogonality Conditions

Before attempting to construct a modal solution of Eqs. (2.7), let us determine the orthogonality conditions which may be used to obtain uncoupled equations. We begin with the homogeneous equations obtained from Eqs. (2.7) by setting the forcing function f equal to zero and assuming a solution in the form

$$
\begin{aligned}
y(x,t) &= \sum_{i=1}^{\infty} a_i \, \Phi_i(x) \, e^{\alpha_i t} \\
v(x,t) &= \sum_{i=1}^{\infty} a_i \, \psi_i(x) \, e^{\alpha_i t}
\end{aligned}
\qquad (2.8)
$$

The $i^{th}$ terms of Eqs. (2.8) represent the activity in the $i^{th}$ normal mode. When the beam is disturbed by some initial conditions, the subsequent vibration in any of the normal modes takes place independently of the activity in any other mode. Consequently, the $i^{th}$ terms of Eqs. (2.8) must satisfy both the homogeneous equations and the boundary conditions. Substituting these terms into Eqs. (2.4) and (2.7) and setting f = 0 leads to

$$
\begin{aligned}
& m\alpha_i \Phi_i - m\psi_i = 0 \\
& (k\Phi_i'')'' + \alpha_i c\Phi_i + \alpha_i m\psi_i = 0
\end{aligned}
\qquad (2.9)
$$

and

$$\Phi_i = 0, \ \Phi_i' = 0 \tag{2.10a}$$

$$\Phi_i = 0, \ k\Phi_i'' = 0 \tag{2.10b}$$

$$k\Phi_i'' = 0, \ (k\Phi_i'')' = 0 \tag{2.10c}$$

The primes indicate differentiation with respect to x. Now, let us multiply the first of Eqs. (2.9) by $\psi_j$ dx, the second by $\Phi_j$ dx, sum the two, and integrate over the length of the beam.

$$\alpha_i \int_0^l [m(\Phi_i\psi_j + \psi_i\Phi_j) + c\Phi_i\Phi_j]\, dx + \int_0^l [(k\Phi_i'')''\, \Phi_j - m\psi_i\psi_j]\, dx = 0 \tag{2.11}$$

Since all the normal modes must satisfy Eqs. (2.9), we may construct another equation from Eq. (2.11) by interchanging the subscripts i and j. Then, subtracting this equation from Eq. (2.11) gives

$$(\alpha_i-\alpha_j) \int_0^l [m(\Phi_i\psi_j + \psi_i\Phi_j) + c\Phi_i\Phi_j]\, dx + \int_0^l [(k\Phi_i'')''\, \Phi_j - (k\Phi_j'')''\, \Phi_i]\, dx = 0 \tag{2.12}$$

The symmetry of the second integral suggests the next step, integrating by parts to evaluate this term. Two cases must be considered. In the first case, damping is distributed, and the shear force is a continuous function of x everywhere along the beam. The second case arises when concentrated damping is applied by dashpots attached to the beam. With concentrated damping, the shear force will have a step discontinuity at each dashpot. In both cases, the beam properties and applied force are assumed to be distributed.

With distributed damping, the second term of Eq. (2.12) may be integrated twice by parts to obtain

$$\int_0^1 [(k\Phi_i'')''\Phi_j - (k\Phi_j'')''\Phi_i]\,dx$$

$$= \int_0^1 (k\Phi_i''\Phi_j'' - k\Phi_j''\Phi_i'')\,dx \tag{2.13}$$

$$+ \left[(k\Phi_i'')'\Phi_j - (k\Phi_j'')'\Phi_i - k\Phi_i''\Phi_j' + k\Phi_j''\Phi_i'\right]\Big|_0^1$$

The integral on the right side of Eq. (2.13) vanishes identically, and the integrated terms are zero when the mode functions satisfy any of the end conditions given by Eqs. (2.10). If, as is usually the case, the normal modes have distinct characteristic values, i.e., $\alpha_i \neq \alpha_j$, Eq. (2.12) reduces to

$$\int_0^1 [m(\Phi_i\psi_j + \psi_i\Phi_j) + c\Phi_i\Phi_j]\,dx = 0 \qquad i \neq j \tag{2.14}$$

Taking Eq. (2.14) into Eq. (2.11) produces a second relation, after integrating by parts.

$$\int_0^1 (k\Phi_i{}'' \Phi_j{}'' - m\psi_i\psi_j)\, dx = 0, \qquad i \neq j \tag{2.15}$$

Equations (2.14) and (2.15) are the orthogonality conditions for the beam with distributed damping.

When the damping is applied through isolated dashpots, shear discontinuities exist at the point where the dashpots act; therefore, $(k\Phi_i{}'')'$ is discontinuous at these points. If we express the damping function, c, in terms of delta functions as suggested earlier, Eq. (2.6), the integrals containing c in Eqs. (2.11) and (2.12) may be replaced by finite sums.

$$\int_0^1 c\Phi_i\Phi_j\, dx = \sum_{r=1}^{M} c_r \int_0^1 \delta(x-x_r)\, \Phi_i\Phi_j\, dx$$

$$= \sum_{r=1}^{M} c_r\Phi_i\,(x_r)\, \Phi_j\,(x_r) \tag{2.16}$$

From the second of Eqs. (2.9) and Eq. (2.6), the elastic force term may be expressed as follows.

$$(k\Phi_i{}'')'' = (k\theta_i{}'')'' - \alpha_i \sum_{r=1}^{M} c_r\delta(x-x_r)\, \Phi_i(x_r) \tag{2.17}$$

$(k\theta_i{}'')'$ is continuous. The first two integrals of Eq. (2.17) are

$$(k\Phi_i'')' = (k\theta_i'')' - \alpha_i \sum_{r=1}^{M} c_r \; H(x-x_r) \; \Phi_i(x_r)$$

$$k\Phi_i'' = k\theta_i'' - \alpha_i \sum_{r=1}^{M} c_r(x-x_r) \; H(x-x_r) \; \Phi_i(x_r)$$

where $H(x-x_r)$ is the unit step function. Eq. (2.17) and its integrals may be used to evaluate the second integral in Eq. (2.12). We consider the first term.

$$\int_0^1 (k\Phi_i'')'' \Phi_j dx = \int_0^1 (k\theta_i'')'' \Phi_j dx \tag{2.18}$$

$$- \alpha_i \sum_{r=1}^{M} c_r \Phi_i(x_r) \int_0^1 \delta(x-x_r) \; \Phi_j dx$$

$$= \int_0^1 (k\theta_i'')'' \Phi_j dx - \alpha_i \sum_{r=1}^{M} c_r \Phi_i(x_r) \Phi_j(x_r)$$

If we integrate by parts twice and use Eq. (2.17) and its integrals to replace terms involving $\theta_i$, we have the same expression which would result if the shear force were continuous.

$$\int_0^1 (k\Phi_i'')'' \Phi_j dx = [(k\Phi_i'')' \Phi_j - k\Phi_i'' \Phi_j'] \Big|_0^1 + \int_0^1 k\Phi_i'' \Phi_j'' dx$$

For homogeneous end conditions, Eqs. (2.10), the integrated terms are zero; and only the symmetric integral remains on the right side. Again the second integral in Eq. (2.12) is zero and the orthogonality conditions are Eqs. (2.14) and

(2.15). Equation (2.14) may be modified using Eq. (2.16).

$$\int_0^1 m(\Phi_i \Psi_j + \Psi_i \Phi_j)dx + \sum_{r=1}^{M} c_r \Phi_i(x_r)\Phi_j(x_r) = 0 \qquad i \neq j \quad (2.19)$$

In some cases, the problem of a beam with concentrated damping may be solved by considering the undamped equations of motion on intervals between the dashpots. The constants in the solutions over these restricted intervals are determined by the end conditions and special conditions which connect the intervals. These special conditions enforce continuity of transverse deflection, slope, and bending moment across the interval boundaries and equate the damping force generated by each dashpot to the step in shear force at its point of application. This approach will be illustrated in detail in the example given later. In this case, the orthogonality conditions may be written in terms of the sums of integrals taken over the individual intervals. To write the resulting orthogonality conditions in compact form, we introduce the notation, $\Phi_{ri}$, representing the $i^{th}$ mode function of the interval, $x_{r-1} \leq x \leq x_r$, where $x_o = 0$, $x_{M+1} = 1$.

Then,

$$\sum_{r=1}^{M+1} \int_{x_{r-1}}^{x_r} m(\Phi_{ri}\psi_{rj} + \psi_{rj}\Phi_{ri})dx + \sum_{r=1}^{M} c_r\Phi_{ri}(x_r)\Phi_{rj}(x_r) = 0$$

$$\sum_{r=1}^{M+1} \int_{x_{r-1}}^{x_r} (k\Phi_{ri}''\Phi_{rj}'' - m\psi_{ri}\psi_{rj})dx = 0 \qquad (2.20)$$

$$i \neq j$$

## Galerkin's Method

When exact solutions for the vibratory response of a continuous system are not accessible, we may seek an approximate solution in terms of a finite number of coordinates. To be useful, an approximate method should provide solutions which converge on the exact solution as the number of coordinates is increased. Obviously we would like to minimize the number of coordinates needed to reduce the error to an acceptable level. Since Galerkin's method is quite efficient in this respect, it has been adopted for the problem being considered here.

Galerkin's method is based on an error weighting procedure. An approximate solution is constructed from a set of assumed mode functions (approximating functions) and an associated set of generalized coordinates. The form of the solution is illustrated by Eqs. (2.21). An approximate solution in a finite number of coordinates will not identically satisfy the exact equation of motion. The error, weighted by each of the approximating functions, in turn, is integrated over the length of the beam, and each of the resulting expressions is set equal to zero. From a slightly different viewpoint, we may say that the error is required to be orthogonal to each of the approximating functions. Enforcing these conditions produces a set of ordinary differential equations in the generalized coordinates. The mechanics of the method are illustrated below; its mathematical foundations have been described in detail by Mikhlin.[10] Bolotin[2] and Kantorovich and Krylov[8] have applied the method to a number of interesting problems.

We begin by assuming that y and v may be expressed in the following forms.

$$y = \sum_{i=1}^{n} \varphi_i(x) q_i(t)$$
$$v = \sum_{i=1}^{n} \varphi_i(x) r_i(t) \tag{2.21}$$

The approximating functions, $\varphi_i(x)$, are elements of a complete set of functions which satisfy all the boundary conditions of the system. The functions, $q_i(t)$ and $r_i(t)$, are the generalized coordinates. Anderson[1] suggests that the approximating functions need only satisfy geometric boundary conditions. In this development, however, we will consider only functions which satisfy all the boundary conditions.

Let us substitute Eqs. (2.21) into Eqs. (2.7). Then,

$$\sum_{i=1}^{n} m\varphi_i \dot{q}_i - \sum_{i=1}^{n} m\varphi_i r_i = D(x,t)$$

$$\sum_{i=1}^{n} (k\varphi_i^{\prime\prime})^{\prime\prime} q_i + \sum_{i=1}^{n} c\varphi_i \dot{q}_i + \sum_{i=1}^{n} m\varphi_i \dot{r}_i \qquad (2.22)$$

$$- f(x,t) = E(x,t)$$

The superscript dots denote differentiation with respect to time. Since the first of Eqs. (2.7) is an identity defining velocity, D will be identically zero. To obtain equations in a convenient form, we will treat D formally in the same manner as the distributed force error E which arises from the equation of motion. So, we enforce the following conditions.

$$\int_0^1 D\varphi_j dx = \int_0^1 E\varphi_j dx \equiv 0 \qquad (2.23)$$

$$j = 1, 2, \ldots, n; \; t > 0.$$

With these conditions, the work done by the force error will be zero during any displacement which can be expressed by Eqs. (2.21). Equations (2.23) lead directly to a set of 2n ordinary differential equations, Eq. (2.24), written in matrix form.

$$\left[\begin{array}{c|c} 0 & M \\ \hline M & C \end{array}\right] \left\{\begin{array}{c} \dot{r} \\ \hline \dot{q} \end{array}\right\} + \left[\begin{array}{c|c} -M & 0 \\ \hline 0 & K \end{array}\right] \left\{\begin{array}{c} r \\ \hline q \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline F \end{array}\right\} \tag{2.24}$$

[C], [K], and [M] are n x n matrices, and {F} is a 1 x n matrix.

Their elements are given by

$$\begin{aligned} C_{ij} &= \int_0^1 c\varphi_i\varphi_j dx \\ K_{ij} &= \int_0^1 (k\varphi_j'')''\varphi_i dx = \int_0^1 k\varphi_i''\varphi_j'' dx \\ M_{ij} &= \int_0^1 m\varphi_i\varphi_j dx \\ F_j(t) &= \int_0^1 f(x,t)\,\varphi_j dx \end{aligned} \tag{2.25}$$

[C] and [M] are symmetric matrices; and since the approximating functions obey homogeneous boundary conditions, [K] is also symmetric.

For concentrated damping by N dashpots,

$$C_{ij} = \sum_{s=1}^{N} c_s \varphi_i(x_s)\ \varphi_j(x_s) \tag{2.26}$$

Equation (2.24) is in the form suggested by Foss[5] for lumped parameter systems. He pointed out that a normal mode solution of this equation is possible if the matrices are symmetric. We denote the square matrices of Eq. (2.24) by [D] and [E], respectively.

$$[D] = \left[\begin{array}{c|c} O & M \\ \hline M & C \end{array}\right]$$

$$[E] = \left[\begin{array}{c|c} -M & O \\ \hline O & K \end{array}\right] \tag{2.27}$$

Then we define

$\{A\}_i$ = column matrix form of the $i^{th}$ normal mode vector,

$\lfloor A \rfloor_i$ = Row matrix form of the $i^{th}$ normal mode vector.

Equations (2.28) are the orthogonality conditions derived by Foss.

$$\lfloor A \rfloor_i\ [D]\ \{A\}_j = 0$$

$$\lfloor A \rfloor_i\ [E]\ \{A\}_j = 0 \tag{2.28}$$

$$i \neq j$$

## Some Observations

If we can find the normal mode functions and frequencies for a particular beam configuration, the orthogonality conditions permit us to construct general solutions for either forced or free damped vibration. There are no dependable general techniques which yield exact mode functions, but we can handle some simple cases.

In Galerkin's method, finding normal mode parameters is not so difficult; we may use an iterative process based on the matrix equation, Eq. (2.24), for this purpose. The significant problems in this case are selecting good approximating functions and calculating the elements of the system matrices.

In any stable linear system with light damping, the characteristic values will occur in complex conjugate pairs with negative real parts. The corresponding mode functions or vectors will also be complex conjugate pairs. The real part of any of the characteristic values is the subsidence or attenuation of free damped vibration in the associated mode; the imaginary part is the corresponding circular frequency. For larger values of damping, some or all of the circular frequencies vanish; and the free motion of the corresponding modes changes from decaying oscillation to exponential decay.

The terms, complex frequency and normal mode function, are not strictly correct when applied to the characteristic parameters of the damped case; eigenvalue and eigenfunction would be more precise. However, these terms will be used to emphasize the analogy between the treatment of damped and undamped problems.

## III. GENERAL SOLUTIONS

### Closed Form

If we have found normal mode functions, $\Phi_i$, and frequencies, $\alpha_i$, which satisfy Eqs. (2.9) and applicable boundary conditions, we may assume the following solution of Eqs. (2.7).

$$y = \sum_{i=1}^{\infty} \Phi_i(x)p_i(t)$$
$$v = \sum_{i=1}^{\infty} \psi_i(x)p_i(t) \qquad (3.1)$$

The first of Eqs. (2.7) determines the form of $\psi_i(x)$.

$$\psi_i(x) = \alpha_i \Phi_i(x)$$

Substituting Eqs. (3.1) into Eqs. (2.7), we have

$$\sum_{i=1}^{\infty} (m\Phi_i \dot{p}_i - m\psi_i p_i) = 0 \qquad (3.2)$$
$$\sum_{i=1}^{\infty} [(k\Phi_i'')'' p_i + c\Phi_i \dot{p}_i + m\psi_i \dot{p}_i] = f(x,t)$$

Equations (3.2) may be transformed to a set of uncoupled equations by using the orthogonality conditions. The first equation is multiplied by $\psi_j dx$, the second by

$\Phi_j dx$; both are integrated over the length of the beam. If the resulting equations are summed,

$$\sum_{i=1}^{\infty} \dot{p}_i \int_0^1 [m(\Phi_i\psi_j + \psi_i\Phi_j) + c\Phi_i\Phi_j]dx$$

$$+ \sum_{i=1}^{\infty} p_i \int_0^1 [k\Phi_i{}'')'' \Phi_j - m\psi_i\psi_j]dx \tag{3.3}$$

$$= \int_0^1 f(x,t)\Phi_j dx$$

Integrating the terms containing k by parts and applying the orthogonality conditions, Eqs. (2.14) and (2.15), the terms where $i \neq j$ drop out. Repeating this process for all the modes, we have Eqs. (3.4).

$$R_j\dot{p}_j + S_j p_j = F_j(t), \quad j = 1, 2, \ldots, \infty \tag{3.4}$$

with

$$R_j = \int_0^1 (2m\Phi_j\psi_j + c\Phi_j^2)dx$$

$$S_j = \int_0^1 [(k\Phi_j{}'')''\Phi_j - m\psi_j^2]dx = \int_0^1 [k(\Phi_j{}'')^2 - m\psi_j^2]dx \tag{3.5}$$

$$F_j(t) = \int_0^1 \Phi_j f(x,t)dx$$

We note that $R_j$ and $S_j$ are the integrals in Eq. (2.11) for the case, $i = j$.

Consequently, we may write

$$S_j = -\alpha_j R_j$$

and

$$\dot{p}_j - \alpha_j p_j = \frac{F_j(t)}{R_j}, \quad j = 1, \ldots, \infty \tag{3.6}$$

Equations (3.6) may be solved by elementary means; the complete solution is

$$p_j(t) = p_{j0}\, e^{\alpha_j t} + \frac{1}{R_j} \int_0^t e^{\alpha_j (t-T)} F_j(T)\, dT \tag{3.7}$$

$$p_{j0} = p_j(0), \quad j = 1, 2, \ldots, \infty$$

The initial values, $p_j(0)$, may be determined from the initial displacement and velocity of the beam by setting $t = 0$ in Eqs. (3.1). If we multiply $y(x,0)$ by $(m\psi_j + c\Phi_j)$ $dx$ and $v(x,0)$ by $m\Phi_j dx$, sum and integrate the result over the beam length, we have

$$\int_0^1 [(m\psi_j + c\Phi_j)\cdot y(x,0) + m\Phi_j v(x,0)]dx$$

$$= \sum_{i=1}^{\infty} p_i(0) \int_0^1 [m(\Phi_i \psi_j + \psi_i \Phi_j) + c\Phi_i \Phi_j]dx$$

Applying the orthogonality condition, Eq. (2.14), only the $j^{th}$ term remains on the right side. We may solve for $p_j(0)$ and apply the first of Eqs. (3.5) to obtain

$$p_j(0) = \frac{1}{R_j} \int_0^1 [(m\psi_j + c\Phi_j) y(x,0) + m\Phi_j v(x,0)] dx \tag{3.8}$$

$$j = 1,\ 2,\ \ldots,\ \infty$$

Finally, the results from Eqs. (3.7) may be substituted into Eqs. (3.1) to give the displacement and velocity response of the beam for the initial conditions and applied forces.

When the damping is concentrated, it may be convenient to represent the mode functions by separate expressions for each of the intervals between dashpots. Then, Eqs. (3.1) may be written for each interval and substituted into the corresponding differential equations for that interval. Following the steps outlined for distributed damping and using the orthogonality conditions, Eqs. (2.20), instead of Eqs. (2.14) and (2.15), Eq. (3.7) again gives the solution. In this case, $R_j$, $F_j$ and $P_{j0}$ have the forms shown in Eqs. (3.9).

$$R_j = \sum_{r=1}^{M+1} \int_{x_{r-1}}^{x_r} 2m\Phi_{rj}\psi_{rj}\,dx + \sum_{r=1}^{M} c_r \Phi_{rj}(x_r)^2$$

$$F_j(t) = \sum_{r=1}^{M+1} \int_{x_{r-1}}^{x_r} \Phi_{rj} f(x,t)\,dx$$

$$p_{j0} = \frac{1}{R_j} \sum_{r=1}^{M+1} \int_{x_{r-1}}^{x_r} m[\psi_{rj} y(x,0) + \Phi_{rj} v(x,0)]dx$$

$$+ \frac{1}{R_j} \sum_{r=1}^{M} c_r \Phi_{rj}(x_r) y(x_r,0) \tag{3.9}$$

## Galerkin's Method

As pointed out earlier, the frequencies and normal mode vectors needed to solve Eq. (2.24) may be obtained by an iterative process. Substituting Eqs. (2.27) into Eq. (2.24) and setting the applied forces to zero, we have

$$[D] \begin{Bmatrix} \dot{r} \\ \hline \dot{q} \end{Bmatrix} + [E] \begin{Bmatrix} r \\ \hline q \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \tag{3.10}$$

We assume

$$\begin{Bmatrix} r \\ \hline q \end{Bmatrix} = \sum_{i=1}^{2n} \{A\}_i \, e^{\alpha_i t}$$

For the terms of this series to represent the contribution of normal modes, each term must satisfy Eq. (3.10).

$$\alpha_i \, [D] \, \{A\}_i \, e^{\alpha_i t} + [E] \, \{A\}_i \, e^{\alpha_i t} = \{0\} \tag{3.11}$$

We may rearrange this equation to obtain Eq. (3.12).

$$\frac{1}{\alpha_i} \{A\}_i = -[E]^{-1} [D] \{A\}_i \tag{3.12}$$

If we select an arbitrary trial vector and premultiply it by $-[E]^{-1}[D]$ a number of times, the resulting vectors converge to a function of the pair of complex conjugate frequencies having the smallest modulus, together with the corresponding pair of mode vectors. Fox[6] and Frazer, Duncan, and Collar[7] explain this iterative technique and describe procedures for finding the mode parameters.

Once the lowest modes are determined, we may force the iteration process to produce the next larger pair of frequencies and the associated vectors. Appendix A describes a sweeping matrix technique which accomplishes this task. We may continue this process until we have established the required number of modes.

The solution of Eq. (2.24) may now be written in terms of normal coordinates, $Q_i$.

$$\left\{\frac{r}{q}\right\} = [A]\,\{Q\} \tag{3.13}$$

The columns of $[A]$ are the normal mode vectors, $\{A\}_i$. We substitute this solution into Eq. (2.24) and premultiply the equation by the transpose of $[A]$.

$$[A]^T[D]\,[A]\,\{\dot{Q}\} + [A]^T[E]\,[A]\,\{Q\} = [A]^T\left\{\frac{0}{F}\right\}$$

The orthogonality conditions, Eqs. (2.28), show that $[A]^T[D]\,[A]$ and $[A]^T[E]\,[A]$ are diagonal matrices; from

Eq. (3.11), their diagonal elements are related by

$$\lfloor A \rfloor_i \, [D] \, \{A\}_i = -\frac{1}{\alpha_i} \lfloor A \rfloor_i \, [E] \, \{A\}_i$$

Consequently, Eq. (2.24) is transformed into a set of 2n uncoupled equations in the normal coordinates.

$$\dot{Q}_i - \alpha_i Q_i = \frac{1}{R_i} \lfloor A \rfloor_i \begin{Bmatrix} 0 \\ \hline F \end{Bmatrix}$$

$$R_i = \lfloor A \rfloor_i \, [D] \, \{A\}_i \tag{3.14}$$

$$i = 1, 2, \ldots, 2n$$

The complete solution of Eqs. (3.14) is:

$$Q_i(t) = Q_{i0} \, e^{\alpha_i t} + \frac{1}{R_i} \int_0^t e^{\alpha_i (t-T)} \left( \lfloor A \rfloor_i \begin{Bmatrix} 0 \\ \hline F \end{Bmatrix} \right) dT \tag{3.15}$$

$$i = 1, 2, \ldots, 2n$$

with

$Q_{i0}$ = initial values of the normal coordinates.

The initial values are derived from the initial displacement and velocity, $y_0(x)$ and $v_0(x)$.

From Eqs. (2.21),

$$y_0(x) = \sum_{i=1}^{n} \varphi_i(x) q_{i0}$$

$$v_0(x) = \sum_{i=1}^{n} \varphi_i(x) r_{i0}$$

We form the functions $Z_j$ and $Z_{n+j}$ and apply Eqs. (2.25), which define [M] and [C].

$$Z_j = \int_0^1 m\varphi_j y_0 dx = \sum_{i=1}^{n} M_{ji} q_{i0} \tag{3.16}$$

$$Z_{n+j} = \int_0^1 (m\varphi_j v_0 + c\varphi_j y_0) dx = \sum_{i=1}^{n} (M_{ji} r_{i0} + C_{ji} q_{i0})$$

$$j = 1, 2, \ldots, n$$

in matrix form, using the first of Eqs. (2.27)

$$\{Z\} = \left[\begin{array}{c|c} O & M \\ \hline M & C \end{array}\right] \left\{\begin{array}{c} r_0 \\ \hline q_0 \end{array}\right\} = [D]\,[A]\,\{Q_0\} \tag{3.17}$$

{Z} is a 2n x 1 matrix whose elements are defined by Eq. (3.16). Premultiplying Eq. (3.17) by $[A]^T$ and applying the first orthogonality condition from Eqs. (2.28), we may solve for the initial values of the normal coordinates.

$$Q_{i0} = \frac{1}{R_i} \lfloor A \rfloor_i \{Z\} \tag{3.18}$$

Taking Eqs. (3.15) and (3.18) together gives the complete solution for the normal coordinates, $Q_i$. To find the velocity and displacement of the beam in terms of the normal coordinates, Eq. (3.13) may be substituted into Eqs. (2.21) after rewriting the latter equations in matrix form. The result is given by Eqs. (3.19).

$$\begin{array}{cccc} 2x1 & 2x2n & 2nx2N & 2Nx1 \\ \begin{Bmatrix} v(x,t) \\ y(x,t) \end{Bmatrix} = & \left[\begin{array}{c|c} \varphi^T(x) & 0 \\ \hline 0 & \varphi^T(x) \end{array}\right] & \begin{bmatrix} A \end{bmatrix} & \begin{Bmatrix} Q(t) \end{Bmatrix} \end{array} \tag{3.19}$$

The number of rows and columns of each matrix are noted above it. $\lfloor \varphi(x) \rfloor$ is a 1 by n row matrix function of x whose elements are the approximating functions. We have considered the first N pairs of modes in constructing the solution. In general, we will wish to supply more approximating functions than the number of pairs of modes needed.

## Harmonic Input

The solutions we have derived permit us to handle arbitrary forcing inputs. Among the inputs which may be considered, those which are periodic in time often occur in practice. We will examine the behavior of the beam under this type of loading by inserting a sinusoidal forcing function, Eq. (3.20) into the Galerkin's method solution.

$$f(x,t) = g(x) \sin\Omega t \tag{3.20}$$

From the definition of $F_j(t)$, Eqs. (2.25),

$$F_j(t) = \sin\Omega t \int_0^1 g(x)\varphi_j dx = h_j \sin\Omega t$$

where

$$h_j = \int_0^1 g(x)\varphi_j dx \tag{3.21}$$

$$j = 1, 2, \ldots, n$$

In matrix form,

$$\begin{Bmatrix} 0 \\ --- \\ F \end{Bmatrix} = \begin{Bmatrix} 0 \\ --- \\ h \end{Bmatrix} \sin\Omega t \tag{3.22}$$

Substituting Eq. (3.22) into Eq. (3.15) gives

$$Q_i(t) = Q_{i0}e^{\alpha_i t} + \frac{1}{R_i} \lfloor A \rfloor_i \begin{Bmatrix} 0 \\ --- \\ h \end{Bmatrix} \int_0^t e^{\alpha_i(t-T)} \sin\Omega T dT$$

We may evaluate the integral by elementary means,

$$\int_0^t e^{\alpha_i(t-T)} \sin\Omega T dT = - \frac{1}{\alpha_i^2+\Omega^2} [\alpha_i \sin\Omega t + \Omega\cos\Omega t - \Omega e^{\alpha_i t}]$$

Since the mode frequencies have negative real parts, the terms containing the exponential will be negligible after some time. Dropping these terms, we have the steady state response.

$$Q_i(t) = - \lfloor A \rfloor_i \begin{Bmatrix} 0 \\ --- \\ h \end{Bmatrix} \frac{\alpha_i \sin\Omega t + \Omega\cos\Omega t}{R_i(\alpha_i^2 + \Omega^2)} \tag{3.23}$$

In matrix form,

$$\{Q\} = - [G][A]^T \begin{Bmatrix} 0 \\ --- \\ h \end{Bmatrix} \sin\Omega t - [H][A]^T \begin{Bmatrix} 0 \\ --- \\ h \end{Bmatrix} \cos\Omega t \tag{3.24}$$

[G] and [H] are diagonal matrices.

$$G_{ii} = \frac{\alpha_i}{R_i(\alpha_i^2+\Omega^2)} \tag{3.25}$$

$$H_{ii} = \frac{\Omega}{R_i(\alpha_i^2+\Omega^2)}$$

Substituting Eq. (3.24) into Eq. (3.19),

$$\begin{Bmatrix} v \\ \hline y \end{Bmatrix} = -\left[\begin{array}{c|c} \varphi^T & 0 \\ \hline 0 & \varphi^T \end{array}\right] [A]\,[G]\,[A]^T \begin{Bmatrix} 0 \\ \hline h \end{Bmatrix} \sin\Omega t \tag{3.26}$$

$$- \left[\begin{array}{c|c} \varphi^T & 0 \\ \hline 0 & \varphi^T \end{array}\right] [A]\,[H]\,[A]^T \begin{Bmatrix} 0 \\ \hline h \end{Bmatrix} \cos\Omega t$$

## Measures of Damping Effectiveness

The primary goal of investigating structural vibration is often to assess the possibility that the structure will fail under the expected vibratory input. When we are free to choose design parameters over a wide range, we may wish to use strain energy as an initial indicator of the internal loads within the structure. When viscous damping is used to control vibratory response, the rate at which energy must be dissipated by the damping elements is also important. We may easily derive expressions for strain energy and dissipation rate from the approximate solution.

Consider a beam undergoing purely flexural deformation. Its strain energy is given in nondimensional form by

$$U = \int_{o}^{1} \tfrac{1}{2}k(y'')^2 dx \tag{3.27}$$

We may relate this to the energy in appropriate dimensional units, $\overline{U}$, using the definitions of Eqs. (2.5).

$$\overline{U} = \frac{E_o I_o}{L} U$$

Differentiating the first of Eqs. (2.21) twice with respect to x, we obtain y".

$$y''(x,t) = \sum_{i=1}^{n} \varphi_i'' q_i$$

substituting this expression into Eq. (3.27), and using Eqs. (2.25)

$$U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \int_{o}^{1} k\varphi_i'' \varphi_j'' dx = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} q_i q_j$$

In matrix form,

$$U = \frac{1}{2} \lfloor q \rfloor [K] \{q\}$$

$$= \frac{1}{2} \lfloor Q \rfloor [A]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right] [A]\{Q\} \tag{3.28}$$

Finally, for the steady state response to the sinusoidal input of Eq. (3.20), we substitute Eq. (3.24) into Eq. (3.28),

$$U = \frac{1}{2}(U_1 \sin^2\Omega t + 2U_2 \sin\Omega t \, \cos\Omega t + U_3 \cos^2\Omega t)$$

$$U_1 = \lfloor 0 \,\vdots\, h \rfloor [A][G][A]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right] [A][G][A]^T \left\{\begin{array}{c} 0 \\ \hline h \end{array}\right\}$$

$$U_2 = \lfloor 0 \,\vdots\, h \rfloor [A][G][A]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right] [A][H][A]^T \left\{\begin{array}{c} 0 \\ \hline h \end{array}\right\} \quad (3.29)$$

$$U_3 = \lfloor 0 \,\vdots\, h \rfloor [A][H][A]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right] [A][H][A]^T \left\{\begin{array}{c} 0 \\ \hline h \end{array}\right\}$$

In nondimensional form, the rate at which work is done by the damping force is

$$\dot{W} = \int_0^1 (-cv)v\,dx \quad (3.30)$$

Using Eqs. (2.5), this may be transformed to the rate in dimensional units, $\dot{\overline{W}}$.

$$\dot{\overline{W}} = \frac{E_oI_o}{L^3} \sqrt{\frac{E_oI_o}{\mu_o}} \; \dot{W}$$

Substituting the second of Eqs. (2.21) into Eq. (3.30),

$$\dot{W} = - \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \int_0^1 c\varphi_i\varphi_j dx = - \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} r_i r_j$$

in matrix form

$$\dot{W} = - \lfloor Q \rfloor \, [A]^T \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} [A]\{Q\} \tag{3.31}$$

For harmonic input, we may substitute Eq. (3.24) into Eq. (3.31).

$$\dot{W} = - (W_1 \sin^2 \Omega t + 2W_2 \sin\Omega t \cos\Omega t + W_3 \cos^2 \Omega t)$$

$$\begin{aligned} W_1 &= \lfloor 0 \mid h \rfloor \, [A][G][A]^T \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} [A][G][A]^T \begin{Bmatrix} 0 \\ h \end{Bmatrix} \\ W_2 &= \lfloor 0 \mid h \rfloor \, [A][G][A]^T \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} [A][H][A]^T \begin{Bmatrix} 0 \\ h \end{Bmatrix} \\ W_3 &= \lfloor 0 \mid h \rfloor \, [A][H][A]^T \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} [A][H][A]^T \begin{Bmatrix} 0 \\ h \end{Bmatrix} \end{aligned} \tag{3.32}$$

Equations (3.28) and (3.32) show that the strain energy and energy dissipation rate are periodic functions of time after transient effects disappear. In general, we will be most interested in the maximum strain energy and the average power dissipation. We may write the following expressions for these quantities.

$$U_{max} = \frac{1}{2}\left(U_1 + U_3 + \sqrt{4U_2^{\,2} + (U_3 - U_1)^2}\,\right) \tag{3.33}$$

$$P_{av} = \frac{-1}{T} \int_0^T \dot{W} dt \ : \ T = \frac{2\pi}{\Omega}$$

Then,

$$P_{av} = \frac{1}{2} (W_1 + W_3) \tag{3.34}$$

# IV. APPLICATION TO A SELECTED EXAMPLE

## Description of the Problem

We will consider a uniform, simply supported beam with a single dashpot attached at the center of its span. Vibration of the beam is forced by sinusoidal translation of the supports without rocking. The configuration is illustrated in Fig. 4.1.

This example is simple enough to permit an exact solution for mode frequencies and functions; however, it exhibits the effects of concentrated damping as well as a more complicated case would. We will use both exact and approximate methods to find the normal mode parameters, but solutions for strain energy and average dissipated power will be obtained by Galerkin's method alone. The exact result would not be substantially better but would require considerable additional labor.

Before writing the equation of motion, the forcing input should be defined more precisely. We have assumed that the support motion is sinusoidal given by

$$\overline{Y}(x,t) = Y_0 \sin\overline{\Omega t}$$

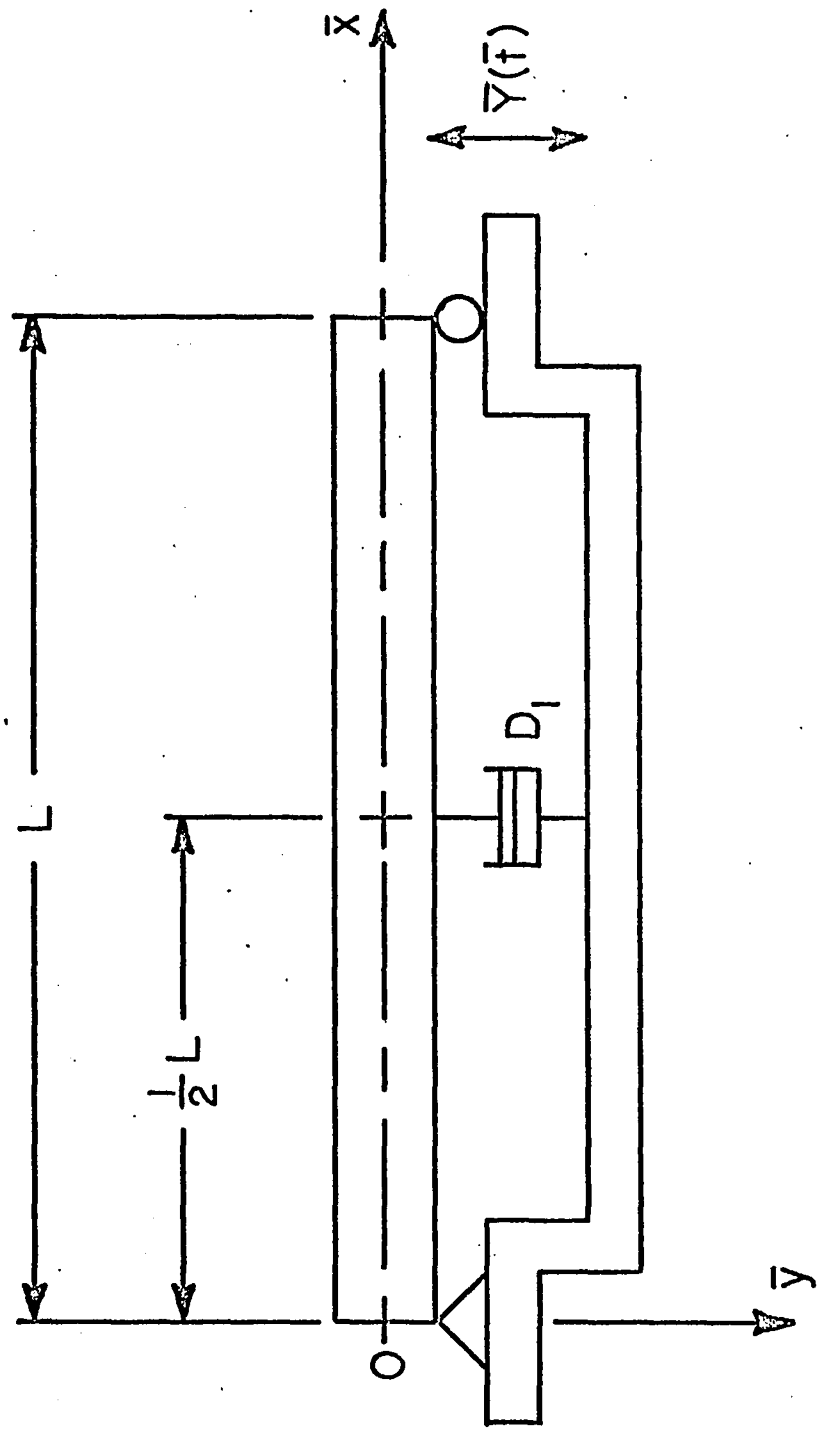



Fig. 4.1 Example Configuration

If we assume that the beam deflection is measured with respect to the support, the inertia term of Eq. (2.1) becomes

$$\mu \frac{\partial^2}{\partial \overline{t}^2} (\overline{y}+\overline{Y}) = \mu\left(\frac{\partial^2 \overline{y}}{\partial \overline{t}^2} - \overline{\Omega}^2 Y_0 \sin\overline{\Omega t}\right)$$

We return the differential equation to the form of Eq. (2.1) by defining $\overline{f}(\overline{x},\overline{t})$ as follows.

$$\overline{f} = \mu \,\overline{\Omega}^2 Y_0 \sin\overline{\Omega t} \tag{4.1}$$

To put Eq. (4.1) in nondimensional form, we observe that the argument of the sine function must be dimensionless. Then, using Eqs. (2.5), we have

$$\Omega = \frac{\overline{\Omega t}}{t} = \sqrt{\frac{\mu_o}{E_o I_o}}\; L^2 \overline{\Omega} \tag{4.2}$$

and

$$f = \frac{m\Omega^2 Y_o}{L} \sin\Omega t \tag{4.3}$$

To simplify the following analysis, we will assume the magnitude of $Y_o$ is such that the coefficient in Eq. (4.3) is equal to one.

With the foregoing assumptions, the nondimensional equation of motion, Eq. (2.3), for the beam illustrated in Fig. 4.1 becomes

$$\frac{\partial^4 y}{\partial x^4} + c_1 \delta(x-\tfrac{1}{2}) \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} = \sin\Omega t \tag{4.4}$$

The reduced form corresponding to Eqs. (2.7) is

$$\frac{\partial y}{\partial t} - v = 0$$

$$\frac{\partial^4 y}{\partial x^4} + c\ \delta\ (x-\tfrac{1}{2})\ \frac{\partial y}{\partial t} + \frac{\partial v}{\partial t} = \sin\Omega t \tag{4.5}$$

The boundary conditions for simply supported ends, Eq. (2.4b), become

$$y = \frac{\partial^2 y}{\partial x^2} = 0,\ x = 0,\ 1 \tag{4.6}$$

## Closed Form Solution for Free Vibration

We define the deflection by two functions, one for the portion of the beam to the left of the dashpot and one to the right. Each function obeys a differential equation without damping included, Eqs. (4.7).

$$\frac{\partial^4 y_1}{\partial x^4} + \frac{\partial^2 y_1}{\partial t^2} = \sin\Omega t,\ 0 < x < \tfrac{1}{2}$$

$$\frac{\partial^4 y_2}{\partial x^4} + \frac{\partial^2 y_2}{\partial t^2} = \sin\Omega t,\ \tfrac{1}{2} < x < 1 \tag{4.7}$$

We may connect the resulting simple solutions through appropriate continuity and shear conditions at the dashpot location, Eqs. (4.8) and (4.9). The first equation expresses continuity of displacement, slope and moment;

the second relates the jump in shear to the concentrated force exerted by the dashpot.

Continuity

$$y_1 - y_2 = \frac{\partial y_1}{\partial x} - \frac{\partial y_2}{\partial x} = \frac{\partial^2 y_1}{\partial x^2} - \frac{\partial^2 y_2}{\partial x^2} = 0,$$

$$x = \frac{1}{2} \tag{4.8}$$

Shear

$$\frac{\partial^3 y_1}{\partial x^3} - \frac{\partial^3 y_2}{\partial x^3} = c \frac{\partial y_1}{\partial t}, \qquad x = \frac{1}{2} \tag{4.9}$$

For convenience we have dropped the subscript from c.

We consider the homogeneous form of Eqs. (4.7) and assume the deflection functions are given in a normal mode series.

$$y_1 = \sum_{i=1}^{\infty} \Phi_{1i}(x)\, e^{\alpha_i t}, \quad 0 \le x \le \frac{1}{2}$$

$$y_2 = \sum_{i=1}^{\infty} \Phi_{2i}(x)\, e^{\alpha_i t}, \quad \frac{1}{2} \le x \le 1 \tag{4.10}$$

The normal mode parameters must each satisfy ordinary differential equations of the following form; the mode subscripts will be dropped for clarity.

$$\frac{d^4 \Phi_1}{dx^4} + \alpha^2 \Phi_1 = 0$$

$$\frac{d^4 \Phi_2}{dx^4} + \alpha^2 \Phi_2 = 0 \tag{4.11}$$

The continuity and shear conditions are

$$\Phi_1\ (\tfrac{1}{2}) - \Phi_2\ (\tfrac{1}{2}) = 0$$

$$\Phi_1'(\tfrac{1}{2}) - \Phi_2'(\tfrac{1}{2}) = 0$$

$$\Phi_1''(\tfrac{1}{2}) - \Phi_2''(\tfrac{1}{2}) = 0$$

$$\Phi_1'''(\tfrac{1}{2}) - c\alpha\Phi_1(\tfrac{1}{2}) - \Phi_2'''(\tfrac{1}{2}) = 0 \qquad (4.12)$$

The solution of Eqs. (4.11) is easily found. After applying the boundary conditions at x = 0,1; we have

$$\Phi_1 = A_1 \sin 2\beta x + B_1 \sinh 2\beta x$$

$$\Phi_1 = A_2 \sin 2\beta(1-x) + B_2 \sinh 2\beta(1-x) \qquad (4.13)$$

$$\alpha = \hat{i}\ (2\beta)^2,\ \hat{i} = \sqrt{-1}$$

Substituting Eqs. (4.13) into Eqs. (4.12) produces a set of four homogeneous algebraic equations in four undetermined constants.

In matrix form,

$$\begin{bmatrix} \sin\beta & -\sin\beta & \sinh\beta & -\sinh\beta \\ \cos\beta & \cos\beta & \cosh\beta & \cosh\beta \\ \sin\beta & \sin\beta & \sinh\beta & -\sinh\beta \\ G & -\cos\beta & H & \cosh\beta \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix} = \{0\} \qquad (4.14)$$

where

$$G = -\cos\beta - \frac{\hat{i}c}{2\beta}\sin\beta$$

$$H = \cosh\beta - \frac{\hat{i}c}{2\beta}\sinh\beta$$

We must now seek the values of $\alpha$ which permit non-zero constants. Accordingly, we require that the determinant of the matrix in Eq. (4.14) be zero. Expanding that determinant leads directly to the characteristic equation.

$$\sin\beta \sinh\beta\left[\frac{\hat{i}c}{\beta}(\sinh\beta \cos\beta - \sin\beta \cosh\beta)\right.$$

$$\left. - 8 \cos\beta \cosh\beta\right] = 0 \qquad (4.15)$$

Before examining the implications of Eq. (4.15), let us define real variables representing the real and imaginary parts of $\alpha$ and $\beta$.

$$\alpha = -\nu + \hat{i}\omega$$
$$\beta = a + \hat{i}b \qquad (4.16)$$

We will call $\nu$ the attenuation and $\omega$ the circular frequency. They are related to a and b by Eq. (4.17).

$$\nu = 8ab$$
$$\omega = 4(a^2 - b^2) \qquad (4.17)$$

Equation (4.15) will be satisfied if either

$$\sin\beta \ \sinh\beta = 0 \tag{4.18}$$

or

$$c = - 8\hat{i}\beta \frac{\cos\beta \ \cosh\beta}{\sinh\beta\cos\beta - \sin\beta\cosh\beta} \tag{4.19}$$

Equation (4.18) leads to purely real or imaginary values of $\beta$. Then, either of the following occurs.

$$a = n\pi, \ b = 0, \ \nu = 0, \ \omega = (2n\pi)^2$$

or

$$a = 0, \ b = n\pi, \ \nu = 0, \ \omega = - (2n\pi)^2$$

$$n = 1,2,3, \ . \ . \ .$$

These cases yield the antisymmetric modes of the undamped beam; each of these modes has a node at midspan and is therefore not affected by the dashpot. Since the assumed load does not excite the antisymmetric modes, we will neglect them.

The symmetric modes of the beam are established by solving Eq. (4.19). The limiting cases are zero and infinite damping. In both cases, $\beta$ is again either real or imaginary; the symmetric modes are respectively those of the undamped simple beam and an undamped beam on three equally spaced supports.

For finite, non-zero values of c, the right-hand side of Eq. (4.19) is a complex function. Since c is real, admissible values of $\beta$ are those which make the imaginary part of this function zero. Then, the real part gives the corresponding value of c. We expand the equation in terms of a and b and set the imaginary part equal to zero. Equations (4.20) define c.

$$
\left.
\begin{aligned}
c &= 4\,\frac{G(a,b)}{H(a,b)} \\
G(a,b) &= (\cosh 2b + \cos 2a)(b \sin 2a - a \sin 2b) \\
&\quad + (\cosh 2a + \cos 2b)(a \sin 2a - b \sin 2a) \\
H(a,b) &= \cosh 2a \cosh 2b - \cos 2a \cos 2b \\
&\quad - \sin 2a \sinh 2a - \sin 2b \sinh 2b
\end{aligned}
\right\} \qquad (4.20)
$$

The parts of $\beta$ must satisfy

$$
\begin{aligned}
&(\cosh 2a + \cos 2b)(a\sin 2a + b\sinh 2b) \\
&\quad - (\cosh 2b + \cos 2a)(b\sin 2b + a\sinh 2a) = 0
\end{aligned}
\qquad (4.21)
$$

After Eqs. (4.20) and (4.21) are solved by some means, we may determine the mode parameters by substituting the solutions (a,b) into Eqs. (4.13) and (4.16). The constants in Eqs. (4.13) are determined by Eq. (4.14). We may express the other three constants in terms of $A_1$ and arbitrarily make $A_1 = 1$; the mode functions are defined by the following.

$$\Phi_{1i} = \sin 2\beta_i x - \frac{\cos\beta_i}{\cosh\beta_i} \sinh 2\beta_i x$$

$$\Phi_{2i} = \sin 2\beta_i (1-x) - \frac{\cos\beta_i}{\cosh\beta_i} \sinh 2\beta_i (1-x) \qquad (4.22)$$

$$\beta_i = a_i + \hat{i} b_i$$

Equation (4.21) must be solved by numerical means. If we survey a range of values for a and b and calculate the function on the left-hand side, we may estimate the loci by noting where the sign changes. Examining the forms of Eqs. (4.20) and (4.21) will be helpful. If $(a_i, b_i)$ is a solution of these equations, $(b_i, a_i)$, $(-a_i, -b_i)$, and $(-b_i, -a_i)$ are also solutions for the same c. If we take each of these solutions into Eqs. (4.17) and (4.22), we observe that the second solution yields mode frequencies and functions which are complex conjugates of those given by the first solution. The third and fourth solutions repeat the results of the first two with only the sign of the mode functions changed. Finally, if $a_i$ and $b_i$ have opposite signs, the attenuation would be negative, implying unstable free-vibration which is not possible for the assumed model. Consequently, solutions of Eqs. (4.20) and (4.21) for values of a and b lying in the region bounded by the positive a axis and the line, a=b, determine all the physically possible sets of mode parameters. The line, a=b, is itself a solution of Eq. (4.21). In this case, the circular frequency is zero; and the solution represents heavy damping in the fundamental mode. For this solution, Eq. (4.20) reduces to

$$c = 8a \frac{\cosh 2a + \cos 2a}{\sinh 2a - \sin 2a} \tag{4.23}$$

## Galerkin's Method

The first and most important task is to select an appropriate set of approximating functions. After this is done, the rest of the solution proceeds in a straightforward and orderly manner.

For the problem considered here, we will use the mode functions of the equivalent undamped beam. This choice is attractive for several reasons. These functions satisfy the boundary conditions and are simple in form. The necessary matrices will be diagonal, which is certainly convenient. Accordingly, the approximating functions are:

$$\varphi_i(x) = \sin(i\pi x)$$

This set includes the antisymmetric modes; in the previous section we remarked that they were unaffected by either the damping or the loading assumed here. Consequently, we consider only the symmetric functions.

$$\varphi_i(x) = \sin(m_i \pi x)$$
$$m_i = 2i - 1 \tag{4.24}$$

We proceed directly to determine the elements of the system matrices using the definitions of Eqs. (2.25), (2.26), and (3.21).

$$C_{ij} = C_{ji} = c \sin(\frac{m_i\pi}{2}) \sin \frac{m_j\pi}{2}) = (-1)^i(-1)^j c$$

$$M_{ij} = M_{ji} = \int_0^1 \sin(m_i\pi x)\sin(m_j\pi x)dx = \tfrac{1}{2}\delta_{ij}$$

$$K_{ij} = K_{ji} = \int_0^1 (m_i\pi)^4 \sin(m_i\pi x)\sin(m_j\pi x)dx = \tfrac{1}{2}(m_i\pi)^4\delta_{ij}$$

$$h_j = \int_0^1 \sin(m_j\pi x)dx = -\frac{1}{m_j\pi}[\cos(m_j\pi)-1] = \frac{2}{m_j\pi}$$

$$\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} \tag{4.25}$$

Finally, in explicit matrix form,

$$[C] = c \begin{bmatrix} 1 & -1 & 1 & \dots \\ -1 & 1 & -1 & \dots \\ 1 & -1 & 1 & \dots \\ & \dots & & \end{bmatrix}$$

$$[M] = \frac{1}{2} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 1 \end{bmatrix} = \frac{1}{2}[I]$$

$$[K] = \frac{\pi^4}{2} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 81 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & & \dots & m_n^4 \end{bmatrix} \tag{4.26}$$

$$\{h\} \quad \frac{2}{\pi} \begin{Bmatrix} 1 \\ 1/3 \\ \vdots \\ 1/m_n \end{Bmatrix}$$

The equation of motion becomes

$$\left[\begin{array}{c|c} 0 & M \\ \hline M & C \end{array}\right] \left\{\begin{array}{c} \dot{r} \\ \hline \dot{q} \end{array}\right\} + \left[\begin{array}{c|c} -M & 0 \\ \hline 0 & M \end{array}\right] \left\{\begin{array}{c} r \\ \hline q \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline h \end{array}\right\} \sin\Omega t \tag{4.27}$$

With these matrices determined, we may calculate the mode vectors and frequencies for various values of c by the iterative process based on Eq. (3.12). The steady state solutions for displacement, strain energy, or power dissipation are easily obtained from the equations presented in Chapter III.

# V. NUMERICAL RESULTS

## Attenuation and Frequency

The characteristic equation, Eq. (4.15), was solved by assuming values of a and increasing b in steps until the sign of the left side of the equation changed. The estimates of b were improved by reducing the step size and examining the interval between the last two estimates. The process was repeated over successively smaller intervals until it reached two values of b which surrounded the solution and differed by less than .001%. Figure 5.1 shows some of the solutions of Eq. (4.15).

As the dashpot strength goes up, the curves representing the first symmetric mode intersect the line, $a=b$. At this intersection, the first mode is critically damped; its free vibration passes from a decaying oscillation to exponential decay. The corresponding dimensionless value of the dashpot strength is: $c \approx 9.59$.

As damping increases beyond this value, the first mode solution has two branches which follow the line, $a=b$. One branch moves from the intersection toward the origin; the second moves outward from the intersection.

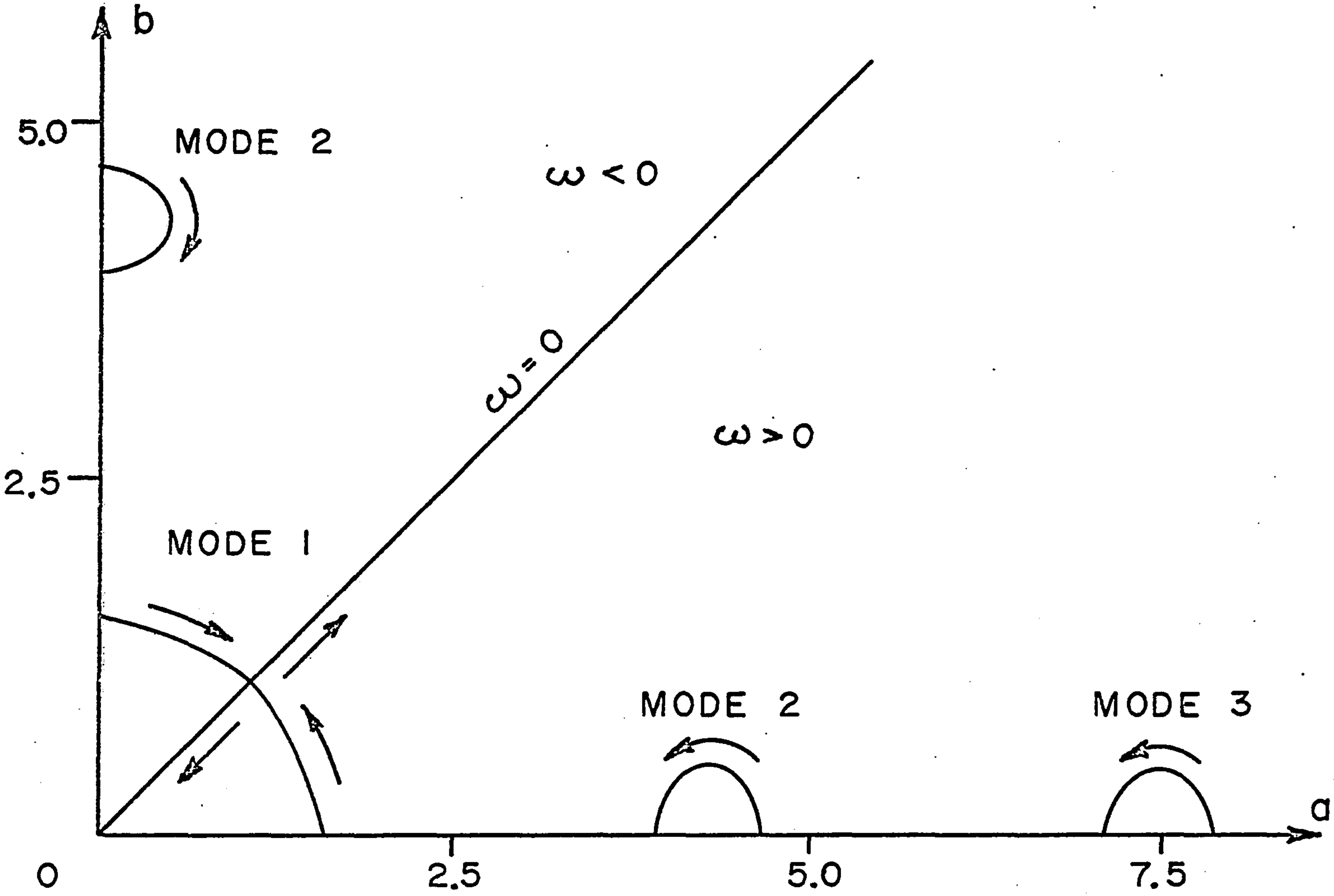



Fig. 5.1 Solutions of the Characteristic Equation

Arrows indicate the direction of the solution as damping increases.

The other modes are not critically damped; their free vibration will be oscillatory for any level of dashpot strength. As Fig. 5.1 shows, their solutions finally return to the a and b axes. In contrast to this situation, if the damping is distributed along the entire length of the beam, all the modes become critically damped at sufficiently high values of the damping parameter. This suggests that to critically damp more of the normal modes, additional dashpots must be employed.

With the solutions of the frequency equation established, we may plot the variations in mode parameters with dashpot strength. Figure 5.2 is a plot of the attenuation in the first four symmetric modes for light damping, i.e., less than the critical value for the first mode. It is interesting that attenuation in all the modes increases almost linearly with the damping. We ordinarily expect free vibration of the higher modes to damp out more rapidly than that of the fundamental mode. Contrary to this expectation, Fig. 5.2 shows little difference in attenuation among the four modes at low damping levels. At slightly higher values of damping, the second mode shows the lowest attenuation; while the fundamental mode has the highest attenuation and will consequently damp out more rapidly than the others.

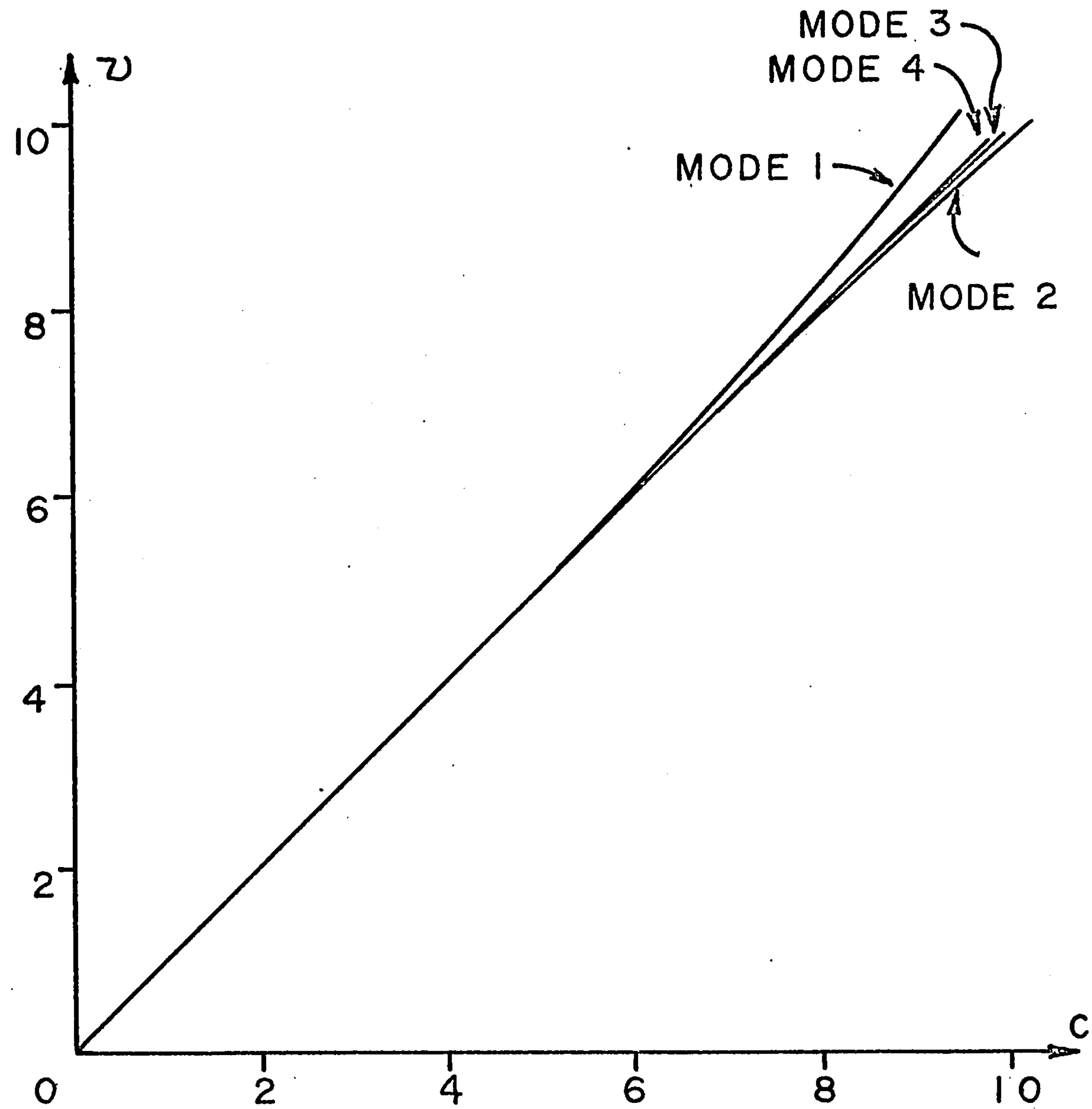



Fig. 5.2 Attenuation for Light Damping

Figure 5.3 plots the circular frequencies of the same modes for light damping. Here, the effect of damping is greater in the first mode, decreasing as the mode number goes up.

In Fig. 5.4 and Fig. 5.5, which extend over a larger range of damping, we can see how the attenuation and frequency approach a limiting case where the dashpot begins to act like a rigid support. Figure 5.4 shows the branching of the attenuation curves for the lowest mode; one branch approaching zero, the other increasing with the damping. In contrast, the attenuation for the other modes rises to a maximum and then gradually drops until it begins to approach the c axis asymptotically. The peak attenuation and the value of dashpot strength at which it occurs both increase as the mode number goes up.

Figure 5.5 shows the variation of circular frequency in the second mode; the curve is also representative of the higher modes. We notice that the curve takes a knee shape, which asymptotically approaches the frequency of the lowest symmetric mode of a beam on three supports ($\omega = 61.67$).

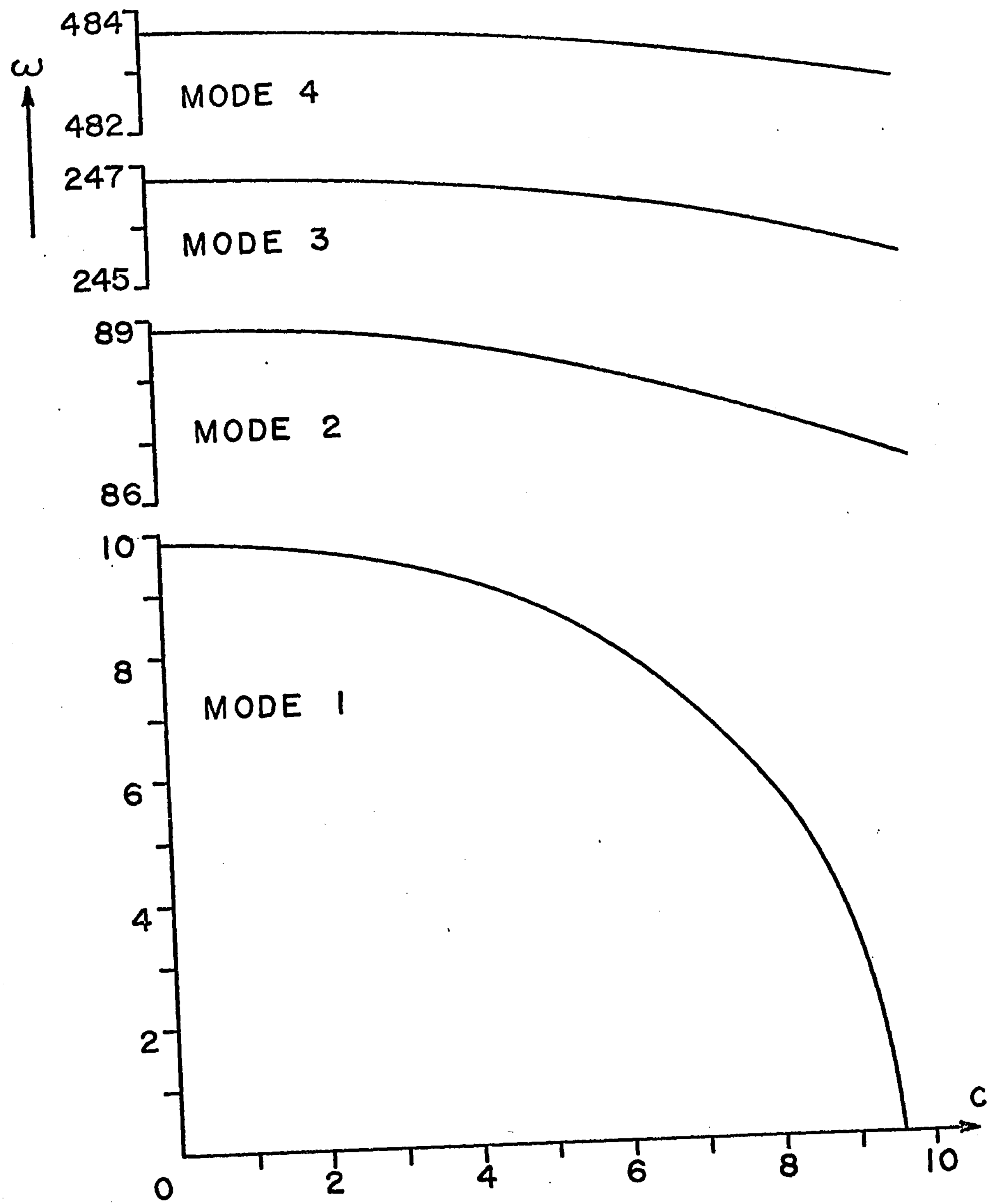



Fig. 5.3 Frequency for Light Damping

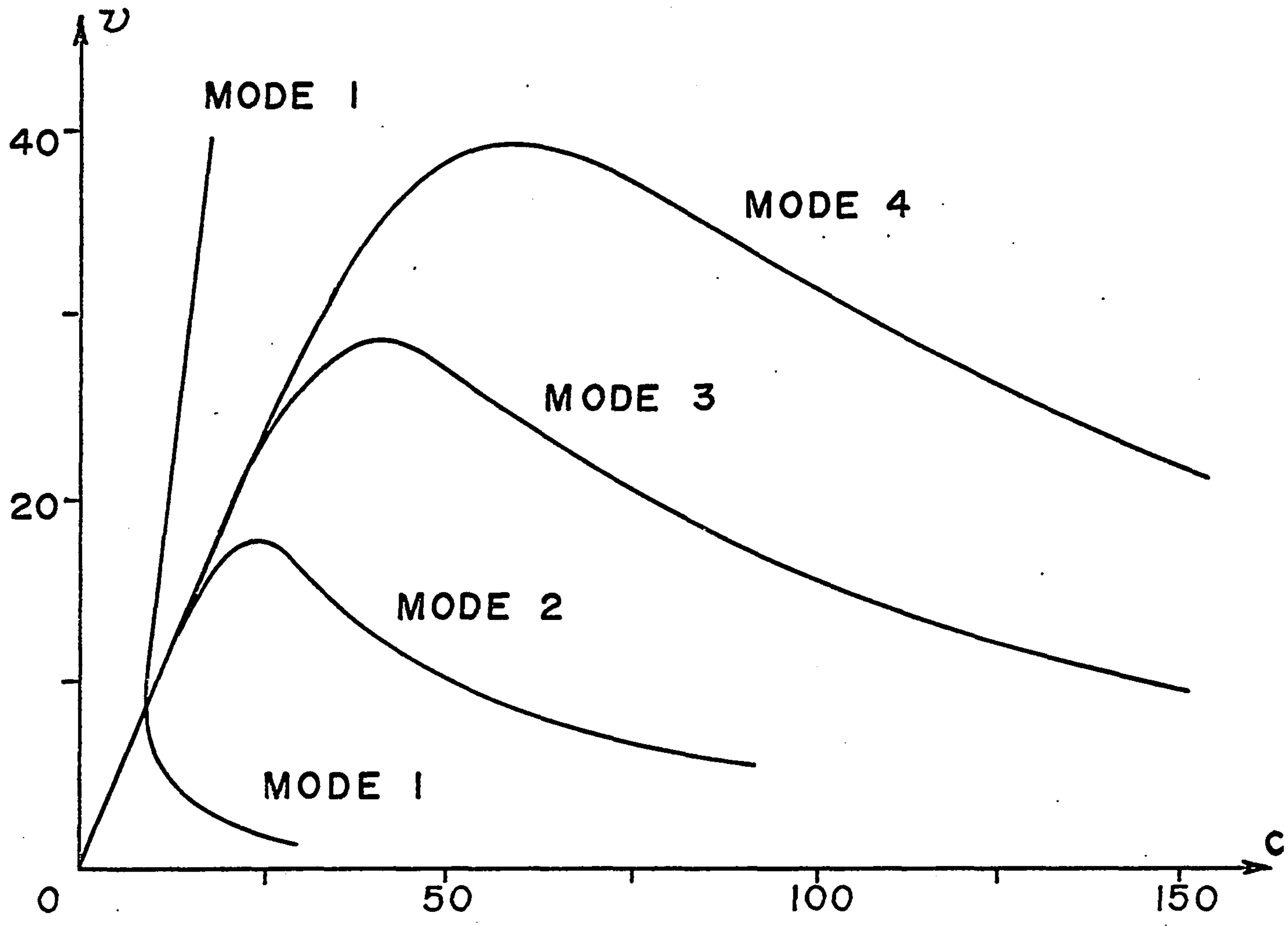



Fig. 5.4 Attenuation for a Wide Range of Dashpot Strength

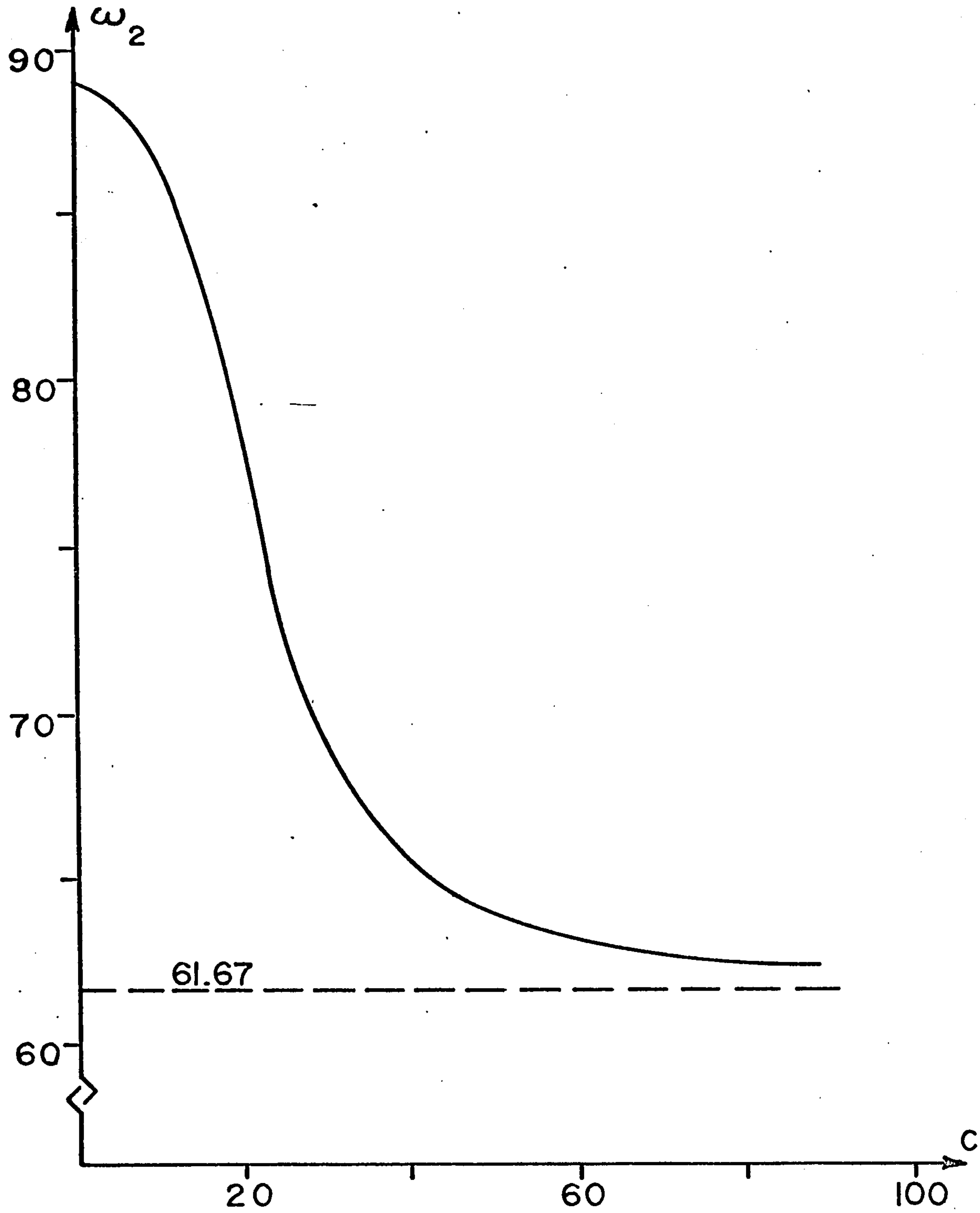



Fig. 5.5 Frequency Variation of the Second Symmetric Mode

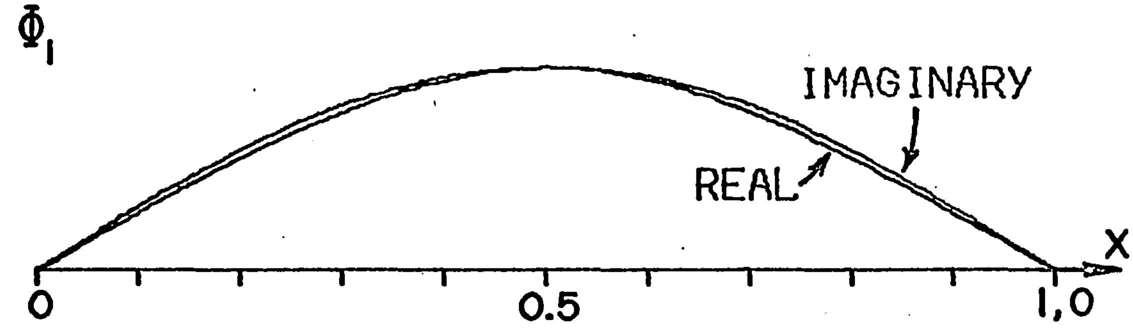
## Mode Functions

Figure 5.6 shows the normalized complex mode functions for the first two modes at a selected value of dashpot strength (c=6). The functions were normalized so that the real and imaginary parts are equal at the midpoint. In the first mode, the real and imaginary parts are nearly identical and each looks very much like the undamped normal mode function. In the second mode, the two parts diverge noticeably, particularly at the peaks away from the midpoint.

To understand the physical significance of these complex mode functions, it will be helpful to examine them from a somewhat different viewpoint. From the form of the solutions derived in Chapter III, we may verify that, for real initial conditions and applied force, the normal coordinates of a conjugate pair of modes are also complex conjugates. Consequently, the sum of the responses of this pair of modes is a real function of the spatial coordinate and time.

Let us assume that we have imposed some initial conditions which excite only the $i^{th}$ mode and its conjugate. From Eqs. (3.1) and (3.7), the resulting displacement is

$$y(x,t) = e^{-\nu_i t}[RE(\Phi_i)\cos\omega_i t - IM(\Phi_i)\sin\omega_i t] \tag{5.1}$$



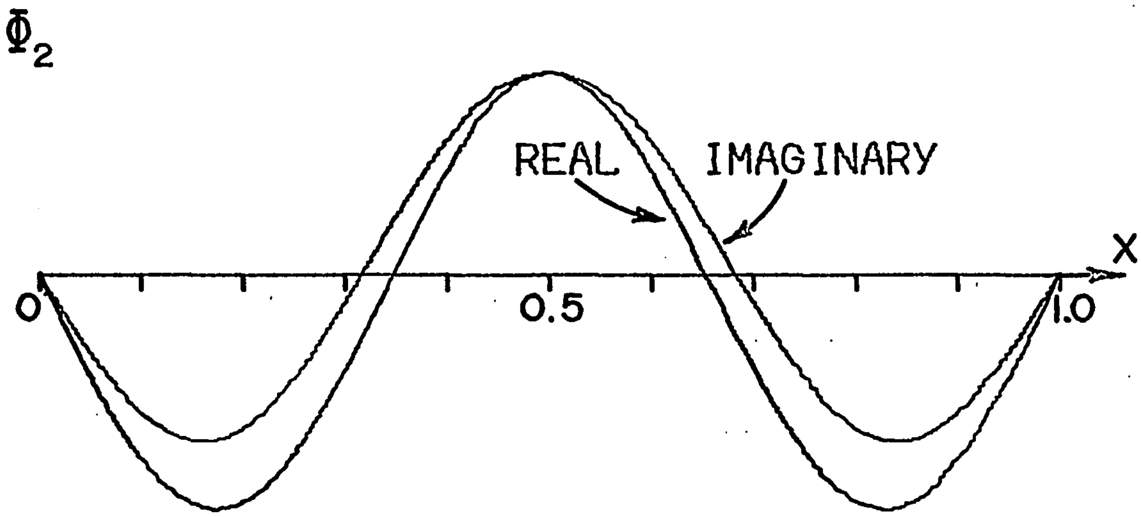




Fig. 5.6 Typical Mode Functions for the First Two Symmetric Modes

For simplicity, we have absorbed the initial values of the normal coordinates into the mode functions. Since the relative values of real and imaginary parts of the mode function vary along the beam, we conclude that the phase of the resulting vibration varies with x. Let us write $\Phi_i(x)$ in the form,

$$\Phi_i(x) = \left|\Phi_i(x)\right| \quad [\cos\theta_i(x) + \hat{i}\,\sin\theta_i(x)]$$

Putting this into Eq. (5.1), the displacement becomes

$$y(x,t) = \left|\Phi_i(x)\right| \; e^{-\nu_i t} \; \cos[\omega_i t + \theta_i(x)] \qquad (5.2)$$

From a physical standpoint, the interesting properties of a complex mode function are its magnitude, which describes the amplitude of the motion at each point along the beam, and its argument, which indicates the phase lead angle at each point. In Figs. 5.7 and 5.8, the magnitude and phase angle of the first two modes are plotted for several values of light damping. The curves have been normalized so that the magnitude is one and the phase angle is zero at midspan.

From Fig. 5.7, we see that the first mode function changes very little under light damping. The change in phase for this mode is quite small. For the highest value of dashpot strength shown (c=9), the phase angle curve falls below the intermediate curve (c=5). This suggests

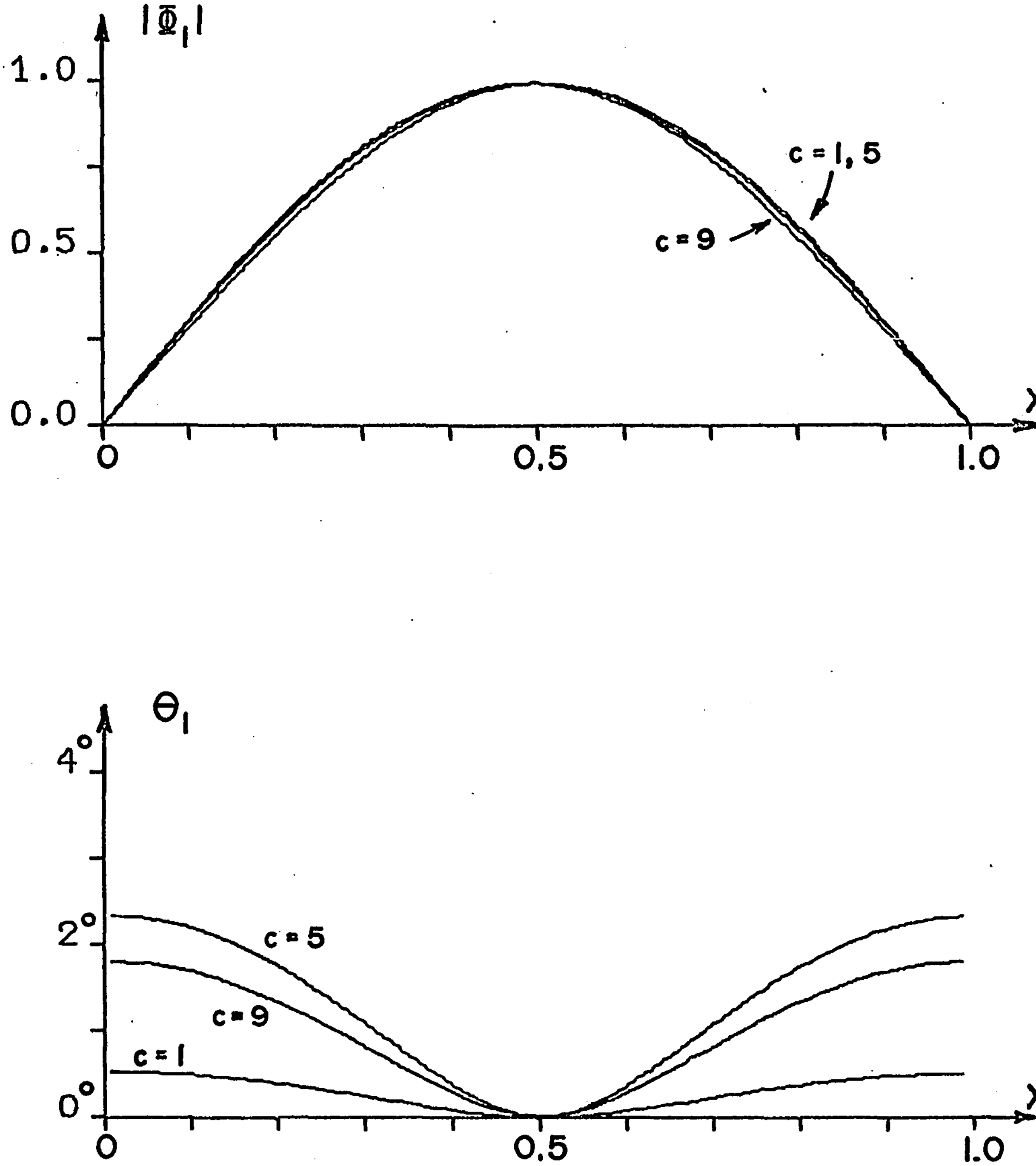
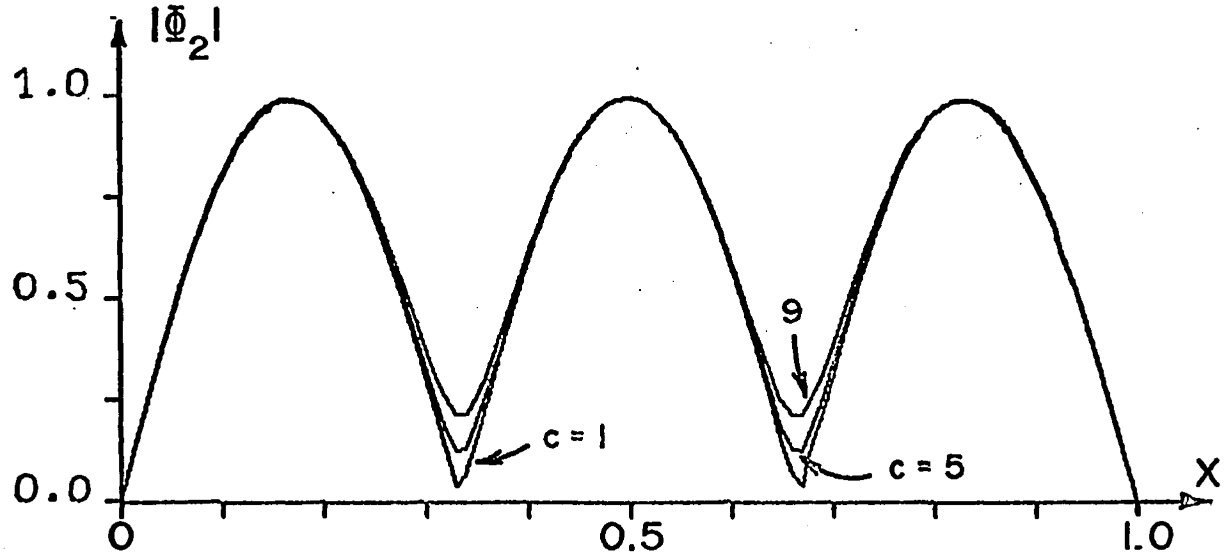



Fig. 5.7 Effect of Dashpot Strength on the First Mode Function



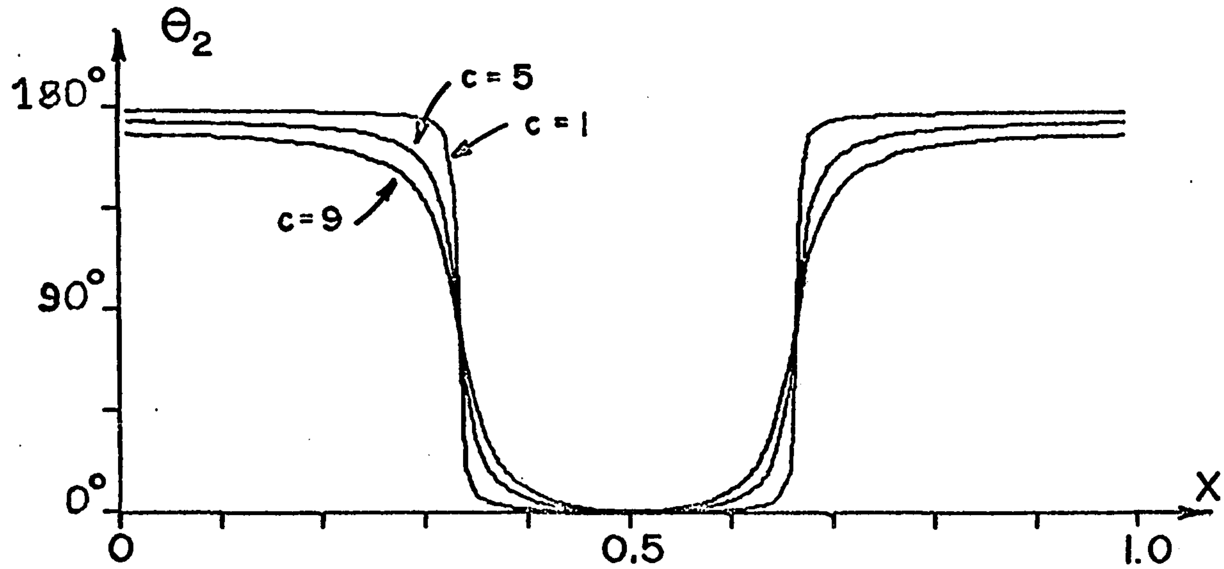




Fig. 5.8 Effect of Dashpot Strength on the Second Mode Function

that at critical damping, the first mode function is again real as in the undamped case. Examining Eqs. (4.22) for the case, a = b, it is possible to verify that the normalized mode function is real for any damping at or above the critical value.

In Fig. 5.8, the effect of damping on the second mode function is most noticeable near the nodes of the undamped mode. With damping, the nodes disappear; and as the dashpot strength goes up, the displacement near these points also grows. At other points along the beam, the magnitude of the function is not greatly affected by the damping. The changes in phase angle are rather straightforward. As damping is increased, the phase curves depart further from the undamped case, for which each point of the beam moves either in phase or 180 degrees out of phase with respect to the reference point.

## Comparison with Results by Galerkin's Method

Complex frequencies and mode functions obtained by Galerkin's method agreed very closely with the exact results. Values for the first four modes at selected dashpot strengths were determined by matrix iteration, using (in turn) the first 4, 5, 6, and 7 approximating functions. Appendix B describes the mechanics of the iteration process.

To compare attenuation and circular frequency calculated at both methods, the ratios of values determined by Galerkin's method to the corresponding exact parameters were plotted as functions of dashpot strength. Figure 5.9 and Fig. 5.10 show comparisons of parameters of the first and fourth modes for light damping and several choices of n, the number of approximating functions used.

We observe that the solutions for successive values of n tend to converge as n goes up. This convergence is an important characteristic of Galerkin's method, which permits us to estimate the number of approximating functions needed. If we construct a sequence of solutions for increasing n and continue until the difference between two successive solutions is negligible, the convergence property suggests that increasing n further will not significantly affect the result.

In Fig. 5.9, the circular frequency ratios increase sharply near critical damping. The reason for this increase is not certain; however, the first mode frequency is approaching zero in this region, and the relative effect of small differences may be exaggerated by taking the ratios.

Using the mode functions of the undamped beam in the Galerkin's method solution, we would expect agreement with the exact solution to be best for small damping and to

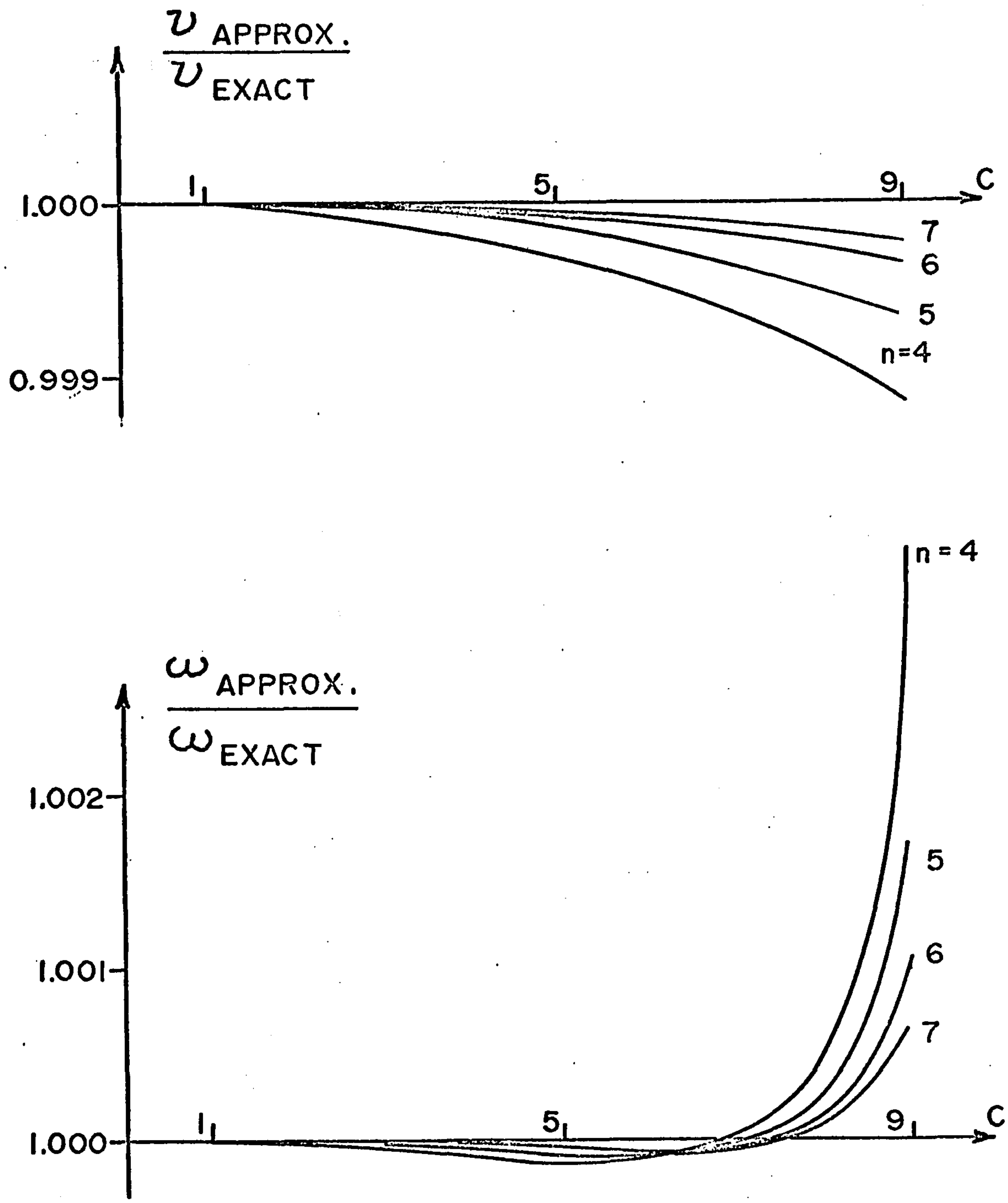



Fig. 5.9 First Mode Attenuation and Frequency Ratios

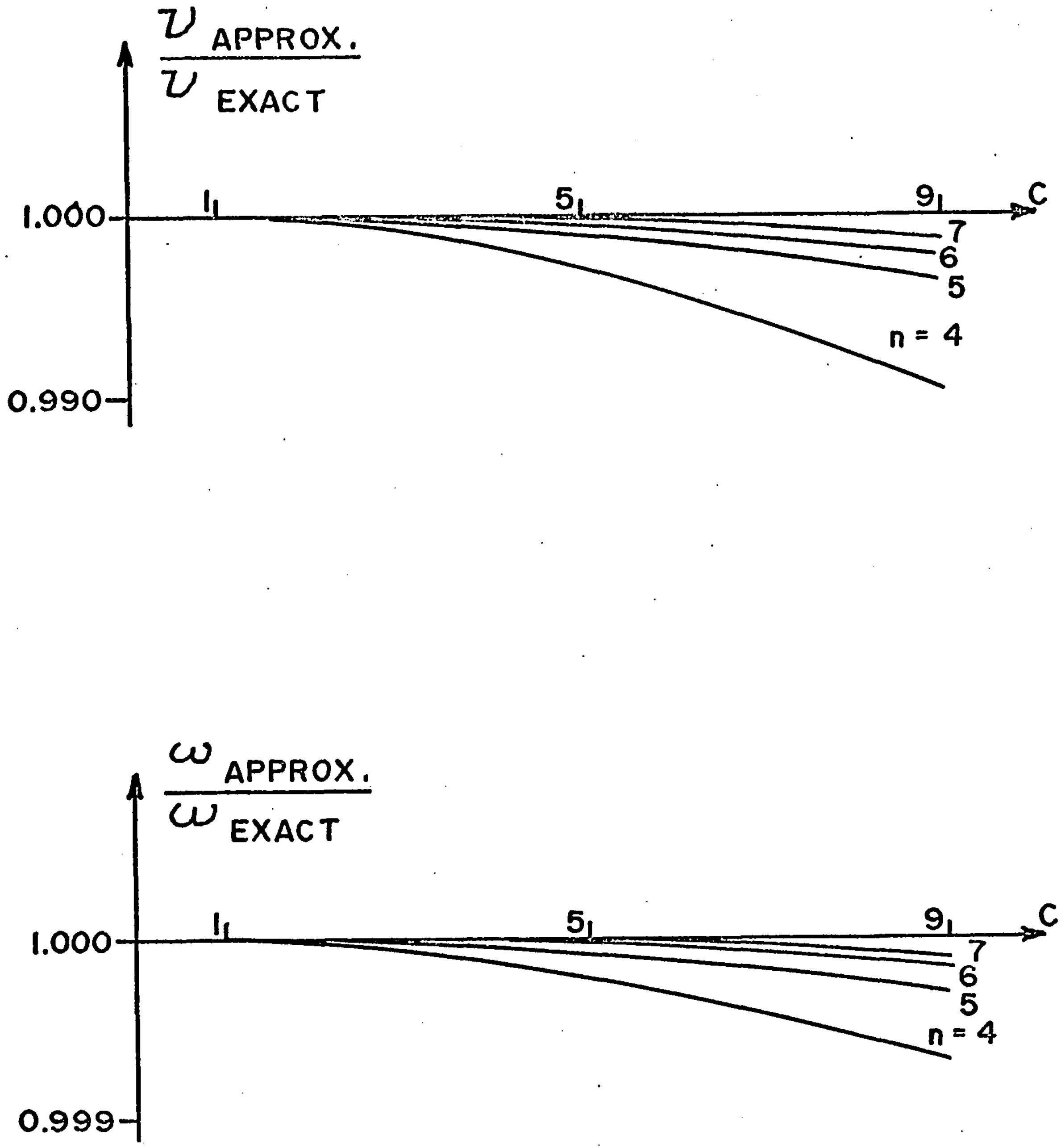



Fig. 5.10 Attenuation and Frequency Ratios for the Fourth Symmetric Mode

deteriorate as dashpot strength goes up. The curves shown in Figs. 5.9 and 5.10 support this assumption and also indicate that using a greater number of approximating functions both improves the agreement and reduces its variation with damping level.

Since the sine functions used here cannot exactly account for the shear discontinuity at the dashpot, convergence of the approximate solution will become rather poor as damping is increased. To improve the convergence under these conditions, we may include an additional approximating function for which the associated shear function has a step discontinuity. A suitable choice would be the static deflection curve of the beam with a concentrated force applied at the location of the damper.

## Forced Motion Results

Maximum strain energy and average power dissipation for a range of input frequencies were calculated using Eqs. (3.33) and (3.34). Figure 5.11 and Fig. 5.12 show curves of these quantities for several values of dashpot strength. In Fig. 5.11, strain energy curves for the undamped beam and the beam on three supports have been added for comparison. In both sets of curves, the acceleration amplitude of the support motion is assumed to be the same

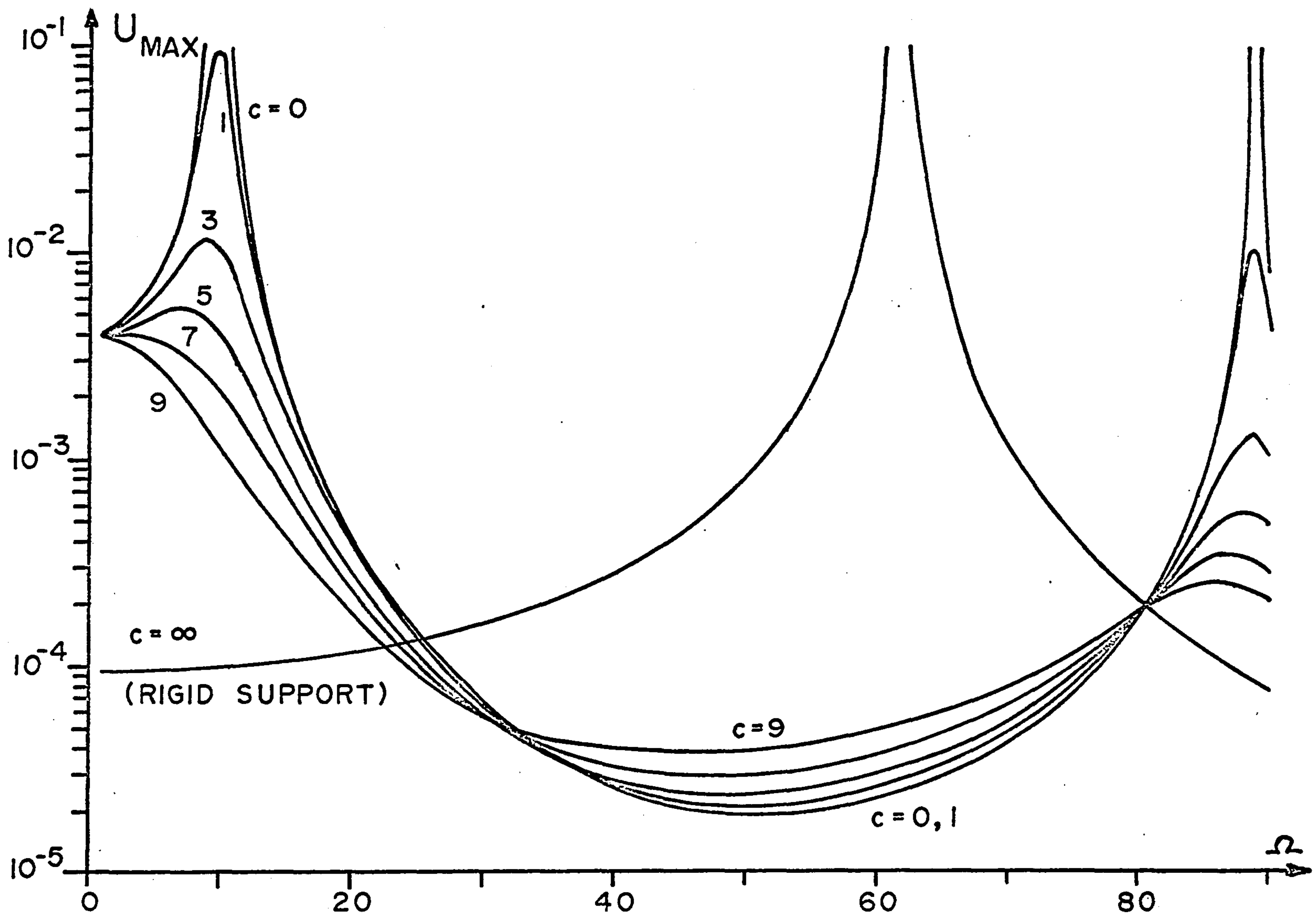



Fig. 5.11 Maximum Strain Energy

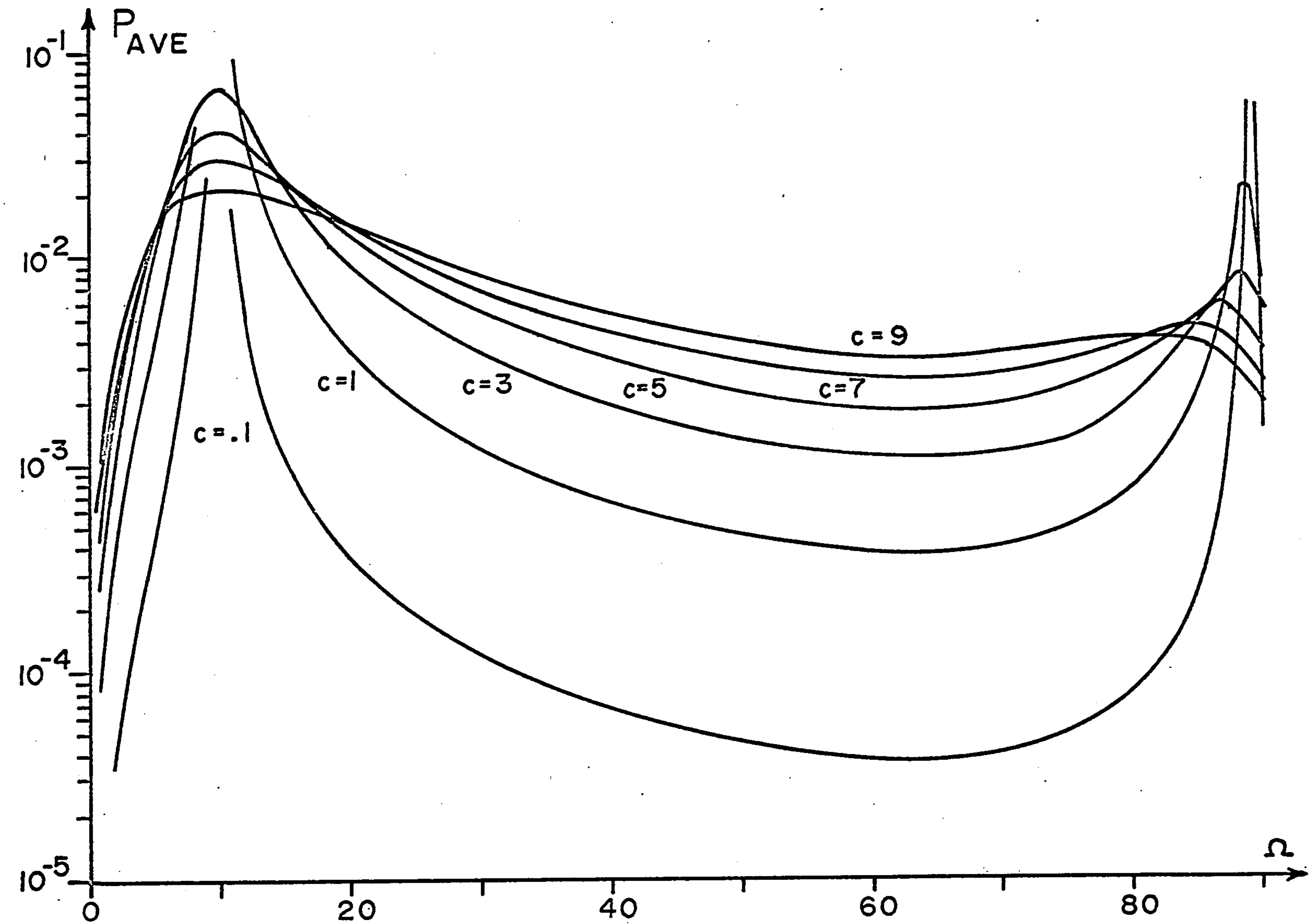



Fig. 5.12 Average Power Dissipation

for any input frequency. Other input limitations could be considered without serious difficulty using the same general equations.

Figure 5.11 shows clearly that the sensitivity of the strain energy to changes in the dashpot strength depends strongly on the input frequency. If we wished to control the response of the beam by adding concentrated damping, our strategy and chances of success will be influenced by the frequency characteristics of the input. For example, in a relatively low frequency range, we may suppress the strain energy most effectively by replacing the dashpot with a rigid support.

The variation of power dissipation with damping also depends on input frequency. Obviously, the power must be zero for the limiting cases and have a maximum for some intermediate value of damping. The curves show that the maxima near the resonance peaks are greater and occur at smaller values of dashpot strength than at other values of input frequency.

For this steady state solution, the power dissipation curve also represents the power being supplied at the support to produce the displacement input.

# VI. CONCLUSIONS

## Methods of Solution

Exact and approximate normal mode solutions for the flexural vibrations of damped beams have been shown. The exact solution is possible in some cases, but it is not very convenient. Even in the rather simple problem considered here, deriving the exact frequency equation required a great deal of tedious algebra.

The approximate solution by Galerkin's method yielded quite accurate results with a relatively small amount of labor. This approach may be applied in a straightforward and orderly manner to a wide variety of assumed damping distributions. The only changes would be in calculating the elements of the damping matrix. For concentrated damping, this is particularly easy since the need for integrating products of several functions is eliminated.

In practical problems, stiffness and inertia distributions are frequently complicated enough to require approximate solutions even for undamped problems. If viscous damping is present in such cases, the approximate solution derived here is particularly attractive.

## Normal Mode Characteristics

It seems reasonable that complex normal mode solutions for problems with viscous damping may have been neglected primarily because of two factors. First, the increase in the number of coordinates and the difficulty of determining mode functions make this approach appear extremely complicated. Second, the physical interpretation of complex mode functions is not so readily apparent as in the case of the usual real functions.

The first factor becomes less important for many problems with the use of approximate methods, such as the one described here, and modern digital computers to carry out the necessary calculations.

We may clarify the meaning of complex mode functions by considering each pair of complex conjugate modes as a mathematically convenient representation of the activity of one physically possible mode of free damped vibration. In this "physical" mode, each part of the system executes damped motion with the same attenuation and circular frequency but, in general, differing phase and amplitude. Systems which may be uncoupled by the classical normal modes become merely a special case where the phase of the motion of each point is the same. Undamped beams are another specialization, characterized by uniform phase and zero attenuation.

## Concentrated Damping

The solution of the example considered here revealed two interesting properties of a beam with concentrated viscous damping.

First, for the beam with a simple dashpot, only the fundamental mode has associated with it a critical level of damping, above which its characteristic free motion changes from damped oscillation to exponential decay. Each of the other modes exhibits oscillatory free motion at all values of dashpot strength.

Second, even below the critical value of damping, the fundamental has the highest attenuation of the modes examined. This implies that the free motion of the fundamental will subside before that of the higher modes. This is at odds with the usual assumption that free vibration of the higher modes of a beam will damp out most rapidly.

## Suggestions for Further Investigation

Any investigation of a relatively neglected topic is bound to produce more new questions than answers; the present work is no exception. The most promising questions stem from the rather interesting consequences of concentrated damping described in Chapter V.

The relatively weak effect of a single dashpot on modes other than the fundamental suggests examining beams having several dampers to explore the possibility of achieving more effective control of beam response. One specific objective should be to test the conjecture that using additional dashpots will force more than one mode to exhibit a critical damping level. The influence of varying the locations at which concentrated damping elements are attached to the beam should also be investigated.

Finally, since the suggested investigations of concentrated damping will depend on efficient approximate methods, attention should be paid to improving the Galerkin's method solution developed here. For example, convergence at higher damping levels may be slow due to the relatively large shear discontinuities. It would be useful to explore ways to improve the convergence under these conditions.

## APPENDIX A

## SWEEPING MATRICES FOR COMPLEX MODES

The matrix iteration process based on Eq. (3.12) produces the pair of modes whose frequencies have the lowest modulus. To find the frequencies and vectors of other modes, we must suppress the effects of the dominant lower modes. To do this, we may insert sweeping matrices into the iteration equation; these matrices receive their name from the idea that they "sweep" the lower modes out of the iteration. We will describe a convenient way to construct these matrices for equations with complex normal mode vectors.

We note that the 2n normal mode vectors are elements of a 2n dimensional vector space and are linearly independent. Consequently, they form a basis of the space, and any vector of that dimension may be expressed as a linear combination of these mode vectors.

We may express the assumed trial vector used in the iteration process as follows.

$$\{A\}^0 = d_1 \ \{A\}_1 + \sum_{i=2}^{2n} d_i \ \{A\}_i$$

We construct the expression,

$$\lfloor A \rfloor_1 \, [E] \, \{A\}^0 = d_1 \lfloor A \rfloor_1 \, [E] \, \{A\}_1$$

$$+ \sum_{i=2}^{2n} d_i \lfloor A \rfloor_1 \, [E] \, \{A\}_i$$

From the orthogonality conditions, Eqs. (2.28), we see that the terms of the series on the right side are each zero. We may solve for the scalar coefficient, $d_1$.

$$d_1 = \frac{\lfloor A \rfloor_1 \, [E] \, \{A\}^0}{\lfloor A \rfloor_1 \, [E] \, \{A\}_1}$$

We must also remove the complex conjugate mode. We arrange the sequence of the modes so that the $(n+i)^{th}$ mode is the conjugate of the $i^{th}$ mode,

$$\{A\}_{n+i} = \{A\}_i^*$$

where the asterisk denotes the complex conjugate. We may write the coefficient, $d_{n+1}$.

$$d_{n+1} = \frac{\lfloor A \rfloor_{n+1} \, [E] \, \{A\}^0}{\lfloor A \rfloor_{n+1} \, [E] \, \{A\}_{n+1}} = \frac{\lfloor A \rfloor_1^* \, [E] \, \{A\}^0}{\lfloor A \rfloor_1^* \, [E] \, \{A\}_1^*}$$

Since $[E]$ and $\{A\}^0$ are real, $d_1$ and $d_{n+1}$ are complex conjugates.

We may now remove the lowest mode terms from the trial vector.

$$
\begin{aligned}
\{\overline{A}\}^{0} &= \{A\}^{0} - \{A\}_{1}d_{1} - \{A\}_{n+1}d_{n+1} \\
&= \{A\}^{0} - \{A\}_{1}d_{1} - \{A\}_{1}^{*}d_{1}^{*} \\
&= \{A\}^{0} - 2RE\left[\{A\}_{1}d_{1}\right]
\end{aligned}
$$

If we substitute for $d_1$ and factor the original trial vector,

$$
\{\overline{A}\}^{0} = \left[ [I] - 2RE\left[\frac{\{A\}_1 \lfloor A \rfloor_1 [E]}{\lfloor A \rfloor_1 [E] \{A\}_1}\right]\right] \{A\}^{0}
$$

The square matrix is the sweeping matrix which will remove the lowest mode components from any real assumed vector.

To use this sweeping matrix, we first define $[V_1]$.

$$
[V_1] = - [E]^{-1}[D]
$$

Now, we premultiply the swept vector by $[V_1]$ and use Eq. (3.12).

$$
\begin{aligned}
[V_1]\{\overline{A}\}^{0} &= \left[ [V_1] - 2RE\left[\frac{\{A\}_1 \lfloor A \rfloor_1 [E]}{\alpha_1 \lfloor A \rfloor_1 [E] \{A\}_1}\right]\right] \{A\}^{0} \\
&= [V_2]\{A\}^{0}
\end{aligned}
$$

If we use $[V_2]$ in the iterative process, the lower modes are swept out at each step. This prevents convergence to these modes if they are accidentally introduced by the numerical process.

The same process may be continued until all modes of interest are established. In general,

$$[V_{j+1}] = [V_j] - 2RE \left[ \frac{\{A\}_j \ \lfloor A \rfloor_j \ [E]}{\alpha_j \ \lfloor A \rfloor_j \ [E] \ \{A\}_j} \right] .$$

# APPENDIX B

## MATRIX ITERATION PROCESS

Normal mode parameters satisfying Eq. (3.12) were calculated through the iterative technique described by Fox.[6] This appendix will summarize the technique briefly and describe the criteria used to decide when to terminate the iteration.

To determine the $i^{th}$ pair of modes, we use the matrix $[V_i]$, defined in Appendix A, and an arbitrary real trial vector $\{A\}^0$. We calculate some of the powers of the iteration matrix and form the following expressions using three successive powers of $[V_i]$,

$$\{Z^s\} = [V_i]^s \{A\}^0$$

$$\{Z^{s+1}\} = [V_i]^{s+1} \{A\}^0$$

$$\{Z^{s+2}\} = [V_i]^{s+2} \{A\}^0$$

We choose two sets of corresponding elements of the resulting vectors and solve the following pair of equations for f and g.

$$Z_j^{s+2} + fZ_j^{s+1} + gZ_j^{s} = 0$$

$$Z_k^{s+2} + fZ_k^{s+1} + gZ_k^{s} = 0$$

Then we may estimate the complex frequency,

$$\alpha_i = - \left(\frac{f}{2g}\right) + \hat{i}\sqrt{\frac{1}{g} - \left(\frac{f}{2g}\right)^2}$$

The elements of the corresponding vector are

$$A_{ji} = Z_j^{s+1} + \hat{i}\,\frac{fZ_j^{s+1} + 2Z_j^{s+2}}{b}$$

$$b = \sqrt{g - \left(\frac{f}{2}\right)^2}$$

$$j = 1, \ldots, 2n$$

If we calculate f and g for several choices of j and k, the values obtained will vary somewhat. As s increases, these values and the resulting estimates of the mode frequencies and vectors converge.

In calculating normal mode parameters for the problem considered here, this convergence of results for distinct choices of j and k was used to indicate when the iteration had been carried far enough. At each step of the iteration, reference values of f and g calculated from one pair of vector element sets were compared with values obtained from other pairs. We identify the reference values with the subscript r and calculate the following expression for each of the other results.

$$B = \sqrt{\frac{(f-f_r)^2 + (g-g_r)^2}{f_r^{\,2} + g_r^{\,2}}}$$

When this expression was less than a predetermined value for each possible choice of vector element sets, the iteration was ended and the mode frequency and vector were calculated. Otherwise, s was increased, and the process repeated.

In problems where the number of vector elements is large, this procedure may become rather cumbersome; the calculation of B must be repeated many times at each step near the end of the iteration.

The matrix iteration process was carried out on a Control Data Corporation model 3600 digital computer. For this problem, the $[V_i]$ were 14 by 14 matrices. The results were first tested at s=4, and s increased by one at each step until all values of B were sufficiently small. Several upper limits on B were examined, and $10^{-5}$ was finally selected to produce the results shown in Chapter V. This limit was deliberately kept quite small to keep errors in the lower modes small. Lower mode errors will affect the sweeping matrices and thus produce errors in the higher modes.

From 15 to 35 iterations were required, depending on which mode was being calculated; higher modes required fewer iterations. As a check on both the iteration process and the sweeping matrices, the estimates of frequencies and mode vectors were inserted into Eq. (3.12). In each case, the difference between the two sides of the equation was negligible, occurring usually in the fifth or sixth significant figure.

Although the iteration process seems rather lengthy, calculating four sets of modes for each of nine values of damping required less than three minutes running time on the computer. This included constructing the iteration matrices, performing the iteration, calculating sweeping matrices, and computing mode functions from the vectors and original approximating functions.

Mode function data were recorded on magnetic tape for subsequent automatic plotting. Figures 5.6, 5.7, and 5.8 were produced from these data using a CALCOMP 565 digital incremental plotter.

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