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I hereby recommend that this dissertation prepared under my direction by James Anthony Scarborough

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Hormoz Mahmoud — July 14, 1969

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After inspection of the final copy of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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SIGNED: James Anthony Scarborough

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ABSTRACT

The bound three-nucleon system is treated in an approach based on the method of K-harmonics. D-state angular momentum eigenfunctions are constructed and included in a nuclear wave function satisfying the generalized Pauli principle.

The Hamiltonian contains central and tensor pair interactions. Both central and tensor forces may be spin-isospin-parity dependent, or both may be average effective forces depending only on the internucleon distances.

A solvable system of second-order ordinary differential equations in one variable is derived for the radial parts of the total wave function. The system is solvable by standard computer methods for arbitrary finite potentials.

CHAPTER I

INTRODUCTION

Calculations pertaining to the bound state of three nucleons are generally quite complicated and often give widely varying results.

Calculations of the binding energy of the triton range from about 6 Mev smaller[1] than the correct value of 8.482 Mev to about 3.5 Mev larger.[2] These results depend both on the method of calculation and on the choice of nucleon-nucleon interaction.

A similar situation exists in determining the Coulomb energy of He^3. One author might get too large a Coulomb energy,[3] another might conclude that it is almost impossible to obtain as large a value as the correct one,[4] and yet another concludes that agreement with the correct value might require a slight charge asymmetry in the neutron-neutron versus the proton-proton interactions.[5]

Part of the difficulty is that, while a large number of potentials describe low-energy nucleon-nucleon scattering data reasonably well, these same potentials do not necessarily lead to the same nuclear properties.[6] In particular, the Hamada-Johnston potentials having infinite cores have often been used in variational calculations for the three-nucleon problem because of their "physically realistic" nature. It is now believed, by some, that extremely strong or infinite cores are not at all necessary to fit the low-energy data.[7] Indeed, such cores might possibly be a source of error, in view of the recent statement that they might not be the limiting case of soft cores in nuclear matter calculations.[8] It is desirable then, that a method be available whereby different proposed two-body potentials could be tested in the three-body system in a realistic way.

It is known that the triton wave function is composed of an S-state, a D-state, and a so-called "mixed symmetry" S-state, usually denoted by S′. All other components of the wave function are completely negligible. The relative contributions of the S, D, and S′ states is fairly well determined, although not without variation from author to author.

The estimates for the S′ state, for example, range from a little over 1%[9,10] to about 4%.[11] Similar variations occur in the literature regarding the S and D contributions. It appears, however, that most triton properties can be described reasonably well if the S, D, and S′ states are present in the approximate amounts of 92%, 6%, and 2%, respectively.[12] For some calculations, it is sufficient to consider only the S state, written as a function of the root-mean-square separation of the nucleons. This is the case for the well-known wave functions of Fetisov[13] and of Gunn and Irving.[14]

The problems briefly outlined here lead us to look for a method whereby the effects of arbitrary potentials in the triton can be determined conveniently. It is further desirable that the method should include the small S′ and D states, and that the effects to be examined should be calculable by widely understood means.

In this dissertation we will extend the method of K-harmonics to include the contributions of the S, D, and S′ states in the triton wave function in such a way that the resulting wave function will satisfy the generalized Pauli principle. The forces used will be the central and tensor types, both of which may be average effective forces, or both of which may be spin, isospin, and parity dependent. Coulomb forces are also included for the case of He^3. The inclusion, along the lines of the method presented here, of the other possible two-body interactions is expected to furnish some insight into the nature of three-body forces.

Total angular momentum eigenfunctions will be given explicitly, and there will result a solvable system of second-order differential equations for the S, D, and S′ radial functions. All terms involved will be calculated in detail, and the system can be solved for arbitrary finite tensor and central forces by well-known methods.

CHAPTER II

GENERAL FORM OF THE WAVE FUNCTION

The H^3 and He^3 nuclei are thought of as being composed of three nucleons. The wave function for a system of three fermions must satisfy the generalized Pauli principle; that is, it must change sign under the interchange of all coordinates (space, spin, and isospin) of any two of the nucleons. This constraint limits the possible forms that the wave function may have.

We shall use the spin functions α_j, β_j where j is the particle number. They are chosen to be in the usual representation of the Pauli spin matrices in which $\sigma_z^{\,j}$ is diagonal and has eigenvalues 1 and -1. The analogous choice is also made for the isospin representation.

The spin and isospin functions are combined to form spin-isospin eigenstates which transform according to a particular representation of the symmetric group on three indices. These eigenstates are then combined with spatial functions of the appropriate symmetries in order to form a totally antisymmetric wave function. The form so constructed necessarily obeys the generalized Pauli principle. The particular two-dimensional representation used for the permutation group on three indices is

$$P_{12} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

$$P_{23} \longleftrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}, \qquad P_{31} \longleftrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix},$$

where P_{jk} is the matrix operator corresponding to the interchange of particles j and k. We will denote by $'$ (prime) and $''$ (double prime) the first and second components, respectively, of a column vector which transforms under permutations like an operand of the above matrices.

The spin functions will be

$$\begin{array}{ll} S=\frac{1}{2}, m_S=\frac{1}{2} & \chi' = \sqrt{\tfrac{2}{3}}\left(\alpha_1\alpha_2\beta_3 - \tfrac{1}{2}\alpha_1\beta_2\beta_3 - \tfrac{1}{2}\beta_1\alpha_2\alpha_3\right) \\ & \chi'' = \sqrt{\tfrac{1}{2}}\left(\alpha_1\beta_2\alpha_3 - \beta_1\alpha_2\alpha_3\right) \\ m_S=-\frac{1}{2} & \chi' = \sqrt{\tfrac{2}{3}}\left(\beta_1\beta_2\alpha_3 - \tfrac{1}{2}\beta_1\alpha_2\beta_3 - \tfrac{1}{2}\alpha_1\beta_2\beta_3\right) \\ & \chi'' = \sqrt{\tfrac{1}{2}}\left(\beta_1\alpha_2\beta_3 - \alpha_1\beta_2\beta_3\right) \end{array}, \tag{2}$$

$$\begin{array}{ll} S=\frac{3}{2}, m_S=\frac{3}{2} & \chi^s = \alpha_1\alpha_2\alpha_3 \\ m_S=\frac{1}{2} & \chi^s = \sqrt{\tfrac{1}{3}}\left(\alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3\right) \\ m_S=-\frac{1}{2} & \chi^s = \sqrt{\tfrac{1}{3}}\left(\alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha\right) \\ m_S=-\frac{3}{2} & \chi^s = \beta_1\beta_2\beta_3 \end{array}. \tag{3}$$

The χ's for $S=\frac{3}{2}$ are completely symmetric under permutations, while the pair (χ', χ'') for a given m_S transforms as indicated by the primes. The χ's are orthonormal such that

$$\chi^{\dagger i}_{S,m_S}\,\chi^{j}_{S',m_S'} = \delta_{ij}\,\delta_{SS'}\,\delta_{m_S m_S'}, \tag{4}$$

where $i,j \in (', '', s)$. There are no antisymmetric three-particle spin functions.

The mixed symmetry isospin states ζ are formed analogously to the $\chi^{','' }$'s and the resulting spin-isospin formalism is similar to that given by Verde.[15,16] Our particles are numbered differently from Verde's, and a different representation has been used for the symmetric group S_3. The appropriate spin-isospin functions in our representation are:

$$\varphi^s = \sqrt{\tfrac{1}{2}}\left(\chi''\zeta'' + \chi'\zeta'\right) \quad , \qquad \varphi' = \sqrt{\tfrac{1}{2}}\left(\chi''\zeta'' - \chi'\zeta'\right) \quad , \tag{5}$$
$$\varphi^a = \sqrt{\tfrac{1}{2}}\left(\chi'\zeta'' - \chi''\zeta'\right) \quad , \qquad \varphi'' = \sqrt{\tfrac{1}{2}}\left(\chi'\zeta'' + \chi''\zeta'\right) \quad .$$

The spin and isospin quantum numbers have been suppressed for clarity, and may be inserted in the obvious way. For example, the antisymmetric spin-isospin state having $S = \frac{1}{2}$, $m_s = -\frac{1}{2}$, $T = \frac{1}{2}$, and $T_3 = \frac{1}{2}$ is

$$\varphi^a = \sqrt{\frac{1}{2}}\left(\chi'_{-\frac{1}{2}}\zeta''_{\frac{1}{2}} - \chi''_{-\frac{1}{2}}\zeta'_{\frac{1}{2}}\right) \ . \tag{6}$$

The convention $T_3 = \frac{1}{2}$ for a proton has been used. Again we have orthonormality such that

$$\varphi^{\dagger(i)}_{sm_sTT_3}\,\varphi^{(j)}_{s'm_s'T'T_3'} = \delta_{ss'}\,\delta_{m_sm_s'}\,\delta_{TT'}\,\delta_{T_3T_3'}\,\delta_{ij} \ . \tag{7}$$

For three-nucleon systems having total isospin $T = \frac{1}{2}$ we arrive at the completely antisymmetric wave functions given by Verde:

$$S = \tfrac{1}{2}: \qquad \Psi_s = \psi^a \varphi^s - \psi^s \varphi^a + \psi' \varphi'' - \psi'' \varphi' ,$$
$$S = \tfrac{3}{2}: \qquad \Psi_D = (\psi_D' \zeta'' - \psi_D'' \zeta') \chi^s \quad . \tag{8}$$

The ψ functions depend only upon the spatial coordinates, and must transform under permutations as indicated by their superscripts. The crux of the problem is, of course, to find the ψ's for a given nucleon-nucleon interaction. Toward this end we make use of recent developments[17-23] whereby the functions ψ^a, ψ^s, ψ', ψ'' can be expanded in terms of "radial" functions and angular momentum eigenstates. A similar approach is applied to ψ_D' and ψ_D'' by explicitly constructing the D-state angular momentum eigenfunctions. Then, upon insertion of arbitrary central and tensor potentials, and projection onto spin, isospin, and orbital angular momentum eigenstates, a system of coupled differential equations results for the radial functions, with the binding energy an unknown parameter.

The variables to be used are those shown in Figure 1.

The Laplacian in terms of r_1, r_2, r_3 may be written as $\nabla_1^2 + \nabla_2^2 + \nabla_3^2$. Upon making the transformations to $\vec{\eta}$, $\vec{\xi}$, and $\vec{R}$, and separating the center of mass motion, the Schrödinger equation becomes

$$\left[\frac{-\hbar^2}{2m}\left(\nabla_\eta^2 + \nabla_\xi^2\right) + u\right] \Psi = \varepsilon \Psi , \tag{9}$$

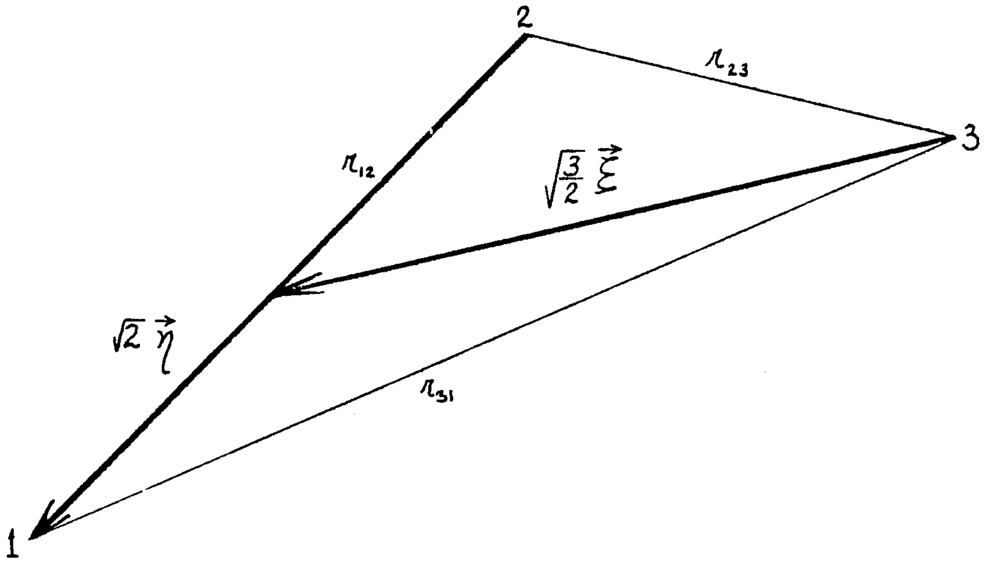

Figure 1. Coordinates for the Three Nucleon Problem.

The mathematical definitions of the coordinates, wherein $\vec{r}_j$ is the radius vector from some fixed origin to particle j, are

$$\vec{R} = \tfrac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \quad ,$$

$$\vec{\eta} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) \quad ,$$

$$\vec{\xi} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \quad .$$

where

$$\nabla_\eta^2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial\eta_j \partial\eta_j} \quad , \qquad \nabla_\xi^2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial\xi_j \partial\xi_j} \quad ,$$

and

$$\Psi = \Psi_S + \Psi_D \ .$$

The potential energy u is a function of the spin, isospin, and space coordinates of all three nucleons. For a bound state, ε is negative, and we write

$$\chi^2 = \frac{2m|\varepsilon|}{\hbar^2} \ . \tag{10}$$

The nucleon mass is m, assumed the same for both protons and neutrons. The Schrodinger equation is then

$$(\nabla_\eta^2 + \nabla_\xi - \chi^2)\Psi = U\Psi , \tag{11}$$

with

$$U = \frac{\hbar^2 u}{2m} \ .$$

The theory of angular harmonics on the sphere in six-dimensional space[18,19] has given the S-state angular functions to be

$$v_K^{\nu} = \sqrt{\frac{K+2}{\pi^3}} \quad \cos\lambda\nu \cdot A^{\nu} \cdot P^{\nu,0}_{\frac{1}{2}(\frac{K}{2}-\nu)}(1-2A^2) \quad , \qquad \nu > 0 \quad , \tag{12}$$

$$v_K^{0} = \sqrt{\frac{K+2}{2\pi^3}} \quad P^{0,0}_{\frac{K}{4}}(1-2A^2) \quad , \qquad \nu = 0 \; ,$$

$$w_K^{\nu} = \sqrt{\frac{K+2}{\pi^3}} \quad \sin\lambda\nu \cdot A^{\nu} \cdot P^{\nu,0}_{\frac{1}{2}(\frac{K}{2}-\nu)}(1-2A^2) \times \left.\begin{matrix} 1 \\ 1 \\ -1 \end{matrix}\right\} \begin{matrix} \nu \equiv 0 \pmod 3 \\ \nu \equiv 1 \pmod 3 \\ \nu \equiv 2 \pmod 3 \; . \end{matrix}$$

The P functions are Jacobi polynomials, and the coordinates used are

$$\rho^2 = \eta^2 + \xi^2 \quad , \qquad 0 \leq \rho < \infty \quad , \tag{13}$$

$$A^2 = \frac{(\eta^2 - \xi^2)^2 + (2\vec{\eta}\cdot\vec{\xi})^2}{\rho^4} \, , \qquad 0 \leq A \leq 1 \; ,$$

$$\tan\lambda = \frac{2\vec{\eta}\cdot\vec{\xi}}{\eta^2 - \xi^2} \quad , \qquad 0 \leq \lambda < \pi \, .$$

The volume element $d\vec{\eta}\, d\vec{\xi}$ can be written symbolically as

$$d\vec{\eta}\, d\vec{\xi} = \rho^5 d\rho \, d\Omega_6 \tag{14}$$

in the six-dimensional space. The element of solid angle, $d\Omega_6$, takes on various forms depending upon how the five variables other than ρ are defined. For integrands which are functions of ρ, A, λ only, it can be written as

$$d\Omega_6 = \pi^2 A\, dA\, d\lambda \ . \qquad (15)$$

The functions v, w then form an orthonormal set with respect to integration over $d\Omega_6$, the orthonormality being with respect to the index ν and, independently, with respect to the index K. In addition, any v is orthogonal to any w.

K may be any even integer or zero, and ν may be any of the positive integers $\frac{1}{2}K, \frac{1}{2}K-2, \frac{1}{2}K-4, \ldots, 0$ (or 1). These so-called K-harmonics are L = 0 angular momentum eigenstates. The presence of only the variables ρ, A, λ, rather than $\vec{\eta}, \vec{\xi}$, is a result of the restriction to zero angular momentum.

It is in general true that for an n-dimensional Laplacian,

$$\nabla_n^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_j} \quad ,$$

the definition

$$\rho^2 = \sum_{j=1}^{n} x_j^2$$

results in the operator having the form

$$\nabla_n^2 = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \rho^{n-1} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \nabla_\Omega^2 . \tag{16}$$

∇_Ω^2 symbolically denotes a second-order partial differential operator containing derivatives with respect to whatever other variables were chosen, but containing no ρ-derivatives and no ρ-dependence.

In terms of ρ, A, λ, the Laplacian undergoes the transformation

$$\nabla_6^2 \longrightarrow \frac{1}{\rho^5}\frac{\partial}{\partial\rho}\rho^5\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left[\frac{4}{A^2}\frac{\partial^2}{\partial\lambda^2} + 4(1-A^2)\frac{\partial^2}{\partial A^2} - 12A\frac{\partial}{\partial A}\right] . \tag{17}$$

Equation (17) is valid for use on operands which depend on ρ, A, λ only. The operator in Eq. (17) is separable in ρ, A, λ, and the solutions may thus be written in the form

$$\rho^{-2} \sum_{K,\nu} R_K^\nu(\rho)\, \mathcal{U}_K^\nu(A,\lambda) . \tag{18}$$

Here $\mathcal{U}_K^\nu$ stands for the angular functions v_K^ν and w_K^ν, and the ρ^{-2} has been factored out for later convenience.

The all-important symmetry properties of the $\mathcal{U}$'s are deduced by noting that the variables A and ρ are invariant under all permutations, and the variable λ behaves as follows:

$$
\begin{aligned}
(12)_R\,\lambda &\longrightarrow -\lambda\,, \\
(23)_R\,\lambda &\longrightarrow -\lambda + \tfrac{2\pi}{3}\,, \\
(31)_R\,\lambda &\longrightarrow -\lambda - \tfrac{2\pi}{3}\,.
\end{aligned}
\tag{19}
$$

Throughout this thesis the notation $(jk)_x$ will denote permutation operation on the coordinates of particles j and k as indicated. The subscript x may be R to indicate spatial coordinates, or it may be σ or τ, or any combination of R, σ, τ.

Since only λ is affected by permutations, the transformation properties of the $\mathcal{U}$'s are contained in the sine and cosine terms. For $\nu \equiv 0$ (mod 3), the V's are symmetric under $(ij)_R$ while the w's are antisymmetric. For $\nu \equiv 1$ (mod 3), the column vector $\big(\cos (3n+1)\lambda\,,\ \sin(3n+1)\lambda\big)$ transforms like $(\vec{\xi}, \vec{\eta})$, and for $\nu \equiv 2$ (mod 3), the pair $\big(\cos(3n+2)\lambda\,,\ -\sin(3n+2)\lambda\big)$ transforms the same way.

Thus the S-state component of the wave function will have the form shown in (18), and the R's must necessarily be such that

$$
\begin{aligned}
R_K^{\nu} &= R_K^{-\nu}\,, \qquad & \nu &\equiv 1\ (\mathrm{mod}\ 3)\,, \\
R_K^{\nu} &= -R_K^{-\nu}\,, \qquad & \nu &\equiv 2\ (\mathrm{mod}\ 3)\,,
\end{aligned}
\tag{20}
$$

in order that Ψ_s be antisymmetric.

CHAPTER III

CONSTRUCTION OF D-STATE ANGULAR FUNCTIONS

We use the "rotational" coordinates defined by

$$\begin{aligned} \xi_+ &= -\frac{1}{\sqrt{2}}(\xi_1 + i\,\xi_2) , \\ \xi_0 &= \xi_3 , \\ \xi_- &= \frac{1}{\sqrt{2}}(\xi_1 - i\,\xi_2) , \end{aligned} \tag{21}$$

and similarly for $\vec{\eta}$. One can then show the following relations to be true:

$$\nabla_\eta^2 + \nabla_\xi^2 = \frac{\partial^2}{\partial \xi_0^2} - 2\frac{\partial^2}{\partial \xi_- \partial \xi_+} - 2\frac{\partial^2}{\partial \eta_- \partial \eta_+} + \frac{\partial^2}{\partial \eta_0^2} , \tag{22}$$

$$\begin{aligned} \hat{L}_z &= -i\hbar\left(\vec{\eta}\times\nabla + \vec{\xi}\times\nabla\right)_z \\ &= \hbar\left[\eta_+\frac{\partial}{\partial \eta_+} - \eta_-\frac{\partial}{\partial \eta_-} + \xi_+\frac{\partial}{\partial \xi_+} - \xi_-\frac{\partial}{\partial \xi_-}\right] , \end{aligned} \tag{23}$$

$$
\begin{aligned}
\frac{-\vec{L}^2}{\hbar^2} &= \left(\vec{\eta}\times\nabla_\eta + \vec{\xi}\times\nabla_\xi\right)\cdot\left(\vec{\eta}\times\nabla_\eta + \vec{\xi}\times\nabla_\xi\right) \qquad (24)\\
&= \eta^2\nabla_\eta^2 + 2(\vec{\eta}\cdot\vec{\xi})(\nabla_\eta\cdot\nabla_\xi) + \xi^2\nabla_\xi^2 - (\vec{\eta}\cdot\nabla_\eta)^2 - (\vec{\xi}\cdot\nabla_\xi)^2 - \left[(\vec{\eta}\cdot\nabla_\xi)(\vec{\xi}\cdot\nabla_\eta) + (\vec{\xi}\cdot\nabla_\eta)(\vec{\eta}\cdot\nabla_\xi)\right]\\
&= \left(\eta_0^2 - 2\eta_-\eta_+\right)\left(\frac{\partial^2}{\partial\eta_0^2} - 2\frac{\partial^2}{\partial\eta_-\partial\eta_+}\right) + \left(\xi_0^2 - 2\xi_-\xi_+\right)\left(\frac{\partial^2}{\partial\xi_0^2} - 2\frac{\partial^2}{\partial\xi_-\partial\xi_+}\right)\\
&\quad - \left(\eta_+\frac{\partial}{\partial\eta_+} + \eta_-\frac{\partial}{\partial\eta_-} + \eta_0\frac{\partial}{\partial\eta_0}\right)^2 - \left(\xi_+\frac{\partial}{\partial\xi_+} + \xi_-\frac{\partial}{\partial\xi_-} + \xi_0\frac{\partial}{\partial\xi_0}\right)^2\\
&\quad + 2\left(\eta_0\xi_0 - \eta_-\xi_+ - \eta_+\xi_-\right)\left(\frac{\partial^2}{\partial\eta_0\partial\xi_0} - \frac{\partial^2}{\partial\eta_-\partial\xi_+} - \frac{\partial^2}{\partial\eta_+\partial\xi_-}\right)\\
&\quad - \Bigg[\left(\xi_0\frac{\partial}{\partial\eta_0} + \xi_-\frac{\partial}{\partial\eta_-} + \xi_+\frac{\partial}{\partial\eta_+}\right)\left(\eta_0\frac{\partial}{\partial\xi_0} + \eta_-\frac{\partial}{\partial\xi_-} + \eta_+\frac{\partial}{\partial\xi_+}\right)\\
&\qquad + \left(\eta_0\frac{\partial}{\partial\xi_0} + \eta_-\frac{\partial}{\partial\xi_-} + \eta_+\frac{\partial}{\partial\xi_+}\right)\left(\xi_0\frac{\partial}{\partial\eta_0} + \xi_-\frac{\partial}{\partial\eta_-} + \xi_+\frac{\partial}{\partial\eta_+}\right)\Bigg].
\end{aligned}
$$

These relations will be used in constructing the D-state angular momentum eigenfunctions.

In the previous equations, and in all of those which follow, $\vec{L}$ always denotes total three-nucleon orbital angular momentum about the center of mass of the nucleus.

The following commutation relations are also valid, wherein ∇_6^2 is defined to be $\nabla_\eta^2 + \nabla_\xi^2$:

$$\left[\nabla_6^2, \hat{L}_z\right] = \left[\nabla_\eta^2 + \nabla_\xi^2, -i\hbar\left(\vec{\eta}\times\nabla_\eta + \vec{\xi}\times\nabla_\xi\right)\right] = 0\,, \qquad (25)$$

$$\left[\nabla_6^2, \vec{L}^2\right] = 0\,, \qquad (26)$$

$$\left[\vec{L}^2, \hat{L}_z\right] = 0\,. \qquad (27)$$

Since these operators commute, simultaneous eigenstates of them exist.

The quantities η^2, ξ^2, $\vec{\eta}\cdot\vec{\xi}$ have eigenvalues $L^2 = 0$ and $L_z = 0$, as may be verified by writing them in rotational coordinate form and operating with L_z and L^2. Hence they are scalars under three-dimensional rotations of the system as a whole. The combination $\left(\frac{1}{2}(\eta^2-\xi^2)\,,\ \vec{\eta}\cdot\vec{\xi}\right)$ can then be shown to transform like $(\prime,\prime\prime)$ under permutations, and to have eigenvalue L = 0.

We are thus led to construct D-state angular functions from the pair

$$\begin{pmatrix} \underset{\approx}{T}' \\ \underset{\approx}{T}'' \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} \frac{1}{2}(\vec{\eta}\,\vec{\eta} - \vec{\xi}\,\vec{\xi}) - \frac{1}{6}(\eta^2-\xi^2)\,\underset{\approx}{I} \\ \frac{1}{2}(\vec{\eta}\,\vec{\xi} + \vec{\xi}\,\vec{\eta}) - \frac{1}{3}(\vec{\eta}\cdot\vec{\xi})\,\underset{\approx}{I} \end{pmatrix}, \qquad (28)$$

where $\underset{\approx}{I}$ is chosen to be the unit dyadic in terms of the rotational unit vectors $\hat{e}^+$, $\hat{e}^o$, $\hat{e}^-$. The factor ρ^{-2} has been inserted for convenience.

We note that the second-rank tensors are symmetric under $(\vec{\xi}, \vec{\eta}) \longrightarrow (-\vec{\xi}, -\vec{\eta})$ and traceless, and therefore correspond to total angular momentum L = 2.

The five pairs of functions corresponding to the five possible values of L_{Ξ} may be projected from T′ and T″ by using the unit vectors. For example,

$$(\hat{e}^-)^* \cdot \underset{\approx}{T}' \cdot (\hat{e}^-)^* = \left[\tfrac{1}{2}(\eta_+\eta_+ - \xi_+\xi_+) - \tfrac{1}{6}(\eta^2 - \xi^2)\right]\rho^{-2}. \qquad (29)$$

This is analogous to the method of generating the ordinary spherical harmonics by taking projections of the tensor $\vec{r}\vec{r} - \frac{1}{3}r^2\underset{\approx}{I}$.

The definitions of the ordinary spherical harmonics in terms of Cartesian coordinates may be used to express these D-state functions in terms of ordinary spherical harmonics of the directions of $\hat{\eta}$ and $\hat{\xi}$.

The calculations to be done later will require integrations over the 6-sphere, and it will then be convenient to use the form

$$d\Omega_6 = \sin^2\theta \cos^2\theta \, d\theta \, d\Omega_\eta d\Omega_\xi \quad , \qquad (30)$$

rather than that given in Eq. (15).

In Eq. (30), $d\Omega_\eta$ is an element of solid angle in the direction of $\hat{\eta}$, and $d\Omega_\xi$ is defined similarly. The angle θ is defined by

$$\rho\cos\theta = \eta \quad , \quad \rho\sin\theta = \xi \quad , \quad 0 \leqslant \theta \leqslant \frac{\pi}{2} . \quad (31)$$

The L = 2 functions expressed in this form are presented in Tables 1 and 2. The results in Table 2 are derived from those of Table 1 by using the definitions of the ordinary spherical harmonics, Y_ℓ^m, in terms of rotational coordinates in 3-space with the Condon-Shortley phase convention. Using the differential operators (22), (23), and (24), one can verify that the $\mathcal{Y}$'s have the following properties:

i. Each $\mathcal{Y}$ satisfies the six-dimensional Laplace equation: $\nabla_6^2 \mathcal{Y} = 0$.

ii. Each $\mathcal{Y}$ is a simultaneous eigenstate of L^2 and L_z, having total orbital angular momentum of 2, and z-projection as indicated in the tables.

iii. Each $\mathcal{Y}$ is symmetric under $(\vec{\xi}, \vec{\eta}) \longrightarrow (-\vec{\xi}, -\vec{\eta})$

iv. The pair $(\mathcal{Y}', \mathcal{Y}'')$ for a given L_z transforms according to the representation of the permutation group being used.

v. The $\mathcal{Y}$'s are orthonormal such that

$$\int \left(\mathcal{Y}_2^{(i),m}\right)^* \mathcal{Y}_2^{(j),m'} \sin^2\theta \cos^2\theta \, d\theta \, d\Omega_\eta \, d\Omega_\xi = \delta_{ij}\delta_{mm'} , \quad (32)$$

with $L_z, L'_z \in (-2,-1,0,1,2)$ and $i,j \in (','')$.

vi. The $\mathcal{Y}$'s are orthogonal to the $\mathcal{U}_K^{\nu}$'s.

The $\mathcal{Y}$ functions thus qualify as D-state angular functions. The total orbital angular momentum about the center of mass may be thought of as the angular momentum of a pair coupled with that of the third particle. Expanding $\mathcal{V}_0^0$ and $\mathcal{V}_2^1$ in terms of spherical harmonics about $\hat{\eta}$ and $\hat{\xi}$, one sees them to be made up of $\ell_{12} = 0$ coupled with $\ell_3 = 0$. Angular momentum S-state functions with higher indices are made up of higher angular momentum component states, such as $\ell_{12} = 1$ coupled to $\ell_3 = 1$ to give L = 0, and so on. It can be shown that the radial functions R_K^{ν} which will eventually multiply the corresponding $\mathcal{U}$'s decrease roughly as $K^{-\frac{7}{2}}$ for $\rho \geq 1$ fermi.[22] Hence, only the angular functions made up of the lowest ℓ-components are of any importance in this problem.

A similar situation exists with the D-state angular functions. These functions may be written in terms of $Y_2^m(\hat{r}_{ij})$ by expanding the $Y_2^m(\hat{\xi})$ and $Y_2^m(\hat{\eta})$ in terms of their Cartesian definitions. In this way, one sees that the $\mathcal{Y}$'s contain two-nucleon relative orbital angular momenta up to $\ell = 2$, and are composed of the combinations $\ell_{12} = 2$, $\ell_3 = 0$; $\ell_{12} = 1$, $\ell_3 = 1$; and $\ell_{12} = 0$, $\ell_3 = 2$.

It is possible to write D-state angular functions involving higher couplings by extending the method of this

Table 1. Normalized $\mathcal{Y}$'s in Rotational Coordinates.

The pair $(\mathcal{Y}', \mathcal{Y}'')$ for fixed m transforms under permutations of particles according to the given representation of the permutation group.

m	$\frac{\rho^2}{4}\sqrt{\frac{\pi^3}{3}}\,\mathcal{Y}_2'^{\,m}$	$\frac{\rho^2}{4}\sqrt{\frac{\pi^3}{3}}\,\mathcal{Y}_2''^{\,m}$
2	$\eta_+\eta_+ - \xi_+\xi_+$	$\eta_+\xi_+$
1	$\frac{\eta_0\eta_+ - \xi_+\xi_0}{\sqrt{2}}$	$\frac{\eta_+\xi_0 + \xi_+\eta_0}{\sqrt{2}}$
0	$\frac{\eta_-\eta_+ - \xi_-\xi_+ + \eta_0^2 - \xi_0^2}{\sqrt{6}}$	$\frac{\xi_-\eta_+ + 2\eta_0\xi_0 + \eta_-\xi_+}{\sqrt{6}}$
−1	$-\,\frac{\eta_0\eta_- - \xi_0\xi_-}{\sqrt{2}}$	$-\,\frac{\eta_-\xi_0 + \xi_-\eta_0}{\sqrt{2}}$
−2	$\eta_-\eta_- - \xi_-\xi_-$	$\eta_-\xi_-$

Table 2. Normalized D-state Angular Functions

	m	
$\frac{\pi}{4}\sqrt{\frac{5}{2}}\,\mathcal{Y}_2^{I\,m}$	2	$\cos^2\theta\; Y_2^2(\hat\eta) \;-\; \sin^2\theta\; Y_2^2(\hat\xi)$
	1	$\cos^2\theta\; Y_2^1(\hat\eta) \;-\; \sin^2\theta\; Y_2^1(\hat\xi)$
	0	$\cos^2\theta\; Y_2^0(\hat\eta) \;-\; \sin^2\theta\; Y_2^0(\hat\xi)$
	-1	$-\left(\cos^2\theta\; Y_2^{-1}(\hat\eta) \;-\; \sin^2\theta\; Y_2^{-1}(\hat\xi)\right)$
	-2	$\cos^2\theta\; Y_2^{-2}(\hat\eta) \;-\; \sin^2\theta\; Y_2^{-2}(\hat\xi)$
	m	
$\frac{\sqrt{3\pi}}{16}\,\mathcal{Y}_2^{II\,m}$	2	$\sin\theta\cos\theta\; Y_1^1(\hat\eta)\, Y_1^1(\hat\xi)$
	1	$\sin\theta\cos\theta\,\frac{1}{\sqrt{2}}\left(Y_1^1(\hat\eta)\,Y_1^0(\hat\xi) \;+\; Y_1^1(\hat\xi)\,Y_1^0(\hat\eta)\right)$
	0	$\sin\theta\cos\theta\left(\frac{1}{\sqrt{6}}Y_1^1(\hat\eta)\,Y_1^{-1}(\hat\xi) + \sqrt{\frac{2}{3}}\,Y_1^0(\hat\eta)\,Y_1^0(\hat\xi) + \frac{1}{\sqrt{6}}Y_1^{-1}(\hat\eta)\,Y_1^1(\hat\xi)\right)$
	-1	$-\sin\theta\cos\theta\,\frac{1}{\sqrt{2}}\left(Y_1^{-1}(\hat\eta)\,Y_1^0(\hat\xi) \;+\; Y_1^0(\hat\eta)\,Y_1^{-1}(\hat\xi)\right)$
	-2	$\sin\theta\cos\theta\; Y_1^{-1}(\hat\eta)\; Y_1^{-1}(\hat\xi)$

The Y's are ordinary spherical harmonics about the directions $\hat\eta$ and $\hat\xi$. The D-state angular functions are normalized for integration over $d\Omega_6 = \sin^2\theta\cos^2\theta\, d\theta\, d\Omega_\eta\, d\Omega_\xi$.

section to higher order tensor products, and selecting from these the components having the correct transformation properties and angular momentum. These higher components of the D-state would, however, correspond to higher order partial waves and more energetic excitations. These are expected to be quite small in the ground state. Since the D-state itself accounts for only a few per cent of the total wave function, these small corrections to it will be ignored.

The expression for the D-state has been presented symbolically as

$$\Psi_D = (\mathcal{Y}_D' \zeta'' - \mathcal{Y}_D'' \zeta')\chi^s \qquad (33)$$

for a total nuclear spin of $J = \frac{1}{2}$. It is implied here that the primed $\mathcal{Y}$-functions ($L = 2$) are to be vector-coupled with the symmetric χ-functions ($S = 3/2$) to give a total $J = \frac{1}{2}$. The explicit form $\mathcal{Y}_D \chi^s$ will then be, for the choice $J_z = -\frac{1}{2}$,

$$\phi = -\sqrt{\tfrac{2}{5}}\,\mathcal{Y}_2^{-2}\chi^s_{\frac{3}{2}} + \sqrt{\tfrac{3}{10}}\,\mathcal{Y}_2^{-1}\chi^s_{\frac{1}{2}} - \sqrt{\tfrac{1}{5}}\,\mathcal{Y}_2^{0}\chi^s_{-\frac{1}{2}} + \sqrt{\tfrac{1}{10}}\,\mathcal{Y}_2^{1}\chi^s_{-\frac{3}{2}} \quad , \qquad (34)$$

where all $\mathcal{Y}$'s may be either primed or double-primed. The inclusion of the ρ-dependence gives, with these ϕ's,

$$\Psi_D = f(\rho)\cdot\zeta''\phi' - f(\rho)\zeta'\phi'' . \qquad (35)$$

The $\mathcal{f}$ and $\mathcal{g}$ functions are clearly scalars under rotations and permutations. The Ψ_D must change sign, however, under a permutation of all coordinates of any pair of particles. Operating on Ψ_D with a permutation operator and setting it equal to its negative discloses that $\mathcal{f}$ and $\mathcal{g}$ must be identical. They will be given the new name $\rho^{-2} F(\rho)$. Then Ψ_D satisfies

$$\Psi_D = \rho^{-2} F(\rho) \left(\zeta'' \phi' - \zeta' \phi'' \right). \qquad (36)$$

Later we shall need to know the effect of $\nabla_6^2 - \chi^2$ on Ψ_D . By using the rotational form for ∇_6^2 and for any $\mathcal{Y}$, one can show that

$$(\nabla_6^2 - \chi^2)\rho^{-2} F(\rho)\mathcal{Y} = \rho^{-2}\mathcal{Y}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\chi^2 + \frac{16}{\rho^2}\right)\right]F(\rho) . \qquad (37)$$

CHAPTER IV

THE ENERGY EIGENVALUE EQUATION

The procedure will be to write the energy eigenvalue equation and to take its projections onto spin, isospin, and orbital angular momentum states, in order to arrive at a system of differential equations for the radial functions only.

The energy eigenvalue equation was given in Eq. (11) to be

$$\left(\nabla_6^2 - \chi^2\right)\Psi = \hat{U}_{(123)}\Psi . \qquad (38)$$

We now write the potential $\hat{U}(123)$ as the sum of central pair potentials U(ij) and tensor potentials V(ij) as follows.

$$\hat{U}(123) = U(123) + V(123) \quad , \qquad (39)$$

$$U(123) = U(12) + U(23) + U(31) ,$$

$$V(123) = V(12)\hat{S}_{12} + V(23)\hat{S}_{23} + V(31)\hat{S}_{31} .$$

The tensor potential will be written as

$$V_{(123)} = \sum_{(ij)} V_{(ij)}\,\hat{S}_{ij} \quad , \qquad (40)$$

where the sum is over all pairs (ij) = (12), (23), (31). $\hat{S}_{ij}$ is the usual tensor operator defined by

$$\hat{S}_{ij} = 3(\vec{\sigma}^i \cdot \hat{r})(\vec{\sigma}^j \cdot \hat{r}) - \vec{\sigma}^i \cdot \vec{\sigma}^j \qquad (41)$$

for particles ij, and $V(ij)$ is defined by

$$V(ij) = V_{33-} P^+_\sigma(ij) P^+_\tau(ij) P^-_R(ij) + V_{31+} P^+_\sigma(ij) P^-_\tau(ij) P^+_R(ij) . \qquad (42)$$

The subscripts on the V's stand for 2s+1, 2t+1, π. π is relative orbital parity, s is total spin, and t is total isospin, all referring to the two-body states. The projection operators P_σ are defined by

$$P^\pm_\sigma(ij) = \frac{1 \pm (ij)_\sigma}{2} , \qquad (43)$$

where $(ij)_\sigma$ is the operator for exchanging the spins of particles i and j. Similar definitions hold for the other projection operators.

The central potential $U(ij)$ is written as

$$U(ij) = U_{13+} P^-_\sigma(ij) P^+_\tau(ij) P^+_R(ij) + U_{31+} P^+_\sigma(ij) P^-_\tau(ij) P^+_R(ij) \qquad (44)$$
$$+ U_{33-} P^+_\sigma(ij) P^+_\tau(ij) P^-_R(ij) + U_{11-} P^-_\sigma(ij) P^-_\tau(ij) P^-_R(ij) .$$

The symbol U will be used exclusively for central interactions, and V exclusively for tensor interactions.

All U's and V's are functions of the indicated particle separations. For example, $U_{31+}(ij)$ is the central potential between particles i and j when they are in a state characterized by total spin of one, total isospin of zero, and even orbital parity. It is a function of r_{ij} only. In the manner, spin-isospin-parity dependent potentials may be used.

The energy eigenvalue equation has the form

$$(\nabla_6^2 - \chi^2)(\Psi_S + \Psi_D) = U\Psi_S + U\Psi_D + V\Psi_S + V\Psi_D \,. \quad (45)$$

The expansions for Ψ_S and Ψ_D are understood to be inserted. Since

$$\nabla_6^2 = \frac{1}{\rho^5}\frac{\partial}{\partial\rho}\,\rho^5\frac{\partial}{\partial\rho} + \frac{1}{\rho^5}\nabla_\Omega^2 \quad , \qquad (46)$$

as already noted, we may use the fact that, for homogeneous polynomial solutions of order K of the n-dimensional Laplace equation,[18]

$$\nabla_\Omega^2 \mathscr{P}_K = -K(K+n-2)\,\mathscr{P}_K \,. \qquad (47)$$

In particular, we have

$$\nabla_\Omega^2 \, u_K^\nu = - K(K+4)\, u_K^\nu \quad , \tag{48}$$

for any of the angular functions v_K^ν and w_K^ν. Relation (47) is recognized as giving the familiar factor $\ell(\ell+1)$ for the one-particle problem in three-dimensional space. It should be pointed out, however, that in contrast with the interpretation given for ℓ, the K here is not the angular momentum in units of $\hbar$.

To arrive at a system of equations in one variable, we begin by projecting Eq. (45) onto the spin-isospin functions, and considering the resulting terms individually.

We see that each term of Ψ_D contains a factor of the form $(\nabla_6^2 - \kappa^2)\,\Psi_D$. Hence, $\rho^{-2} F(\rho)\, \mathcal{Y}\, \zeta\, \chi^s$ has a factor χ^s remaining in it, and is orthogonal to all of the φ^α due to the orthogonality of the spin functions for different spins. This term therefore is zero.

In the term involving $V\Psi_S$, the expansion gives a sum of terms containing factors of the form $\varphi^{\dagger\alpha} \hat{S}_{ij} \varphi^\beta$. The evaluation of these terms is made manageable by using the following expansion for $\hat{S}_{ij}$:

$$\hat{S}_{ij} = \sqrt{\frac{24\pi}{5}} \sum_{m=-2}^{2} \sum_{\mu+\mu'=-m} Y_2^m(\hat{r}_{ij})\, C(1\,1\,\mu\,\mu' \,|\, 2\, {-m})\, \sigma_\mu^i\, \sigma_{\mu'}^j \, . \tag{49}$$

In Eq. (49), $\hat{r}_{ij}$ is a unit vector directed along the line connecting particles i and j, $C(1\,1\,\mu\mu'|\,2\,-m)$ is a Clebsch-Gordan coefficient in the form $C(jj'mm'|JM)$ and the spin operators are defined by

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + i\sigma_y)\,, \qquad \sigma_{-1} = \tfrac{-1}{2}(\sigma_x - i\sigma_y)\,, \qquad \sigma_0 = \sigma_z\,. \tag{50}$$

These spin operators have the following effects on the spin functions:

$$\begin{array}{lll} \sigma_1\alpha = 0 & \sigma_{-1}\alpha = -\sqrt{2}\,\beta & \sigma_0\alpha = \alpha \\ \sigma_1\beta = \sqrt{2}\,\alpha & \sigma_{-1}\beta = 0 & \sigma_0\beta = -\beta\,. \end{array} \tag{51}$$

Equation (46) was derived by algebraic manipulation of these quantities. From that equation, it is clear that the spin operators enter into $\hat{S}_{ij}$ in the form of a pure spherical tensor of second rank. Accordingly, the Wigner-Eckart theorem gives

$$\varphi^{\dagger\alpha}\,\hat{S}_{ij}\,\varphi^{\beta} = 0\,. \tag{52}$$

The other matrix elements of $\hat{S}_{ij}$ which will be needed are listed in Table 3. They were derived algebraically by using the expansion for $\hat{S}_{ij}$ and the definitions of the φ's.

To evaluate expressions of the type $\varphi^{\dagger\alpha} U\,\Psi_D$ we note that the projection operators pick out the part of Ψ_D which is even (+) or odd (−) under interchange of

Table 3. Matrix Elements of $\hat{S}_{ij}$ between Spin States

		$\lvert s, \tfrac{3}{2}\rangle$	$\lvert s, \tfrac{1}{2}\rangle$	$\lvert s, -\tfrac{1}{2}\rangle$	$\lvert s, -\tfrac{3}{2}\rangle$
$\hat{S}_{12}$	$\langle {}', -\tfrac{1}{2}\rvert$	$2\sqrt{\frac{2}{3}}\, Y_2^2$	$\sqrt{2}\, Y_2^1$	$\frac{2}{\sqrt{3}}\, Y_2^0$	$\frac{\sqrt{2}}{\sqrt{3}}\, Y_2^{-1}$
	$\langle {}'', -\tfrac{1}{2}\rvert$	0	0	0	0
$\hat{S}_{23}$	$\langle {}', -\tfrac{1}{2}\rvert$	$-\frac{\sqrt{2}}{\sqrt{3}}\, Y_2^2$	$-\frac{1}{\sqrt{2}}\, Y_2^1$	$-\frac{1}{\sqrt{3}}\, Y_2^0$	$-\frac{1}{\sqrt{6}}\, Y_2^{-1}$
	$\langle {}'', -\tfrac{1}{2}\rvert$	$-\sqrt{2}\, Y_2^2$	$-\frac{\sqrt{3}}{\sqrt{2}}\, Y_2^1$	$-Y_2^0$	$-\frac{1}{\sqrt{2}}\, Y_2^{-1}$
$\hat{S}_{31}$	$\langle {}', -\tfrac{1}{2}\rvert$	$-\frac{\sqrt{2}}{\sqrt{3}}\, Y_2^2$	$-\frac{1}{\sqrt{2}}\, Y_2^1$	$-\frac{1}{\sqrt{3}}\, Y_2^0$	$-\frac{1}{\sqrt{6}}\, Y_2^{-1}$
	$\langle {}'', -\tfrac{1}{2}\rvert$	$\sqrt{2}\, Y_2^2$	$\frac{\sqrt{3}}{\sqrt{2}}\, Y_2^1$	Y_2^0	$\frac{1}{\sqrt{2}}\, Y_2^{-1}$
$\hat{S}_{ij}$	$\langle s, \tfrac{3}{2}\rvert$	$\sqrt{\frac{16\pi}{5}}\, Y_2^0$	$\sqrt{\frac{32\pi}{5}}\, Y_2^{-1}$	$\sqrt{\frac{32\pi}{5}}\, Y_2^{-2}$	0
	$\langle s, \tfrac{1}{2}\rvert$	$-\sqrt{\frac{32\pi}{5}}\, Y_2^1$	$-\sqrt{\frac{16\pi}{5}}\, Y_2^0$	0	$\sqrt{\frac{32\pi}{5}}\, Y_2^{-2}$
	$\langle s, -\tfrac{1}{2}\rvert$	$\sqrt{\frac{32\pi}{5}}\, Y_2^2$	0	$-\sqrt{\frac{16\pi}{5}}\, Y_2^0$	$-\sqrt{\frac{32\pi}{5}}\, Y_2^{-1}$
	$\langle s, -\tfrac{3}{2}\rvert$	0	$\sqrt{\frac{32\pi}{5}}\, Y_2^2$	$\sqrt{\frac{32\pi}{5}}\, Y_2^1$	$\sqrt{\frac{16\pi}{5}}\, Y_2^0$

The entries here are the results of

$$\left\langle \text{bra from left column} \,\middle|\, \text{as labeled} \,\middle|\, \text{ket from top row} \right\rangle .$$

σ (or τ, or R) for particles ij. For Ψ_D, the spin states are completely symmetric under any interchange of spins, and hence

$$P_\sigma^-(ij)\Psi_D = 0 \quad \text{and} \quad P_\sigma^+(ij)\Psi_D = \Psi_D . \tag{53}$$

Therefore, one has

$$U(ij)\Psi_D = U_{31+}(ij)P_\tau^-(ij)P_R^+(ij)\Psi_D + U_{33-}(ij)P_\tau^+(ij)P_R^-(ij)\Psi_D . \tag{54}$$

The U's are simply scalar functions, and the P_τ's operate only on the isospins. Therefore each term of the result of the operation contains a factor of χ^s from Ψ_D, and it immediately follows that

$$\varphi^{\dagger\alpha}U\Psi_D = 0 , \tag{55}$$

owing to the orthogonality of the spin functions.

The remaining terms from Eq. (45), projected onto $\varphi^{\dagger\alpha}$, are

$$\varphi^{\dagger\alpha}(\nabla_6^2 - \varkappa^2)\Psi_S - \varphi^{\dagger\alpha}U\Psi_S - \varphi^{\dagger\alpha}V\Psi_D = 0 . \tag{56}$$

In the first term we can replace

$$(\nabla_6^2 - \varkappa^2)\rho^{-2}R_K^\nu u_K^\nu$$

in each term of the expansion for Ψ_S by

$$\frac{1}{\rho^2} \mathcal{U}_K^{\nu}(A,\lambda) \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\chi^2 + \frac{(K+2)^2}{\rho^2}\right)\right] R_K^{\nu} , \quad (57)$$

obtained by using Eqs. (46) and (47). Multiplication of Eq. (56) by any S-state angular function and integration over Ω_6 results in a system of differential equations in ρ alone for the radial functions R_K^{ν} and the binding energy χ^2.

The matrix elements $\varphi^{\dagger\alpha} U \varphi^{\beta}$ will be needed for the second term of (56). They have been calculated and listed in Table 4.

The most troublesome term is $\varphi^{\dagger\alpha} V \Psi_D$. To evaluate it, we first note that $\hat{S}_{ij}$ commutes with

$$P^{\pm}_{\sigma(ij)} , \quad P^{\pm}_{\tau(ij)} , \quad P^{\pm}_{R(ij)} ,$$

and

$$\vec{\tau}^{\,i}\cdot\vec{\tau}^{\,j} , \quad \vec{\sigma}^{\,i}\cdot\vec{\sigma}^{\,j} ,$$

so that the location of $\hat{S}_{ij}$ in the expression for $V(123)$ is of no consequence. Hence, in expressions involving

$$(\vec{\tau}^{\,i}\cdot\vec{\tau}^{\,j})(\vec{\sigma}^{\,i}\cdot\vec{\sigma}^{\,j})\,\hat{S}_{ij} ,$$

such as those occurring in the Hamada-Johnston potentials, one may simply replace $\vec{\sigma}^{\,i}\cdot\vec{\sigma}^{\,j}$ (and $\vec{\tau}^{\,i}\cdot\vec{\tau}^{\,j}$) by -3 or 1,

Table 4. Matrix Elements of State-Dependent Central Potentials between Spin-Isospin States

$$\varphi^{a\dagger} U \varphi^{s} = 0$$

$$\varphi^{a\dagger} U \varphi' = \frac{\sqrt{3}}{4}\left(U^{(31)}_{13+} - U^{(31)}_{31+}\right)P^{+}_{R(31)} - \frac{\sqrt{3}}{4}\left(U^{(23)}_{13+} - U^{(23)}_{31+}\right)P^{+}_{R(23)}$$

$$\varphi^{s\dagger} U \varphi'' = \frac{\sqrt{3}}{4}\left(U^{(23)}_{33-} - U^{(23)}_{11-}\right)P^{-}_{R(23)} - \frac{\sqrt{3}}{4}\left(U^{(31)}_{33-} - U^{(31)}_{11-}\right)P^{-}_{R(31)}$$

$$\begin{aligned}\varphi^{a\dagger} U \varphi'' = {} & \frac{1}{2}\left(U^{(12)}_{31+} - U^{(12)}_{13+}\right)P^{+}_{R(12)} \\ & - \frac{1}{4}\left(U^{(23)}_{31+} - U^{(23)}_{13+}\right)P^{+}_{R(23)} - \frac{1}{4}\left(U^{(31)}_{31+} - U^{(31)}_{13+}\right)P^{+}_{R(31)}\end{aligned}$$

$$\begin{aligned}\varphi^{s\dagger} U \varphi' = {} & -\frac{1}{2}\left(U^{(12)}_{33-} - U^{(12)}_{11-}\right)P^{-}_{R(12)} \\ & + \frac{1}{4}\left(U^{(23)}_{33-} - U^{(23)}_{11-}\right)P^{-}_{R(23)} + \frac{1}{4}\left(U^{(31)}_{33-} - U^{(31)}_{11-}\right)P^{-}_{R(31)}\end{aligned}$$

$$\begin{aligned}\varphi'^{\dagger} U \varphi'' = {} & \frac{\sqrt{3}}{8}\left(U^{(31)}_{31+} + U^{(31)}_{13+}\right)P^{+}_{R(31)} - \frac{\sqrt{3}}{8}\left(U^{(31)}_{33-} + U^{(31)}_{11-}\right)P^{-}_{R(31)} \\ & - \frac{\sqrt{3}}{8}\left(U^{(23)}_{31+} + U^{(23)}_{13+}\right)P^{+}_{R(23)} + \frac{\sqrt{3}}{8}\left(U^{(23)}_{33-} + U^{(23)}_{11-}\right)P^{-}_{R(23)}\end{aligned}$$

$$\begin{aligned}\varphi'^{\dagger} U \varphi' = {} & \frac{1}{2}\left(U^{(12)}_{11-} + U^{(12)}_{33-}\right)P^{-}_{R(12)} + \frac{1}{8}\left(U^{(23)}_{33-} + U^{(23)}_{11-}\right)P^{-}_{R(23)} + \frac{1}{8}\left(U^{(31)}_{33-} + U^{(31)}_{11-}\right)P^{-}_{R(31)} \\ & + \frac{3}{8}\left(U^{(23)}_{31+} + U^{(23)}_{13+}\right)P^{+}_{R(23)} + \frac{3}{8}\left(U^{(31)}_{31+} + U^{(31)}_{13+}\right)P^{+}_{R(31)}\end{aligned}$$

$$\begin{aligned}\varphi''^{\dagger} U \varphi'' = {} & \frac{1}{2}\left(U^{(12)}_{31+} + U^{(12)}_{13+}\right)P^{+}_{R(12)} + \frac{1}{8}\left(U^{(31)}_{31+} + U^{(31)}_{13+}\right)P^{+}_{R(31)} + \frac{1}{8}\left(U^{(23)}_{31+} + U^{(23)}_{13+}\right)P^{+}_{R(23)} \\ & + \frac{3}{8}\left(U^{(31)}_{33-} + U^{(31)}_{11-}\right)P^{-}_{R(31)} + \frac{3}{8}\left(U^{(23)}_{33-} + U^{(23)}_{11-}\right)P^{-}_{R(23)}\end{aligned}$$

$$\varphi^{s\dagger} U \varphi^{s} = \frac{1}{2}\sum_{(ij)}\left(U^{(ij)}_{33-} + U^{(ij)}_{11-}\right)P^{-}_{R(ij)}$$

$$\varphi^{a\dagger} U \varphi^{a} = \frac{1}{2}\sum_{(ij)}\left(U^{(ij)}_{31+} + U^{(ij)}_{13+}\right)P^{+}_{R(ij)}$$

The summations are taken over all pairs of particles.

depending upon whether a singlet or triplet two-body spin (or isospin) state is being projected.

Since Ψ_D is symmetric under $(ij)_\sigma$, we replace $P_\sigma^+(ij)$ by 1, and proceed to evaluate $\phi^{\dagger\alpha} V \Psi_D$.

The equations involving $\phi^{\dagger\alpha} V \Psi_D$ will eventually be multiplied by an angular function symmetric under $(ij)_R$ and integrated over $d\Omega_6$. The action of $(ij)_R$ on the variables ρ, A, λ has already been given. By recognizing that

$$\begin{aligned} r_{12} &= \rho\sqrt{1 + A\cos\lambda}\,, \\ r_{23} &= \rho\sqrt{1 + A\cos(\lambda + \tfrac{2\pi}{3})}\,, \\ r_{31} &= \rho\sqrt{1 + A\cos(\lambda - \tfrac{2\pi}{3})}\,, \end{aligned} \qquad (58)$$

and that a symmetric S-state function depends on λ through $\sin(\nu\lambda)$ with $\nu \equiv 0$ (mod 3), we see that the integrand will always be periodic in λ with period $2\pi/n$, n being some integer. The λ integral is over the region $0 \leq \lambda < 2\pi$, and one can show that the same integral is obtained if the operators $P_R^{\pm}(ij)$ are considered to operate to the left in the integrand rather than to the right. Hence, the terms of involving $P_R^{-}(ij)$ will all integrate to zeros, and $P_R^{+}(ij)$ acting to the left behaves like the identity operator.

It is shown in Appendix A that

$$\varphi^{\dagger 2} V \Psi_D = \left[(12)_R + (23)_R + (31)_R\right] \frac{F(\rho)\, V_{31+}^{(12)}}{\rho^2 \sqrt{30}} \sum_{m=-1}^{2} (-)^m (m+2)\, \mathcal{Y}_2^{1-m}\, Y_{2\,(12)}^{m} \; . \qquad (59)$$

Other relations needed are listed here for reference.

$$\left[-(12)+\tfrac{1}{2}(23)+\tfrac{1}{2}(31)\right]\begin{pmatrix} f' \\ f'' \end{pmatrix} = \left[-2P^+_{(12)} + P^+_{(23)} + P^+_{(31)}\right]\begin{pmatrix} f' \\ f'' \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} f' \\ \frac{3}{2} f'' \end{pmatrix} \qquad (60)$$

$$\left[(23) - (31)\right]\begin{pmatrix} f' \\ f'' \end{pmatrix} = 2\left[P^+_{(23)} - P^+_{(31)}\right]\begin{pmatrix} f' \\ f'' \end{pmatrix} = \sqrt{3}\begin{pmatrix} f'' \\ f' \end{pmatrix} \qquad (61)$$

$$\left[-(12)+\tfrac{1}{2}(23)+\tfrac{1}{2}(31)\right]\begin{pmatrix} f^a \\ f^s \end{pmatrix} = \left[(23) - (31)\right]\begin{pmatrix} f^a \\ f^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (62)$$

$$\left[(12) + (23) + (31)\right]\begin{pmatrix} f^a \\ f^s \end{pmatrix} = \begin{pmatrix} -3f^a \\ 3f^s \end{pmatrix} \qquad (63)$$

$$\left[(12) + (23) + (31)\right]\begin{pmatrix} f' \\ f'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (64)$$

$$\begin{aligned}
P^+_{(12)} f' &= f' & P^+_{(12)} f'' &= 0 \\
P^-_{(12)} f' &= 0 & P^-_{(12)} f'' &= f'' \\
P^+_{(23)} f' &= -\tfrac{1}{2}(31) f' & P^+_{(23)} f'' &= -\tfrac{\sqrt{3}}{2}(31) f' \\
P^-_{(23)} f' &= -\tfrac{\sqrt{3}}{2}(31) f'' & P^-_{(23)} f'' &= \tfrac{1}{2}(31) f'' \\
P^+_{(31)} f' &= -\tfrac{1}{2}(23) f' & P^+_{(31)} f'' &= \tfrac{\sqrt{3}}{2}(23) f' \\
P^-_{(31)} f' &= \tfrac{\sqrt{3}}{2}(23) f'' & P^-_{(31)} f'' &= \tfrac{1}{2}(23) f'' \; .
\end{aligned} \qquad (65)$$

The complete symmetry of $\varphi^{\dagger a} V \Psi_D$ is explicitly displayed in the form given in Eq. (59). In general, the symmetry of $\varphi^{\dagger \alpha} V \Psi_D$, which is a function of spatial coordinates only, turns out to be conjugate to that designated by α. That is,

$$\begin{aligned} &\varphi^{\dagger a} V \Psi_D \quad \text{is symmetric under } (ij)_R\,, \\ &\varphi^{\dagger s} V \Psi_D \quad \text{is antisymmetric,} \\ &\varphi^{\dagger\prime} V \Psi_D \quad \text{has } {}'' \text{ symmetry, and} \\ &\varphi^{\dagger\prime\prime} V \Psi_D \quad \text{has } {}' \text{ symmetry.} \end{aligned} \tag{66}$$

These symmetries are quite important for setting many integrals equal to zero exactly for an arbitrary potential.

For example, let the spatial functions f,g,h transform under permutations as their indices will imply. Then one can show by simple calculus that integrals having integrands like $f^a g^s$, $f' g''$, $f' g'' h^s$, and many more, must integrate to zero identically over $d\Omega_6$ and $d\vec{\eta}\, d\vec{\xi}$. In particular, when multiplied by symmetric S-state functions and integrated over $d\Omega_6$, all terms in (66) necessarily give zero except for the first term.

This point is also quite important regarding the integrals which arise in deriving the equations for the R_K^ν. It is shown in the same way that many of the terms arising there, such as

$$\iint v_K^{\nu}\,\varphi^{\dagger s}\,U\,\varphi^{s}\,w_{K'}^{\nu'}\,d\Omega_6 \quad , \tag{67}$$

must be identically zero.

The integrals involving Eq. (59) can be done by noting that operating to the left on a symmetric function f^S simply gives $3f^S$, and by performing the integrations with $d\Omega_6$ in the form

$$d\Omega_6 = \cos^2\theta\,\sin^2\theta\,d\theta\,d\Omega_\eta\,d\Omega_\xi . \tag{68}$$

The integrals over $d\Omega_\eta d\Omega_\xi$ can be done by inspection, and only an integral over θ remains to be evaluated for the particular potential function being used. For example, it is shown in Appendix B that

$$\iiiint v_0^{\circ}\,\varphi^{\dagger a}\,V\,\Psi_D\,d\Omega_6 = \sqrt{\frac{3}{\pi^3}}\,\frac{64F(\rho)}{5\rho^2}\int_0^{\frac{\pi}{2}} V_{31+}(r_{12})\cos^4\theta\,\sin^2\theta\,d\theta \;, \tag{69}$$

where $r_{12} = \sqrt{2}\,\eta = \sqrt{2}\,\rho\cos\theta$ has been used. The single integral in this equation can be evaluated as a function of the parameter ρ.

Rather than using the most general form for an interaction, it is useful to be able to also use an average effective interaction as done in some nuclear matter calculations.

In this case, however, the tensor terms take on a different appearance. After a derivation similar to the one just outlined (see Appendix B), one finds the following:

$$\varphi^{\dagger a} V \Psi_D = -\frac{\sqrt{6\pi}\, F(\rho)}{5\rho^2} \sum_{m=-1}^{2} (-)^m (m+2) \left(\mathcal{Y}_2^{\prime m}\, \mathcal{Y}_2^{\prime -m} \pm \mathcal{Y}_2^{\prime\prime m}\, \mathcal{Y}_2^{\prime\prime -m} \right), \text{ upper sign,} \tag{70}$$

$$\varphi^{\dagger\prime} V \Psi_D = \text{ above expression with lower sign ,} \tag{71}$$

$$\varphi^{\dagger s} V \Psi_D = \frac{\sqrt{6\pi}\, F(\rho)}{5\rho^2} \sum_{m=-1}^{2} (-)^m (m+2) \left(\mathcal{Y}_2^{\prime m}\, \mathcal{Y}_2^{\prime\prime -m} \mp \mathcal{Y}_2^{\prime\prime m}\, \mathcal{Y}_2^{\prime -m} \right), \text{ upper sign,} \tag{72}$$

$$-\varphi^{\dagger\prime} V \Psi_D = \text{ above expression with lower sign.} \tag{73}$$

The $\mathcal{Y}$ functions are defined in terms of spherical harmonics, Y_ℓ^m, and the effective nucleon-nucleon tensor interaction, $v(ij)$, to be

$$\mathcal{Y}_2^{\prime m} = \sqrt{\frac{2}{3}} \left[-(12) + \tfrac{1}{2}(23) + \tfrac{1}{2}(31) \right] \left[Y_2^m(12)\, v(12) \right], \tag{74}$$

$$\mathcal{Y}_2^{\prime\prime m} = \sqrt{\frac{1}{2}} \left[(31) - (23) \right] \left[Y_2^m(12)\, v(12) \right]. \tag{75}$$

The functions of relations (70)-(73) are observed to have the same symmetries as the more general expression using state-dependent potentials.

For average effective central forces we still write

$$U(123) = \sum_{(ij)} U(ij) , \tag{76}$$

where no projection operators are involved. In this case the terms resulting from projection onto φ^{α} are evaluated trivially, owing to the orthonormality of the φ's. The resulting equations will be given in the next chapter.

CHAPTER V

THE D-STATE EQUATION

The equation for the D-state radial function $F(\rho)$ is derived from the general Schrodinger equation by projecting it onto χ_μ^s, ζ' or ζ'', and a $\mathcal{Y}$. The derivation is again lengthy, as are most of those in this problem, and only the salient features are outlined here. Further details are given in Appendix C.

Briefly, since none of ∇_6^2, $\varkappa^2$, U changes the relative numbers of α's and β's in the spin functions of Ψ_s, we have

$$\chi_\mu^{s\dagger}(\nabla_6^2 - \varkappa^2 - U)\Psi_s = 0 . \qquad (77)$$

The remaining terms are

$$\zeta'^\dagger \chi_\mu^{s\dagger}(\nabla_6^2 - \varkappa^2)\Psi_D = \zeta'^\dagger \chi_\mu^{s\dagger} V \Psi_s + \zeta'^\dagger \chi_\mu^{s\dagger} U \Psi_D + \zeta'^\dagger \chi_\mu^{s\dagger} V \Psi_D . \qquad (78)$$

Here the equation has been projected onto ζ' arbitrarily. The same result is obtained by projection onto ζ''. The last remark should perhaps be explained further, since the resulting equations do appear different. This is due to the occurence of integrals involving the v and w functions.

Suppose one has an integral over Ω_6 of an integrand having the form $f'g'$, for example. Then by noting that the combination $f'g' - f''g'' = \mathrm{X}'$, some function having $'$ symmetry, one has that

$$\iiiint\!\!\int \mathrm{X}' \, d\Omega_6 = 0 \,, \tag{79}$$

so that

$$\iiiint\!\!\int f'g' \, d\Omega_6 = \iiiint\!\!\int f''g'' \, d\Omega_6 \,. \tag{80}$$

Repeated use of this and related equalities eventually leads to the deduction that the differential equations for $F(\rho)$ are all equivalent, as they should be, regardless of which spin state χ^s_μ and which isospin state, ζ' or ζ'', are used in projecting the Schrödinger equation. For definiteness, $\mu = -\frac{1}{2}$ and ζ' have been used. Thus the left-hand side of (78) reduces to

$$\zeta'^{\dagger} \chi^{s\,\dagger}_{-\frac{1}{2}} (\nabla_6^2 - \chi^2) \Psi_D = -\sqrt{\tfrac{1}{5}} \frac{1}{\rho^2} \mathcal{Y}_2^{\prime\prime 0} \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\chi^2 + \frac{16}{\rho^2}\right) \right] F(\rho) \,, \tag{81}$$

where $-\sqrt{\frac{1}{5}}$ is a Clebsch-Gordan coefficient.

The remaining terms are treated as were those for the S-state radial functions, with a view toward having projection operators factored out of them. There results from this treatment the equation

$$-\sqrt{\frac{1}{5}}\,\mathcal{Y}_2^{0\,\prime\prime}\frac{1}{\rho^2}\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\left(\chi^2+\frac{16}{\rho^2}\right)\right]F(\rho)= \tag{82}$$

$$\left[-2\,\underset{R}{P}^{+}_{(12)}+\underset{R}{P}^{+}_{(23)}+\underset{R}{P}^{+}_{(31)}\right]\left\{\sqrt{\frac{2}{3}}\,V_{\substack{(12)\\33-}}\left(\psi^{a}+\psi''\right)Y_2^0(\hat{\eta})\,+\right.$$

$$\left.\frac{F(\rho)}{\rho^2}\left[-\sqrt{\frac{1}{5}}\,U_{\substack{(12)\\33-}}\mathcal{Y}_2^{0\,\prime\prime}+V_{\substack{(12)\\33-}}\sum_{m+m'=-\frac{1}{2}}C\left(\tfrac{3}{2}\,m'\,2\,m\,|\,\tfrac{1}{2}\,-\tfrac{1}{2}\right)\mathcal{Y}_2^{m\,\prime\prime}\chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{m'}\right]\right\}$$

$$+\left[\underset{R}{P}^{+}_{(23)}-\underset{R}{P}^{+}_{(31)}\right]\left\{-\sqrt{2}\,V_{\substack{(12)\\31+}}\left(-\psi^{s}+\psi'\right)Y_2^0(\hat{\eta})\,+\right.$$

$$\left.\frac{F(\rho)}{\rho^2}\left[-\sqrt{\frac{3}{5}}\,U_{\substack{(12)\\31+}}\mathcal{Y}_2^{0\,\prime}+\sqrt{3}\,V_{\substack{(12)\\31+}}\sum_{m+m'=-\frac{1}{2}}C\left(\tfrac{3}{2}\,m'\,2\,m\,|\,\tfrac{1}{2}\,-\tfrac{1}{2}\right)\mathcal{Y}_2^{m\,\prime}\chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{m'}\right]\right\}.$$

The presence of the projection operators considerably simplifies the integration of this equation. After multiplication by $(\mathcal{Y}_2^{0\,\prime\prime})^*$, one lets the P's operate to the left and uses the relationships from Eqs. (60) and (61).

Finally, we can substitute the following

$$\chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{\frac{3}{2}}=\sqrt{\frac{32\pi}{5}}\,Y_2^2(\hat{\eta}),\qquad \chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{-\frac{1}{2}}=-\sqrt{\frac{16\pi}{5}}\,Y_2^0(\hat{\eta}), \tag{83}$$

$$\chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{\frac{1}{2}}=0,\qquad \chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{-\frac{3}{2}}=-\sqrt{\frac{32\pi}{5}}\,Y_2^{-1}(\hat{\eta}).$$

Then we multiply by $(\mathcal{Y}_2^{\prime\prime 0})^*$ and integrate over Ω_6, setting some integrals identically equal to zero by symmetry arguments such as those already advanced. Omitting the small terms ψ^a, ψ', and ψ'', and keeping only the R_0^0 term in ψ^s, we have

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\left(\chi^2+\frac{16}{\rho^2}\right)\right]F(\rho) = -\sqrt{\frac{15}{2\pi^3}}\,R_0^0\iint(\mathcal{Y}_2^{\prime 0})^*\,V_{31+}^{(12)}\,Y_2^0(\hat{\eta})\,d\Omega_6 + \tag{84}$$

$$F(\rho)\Bigg\{\frac{3}{2}\iint\left[(\mathcal{Y}_2^{\prime\prime 0})^*U_{33-}^{(12)}\mathcal{Y}_2^{\prime\prime 0} + (\mathcal{Y}_2^{\prime 0})^*U_{31+}^{(12)}\mathcal{Y}_2^{\prime 0}\right] +$$

$$\frac{3}{2}\sum_{m+m'=-\frac{1}{2}}C(\tfrac{3}{2}m'2m|\tfrac{1}{2}\,\tfrac{-1}{2})\iint\left[(\mathcal{Y}_2^{\prime\prime 0})^*V_{33-}^{(12)}\mathcal{Y}_2^{\prime\prime m} + (\mathcal{Y}_2^{\prime 0})^*V_{31+}^{(12)}\mathcal{Y}_2^{\prime m}\right]\chi_{-\frac{1}{2}}^{t_s}\hat{S}_{12}\chi_{m'}^{s}\,d\Omega_6\Bigg\}.$$

Despite the formidable appearance of the integrals, they are manageable. Integration over all directions of $\hat{\eta}$ and $\hat{\xi}$ leaves only integrals over θ to be done. The result, from Appendix C, is

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\left(\chi^2+\frac{16}{\rho^2}\right)\right]F(\rho) = -16\sqrt{\frac{3}{\pi^3}}\,R_0^0\int_0^{\frac{\pi}{2}}V_{31+}\cos^4\theta\sin^2\theta\,d\theta \tag{85}$$

$$+F(\rho)\Bigg[\frac{192}{5\pi}\int_0^{\frac{\pi}{2}}U_{31+}\,(\cos^4\theta+\sin^4\theta)\cos^2\theta\sin^2\theta\,d\theta$$

$$-\frac{1248}{35\pi^2}\int_0^{\frac{\pi}{2}}V_{31+}\cos^6\theta\sin^2\theta\,d\theta - \frac{128}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{7}{5}V_{33-}-U_{33-}\right)\cos^4\theta\sin^4\theta\,d\theta\Bigg].$$

The U's and V's are functions of $r_{12}=\sqrt{2}\,\rho\cos\theta$.

The only approximations made are the ones which have been explicitly mentioned. For practical calculations, Eq. (85) is probably more accurate than is needed. Due to the smallness of $F(\rho)$ as compared with $R_0^\circ(\rho)$, and due to the presence of higher powers of the sine and cosine functions in the last two integrals than in the first one, it is expected that sufficient accuracy would be maintained if the $F(\rho)$ term were deleted from the right-hand side of Eq. (85). This is somewhat dependent, however, upon the exact nature of the U's and V's, and should be checked for the particular potentials used.

So far, only nuclear forces have been considered, and to this extent the equations derived are valid for both the H^3 and the He^3 nuclei.

The Coulomb force may be included by means of the electrostatic potential energy operator ,

$$E = \sum_{(ij)} \frac{e^2}{r_{ij}} \left(\frac{1+\tau_3}{2}\right)_i \left(\frac{1+\tau_3}{2}\right)_j \quad . \tag{86}$$

The sum is again over all pairs of nucleons, with $\tau_3 = +1$ corresponding to a proton. The matrix elements $\varphi^{\dagger\alpha} E \varphi^\beta$ are calculated directly and projection operators factored out in a way by now familiar.

The results of the calculation are presented in forms involving factored projection operators, for convenience in later uses. The results are

$$\varphi^{\dagger s} E \varphi^{s} = \varphi^{\dagger a} E \varphi^{a} = \varphi^{\dagger\prime} E \varphi^{\prime} = \varphi^{\dagger\prime\prime} E \varphi^{\prime\prime} \tag{87}$$

$$= \left[\frac{2}{3}\left(P^{+}_{R(12)} + P^{+}_{R(23)} + P^{+}_{R(31)}\right) - 1\right] \frac{e^2}{r_{12}} ,$$

$$\varphi^{\dagger s} E \varphi^{\prime} = \varphi^{\dagger a} E \varphi^{\prime\prime} = \frac{1}{3}\left[-2 P^{+}_{R(12)} + P^{+}_{R(23)} + P^{+}_{R(31)}\right] \frac{e^2}{r_{12}} , \tag{88}$$

$$\varphi^{\dagger s} E \varphi^{a} = \varphi^{\dagger\prime} E \varphi^{\prime\prime} = 0 , \tag{89}$$

$$\varphi^{\dagger s} E \varphi^{\prime\prime} = \varphi^{\dagger a} E \varphi^{\prime} = -\frac{4}{\sqrt{3}}\left[P^{+}_{R(23)} - P^{+}_{R(31)}\right] \frac{e^2}{r_{12}} . \tag{90}$$

The derivation of the equations for the radial functions is similar to the one already given. Since no new techniques are involved there, the work will not be shown. The results are:

$$\hat{D}_2 R^{0}_{0} = R^{0}_{0} \iint v^{0}_{0} \left[\varphi^{\dagger a} U \varphi^{a} + \frac{e^2}{r_{12}}\right] v^{0}_{0}\, d\Omega_6 + \tag{91}$$

$$R^{0}_{4} \iint v^{0}_{0} \left[\varphi^{\dagger a} U \varphi^{a} + \frac{e^2}{r_{12}}\right] v^{0}_{4}\, d\Omega_6 - \rho^2 \iint v^{0}_{0} \varphi^{\dagger a} V \Psi_D\, d\Omega_6 ,$$

$$\hat{D}_6 R^{0}_{4} = R^{0}_{0} \iint v^{0}_{4} \left[\varphi^{\dagger a} U \varphi^{a} + \frac{e^2}{r_{12}}\right] v^{0}_{0}\, d\Omega_6 + \tag{92}$$

$$R^{0}_{4} \iint v^{0}_{4} \left[\varphi^{\dagger a} U \varphi^{a} + \frac{e^2}{r_{12}}\right] v^{0}_{4}\, d\Omega_6 - \rho^2 \iint v^{0}_{4} \varphi^{\dagger a} V \Psi_D\, d\Omega_6 ,$$

$$\hat{D}_4 R_2^1 = R_0^0 \iint v_2^1 \varphi^{\dagger\prime\prime} V \varphi^a v_0^0 \, d\Omega_6 + \rho^2 \iint \omega_2^1 \varphi^{\dagger\prime} V \Psi_D \, d\Omega_6 \tag{93}$$

$$+ R_2^1 \iint \left[v_2^1 \varphi^{\dagger\prime\prime} U \varphi'' v_2^1 - v_2^1 \varphi^{\dagger\prime\prime} U \varphi' w_2^1 \right] d\Omega_6 \; ,$$

$$\hat{D}_8 R_6^{-3} = - R_6^{-3} \iint w_6^3 \frac{e^2}{r_{12}} w_6^3 \, d\Omega_6 + \rho^2 \iint w_6^3 \varphi^{\dagger_3} V \Psi_D \, d\Omega_6 \, , \tag{94}$$

$$\hat{D}_2 F(\rho) = - 16 \sqrt{\frac{3}{\pi^3}} R_0^0 \int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^2\theta \, d\theta \, . \tag{95}$$

The abbreviation $\hat{D}_N = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\chi^2 + \frac{N^2}{\rho^2}\right)$ has been used, and terms proportional to $F(\rho)$ have been dropped from the right-hand side of Eq. (95).

The system of equations has been truncated at K = 6, since the R_K's of higher order constitute of the order of 1% of the total wave function. This figure is based on estimates made in references 12, 22, 23 and which are essentially unchanged for the present treatment of the problem.

The equations are valid for He^3 and, by putting e = 0, for H^3. All terms necessary for the solution of this system of equations have been prepared in this paper in a form such that arbitrary potentials may be immediately used. Eqs. (91)-(95) are written out in detail for reference, but may in fact be truncated further.

In terms of the functions kept, the final wave functions have the form

$$\rho^2 \Psi = -(R_0^0 v_0^0 + R_4^0 v_4^0)\varphi^a + R_2^1(v_2^1 \varphi'' - w_2^1 \varphi') + R_6^{-3} w_6^3 \varphi^s + F(\varphi) \sum_{m+m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}\tfrac{-1}{2})(\zeta' \mathcal{Y}_2''^m - \zeta'' \mathcal{Y}_2'^m)\chi_{m'}^s . \tag{96}$$

The completely antisymmetric spatial part, $R_6^{-3} w_6^3$, has been widely used as being zero. Equation (94) is unlike the other equations in that it is not a truncated expression.

It is clear then, that in the absence of tensor or electrostatic forces the radial function $R_6^{-3}(\rho)$ would be identically zero. That is, there is no continuous function of ρ, finite everywhere in $0 \leq \rho < \infty$ and satisfying the homogeneous part of (94) except the trivial solution $R_6^{-3}(\rho) = 0$. This agrees well with previous beliefs.

In the present case, however, a small R_6^{-3} can exist, and is seen to be due to transitions from the D-state via the tensor interaction.

The radial function $R_2^1(\rho)$, the so-called "mixed-symmetry" term, is unaffected by the presence of the Coulomb interaction. Electrostatic potentials do not occur at all in Eq. (93).

Accordingly, the mixed-symmetry components of the wave functions of H^3 and He^3 are essentially identical.

They differ only slightly, the difference being due to the slightly different solutions for R_0^0 in the two cases.

The D-state owes its existence essentially to transitions from the symmetric S-state by means of the "triplet-even" tensor interaction, V_{31+}.

In convenient units, e^2/r is $1.44/r$ Mev if r is in fermis. For a typical cutoff radius of one fermi, the Coulomb potential is never more than about one Mev. Hence, for particle separations such as occur in the nucleus, the nuclear forces are typically one to two orders of magnitude larger than the Coulomb force. The Coulomb interaction may therefore be generally neglected in the equations for the radial functions, and in that approximation the spatial part of the nuclear wave functions for H^3 and He^3 should be identical.

Another interesting feature of the equations for the radial functions is the particular form of the matrix elements $\varphi^{\dagger\alpha} U \varphi^{\beta}$. From Table 4 we see that

$$\varphi^{\dagger a} U \varphi^{a} = \frac{1}{2} \sum_{(ij)} \left[U_{31+}^{(ij)} + U_{13+}^{(ij)} \right] P_{R(ij)}^{+} .$$

This is the only matrix element which occurs in the truncated equations, except for the equation for R_2^1. Only in the latter equation are the potentials U_{33-} and U_{11-} involved. Since R_2^1 is only a few per cent of the total wave function, it is clear that U_{33-} and U_{11-} affect the nuclear wave function insignificantly.

This conclusion justifies the setting of U_{33-} and U_{11-} to zero, as has been a common custom in the literature.

The potentials which account almost by themselves for the entire behavior of the nuclear wave function are then the central potentials U_{31+} and U_{13+}, and the tensor potential V_{31+}. All other potentials make their effects felt by what amount to small perturbations in the radial wave functions.

Finally, the normalization integral

$$\int_{\rho=0}^{\infty} \cdot \iiiint\!\!\!\iint_{\Omega_6} \Psi^{\dagger} \Psi \, d\Omega_6 \cdot \rho^5 d\rho = 1 \qquad (97)$$

implies the normalization of the radial functions to be

$$\int_0^{\infty} \left[(R_0^0)^2 + (R_4^0)^2 + 2(R_2^1)^2 + (R_6^{-3})^2 + 2(F\varphi)^2 \right] \rho \, d\rho = 1 . \qquad (98)$$

Since

$$\rho^2 = \frac{1}{3}\left(r_{12}^2 + r_{23}^2 + r_{31}^2 \right) ,$$

the boundary conditions must be:

$$R_K^{\nu}(0) = F(0) = 0 , \qquad R_K^{\nu}(\infty) = F(\infty) = 0 . \qquad (99)$$

As has been pointed out, the present method may also be used with average effective pair interactions. Typical of these are the interactions of Nestor <u>et al</u>,[24] which are taken to be the same between any two nucleons. The inclusion of this type of force is done by similar methods to those already presented, and one finds the radial equations for average effective central and tensor forces to be

$$\hat{D}_2 R_0^0 = 3R_0^0 \iiiint\!\!\iint v_0^0 \bar{U} v_0^0 \, d\Omega_6 \; - \; 3\rho^2 \iiiint\!\!\iint v_0^0 \varphi^{t_2} \bar{V} \Psi_D \, d\Omega_6 \,, \tag{100}$$

$$\hat{D}_6 R_4^0 = 3R_4^0 \iiiint\!\!\iint v_4^0 \bar{U} v_4^0 \, d\Omega_6 \; - \; 3\rho^2 \iiiint\!\!\iint v_4^0 \varphi^{t_2} \bar{V} \Psi_D \, d\Omega_6 \,, \tag{101}$$

$$\hat{D}_8 R_6^{-3} = 3R_6^{-3} \iiiint\!\!\iint w_6^3 \bar{U} w_6^3 \, d\Omega_6 \; + \; 3\rho^2 \iiiint\!\!\iint w_6^3 \varphi^{t_5} \bar{V} \Psi_D \, d\Omega_6 \,, \tag{102}$$

$$\hat{D}_4 R_2^1 = 3R_2^1 \iiiint\!\!\iint v_2^1 \bar{U} v_2^1 \, d\Omega_6 \; + \; 3\rho^2 \iiiint\!\!\iint v_2^1 \varphi^{t_{ll}} \bar{V} \Psi_D \, d\Omega_6 \,, \tag{103}$$

$$\hat{D}_4 F(\rho) = 3F(\rho) \iiiint\!\!\iint (\mathcal{Y}_2^{\prime 0})^* (\bar{U} + \bar{V}) \mathcal{Y}_2^{\prime 0} \, d\Omega_6 \,. \tag{104}$$

These equations have not been truncated, although the system is cut off at R_6^{-3} .

In each equation, $\bar{U}$ and $\bar{V}$ are the given average potentials written as functions of $r_{12} = \sqrt{2}\,\rho \cos\theta$. The integrations are carried out over $\cos^2\theta \sin^2\theta \, d\theta \, d\Omega_\eta \, d\Omega_\xi$. In the case using average interactions, the equations for the radial functions are coupled only through the tensor interaction and the D-state. The D-state function $F(\rho)$ is determined independently of the R's. Since it is the connecting link between the R-equations via the tensor force, the small integrals which are the last terms in Eqs. (100)-(104) may not be dropped, as they are necessary for establishing the relative magnitudes of the R's.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Central and tensor forces have been included for the bound three-nucleon system containing S, D, and S′ components in its wave function. The central and the tensor forces may be spin-isospin-parity dependent, or they may be average effective forces depending only upon the separation of two nucleons.

A system of equations for each of these cases is given, from which the radial functions and binding energies can be found. The systems of equations have been given in explicit form with instructions for calculating all terms, and are solvable for arbitrary potentials by relatively standard methods on an electronic computer. This is in sharp contrast to the usual treatments of the three-nucleon problem.

These radial functions can be inserted immediately into the expansion given for the total wave function, and the resulting wave function will obey the generalized Pauli principle.

The action of a given set of potentials in a problem other than a scattering problem can be deduced from the solutions of the equations given. This furnishes an important added criterion that potential functions must satisfy. Since many potentials describe low-energy nucleon-nucleon scattering well, but the nuclear state of matter is sensitive to details in the potentials that low-energy scattering is not, a possible way of choosing from among these potentials has been presented.

It has further been shown that, except for the change of isospin functions, the wave functions of He^3 and H^3 will be virtually identical. In each case, they are dominated by the symmetric space function depending only upon the rms separation of nucleons. This provides a justification from first principles of the widely-used phenomenological Irving-Gunn wave function, which depends only on the variable ρ.

The spatially antisymmetric part of the wave function has been shown in the present approximation to be identically zero in the absence of tensor and Coulomb forces. When these forces, or the tensor force alone, are present, the spatially antisymmetric state is connected to the other states through transitions from the D-state via the tensor interaction.

The "mixed symmetry" radial function $'R$ is unaffected by the Coulomb interaction, and is therefore identical in the H^3 and He^3 nuclei.

The central potentials U_{31+} and U_{13+}, and the tensor potential V_{31+}, have been shown to be the principal interactions needed for describing the behavior of these nuclei. All other central and tensor potentials enter into the problem only in relatively insignificant ways.

APPENDIX A

We write the tensor interaction as the sum of pair interactions:

$$V = \sum_{(ij)} V(ij) \tag{A1}$$
$$= \sum_{(ij)} \left(V_{33-}(ij) P^{+}_{\sigma}(ij) P^{+}_{\tau}(ij) P^{-}_{R}(ij) + V_{31+}(ij) P^{+}_{\sigma}(ij) P^{-}_{\tau}(ij) P^{+}_{R}(ij) \right) \hat{S}_{ij} .$$

All of the operators commute. The action of $P^{+}_{\sigma}(ij)$ on Ψ_D is that of the identity operator, since Ψ_D contains the factor χ^s.

To evaluate $\varphi^{\dagger a} V \Psi_D$, we recall that it will eventually be multiplied by a symmetric angular function and integrated over Ω_6. The parts of the integrand which have $P^{-}_{R}(ij)$ as a factor therefore give no contribution to the integrals, and we consider only the part

$$\varphi^{\dagger a} \sum_{(ij)} V_{31+}(ij) P^{-}_{\tau}(ij) \hat{S}_{ij} \Psi_D . \tag{A2}$$

We substitute the following expressions ,

$$\varphi^{\dagger a} = \frac{1}{\sqrt{2}} \left(\chi^{\dagger\prime} \zeta^{\dagger\prime\prime} - \chi^{\dagger\prime\prime} \zeta^{\dagger\prime} \right) , \tag{A3}$$

$$\Psi_D = \frac{F(\rho)}{\rho^2} \sum_{m+m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}\tfrac{-1}{2})\left(\zeta'\mathcal{Y}_2^{''m} - \zeta''\mathcal{Y}_2^{'m}\right)\chi^s_{m'}\,, \quad \text{(A4)}$$

into Eq. (A2), and carry out the algebra. A partial result is

$$\phi^{\dagger a} V(ij)\Psi_D = \frac{F(\rho)}{\sqrt{2}\rho^2} V_{31+}^{(ij)} \sum_{m+m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}\tfrac{-1}{2}) \times \quad \text{(A5)}$$
$$\Big\{\Big[(\zeta^{\dagger ''}P^-_\tau(ij)\zeta')\mathcal{Y}_2^{''m} - (\zeta^{\dagger ''}P^-_\tau(ij)\zeta'')\mathcal{Y}_2^{'m}\Big]\chi^{\dagger\prime}\hat{S}_{ij}\chi^s_{m'}$$
$$+\Big[(\zeta^{\dagger\prime}P^-_\tau(ij)\zeta'')\mathcal{Y}_2^{'m} - (\zeta^{\dagger\prime}P^-_\tau(ij)\zeta')\mathcal{Y}_2^{''m}\Big]\chi^{\dagger ''}\hat{S}_{ij}\chi^s_{m'}\Big\}.$$

The matrix elements $\zeta^\dagger P^-_\tau \zeta$ can be evaluated by tedious algebra directly from the definitions. Substitution of these numbers gives terms like

$$\phi^{\dagger a} V(23)\Psi_D = \frac{F(\rho)}{\sqrt{2}\rho^2} V_{31+}^{(23)} \sum_{m+m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}\tfrac{-1}{2}) \times \quad \text{(A6)}$$
$$\left[\left(-\tfrac{\sqrt{3}}{4}\mathcal{Y}_2^{''m} - \tfrac{1}{4}\mathcal{Y}_2^{'m}\right)\chi^{\dagger\prime}\hat{S}_{23}\chi^s_{m'} - \left(\tfrac{3}{4}\mathcal{Y}_2^{''m} + \tfrac{\sqrt{3}}{4}\mathcal{Y}_2^{'m}\right)\chi^{\dagger ''}\hat{S}_{23}\chi^s_{m'}\right].$$

By referring to the transformation matrices of Eq. (1), we see that the combinations of $\mathcal{Y}$'s in (A6) are given in terms of permutation operators to be

$$-\tfrac{\sqrt{3}}{4}\mathcal{Y}_2^{''m} - \tfrac{1}{4}\mathcal{Y}_2^{'m} = \tfrac{1}{2}(31)_R\mathcal{Y}_2^{'m}\,, \quad \text{(A8)}$$

$$\tfrac{3}{4}\mathcal{Y}_2^{''m} + \tfrac{\sqrt{3}}{4}\mathcal{Y}_2^{'m} = -\tfrac{\sqrt{3}}{2}(31)_R\mathcal{Y}_2^{'m}\,. \quad \text{(A9)}$$

The matrix elements $\chi^{\dagger} \hat{S}_{ij} \chi^{s}$ have been given in Table 3, and are proportional to spherical harmonics about the directions $\hat{r}_{ij}$. It follows that

$$\left(-\frac{\sqrt{3}}{4} \mathcal{Y}_2^{\prime\prime m} - \frac{1}{4} \mathcal{Y}_2^{\prime m}\right) \chi^{\dagger\prime} \hat{S}_{23} \chi^{s}_{m'} \propto Y_{2(23)}^{m'+\frac{1}{2}} \left[(31)_R \mathcal{Y}_2^{\prime m} \right] \tag{A10}$$
$$= (31)_R \left[\mathcal{Y}_2^{\prime m} Y_{2(12)}^{m'+\frac{1}{2}} \right].$$

In this manner one finds that

$$\varphi^{\dagger a} V_{(23)} \Psi_D = (31)_R \left[\frac{F(\rho)}{\rho^2 \sqrt{30}} V_{31+}^{(12)} \left(4 \mathcal{Y}_2^{\prime -2} Y_{2(12)}^{2} - \right.\right. \tag{A11}$$
$$\left.\left. 3 \mathcal{Y}_2^{\prime -1} Y_{2(12)}^{1} + 2 \mathcal{Y}_2^{\prime 0} Y_{2(12)}^{0} - \mathcal{Y}_2^{\prime 1} Y_{2(12)}^{-1} \right) \right].$$

The expressions for other (ij) differ from this one only in the permutation operator. The result turns out to be

$$\varphi^{\dagger a} V \Psi_D = \sum_{(ij)} \varphi^{\dagger a} V_{(ij)} \Psi_D \tag{A12}$$
$$= \left[(12)_R + (23)_R + (31)_R \right] \frac{F(\rho) V_{31+}^{(12)}}{\rho^2 \sqrt{30}} \sum_{m=-1}^{2} (-)^m (m+2) \mathcal{Y}_2^{\prime -m} Y_{2(12)}^{m}.$$

The complete symmetry of $\varphi^{\dagger a} V \Psi_D$ is explicitly shown by the presence of the permutation operators. The five-fold integrals which occur can be reduced to single integrals as in the following example.

$$\iiiint \mathcal{V}_o^{\circ} \mathcal{P}^{\dagger_2} V \Psi_D \, d\Omega_6 = \iiiint 3\mathcal{V}_o^{\circ} \frac{F(\rho)\, V_{31+}(12)}{\rho^2 \sqrt{30}} \sum_{m=-1}^{2} (-)^m (m+2) \mathcal{Y}_2^{1-m} Y_2^m(12)\, d\Omega_6 \quad \text{(A13)}$$

We operate to the left on $\mathcal{V}_o^{\circ}$ with $(12)_R + (23)_R + (31)_R$ to replace $\mathcal{V}_o^{\circ}$ with $3\mathcal{V}_o^{\circ}$, and substitute $\sqrt{2}\rho\cos\theta$ for r_{12} to get

$$\sqrt{\frac{3}{10\pi^3}} \frac{F(\rho)}{\rho^2} \sum_{m=-1}^{2} (-)^m (m+2) \int_{\theta=0}^{\frac{\pi}{2}} \iint_{\Omega_\eta} \iint_{\Omega_\xi} \mathcal{Y}_2^{1-m} Y_2^m(\hat{\eta}) V_{31+}(12) \cos^2\theta \sin^2\theta \, d\Omega \, d\Omega \, d\theta. \text{(A14)}$$

Since $\mathcal{Y}_2^{1-m} \propto \cos^2\theta \, Y_2^{-m}(\hat{\eta}) - \sin^2\theta \, Y_2^{-m}(\hat{\xi})$, from Table 2, and $Y_2^{-m} = (-)^m (Y_2^m)^*$, we can use the orthonormality of the ordinary spherical harmonics Y_2^m and do the integrations over all directions of $\hat{\eta}$ and $\hat{\xi}$. Both ρ and θ are constants in the latter integrations. Thus, the integral in (A14), for m = 2, is

$$\int_{\theta=0}^{\frac{\pi}{2}} \iint_{\Omega_\eta} \iint_{\Omega_\xi} \frac{4}{\pi}\sqrt{\frac{2}{5}} \left[\cos^2\theta \, Y_2^{-2}(\hat{\eta}) - \sin^2\theta \, Y_2^{-2}(\hat{\xi})\right] V_{31+}(12) \left(Y_2^{-2}(\hat{\eta})\right)^* \cos^2\theta \sin^2\theta \, d\theta \, d\Omega_\eta \, d\Omega_\xi \quad \text{(A15)}$$

$$= 16\sqrt{\frac{2}{5}} \int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^2\theta \, d\theta.$$

Similar results are obtained using the other values of m, and the final expression is

$$\iiiint \mathcal{V}_o^{\circ} \mathcal{P}^{\dagger_2} V \Psi_D \, d\Omega_6 = \frac{64}{5}\sqrt{\frac{3}{\pi^3}} \frac{F(\rho)}{\rho^2} \int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^2\theta \, d\theta. \quad \text{(A16)}$$

APPENDIX B

The expression

$$\Psi_D = \rho^{-2} F(\rho) \sum_{m+m'=-\frac{1}{2}} C(2 m \tfrac{3}{2} m' | \tfrac{1}{2} \tfrac{-1}{2}) (\zeta' \mathcal{Y}_2^{''m} - \zeta'' \mathcal{Y}_2^{'m}) \chi_{m'}^{s} , \tag{B1}$$

multiplied by $\varphi^{\dagger s} V$, is

$$\varphi^{\dagger s} V \Psi_D = \frac{F(\rho)}{\sqrt{2}\rho^2} (\chi^{\dagger ''} \zeta^{\dagger ''} + \chi^{\dagger '} \zeta^{\dagger '}) V \sum_{m+m'=-\frac{1}{2}} C(2 m \tfrac{3}{2} m' | \tfrac{1}{2} \tfrac{-1}{2}) (\zeta' \mathcal{Y}_2^{''m} - \zeta'' \mathcal{Y}_2^{'m}) \chi_{m'}^{s} \tag{B2}$$

$$= \frac{F(\rho)}{\sqrt{2}\rho^2} \sum_{m-m'=-\frac{1}{2}} C(2 m \tfrac{3}{2} m' | \tfrac{1}{2} \tfrac{-1}{2}) \left[\chi^{\dagger '} V \chi_{m'}^{s} \mathcal{Y}_2^{''m} - \chi^{\dagger ''} V \chi_{m'}^{s} \mathcal{Y}_2^{'m} \right] .$$

For average tensor forces, V is given by

$$V = \sum_{(ij)} V(ij) \hat{S}_{ij} . \tag{B3}$$

Using this definition and the results from Table 3, we find that

$$\chi^{\dagger '} V \chi_{\frac{3}{2}}^{s} = 2 \sqrt{\frac{2}{3}} \left[Y_2^2(12) V(12) - \frac{1}{2} Y_2^2(23) V(23) - \frac{1}{2} Y_2^2(31) V(31) \right] , \tag{B4}$$

and

$$\chi^{\dagger\prime\prime} V \chi^{s}_{\frac{3}{2}} = \sqrt{2}\left[\mathcal{Y}_{2}^{2}(31) V(31) - \mathcal{Y}_{2}^{2}(23) V(23) \right], \tag{B5}$$

are typical of the forms that result.

We define the Υ functions to be

$$\Upsilon^{\prime m} = \sqrt{\frac{2}{3}}\left[-\mathcal{Y}_{2}^{m}(12) V(12) + \frac{1}{2}\mathcal{Y}_{2}^{m}(23) V(23) + \frac{1}{2}\mathcal{Y}_{2}^{m}(31) V(31) \right], \tag{B6}$$

$$\Upsilon^{\prime\prime m} = \sqrt{\frac{1}{2}}\left[\mathcal{Y}_{2}^{m}(23) V(23) - \mathcal{Y}_{2}^{m}(31) V(31) \right]. \tag{B7}$$

The V's are identical functional forms of the different r_{ij}'s. Using this fact, one can show that the pair $(\Upsilon^{\prime m}, \Upsilon^{\prime\prime m})$ transforms under permutations of spatial coordinates according to the two-dimensional representation being used.

Substituting (B4)-(B7) into the last expression in (B2), and rearranging terms leads to

$$\varphi^{\dagger s} V \Psi_{D} = \frac{\sqrt{6\pi}}{5}\frac{F(\rho)}{\rho^{2}} \sum_{m=-1}^{2} (-)^{m}(m+2)\left(\Upsilon^{\prime m} \mathcal{Y}_{2}^{\prime\prime -m} - \Upsilon^{\prime\prime m} \mathcal{Y}_{2}^{\prime -m} \right). \tag{B8}$$

It is clear that the transformation properties of $\varphi^{\dagger s} V \Psi_{D}$ under permutations are contained completely in the bracketed term in the summand. Comparison of that term with the expression

$$\varphi^{a} = \sqrt{\tfrac{1}{2}}\left(\chi'\zeta'' - \chi''\zeta'\right) \tag{B9}$$

shows that $\varphi^{\dagger s} V \Psi_{D}$ is antisymmetric under permutations.

The other $\varphi^{\dagger} V \Psi_{D}$'s are expanded similarly to get Eqs. (70) through (73).

APPENDIX C

We start with Eq. (82):

$$-\frac{\mathcal{Y}_2^{''0}}{\rho^2\sqrt{5}}\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho^2}\frac{d}{d\rho}-\left(\chi^2+\frac{16}{\rho^2}\right)\right]F(\rho) = \tag{C1}$$

$$\left[-2P^{+}_{R(12)}+P^{+}_{R(23)}+P_{R(31)}\right]\left\{\sqrt{\frac{2}{3}}\,V^{(12)}_{33-}(\psi^{a}+\psi'')\,Y^{0}_{2}(\hat{\eta})+\right.$$

$$\left.\frac{F(\rho)}{\rho^2}\left[-\sqrt{\frac{1}{5}}\,U^{(12)}_{33-}\,\mathcal{Y}_2^{''0}+V^{(12)}_{33-}\sum_{m+m'=-\frac{1}{2}}C(2m\tfrac{3}{2}m'|\tfrac{1}{2}-\tfrac{1}{2})\,\mathcal{Y}_2^{''m}\,\chi^{t_s}_{-\frac{1}{2}}\,\hat{S}_{12}\,\chi^{s}_{m'}\right]\right\}$$

$$+\left[P^{+}_{R(23)}-P^{+}_{R(31)}\right]\left\{-\sqrt{2}\,V^{(12)}_{31+}(-\psi^{s}+\psi')\,Y^{0}_{2}(\hat{\eta})+\right.$$

$$\left.\frac{F(\rho)}{\rho^2}\left[-\sqrt{\frac{3}{5}}\,U^{(12)}_{31+}\,\mathcal{Y}_2^{'0}+\sqrt{3}\,V^{(12)}_{31+}\sum_{m+m'=-\frac{1}{2}}C(2m\tfrac{3}{2}m'|\tfrac{1}{2}-\tfrac{1}{2})\,\mathcal{Y}_2^{'m}\,\chi^{s}_{-\frac{1}{2}}\,\hat{S}_{12}\,\chi^{s}_{m'}\right]\right\}.$$

We then multiply by $(\mathcal{Y}_2^{''0})^*$ and integrate over Ω_6. The projection operators in the integrand may operate to the left, and we substitute

$$\left(-2P^{+}_{R(12)}+P^{+}_{R(23)}+P^{+}_{R(31)}\right)\left(\mathcal{Y}_2^{''0}\right)^* = \frac{3}{2}\left(\mathcal{Y}_2^{''0}\right)^*, \tag{C2}$$

$$\left(P_{R(23)}-P_{R(31)}\right)\left(\mathcal{Y}_2^{''0}\right)^* = \frac{\sqrt{3}}{2}\left(\mathcal{Y}_2^{'0}\right)^*. \tag{C3}$$

Discarding the small terms ψ^{a} and ψ'', we find the factors from the right-hand side of (C1) resulting from the use of (C2) to be

$$-\sqrt{\frac{1}{5}}\,\frac{F(\rho)}{\rho^2}\iiiint\!\!\iint \frac{3}{2}\left(\mathcal{Y}_2^{\prime\prime 0}\right)^{*} U_{\substack{(12)\\33-}}\,\mathcal{Y}_2^{\prime\prime 0}\sin^2\theta\cos^2\theta\, d\theta\, d\Omega_\eta\, d\Omega_\xi \qquad (C4)$$

$$+\frac{F(\rho)}{\rho^2}\iiiint\!\!\iint \frac{3}{2}\left(\mathcal{Y}_2^{\prime\prime 0}\right)^{*} V_{\substack{(12)\\33-}}\sum_{m-m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}\tfrac{-1}{2})\,\mathcal{Y}_2^{\prime\prime m}\chi^{\dagger s}_{\frac{1}{2}}\hat{S}_{12}\chi^{s}_{m'}\sin^2\theta\cos^2\theta\, d\theta\, d\Omega_\xi\, d\Omega_\eta .$$

In the first term of (C4), the integrals over $d\Omega_\eta\, d\Omega_\xi$ can be done with the aid of Table 2, and there results the expression

$$-\frac{128}{\pi\sqrt{5}}\,\frac{F(\rho)}{\rho^2}\int_0^{\frac{\pi}{2}} U_{33-}(\sqrt{2}\,\rho\cos\theta)\sin^4\theta\cos^4\theta\, d\theta . \qquad (C5)$$

In the second term of (C4), the factors $\chi^{\dagger s}_{-\frac{1}{2}}\hat{S}_{12}\chi^{s}_{m'}$ are proportional to the spherical harmonics $Y_2^{m'+\frac{1}{2}}(\hat{\eta})$, where the direction of $\hat{\eta}$ is the same as the direction of $\hat{r}_{12}$. The separate integrals occurring in the second term are multiplied by Clebsch-Gordan coefficients and summed over $m + m' = -\frac{1}{2}$. Typical of the integrals is

$$\iiiint \left(\mathcal{Y}_2^{\prime\prime 0}\right)^* \underset{33-}{V_{(12)}} \mathcal{Y}_2^{\prime\prime -2} \chi^{\dagger s}_{-\frac{1}{2}} \hat{S}_{12} \chi^{s}_{-\frac{3}{2}} \, d\Omega_6 \tag{C6}$$

$$= \sqrt{\frac{32\pi}{5}} \iiiint \underset{33-}{V}(\sqrt{2}\rho\cos\theta) \left(\mathcal{Y}_2^{\prime\prime 0}\right)^* \mathcal{Y}_2^{\prime\prime -2} Y_2^2(\hat{\eta}) \sin^2\theta \cos^2\theta \, d\theta \, d\Omega_\eta \, d\Omega_\xi$$

$$= \frac{1024}{3\pi}\sqrt{\frac{\pi}{5}} \iint \overset{*}{Y_1^1}(\hat{\eta}) \, Y_1^{-1}(\hat{\eta}) \, Y_2^2(\hat{\eta}) \, d\Omega_\eta \cdot \int_0^{\frac{\pi}{2}} \underset{33-}{V}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^4\theta \, d\theta$$

$$= -\frac{1024}{15\pi\sqrt{2}} \int_0^{\frac{\pi}{2}} \underset{33-}{V}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^4\theta \, d\theta .$$

The second term of (C4), with results of this kind substituted for each term of the sum, has the form

$$\frac{896}{5\pi\sqrt{5}} \cdot \frac{F(\rho)}{\rho^2} \int_0^{\frac{\pi}{2}} \underset{33-}{V}(\sqrt{2}\rho\cos\theta) \cos^4\theta \sin^4\theta \, d\theta . \tag{C7}$$

Expression (C4) thus equals

$$\frac{128}{\pi\sqrt{5}} \cdot \frac{F(\rho)}{\rho^2} \int_0^{\frac{\pi}{2}} \left(\frac{7}{5} V(\sqrt{2}\rho\cos\theta) - U(\sqrt{2}\rho\cos\theta) \right) \sin^4\theta \cos^4\theta \, d\theta . \tag{C8}$$

We now consider the second term on the right of Eq. (C1). This term has a factor of $P^+_{R(23)} - P^+_{R(31)}$

in front of it. We multiply by $\left(\mathcal{Y}_2^{''0}\right)^*$ and operate to the left in the integrand with the projection operators, and use

$$\left(P^+_{R(23)} - P^+_{R(31)}\right)\left(\mathcal{Y}_2^{''m}\right)^* = \frac{\sqrt{3}}{2}\,\mathcal{Y}_2^{'m} \tag{C9}$$

to get the expression

$$-\sqrt{\frac{3}{2}}\iiiint\!\!\iint\left(\mathcal{Y}_2^{'0}\right)^* \underset{31+}{V_{(12)}}\left(-\psi^s + \psi'\right) Y_2^0(\hat{\eta})\, d\Omega_6 \tag{C10}$$

$$-\frac{3}{2\sqrt{5}}\frac{F(\rho)}{\rho^2}\iiiint\!\!\iint\left(\mathcal{Y}_2^{'0}\right)^* \underset{31+}{U_{(12)}}\,\mathcal{Y}_2^{'0}\, d\Omega_6$$

$$+\frac{3}{2}\frac{F(\rho)}{\rho^2}\sum_{m+m'=-\frac{1}{2}} C(2m\tfrac{3}{2}m'|\tfrac{1}{2}-\tfrac{1}{2})\iiiint\!\!\iint\left(\mathcal{Y}_2^{'0}\right)^* \underset{31+}{V_{(12)}}\,\mathcal{Y}_2^{'m}\,\chi^{\dagger s}_{-\frac{1}{2}}\,\hat{S}_{12}\,\chi^s_{m'}\, d\Omega_6 .$$

In the first term of this expression, we discard the small quantity ψ', and substitute $\pi^{-\frac{3}{2}}\rho^{-2}R_0^0$ for ψ^s. This term then becomes

$$\frac{4R_0^0}{\pi\rho^2}\sqrt{\frac{3}{5\pi^3}}\iiiint\!\!\iint\Big[\left(\cos^2\theta\; Y_2^{0*}(\hat{\eta}) - \sin^2\theta\; Y_2^{0*}(\hat{\xi})\right)\underset{31+}{V}(\sqrt{2}\rho\cos\theta)\times \tag{C11}$$

$$Y_2^0(\hat{\eta})\sin^2\theta\cos^2\theta\, d\theta\, d\Omega_\eta\, d\Omega_\xi\Big]$$

$$= \frac{16R_0^0}{\rho^2}\sqrt{\frac{3}{5\pi^3}}\int_0^{\frac{\pi}{2}} \underset{31+}{V}(\sqrt{2}\rho\cos\theta)\cos^4\theta\sin^2\theta\, d\theta .$$

The second term in (C10) is equally simple, and becomes

$$-\frac{192\,F(\rho)}{5\pi\sqrt{5}\,\rho^2}\int_0^{\frac{\pi}{2}} U_{31+}(\sqrt{2}\rho\cos\theta)\left(\cos^4\theta+\sin^4\theta\right)\cos^2\theta\,\sin^2\theta\,d\theta. \tag{C12}$$

The final part of (C10) is evaluated as will be illustrated here for the term m = -2, m′= 3/2:

$$\iiiint\!\!\iint \left(\mathcal{Y}_2^{\prime 0}\right)^* V_{31+}(12)\,\mathcal{Y}_2^{\prime -2}\,\chi^{\dagger s}_{-\frac{1}{2}}\,\hat{S}_{12}\,\chi^{s}_{\frac{3}{2}}\,d\Omega_6 \tag{C13}$$

$$=\frac{32}{5\pi^2}\iiiint\!\!\iint\left(\cos^2\theta\,\overset{*}{Y_2^0}(\hat{\eta})-\sin^2\theta\,\overset{*}{Y_2^0}(\hat{\xi})\right)V_{31+}(12)\left(\cos^2\theta\,Y_2^{-2}(\hat{\eta})-\sin^2\theta\,Y_2^{-2}(\hat{\xi})\right)\sqrt{\frac{32\pi}{5}}\,Y_2^2(\hat{\eta})\sin^2\theta\cos^2\theta\,d\theta\,d\Omega_\eta\,d\Omega_\xi$$

$$=\frac{32}{5\pi^2}\sqrt{\frac{32\pi}{5}}\int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta)\cos^6\theta\,\sin^2\theta\,d\theta\cdot\iint_{\Omega_\eta}\overset{*}{Y_2^0}(\hat{\eta})\,Y_2^{-2}(\hat{\eta})\,Y_2^2(\hat{\eta})\,d\Omega_\eta\,.$$

The integral over the spherical harmonics is found to be $-\frac{1}{7}\sqrt{\frac{5}{\pi}}$. Hence, (C13) is equal to

$$-\frac{128\sqrt{2}}{35\pi^2}\int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta)\cos^6\theta\,\sin^2\theta\,d\theta\,. \tag{C14}$$

The terms for other values of m and m' are evaluated the same way, multiplied by Clebsch-Gordan coefficients, and summed. The last term of (C10) is then equal to

$$\frac{1248\, F(\rho)}{35\pi^2\sqrt{5}\,\rho^2}\int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta)\cos^6\theta\,\sin^2\theta\,d\theta. \tag{C15}$$

Combining the results of this appendix gives the equation for $F(\rho)$ to be

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\left(\chi^2+\frac{16}{\rho^2}\right)\right]F(\rho) = -16\sqrt{\frac{3}{\pi^3}}\,R_0^0\int_0^{\frac{\pi}{2}} V_{31+}(12)\cos^4\theta\,\sin^2\theta\,d\theta + \tag{C16}$$

$$F(\rho)\left[\frac{192}{5\pi}\int_0^{\frac{\pi}{2}} U_{31+}(12)\left(\cos^4\theta+\sin^4\theta\right)\cos^2\theta\,\sin^2\theta\,d\theta\right.$$

$$-\frac{1248}{35\pi^2}\int_0^{\frac{\pi}{2}} V_{31+}(12)\cos^6\theta\,\sin^2\theta\,d\theta$$

$$\left.-\frac{128}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{7}{5}V_{33-}(12)-U_{33-}(12)\right)\cos^4\theta\,\sin^4\theta\,d\theta\right].$$

The trigonometric functions in the integral multiplying R_0^0 are, loosely speaking, collectively to the sixth power. In the integrals multiplying $F(\rho)$, the trigonometric functions are, in this sense, to the eighth power. Hence these latter integrals will be somewhat smaller than the former one, if all the V's and U's are about the same size.

Further, R_0^0 is expected to be roughly 20 times as large as $F(\rho)$, and is multiplied by $16\sqrt{3} \approx 28$. It appears, then, that the dominant inhomogeneous term appearing in (C16) is the term containing R_0^0. If we keep only this term, we have

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left(\chi^2 + \frac{16}{\rho^2}\right)\right] F(\rho) \approx -16\sqrt{\frac{3}{\pi^3}}\, R_0^0 \int_0^{\frac{\pi}{2}} V_{31+}(\sqrt{2}\rho\cos\theta)\cos^4\theta \sin^2\theta \, d\theta . \qquad \text{(C17)}$$

The validity of this equation is, however, contingent upon the small size of the total term multiplying $F(\rho)$ in Eq. (C16). Hence, when calculations are to be made, the size of that term should be investigated for the potentials being used.

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