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THE EFFECT OF UNIFORMLY SPACED ELASTIC SUPPORTS ON THE MODAL PATTERNS AND NATURAL FREQUENCIES OF A UNIFORM CIRCULAR RING

by

Joseph William McKinley

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I hereby recommend that this dissertation prepared under my direction by Joseph William McKinley

entitled THE EFFECT OF UNIFORMLY SPACED ELASTIC SUPPORTS ON THE MODAL PATTERNS AND NATURAL FREQUENCIES OF A UNIFORM CIRCULAR RING

be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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SIGNED: Joseph William McKinley

PREFACE

The purpose of this study was to investigate the behavior of the modal patterns of a circular ring supported by equally spaced radial springs. The introduction of radial springs has the effect of elastically coupling the equations of motion of the system and the question arises as to whether there are still an infinity of mode shapes for the supported ring. LaGrange's equations were used in conjunction with Fourier series to provide the coupled equations of motion in matrix form. Resulting solutions were found with the aid of a digital computer.

The facilities of the University Computer Center at The University of Arizona were used, and their contributions, in facilities and personal advice, are gratefully acknowledged.

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ABSTRACT

For an axisymmetric body the vibration modal patterns may be rotated to any position around the axis of symmetry. There are, therefore, an infinity of modal patterns associated with each natural frequency. When uniformly spaced supports of equal stiffness are introduced around the periphery, the elastic properties of the body are no longer axisymmetric. The primary objective of this study is to determine the effects of uniformly spaced elastic supports of equal stiffness on the modal patterns of an axisymmetric body, and in particular to find if there are still an infinity of modal patterns associated with each natural frequency. A relatively minor secondary objective is to investigate the effect of the supports on the natural frequencies of the body.

A uniform circular ring is considered as an example of an axisymmetric body. The mode shapes and natural frequencies for in-plane bending vibration are investigated. The investigation is restricted to inextensional behavior of a thin ring. Rotary inertia and shear distortion of the ring are neglected. Uniformly spaced springs of equal stiffness which restrain motion of the ring in the radial direction are introduced. The number of springs is arbitrary. A Fourier series for radial and transverse displacements is assumed, and energy methods are used to derive the equations of motion in the Fourier coefficients.

Because of the completeness property of Fourier series, no modes will be missed. The equations of motion in matrix form are solved for the natural frequencies and mode shapes using a digital computer.

The stiffness influence coefficients resulting from the springs produce elastic coupling of the equations of motion. Analysis of the coupling patterns reveals that the equations of motion in the cosine series are uncoupled from the equations of motion in the sine series. For each of the series there are several distinct families of modes, each of the families being constructed of a specific set of the harmonics. Given the number of supporting springs, it is easy to predict the number and type of modal families present. Since the specific harmonics present in each family of modes can be identified, it becomes possible to visualize the character of the mode shapes of each family.

A procedure for improving the convergence of the series representation of the mode shapes is developed. This procedure involved the addition of special functions to the Fourier series. The special functions incorporate appropriate discontinuities in certain of the derivatives which makes it possible to represent the shear force discontinuities at the springs better. The procedure is applied to a simply supported beam with a transverse spring at the midpoint. The special function is formed from the solution for the displacement of the beam with a concentrated, static force applied at the midpoint in the transverse direction. The utilization of this function improves the results for mode shape and natural frequency. It especially improves the representation of the shear force distribution. For the ring the primary interest is in the kinds of natural modes and the general character of the mode shapes. Thus, the procedure for improving convergence was not used in the calculations for the supported ring. However, the results were used as a guide in selecting the number of terms needed in the Fourier series solution for the ring.

Analysis of results for the ring problem reveals that for some families of modes an infinity of mode shapes is associated with each of the natural frequencies. For other families, a single modal pattern is associated with each natural frequency. The procedure for finding the modal patterns and natural frequencies for any number of springs and any spring stiffness is given. The results for natural frequency agree with those for the unsupported ring when the spring stiffnesses are equal to zero.

CHAPTER 1

INTRODUCTION

1.1 Motivation for the Study

For an axisymmetric body the vibration modal patterns may be rotated to any position around the axis of symmetry. There are, therefore, an infinity of modal patterns associated with each natural frequency. When uniformly spaced supports of equal stiffness are introduced around the periphery, the elastic properties of the body are no longer axisymmetric. The primary objective of this study is to determine the effects of uniformly spaced elastic supports of equal stiffness on the modal patterns of an axisymmetric body, and in particular to find if there are still an infinity of modal patterns associated with each natural frequency. A relatively minor secondary objective is to investigate the effect of the supports on the natural frequencies of the body.

The modal patterns and natural frequencies of an unconstrained complete ring were found by Love (Ref. 1),[1] who found that the mode shapes are harmonic, having an integer number of wavelengths. As a consequence of the axisymmetry of the ring, the modal patterns may be rotated to any position around the axis of symmetry. Thus, an infinity of modal patterns share the same natural frequency. Love derived the equations of motion by considering the dynamic equilibrium of a ring

[1] Numbers appearing in parenthesis refer to references which are given at the end of this dissertation.

element, neglecting rotary inertia and shear deformation. Solution of these equations of motion yields the natural frequencies ω_i.

$$\omega_i^2 = \frac{i^2(i^2 - 1)^2}{1 + i^2} \frac{EI}{A\rho r^4} \qquad i = 2, 3, \ldots \qquad (1.1)$$

where the mode shapes are given by $\sin i\theta$ and $\cos i\theta$ and $i = 1$ is the rigid body mode.

Wenk (Ref. 2) investigated a ring with a uniform radial elastic support. The ring was subjected to a concentrated impulsive radial load and the resulting radial displacements and in-plane bending moments were found. Wenk showed good agreement between theoretical and experimental results. Wenk was primarily interested in the response of the ring, and did not comment on the nature of the mode shapes. However, since the circular ring with a uniform radial elastic support is an axisymmetric body with an axisymmetric support, there are an infinite number of modes for each natural frequency in this problem. Wenk gave for the natural frequencies the following equation

$$\omega_i^2 = \frac{1}{\rho A} \frac{i^2}{1 + i^2} \left[\frac{EI}{r^4} (1 - i^2)^2 + k \right] \qquad (1.2)$$

where k = the support stiffness

Carrier (Ref. 3) investigated the problem of the in-plane bending vibrations of a rotating circular ring with evenly spaced radial supports which rotate along with the ring. The differential equations of the spinning ring were given, and the solution for the case when the radial supports are rigid was given in detail. Carrier showed that the problem may be solved by considering that a periodically varying force acts at each support. This force will be that which is required to prevent radial displacement of the ring at any time. He then considered the elastic support, and by using the same technique, determined the natural frequencies and mode shapes at which the system may vibrate. Although this work by Carrier will (for the special case of zero angular velocity) give the mode shapes and natural frequencies being considered in this study, Carrier did not investigate in detail the nature of the modes. Carrier does not explicitly solve for natural frequencies or mode shapes. Nor does he determine which harmonic components are present in a mode shape. Carrier's use of a forcing function at the support makes it more difficult to generalize results than this present study, which utilizes a Fourier series approach. Carrier's most interesting result is for the unconstrained, spinning ring, for which he shows the existence of modal patterns which rotate relative to coordinates fixed in the ring. The nodal points thus rotate with respect to the rotating ring.

Lang (Ref. 4) gave the analysis of in-plane vibration of thin circular rings, considering both inextensional and extensional deformation theories. He was concerned with defining the range of validity of the two theories in terms of mode number and thickness to radius ratios. He concluded that inextensional theory is adequate for mode numbers of nine or less if the thickness to radius ratio is less than 0.01. This report gave an excellent basic discussion of in-plane vibrations of complete circular rings.

Many other references exist, but none have been found which discuss the problem of this dissertation.

1.2 Statement of Problem

The effect of uniformly spaced elastic supports on a particular axisymmetric body, a uniform circular ring, will be investigated. The in-plane bending vibrations of the ring will be considered. Since the object of interest is the effect of the supports on the character of the mode shapes, nothing is lost in further restricting the ring to the following:

1. a thin ring
2. inextensional deformation
3. constant cross section
4. elastic supports will constrain radial motion only
5. weight of the elastic supports will be neglected
6. damping is neglected
7. rotary inertia and shear deformation are not considered

This ring is shown in Fig 1.[1]

The primary objective will be to find the effect of the elastic supports on the mode shapes of the ring. In particular, we wish to find if there are still an infinity of modal patterns associated with each natural frequency. A secondary objective will be to find the effect of these supports on the natural frequencies of the ring.

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[1]Figures appear on the page following the page where they are first mentioned.

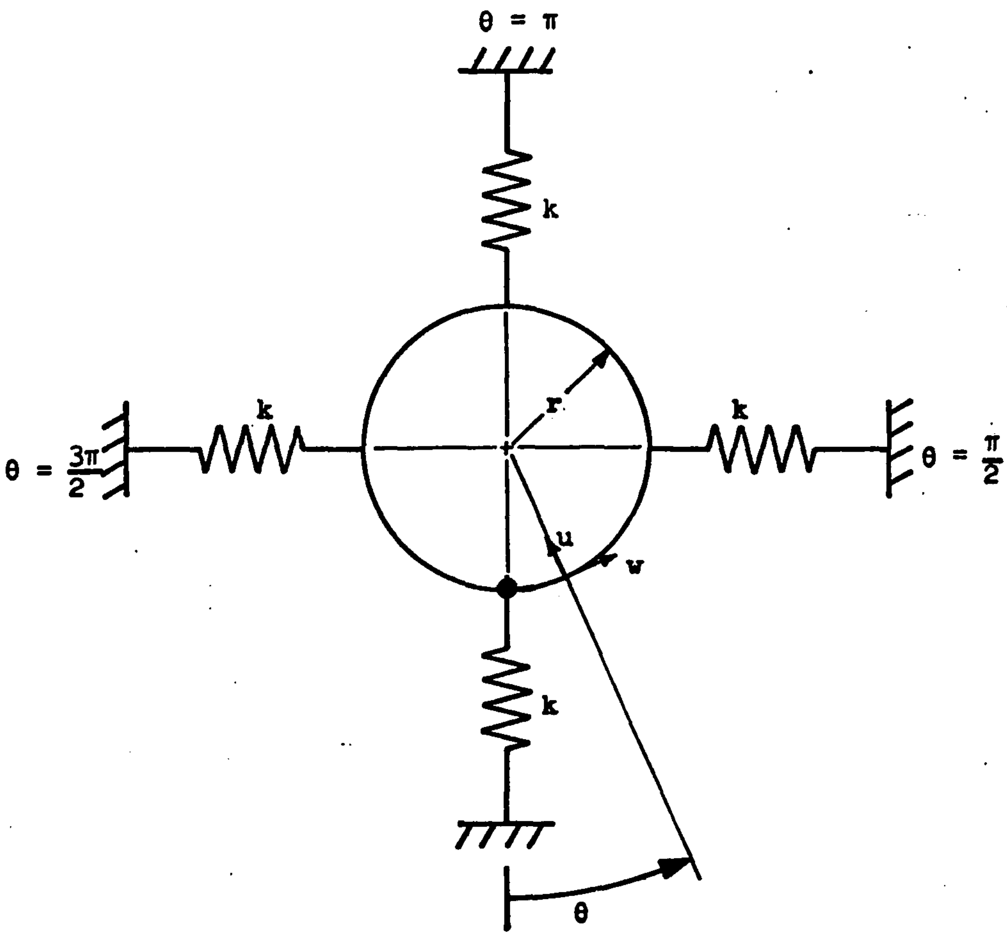


Figure 1. Sign Convention and Notation for Ring With Radial Springs.

1.3 Approach to the Solution

A Fourier series will be used to represent the radial and transverse displacements of the ring. The use of Fourier series and LaGrange's equations will lead to the equations of motion, which will be given in matrix form. Examination of these equations permits conclusions to be reached about the nature of the modal patterns before the equations of motion are solved. Because of the completeness property of the Fourier series, none of the natural modes will be overlooked. If the Fourier series contains n terms, n modes will be found. These modes approximate the actual natural modes of the ring, and only the first few are good approximations.

A method for improving the convergence of the Fourier series representation of the mode shapes will be examined. The approximate results for mode shape and natural frequency can be improved if special functions are added to the Fourier series. The special functions incorporate appropriate discontinuities in certain of the derivatives. This procedure will be applied to a simply supported beam with a transverse spring at the mid-point. The special function is formed from the solution for the displacement response of the beam to the static application of a concentrated transverse force at the beam mid-point. This procedure will be shown to improve the results for mode shapes and natural frequencies substantially as well as the results for the shear force distributions. For the ring, the primary interest is in the kinds of natural modes and the general character of the mode shapes. Thus, the procedure for improving convergence will not be used in the ring calculations. The results will be used as a guide in selecting the number of terms needed in the Fourier series.

The equations of motion will be solved with the aid of a digital computer. The resulting modal patterns will be shown for several example cases.

CHAPTER 2

EQUATIONS OF MOTION

2.1 Introduction

The equations of motion are derived in this Chapter. LaGrange's equations of motion are used along with the expressions for the kinetic energy and the potential energy of ring bending and spring deflection.

Figure 1 shows a ring with four equally spaced radial springs. Symbols, origin, and sign conventions to be used are shown. The radial displacement u of the ring is approximated by the first n terms of a Fourier series, written as

$$u(\theta,t) = \sum_{i=1}^{n} [a_i(t)\cos i\theta + b_i(t)\sin i\theta] \qquad (2.1)$$

in which $a_i(t)$ and $b_i(t)$ are the Fourier coefficients, or the generalized coordinates. The radial displacement, u, and the transverse displacement, w, are functions of the angular coordinate, θ, and time, t.

For an inextensional ring, the radial and transverse displacements u and w are related by

$$u = \frac{\partial w}{\partial \theta}$$

Thus the radial displacement w is approximated by

$$w(\theta,t) = \sum_{i=1}^{n} \left(\frac{a_i}{i}\sin i\theta - \frac{b_i}{i}\cos i\theta\right) \tag{2.2}$$

Equations 2.1 and 2.2 describe the radial and transverse displacements of the ring. The completeness of the description depends on the number of terms n used. The resulting system will have 2n degrees of freedom.

2.2 Kinetic Energy

If rotary inertia is neglected, the kinetic energy T of the in-plane motion of the ring can be written as

$$T = \frac{A\rho r}{2} \int_0^{2\pi} (\dot{u}^2 + \dot{w}^2)\,d\theta \tag{2.3}$$

in which A, ρ, and r represent the cross sectional area, the mass per unit volume, and the radius of the ring, respectively. Substitution of Eq's 2.1 and 2.2 into Eq 2.3 leads to

$$T = \frac{A\rho r}{2} \int_0^{2\pi} \sum_{i=1}^{n} [(\dot{a}_i \cos i\theta + \dot{b}_i \sin i\theta)^2 + \left(\frac{\dot{a}_i}{i}\sin i\theta - \frac{\dot{b}_i}{i}\cos i\theta\right)^2]\,d\theta \tag{2.4}$$

Integrating Eq 2.4 and using the orthogonality properties of the sine and cosine functions

$$T = \frac{A\rho r\pi}{2} \sum_{i=1}^{n} [(\dot{a}_i^2 + \dot{b}_i^2)\ (1 + \frac{1}{i^2})] \tag{2.5}$$

Since the coefficients of $\dot{a}_i \times \dot{b}_i$ are zero, the equations of motion will not be inertially coupled. Thus the inertia matrix will be diagonal.

2.3 Mass Matrix Coefficients

The coefficients of the mass matrix may now be determined by using LaGrange's equation together with the expression for kinetic energy, Eq 2.5. The generalized inertia forces associated with the generalized coordinates a_i are

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_i}\right) = -A\rho r\pi \left(1 + \frac{1}{i^2}\right) \ddot{a}_i$$

Then the generalized masses associated with a_i are

$$m_{a_i a_i} = \overline{M}\left(1 + \frac{1}{i^2}\right) \tag{2.6}$$

where $\overline{M} = A\rho r\pi$.

The generalized inertia forces associated with the generalized coordinates b_i are

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{b}_i}\right) = -A\rho r\pi \left(1 + \frac{1}{i^2}\right) \ddot{b}_i$$

Then the generalized masses associated with b_i are

$$m_{b_i b_i} = \overline{M}\left(1 + \frac{1}{i^2}\right) \tag{2.7}$$

The mass matrix, $\underline{M}$, is seen to consist of two diagonal submatrices, $\underline{M}_a$ and $\underline{M}_b$ as follows:

$$\underline{M} = \left[\begin{array}{c:c} \underline{M}_a & 0 \\ \hdashline 0 & \underline{M}_b \end{array}\right] \tag{2.8a}$$

where

$$\underline{M}_a = \underline{M}_b = \begin{bmatrix} 2\bar{M} & & & \\ & \frac{5}{4}\bar{M} & & \\ & & \frac{10}{9}\bar{M} & \\ & & & \ddots \end{bmatrix} \tag{2.8b}$$

The upper left submatrix is associated with the cosine series and the lower right submatrix is associated with the sine series.

2.4 The Bending Strain Energy

The bending strain energy V_B of the ring, associated with the ring bending moment M, is given by the integral

$$V_B = \frac{1}{2} \int_L \frac{M^2}{EI} \, r d\theta \tag{2.9}$$

in which EI represents the flexural ridgidity of the ring. The bending moment--displacement relation for an inextensional ring, neglecting shear displacements, is

$$M = \frac{EI}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} + u\right) \tag{2.10}$$

Substitution of Eq 2.10 into 2.9 results in

$$V_B = \frac{EI}{2r^4} \int_0^{2\pi} \left[\left(\frac{\partial^2 u}{\partial \theta^2} + u\right)^2\right] r d\theta \tag{2.11}$$

Substitution of Eq 2.1 into 2.11 leads to

$$V_B = \frac{EI}{2r^3} \int_0^{2\pi} \sum_{i=1}^{n} [a_i(1-i)^2 \cos i\theta + b_i(1-i)^2 \sin i\theta]^2 \, d\theta \tag{2.12}$$

Or, using the orthogonality properties of the cosine and sine functions and integrating,

$$V_B = \frac{\pi EI}{2r^3} \sum_{i=1}^{n} [(1-i^2)^2(a_i^2 + b_i^2)] \tag{2.13}$$

Since the coefficients of $a_i \times b_i$ are zero, the equations of motion will not be elastically coupled by ring bending. Thus this portion of the stiffness matrix will be diagonal.

2.5 The Bending Stiffness Coefficients

The bending stiffness coefficients may be evaluated by using LaGrange's equation and the expression for the strain energy in bending, Eq 2.13. The generalized bending forces associated with the generalized coordinates a_i and b_i are

$$-\frac{\partial V_B}{\partial a_i} = -\frac{\pi EI}{r^3}(1-i^2)^2 a_i = -\overline{K}\, a_i(1-i^2)^2$$

$$-\frac{\partial V_B}{\partial b_i} = -\frac{\pi EI}{r^3}(1-i^2)^2 b_i = -\overline{K}\, b_i(1-i^2)^2$$

where

$$\overline{K} = \frac{\pi EI}{r^3} \tag{2.14}$$

Then the generalized bending stiffnesses associated with the generalized coordinates a_i and b_i are

$$k_{a_i a_i} = \overline{K}\,(1-i^2)^2 \tag{2.15a}$$

$$k_{b_i b_i} = \overline{K}\,(1-i^2)^2 \tag{2.15b}$$

The bending stiffness matrix, $\underline{K}_B$ is seen to be a diagonal matrix, comprised of the two identical submatrices, $\underline{K}_{B_a}$ and $\underline{K}_{B_b}$ as follows:

$$\underline{K}_B = \left[\begin{array}{c|c} \underline{K}_{B_a} & 0 \\ \hline 0 & \underline{K}_{B_b} \end{array}\right] \tag{2.16a}$$

where:

$$\underline{K}_{B_a} = \underline{K}_{B_b} = \begin{bmatrix} 0 & & & \\ & 9\overline{K} & & \\ & & 64\overline{K} & \\ & & & \ddots \end{bmatrix} \tag{2.16b}$$

The upper left submatrix is associated with the cosine series and the lower right submatrix is associated with the sine series. The zero coefficient on the diagonal results from the fact that the first terms in the cosine and sine series represent a rigid body displacement and do not involve any bending deformation.

2.6 The Spring Strain Energy

The strain energy V_s of a spring which is deflected by u_s is

$$V_s = \frac{1}{2} k u_s^2$$

in which k is the stiffness of the spring.

Substitution of Eq 2.1, evaluated at $\theta = \theta_s$, into this equation results in

$$V_s = \frac{1}{2} k \left[\sum_i (a_i \cos i\theta_s + b_i \sin i\theta_s) \times \sum_j (a_j \cos j\theta_s + b_j \sin j\theta_s)\right] \tag{2.17}$$

For a number N of springs, the spring strain energy is

$$V_s = \frac{1}{2} k \sum_{s=1}^{N} \left[\sum_i (a_i \cos i\theta_s + b_i \sin i\theta_s) \times \sum_j (a_j \cos j\theta_s + b_j \sin j\theta_s)\right] \tag{2.18}$$

2.7 The Spring Stiffness Coefficients

The generalized spring forces associated with the generalized coordinates a_i are

$$-\frac{\partial V_s}{\partial a_i} = -k \sum_{s=1}^{N} \sum_{j=1}^{n} a_j \cos i\theta_s \cos j\theta_s - k \sum_{s=1}^{N} \sum_{j=1}^{n} b_j \cos i\theta_s \sin j\theta_s$$

Then the generalized spring stiffnesses associated with a_i are

$$k_{a_i a_j} = k \sum_{s=1}^{N} \cos i\theta_s \cos j\theta_s \tag{2.19}$$

$$k_{a_i b_j} = k \sum_{s=1}^{N} \cos i\theta_s \sin j\theta_s \tag{2.20}$$

The generalized spring forces associated with the generalized coordinates b_i are

$$-\frac{\partial V_s}{\partial b_i} = -k \sum_{s=1}^{N} \sum_{j=1}^{n} a_j \sin i\theta_s \cos j\theta_s - k \sum_{s=1}^{N} \sum_{j=1}^{n} b_j \sin i\theta_s \sin j\theta_s$$

Then the generalized spring stiffnesses associated with b_i are

$$k_{b_i a_j} = k \sum_{s=1}^{N} \sin i\theta_s \cos j\theta_s \tag{2.21}$$

$$k_{b_i b_j} = k \sum_{s=1}^{N} \sin i\theta_s \sin j\theta_s \tag{2.22}$$

The spring stiffness matrix will have the form

$$\underline{K_s} = \left[\begin{array}{c|c} \underline{K_s}_{aa} & \underline{K_s}_{ab} \\ \hline \underline{K_s}_{ba} & \underline{K_s}_{bb} \end{array}\right]$$

in which the elements of the submatrices $\underline{K_s}_{aa}$ and $\underline{K_s}_{ab}$ are given by Eq's 2.19 and 2.20, and the elements of submatrices $\underline{K_s}_{ba}$ and $\underline{K_s}_{bb}$ are Eq's 2.21 and 2.22. In Appendix B it is shown that the submatrices $\underline{K_s}_{ab}$ and $\underline{K_s}_{ba}$ are zero matrices. The remaining submatrices $\underline{K_s}_{aa}$ and $\underline{K_s}_{bb}$ are symmetric but will have off-diagonal terms.

2.8 Equations of Motion

This completes the derivation of the mass and stiffness coefficients. The coupling of the equations of motion resulting from the springs will be discussed further in Chapter 3.

The general form of the equations of motion involves a partitioned mass and a partitioned stiffness matrix, one part to be associated with the cosine series and one part to be associated with the sine series. This is shown below.

$$\left[\begin{array}{c|c} \underline{M_a} & 0 \\ \hline 0 & \underline{M_b} \end{array}\right] \left\{\begin{array}{c} \ddot{a}_i \\ \hline \ddot{b}_i \end{array}\right\} + \left[\begin{array}{c|c} \underline{K_a} & 0 \\ \hline 0 & \underline{K_b} \end{array}\right] \left\{\begin{array}{c} a_i \\ \hline b_i \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \quad (2.23)$$

CHAPTER 3

ANALYSIS OF ELASTIC COUPLING

3.1 Introduction

The spring stiffness matrices for a number of cases are presented in Appendix A. These are given for spring stiffness k = unity. Figures 2 and 3 are shown as examples on the following pages for easy referral. These figures show a case with even number of springs and a case with odd number of springs. The elastic coupling patterns can be discussed completely by considering a typical even case and a typical odd case. Thus these two examples will be sufficient.

The presence of radial springs supporting the ring leads to elastic coupling of the equations of motion. Because the springs are of equal stiffness and because they are equally spaced, the off-diagonal stiffness coefficients will form regular patterns. An analysis of the patterns results in methods for predicting the coupling patterns for rings with any number of equally spaced springs of equal stiffness. The magnitudes of the stiffness coefficients can also be predicted.

The advantage of using the Fourier series is that important conclusions about the nature of the normal modes can be made now without solving the equations of motion.

$$\left[\begin{array}{c|c}
\begin{matrix}
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4
\end{matrix} & 0 \\
\hline
0 & \begin{matrix}
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\end{array}\right]$$

Figure 2. Spring Stiffness Coefficients, Four-Spring Case, Unit k

$$
\left[\begin{array}{ccccccccc|ccccccccc}
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 & & & & & & & & & \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 & & & & & & & & & \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 & & & & & & & & & \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 & & & & & & & & & \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & & & & & 0 & & & & \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 & & & & & & & & & \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 & & & & & & & & & \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 & & & & & & & & & \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 & & & & & & & & & \\
\hline
 & & & & & & & & & 2.5 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & -2.5 \\
 & & & & & & & & & 0 & 2.5 & -2.5 & 0 & 0 & 0 & 2.5 & -2.5 & 0 \\
 & & & & & & & & & 0 & -2.5 & 2.5 & 0 & 0 & 0 & -2.5 & 2.5 & 0 \\
 & & & & & & & & & -2.5 & 0 & 0 & 2.5 & 0 & -2.5 & 0 & 0 & 2.5 \\
 & & & & 0 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 2.5 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & -2.5 \\
 & & & & & & & & & 0 & 2.5 & -2.5 & 0 & 0 & 0 & 2.5 & -2.5 & 0 \\
 & & & & & & & & & 0 & -2.5 & 2.5 & 0 & 0 & 0 & -2.5 & 2.5 & 0 \\
 & & & & & & & & & -2.5 & 0 & 0 & 2.5 & 0 & -2.5 & 0 & 0 & 2.5
\end{array}\right]
$$

Figure 3. Spring Stiffness Coefficients, Five-Spring Case, Unit k

3.2 Prediction of Coupling Patterns

An analysis of the coupling patterns permits several important conclusions to be reached about the nature of the normal modes. For example, consider Fig 2, which is the spring stiffness matrix for the ring with four springs. The submatrix at the upper left is associated with the generalized coordinates a_i, which are the coefficients of the cosine series. In this submatrix, odd numbered columns (and rows) have zero even numbered elements. Thus all the odd numbered a_i are coupled to each other, but they are not coupled with the even numbered a_i. As a result of this coupling, there will be a "family" of normal modes made up of the odd numbered harmonics of the cosine series. Consider now the even numbered columns (and rows) 2, 6, 10, ... These have zeroes except at the positions 2, 6, 10, ... Thus the even numbered coefficients a_i with $i = 2, 6, 10, \ldots$, are coupled only to each other. As a result there will be a "family" of normal modes constructed of the harmonics $i = 2, 6, 10, \ldots$, of the cosine series. The remaining even numbered columns (and rows) of this submatrix are composed of the multiples of the number of springs N, i.e., 4, 8, 12, ..., and these have zeroes except at positions 4, 8, 12, ... Thus the even numbered a_i with $i = 4, 8, 12, \ldots$, are coupled to each other only. As a result there will be a "family" of normal modes made up of the harmonics $i = 4, 8, 12, \ldots$, of the cosine series. For the case of four springs, there will therefore be three families of normal modes constructed from the cosine series.

The submatrix at the lower right in Fig 2 is associated with the generalized coordinates b_i, which are the coefficients of the sine series. Odd numbered columns (and rows) have zero even numbered elements. Thus all odd numbered b_i are coupled to each other only. As a result of this there will be a "family" of normal modes made up of the odd numbered harmonics of the sine series. The odd numbered columns (and rows) of the upper left and lower right submatrices in Fig 2 are identical except for an alternation in sign. Thus an eigenvector {a} which satisfies the first submatrix will be identical except for alternating minus signs with an eigenvector {b} which satisfies the second submatrix. (Since the mass submatrices and bending stiffness submatrices associated with each of them are identical.) Consequently, the natural frequencies of the normal modes constructed of the odd numbered harmonics of the cosine series will be identical to those constructed of the odd numbered harmonics of the sine series. For any one of these natural frequencies, the associated normal mode shape can be given by <u>any</u> linear combination of the cosine series and sine series mode shapes. Thus an infinity of mode shapes are associated with each of these natural frequencies. The even numbered columns (and rows) of the lower right submatrix are all zeroes. Therefore all even numbered b_i are completely uncoupled. The presence of bending stiffness terms on the diagonal of the total stiffness matrix will lead to a normal mode of the system for each even numbered harmonic of the sine series. For these normal modes, the nodes are at the springs.

Consider now Fig 3, which is typical of the cases for odd numbers of springs. The submatrix at the upper left in this figure is again associated with the generalized coordinate a_i, the coefficients of the cosine series. The columns (and rows) numbered 1, 4, 6, ..., have zeroes except at positions 1, 4, 6, ..., and as a result the a_i with i = 1, 4, 6, ..., are coupled to each other only. Consequently, there will be a "family" of normal modes constructed of the harmonics i = 1, 4, 6, ..., of the cosine series. The columns (and rows) numbered 2, 3, 7, ..., have zeroes except at positions 2, 3, 7, ..., and as a result, those a_i with i = 2, 3, 7, ..., are coupled to each other only. Thus there will be a "family" of normal modes constructed of the harmonics i = 2, 3, 7, ... of the cosine series. The remaining columns (and rows) are composed of the integer multiples of the number of springs, i.e., 5, 10, 15, ... These columns (and rows) have all zeroes except at positions 5, 10, 15, ..., and as a result there will be a family of modes made up of the harmonics i = 5, 10, 15, ..., of the cosine series. There will be three families of normal modes constructed from the cosine series. The submatrix at the lower right in Fig 3 is associated with the generalized coordinates b_i, the coefficients of the sine series. Columns (and rows) numbered 1, 4, 6, ..., in this submatrix have zeroes except at positions 1, 4, 6, ..., and thus the b_i with i = 1, 4, 6, ..., are coupled to each other only. There will be a family of normal modes made up of the harmonics i = 1, 4, 6, ..., of the sine series. These columns (and rows) are the same for the upper left and lower right submatrices except for an alternation in sign. Thus, as in the discussion of the four spring case, and eigenvector {a} which satisfies the first submatrix will be identical except for alternating minus sign with eigenvector {b} which satisfies the second submatrix. As a result the natural frequencies of the normal modes constructed from the harmonics of the cosine series will be the same as those constructed from the harmonics of the sine series. For any one of the natural frequencies in this case, the associated normal mode shape is given by <u>any</u> linear combination of the cosine series and sine series mode shape. Thus there are an infinity of mode shapes associated with each of these natural frequencies. Similarly, the columns (and rows) numbered 2, 3, 7, ..., in the lower right submatrix have zeroes except at positions 2, 3, 7, ..., and as a result, the b_i with $i = 2, 3, 7, \ldots$, will be coupled only to each other. There will be a family of normal modes constructed from the harmonics $i = 2, 3, 7, \ldots$, of the sine series. These columns (and rows) are also identical in the upper right and lower left submatrices, except for an alternation in sign. The same remarks that were made above for the columns 1, 4, 6, ..., are appropriate for the columns 2, 3, 7, ... The normal mode shapes can be given by any linear combination of the cosine series and sine series mode shapes. Thus there will be an infinity of mode shapes associated with each of these natural frequencies also.

The columns (and rows) which have numbers corresponding to integer multiples of the number of springs, i.e., 5, 10, 15, ..., are all zeroes. Therefore, the b_i for $i = 5, 10, 15, \ldots$, are all uncoupled. The presence of bending stiffness terms on the diagonal of the total stiffness matrix will lead to a normal mode of the system for each of these harmonics of the sine series. For these normal modes, the nodes are also at the springs.

These observations may now be extended to other cases. The following discussion is based on observation of patterns for three, four, five, six and seven springs. This is felt to be a sufficient number of cases to observe the general patterns that result and to form general conclusions from them. It has been pointed out that the upper right and lower left submatrices of the spring stiffness matrix are identically zero for all cases. A proof of this is given in Appendix B. Thus the generalized coordinates a_i and b_i are not coupled to each other. This permits separation of the equations of motion into two sets of equations, as will be discussed in Chapter 5.

As discussed above, each family of normal modes is constructed of a specific set of the harmonics of the cosine series or the sine series. Examination of the patterns of non-zero elements in the columns (and rows) of the spring stiffness matrix reveals the number of modal families present and the set of harmonics involved in each family. Consider first the upper left submatrix of the stiffness matrix. Let us assign a number n, a positive integer or zero, to represent each of the modal families. Observation of the submatrix for several different numbers of springs N indicates that the modal family n involves the columns (and rows) numbered

$$i = n,\ 1 \cdot N \pm n,\ 2 \cdot N \pm n,\ \ldots,\ M \cdot N \pm n \tag{3.1}$$

in which M is a positive integer. Then the modal family n is constructed of the harmonics of the cosine series numbered by Eq 3.1. The positions of the non-zero elements in the given columns (or rows) are also given by Eq 3.1. The number of families present is indicated by the range of n, given by

$$n = 0, 1, \ldots, \frac{N}{2} \quad \text{for an even number of springs} \tag{3.2a}$$

$$n = 0,1, \ldots, \left(\frac{N-1}{2}\right) \text{ for an odd number of springs} \qquad (3.2b)$$

For example, in the four-spring case, the range of n is, from Eq 3.2a

$$n = 0, 1, 2$$

and the resulting modal families are, from Eq 3.1, made up of

Harmonics of Cosine Series	Family
$i = 4, 8, 12, \ldots,$	$n = 0$
$i = 1, 3, 5, 7, \ldots,$	$n = 1$
$i = 2, 6, 10, 14, \ldots,$	$n = 2$

Equation 3.1 may give repetitions in the coupling patterns. For example, for $n = 2$, the harmonics of the cosine series are numbered

$$i = 2,2,6,6,10,10, \ldots$$

The repeated numbers may be discarded to give the sequence

$$i = 2, 6, 10, 14, \ldots$$

For five springs, the range of n is, from Eq 3.2b

$$n = 0, 1, 2$$

and the resulting modal families are made up of

Harmonics of Cosine Series	Family
$i = 5, 10, 15, 20, \ldots,$	$n = 0$
$i = 1, 4, 6, 9, \ldots,$	$n = 1$
$i = 2, 3, 7, 8, \ldots,$	$n = 2$

Equation 3.1 gives the coupling patterns and Eq's 3.2 gives the number of modal families which are constructed from the cosine series.

Consider now the lower right submatrix of the spring stiffness matrix. The rules given for the upper left submatrix are still appropriate, but further comments are necessary. For an odd number of springs N, the elements of the columns (or rows) for the modal family $n = 0$ are all zero. Thus the normal modes of the modal family $n = 0$ are the pure harmonics of the sine series, numbered by

$$i = N, 2N, 3N, \ldots .$$

Note that these mode shapes have nodal points at the springs and that these are identical with the corresponding mode shapes of the unsupported ring. For an even number of springs N, the elements of the columns (or rows) for the modal families $n = 0$ and $n = \frac{N}{2}$ are all zero. As a result the normal modes of the modal families $n = 0$ and $n = \frac{N}{2}$ are the pure harmonics of the sine series, numbered by

$$i = N, 2N, 3N, \ldots ,$$

and

$$i = N/2, 3N/2, \ldots .$$

These mode shapes have nodal points at the springs and are identical with the corresponding mode shapes of the unsupported ring.

3.3 Magnitudes of the Coefficients

The magnitudes of the spring stiffness coefficients also follow an observable pattern. Consider first the coefficients of the upper left submatrix of the spring stiffness matrix. For an even number of springs N, the columns (or rows) associated with the families $n = 0$, and $n = N/2$ have non-zero coefficients of magnitude N. This is seen in Fig 2, where the columns (or rows) numbered 2, 4, 6, 8, . . . , have coefficients of magnitude 4, corresponding to the number of springs $N = 4$. The columns (or rows) associated with the other families have non-zero coefficients of magnitude $N/2 = 2$. This is also seen in Fig 2, where the columns (or rows) numbered 1, 3, 5, 7, . . . , have coefficients (non-zero) of $N/2 = 2$. For an odd number of springs N, the columns (or rows) associated with the family $n = 0$ have non-zero coefficients of magnitude N. The remaining columns have non-zero coefficients of magnitude $N/2$. This can be seen in Fig 3, where it is observed that the columns numbered 5, 10, 15, . . . , have magnitude of $N = 5$. The remaining columns have non-zero coefficients of magnitude $N/2 = 2.5$. Considering the lower right submatrix, all of the non-zero coefficients are of magnitude $N/2$. This is seen in both Fig 2 and Fig 3, where the coefficients of the lower right submatrices which are not zero are all of magnitude $N/2$; $N/2 = 2$ in the four spring case and $N/2 = 2.5$ in the five spring case. The signs of the coefficients alternate in a plus to minus fashion in the lower right submatrix. This is true for both even number of springs and odd number of springs. Utilizing the above observations, the entire spring stiffness matrix for a ring with any number of equally spaced springs of equal stiffness can be easily constructed.

This completes the discussion of the coupling patterns, modal families, and magnitudes of the coefficients for the spring stiffness matrix.

CHAPTER 4

IMPROVEMENT IN CONVERGENCE OF SERIES SOLUTION

4.1 Introduction

As the ring vibrates, the supporting springs will each exert a radial force on the ring. Thus there will be discontinuities in the shear force distribution in the ring. Since the Fourier series and its derivatives do not have discontinuities, the representation of the shear force will be poor, especially near the springs. Although a Fourier series can be used to represent a function with a discontinuity, the representation in the region of the discontinuity is necessarily poor. At the discontinuity the series will converge to the mean value of the function being represented (Ref. 5). As a result of this behavior of the Fourier series, the ring bending moment distribution and the strain energy will also be in error.

A scheme for improving the series solution to give better results for the shear force distribution will be discussed in this Chapter. In this scheme, a special function (or functions) will be added to the Fourier series. The special functions are selected to permit the shear force distribution to be discontinuous at the springs. A special function will be referred to as a "modifying function." The modifying function, when used in addition to the Fourier series, will provide a much improved representation of the shear force distribution, especially at the discontinuities. Consequently, the bending moment representation and bending strain energy will be improved. This will lead to an improvement in natural frequencies.

An example of the method will be worked out for a simply supported uniform beam with a transverse spring at the midpoint as shown in Fig 4. The normal mode shapes of the beam without the spring are given by $\phi_i(x) = \sin \frac{i\pi x}{L}$, in which i is the mode number. Since the even numbered modes, i = 2, 4, ..., all have a nodal point at the midpoint, the spring will not affect them. Only the odd numbered modes, i = 1, 3, ..., will be needed. Because of the complexity of applying this method to the ring problem, a simpler example is used initially.

4.2 Example for the Beam

The choice of an appropriate function to represent the shear force discontinuity is an important step. A natural choice in this case is the displacement function for a simply supported beam with a concentrated load at the center. This result is well known from elementary strength of materials, and, when normalized to have a unit displacement at x = L/2, is for $0<x<\frac{L}{2}$

$$\phi_o(x) = \frac{3x}{L} - 4\frac{x^3}{L^3} \tag{4.1}$$

The symbol $\phi_o(x)$ is given to the modifying function to distinguish it from the modes $\phi_i(x) = \sin \frac{i\pi x}{L}$.

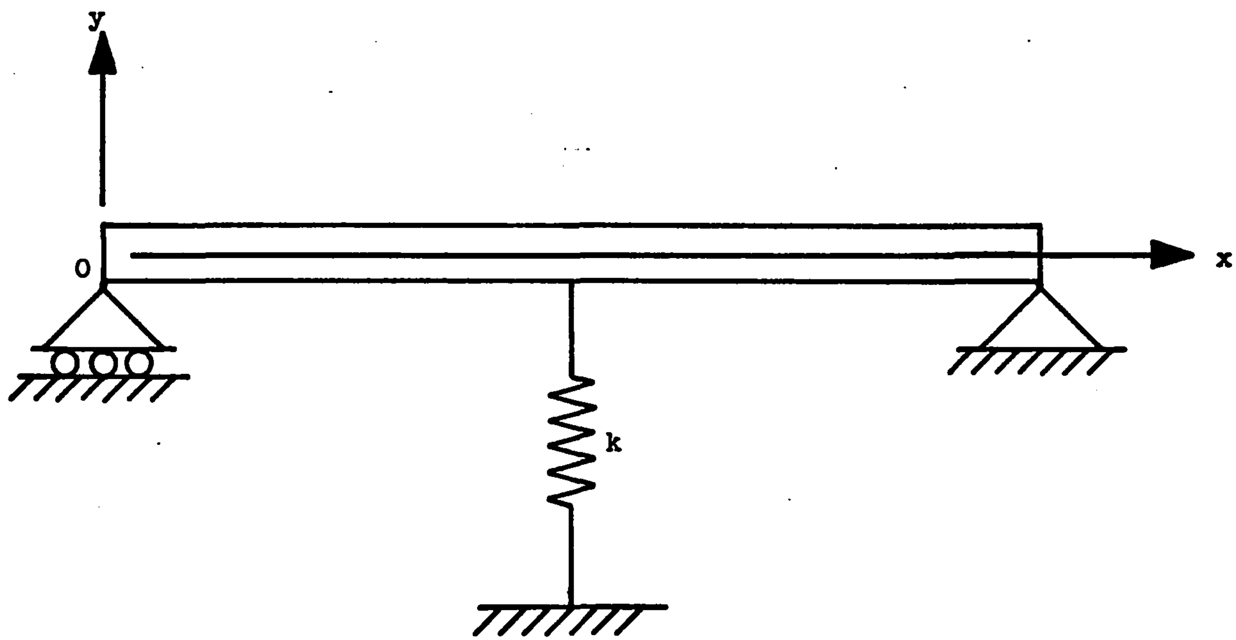


Figure 4. Simply Supported Uniform Beam With Transverse Spring at Midpoint.

The appropriate Fourier series for representing the transverse displacements of the beam is the sine series

$$y = \sum_{i=1}^{n} q_i(t) \sin \frac{i\pi x}{L} \tag{4.2}$$

$$i = 1, 3, 5, \ldots,$$

where the $q_i(t)$ are the generalized coordinates. Adding the modifying function $\phi_o(x)$ to the sine series we have

$$y(x,t) = \phi_o(x)q_o(t) + \sum_{i=1}^{n} \sin \frac{i\pi x}{L} q_i(t) \tag{4.3}$$

$$i = 1, 3, 5, \ldots.$$

The equations of motion in the generalized coordinates q_o, q_1, q_3, ..., can be obtained by using LaGrange's equations along with the expressions for the kinetic and potential energy of the beam.

Neglecting the rotary inertia of the beam, the kinetic energy T is

$$T = \tfrac{1}{2}\mu\int_o^L \dot{y}^2 dx$$

in which μ is mass per unit length of the beam. Using y from Eq 4.3

$$T = \tfrac{1}{2}\mu\, \dot{q}_o^2 \int_o^L \phi_o^2 dx + \mu\dot{q}_o \sum_{i=1}^{n} \dot{q}_i \int_o^L \phi_o \sin \frac{i\pi x}{L}\, dx$$

$$+ \tfrac{1}{2}\mu \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{q}_i\dot{q}_j \int_o^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L}\, dx$$

The generalized inertia force associated with the generalized coordinate q_o is

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_o}\right) = -\mu\int_o^L \phi_o^2 dx\, \ddot{q}_o - \sum_{i=1}^{n} \mu\int_o^L \phi_o \sin \frac{i\pi x}{L}\, dx\, \ddot{q}_i$$

Then the generalized masses which make up the first row of the mass matrix are

$$m_{oo} = \mu\int_o^L \phi_o^2 dx \tag{4.5}$$

$$m_{oi} = \mu\int_o^L \phi_o \sin \frac{i\pi x}{L}\, dx$$

$$i = 1, 3, 5, \ldots .$$

Since the functions ϕ_o and $\sin \frac{i\pi x}{L}$ are even functions about the midpoint of the beam, we can evaluate the integrals over the interval 0 to $\frac{L}{2}$ and double the results. Substituting the expression for ϕ_o, Eq 4.1, into Eq's 4.5 and integrating, we obtain

$$m_{oo} = \frac{17}{35}\mu L$$

$$m_{oi} = \frac{48\mu L}{(i\pi)^4}(-1)^{\frac{i-1}{2}} \tag{4.6}$$

$$i = 1, 3, 5, \ldots .$$

The generalized inertia forces associated with the generalized coordinates q_i are

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = -\mu\int_o^L \phi_o \sin \frac{i\pi x}{L}\, dx\, \dot{q}_o$$

$$-\sum_{j=1}^{n} \mu\int_o^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L}\, dx\, \dot{q}_j$$

Then the generalized masses which make up the ith row of the mass matrix are

$$m_{io} = \mu\int_o^L \phi_o \sin \frac{i\pi x}{L}\, dx$$

$$m_{ij} = \mu\int_o^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L}\, dx \tag{4.7}$$

Integrating these expressions with the aid of Eq 4.1 we obtain

$$m_{io} = \frac{48\mu L}{(i\pi)^4} (-1)^{\frac{i-1}{2}}$$

$$m_{ij} = 0 \qquad\qquad i \neq j \qquad\qquad (4.8)$$

$$= \frac{\mu L}{2} \qquad\qquad i = j$$

The combined strain energy V of the beam and the spring is

$$V = \tfrac{1}{2}EI\int_0^L y''^2 dx + \tfrac{1}{2}k\,[y(\tfrac{L}{2})]^2$$

in which EI is the flexural rigidity of the beam and k is the stiffness of the spring. Using y from Eq 4.3

$$V = \tfrac{1}{2}q_o^2\,[EI\int_0^L \phi_o''^2 dx + k\phi_o^2(\tfrac{L}{2})]$$

$$+ q_o \sum_{i=1}^{n} q_i\,[-EI(\tfrac{i\pi}{L})^2 \int_0^L \phi_o'' \sin\frac{i\pi x}{L}\,dx + k\phi_o(\tfrac{L}{2}) \sin\frac{i\pi}{2}]$$

$$+ \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j [EI(\tfrac{i\pi}{L})^2 (\tfrac{j\pi}{L})^2 \int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}\,dx$$

$$+ k \sin\frac{i\pi}{2} \sin\frac{j\pi}{2}]$$

The generalized elastic force associated with q_o is

$$-\frac{\partial V}{\partial q_o} = -\,[EI\int_0^L \phi_o''^2 dx + \phi_o^2(\tfrac{L}{2})]\,q_o$$

$$-\sum_{i=1}^{n} [-EI(\tfrac{i\pi}{L})^2 \int_0^L \phi_o'' \sin\frac{i\pi x}{L}\,dx$$

$$+ k\phi_o(\tfrac{L}{2}) \sin\frac{i\pi}{2}]\,q_i$$

Then the generalized stiffnesses which make up the first row of the stiffness matrix are

$$k_{oo} = EI\int_o^L \phi_o''^2 dx + k\phi_o^2(\tfrac{L}{2})$$
$$k_{oi} = -EI\,(\tfrac{i\pi}{L})^2 \int_o^L \phi_o'' \sin\tfrac{i\pi x}{L}\,dx + k\phi_o(\tfrac{L}{2})\sin\tfrac{i\pi}{2} \qquad (4.9)$$

$$i = 1, 3, 5, \ldots.$$

Substituting the expression for ϕ_o into Eq 4.9 and integrating, we obtain

$$k_{oo} = \frac{48EI}{L^3} + k$$
$$k_{oi} = \frac{48EI}{L^3}(-1)^{\frac{i-1}{2}} + k\,(-1)^{\frac{i-1}{2}} \qquad (4.10)$$

$$i = 1, 3, 5, \ldots.$$

The generalized elastic forces associated with the generalized coordinates q_i are

$$-\frac{\partial V}{\partial q_i} = -\,[-EI\,(\tfrac{i\pi}{L})^2 \int_o^L \phi_o'' \sin\tfrac{i\pi x}{L}\,dx$$
$$+ k\phi_o(\tfrac{L}{2})\sin\tfrac{i\pi}{2}]\,q_o$$
$$-\sum_{j=1}^{n} [EI\,(\tfrac{i\pi}{L})^2(\tfrac{j\pi}{L})^2 \int_o^L \sin\tfrac{i\pi x}{L}\sin\tfrac{j\pi x}{L}\,dx$$
$$+ k\sin\tfrac{i\pi}{2}\sin\tfrac{j\pi}{2}]\,q_j$$

Then the generalized stiffnesses which make up the ith row of the stiffness matrix are

$$k_{io} = -EI\left(\frac{i\pi}{L}\right)^2\int_o^L \phi_o'' \sin\frac{i\pi x}{L}\,dx + k\,\phi_o\left(\frac{L}{2}\right)\sin\frac{i\pi}{2}$$

$$k_{ij} = EI\left(\frac{i\pi}{L}\right)^2\left(\frac{j\pi}{L}\right)^2\int_o^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}\,dx + k\sin\frac{i\pi}{2}\sin\frac{j\pi}{2} \qquad i, j = 1, 3, \ldots. \tag{4.11}$$

Integration of these expressions yields

$$k_{io} = \frac{48EI}{L^3}(-1)^{\frac{i-1}{2}} + k\,(-1)^{\frac{i-1}{2}}$$

$$k_{ij} = k\,(-1)^{\frac{i-1}{2}}(-1)^{\frac{j-1}{2}} \qquad i \neq j$$

$$= \frac{(i\pi)^4}{2}\cdot\frac{EI}{L^3} + k \qquad i = j \tag{4.12}$$

Having obtained the mass and stiffness coefficients the equations of motion for a free vibration of the beam can be written in the form

$$\underline{M}\ \{\ddot{q}\} + \underline{K}\ \{q\} = \{0\}$$

In attempting to solve these equations of motion it was discovered that the mass matrix, $\underline{M}$, is nearly singular. It is so nearly singular that the numerical scheme used to find eigenvalues and eigenvectors in the CDC 6400 MATRIX routines would not work. This undesirable result can be remedied by using a slight modification of the above scheme. The singularity of the mass matrix results from the fact that the modifying function $\phi_o(x)$, can be represented by a rapidly converging series in $\sin\frac{i\pi x}{L}$. To improve this scheme, let us replace $\phi_o(x)$ by $\Psi(x)$.

The function $\overline{\Phi}(x)$ is formed by subtracting the first n Fourier series components from $\phi_o(x)$, where n is the number of Fourier series terms being used in the representation of the displacement. The modifying function shall now have the form

$$\overline{\Phi}(x) = \phi_o(x) - \sum_{i=1}^{n} a_i \sin \frac{i\pi x}{L} \qquad (4.13)$$

$$i = 1, 3, 5, \ldots,$$

where a_i are the Fourier series coefficients of $\phi_o(x)$. The use of $\overline{\Phi}(x)$ rather than $\phi_o(x)$ as a modifying function will cast the equations into a form in which the near singularity can be circumvented.

The function $\overline{\Phi}(x)$ has some interesting properties. Since $\phi_o(x)$ is a continuous function, its Fourier series representation is very convergent. Thus $\overline{\Phi}(x)$ is almost zero for all x. However, the third derivative of $\overline{\Phi}(x)$ still has the same jump at $x = \frac{L}{2}$ that $\phi_o(x)$ had. Thus the representation of the shear force will have a discontinuity at $x = \frac{L}{2}$ as required. Actually, $\phi_o(x)$ represents the response of the beam to the static load applied at the midpoint. Then $\overline{\Phi}(x)$ represents the response of the higher modes to this force.

The Fourier coefficients can be determined directly from Eq 4.13 by requiring $\overline{\Phi}(x)$ to be orthogonal to $\sin \frac{i\pi x}{L}$. Rearranging this equation, multiplying through by $\sin \frac{i\pi x}{L}$ and integrating over the length of the beam gives

$$\int_0^L \phi_o(x) \sin \frac{i\pi x}{L}\, dx = \int_0^L \overline{\Phi}(x) \sin \frac{i\pi x}{L}\, dx + a_i \int_0^L \sin^2 \frac{i\pi x}{L}\, dx$$

Since $\overline{\Phi}(x)$ is orthogonal to $\sin \frac{i\pi x}{L}$ for i from 1 to n, the first integral on the right hand side is zero.

Then

$$a_i = \frac{\int_0^L \phi_o(x) \sin \frac{i\pi x}{L} dx}{\int_0^L \sin^2 \frac{i\pi x}{L} dx} \tag{4.14}$$

From Eq 4.8

$$a_i = \frac{m_{io}}{m_{ii}} = \frac{96}{(i\pi)^4} (-1)^{\frac{i-1}{2}} \tag{4.15}$$

Replacing the modifying function $\phi_o(x)$ by the new modifying function $\overline{\Phi}(x)$ the new generalized masses and stiffnesses can be written by examination of Eq's 4.5, 4.7, 4.9, and 4.11. Then

$$\begin{aligned}
\overline{m}_{oo} &= \mu \int_0^L \overline{\Phi}^2 \, dx \\
\overline{m}_{oi} &= \overline{m}_{io} = \mu \int_0^L \overline{\Phi} \sin \frac{i\pi x}{L} dx \\
\overline{m}_{ij} &= \mu \int_0^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L} dx \\
\overline{k}_{oo} &= EI \int_0^L \overline{\Phi}''^2 \, dx + k \, \overline{\Phi}^2 (\tfrac{L}{2}) \\
\overline{k}_{oi} &= \overline{k}_{io} = -EI \left(\frac{i\pi}{L}\right)^2 \int_0^L \overline{\Phi}'' \sin \frac{i\pi x}{L} dx + k \overline{\Phi} (\tfrac{L}{2}) \sin \frac{i\pi}{L} \\
\overline{k}_{ij} &= EI \left(\frac{i\pi}{L}\right)^2 \left(\frac{j\pi}{L}\right)^2 \int_0^L \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L} dx + k \sin \frac{i\pi}{2} \sin \frac{j\pi}{2}
\end{aligned} \tag{4.16}$$

in which the bar over the quantities indicates that they are associated with $\overline{\Phi}(x)$ rather than $\phi_o(x)$. Since $\overline{\Phi}(x)$ is nearly zero for all x, $\overline{m}_{oo}$ will be very nearly zero. Because of the orthogonality of $\overline{\Phi}(x)$ and $\sin \frac{i\pi x}{L}$, $\overline{m}_{oi} = \overline{m}_{io} = 0$. Thus the first row and column of the new inertia matrix will be all zeroes except for the extremely small generalized mass $\bar{m}_{oo}$. Comparing the third of Eq's 4.16 with the second of Eq's 4.7, $\bar{m}_{ij} = m_{ij}$. Making use of Eq 4.8 the new generalized masses can be summarized as

$$\bar{m}_{oo} \approx 0$$

$$\bar{m}_{io} = \bar{m}_{oi} = 0 \qquad (4.17)$$

$$\bar{m}_{ij} = 0 \qquad i \neq j$$

$$= \frac{\mu L}{2} \qquad i = j$$

Comparing the sixth of Eq's 4.16 with the second of Eq's 4.11, $\bar{k}_{ij} = k_{ij}$. Substitution of Eq's 4.13 and 4.15 into the fourth and fifth of Eq's 4.16 followed by integration yields the new generalized stiffnesses. Making use of Eq 4.12 the new generalized stiffnesses can be summarized as

$$\bar{k}_{oo} = \left(\frac{48EI}{L^3} + k\right)\left(1 - \sum_{i=1,3,\ldots}^{n} \frac{96}{(i\pi)^4}\right)$$

$$\bar{k}_{oi} = \bar{k}_{io} = k\left(1 - \sum_{i=1,3,\ldots}^{n} \frac{96}{(i\pi)^4}\right) \qquad (4.18)$$

$$\bar{k}_{ij} = k\,(-1)^{\frac{i-1}{2}}\,(-1)^{\frac{j-1}{2}} \qquad i \neq j$$

$$= \frac{(i\pi)^4}{2} \cdot \frac{EI}{L^3} + k \qquad i = j$$

Consider now the equations of motion appropriate for the modifying function $\bar{\Phi}(x)$. They can be written as

$$\begin{bmatrix} \bar{m}_{0} & 0 & 0 & \cdots \\ 0 & \bar{m}_{11} & & \\ 0 & & \bar{m}_{33} & \\ \vdots & & & \ddots \end{bmatrix} \begin{Bmatrix} \ddot{q}_o \\ \vdots \\ \ddot{q}_i \\ \vdots \end{Bmatrix} + \begin{bmatrix} \bar{k}_{oo} & \bar{k}_{o1} & \bar{k}_{o3} & \cdots \\ \bar{k}_{10} & \bar{k}_{11} & \bar{k}_{13} & \cdots \\ \bar{k}_{30} & \bar{k}_{31} & \bar{k}_{33} & \\ \vdots & \vdots & & \ddots \end{bmatrix} \begin{Bmatrix} q_o \\ \vdots \\ q_i \\ \vdots \end{Bmatrix} = \{0\} \qquad (4.19)$$

which, ignoring the nearly zero term $\bar{m}_{oo}$, can be represented as the two equations

$$\left[\diagdown_{M}\right] \left\{\ddot{q}_i\right\} + \left[-\frac{1}{\bar{k}_{oo}} \left\{\bar{k}_{io}\right\} \left\lfloor \bar{k}_{oi} \right\rfloor + \left[\bar{k}_{ij}\right] \right] \left\{q_i\right\} = \{0\} \qquad (4.20)$$

$$q_o = -\frac{1}{\bar{k}_{oo}} \cdot \left\lfloor \bar{k}_{oi} \right\rfloor \cdot \left\{q_i\right\} \qquad (4.21)$$

Equation 4.20 represents a set of reduced equations of motion in the generalized coordinates q_i and Eq 4.21 provides a method for finding the generalized coordinate q_o after Eq 4.20 is solved.

Let us compare Eq 4.20 with the equations of motion appropriate for the Fourier series representation in which a modifying function is not added. The mass matrix is unchanged. However, the stiffness matrix is augmented by $-\frac{1}{\bar{k}_{oo}} \cdot \{\bar{k}_{io}\} \cdot \left\lfloor \bar{k}_{oi} \right\rfloor$. Thus if we wish to introduce a modifying function we need only augment the stiffness matrix, a simple operation.

4.3 Comparison of Results With and Without Modifying Function

Only limited data are presented in this section. The method of the modifying function will not be used on the ring. For the ring we are interested only in modal patterns and the geometry of the modes. Thus we do not need precise mode shapes or frequencies. The method of the modifying function is very interesting, and deserves much further attention. In itself, it could be the subject of a separate research study.

A computer program was written to assemble the mass and stiffness matrices and formulate and solve the eigenvalue problem for the beam shown in Fig 4. The program, which is discussed in more detail in Appendix C, produced results for examples with and without the modifying function, $\overline{\Phi}(x)$. The results have been obtained for unit E, I, μ, and L. The relative spring stiffness, k_r, which is the ratio of the spring stiffness to the bending stiffness ($k_r = \frac{kL^3}{EI}$) was permitted to vary. The stiffness number, m, is another measure of relative spring stiffness. The stiffness number relates the stiffness of the spring to the generalized bending stiffness of the n^{th} Fourier component. For m = 1, the spring is as stiff as the generalized bending stiffness of the fundamental Fourier component, n = 1, etc.

Figure 5 shows the improvement in natural frequency, ω, vs. the relative stiffness k_r and modal stiffness number, m. The modifying function was used with a six-term Fourier series. The frequency improvement is given by

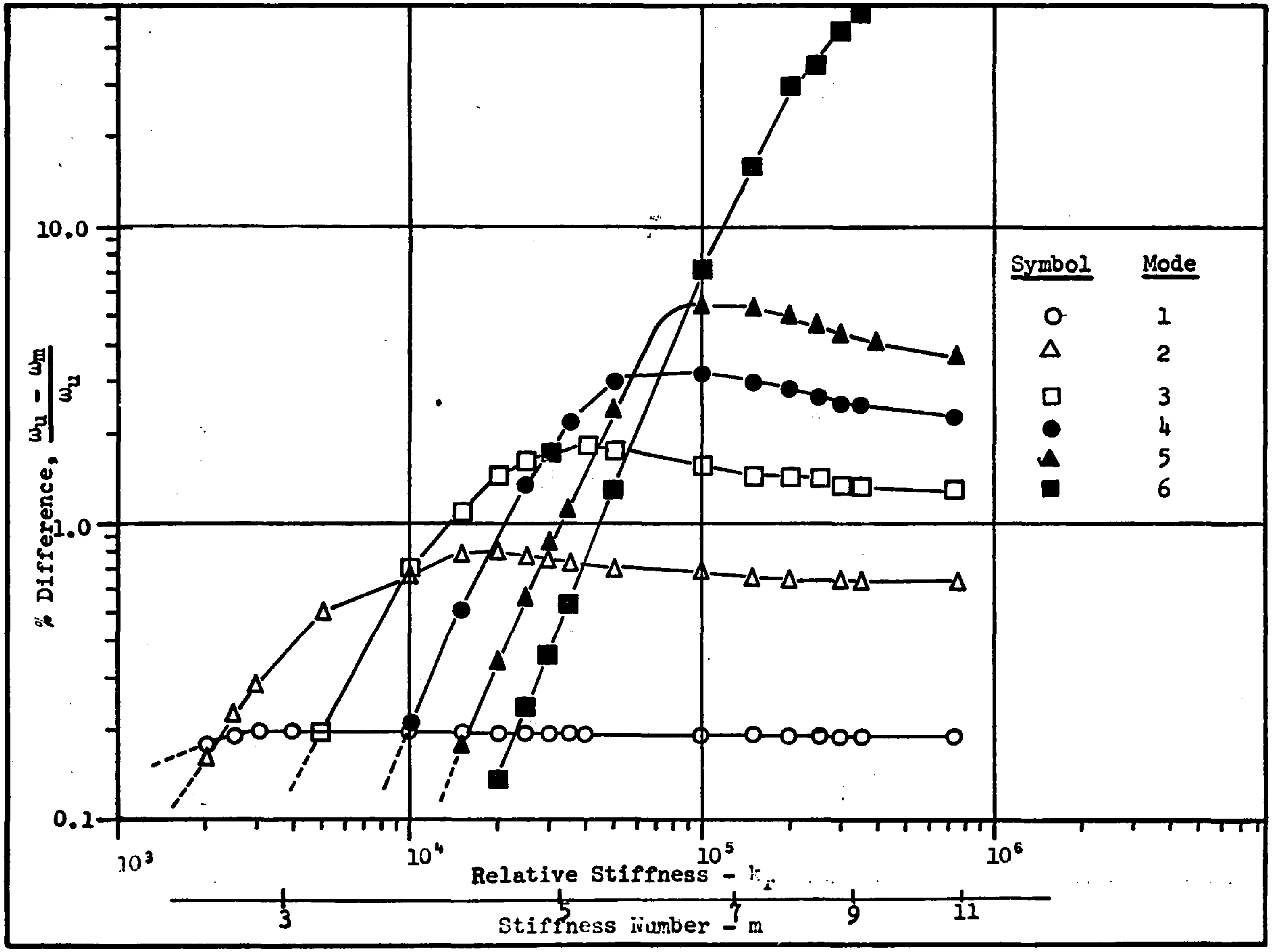


Figure 5. Improvement in Natural Frequencies for Beam Shown in Fig 4 Resulting from the use of the Modifying Function With a Six-Term Fourier Series.

$$\frac{\omega_{unmodified} - \omega_{modified}}{\omega_{modified}}$$

or

$$\frac{\omega_u - \omega_m}{\omega_m}$$

The frequencies ω_m, resulting from the use of the modifying function, are lower in each case and represent a better approximation to the exact natural frequencies.

The first important result from Fig 5 is that the improvement is less than 2 percent for all values of relative spring stiffness for the first three natural frequencies. Thus if we use an n term series and wish to give credence to the first $\frac{n}{2}$ natural frequencies there is little advantage in introducing a modifying function. The second result shown is that marked improvements are made in the higher mode natural frequencies when the modifying function is used. The improvement in the sixth natural frequency approaches 50 to 60 percent with stiffer springs.

An interesting feature of Fig 5 is the crossovers of the mode frequency curves. It can be argued that the presence of a spring at the beam midpoint will be felt first of all by the lower modes. It takes a stiffer spring to have a similar effect on the higher modes. However, the limiting value of percentage improvement is lower for the lower modes. As a result of this, the sixth frequency, for example, begins to show its improvement at a relatively high spring stiffness, but it shows the greatest improvement ultimately. Therefore the frequency curve for the sixth mode crosses over all of the other curves.

In Fig 5 we examined the improvement resulting from use of the modifying function. However Fig 5 does not show how nearly exact the results for the natural frequencies are. In Figure 6 the improvement of the first eigenvalue is shown as the number of terms in the Fourier series is increased. With the modifying function, the error in the eigenvalue will be less than one percent even if only a one-term series is used. This is true for a spring of given stiffness.

Along with the improvement in the natural frequencies we can expect a more dramatic improvement in the modal patterns and in particular the shear force distribution. Figure 7 shows the shear force distribution in the fundamental mode obtained using a six-term Fourier series, with or without the modifying function. The modifying function gives a much better representation, and, because of the nature of $\overline{\Phi}(x)$, the shear force will have a discontinuity at $\frac{x}{L} = 0.5$. Because the Fourier series without the modifying function must converge to the mean value of the discontinuity at $\frac{x}{L} = 0.5$, the pure Fourier series gives a poor representation of shear force in the region of $\frac{x}{L} = 0.5$, including a false peak in the vicinity of $\frac{x}{L} = 0.4$. The improvement in the shear force distribution is shown even more vividly in Fig 8. Here a two-term Fourier series, with and without the modifying function, has been used to represent the shear force distribution in the fundamental mode. The two-term Fourier series solution represents the shear force distribution very poorly. The two-term Fourier series plus a modifying function, however, represents the shear force distribution very well.

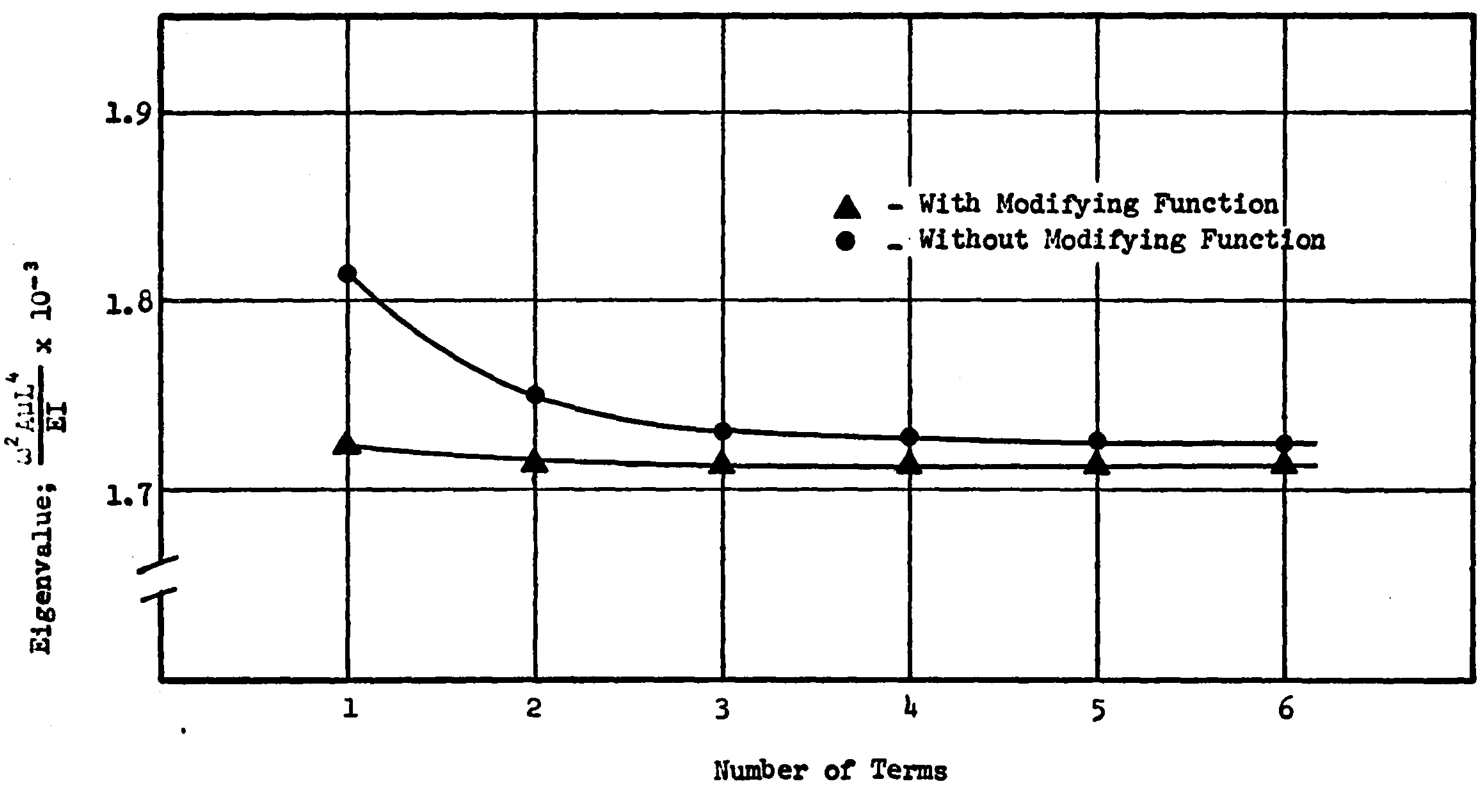


Figure 6. Comparison of Fundamental Eigenvalue vs. Number of Terms With and Without Modifying Function For the Beam of Figure 4. Relative Stiffness = 10,000.

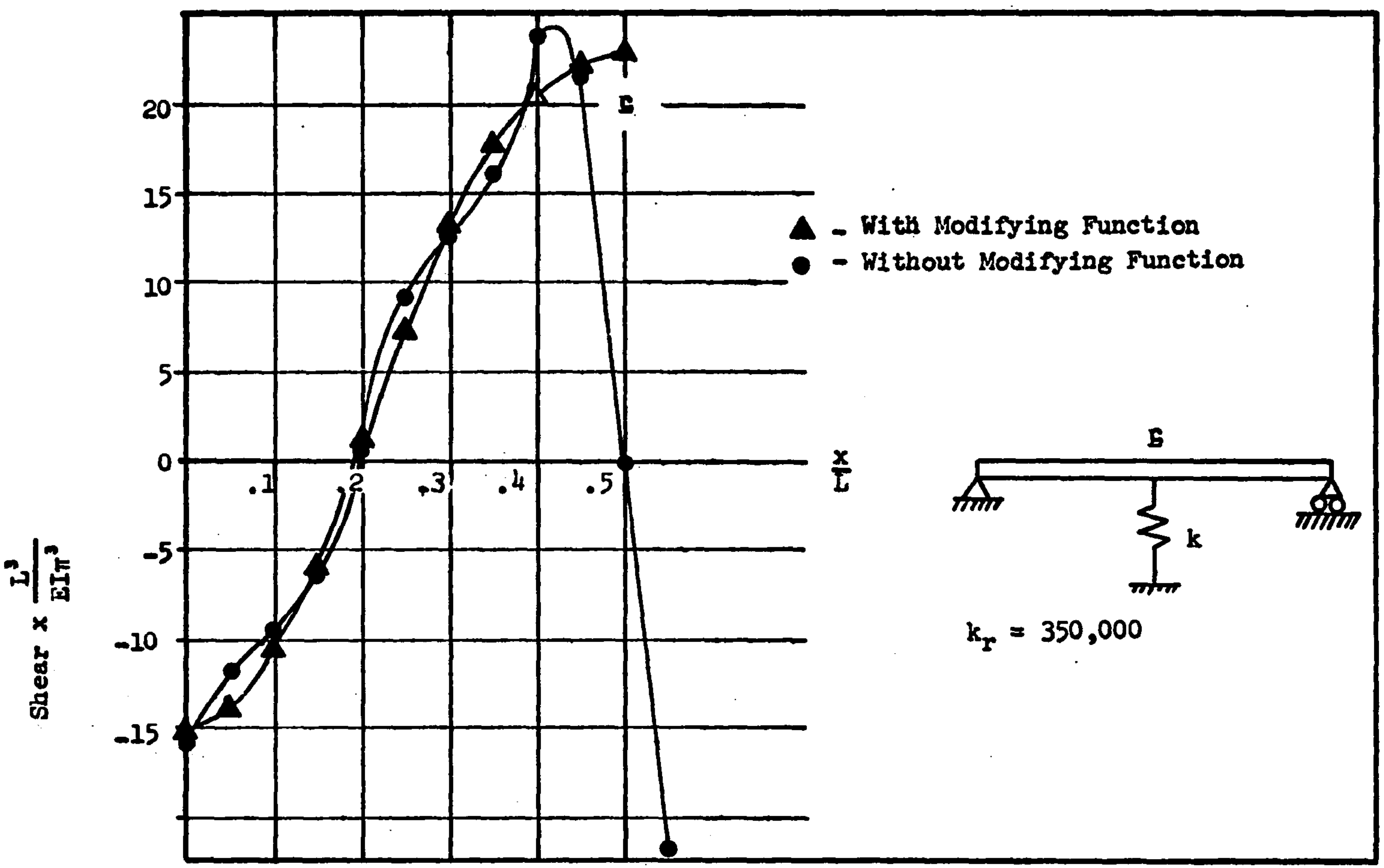


Figure 7. Comparison of Shear Force Distribution for Six-Term Fourier Series With and Without Modifying Function. Fundamental Mode.

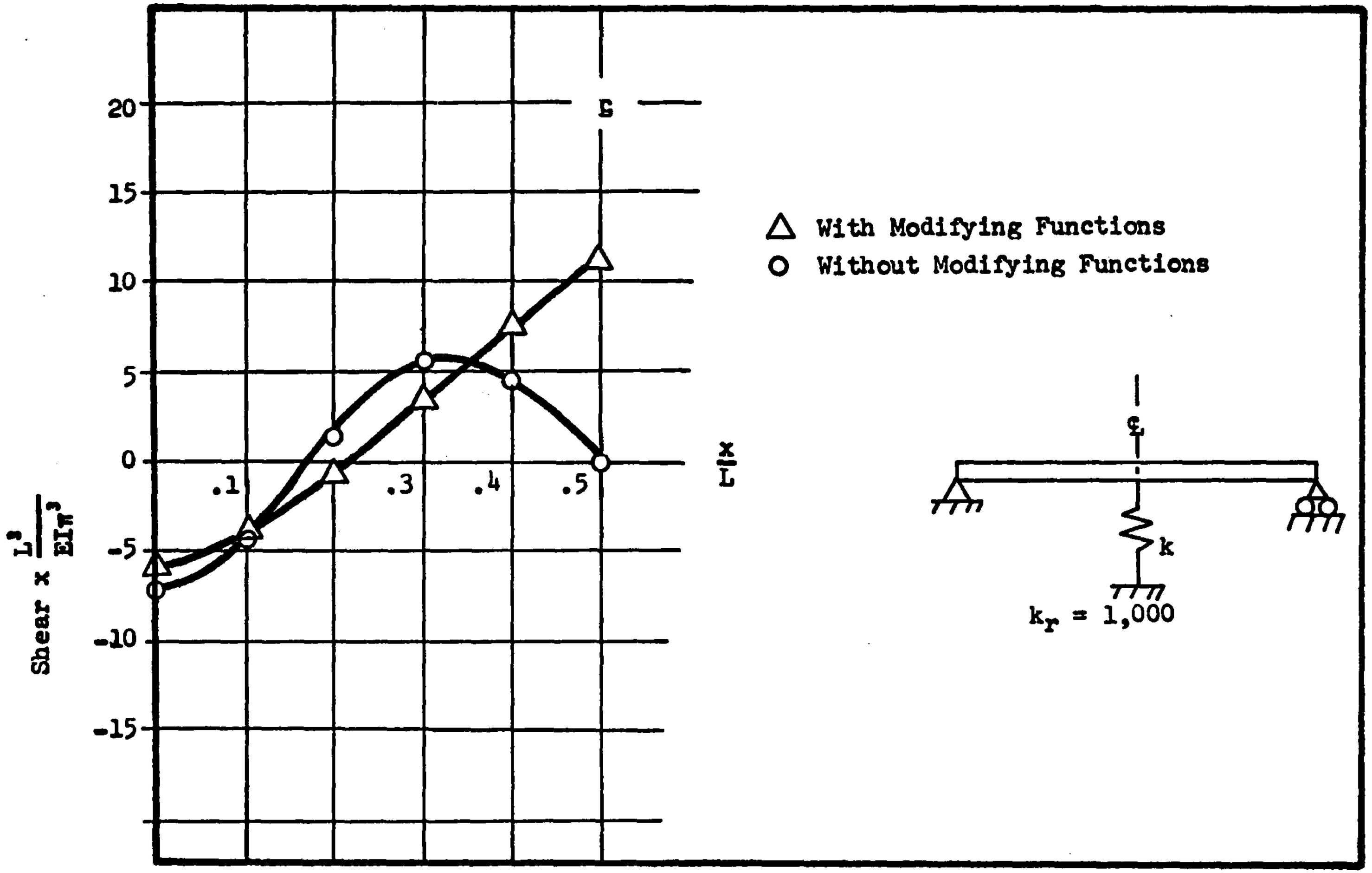


Figure 8. Comparison of Shear Force Distribution for Two-Term Fourier Series With and Without Modifying Function. Fundamental Mode.

It is possible to obtain an exact expression for the shear force distribution for this problem. The differential equation for the uniform simply supported beam is (Ref. 6).

$$EIy^{IV} + \mu\omega^2 y = 0$$

The general solution for this equation is

$$y(x) = B_1 \sin \sqrt[4]{\frac{\mu\omega^2}{EI}}\, x + B_2 \cos \sqrt[4]{\frac{\mu\omega^2}{EI}}\, x$$

$$+ B_3 \sinh \sqrt[4]{\frac{\mu\omega^2}{EI}}\, x + B_4 \cosh \sqrt[4]{\frac{\mu\omega^2}{EI}}\, x$$

When the boundary conditions

$$y(o) = 0$$

$$EIy''(o) = 0$$

$$EIy'''(\tfrac{L}{2}) = \tfrac{k}{2}\, y(\tfrac{L}{2})$$

$$y'(\tfrac{L}{2}) = 0$$

are substituted into the general solution, an exact expression for the shear force is obtained as follows:

$$EIy''' = C \left[\sqrt[3/4]{\frac{\mu\omega^2}{EI}}\; \frac{\cosh \sqrt[4]{\frac{\mu\omega^2}{EI}}\,\frac{L}{2}}{\cos \sqrt[4]{\frac{\mu\omega^2}{EI}}\,\frac{L}{2}} \quad \cos \sqrt[4]{\frac{\mu\omega^2}{EI}} \cdot x \right.$$

$$\left. + \sqrt[3/4]{\frac{\mu\omega^2}{EI}} \quad \cosh \sqrt[4]{\frac{\mu\omega^2}{EI}} \cdot x \right]$$

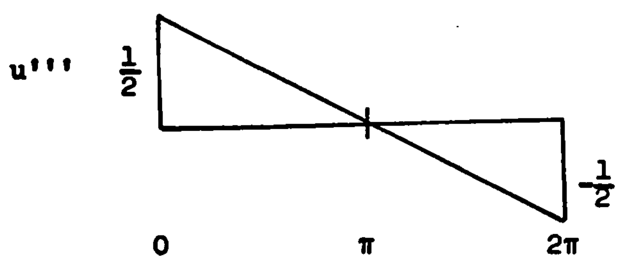
When this expression is evaluated, and the results are scaled to the same maximum shear force shown in Fig 8, the exact results are indistinguishable from the two-term Fourier series with the modifying function which is plotted in Fig 8. Thus we see that the use of the modifying function with only a two-term Fourier series produces nearly exact results for this particular example.

4.4 Application to the Ring

The use of the modifying function has been shown to improve results for the natural frequencies and shear force distribution for an example beam. The question of the application of this method to the ring problem now arises. As previously mentioned, the modifying function will not be applied to the ring. For the ring, a Fourier series approximation for the displacement will be used, and one of the results of the investigation of the effect of the modifying function on the beam. Namely, if we consider n terms for a sine series and n terms for a cosine series representation, only the first $\frac{n}{2}$ of the resulting frequencies and mode shapes will be accepted, and the $\frac{n}{2}$ higher frequencies and mode shapes will be discarded.

At this point, however, the method for extending the modifying function approach to the ring problem will be briefly discussed. The choice of an appropriate modifying function is the first concern. This function must have discontinuities in shear at each spring, and should be as simple as possible, in order to reduce the labor of calculating the mass and stiffness coefficients. One choice for such a function is the radial displacement function, u, for a free ring which is being accelerated by a concentrated force. However, a solution to this problem by LaPlace transform methods leads to very complicated functions for the displacement.

Perhaps the simplest shear function, u''', with the required discontinuity in shear, is depicted below.



The shear force has a unit jump at the origin, and is represented by

$$u''' = \frac{1}{2} - \frac{\theta}{2\pi} \qquad 0 \leq \theta < 2\pi$$

Integrating this shear distribution three times gives

$$u = -\frac{\theta^4}{48\pi} + \frac{\theta^3}{12} - \frac{\theta^2\pi}{12} + 1 \qquad 0 \leq \theta < 2\pi \tag{4.22}$$

The constants of integration are evaluated from continuity considerations. The relationship $u = \frac{\partial w}{\partial \theta}$ will give the transverse displacement, w.

$$w = \frac{\theta^5}{240\pi} + \frac{\theta^4}{48} - \frac{\theta^3\pi}{36} + \theta \qquad 0 \leq \theta < 2\pi. \tag{4.23}$$

Equations 4.22 and 4.23 give functions which have shear discontinuity at the origin. For a ring with N springs, N such functions will be needed. For each spring there will be a function with a shear discontinuity at the location of that spring.

Figure 9 shows the unfolded circumference of a ring with N equally spaced springs. Let us denote by ϕ_i the modifying function for the radial displacement, and by Φ_i the modifying function for the

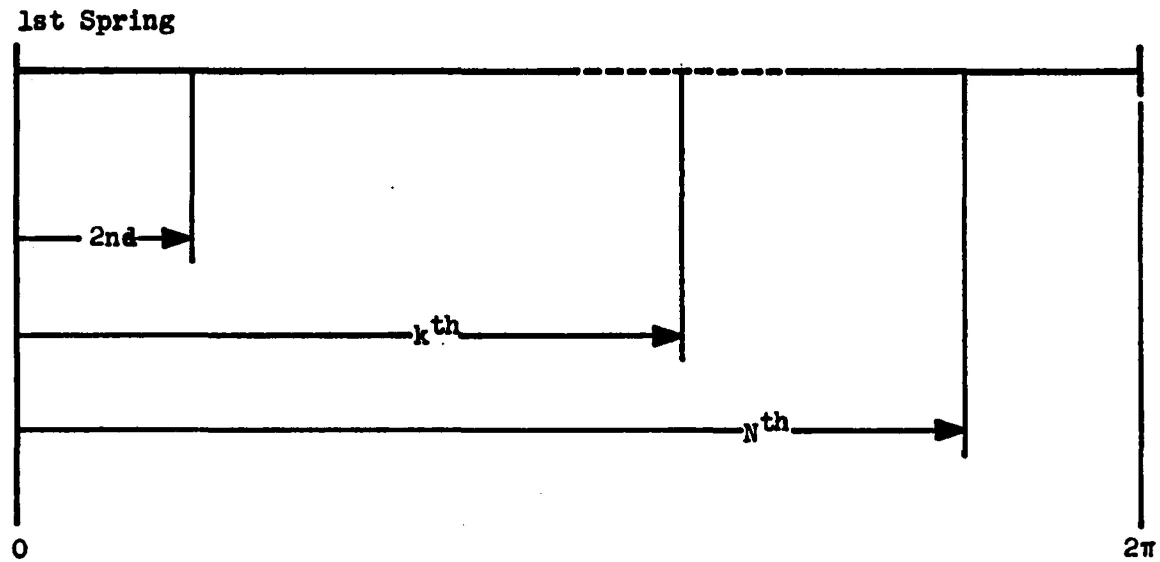


Figure 9. Unfolded View of Circumference of Ring with N Equally-Spaced Springs.

transverse displacement. Then, for the kth spring, we can show a function with a jump at this location by

$$\phi_k = u\,(\theta - 2\pi \cdot \frac{k-1}{N}) \qquad \text{for } \theta \geq \frac{2\pi}{N}(k-1)$$

and $$\phi_k = u\,(\theta + 2\pi \cdot \frac{N-k+1}{N}) \qquad \text{for } \theta < \frac{2\pi}{N}(k-1)$$

Similar expressions may be written for Φ_k. Then the total radial displacement, u, of the ring is given by

$$u(\theta) = \phi_1 + \phi_2 + \ldots + \phi_N + \sum_{i=1}^{n} (a_i \cos i\theta + b_i \sin i\theta) \qquad (4.24)$$

Also, the transverse displacement, w, is

$$w(\theta) = \Phi_1 + \Phi_2 + \ldots + \Phi_N + \sum_{i=1}^{n} (\frac{a_i}{i} \sin i\theta - \frac{b_i}{i} \cos i\theta) \qquad (4.25)$$

These displacement relationships can be used in expressions for kinetic and potential energies to obtain the mass and stiffness coefficients. The details of this will not be given, since the results are not going to be utilized, but it should be mentioned that the derivation of these coefficients, for a general case (any number of springs, N) is quite laborious. It is also essential, based on the experience in Section 4.3, to subtract out the first n Fourier coefficients (n being the number of cosine and sine terms considered in the problem) of the ϕ and Φ functions. This will result in the final equations of motion having the same number of degrees of freedom, 2n, as the original ones. The stiffness matrix will be augmented in a manner similar to that of the beam example problem.

4.5 Conclusion

We have seen the very significant improvements in natural frequency and shear force distribution which result from introducing a modifying function in addition to the usual Fourier series representation for the displacement. The modifying function method is simple to use, requiring only an augmentation of the stiffness matrix.

This method should find wide application wherever the presence of springs cause discontinuities in shear force or bending moment distribution of otherwise continuous systems. The method is by no means restricted to the simple example of the beam given in this Chapter. The results presented in this Chapter are limited, and the modifying function method deserves further study.

CHAPTER 5

SOLUTION OF EQUATIONS OF MOTION
AND PRESENTATION OF RESULTS

5.1 Introduction

In this Chapter, solution of the equations of motion will be discussed. In Chapter 3 it was shown that the equations of motion are not completely coupled, and that they may be written as several independent sets of equations. Each of the independent sets of equations yields a family of normal modes. The families of normal modes will be identified as the "spring families" and the "intermediate families." For an odd number N of springs, the spring families are made up of the N^{th}, $2N^{th}$, $3N^{th}$, ... harmonics. The remaining families are referred to as the intermediate families. For an even number N of springs, there are two spring families; one made up of the $\frac{N}{2}^{th}$, $\frac{3N}{2}^{th}$, $\frac{5N}{2}^{th}$, ... harmonics and the other made up of the N^{th}, $2N^{th}$, $3N^{th}$, ... harmonics. The remaining families will again be referred to as the intermediate families.

Referring to Fig 2, the spring stiffness matrix for the case of N = 4 springs, one spring family is made up of the 2^{nd}, 6^{th}, 10^{th}, ... harmonics. A second spring family is made up of the 4^{th}, 8^{th}, 12^{th}, ... harmonics. Referring to Fig 3, the case of N = 5 springs, the spring family is made up of the 5^{th}, 10^{th}, 15^{th}, ... harmonics. One intermediate family is made up of the 1^{st}, 4^{th}, 6^{th}, 9^{th}, ... harmonics, and the other intermediate family is made up of the 2^{nd}, 3^{rd}, 7^{th}, 8^{th}, ... harmonics.

After discussing the solution of the equations of motion, the effect of varying spring stiffnesses on natural frequency will be discussed. A method for representing the resulting mode shape will be developed and a discussion of the resulting mode shape will be given.

5.2 Equations of Motion

Let us summarize the general form found for the equation of motion. Recall that the mass matrix was diagonal, and that the stiffness matrix had identically zero submatrices in the upper right and lower left positions. Therefore, we may write

$$\left[\begin{array}{c|c} \underline{M_a} & \underline{0} \\ \hline \underline{0} & \underline{M_b} \end{array}\right] \left\{\begin{array}{c} \ddot{\underline{a}} \\ \hline \ddot{\underline{b}} \end{array}\right\} + \left[\begin{array}{c|c} \underline{K_a} & \underline{0} \\ \hline \underline{0} & \underline{K_b} \end{array}\right] \left\{\begin{array}{c} \underline{a} \\ \hline \underline{b} \end{array}\right\} = \{\underline{0}\} \qquad (5.1)$$

If we assume a harmonic solution for {a} and {b} with respect to time,

$$a = a \sin \omega t$$

$$b = b \sin \omega t$$

then we have

$$\ddot{a} = -\omega^2 a \sin \omega t$$

$$\ddot{b} = -\omega^2 b \sin \omega t$$

Equations of motion 5.1 may then be written as

$$\begin{bmatrix} \underline{K_a} & \underline{0} \\ \underline{0} & \underline{K_b} \end{bmatrix} \begin{Bmatrix} \underline{a} \\ \underline{b} \end{Bmatrix} = \omega^2 \begin{bmatrix} \underline{M_a} & \underline{0} \\ \underline{0} & \underline{M_b} \end{bmatrix} \begin{Bmatrix} \underline{a} \\ \underline{b} \end{Bmatrix} \tag{5.2}$$

Because of the zero submatrices in the stiffness matrix, Eq 5.2 separates into the two matrix equations

$$\underline{K_a}\,\{a\} = \omega_a^2\,\underline{M_a}\,\{a\}$$

$$\underline{K_b}\,\{b\} = \omega_b^2\,\underline{M_b}\,\{b\} \tag{5.3}$$

Let us now discuss the sets of equations which may be obtained from Eq's 5.3. Consider first the sets of equations associated with the spring family or families of modes. The spring family equations in the coefficients {a} of the cosine series are coupled elastically by the springs. For an odd number N of springs, each of the spring family modes is represented by a cosine series involving the N^{th}, $2N^{th}$, $3N^{th}$, ... harmonics. For an even number of springs there will be an additional spring family mode given by a cosine series composed of the $\frac{N}{2}$th, $\frac{3N}{2}$th, $\frac{5N}{2}$th, ... harmonics. The spring family equations in the coefficients {b} of the sine series are not coupled. Thus for an odd number N of springs, the spring family modes are the individual sine functions, the pure N^{th}, $2N^{th}$, $3N^{th}$, ... harmonic. For an even number of springs, there will be additional spring family modes given by the individual sine functions, the pure $\frac{N}{2}$th, $\frac{3N}{2}$th, $\frac{5N}{2}$th, ... harmonic. The pure sine function modes are the spring family of modes and have nodal points at all of the springs. Thus these modes are identical with those of the ring unsupported by springs. The natural frequencies of the spring family of normal modes which are made up of the cosine functions will obviously be different from those made up of the sine functions. The natural frequencies resulting from the solution of the equations of motion in the cosine series (the {a} terms) will therefore be distinct from those associated with the sine series (the {b} terms).

Let us now consider the sets of equations associated with the intermediate families of modes. All of the equations of motion in these sets are coupled by the springs. For a particular intermediate family, the equations in the coefficients {a} of the cosine series differ in only one respect from the coefficients {b} of the sine series. Although the stiffness matrices are term by term identical in magnitude, the stiffnesses associated with the coefficients {a} are all positive while those associated with {b} are alternately positive and negative. As a result the natural frequencies of the intermediate family made up of the cosine series are identical with those of the same intermediate family made up of the sine series. Thus for a given natural frequency any linear combination of the cosine series and the sine series represents a normal mode shape. This results in the important conclusion that there are an infinity of mode shapes associated with each natural frequency for an intermediate family.

Consider a specific example for the case of $N = 4$ springs. The first spring family of modes in the coefficients {a} will be represented by a cosine series of the 2^{nd}, 6^{th}, 10^{th}, ... harmonics. The first spring family in the coefficients {b} will be composed of the pure sine harmonic 2, or 6, or 10, or ..., depending on the mode in the family being examined. The modal representation for the {a} terms and the {b} terms will be different, and the corresponding natural frequency for the equations of motion in the coefficients {a} will be distinct from the natural frequency resulting from the equation of motion in the coefficient {b}. This will be referred to later as a "distinct-frequency" case. The second spring family of modes for the case of $N = 4$ springs will be represented as follows: in the coefficients {a} by a cosine series composed of the 4^{th}, 8^{th}, 12^{th}, ... harmonics, and in the coefficient {b} by a pure sine harmonic--the 4^{th}, or 8^{th}, or 12^{th}, or .... This will also be a distinct-frequency case. For both of these spring families, the sine harmonics will have a nodal point at each of the springs. The mode shape and natural frequency for these sine-term spring families will be the same as for an unsupported ring. The intermediate family in the case of $N = 4$ springs will be composed of a linear combination of the cosine series in the 1^{st}, 3^{rd}, 5^{th}, ... harmonics and the sine series in the 1^{st}, 3^{rd}, 5^{th}, ... harmonics. As previously discussed, the natural frequency associated with the equations of motion in the {a} coefficients is equal to the natural frequency associated with the equations of motion in the {b} coefficients for the intermediate families. This will be referred to as an "equal-frequency" case.

Consider now the specific example of $N = 5$ springs. The one spring family of modes in this case will be composed of the cosine series in the 5^{th}, 10^{th}, or 15^{th}, ... harmonics plus a pure sine term of the 5^{th}, or 10^{th}, or 15^{th}, or ... harmonic, depending on whether we are discussing the first, second, etc. mode in the family. This will be a distinct-frequency case. One of the intermediate families of modes will be composed of a linear combination of a cosine series in the 1st, 4th, 6th, 9th, ... harmonics plus a sine series in the 1st, 4th, 6th, 9th, ... harmonics. This will be an equal-frequency case. The second intermediate family of modes will be composed of a linear combination of a cosine series in the 2nd, 3rd, 7th, 8th, ... harmonics plus a sine series in the 2nd, 3rd, 7th, 8th, ... harmonics. This will also be an equal-frequency case. The examples for the case of N = 4 springs and N = 5 springs are typical of all cases.

5.3 Non-Dimensionalization of the Mass and Stiffness Coefficients

It is convenient for computational purposes and for presentation of results to non-dimensionalize Eq's 5.3. Either of these equations may be chosen as a typical example, and, dropping the subscripts, this gives

$$\overline{M}\,\omega^2\,\underline{M}'\,\{a\} = \overline{K}\,\underline{K}'\,\{a\} \tag{5.4}$$

where $\overline{M}$ and $\overline{K}$ have been defined in Eq's 2.6 and 2.14, and $\underline{M}'$ and $\underline{K}'$ are the non-dimensionalized mass and stiffness matrices. Dividing through by $\overline{K}$, we may write Eq 5.4 as

$$\frac{\overline{M}}{\overline{K}}\,\omega^2\,\underline{M}'\,\{a\} = \underline{K}'\,\{a\} \tag{5.5}$$

Rearranging, and defining $\omega_r^2 = \omega^2\frac{\overline{M}}{\overline{K}}$, we have

$$\underline{K}'\,\{a\} = \omega_r^2\,\underline{M}'\,\{a\} \tag{5.6}$$

Equation 5.6 represents the non-dimensionalized equations of motion. When this Equation is solved for eigenvalues ω_r^2, the eigenvalues ω^2 may be computed from

$$\omega^2 = \omega_r^2 \frac{\bar{K}}{\bar{M}} = \omega_r^2 \frac{EI}{A\rho r^4} \tag{5.7}$$

5.4 Solution of the Eigenvalue Problem

Any of the typical eigenvalue problems discussed above can be solved numerically in the following way: The University of Arizona CDC 6400 MATRIX routines, which are contained in the computer system library, will find the eigenvalues and eigenvectors for a real symmetric matrix. Equation 5.6 can be written with a symmetric characteristic matrix by using a simple transformation.[1] Premultiply Eq 5.6 by $\underline{M}'^{-1/2}$

$$\underline{M}'^{-1/2}\,\underline{K}'\,\{a\} = \omega_r^2\,\underline{M}'^{1/2}\,\{a\} \tag{5.8}$$

Insert $\underline{M}'^{-1/2} \cdot \underline{M}'^{1/2}$ in the left-hand side

$$[\underline{M}'^{-1/2}\,\underline{K}'\underline{M}'^{-1/2}]\,\underline{M}'^{1/2}\,\{a\} = \omega_r^2\,\underline{M}'^{1/2}\,\{a\} \tag{5.9}$$

Letting

$$\underline{M}'^{-1/2}\,\underline{K}'\underline{M}'^{-1/2} = \underline{D} \tag{5.10}$$

and

$$\underline{M}'^{1/2}\,\{a\} = \{r\}$$

The eigenvalue problem becomes

$$\underline{D}\,\{r\} = \omega_r^2\,\{r\} \tag{5.11}$$

The characteristic matrix $\underline{D}$ for Eq 5.11 will now be symmetric. The eigenvalues of Eq 5.11 will be the same as those for Eq 5.6. The

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[1]This procedure was brought to the writer's attention by Dr. H. A. Kamel.

eigenvectors for Eq 5.6 may be found from the eigenvectors of Eq 5.11 by using the second of Eq's 5.10 in the form

$$\{a\} = \underline{M}^{-1/2}\ \{r\} \tag{5.12}$$

5.5 Solution for Natural Frequencies

The behavior of natural frequencies for number of springs N = 3, 4, and 5 has been examined in detail. This is felt to be a sufficient number of cases for a representative sample. A computer program, which is discussed more thoroughly in Appendix D, was used to obtain the results. Figures 10, 11, and 12 on the following pages show the behavior of eigenvalues ω_r^2 for varying relative spring stiffness, k_r. These results have been determined for unit value of E, I, A, ρ, and r. The size of total mass and stiffness matrix used to obtain the results is S = 2N + 2. This is used to assure at least one repetition of the basic coupling pattern among the equations of motion, regardless of the value of N. The equations of motion in the cosine coefficients and sine coefficients will be of size N + 1. However, in accordance with the conclusions in Sec 4.3, and the discussion of Fig 5, only $\frac{N}{2}$ of the resulting eigenvalues have been used. The $\frac{N}{2}$ highest eigenvalues have been discarded. The results of Sec 4.3 apply to the problem of a beam with a transverse spring at the midpoint. The behavior of the thin ring in the vicinity of any one of its radially acting springs should be very similar. Therefore the results of Sec 4.3 are assumed to be valid for the ring.

The following notation has been adopted for the presentation of the eigenvalues. The subscript r will be dropped in Figs 10, 11, and 12, and the eigenvalue will be given as $\omega^2_{n,m}$, where n is the family

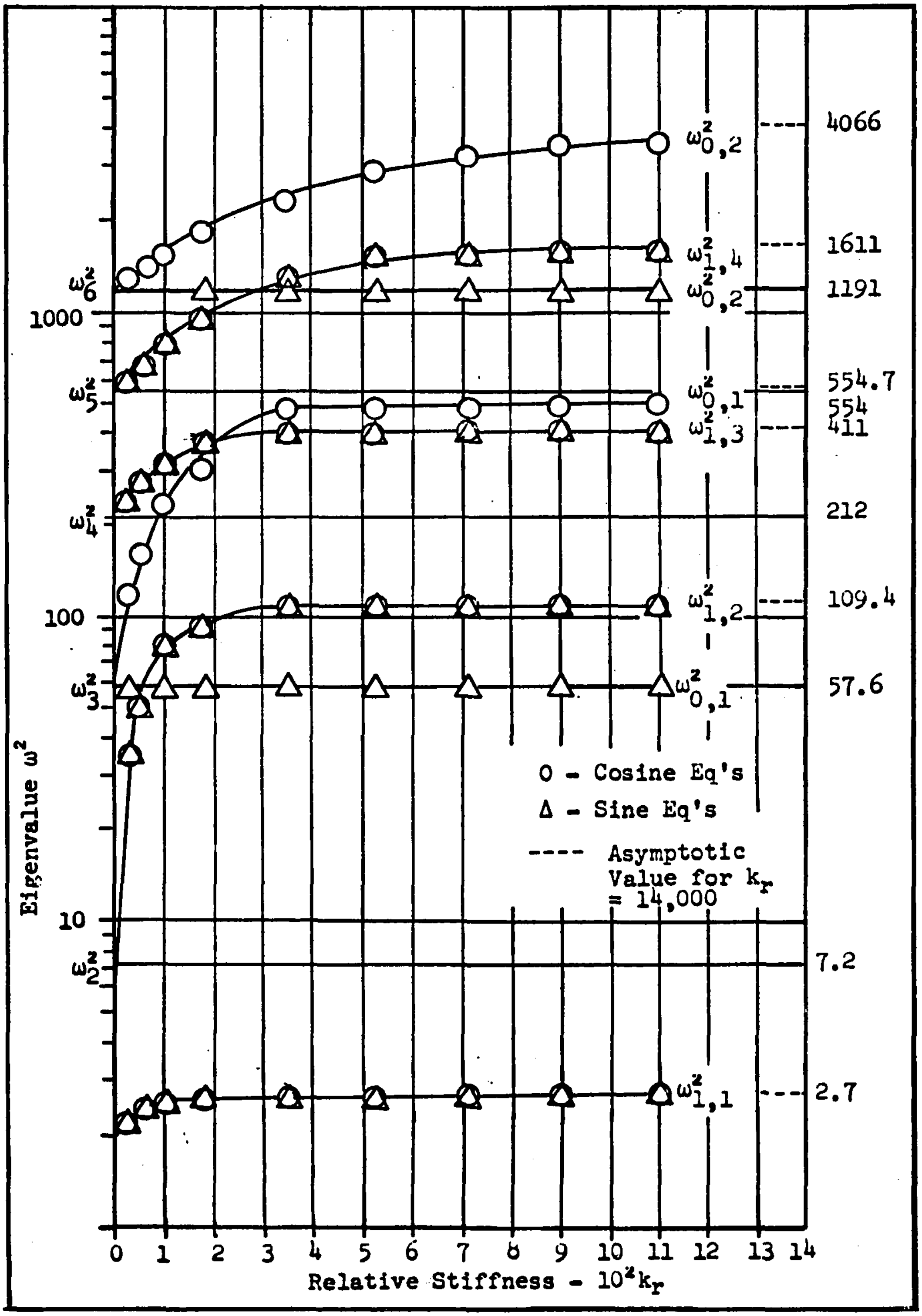

Figure 10. Eigenvalues vs. Relative Stiffness for Ring With Three Springs. ω^2 With Single Subscripts Are for a Free Ring.

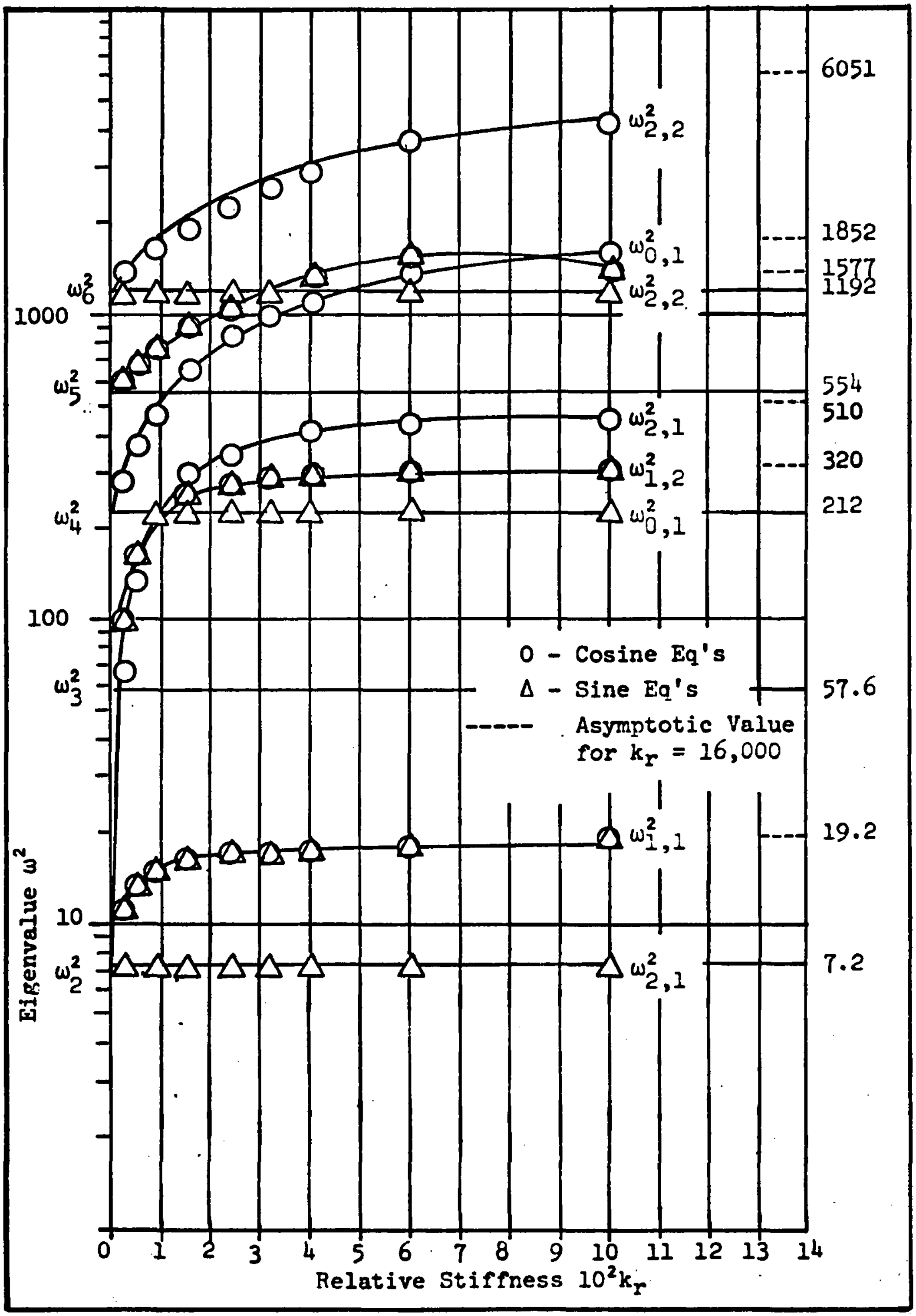

Figure 11. Eigenvalues vs. Relative Stiffness for Ring With Four Springs. ω^2 With Single Subscripts Are for a Free Ring.

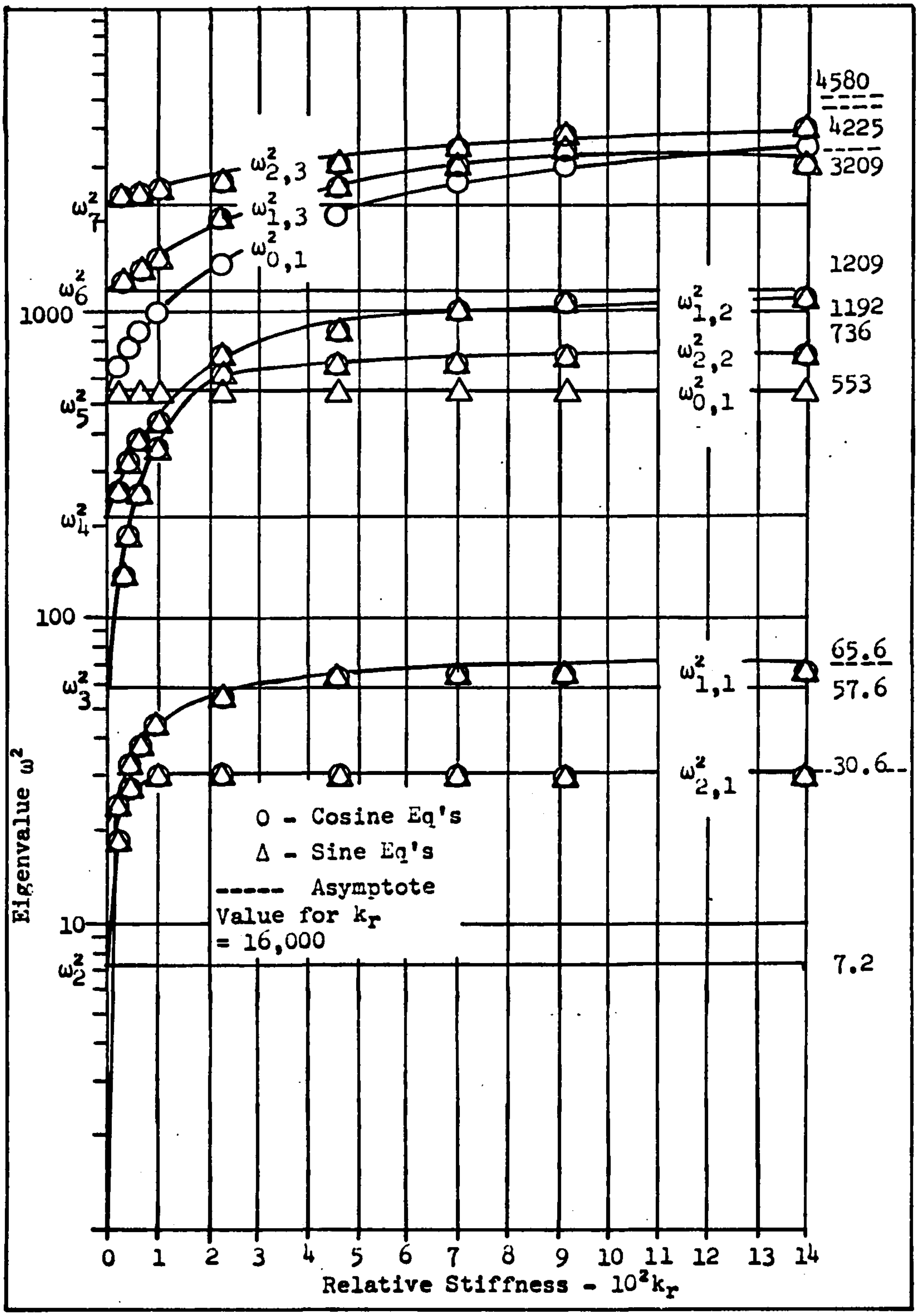

Figure 12. Eigenvalue vs. Relative Stiffness k_r for a Ring With Five Springs. ω^2 With Single Subscripts Are for a Free Ring.

number, and m = 1, 2, 3, ... will indicate the ascending eigenvalues within the family. The values ω_n^2, with a single subscript, which are given on the ordinate, are the eigenvalues for a ring without springs.

Figure 10, for N = 3 springs, shows the behavior of the eigenvalues. The spring family in this case is composed of harmonics 3, 6, 9, ... This is family number n = 0. The behavior of spring family eigenvalues $\omega_{0,1}^2$ and $\omega_{0,2}^2$, the two lowest eigenvalues in this family are shown. Note that this is a distinct-eigenvalue case. The eigenvalues resulting from the equations of motion in the cosine series for this family are distinct from those resulting from the equations of motion in the sine series. The eigenvalues for the equations of motion in the sine series are equal to the third and sixth eigenvalues, ω_3^2 and ω_6^2, for the unsupported ring for all values of relative stiffness k_r. The eigenvalues for the cosine series increase above the sine series values for increasing values of k_r; until, for high values of k_r, they approach asymptotically an eigenvalue which represents that of a ring pinned at the springs. These asymptotes are indicated by dashed lines, and the associated value of k_r is given in the legend of the figure. The intermediate family n = 1, is composed of the 1st, 2nd, 4th, 5th,..., harmonics. Four of its eigenvalues, $\omega_{1,1}^2$, $\omega_{1,2}^2$, $\omega_{1,3}^2$, and $\omega_{1,4}^2$ are shown in Fig 10. This is an intermediate family, and represents an <u>equal-eigenvalue</u> case. The eigenvalues resulting from the equations of motion in the cosine series are equal to the eigenvalues resulting from the equations of motion in the sine series. For increasing values of k_r, the eigenvalues increase rapidly at first, and then asymptotically approach a limiting eigenvalue which represents that of a ring which is pinned at each of the springs. In all cases, when an eigenvalue approaches an asymptotic value, this will be the eigenvalue for a ring which is pinned at each of the springs. It is interesting to note the rapid increase in eigenvalue for small values of k_r, indicating that the introduction of relatively flexible supports may affect the ring eigenvalues (and natural frequencies) significantly.

Figure 11 shows the same plot for an even number of springs, $N = 4$. There are two spring families in this case. The spring family $n = 0$ is composed of the 4^{th}, 8^{th}, 12^{th}, ... harmonics. This is a distinct-eigenvalue case, and only the lowest eigenvalue, $\omega^2_{0,1}$, is shown. The eigenvalue resulting from the sine series equations of motion are equal to the fourth eigenvalue ω^2_4 for the unsupported ring. This eigenvalue is independent of k_r. The eigenvalue associated with the cosine series for this family increases rapidly with increasing k_r and then approaches an asymptote. The second spring family, $n = 2$, is composed of the 2^{nd}, 6^{th}, 10^{th}, ... harmonics. This is another distinct-eigenvalue case. Two of the eigenvalues for this family, $\omega^2_{2,1}$ and $\omega^2_{2,2}$ are shown in Fig 11. The eigenvalues associated with the sine series equations of motion are constant for all values of k_r. These eigenvalues are equal to the second and sixth eigenvalues, ω^2_2 and ω^2_6, for the unsupported ring. The eigenvalues associated with the cosine series equations of motion increase rapidly with increasing values of k_r and then approach an asymptotic value. The intermediate family in this case is family number $n = 1$ which is composed of harmonics 3, 5, 7, ...

Three of its eigenvalues are shown in Fig 11. These are the eigenvalues $\omega^2_{1,1}$, $\omega^2_{1,2}$, $\omega^2_{1,3}$. Each of these is an equal-eigenvalue case. The eigenvalues associated with the cosine series equations of motion are equal to those associated with the sine series equations of motion.

The case of number of springs N = 5 is shown in Fig 12. The one spring family in this case, family number n = 0, is composed of the 5^{th}, 10^{th}, 15^{th}, ..., harmonics. This is a distinct-eigenvalue case. One of its eigenvalues $\omega^2_{0,1}$, is shown in Fig 12. The eigenvalue associated with the sine series equations of motion is constant for all values of relative stiffness k_r and is equal to the fifth eigenvalue, ω^2_5, for the unsupported ring. The eigenvalue associated with the cosine series increases rapidly for the lower values of k_r and then approaches an asymptotic value. The first intermediate family, family number n = 1, is an equal-eigenvalue case. This family is composed of the 1^{st}, 4^{th}, 6^{th}, ..., harmonics. The eigenvalues $\omega^2_{1,1}$, $\omega^2_{1,2}$, $\omega^2_{1,3}$ are shown in Fig 12. The eigenvalues associated with the cosine series equations of motion are equal to the eigenvalues associated with the sine series equations of motion. The eigenvalues increase rapidly with increasing k_r and then approach asymptotic values. The second intermediate family, family number n = 2, is composed of the 2^{nd}, 3^{rd}, 7^{th}, 8^{th}, ..., harmonics. Three eigenvalues for this family, shown in Fig 12, are $\omega^2_{2,1}$, $\omega^2_{2,2}$, $\omega^2_{2,3}$. Each of these is also an equal-frequency case. These eigenvalues increase with increasing k_r and approach asymptotic values also.

The examples for N = 3, 4, and 5 are typical for all results. The spring families will always be distinct-eigenvalue cases and the intermediate families will always be equal-eigenvalue cases. The effects of relative spring stiffness k_r will be initially high, then those eigenvalues which are functions of k_r will approach asymptotic values which are the eigenvalues for a ring which is pinned at all the springs. It is interesting to note that for all cases, for value of $k_r = 0$, all eigenvalues are equal to the eigenvalues for the unsupported ring as given by (Ref. 7, p. 429).

$$\omega_i^2 = \frac{EI}{\rho A r^4} \frac{i^2(1-i^2)^2}{1 + i^2} \qquad i = 2, 3, \ldots.$$

This discussion summarizes the effects of the springs on the eigenvalues (and hence natural frequencies) of the ring.

5.6 Representation of Equal-Eigenvalue Mode Shapes

When equal eigenvalues occur within a problem, any linear combination of the eigenfunctions corresponding to the equal eigenvalues is also an eigenfunction (Ref. 8, p. 79). Let us consider the problem of selecting a linear combination of eigenfunctions resulting from equal eigenvalues, the object being to provide a convenient representation of mode shape. This will provide a meaningful representation for the modes of the intermediate families.

Begin with a particular example where the number of springs, N, is five, and the family number, n, is one. The modal matrix from the solution of one case for N = 5 is shown in Fig 13. These result from the solution of Eq 5.1 and are, of course, the a and b coefficients of Eq 2.1. A linear combination of the first modal cosine and sine columns is the modal function $f(\theta)$.

$$\underline{a} = \begin{bmatrix} 0.7714 & 0. & 0. & 0.2467 & 0. & 0.2170 \\ 0. & 0.7183 & 0.5019 & 0. & 0. & 0. \\ 0. & -0.6956 & 0.5919 & 0. & 0. & 0. \\ -0.6304 & 0. & 0. & 0.6622 & 0. & 0.4336 \\ 0. & 0. & 0. & 0. & 0.8991 & 0. \\ -0.0853 & 0. & 0. & -0.7043 & 0. & 0.6262 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0.7714 & 0. & 0. & -0.2467 & 0. & 0.2170 \\ 0. & 0.7183 & -0.5019 & 0. & 0. & 0. \\ 0. & 0.6956 & 0.5919 & 0. & 0. & 0. \\ 0.6304 & 0. & 0. & 0.6622 & 0. & -0.4336 \\ 0. & 0. & 0. & 0. & 1.0 & 0. \\ -0.0853 & 0. & 0. & 0.7043 & 0. & 0.6262 \end{bmatrix}$$

Figure 13. Components of Modal Matrix (Coefficients a_{ij} and b_{ij}) for N = 5 and k_r = 1400. Truncated After Six Terms. Rounded to Four Decimal Places.

$$f(\theta) = A(a_1\cos\theta + a_4\cos4\theta + a_6\cos6\theta + a_9\cos9\theta + a_{11}\cos11\theta + \ldots)$$
$$+ B(b_1\sin\theta + b_4\sin4\theta + b_6\sin6\theta + b_9\sin9\theta + b_{11}\sin11\theta$$
$$+ \ldots) \qquad (5.13)$$

Utilizing the fact that the $a_i = \pm b_i$ in the case of intermediate modes (with an alternation in sign of the b_i as previously discussed) Eq 5.13 can be written

$$f(\theta) = A(a_1\cos\theta + a_4\cos4\theta + a_6\cos6\theta + a_9\cos9\theta + a_{11}\cos11\theta + \ldots)$$
$$+ B(a_1\sin\theta - a_4\cos4\theta + a_6\sin6\theta - a_9\sin9\theta + a_{11}\sin11\theta + \ldots)$$
$$(5.14)$$

Now, it is desired to express this mode in a convenient series of cos $mN\theta$ and sin $mN\theta$ for integer values of m. Standard trigonometric identities permit us to write

$$\cos4\theta = \cos(5\theta - \theta) = \cos5\theta\ \cos\theta + \sin5\theta\ \sin\theta$$
$$\cos6\theta = \cos(5\theta + \theta) = \cos5\theta\ \cos\theta - \sin5\theta\ \sin\theta$$

etc.

and

$$\sin4\theta = \sin(5\theta - \theta) = \sin5\theta\ \cos\theta - \cos5\theta\ \sin\theta$$
$$\sin6\theta = \sin(5\theta + \theta) = \sin5\theta\ \cos\theta + \cos5\theta\ \sin\theta$$

etc.

Utilizing the above identities, Eq 5.14 may be written, after factoring and rearrangement

$$f(\theta) = A\{[\cos\theta + \frac{B}{A}\sin\theta] \cdot [a_1 + (a_4 + a_6)\cos5\theta + (a_9 + a_{11})\cos10\theta + \ldots]$$
$$+ [\sin\theta - \frac{B}{A}\cos\theta] \cdot [(a_4 - a_6)\sin5\theta + (a_9 - a_{11})\sin10\theta + \ldots]\}$$
$$(5.15)$$

The modal function $f(\theta)$ is now expressed in terms of the harmonics of the number of springs N. Now defining

$$a_0^* = a_1$$

$$a_5^* = a_4 + a_6$$

$$a_{10}^* = a_9 + a_{11}$$

$$\vdots$$

etc.

and

$$b_5^* = a_4 - a_6$$

$$b_{10}^* = a_9 - a_{11}$$

$$\vdots$$

etc.

Eq 5.15 may be written

$$f(\theta) = A\{[\cos\theta + \frac{B}{A}\sin\theta] \cdot [a_0^* + a_5^* \cos 5\theta + a_{10}^* \cos 10\theta + \ldots]$$

$$+ [\sin\theta - \frac{B}{A}\cos\theta] \cdot [b_5^* \sin 5\theta + b_{10}^* \sin 10\theta + \ldots]\} \qquad (5.16)$$

The following notation will be adopted

$$\Phi(\theta) = a_0^* + a_5^* \cos 5\theta + \ldots + a_n^* \cos n\theta$$

$$\Psi(\theta) = b_5^* \sin 5\theta + b_{10}^* \sin 10\theta + \ldots + b_n^* \sin n\theta$$

A particular choice of A = 1 and B = 0 will permit expression of Eq 5.16 as a particular mode function, $f_1(\theta)$, given by

$$f_1(\theta) = \cos\theta\ \Phi(\theta) + \sin\theta\ \Psi(\theta) \qquad (5.17)$$

Another choice of A = 0 and B = 1 will permit representation of a particular mode function, $f_2(\theta)$ as

$$f_2(\theta) = \sin\theta\ \Phi(\theta) - \cos\theta\ \Psi(\theta) \tag{5.18}$$

The final expression adoped for a general mode shape, $\phi(\theta)$, is

$$\phi(\theta) = Cf_1(\theta) + Df_2(\theta) \tag{5.19}$$

Equation 5.19 gives a general modal representation for this particular example. The constants C and D are arbitrary. Thus there are an infinity of modes associated with this particular solution. The mode shape $\phi(\theta)$ is expressed in terms of integer multiples of N, that is

$$\Phi(\theta) = \sum_{m=0}^{k} a_{mN} \cos mN\theta \tag{5.20}$$

and

$$\Psi(\theta) = \sum_{m=1}^{k} b_{mN} \sin mN\theta \tag{5.21}$$

The above expressions were derived for a family number n = 1. Further derivations have led to the deduction of more general expressions which permit expressing of all of the families of modes in a similar fashion. Also, of course, there is no restriction on number of springs N = 5, so that a general expression can be written for the modal representation for any number of springs, N. Combining these generalities for the representation of the intermediate modes will give

$$\Phi_{N,n}(\theta) = a_o^* + a_N^* \cos N\theta + a_{2N}^* \cos 2N\theta + \ldots + a_{mN} \cos mN\theta \tag{5.22}$$

where:

$$a_o^* = a_1$$

$$a_N = a_{N-n} + a_{N+n}$$

N = No. of springs

n = family number

and

$$\Psi_{N,n}(\theta) = b_N^* \sin N\theta + b_{2N}^* \sin 2N\theta + \ldots + b_{mN}^* \sin mN\theta \qquad (5.23)$$

where:

$$b_N^* = b_{N-n} - b_{N+n}$$

N = No. of springs

n = family number

Recall that the a_i and b_i without the stars are the normal modal coefficients shown in Fig 13. Also recall that these representations are valid for the intermediate families only.

Now, the family number n will appear to modify Eq's 5.17 and 5.18.

$$f_1(\theta)_{N,n} = \cos n\theta \, \Phi_{N,n}(\theta) + \sin n\theta \, \Psi_{N,n}(\theta) \qquad (5.24)$$

$$f_2(\theta)_{N,n} = \sin n\theta \, \Phi_{N,n}(\theta) - \cos n\theta \, \Psi_{N,n}(\theta) \qquad (5.25)$$

The final representation of a modal function $\phi_{N,n}$ is given by

$$\phi_{N,n} = Cf_1(\theta)_{N,n} + Df_2(\theta)_{N,n} \qquad (5.26)$$

Results for the functions $\Phi_{5,n}(\theta)$ and $\Psi_{5,n}(\theta)$ are shown in Figs 14, 15, 16 and 17. Several intermediate modes are shown for varying values of spring stiffness, k_r. The mode designation is the same as the frequency designation for $\omega^2_{n,m}$ in Sec 5.4, that is

$$\phi_{c,d}$$

is a mode of family number = c, and d is the mode number with d = 1 the lowest mode in that family. In these plots, both $\Phi_{N,n}(\theta)$ and $\Psi_{N,n}(\theta)$ have been normalized independently. This is done to emphasize the effect of k_r on the shapes. If $\Phi_{N,n}(\theta)$ and $\Psi_{N,n}(\theta)$ were used to represent the final shapes, they would be dependent on each other and the normalization of either one would affect the other.

These representative plots illustrate an interesting feature. As the relative stiffness k_r increases, the shapes draw down to zero displacement at the spring. This is taken as further evidence that the solution approaches that of a ring which is pinned at the spring positions as k_r gets large. No solutions for this particular problem have been found in the literature.

Some summarizing statements about the types of modes found, and their significance, are now appropriate. Let us think once again of the free ring and the mode shapes which it has and ask what effect the springs have when they are introduced. The free ring has an infinity of mode shapes given by sin mθ (or cos mθ) where m is an integer. However, there exists for the free ring a double infinity of modes, since each of the above modes is free to take any orientation with respect to θ which is desired. This is the same as saying that the choice of the origin is

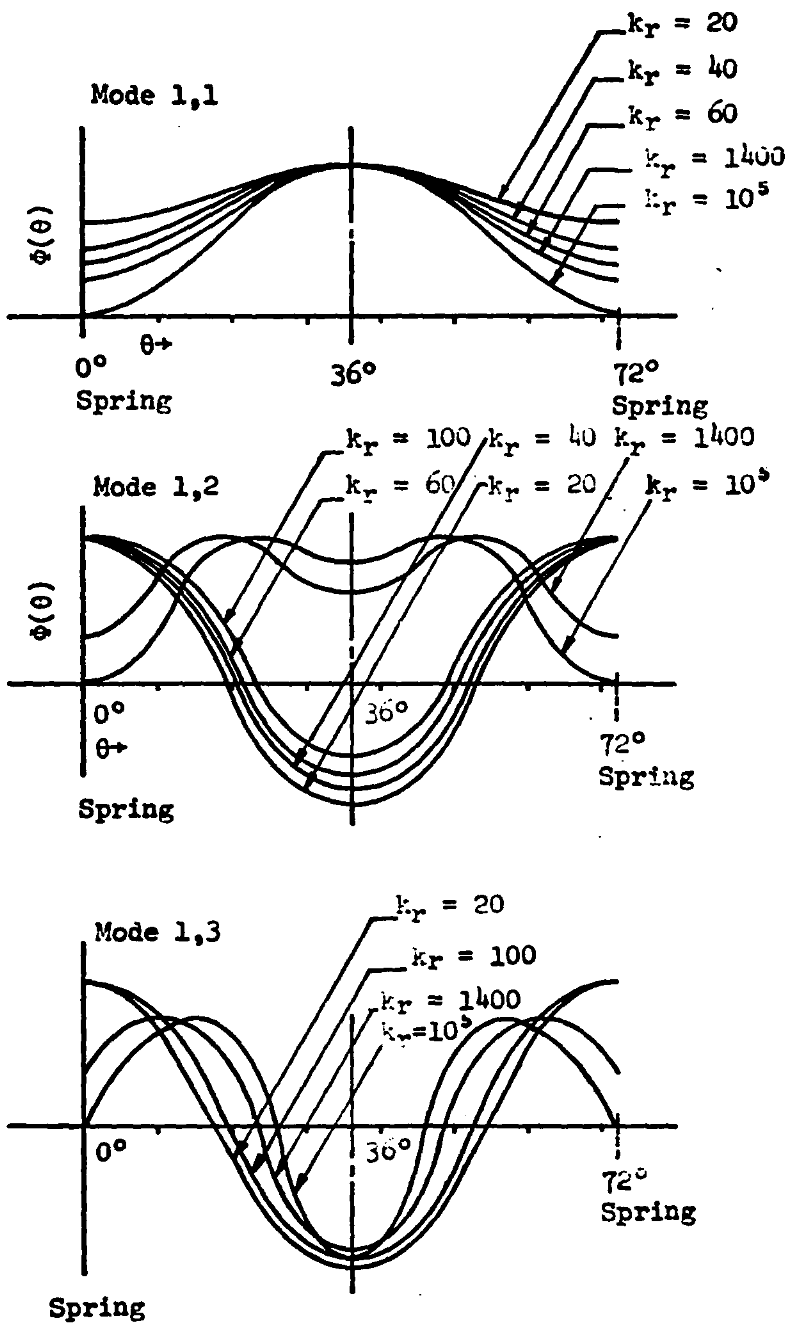

Figure 14. $\Phi(\theta)_{5,1}$ vs. θ for N = 5 Springs. Family Number n = 1.

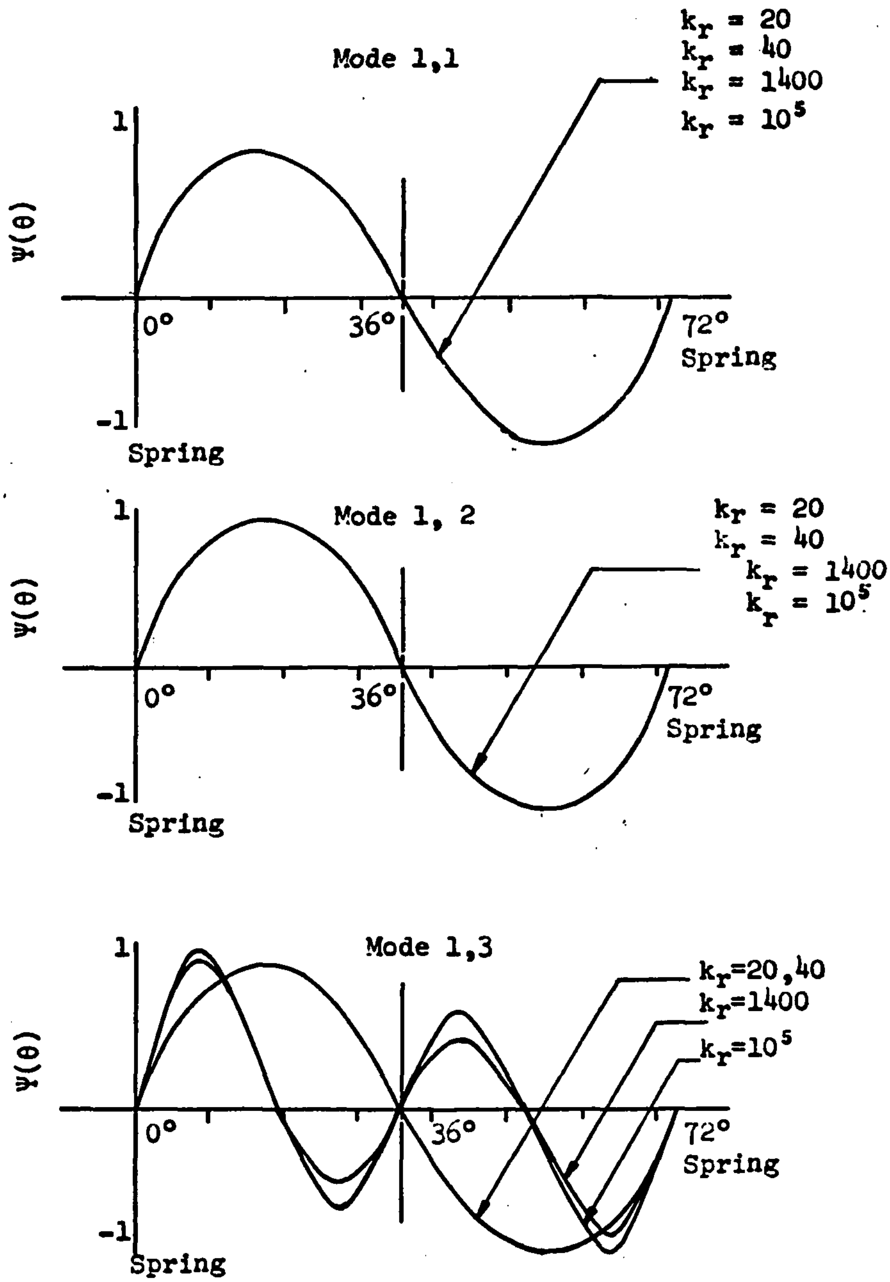

Figure 15. $\Psi(\theta)_{5,1}$ vs. θ for N = 5 Springs. Family Number n = 1.

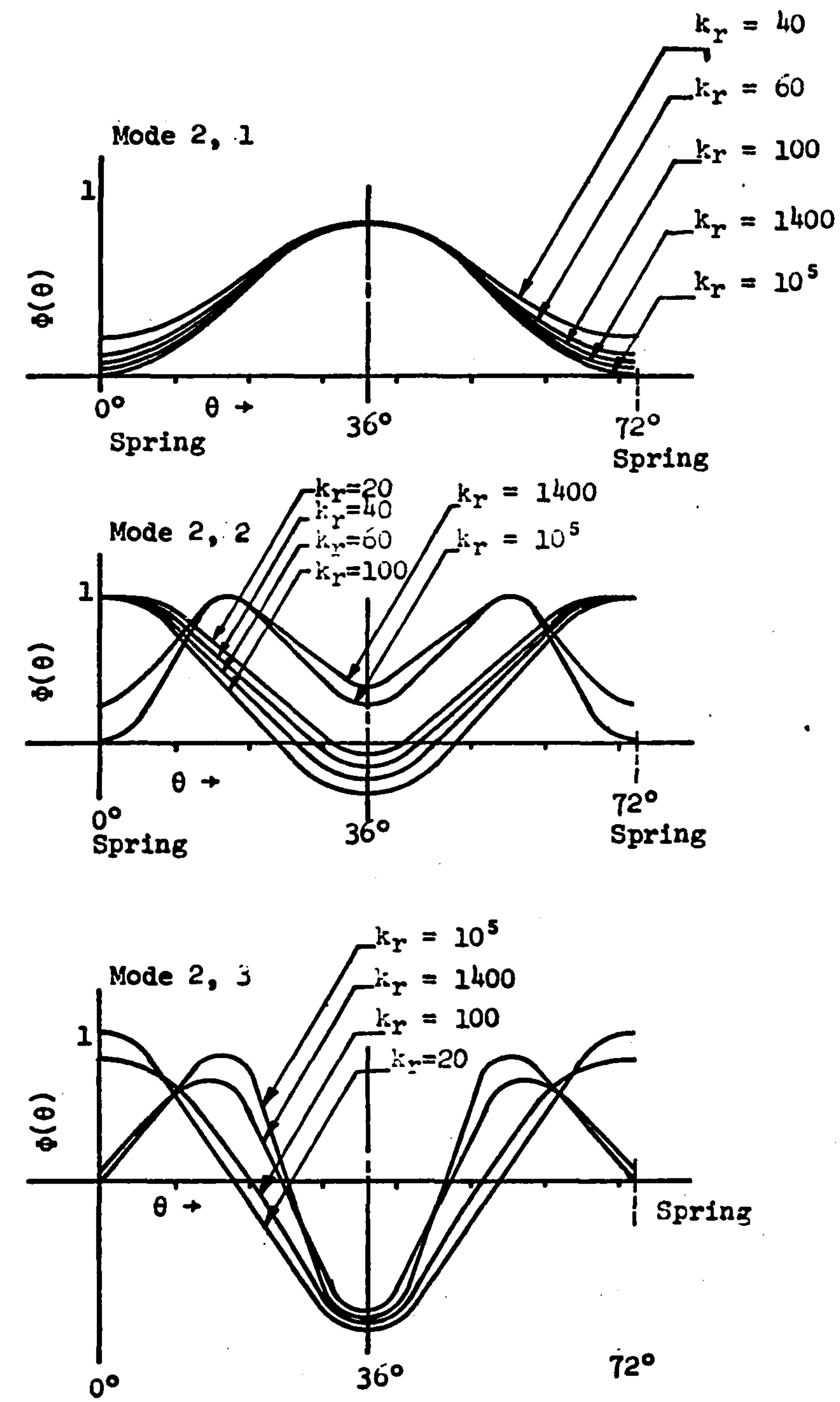

Figure 16. $\Phi(\theta)_{5,2}$ vs. θ for N = 5 Springs and Family No. n = 2.

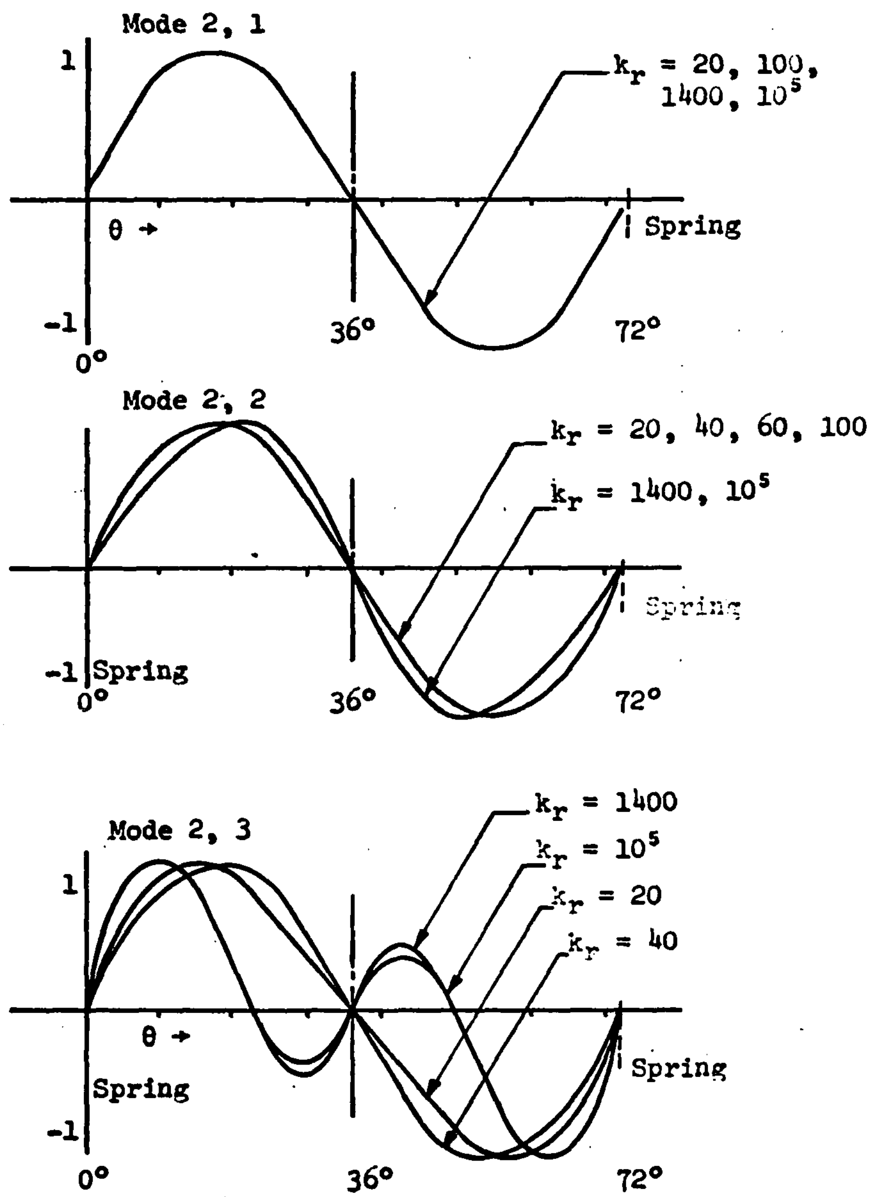

Figure 17. $\Psi(\theta)_{5,2}$ vs. θ for N = 5 Springs. Family Number n = 2.

arbitrary. These modes might be termed "free modes." Such modes are typical of an axially symmetric body.

When the springs are introduced, and the ring is supported, it might be thought that the freedom of the modes is lost. This is not true because the intermediate family modes described above may be formed by <u>any</u> linear combination of $f_1(\theta)_{N,n}$ and $f_2(\theta)_{N,n}$. By choosing this combination in an arbitrary fashion, the intermediate family modes may be made to assume an infinity of orientations. A few of these are illustrated in Fig 18. Although these modes are able to assume any orientation, they change shape as they do.

Therefore, these modes are not completely free, and they will be called "floating modes." These floating modes, which are able to assume an infinity of orientations, are the counterpart of the free modes which occur in the free ring. The existence of such modes for a constrained ring is felt to be a very significant discovery.

The ring has other modes in addition to the floating modes. The floating modes come from the intermediate modal families. The spring families of modes give rise to a different type of mode. This might be described as a "fixed mode." The fixed mode, arising from the spring families, has two components. One component is associated with the cosine series and one with the sine series. The component of the fixed modes associated with the sine series has a nodal point at each spring. This component is unaffected by a change in spring stiffness. The second component, associated with the cosine series, is also fixed with respect to θ. For low values of k_r, it has an antinode at the spring, but for large values of k_r, it develops a node at the spring positions.

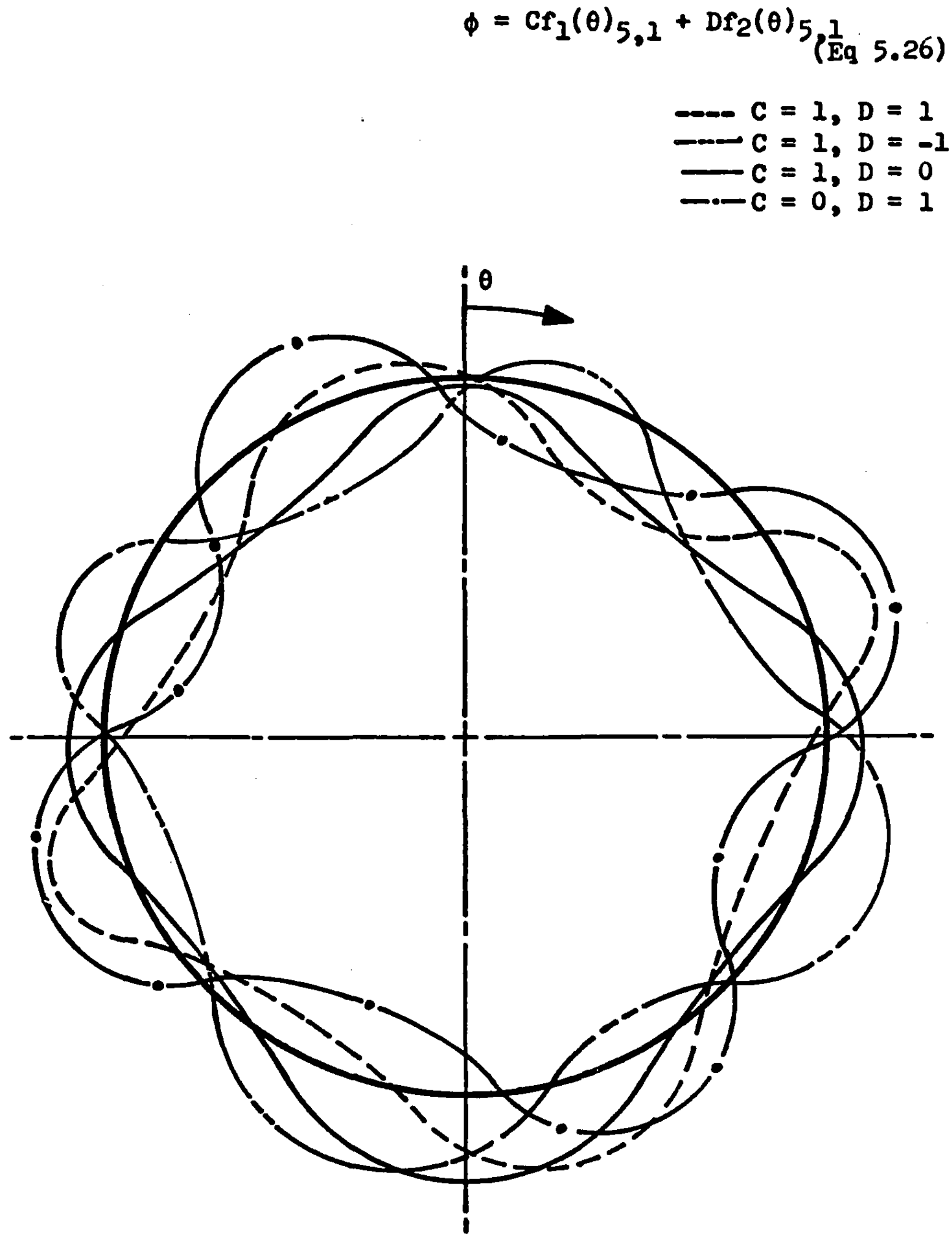

Figure 18. Representation of Equal-Frequency "Floating" Modes. $N = 5$. $n = 1$.

This is a change in shape as a function of k_r; however, its orientation is fixed. The fixed modes are modes of a type which are not present in the free ring, but the fact that some modes are fixed when restraints are introduced is perhaps not too surprising.

To summarize, a study of geometry and mode shape for the ring supported by equally-spaced springs--one of our fundamental aims--has led us to the conclusion that there are two types of modes:

1. Floating
2. Fixed

    Cosine component

    Sine component

5.7 Limiting Cases

There are two limiting cases worth discussion. First, for $k_r = 0$, the frequencies approach that of a free ring as given by standard equations from other sources. (See discussion Sec 5.4.) This is considered to be a good check on the computer program. The program does a great deal of management, computation, and solution, and such verification is good to have.

Second, for large values of k_r, the solution is taken to approach that of a ring which is pinned at the spring positions. The literature survey did not reveal any previous solutions of this problem for comparison.

5.8 Conclusion

The methods discussed in this Chapter provide a means for calculating and presenting the frequencies and mode shapes for a ring with any number of equally spaced radial springs of equal stiffness. The results are subject to the initial assumptions and limitations given in Chapter 1. Thus a basis is provided for presenting complete solutions to all problems of this type.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

The primary objective of this study was to determine if there are an infinity of modal patterns associated with each natural frequency for a ring supported by equally-spaced radial springs. We have shown that an infinity of modal patterns are associated with some of the natural frequencies. These are termed "floating modes" and they are associated with the natural frequencies resulting from the equations of motion for the intermediate families. There are also "fixed modes" which are associated with the remaining natural frequencies. These are the natural frequencies resulting from the equations of motion for the spring families. For both floating modes and fixed modes, the effect of increasing relative spring stiffness, k_r, is to cause the modes to approach a displacement of zero at the spring locations. Thus the mode shapes, and associated natural frequencies, are taken to approach those of a ring which is pinned at each of the springs.

The secondary objective of the study was to investigate the effect of the supports on the natural frequencies of the ring. Here it was found that increasing values of spring stiffness cause the natural frequencies associated with the equations of motion for the intermediate families to increase--rapidly at first--and then to approach an asymptotic value which represents the natural frequencies of a ring pinned at each of the spring locations. The natural frequencies associated with the cosine series equations of motion are identical to those associated with the sine series equations of motion when we consider the intermediate families. This is referred to as an equal-eigenvalue case. It is because of these equal eigenvalues that there are an infinity of modal patterns associated with each natural frequency for the intermediate families. For equal eigenvalues such as these, any linear combination of mode shapes will also be a mode shape.

The natural frequencies associated with the spring families are of two types. Those associated with the cosine series equations of motion increase with increasing values of spring stiffness and then asymptotically approach values which are the natural frequencies for a ring which is pinned at the spring locations. The spring family natural frequencies which are associated with the sine series equations of motion are equal to the natural frequencies for a free ring. Because the natural frequencies for the sine series and cosine series equations of motion are different for the spring families, this is referred to as a distinct-frequency or distinct-eigenvalue case.

The use of a procedure for improving the convergence of the series representation of the mode shapes has also been developed and discussed. Because refined modes were not needed for the ring, the procedure was applied to a beam with a transverse spring at the midpoint. This procedure utilizes a modifying function which is formed by subtracting the first n Fourier series terms (where n is the number of degrees of freedom used in the problem) from the polynomial expression for the displacement of a beam with a concentrated force at the midpoint. The implementation of this procedure was shown to involve only a simple augmentation of the unmodified stiffness matrix for the same problem. The use of the modifying function procedure was shown to improve the mode shape and natural frequency estimates substantially, and to improve the representation of shear force distribution very significantly for several specific example problems. The use of a modifying function with only a two-term Fourier series produced nearly exact results for one example beam. This modifying function procedure deserves much more attention in another study.

Another result of this current study is that the procedure for obtaining the natural frequencies and mode shapes for any uniform circular ring which is supported by any number of uniformly spaced radial springs has been obtained. The results are subject to the limitations on the ring given in Chapter 1. These results are of potential use in engineering design work.

6.2 Recommendations

Possible practical applications for results from this investigation would include structural rings, such as those found in aircraft, missiles, spacecraft or industrial buildings, which are radially stiffened by equally spaced supporting members such as stringers or columns. Based on the results of this investigation, a comprehensive set of design curves for frequencies and mode shapes for various values of relative spring stiffness, k_r, and number of springs, N, could be generated, including the limiting case where the ring is pinned at the springs.

Future studies which could be based on the current work include:

1. A more comprehensive investigation of the general use of modifying functions to hasten the convergence of series solutions in cases where discontinuities are present. This could include the effects of both discrete springs and discrete masses on continuous systems.
2. Extensions of the present problem to include the presence of springs which resist rotation as well as radial displacement. The limiting case for large k_r would then be a built-in ring segment.
3. An investigation of the ring problem where the spring spacing and/or spring stiffnesses are not equal. Such a problem is not beyond the realm of practical application, and results could be interesting.
4. The results from Chapter 5 of this study could be easily extended to produce a production program and design curve for engineering design use.

APPENDIX A

SPRING
STIFFNESS MATRICES (k = UNITY)

$$
\left[\begin{array}{c|c}
\begin{array}{cccccccc}
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5 \\
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5 \\
0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5 \\
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5 \\
0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5 \\
1.5 & 1.5 & 0 & 1.5 & 1.5 & 0 & 1.5 & 1.5
\end{array} & 0 \\
\hline
0 & \begin{array}{cccccccc}
1.5 & -1.5 & 0 & 1.5 & -1.5 & 0 & 1.5 & -1.5 \\
-1.5 & 1.5 & 0 & -1.5 & 1.5 & 0 & -1.5 & 1.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.5 & -1.5 & 0 & 1.5 & -1.5 & 0 & 1.5 & -1.5 \\
-1.5 & 1.5 & 0 & -1.5 & 1.5 & 0 & -1.5 & 1.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.5 & -1.5 & 0 & 1.5 & -1.5 & 0 & 1.5 & -1.5 \\
-1.5 & 1.5 & 0 & -1.5 & 1.5 & 0 & -1.5 & 1.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.5 & -1.5 & 0 & 1.5 & -1.5 & 0 & 1.5 & -1.5
\end{array}
\end{array}\right]
$$

Figure 19. Spring Stiffness Coefficients Three-Spring Case, Odd and Even, Unit k.

$$
\left[\begin{array}{c:c}
\begin{matrix}
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4
\end{matrix} & 0 \\
\hdashline
0 & \begin{matrix}
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\end{array}\right]
$$

Figure 20. Spring Stiffness Coefficients Four-Spring Case, Odd and Even, Unit k.

$$\left[\begin{array}{c|c}
\begin{matrix}
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5 \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 \\
0 & 2.5 & 2.5 & 0 & 0 & 0 & 2.5 & 2.5 & 0 \\
2.5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 0 & 2.5
\end{matrix} & 0 \\
\hline
0 & \begin{matrix}
2.5 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & -2.5 \\
0 & 2.5 & -2.5 & 0 & 0 & 0 & 2.5 & -2.5 & 0 \\
0 & -2.5 & 2.5 & 0 & 0 & 0 & -2.5 & 2.5 & 0 \\
-2.5 & 0 & 0 & 2.5 & 0 & -2.5 & 0 & 0 & 2.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.5 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & -2.5 \\
0 & 2.5 & -2.5 & 0 & 0 & 0 & 2.5 & -2.5 & 0 \\
0 & -2.5 & 2.5 & 0 & 0 & 0 & -2.5 & 2.5 & 0 \\
-2.5 & 0 & 0 & 2.5 & 0 & -2.5 & 0 & 0 & 2.5
\end{matrix}
\end{array}\right]$$

Figure 21. Spring Stiffness Coefficients Five-Spring Case, Odd and Even, Unit k.

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 3 \\ 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 3 \end{bmatrix}$$

NOTE: Even and odd coefficient matrices have been split beginning with this page.

Figure 22. Spring Stiffness Coefficients for Cosine (Even) Terms, Six-Spring Case, Unit k.

$$\begin{bmatrix} 3 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 3 \end{bmatrix}$$

Figure 23. Spring Stiffness Coefficients for Sine (Odd) Terms, Six-Spring Case, Unit k.

$$\left[\begin{array}{cccccccccccccc}
3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 \\
0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 \\
0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 \\
0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 \\
3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 \\
0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 \\
0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & 3.5 & 0 & 0 & 0 \\
0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & 3.5 & 0 & 0 \\
3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{array}\right]$$

Figure 24. Spring Stiffness Coefficients for Cosine (Even) Terms, Seven-Spring Case, Unit k.

$$\begin{bmatrix}
3.5 & 0 & 0 & 0 & 0 & -3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & -3.5 & 0 \\
0 & 3.5 & 0 & 0 & -3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & -3.5 & 0 & 0 \\
0 & 0 & 3.5 & -3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & -3.5 & 0 & 0 & 0 \\
0 & 0 & -3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 & 0 & 0 \\
0 & -3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & -3.5 & 0 & 0 & 3.5 & 0 & 0 \\
-3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & -3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3.5 & 0 & 0 & 0 & 0 & -3.5 & 0 & 3.5 & 0 & 0 & 0 & 0 & -3.5 & 0 \\
0 & 3.5 & 0 & 0 & -3.5 & 0 & 0 & 0 & 3.5 & 0 & 0 & -3.5 & 0 & 0 \\
0 & 0 & 3.5 & -3.5 & 0 & 0 & 0 & 0 & 0 & 3.5 & -3.5 & 0 & 0 & 0 \\
0 & 0 & -3.5 & 3.5 & 0 & 0 & 0 & 0 & 0 & -3.5 & 3.5 & 0 & 0 & 0 \\
0 & -3.5 & 0 & 0 & 3.5 & 0 & 0 & 0 & -3.5 & 0 & 0 & 3.5 & 0 & 0 \\
-3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 & -3.5 & 0 & 0 & 0 & 0 & 3.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Figure 25. Spring Stiffness Coefficients for Sine (Odd) Terms, Seven-Spring Case, Unit k.

APPENDIX B

A PROOF THAT THE OFF-DIAGONAL SUBMATRICES OF THE SPRING STIFFNESS MATRIX ARE ZERO MATRICES

The expression for the spring stiffness coefficients for these particular submatrices is

$$k_{mn} = k \sum_{s=1}^{N} \cos m\theta_s \sin n\theta_s \tag{B.1}$$

where: θ_s is the angular coordinate of each spring

N = the total number of springs.

Consider first an even number of springs N. There will be a spring at $\theta = 0$ and $\theta = \pi$. (See Fig 1.) These two terms will have zero contribution to the summation in Eq B.1, since:

$$\sin (n\theta) = 0$$

and

$$\sin (n2\pi) = 0$$

for all integer values of n. There will be left N-2, or an even number of springs, to be summed over. For every spring located at $+ \theta_s$, there will be a corresponding spring located at $2\pi - \theta_s$. These will sum pair-wise to zero as seen below:

$$\cos m\theta_s \sin n\theta_s + \cos m(2\pi - \theta_s) \sin n(2\pi - \theta_s)$$

$$= \cos m\theta_s \sin n\theta_s + \cos m2\pi \cos m\theta_s + \sin m2\pi \sin m\theta_s \cdot$$

$$\sin n2\pi \cos n\theta_s - \cos n2\pi \sin n\theta_s$$

$$= \cos m\theta_s \sin n\theta_s - \cos m\theta_s \sin n\theta_s$$

$$= 0$$

for all values of m, n, and θ_s.

Consider now the case when N is odd. There will be one spring at $\theta = 0$. This term will have zero contribution to Eq B.1. There will be N-1, an even number, of springs remaining. For every spring located at $+\theta_s$, there will be a corresponding spring at $2\pi - \theta_s$. These will sum pairwise to zero as shown above. Q.E.D.

APPENDIX C

COMPUTER PROGRAM FOR BEAM PROBLEM

The computer program for the beam problem is written in FORTRAN IV language, and is primarily intended to run on CDC 6400 SCOPE system. However, it should run on any FORTRAN system with minor modifications.

The mass and stiffness matrices are assembled. For this particular problem, the even modes are thrown out. The eigenvalue problem is put in a form to have a symmetric characteristic matrix. The eigenvalues and eigenvectors are calculated using CDC's MATRIX System Library Routines. Answers for eigenvalues, eigenvectors, and displacement are printed for both the normal and modified methods. This can be repeated for different values of relative stiffness k_r, and for different values of N, by using DO loops. A flow chart on the following page illustrates the program.

The program is quite straightforward, and the listing will not be given here. The function of some of the more important subroutines is given below.

FORMEIG: Assembles the eigenvalue problem by forming the appropriate $\underline{M}^{-1/2}\ \underline{K}\ \underline{M}^{-1/2}$ product (modified or unmodified $\underline{K}$), calling MATRIX to solve the resulting symmetric characteristic matrix for eigenvalues and eigenvectors, and gets the final eigenvector $\underline{a} = \underline{M}^{-1/2}\ \underline{r}$.

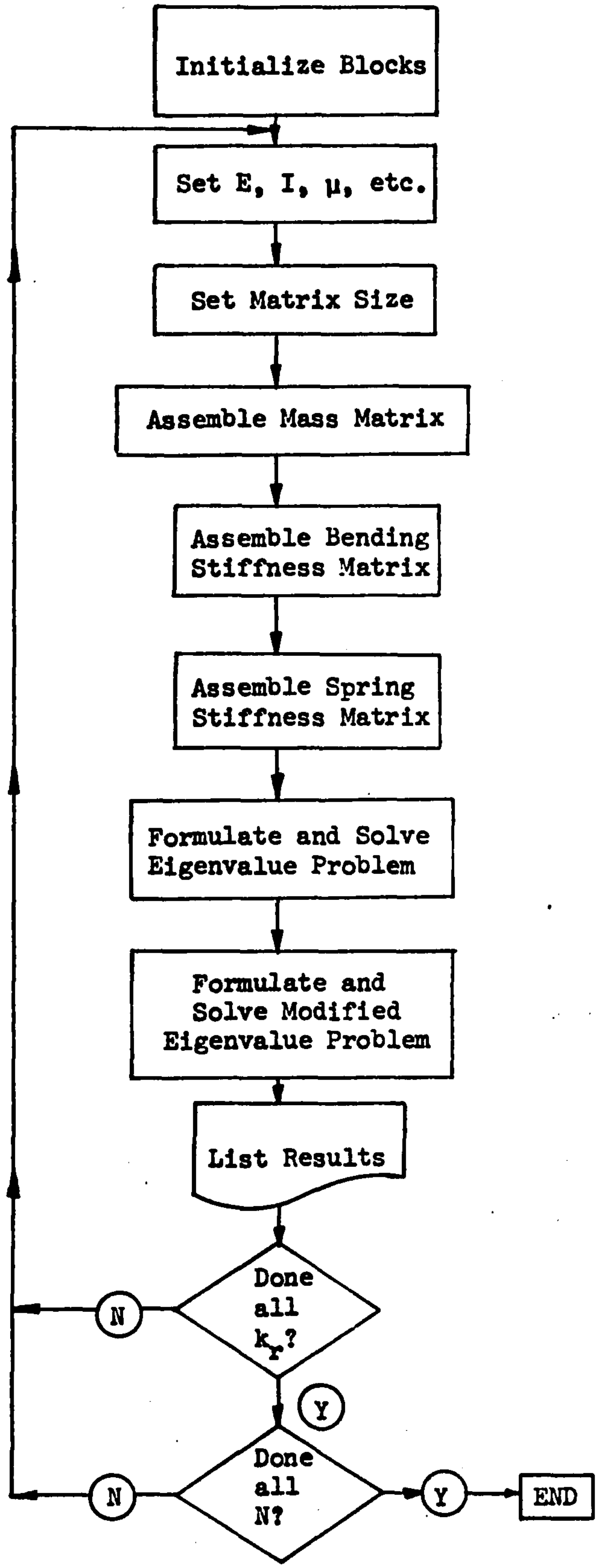

Figure 26. Flow Chart for Beam Program

PRERES: Presents the deflection results for normal and modified problems. The deflection is found by summing an appropriate series for various values of x/L.

PREIGEN: Prints eigenvalue results.

PRID: Prints the characteristic matrix.

SPRK: Assembles spring stiffness matrix.

STIFF: Assembles bending stiffness matrix.

MASM: Assembles mass matrix.

PREIVEC: Prints eigenvector results.

MODPROB: Calculates coefficients and modifies the stiffness matrix for the use of the modifying function method.

TOENM: Throws out even rows and columns in the mass matrix.

TOENK: Throws out even rows and columns for the stiffness matrix.

MATRIX: A CDC system library routine which will solve the eigenvalue problem with a simple call to the system.

A matrix handling scheme--LIMAS--which will add, subtract, transpose, print, etc., any size matrices was provided to the author by Dr. H. A. Kamel. With the exception of LIMAS and MATRIX, programming was done by the author.

APPENDIX D

COMPUTER PROGRAM FOR RING PROBLEM

The computer program for the ring problem is similar to the one for the beam. It is also written in FORTRAN IV and is primarily intended to run on CDC 6400 SCOPE. Experience has indicated that few basic changes are required to make it run on other systems, such as Lawrence Radiation Laboratory's Monitor 400.

The program was originally segmented, but the addition of a second 32 K memory bank on The University of Arizona's CDC 6400 made segmentation unnecessary. The only inputs required to the program are the relative spring stiffness, k_r, and the number of springs, N. If these are the only two inputs, then E, I, ρ, etc., are equal to unity. An option also exists for putting the parameters in via the input.

Based on the number of springs, the spring locations in radians are calculated and printed. All input information and all physical parameters are also printed. The matrix size is specified by a parameter card and is currently limited to 40 x 40.

The mass and stiffness matrices are assembled. Then the eigenvalue problem is manipulated so as to give a symmetric characteristic matrix. MATRIX is called, and it provides the solution for the eigenvalues, eigenvectors, $\Phi(\theta)$, $\psi(\theta)$, $f_1(\theta)$ and $f_2(\theta)$ which are discussed in Chapter 5.

Upon the call to one of the options, EIGCHK, the orthogonality of the eigenvectors with respect to the mass and stiffness matrices is checked.

The program has been found to operate quite satisfactorily, providing the results for six different values of k_r--the equivalent of assembling, solving and listing results for twelve 20 x 20 eigenvalue problems--in 116 CPU seconds. The program is not a production model, and would need some improvements in the output format as well as some additional features in order to be considered one. But it is a good running system and served quite well for the purpose of this study. The program does not contain any unique features, and a listing will not be given here.

A word is in order here regarding the CDC system library routine MATRIX. MATRIX is contained in the system, and only a call to it is required to have it operational in the user's program. This routine uses the Givens-Householder method (Ref. 9) to simultaneously find the n eigenvalues and eigenvectors of a symmetric characteristic matrix. It was used for this study because it was known to be a rapid routine. Convergence to nine decimal places is inherent in the routine. MATRIX can also be used for multiplication, addition, inversion, etc., of matrices. However, since the calls to MATRIX are long, LIMAS (discussed earlier) was used for these matrix operations.

A flow chart for the program is given on page 101. Some of the more important subroutines and their function are given below:

FORMEIG: Forms the symmetric characteristic matrix by taking $\underline{M}^{-1/2}\ \underline{K}\ \underline{M}^{-1/2}$, solves for eigenvalues and eigenvectors and normalizes by taking $\sum_i a_i^2 = 1$.

EIGCHK: Provides a check on the orthogonality by taking $\underline{a}^T\ \underline{K}\ \underline{a}$, $\underline{b}^T\ \underline{K}\ \underline{b}$, $\underline{a}^T\ \underline{M}\ \underline{a}$ and $\underline{b}^T\ \underline{M}\ \underline{b}$ and printing the results.

REAR: Rearranges the eigenvalues and eigenvectors found in FORMEIG, since MATRIX may not put them out in the right order. A user operation which calls REAR and specifies the desired order is required. This is one feature which would need to be cleared up for a production program.

PRERES: Presents displacement results for $\Phi(\theta)$, $\psi(\theta)$, $f_1(\theta)$ and $f_2(\theta)$ every ten degrees around the ring.

Other routines, similar to those explained in Appendix C, are also used.

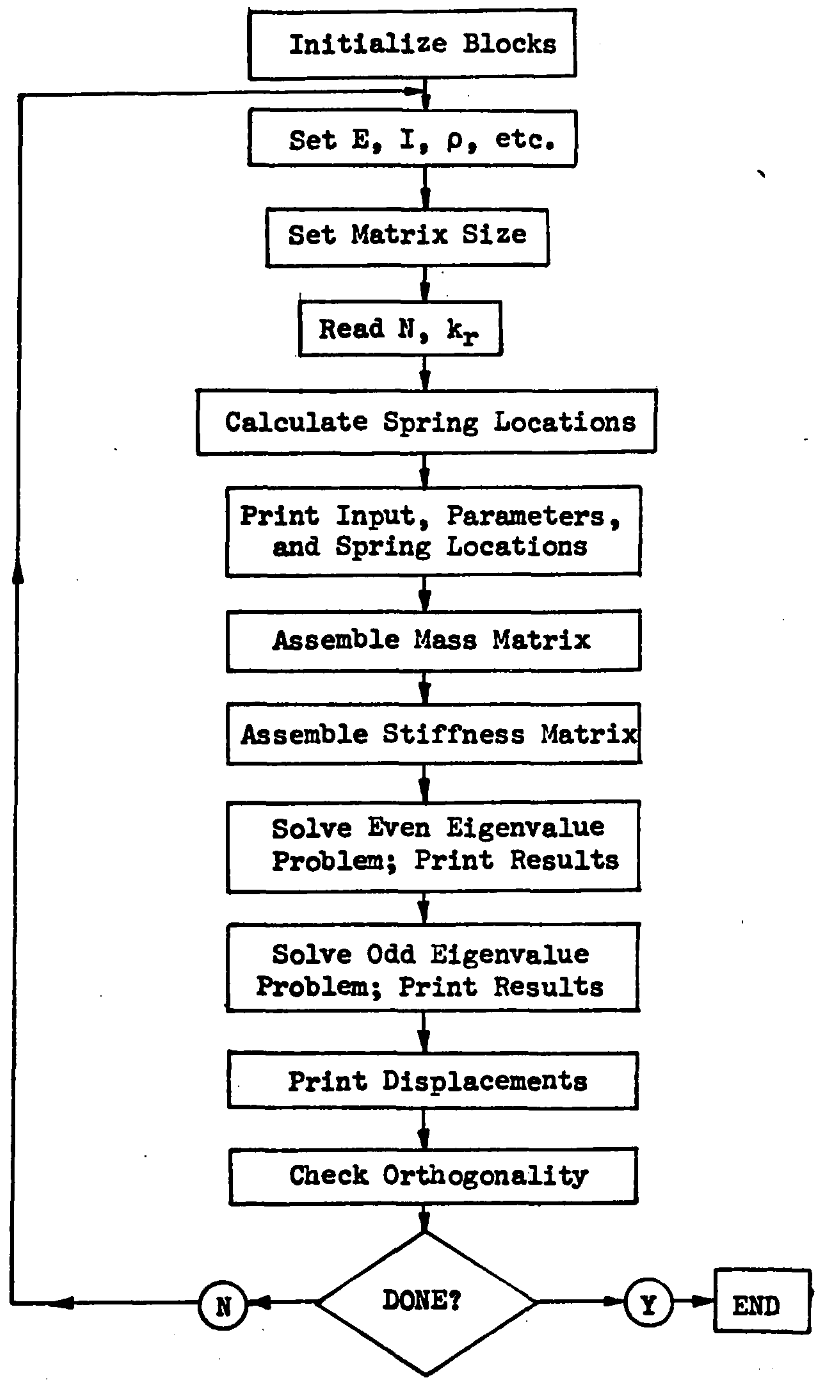

Figure 27. Flow Chart for Ring Program

NOMENCLATURE - ROMAN

A	Ring or beam cross-sectional area; constant
a	Subscript referring to cosine series, or even, coefficients
a_i	Fourier series coefficient for ring displacement; general Fourier series coefficient
a^*	Coefficient of combined eigenfunction--even
B	Subscript, bending; constant
b	Coefficient referring to sine series, or odd, terms
b_i	Fourier series coefficient for ring displacement
b^*	Coefficient of combined eigenfunction--odd
$\underline{D}$	Symmetric characteristic matrix
E	Young's Modulus
$f(\theta)$	Ring modal displacement function
$f_1(\theta)$	Ring modal displacement function
$f_2(\theta)$	Ring modal displacement function
I	Ring or beam cross-sectional moment of inertia
i	Summation index
j	Summation index
$\overline{K}$	$\pi EI/r^3$
$\underline{K}$	Stiffness matrix, total
$\underline{K}_B$	Stiffness matrix, bending
$\underline{K}_s$	Stiffness matrix, spring
k	Spring stiffness, spring location
k_r	relative stiffness

L	Length of beam
M	Bending moment; general integer
$\underline{M}$	Mass matrix
$\bar{M}$	$A\rho r\pi$
m	Coefficient of mass matrix; general subscript or index; modal stiffness number; frequency order number
N	Number of springs
n	Number of terms; general index or subscript; family number
q_i	Coefficient of generalized coordinate
q_o	Coefficient of generalized coordinate--modifying function
r	Radius of ring
S	Number of terms, or degrees of freedom
s	General arc length
T	Kinetic energy
t	Time
u	Radial displacement of ring
V	Strain energy
w	Transverse displacement of ring
x	Beam coordinate
y	Beam displacement

NOMENCLATURE - GREEK

θ	Angular coordinate for ring
μ	Poisson's ratio; mass per unit length of beam
ρ	Mass per unit volume of ring
$\Phi_{(\theta)}$	Cosine series for modal displacement
Φ_k	Modifying function for ring tangential displacement--kth spring
ϕ	Eigenfunction; total or final function for representation of mode
$\bar{\Phi}$	Modifying (displacement) function
ϕ_k	Modifying function for ring radial displacement--kth spring
ϕ_o	Modifying (displacement) function
$\Psi_{(\theta)}$	Sine series for modal displacement
ω	Natural frequency
ω^2	Eigenvalue

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