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PRESCALAR ELEMENTS IN BANACH ALGEBRAS

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Gary Marshall Thomas

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I hereby recommend that this dissertation prepared under my direction by Gary Marshall Thomas

entitled "Prescalar Elements in Banach Algebras"

be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Donald E. Myers — Dec. 11, 1970

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After inspection of the final copy of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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ABSTRACT

Let A be a Banach algebra with identity e. An element a of A is said to be prescalar if there exists a continuous homomorphism $h: C(\sigma_A(a)) \to A$ such that $h(1) = e$ and $h(z) = a$, where z is the function such that $z(\lambda) = \lambda$. If A is commutative, a subset $\{a_\lambda\}_{\lambda \in \Lambda}$ of A is said to be jointly prescalar if there exists a continuous homomorphism $h: C(\sigma_A(\{a_\lambda\})) \to A$ with $h(1) = e$ and $h(z_\lambda) = a_\lambda$ for each λ in Λ; here $\sigma_A(\{a_\lambda\})$ is the joint spectrum and z_λ is the λth coordinate function. This paper is devoted to exploring some of the properties of prescalar elements and jointly prescalar sets, and applying the results to other problems.

The basic properties of prescalar elements are examined, and it is shown that every element of a jointly prescalar set is prescalar (but not conversely). Every finite subset of a commutative Banach algebra A with identity is proved to be jointly prescalar, if and only if A is isomorphic to the algebra of all continuous functions on its maximal ideal space. The subalgebra $B = h[C(\sigma_A(\{a_\lambda\}))]$ is considered; it is shown that B is isomorphic to $C(\sigma_A(\{a_\lambda\}))$, that a form of the Spectral Mapping Theorem holds, and that h is an isometry with respect to spectral radius on A. We also prove that each element of B is prescalar, and use this fact to show that a commutative Banach algebra with a jointly prescalar generating set is

isomorphic to the algebra of continuous functions on its maximal ideal space. Conditions for an element to be prescalar are discussed, as well as some examples.

Motivated by the theory of spectral operators, an element a of A is defined to be prespectral if it is expressible as the sum of a prescalar element and a commuting topological nilpotent. Spectral homomorphisms are introduced as continuous homomorphisms $h: C(\sigma_A(a)) \to A$ with $h(1) = e$, such that $[h(f)]^\wedge = f \circ \hat{a}$ for some a in A and all f in $C(\sigma_A(a))$. Prescalar and prespectral elements are then characterized in terms of spectral homomorphisms. The theory of prespectral elements is applied to strongly decomposable algebras (those which can be written as the direct sum of a closed subalgebra and their radical) to prove a generalization of a theorem of Katznelson; our result states that a strongly decomposable commutative Banach algebra with identity has the property that continuous functions operate on A, if and only if each element of A is prespectral.

Some of our results are applied to operator theory; the main theorem states that a bounded linear operator T on a Hilbert space is prescalar if and only if T is similar to a normal operator. Representations of prescalar elements are discussed, and examples are given.

CHAPTER 1

INTRODUCTION

One of the most fully developed branches of operator theory is the study of normal operators on a Hilbert space H; that is, bounded linear mappings T from H to itself such that $T^*T = TT^*$. Probably the most useful tool in the study of these operators is the Spectral Theorem [12, p. 288], which asserts that any normal operator T is the limit in norm of linear combinations of projections. Stated another way, there exists a projection-valued plane Borel measure $E(\cdot)$ such that $T = \int_{\sigma(T)} \lambda dE(\lambda)$. In a 1954 paper Dunford [6] has generalized this idea to operators on a Banach space X. He defined an operator T to be scalar-type if there exists a projection-valued, multiplicative Borel measure $E(\cdot)$, satisfying a boundedness condition, such that

$T = \int_{\sigma(T)} \lambda dE(\lambda)$. It has been shown by Wermer [14] that an operator T on a Hilbert space is scalar-type if and only if T is similar to a normal operator. Since 1954 the theory of scalar-type operators, and their generalizations, has received much attention (see, for example, [3], [5], and [8]).

What is interesting from the standpoint of the present paper is the fact that integration with respect to a projection-valued measure induces a continuous homomorphism h from the Banach algebra of continuous functions on the (compact) spectrum of T, to the Banach algebra of all operators on X, such that $h(1) = I$ and $h(z) = T$. Here 1 and z denote respectively the functions $1(\lambda) = 1$ and $z(\lambda) = \lambda$ for all λ in $\sigma(T)$. Using this idea, Walsh [13] has defined an element a of the Banach algebra A to be prescalar if there exists a continuous homomorphism h from the algebra of continuous functions on the spectrum of a, to A, such that $h(z) = a$. His principal theorem [13, p. 1170] shows that if each element a of the Banach algebra A is prescalar, then A is commutative, semisimple, and in fact isomorphic to the algebra of continuous functions on its maximal ideal space.

This paper, then, is intended to further study the concept of prescalarity of a Banach algebra element, together with the related idea of joint prescalarity of a set of commuting elements, and to apply the conclusions drawn to some other problems. In Chapter 2 some consequences of, and conditions for, prescalarity are discussed. It is shown, for example, that each finite subset of a commutative Banach algebra A with identity is jointly prescalar if and only if A is isomorphic to $C(M_A)$. The subalgebra $B = h[C(\sigma_A(\{a_\lambda\}))]$ is examined; we show that B is isomorphic to $C(\sigma_A(\{a_\lambda\}))$, and that a form of the Spectral Mapping Theorem holds. Elements with certain restricted types of spectra are proved to be prescalar, and some of these elements are also shown to be prescalar regardless of the subalgebra in which they are embedded. Examples are given.

Chapter 3 deals with prespectral elements, defined as sums of prescalars and commuting topological nilpotents. It is shown that an element a is prescalar if and only if a belongs to the range of a spectral homomorphism (roughly, a homomorphism which commutes with the Gelfand mapping); and a is prespectral if and only if there is a spectral homomorphism for a. Applications to decomposable algebras are given, and we show that if A is strongly decomposable, then continuous functions operate on A if and only if each element a of A is prespectral. This generalizes a theorem of Katznelson [9, p. 904]. A sufficient condition in terms of its prespectral elements is given for A to be strongly decomposable. Finally, in Chapter 4 we discuss some applications to operator theory; the main result is that an operator on a Hilbert space is prescalar if and only if it is similar to a normal operator. We also give conditions sufficient to guarantee that representations of Banach algebra elements be scalar-type operators, and consider some examples.

The ideas presented here seem to offer a unified approach to problems which before have been treated either by highly operator-theoretic, or Banach algebra-theoretic, methods. The work of Douglas [5], for example, on operators similar to normal operators is operator-theoretic in tone; on the other hand, the paper of Bade and Curtis [2] on decomposable algebras uses algebraic methods.

We shall establish the following notational conventions for use throughout this paper. The letter A will denote a nonzero complex Banach algebra; that is, a nonzero algebra over the complex field which is also a Banach space under a norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\,\|y\|$ for all x and y in A. The letter $\mathbb{C}$ is reserved for the Banach algebra of complex numbers with their usual absolute value topology. If X is a Banach space, $B(X)$ will denote the Banach algebra of bounded linear operators on X with the operator norm; in particular, if $X = H$ is a Hilbert space, $B(H)$ will be the algebra of bounded operators on H. For any Banach algebra A and any element a of A, $\sigma_A(a)$ will denote the spectrum of a in A and $\rho(a)$ the spectral radius of a; if A is commutative and $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ is a subset of A, $\sigma_A(\{a_\lambda\})$ will denote the joint spectrum of the set.

If Λ is any index set, $\mathbb{C}^\Lambda$ will denote the product of Λ many copies of $\mathbb{C}$; a typical point of $\mathbb{C}^\Lambda$ will be written in the form $\langle\zeta_\lambda\rangle_{\lambda\varepsilon\Lambda}$. $\mathbb{C}^\Lambda$ is given the usual product topology.

If A and B are Banach algebras, the statement that A is isomorphic to B will be taken to mean that there exists a linear homeomorphism from A onto B.

The letter Ω will always denote a compact Hausdorff space, and $C(\Omega)$ will be the Banach algebra of continuous complex-valued functions on Ω with the supremum norm: $\|f\| = \sup\limits_{x\varepsilon\Omega}|f(x)|$. In $C(\Omega)$, 1 and z will always denote respectively the functions such that $1(x) = 1$ and $z(x) = x$ for all x in Ω. If Ω is a compact subset of $\mathbb{C}^\Lambda$ for some Λ and the origin 0 belongs to Ω, we let

$C_0(\Omega)$ be the ideal determined by 0; that is

$$C_0(\Omega) = \{f \in C(\Omega): \ f(0) = 0\}.$$

Furthermore, z_λ will denote the λth coordinate function in $C(\Omega)$; that is $z_\lambda[\langle\zeta_\lambda\rangle_{\lambda\in\Lambda}] = \zeta_\lambda$.

See Appendix 1 for further definitions and results concerning Banach algebras.

CHAPTER 2

PRESCALAR ELEMENTS

If T is an operator on A Banach space X which is scalar-type in the sense of Dunford [6], then integration with respect to the spectral measure of T induces a continuous homomorphism $h: C(\sigma(T)) \to A$ such that $h(1) = I$ and $h(z) = T$. Similarly, any normal element a of the B*-algebra A with identity e also has associated with it a continuous homomorphism $h: C(\sigma_A(a)) \to A$, such that $h(1) = e$ and $h(z) = z$. Each of these is an example of the type of Banach algebra element which we shall call prescalar.

Properties of Prescalar Elements and Jointly Prescalar Sets

Definition 2.1: Let A be a Banach algebra with identity e. An element a of A is said to be prescalar if there exists a continuous homomorphism $h: C(\sigma_A(a)) \to A$ such that $h(1) = e$ and $h(z) = a$.

Definition 2.2: Let A be a commutative Banach algebra with identity e and let $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ be a subset of A. Then $\{a_\lambda\}$ is said to be a jointly prescalar set if there exists a continuous homomorphism $h: C(\sigma_A(\{a_\lambda\})) \to A$ such that $h(1) = e$ and $h(z_\lambda) = a_\lambda$ for each λ in Λ.

Theorem 2.1: Let A be a commutative Banach algebra with identity e and let $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ be a jointly prescalar subset of A. Then each a_λ is a prescalar element of A.

Proof: For any λ in Λ, the range of the λ^{th} coordinate function z_λ on $\sigma_A(\{a_\lambda\})$ is exactly the set $\sigma_A(a_\lambda)$. Hence, for any f in $C(\sigma_A(a_\lambda))$, $f \circ z_\lambda$ is defined and belongs to $C(\sigma_A(\{a_\lambda\}))$. For each such f we may therefore define $h_\lambda(f) = h(f \circ z_\lambda)$, where h is the homomorphism which makes $\{a_\lambda\}$ a jointly prescalar set. Clearly h_λ is a continuous homomorphism since h is, $h_\lambda\colon C(\sigma_A(a_\lambda)) \to A$, $h_\lambda(1) = h(1 \circ z_\lambda) = h(1) = e$, and $h_\lambda(z) = h(z \circ z_\lambda) = h(z_\lambda) = a_\lambda$. Thus, for all λ in Λ, h_λ makes a_λ a prescalar element of A.

Remark 2.1: The converse of Theorem 2.1 is false; a later example will be given to show that the prescalarity of each element of a given subset of a Banach algebra does not necessarily imply the joint prescalarity of the set (see Example 2.1).

Walsh [13, p. 1170] has proved a theorem which we will use: if A is a Banach algebra with identity each element of which is prescalar, then A is commutative, semisimple, and isomorphic to $C(M_A)$.

Theorem 2.2: Let A be a commutative Banach algebra with identity e, and let M_A be the maximal ideal space of A. Then A is isomorphic to $C(M_A)$ if and only if each finite subset $\{a_i\}_{i=1}^n$ of A is jointly prescalar.

Proof: If each finite subset of A is jointly prescalar, then, in particular, every element of A is prescalar. The fact that A is isomorphic to $C(M_A)$ now follows from the theorem of Walsh [13, p. 1170] quoted above.

Conversely, suppose that A is isomorphic to $C(M_A)$. Note that, in general, if A and B are commutative Banach algebras with identity and Θ is an isomorphism from A onto B, then a subset $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ of A is jointly prescalar if and only if the corresponding subset $\{\Theta(a_\lambda)\}_{\lambda\varepsilon\Lambda}$ of B is jointly prescalar. This follows from the fact that $\{a_\lambda\}$ and $\{\Theta(a_\lambda)\}$ have homeomorphic joint spectra and so Θoh: $C(\sigma_A(\{a_\lambda\})) = C(\sigma_B(\{\Theta(a_\lambda)\})) \to A \to B$, where h is the homomorphism making $\{a_\lambda\}$ jointly prescalar, is a continuous homomorphism making $\{\Theta(a_\lambda)\}$ jointly prescalar in B. Clearly the roles of A and B are reversible. Thus it suffices to prove that, if Ω is a compact Hausdorff space, then every finite subset $\{f_i\}_{i=1}^n$ of $C(\Omega)$ is jointly prescalar.

Now the mapping ψ: $M_{C(\Omega)} = \Omega \to \sigma_{C(\Omega)}(\{f_i\})$ with $\psi(x) = \langle f_i(x)\rangle_{i=1}^n$ is a continuous surjection, by definition of joint spectrum. Define a mapping h: $C(\sigma_{C(\Omega)}(\{f_i\})) \to C(\Omega)$ by $[h(g)](x) = g[\psi(x)]$ for each x in Ω and each g in $C(\sigma_{C(\Omega)}(\{f_i\}))$. It is clear that h is a homomorphism, and h is continuous since

$$\|h(g)\| = \sup_{x\varepsilon\Omega}|[h(g)](x)| = \sup_{x\varepsilon\Omega}|g[\psi(x)]| = \sup_{\zeta\varepsilon\sigma_{C(\Omega)}(\{f_i\})}|g(\zeta)| = \|g\|.$$

Finally, $[h(1)](x) = 1[\psi(x)] = 1$ for all x in Ω, so $h(1) = 1$, the identity of $C(\Omega)$; and $[h(z_i)](x) = z_i[\psi(x)] = z_i[\langle f_i(x)\rangle_{i=1}^n] = f_i(x)$ so that $h(z_i) = f_i$ for $i = 1,\ldots,n$. Thus h makes $\{f_i\}$ a jointly prescalar subset of $C(\Omega)$.

Theorem 2.3: Let A be a Banach algebra with identity and let a be a nonzero prescalar element of A. Then a is not a topological nilpotent.

Proof: If a is a topological nilpotent, then $\sigma_A(a) = \{0\}$ and the functions 0 and z are the same in $C(\sigma_A(a))$. Hence, if a is prescalar and a topological nilpotent, then $a = h(z) = h(0) = 0$.

Corollary 2.1: Let A be a commutative Banach algebra with identity and let $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ be a jointly prescalar subset such that a_λ is nonzero for all λ in Λ. Then no a_λ is a radical element.

Proof: Since A is commutative, the radical R of A consists precisely of all the topological nilpotents belonging to A. Now if $\{a_\lambda\}$ is a jointly prescalar subset of A, then each a_λ is prescalar by Theorem 2.1 and so not a topological nilpotent by Theorem 2.3 above. Hence no a_λ belongs to R for λ in Λ.

Remark 2.2: Theorem 2.3 also implies the interesting (and well-known) fact that, if T is a normal operator on a Hilbert space which is quasinilpotent, then $T = 0$. (For operators, the term quasinilpotent is used instead of topological nilpotent.) Later results in this paper will show that the same conclusion holds for other types of operators (see Chapter 4).

Recall that, if Ω is a compact subset of $\mathbb{C}^\Lambda$ for some Λ and the origin 0 belongs to Ω, then $C_0(\Omega) = \{f \varepsilon C(\Omega): f(0) = 0\}$.

Theorem 2.4: Let A be a Banach algebra with identity and let a be a prescalar element of A with associated homomorphism

h. Assume that a is not invertible, so that 0 belongs to $\sigma_A(a)$. Then if I is any closed two-sided ideal of A containing a, $h(f)$ also belongs to I for any f in $C_0(\sigma_A(a))$.

Proof: Since h is a continuous homomorphism, $h^{-1}(I)$ is a closed ideal in $C(\sigma_A(a))$. Hence there is a unique closed subset F of $\sigma_A(a)$ such that f belongs to $h^{-1}(I)$ if and only if $f(F) = 0$. But since a belongs to I, $h^{-1}(a) = z$ belongs to $h^{-1}(I)$. Since z vanishes only at 0, $F = \{0\}$ and so $h^{-1}(I) = C_0(\sigma_A(a))$. Thus $h(f)$ belongs to I for all f in $C_0(\sigma_A(a))$.

A partial converse to Theorem 2.4 is given by the following proposition of Walsh [13, p. 1168]:

Theorem 2.5: Let A be a Banach algebra with identity and let a be a prescalar element of A. Then if I is any closed ideal of A which contains a^k for some positive integer k, I also contains a.

The proof of this theorem is almost the same as that of Theorem 2.4; one considers $h^{-1}(I)$ as a closed ideal in $C(\sigma_A(a))$ and notes that the functions z and z^k have the same set of zeros.

In Definitions 2.1 and 2.2 we assumed that A was a Banach algebra with identity e, and that any homomorphism $h: C(\Omega) \to A$, where Ω was compact Hausdorff, sent 1 to e. We shall now show that this assumption was justified.

If A is a Banach algebra without identity, the canonical embedding of A in a Banach algebra A_e having identity is described in Appendix 1. In particular, notice that, if Ω is a compact subset of $\mathbb{C}^\Lambda$ for some Λ with the origin 0 belonging to Ω, then $C_0(\Omega)$ is a Banach algebra; and the canonical extension of $C_0(\Omega)$ to a Banach algebra with identity is (isomorphic to) $C(\Omega)$. This follows since $C(\Omega)$ may be written as the topological direct sum $C(\Omega) = C_0(\Omega) \oplus \{\lambda\cdot 1: \lambda \in \mathbb{C}\}$.

Theorem 2.6: Let A be a Banach algebra without identity and A_e its canonical extension to an algebra with identity e. Let Ω be a compact subset of $\mathbb{C}^\Lambda$ with 0 in Ω, and let $h: C(\Omega) \to A$ be a continuous homomorphism. Then there is a continuous homomorphism $\overline{h}: C(\Omega) \to A_e$ such that $\overline{h}(1) = e$ and $\overline{h}(f) = h(f)$ for any f in $C_0(\Omega)$.

Proof: Let h' be the restriction of h to $C_0(\Omega)$. Since, by the preceding remarks, $C(\Omega)$ is just $C_0(\Omega)$ with identity adjoined, we have only to apply the commutative diagram of Appendix 1, to obtain the desired homomorphism $\overline{h} = \overline{\phi}$.

This theorem tells us that, if a is prescalar as an element of A, then a remains prescalar when an identity is adjoined to A and the resulting homomorphism sends identity to identity. If, conversely, we know that an element a of A is prescalar in A_e, we may assert that a is "almost" prescalar as an element of A, in the sense of the next result.

Theorem 2.7: If A is a Banach algebra without identity and an element a of A is prescalar in A_e under a continuous homomorphism $\overline{h}$, then there is a continuous function $h: C_0(\sigma_A(a)) \to A$ with $h(z) = a$, which is a homomorphism.

Proof: Since A is embedded in A_e as a closed maximal ideal, Theorem 2.4 implies that $\overline{h}[C_0(\sigma_A(a))] = \overline{h}[C_0(\sigma_{A_e}(a))] \subseteq A$ since by definition $\overline{h}(z) = a$ belongs to A. Thus the restriction h of $\overline{h}$ to $C_0(\sigma_A(a))$ is the desired continuous homomorphism.

Properties of the Subalgebra B

In this section, let A be a commutative Banach algebra with identity e, and let $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ be a jointly prescalar subset of A with associated homomorphism h: $C(\sigma_A(\{a_\lambda\})) \to A$. We assume as before that $h(1) = e$. We wish to investigate some of the properties of $B = h[C(\sigma_A(\{a_\lambda\}))]$.

Theorem 2.8: B is a closed semisimple subalgebra of A.

Proof: Since the domain of h is the continuous functions on a compact Hausdorff space, we may decompose h uniquely [1, p. 599] into $u + v$ where u and v are homomorphisms such that u is continuous, range (u) is closed in A, and v maps into the radical of B. Hence, since h is continuous, $h = u$ and $v = 0$ so that B is closed. Also, the radical of B is precisely the closure of the range of v, so that B is semisimple. Since B is obviously a subalgebra of A, the result follows.

If $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ is a jointly prescalar subset of A, we denote by h_λ the homomorphism constructed in Theorem 2.1 to show that a_λ is a prescalar element. We denote by $a \to \hat{a}$ the Gelfand mapping on A. Other notation is the same as that preceding Theorem 2.8.

Theorem 2.9: For each f belonging to $C(\sigma_A(\{a_\lambda\}))$ we have $\sigma_A(h(f)) = \text{range } (f) = f[\sigma_A(\{a_\lambda\})]$. Furthermore, for each λ in Λ, and each g in $C(\sigma_A(a_\lambda))$, $[h_\lambda(g)]\hat{} = g\circ\hat{a}_\lambda$.

Remark 2.3: For a single operator, this theorem is usually referred to as the Spectral Mapping Theorem. If T is a normal operator in $B(H)$, for example, the result of the theorem takes the form $\sigma(f(T)) = f(\sigma(T))$ for all continuous functions f.

Proof of Theorem 2.9: For any m belonging to M_A and any f in $C(\sigma_A(\{a_\lambda\}))$ we have

$$[h(f)]\hat{}(m) = m[h(f)] = [h^*(m)](f),$$

where $*$ denotes the adjoint map. Since $h(1) = e$ and h is a continuous homomorphism, $h^*(m)$ belongs to the maximal ideal space of $C(\sigma_A(\{a_\lambda\}))$ and as such is evaluation at some point ζ_0 of $\sigma_A(\{a_\lambda\})$. But ζ_0 is of the form $\zeta_0 = \langle\hat{a}_\lambda(m_0)\rangle_{\lambda\varepsilon\Lambda}$ for some m_0 in M_A; and we know that, for any λ in Λ,

$$[h^*(m)](z_\lambda) = \zeta_\lambda = m[h(z_\lambda)] = m(a_\lambda) = \hat{a}_\lambda(m)$$

for our given m in M_A. Thus $\zeta_0 = \langle\hat{a}_\lambda(m)\rangle_{\lambda\varepsilon\Lambda}$ (that is, $m = m_0$). Therefore

$$[h(f)]\hat{}(m) = [h^*(m)](f) = f(\zeta_0) = f(\langle\hat{a}_\lambda(m)\rangle).$$

so that range (f) = range ($[h(f)]^\wedge$). But, for any element a of a commutative Banach algebra A, the range of $\hat{a}$ is exactly $\sigma_A(a)$. Thus

$$\text{range } ([h(f)]^\wedge) = \sigma_A(h(f)) = \text{range } (f).$$

The same proof shows that, if a_λ is a prescalar element of A, then

$$[h_\lambda(g)]^\wedge(m) = g[\hat{a}(m)]$$

for all m in M_A and all g in $C(\sigma_A(a_\lambda))$. Thus $[h_\lambda(g)]^\wedge = g \circ \hat{a}$.

Corollary 2.2: If A, $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$, and h are as above, then h is an isometry with respect to spectral radius on A.

Proof: By definition, the spectral radius $\rho(a)$ of any element a of A is given by $\rho(a) = \sup_{\zeta\varepsilon\sigma_A(a)} |\zeta|$. Thus for any f in $C(\sigma_A(\{a_\lambda\}))$ we have, using Theorem 2.9,

$$\|f\| = \sup_{x\varepsilon\sigma_A(\{a_\lambda\})} |f(x)| = \sup_{\zeta\varepsilon\sigma_A(h(f))} |\zeta| = \rho(h(f)).$$

Corollary 2.3: With A, B, $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$, and h as above, h is an isomorphism of $C(\sigma_A(\{a_\lambda\}))$ onto B.

Proof: By definition, h is a continuous homomorphism onto B. If $h(f) = 0$ for some f in $C(\sigma_A(\{a_\lambda\}))$, then

$$0 = h(f) = \sigma_A(h(f)) = f[\sigma_A(\{a_\lambda\})]$$

so that $f = 0$. Thus h is one-to-one. Since B is a Banach algebra by Theorem 2.8, h is an isomorphism by the Open Mapping Theorem.

Remark 2.4: Theorem 2.9 is a basic tool in our study. It may be seen now why it was necessary to choose homomorphisms carrying identity to identity, and to prove that this could be done in general. For example, let A be a Banach algebra with identity e and let p be an element of A such that $p^2 = p$ and $p \neq 0, e$. Then $\sigma_A(p) = \{0,1\}$ and we can define a homomorphism h from $C(\sigma_A(p))$ to A by requiring that $h(1) = h(z) = p$. It is not hard to show that 1 and z are a basis for $C(\sigma_A(p))$ and that the h so constructed is a continuous homomorphism, $h\colon C(\sigma_A(p)) \to A$, with $h(z) = p$ but $h(1) \neq e$. In this case Theorem 2.9 and its Corollaries are no longer valid; h is not even one-to-one, much less an isometry with respect to spectral radius.

The notation for the next several results is the same as that preceding Theorem 2.8.

Theorem 2.10: Each element b of B is prescalar (considered as an element of A).

Proof: Let b in B be arbitrary. By definition b belongs to the range of h, so we may fix a function f in $C(\sigma_A(\{a_\lambda\}))$ such that $h(f) = b$ (actually, Corollary 2.3 shows that there is a unique such f). By Theorem 2.9, $\sigma_A(b) = \text{range } (f)$. Thus, for any g in $C(\sigma_A(b))$, $g \circ f$ is defined and belongs to $C(\sigma_A(\{a_\lambda\}))$.

Set $h_b(g) = h(g \circ f)$, so that h_b is a continuous homomorphism, $h_b\colon C(\sigma_A(b)) \to A$, $h_b(1) = h(1 \circ f) = h(1) = e$, and $h_b(z) = h(z \circ f) = h(f) = b$. Hence h_b makes b prescalar as an element of A.

Theorem 2.11: Let $A(\{a_\lambda\})$ be the closed subalgebra of A generated by $\{a_\lambda\}$. Then each element b of $A(\{a_\lambda\})$ is prescalar in A. In particular, if there exists a jointly prescalar set $\{a_\lambda\}$ which generates A, then A is isomorphic to $C(M_A)$.

Proof: Since $A(\{a_\lambda\})$ is, by definition, the smallest closed subalgebra of A containing each a_λ, we have $A(\{a_\lambda\}) \subseteq B = h[C(\sigma_A(\{a_\lambda\}))]$. Thus by Theorem 2.10 each element b of $A(\{a_\lambda\})$ is prescalar as an element of A. Hence if A has a jointly prescalar generating set, then A is isomorphic to $C(M_A)$ by Theorem 2.2.

Remark 2.5: Theorem 2.11 shows that if $\{a_\lambda\}$ is a jointly prescalar subset of A, then the closed subalgebra $A(\{a_\lambda\})$ generated by $\{a_\lambda\}$ consists entirely of prescalar elements of A. It does not, however, follow that $A(\{a_\lambda\})$ is isomorphic to $C(M_{A(\{a_\lambda\})})$. For example, if $C(D)$ is the algebra of continuous functions on the closed unit disc, the function z, such that $z(\lambda) = \lambda$, is prescalar by Theorem 2.2. Therefore each element of $A(z)$ is prescalar by Theorem 2.11. But the maximal ideal space of $A(z)$ is D, the spectrum of z; and $A(z)$ is certainly not isomorphic to $C(D)$ since each function in $A(z)$ is holomorphic on the interior of D. The point is that care must be taken to specify the algebra with respect to which a given set is jointly prescalar.

Remark 2.6: Even though, by Corollary 2.3, B is always isomorphic to $C(\sigma_A(\{a_\lambda\}))$, it is unknown whether or not different choices of the homomorphism h may yield different (although isomorphic) subalgebras B. This question has been raised in the case where h arises via integration with respect to a spectral measure (see Appendix 2 and [3, p. 295]).

Conditions for Prescalarity, Prescalarity in Subalgebras, and Examples

Throughout this section, A will denote a Banach algebra with identity e, and a will be an element of A. We reserve the letter D for a closed subalgebra of A which contains a and e.

Theorem 2.12: Let a be an element of A such that the polynomials in z are uniformly dense in $C(\sigma_A(a))$. Then a is prescalar if and only if there exists a positive constant M such that $\|p(a)\| \leq M\|p(z)\|$ for every polynomial p.

Proof: If a is prescalar, then the constant M exists since the homomorphism $h\colon C(\sigma_A(a)) \to A$ is bounded by definition.

Conversely, if such an M exists, then the mapping which sends $p(z)$ to $p(a)$ is bounded on the polynomials and so has an extension to a continuous homomorphism $h\colon C(\sigma_A(a)) \to A$, since the polynomials are dense. By the definition of h we have $h(1) = e$ and $h(z) = a$, so that a is prescalar.

Theorem 2.13: Let a have the property that polynomials are dense in $C(\sigma_A(a))$ and further assume that $\sigma_A(a)$ does not separate the plane. If there exists a positive constant M such that $\|p(a)\| \leq M\|p(z)\|$ for every polynomial p, then a is prescalar as an element of D, for any subalgebra D.

Proof: Recall the theorem [11, p. 34] to the effect that $\sigma_A(a) = \sigma_D(a)$ for every closed subalgebra D of A containing a and e if and only if $\sigma_A(a)$ does not separate the plane. Hence if $\sigma_A(a)$ does not separate the plane, then $\sigma_A(a) = \sigma_{A(a)}(a)$, where $A(a)$ is the closed subalgebra generated by a. Therefore the homomorphism h constructed in the proof of Theorem 2.12 maps $\sigma_A(a) = \sigma_{A(a)}(a)$ into $A(a)$, since $A(a)$ consists of polynomials in a and their limits. Finally, $A(a) \subseteq D$ and

$$h: \; C(\sigma_A(a)) = C(\sigma_{A(a)}(a)) = C(\sigma_D(a)) \to A(a) \subseteq D$$

so that a is prescalar as an element of D.

<u>Theorem 2.14</u>: Assume that $\sigma_D(a)$ is nowhere dense. Then if a is prescalar as an element of D, a is also prescalar as an element of A.

Proof: We always have the containments

$$\sigma_D(a) \supseteq \sigma_A(a)$$

and

$$\partial\sigma_D(a) \subseteq \partial\sigma_A(a),$$

where ∂ denotes topological boundary. Since $\sigma_D(a)$ is assumed to be nowhere dense, $\partial\sigma_D(a) = \sigma_D(a)$. Hence

$$\partial\sigma_A(a) \supseteq \partial\sigma_D(a) = \sigma_D(a) \supseteq \sigma_A(a) \supseteq \partial\sigma_A(a),$$

the last containment holding since $\sigma_A(a)$ is compact. Therefore all the containments are equalities and so $\sigma_D(a) = \sigma_A(a)$. By hypothesis there is a continuous homomorphism $h\colon C(\sigma_D(a)) \to D$ with $h(1) = e$ and $h(z) = a$, so that

$$h\colon C(\sigma_D(a)) = C(\sigma_A(a)) \to D \subseteq A$$

also makes a prescalar in A.

Theorem 2.15: Let p be a proper idempotent of A; that is, p is an element of A such that $p^2 = p$ and $p \neq 0,e$. Then p is prescalar as an element of any closed subalgebra D containing p and e.

Proof: For any λ in $\mathbb{C}$ except 0 and 1, $(p - \lambda e)^{-1} = [1/\lambda(1-\lambda)][p-(1-\lambda e)]$ so that $\sigma_A(p) \subseteq \{0,1\}$; and $\{0,1\} \subseteq \sigma_A(p)$ since both p and $e - p$ are proper idempotents and so not invertible. Thus $C(\{0,1\})$ is a two dimensional space with the functions 1 and z as a basis [7, Exercise 15, p. 340], and the homomorphism h which sends 1 to e and z to p is continuous [12, Lemma 1, p. 216]. Hence p is prescalar in A; and since $\{0,1\}$ clearly does not separate the plane, Theorem 2.13 shows that p is prescalar in D.

Example 2.1: We may now give an example of a set of elements of a Banach algebra each of which is prescalar, but which do not form a jointly prescalar set (compare Theorem 2.1). Bade and Curtis [1, p. 606], using an example due to Feldman, have exhibited a commutative Banach algebra A having a nonzero radical R, such that A is generated by a set of orthogonal idempotents. By Theorem 2.15 each of these idempotents is prescalar; but the set could not be jointly prescalar in view of Theorems 2.11 and 2.3, since the algebra generated by the set contains nonzero topological nilpotents.

There is one more situation in which prescalarity of an element may be concluded by considering properties of its spectrum.

Definition 2.3: A compact subset Ω of the plane is said to be an R-set if the set of rational functions with poles outside Ω is dense in $C(\Omega)$.

Remark 2.7: The following facts are known about R-sets [3, p. 306]:

(i) every R-set is nowhere dense, but there exist nowhere dense compact sets which are not R-sets

(ii) if Ω is a compact set with plane Lebesgue measure zero, then Ω is an R-set

(iii) if Ω is a compact set whose complement in the plane has at most finitely many components, then Ω is an R-set.

<u>Theorem 2.16</u>: Let a be an element of A such that $\sigma_A(a)$ is an R-set. Then a is prescalar if and only if there exists a positive constant M such that $\|r(a)\| \leq M\|r(z)\|$ for every rational function r with poles outside $\sigma_A(a)$.

Proof: If a is prescalar and $\sigma_A(a)$ is an R-set, the constant M exists by the continuity of the homomorphism h.

Conversely, if such a constant can be found, then the mapping h which sends $p(z)/q(z)$ to $p(a)q(a)^{-1}$ is continuous by the boundedness condition, and well-defined since if $q(z)$ is a polynomial which never vanishes on $\sigma_A(a)$, then $\sigma_A(q(a)) = q(\sigma_A(a)) \neq 0$ so that $q(a)$ is invertible. The mapping h is a homomorphism since the polynomials in a and their inverses all commute. Thus h has a continuous extension to a homomorphism h_1: $C(\sigma_A(a)) \to A$, since $\sigma_A(a)$ is an R-set. By definition, $h_1(1) = e$ and $h_1(z) = a$ so that a is prescalar.

CHAPTER 3

PRESPECTRAL ELEMENTS AND APPLICATIONS

From the theory of spectral operators we know that the only operators on a Hilbert space which possess a resolution of the identity are those expressible as the sum of a scalar-type operator and a commuting quasinilpotent (see, for example, [6] and Appendix 2). Thus it seems natural to investigate the behavior of Banach algebra elements expressible as the sum of a prescalar and a commuting topological nilpotent.

Prespectral Elements, Spectral Homomorphisms, and Continuous Functions

Definition 3.1: Let A be a Banach algebra with identity e and let a be an element of A. Then a is said to be prespectral if there exist elements a_0 and r_0 of A such that a_0 is prescalar, r_0 is a topological nilpotent commuting with a_0, and $a = a_0 + r_0$. We shall refer to a_0 as the prescalar part of a, and to r_0 as the topological nilpotent part.

Definition 3.2: Let A be a commutative Banach algebra with identity e, and let a be an element of A. A continuous homomorphism h is said to be a spectral homomorphism for a if $h\colon C(\sigma_A(a)) \to A$, $h(1) = e$, and $[h(f)]\hat{} = f \circ \hat{a}$ for all f belonging to $C(\sigma_A(a))$.

Lemma 3.1: Let A be a Banach algebra with identity, let a be a prespectral element of A, and let $a = a_0 + r_0$ be any representation of a as the sum of prescalar and topological nilpotent parts. Then:

(i) $\sigma_A(a) = \sigma_A(a_0)$

(ii) if A is commutative, $\hat{a} = \hat{a}_0$.

Proof: The first statement follows from the definition of prespectral element and the fact that commuting Banach algebra elements which differ by a topological nilpotent have the same spectra.

To prove (ii), let m be any element of M_A. Then

$$\hat{a}(m) = m(a) = m(a_0 + r_0) = m(a_0) + m(r_0) = m(a_0) = \hat{a}_0(m)$$

since r_0 is a topological nilpotent and hence belongs to the radical of A. Thus $\hat{a} = \hat{a}_0$.

Theorem 3.1: Let A be a commutative Banach algebra with identity e and let a be any element of A. Then a is prescalar if and only if there is an element b of A and a spectral homomorphism h for b, such that a belongs to the image of h.

Proof: Assume that a is prescalar with associated homomorphism h. Then Theorem 2.9 shows that h is a spectral homomorphism for a; and, by definition, $a = h(z)$.

Conversely, suppose that h is a spectral homomorphism for some element b of A and assume that $a = h(f)$ for some f in

$C(\sigma_A(b))$. Then $\sigma_A(a) = \text{range } (\hat{a})$, and

$$\hat{a} = [h(f)]\hat{} = f\circ\hat{b},$$

so that

$$\text{range } (\hat{a}) = \text{range } (f) = \sigma_A(a).$$

Therefore, for any g in $C(\sigma_A(a))$, $g\circ f$ is defined and belongs to $C(\sigma_A(b))$. Setting $h_a(g) = h(g\circ f)$ for g in $C(\sigma_A(a))$, it is clear that h_a is a continuous homomorphism, $h_a\colon C(\sigma_A(a)) \to A$, $h_a(1) = e$, and

$$h_a(z) = h(z\circ f) = h(f) = a.$$

Hence a is prescalar with associated homomorphism h_a.

Theorem 3.2: Let A be a commutative Banach algebra with identity e and let a be any element of A. Then a is prespectral if and only if there exists a spectral homomorphism for a.

Proof: Assume that a is prespectral and that $a = a_0 + r_0$ is any representation of a, where a_0 is prescalar and r_0 is a topological nilpotent. Let h be the homomorphism associated with a_0. By Lemma 3.1, $\sigma_A(a) = \sigma_A(a_0)$, so that

$$h: C(\sigma_A(a_0)) = C(\sigma_A(a)) \to A$$

and $h(1) = e$. By Theorem 2.9, $[h(f)]\hat{} = f\circ\hat{a}_0$ for all f in $C(\sigma_A(a_0))$. But again using Lemma 3.1, $\hat{a}_0 = \hat{a}$, so $[h(f)]\hat{} = f\circ\hat{a}$ for all f in $C(\sigma_A(a))$. Thus h is a spectral homomorphism for a.

Conversely, suppose h is a spectral homomorphism for a. Set $a_0 = h(z)$, where as before z denotes the function in $C(\sigma_A(a))$ such that $z(\lambda) = \lambda$. By Theorem 3.1 a_0 is prescalar, and

$$\hat{a}_0 = [h(z)]\hat{} = z\circ\hat{a} = \hat{a}$$

since h is a spectral homomorphism. Thus a and a_0 are elements of A having the same Gelfand transform and so $a - a_0$ is a topological nilpotent r_0. Thus $a = a_0 + r_0$ with a_0 prescalar and r_0 a topological nilpotent, implying that a is prespectral.

We shall next give some definitions and results concerning continuous functions of Banach algebra elements, the ultimate purpose of which will be to prove a generalization of a theorem due to Katznelson [9, p. 904] (see also Theorem 3.4 of this chapter). Hereafter, A will be assumed to be a commutative Banach algebra with identity e.

Definition 3.3: Let a and b be elements of A. Then b is said to be a continuous function of a if there is a function f belonging to $C(\sigma_A(a))$ such that $\hat{b} = f\circ\hat{a}$.

Definition 3.4: Let a be an element of A. Then A is said to be closed under continuous functions of a if, for each f in $C(\sigma_A(a))$, there exists an element b of A such that $\hat{b} = f\circ\hat{a}$.

Definition 3.5: A is said to have the property that continuous functions operate on A if A is closed under continuous functions of a, for each a belonging to A.

Theorem 3.3: Let a be a fixed prescalar element of A with associated homomorphism h, let $B = h[C(\sigma_A(a))]$, and let b be any element of A. Then b is a continuous function of a if and only if b is prespectral with prescalar part belonging to B.

Proof: Assume that b is a continuous function of a. Then there exists an f in $C(\sigma_A(a))$ with $\hat{b} = f\circ\hat{a}$. Set $b_0 = h(f)$. By definition of B, b_0 belongs to B; and

$$\hat{b}_0 = [h(f)]\hat{} = f\circ\hat{a} = \hat{b},$$

using Theorem 2.9. Thus, since b and b_0 have the same Gelfand transform, their difference $r_0 = b - b_0$ is a topological nilpotent. Finally, b_0 is prescalar by Theorem 2.10. Hence $b = b_0 + r_0$ is prespectral.

Conversely, suppose that $b = b_0 + r_0$ is prespectral and that b_0 belongs to B. Then $b_0 = h(f)$ for some f in $C(\sigma_A(a))$; and

$$\hat{b} = \hat{b}_0 = [h(f)]\hat{} = f\circ\hat{a},$$

using Lemma 3.1 and Theorem 2.9. Hence b is a continuous function of a.

Applications to Decomposable Algebras

Throughout this section, A will denote a commutative Banach algebra with identity e, and R will denote the radical of A.

Definition 3.6: A is said to be decomposable if there exists a subalgebra D of A such that $A = D \oplus R$. A is said to be strongly decomposable if there exists a closed subalgebra D of A such that $A = D \oplus R$.

Basic definitions and results about decomposable algebras have been given by Bade and Curtis [2].

Remark 3.1: Note that if A is strongly decomposable as $A = D \oplus R$, then the natural projection from A onto D induces an isomorphism between A/R and D. Hence D is necessarily a commutative semisimple Banach algebra with identity.

Remark 3.2: Katznelson [9, p. 904] has proved an interesting and well-known theorem which states that if A is a commutative semisimple Banach algebra with identity, then A is isomorphic to $C(M_A)$ if and only if continuous functions operate on A (in the sense of Definition 3.5). In view of Theorem 2.2 of this paper, we could restate this theorem to read: a commutative semisimple Banach algebra A with identity has the property that continuous functions operate on A, if and only if each element a of A is prescalar. It is this form of Katznelson's Theorem which we shall generalize.

Theorem 3.4: Let A be a commutative Banach algebra with identity which is strongly decomposable as $A = D \oplus R$. Then A has the property that continuous functions operate on A, if and only if each element a of A is prespectral.

Note that this is actually a generalization since any semi-simple Banach algebra is strongly decomposable as $A = A \oplus \{0\}$. The proof of Theorem 3.4 will require two lemmas.

Lemma 3.2: Assume that A is strongly decomposable as $A = D \oplus R$. Then the respective maximal ideal spaces M_A and M_D are homeomorphic.

Proof: There is a natural correspondence between M_A and M_D defined by $m_A \leftrightarrow m_D$, where $m_A(d + r) = m_D(d)$ for any $a = d + r$ in A, d in D, and r in R. Since, by hypothesis, each a in A has a unique representation in this form, the correspondence is a bijection. Furthermore, if α belongs to some directed set, then $m_A^\alpha \to m_A$ in M_A if and only if $m_A^\alpha(a) \to m_A(a)$ for each a in A, by definition of the topology on M_A. Hence if m_D^α and m_D are the corresponding elements of M_D,

$$m_D^\alpha(d) = m_A^\alpha(d + r) \to m_A(d + r) = m_D(d)$$

for each d in D, so that $m_D^{\alpha} \to m_D$ in M_D. Thus the bijection is continuous, and therefore a homeomorphism since M_A is compact and M_D is Hausdorff.

Lemma 3.3: Assume that A is strongly decomposable as $A = D \oplus R$. Then if continuous functions operate on A, continuous functions also operate on D.

Proof: Let d be an arbitrary element of D and let f belong to $C(\sigma_D(d))$. We must find an element d_0 of D such that $\hat{d}_0 = f \circ \hat{d}$. Let Θ be the homeomorphism constructed in Lemma 3.2 above. Then $\hat{d}(m_A) = \hat{d}[\Theta(m_A)] = \hat{d}(m_D)$, where $d = d + 0$ is considered first as an element of A and then as an element of D. If r is any element of the radical R and $a = d + r$, then $\hat{a}(m_A) = \hat{d}(m_A)$ for each m_A in M_A. Since continuous functions operate on A, we can find an element a_0 of A such that $\hat{a}_0 = f \circ \hat{a} = f \circ \hat{d}$. Because A is strongly decomposable, we may write a_0 uniquely as $a_0 = d_0 + r_0$ for some d_0 in D and r_0 in R, with $\hat{a}_0 = \hat{d}_0$. Thus $\hat{d}_0 = f \circ \hat{d}$ as continuous functions on M_A and so $\hat{d}_0 = f \circ \hat{d}$ as continuous functions on M_D. Hence continuous functions operate on D.

Proof of Theorem 3.4: Assume that continuous functions operate on A, where A is strongly decomposable as $A = D \oplus R$. By Lemma 3.3, continuous functions operate on D. Thus D is isomorphic to $C(M_D) = C(M_A)$ by Katznelson's Theorem [9, p. 904] and Lemma 3.2. Hence each element d of D is prescalar as an element of D, and so as an element of A, since the proof of Lemma 3.3 shows that $\sigma_A(d) = \sigma_D(d)$ for any d in D. Thus each a in A has a (unique) representation as $a = d + r$ where d is prescalar and r is a topological nilpotent, so that A consists entirely of prespectral elements.

Conversely, suppose that each a in A is prespectral and that $A = D \oplus R$ is strongly decomposable. If $a = a_0 + r_0$ is any element of A and f belongs to $C(\sigma_A(a))$, set $a_1 = h(f)$, where h is the homomorphism associated with the prescalar a_0. Then, using Lemma 3.1 and Theorem 2.9, we have

$$\hat{a}_1 = [h(f)]\hat{} = f \circ \hat{a}_0 = f \circ \hat{a},$$

so that continuous functions operate on A.

As one more application of these ideas we give the following:

<u>Theorem 3.5</u>: Assume that each element a of A is prespectral, and assume further that the corresponding prescalar parts form a jointly prescalar set. Then A is strongly decomposable.

Proof: Let $\{a_\lambda\}_{\lambda \varepsilon \Lambda}$ be the set of all prescalar parts of elements of A and let $D = A(\{a_\lambda\})$ be the closed subalgebra generated by $\{a_\lambda\}$. Since, by hypothesis, $\{a_\lambda\}$ is a jointly prescalar set, D consists entirely of prescalar elements of A by Theorem 2.11. In particular, $D \cap R = \{0\}$ by Theorem 2.3, where R is the radical of A. Since each element a of A is expressible as $a = d + r$ for some d in D and r in R, and $D \cap R = \{0\}$, we have $A = D \oplus R$ so that A is strongly decomposable.

CHAPTER 4

SOME APPLICATIONS TO OPERATOR THEORY

We shall now apply some of the ideas of Chapters 2 and 3 to the Banach algebra $B(X)$, where X is a Banach space. In particular, we characterize the prescalar elements of $B(H)$, where H is a Hilbert space, as exactly those operators which are similar to normal operators. Additional results are concerned with the spectral behavior of representations of Banach algebra elements in terms of the prescalarity of the elements themselves.

Basic definitions and results concerning scalar-type operators are given in Appendix 2.

Theorem 4.1: Let X be a Banach space with conjugate space X^*, and let T be an operator in $B(X)$. Then:

(i) if T is a scalar-type operator of any class Y, then T is prescalar

(ii) if T is prescalar, then T^* is a scalar-type operator in $B(X^*)$ of class X

(iii) if X is weakly complete and T is prescalar, then T is a scalar-type operator of class X^*

(iv) if $X = H$ is a Hilbert space, then an operator T in $B(H)$ is prescalar if and only if T is similar to a normal operator.

Proof: Statements (i), (ii), and (iii) follow respectively from Theorems C, D, and E quoted in Appendix 2. Statement (iv) follows from the fact that any Hilbért space is reflexive and hence weakly complete, so that by (i) and (iii) T is prescalar if and only if T is scalar-type. But then the theorem of Wermer [14, p. 355], to the effect that an operator T in $B(H)$ is scalar-type if and only if T is similar to a normal operator, implies that T is prescalar if and only if T is similar to a normal operator.

Example 4.1: It is not true that if an operator T in $B(X)$ is prespectral (in the sense of Definition 3.1), then T is a prespectral operator in the sense of [3]. Indeed, Berkson and Dowson [3, p. 308] give an example of an operator T on ℓ^∞ such that $T = S + N$, where S is a scalar-type operator on ℓ^∞ of class ℓ^1 and N is a nilpotent commuting with S, but such that T is not a prespectral operator of any class. The trouble in this case arises from the fact that even though N commutes with S, N does not commute with any resolution of the identity for S.

Remark 4.1: It should be noted that Theorem E of Appendix 2, referred to above, actually implies that if an operator T in $B(H)$ is prescalar as an element of any closed subalgebra of $B(H)$, then T is scalar-type and hence similar to a normal operator. Thus, in view of Theorem 2.2, if T is contained in any closed subalgebra of $B(H)$ which is isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω, then T is similar to a normal operator. This result has recently been proved (by different methods) by Douglas [5, p. 196].

<u>Theorem 4.2</u>: Let H be a Hilbert space and $\{T_i\}_{i=1}^n$ a finite set of commuting operators in $B(H)$. If $\{T_i\}_{i=1}^n$ is jointly prescalar as a subset of the double commutant $(\{T_i\})''$, then there exists a single invertible operator A in $B(H)$ such that AT_iA^{-1} is normal for each $i = 1,\ldots,n$.

Proof: Since $(\{T_i\})''$ is a commutative Banach algebra with identity, it makes sense to speak of $\{T_i\}$ as being a jointly prescalar subset. By Theorem 2.1, each T_i is prescalar as an element of $(\{T_i\})''$. But for each $i = 1,\ldots,n$, $\sigma_{(\{T_i\})''}(T_i) = \sigma_{B(H)}(T_i)$, so the homomorphism which makes T_i prescalar in $(\{T_i\})''$ also makes T_i prescalar in $B(H)$. Thus by Theorem 4.1 each T_i is a scalar-type operator. Hence by the theorem of Wermer [14, p. 355] there is an invertible operator A in $B(H)$ such that AT_iA^{-1} is normal for each $i = 1,\ldots,n$.

<u>Theorem 4.3</u>: Let H be a Hilbert space and let T be an operator in $B(H)$ such that $\sigma_{B(H)}(T)$ is an R-set. Then T is similar to a normal operator if and only if there exists a positive constant M such that $\|r(T)\| \leq M\|r(z)\|$ for each rational function r with poles outside $\sigma_{B(H)}(T)$.

Proof: By Theorem 2.16, T is prescalar if and only if the constant M exists. The conclusion now follows from Theorem 4.1 part (iv).

Definition 4.1: Let Ω be a compact subset of the plane, T an operator in $B(H)$, and K a constant, $K \geq 1$. Then Ω is said to be a K-spectral set for T if $\|r(T)\| \leq K\|r(z)\|_\Omega$ for every rational function r with poles outside Ω; here $\|r(z)\|_\Omega = \sup_{x\in\Omega}|[r(z)](x)|$.

Remark 4.2: In view of the previous definition we may restate Theorem 4.3 to say that if $\sigma_{B(H)}(T)$ is an R-set, then T is similar to a normal operator if and only if $\sigma_{B(H)}(T)$ is a K-spectral set for T. In particular, the result holds for any compact operator T, since it is well-known [7, Theorem 5, p. 579] that the spectrum of such an operator is countable with no accumulation points other than possibly zero. Results similar to this one have also been obtained by Douglas [5, p. 196].

The fundamental properties of the left regular representation of a Banach algebra A are given in Appendix 1. In particular, we shall denote by T_a the operator in $B(A)$ defined by $T_a x = ax$ for all x in A. If A has an identity e, we assume that A has been given an equivalent norm such that $\|e\| = 1$.

Theorem 4.4: Let A be a Banach algebra with identity e and let a be a prescalar element of A with associated homomorphism h; as before, we assume $h(1) = e$. Then:

(i) T_a^* is a scalar-type operator in $B(A^*)$ of class A

(ii) if A is weakly complete as a Banach space, then T_a is a scalar-type operator in $B(A)$.

Proof: Since $\sigma_A(a) = \sigma_{B(A)}(T_a)$, we may let $\overline{h}$ be the composition of the left regular representation with h, so that

$$\overline{h}: C(\sigma_A(a)) = C(\sigma_{B(A)}(T_a)) \to A \to B(A)$$

is a continuous homomorphism with $\overline{h}(1) = I$, and $\overline{h}(z) = T_a$. Thus T_a is a prescalar element of $B(A)$, and (i) and (ii) follow respectively from (ii) and (iii) of Theorem 4.1.

Example 4.2: If A is not weakly complete, then the fact that an element a of A is prescalar need not imply that T_a is a scalar-type operator. Fixman [8, p. 1029] has shown that, if Ω is a compact Hausdorff space and $C(\Omega)$ is the usual continuous function algebra, then multiplication by the function f in $C(\Omega)$ is not in general a scalar-type operator. In particular, any f which is nonconstant on a connected component of Ω furnishes an example. But this multiplication operator is just the image of f under the left regular representation of $C(\Omega)$, and Theorem 2.2 shows that f is prescalar as an element of $C(\Omega)$. Of course, $C(\Omega)$ is in general extremely non-weakly complete (see, for instance, [7, Exercise 15, p. 340]).

Example 4.3: A natural question to ask might be whether or not every prescalar element arises from integration with respect to an idempotent-valued measure. That this need not be the case is shown by considering $C(\Omega)$ for a connected compact Hausdorff space Ω.

Every element of $C(\Omega)$ is prescalar by Theorem 2.2; but the only idempotents are the functions 0 and 1, so that no non-constant f in $C(\Omega)$ could be the limit of linear combinations of idempotents.

APPENDIX 1

ELEMENTARY DEFINITIONS AND RESULTS ON BANACH ALGEBRAS

Since notation and terminology concerning Banach algebras are not entirely standardized, we shall give here some of the relevant definitions and results. As general references, see Rickart [11], Loomis [10], and Bourbaki [4].

Adjunction of Identity

If A is a Banach algebra, we may embed A in $A \times \mathbb{C} = \{(a,\zeta): \ a \in A, \ \zeta \in \mathbb{C}\}$ by $a \to (a,0)$. Defining operations in $A \times \mathbb{C}$ by

$$(a,\zeta) + (b,\xi) = (a + b, \zeta + \xi),$$

$$(a,\zeta)(b,\xi) = (ab + \xi a + \zeta b, \zeta\xi),$$

and

$$\xi(a,\zeta) = (\xi a, \xi\zeta),$$

and a norm

$$\|(a,\zeta)\| = \|a\| + |\zeta|,$$

we obtain a new Banach algebra A_e with identity $e = (0,1)$. This algebra will be called the canonical extension of A to a Banach algebra with identity. If we identify A with its image in A_e, then A is embedded in A_e as a closed maximal ideal of deficiency one [10, p. 59]. Furthermore, this extension has the property that, if A' is a second Banach algebra with canonical extension A'_e and ϕ is a homomorphism from A to A', then there exists a unique homomorphism $\overline{\phi}$ from A_e to A'_e with $\overline{\phi}(e) = e'$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A_e \\ \phi \downarrow & & \downarrow \overline{\phi} \\ A' & \xrightarrow[i']{} & A'_e \end{array}$$

commutes [4, p. 1].

The Regular Representations

Any Banach algebra A may be thought of as an algebra of operators on itself. For each a in A, the left regular representation represents a as the operator T_a, where $T_a x = ax$ for all x in A. Clearly $\|T_a\| \leq \|a\|$ for any a in A; and, if A has an identity e such that $\|e\| = 1$, then the left regular representation is an isometry.

The Spectrum of an Element

If A is a Banach algebra with identity e and a is an element of A, the spectrum of a is defined to be the set $\sigma_A(a)$ of complex numbers λ such that $a - \lambda e$ is not invertible in A. If A does not have identity, $\sigma_A(a)$ is defined to be $\sigma_{A_e}(a)$; that is, the spectrum of a when considered as an element of A_e.

Important properties of the spectrum are:

(i) $\sigma_A(a)$ is a nonempty compact subset of $\mathbb{C}$

(ii) if A has identity e, and D is any closed subalgebra of A containing a and e, then $\sigma_D(a) \supseteq \sigma_A(a)$ and $\partial\sigma_D(a) \subseteq \partial\sigma_A(a)$, where ∂ denotes topological boundary

(iii) if A has identity e and T_a is the image of a under the left regular representation, then $\sigma_A(a) = \sigma_{B(A)}(T_a)$ [4, Exercise 14, p. 89].

Spectral Radius

For any element a of the Banach algebra A with identity, we define the spectral radius of a as the number $\rho(a) = \sup\limits_{\lambda\in\sigma_A(a)} |\lambda|$. Alternatively, $\rho(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. It is always true that $\rho(a) \leq \|a\|$; also, $\rho(a) = \|a\|$ if and only if $\|a^2\| = \|a\|^2$ [4, p. 16]. If a and b are commuting elements of A, then $\rho(ab) \leq \rho(a)\rho(b)$, and $\rho(a + b) \leq \rho(a) + \rho(b)$. An element a of A is said to be a topological nilpotent if $\rho(a) = 0$. If A has identity and a is a topological nilpotent commuting with some element b of A, then $\sigma_A(a + b) = \sigma_A(b)$. If we define the radical R of A to be the intersection of all the maximal two-sided ideals of A, then any element r of R is a topological nilpotent. If, in addition, A is commutative, then R consists precisely of the set of topological nilpotents in A. If R is the zero ideal, A is said to be semisimple. If A has identity, then A/R is a semisimple Banach algebra with identity under the norm

$$\|[a]\| = \inf\{\|a\|: \ a \in [a]\}$$

where $[a]$ denotes the equivalence class of a [10, p. 63]. Furthermore, $\sigma_A(a) = \sigma_{A/R}([a])$ for any a in A [4, p. 3].

The Gelfand Mapping

Let A be a commutative Banach algebra with identity and denote by M_A the set of all maximal ideals of A. Then M_A may be identified with the set of all continuous homomorphisms ϕ of A onto $\mathbb{C}$, by identifying ϕ with its kernel. Given the relative weak-* topology from the conjugate space of A, M_A becomes a compact Hausdorff space. Then A may be represented as an algebra of continuous functions on M_A by the Gelfand Mapping $a \to \hat{a}$, where $\hat{a}(\phi) = \phi(a)$ for ϕ in M_A. M_A will be called the maximal ideal space of A. We then have the following:

(i) the range of $\hat{a}$ is exactly the set $\sigma_A(a)$, for any a in A; in particular, a is invertible if and only if $\hat{a}$ never vanishes on M_A.

(ii) if the algebra $C(M_A)$ of continuous complex-valued functions on M_A is normed by $\|f\| = \sup_{x \in M_A} |f(x)|$, then $\hat{a}$ belongs to $C(M_A)$ for each a in A, and $\|\hat{a}\| = \rho(a)$

(iii) the radical R of A is the kernel of the Gelfand mapping; that is

$$\begin{aligned} R &= \{a \in A: \ \hat{a} = 0\} \\ &= \{a \in A: \ \rho(a) = 0\} \\ &= \{a \in A: \ \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0\} \\ &= \{a \in A: \ \sigma_A(a) = \{0\}\}. \end{aligned}$$

Joint Spectrum

Let A be a commutative Banach algebra with identity e and let $\{a_\lambda\}_{\lambda\varepsilon\Lambda}$ be an indexed subset of A. Then there is a natural mapping of M_A into $\mathbb{C}^\Lambda$ defined by $\phi \to \langle \hat{a}_\lambda(\phi)\rangle_{\lambda\varepsilon\Lambda}$. This mapping is continuous, so the image is a compact subset of $\mathbb{C}^\Lambda$; it is called the joint spectrum of the set of elements $\{a_\lambda\}$ and we denote it by $\sigma_A(\{a_\lambda\})$. Alternatively, a point $\langle \zeta_\lambda\rangle_{\lambda\varepsilon\Lambda}$ of $\mathbb{C}^\Lambda$ belongs to $\sigma_A(\{a_\lambda\})$ if and only if the set of elements $\{a_\lambda - \zeta_\lambda e\}_{\lambda\varepsilon\Lambda}$ of A together generate a proper ideal of A. If z_λ denotes the λth coordinate function on $\mathbb{C}^\Lambda$, then the range of z_λ restricted to $\sigma_A(\{a_\lambda\})$ is exactly $\sigma_A(a\ )$, for any λ in Λ. If the original set $\{a_\lambda\}$ generates A (that is, the smallest closed subalgebra of A containing each a_λ and e is A itself), then the mapping of M_A onto $\sigma_A(\{a_\lambda\})$ is a homeomorphism. Thus we may identify the maximal ideal space of A with the joint spectrum of any generating subset.

The Banach Algebra $C(\Omega)$

If Ω is any compact Hausdorff space, we denote by $C(\Omega)$ the Banach algebra of continuous complex-valued functions on Ω with point-wise operations and the supremum norm: $\|f\| = \sup_{x\varepsilon\Omega}|f(x)|$, f in $C(\Omega)$.

We then have the following facts:

(i) $C(\Omega)$ is a commutative Banach algebra with identity 1 whose maximal ideal space is Ω. For any I in $M_{C(\Omega)}$ there is a unique x_I in Ω such that f belongs to I if and only if $f(x_I) = 0$.

(ii) any closed ideal I of $C(\Omega)$ is equal to the kernel of its hull; that is, I is equal to the intersection of all the (closed) maximal ideals containing I. Thus for any closed ideal I of $C(\Omega)$ there exists a unique closed subset F of Ω such that f belongs to I if and only if $f(F) = 0$.

(iii) the Gelfand representation of $C(\Omega)$ is the identity map.

(iv) for any f in $C(\Omega)$, $\sigma_{C(\Omega)}(f) = \text{range } (f)$, and for any subset $\{f_\lambda\}_{\lambda\varepsilon\Lambda}$ of $C(\Omega)$,

$$\sigma_{C(\Omega)}(\{f_\lambda\}) = \{\langle\zeta_\lambda\rangle \varepsilon \mathbb{C}^\Lambda: \ \zeta_\lambda = f(x_0) \text{ for some fixed } x_0 \text{ in } \Omega\}.$$

If Ω is a compact subset of $\mathbb{C}^\Lambda$ for some cardinal number Λ, and Ω contains the origin 0 of $\mathbb{C}^\Lambda$, we shall denote by $C_0(\Omega)$ the maximal ideal of $C(\Omega)$ determined by 0; that is

$$C_0(\Omega) = \{f \varepsilon C(\Omega): \ f(0) = 0\}.$$

B*-Algebras

Let $B(H)$ denote the Banach algebra of all bounded linear operators on H, where H is a Hilbert space. This is the most common example of a B*-algebra; that is, a Banach algebra A with an involution $a \to a^*$ which satisfies:

(i) $(a^*)^* = a$, a in A

(ii) $(a + b)^* = a^* + b^*$, a and b in A

(iii) $(ab)^* = b^*a^*$, a and b in A

(iv) $(\zeta a)^* = \overline{\zeta} a^*$, ζ in $\mathbb{C}$ and a in A

(v) $\|a^*a\| = \|a\|^2$, a in A

In the case of $B(H)$, T^* is the adjoint of T.

If A is a B*-algebra, then any a satisfying $a^*a = aa^*$ is said to be normal; this follows the usual terminology of operator theory. The following theorem is of interest to us [11, p. 241]: For every normal element a of A there is an isometric *-isomorphism $f \to f(a)$ of $C(\sigma_A(a))$ or $C_0(\sigma_A(a))$, according as A does or does not have identity, to A, such that:

(a) if $f(\zeta) = \zeta$ for ζ in $\sigma_A(a)$, then $f(a) = a$

(b) $\sigma_A(f(a)) = f(\sigma_A(a)) = \text{range } (f)$

(c) $f(a)$ is contained in every *-subalgebra of A containing a and the identity, if one exists.

This theorem is a generalization of the operational calculus for normal operators in $B(H)$ constructed via the Spectral Theorem.

APPENDIX 2

SPECTRAL AND SCALAR-TYPE OPERATORS

We shall collect here some basic definitions and results concerning spectral operators, taken mainly from Dunford [6] and Berkson and Dowson [3]. In the following X will denote a Banach space, X^* the conjugate space, and $B(X)$ the Banach algebra of bounded linear operators on X.

Definition A: A subset Y of X^* is said to be total if x in X and $y(x) = 0$ for each y in Y together imply $x = 0$.

Definition B: Let Σ be a σ-algebra of subsets of a set Ω, with Ω belonging to Σ. A mapping $E(\cdot)$ from Σ to the set of all projections in $B(X)$ is said to be a spectral measure of class (Σ, Y) if the following conditions are satisfied:

(i) $E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2) = E(\delta_1 \cup \delta_2)$, δ_1 and δ_2 in Σ

(ii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$, δ_1 and δ_2 in Σ

(iii) $E(\Omega/\delta) = I - E(\delta)$, δ in Σ

(iv) $E(\Omega) = I$

(v) there exists a positive constant M such that $\|E(\delta)\| \leq M$ for all δ in Σ

(vi) there exists a total linear manifold Y in X^* such that $yE(\cdot)x$ is countably additive on Σ for each x in X and each y in Y.

<u>Definition C</u>: An operator T in $B(X)$ is said to be prespectral of class Y if the following two conditions hold:

(i) there is a spectral measure $E(\cdot)$ of class (B,Y), where B is the σ-algebra of Borel sets in the plane, taking values in $B(X)$, such that $TE(\delta) = E(\delta)T$ for all δ in B.

(ii) if T_δ is the restriction of T to the range of $E(\delta)$ (which makes sense in view of (i)), then $\sigma_{B(X)}(T_\delta) \subseteq \overline{\delta}$ for all δ in B.

In this case, $E(\cdot)$ is said to be a resolution of the identity of class Y for T.

<u>Definition D</u>: An operator T in $B(X)$ is said to be a spectral operator if T has a resolution of the identity of class $Y = X^*$; that is, T is a spectral operator if T is prespectral of class X^*.

See Dunford [6, p. 330 and p. 340] for a discussion of integration with respect to a spectral measure. For our purposes it will be sufficient to consider $\int_\Omega f(\lambda)dE(\lambda)$, for continuous functions f on Ω, to be a Riemann integral in the uniform operator topology.

<u>Definition E</u>: Let S be a prespectral operator in $B(X)$ with resolution of the identity $E(\cdot)$ of class Y. Then S is said to be a scalar-type operator of class Y if $S = \int_{\sigma(S)} \lambda dE(\lambda)$. If $Y = X^*$, then S is simply said to be a scalar-type operator.

Theorem A (Dunford [6, Theorem 8, p. 333]): An operator T is spectral if and only if $T = S + N$, where S is scalar-type and N is a quasinilpotent operator commuting with S.

Theorem B (Wermer [14, Theorem 1, p. 355]): Let $T_1,\ldots,T_n$ be a finite set of commuting scalar-type operators in $B(H)$, where H is a Hilbert space. Then there exists an invertible operator A in $B(H)$ such that the operator AT_iA^{-1} is normal for each $i = 1,\ldots,n$. In particular, an operator T in $B(H)$ is scalar-type if and only if T is similar to a normal operator.

Theorem C (Berkson and Dowson [3, p. 294]): If $E(\cdot)$ is a spectral measure of class (Σ,Y), then the mapping $\psi\colon C(\Omega) \to B(X)$ defined by $\psi(f) = \int_\Omega f(\lambda)dE(\lambda)$ is a bicontinuous algebra isomorphism onto a subset of $B(X)$.

Theorem D (Berkson and Dowson [3, Theorem 3.7, p. 297]): Let Ω be a compact Hausdorff space and let ψ be a continuous algebra homomorphism from $C(\Omega)$ into $B(X)$ with $\psi(1) = I$. Then there exists a spectral measure $E(\cdot)$ of class (B,X), where B denotes the Borel sets of the plane, with values in $B(X^*)$, such that

$$\psi(f)^* = \int_\Omega f(\lambda)dE(\lambda), \quad f \text{ in } C(\Omega).$$

Theorem E (Berkson and Dowson [3, Theorem 3.9, p. 298]): Let X be a weakly complete Banach space, let Ω be a compact Hausdorff space, and let ψ be a continuous algebra homomorphism from $C(\Omega)$ to $B(X)$ with $\psi(1) = I$. Then there exists a spectral measure $E(\cdot)$

of class $(\mathcal{B},X^*)$, $\mathcal{B}$ the σ-algebra of plane Borel sets, such that

$$\psi(f) = \int_\Omega f(\lambda)\,dE(\lambda), \quad f \text{ in } C(\Omega).$$

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