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A PRECISE MEASUREMENT OF THE
SUN'S VISUAL OBLATENESS

by
Paul Douglas Clayton

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF PHYSICS
In Partial Fulfillment of the Requirements
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1973
I hereby recommend that this dissertation prepared under my
direction by Pauli Douglas Clayton
entitled A Precise Measurement of the Sun's Visual
Oblateness
be accepted as fulfilling the dissertation requirement of the
degree of Doctor of Philosophy

Dissertation Director

After inspection of the final copy of the dissertation, the
following members of the Final Examination Committee concur in
its approval and recommend its acceptance:

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Roy E. Heath
Robert N. Cerny

This approval and acceptance is contingent on the candidate's
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SIGNED:  Paul D. Clayton
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The question of solar oblateness is important in relationship to the experimental verification of gravitational theories. The observed perihelion precession of Mercury is presently the only test of sufficient accuracy to verify Einstein's General Theory of Relativity. The agreement between theory and observation in this instance implicitly assumes that the sun does not have a gravitational quadrupole moment. If there is no visual solar oblateness, it is difficult to conceive that a gravitational quadrupole could exist. On the other hand a visual oblateness would not indicate unequivocally that a quadrupole moment exists; one must still ascertain that the visual edge definition was not influenced by latitude dependent variations in the solar limb intensity profile and gain additional understanding about the relationship between the solar photosphere and the gravitational equipotential surface.

Previous reliable observations consist of an impressive eleven year undertaking by Schur and Ambroon in which visual observations using a heliometer indicated a small oblateness consistent with the rotation of the sun, and a modern measurement in which Dicke and Goldenberg used electronic techniques to obtain a visual oblateness of
(ΔR/R) ~ 5 \times 10^{-5}. A gravitational quadrupole moment as large as the visual result of Dicke and Goldenberg would be of significance in gravitational theory.

The present oblateness measurement was made using the astrometric telescope at SCLERA (Santa Catalina Laboratory for Experimental Relativity by Astrometry) about 40 miles northeast of Tucson, Arizona. This telescope was designed to minimize optical aberrations and allow quantitative analysis of any residual errors.

Of primary interest is the problem of edge definition. In the present measurement the edge is defined using phase sensitive detection in which a slit is caused to oscillate across the extreme solar limb with relatively large amplitude (~ 7 arc seconds). The intensity of the solar limb profile is measured by a photomultiplier mounted directly beneath the slit.

Using FCT (Fourier Coefficient Technique) this instantaneously defined edge can be related to a corresponding uniquely defined point on the intrinsic solar limb profile so that one can overcome the effects of atmospheric "seeing." In addition, the FCT formalism, which was developed by H. A. Hill, J. R. Oleson, and R. T. Stebbins, provides crucial insight into the problem of latitude dependence of the limb profile. However, this insight only
becomes crucial to the interpretation of a visual oblateness and not in making the measurement itself.

After the edge has been defined, a measure of the relative difference between any two solar diameters is made by interferometrically measuring the change in separation between two slits locked onto diametrically opposite edges of the sun as the detector is rotated between diameters. The pole of the solar image rotates diurnally in the field of view with respect to the semi-minor axis of any oblateness which is caused by optical aberrations or atmospheric differential refraction. This allows one to quantitatively remove systematic errors via a least squares fit.

It is shown that the edge definition does in practice agree with the theoretical predictions of FCT and that "seeing" errors can be eliminated. It was determined that the visual equatorial diameter exceeds the polar diameter by 0.121 ± 0.029 arc seconds. However, closely allied work performed concurrently at SCLERA by Douglas Patz indicates that any attempt to equate this observed visual oblateness with a gravitational quadrupole moment of similar magnitude would not appear justified at the present time.
CHAPTER I

INTRODUCTION TO THE PROBLEM OF SOLAR OBLATENESS

If the magnitude of the solar equatorial radius exceeds that of the polar radius, the sun is said to be oblate. This departure of the visible solar photosphere from spherical shape can be measured using methods which compensate for the fact that the intrinsic shape of the sun is viewed through the earth's atmosphere and an appropriate optical system. A precise measurement of visual solar oblateness would be of significance in determining accurately the gravitational field of the sun and in the subsequent decisions pertaining to proper formulation of the laws of gravitation.

Gravitational Theory

Einstein's theory of General Relativity is presently accepted as the most complete explanation of gravitational phenomena. In this chapter it will be shown that the experimental evidence supporting General Relativity is based mainly on the precession of Mercury's perihelion and that if the sun has an oblate distribution of matter the present agreement between Einstein's theory and observational data would be invalidated.
To emphasize the significance of the effects which could be expected if the sun does not have spherical distribution of mass and to see the importance of a measurement of the visual oblateness, it is helpful to briefly review the historical development of gravitational theory.

Newtonian Gravitation

A major triumph of Newton's theory of gravitation was the ability to theoretically explain Kepler's theretofore empirical laws. This achievement also indicated that the same laws were applicable in celestial as well as terrestrial realms, and that some effects too small to be measured in a laboratory are readily observable in the heavens. As increased observations soon indicated, Newton's laws not only increased our understanding of gravitation, but were far superior to Kepler's laws in accounting for perturbations from the closed and stationary Keplerian elliptical orbits.

So powerful was the relationship between planetary observations and Newtonian gravitational theory that scientists were able to predict the locations of previously unobserved planets. A historical sketch of the process which led to the discovery of Neptune 1846 is given by Abell (1969, p. 318), and he concludes that "the discovery of the eighth planet, now known as Neptune . . . was a major triumph for
gravitational theory and ranks as one of the great scientific achievements . . . "

Calculations similar to those predicting the existence of Neptune and involving huge quantities of planetary observations showed, however, that there were secular deviations of the orbital elements of the inner planets which could not be explained. The greatest discordance was due to the precession of the perihelion of the planet Mercury and was first reported in 1859 by Leverrier who had already been instrumental in the discovery of Neptune. Subsequent improvement in the measurement of this precession (Clemence 1947, Duncombe 1958) served to confirm the anomaly. Futile attempts to explain the effect using Newtonian physics (Newcomb 1895) will be discussed later in this chapter. Thus the planetary observations which had originally indicated that Newton's laws had universal applicability were, as observational accuracy increased, also instrumental in uncovering the deficiencies in his theory of gravitation.

General Relativity

The perihelion precession did not receive a satisfactory explanation until 1915, when Einstein published his theory of gravitation known as General Relativity. Although Newton's laws were shown to be a suitable approximation to his theory in most instances, Einstein (1916) pointed out three now famous tests for which General Relativity
predicted results differing from Newtonian gravitation by an amount large enough to be observable. These tests are: a) the gravitational red shift of light emitted from massive bodies, b) deflection of light in a gravitational field (which effect would be best observable as light from a distant star passes close to the sun), and c) a precession of Mercury's perihelion amounting to 43 seconds of arc per century. The precession predicted by Einstein applies to any orbiting body, but due to the short period of revolution and relatively large eccentricity of the orbit, the effect is most apparent for Mercury. The formula for this precession as given by Einstein (1916) is:

\[ \psi = \frac{24\pi^3 a^2}{T^2 c^2 (1-e^2)} \]  

where \( \psi \) is the precession per revolution, \( T \) the period of the orbit, \( c \) the velocity of light, \( a \) the semi-major axis of the orbit, and \( e \) the eccentricity. This prediction was able to account for the excess precession of Mercury's perihelion of 42 seconds of arc per century (Eddington 1921, p. 124) with astounding success and was a major factor attracting attention to the new theory.

Eclipse measurements of the deflection of starlight were made as soon as possible and, although the accuracy was
poor, indicated that there was indeed such an effect (Eddington 1920). And so General Relativity came to be regarded by most scientists as a superior formulation of the theory of gravitation because it could explain observed gravitational phenomena more adequately than other theories.

Partly due to the lack of other experiments and the difficulty in obtaining precise results for the light deflection and red shift experiments, general relativity developed as a largely mathematical discipline. Many theories were developed and cosmological models proposed which were apparently self consistent and significant in understanding the origins of our universe, but could not be experimentally verified. Professor R. H. Dicke of Princeton University has been one of the leading advocates of the need for experimental verification of the fundamental assumptions upon which various relativistic theories of gravitation are based (Dicke 1964a). Since it appears unfruitful to attempt a differentiation between these theories solely on a philosophical basis, the need for precise experimental results is obvious. The present interpretation of the perihelion precession of Mercury strongly supports Einstein's original theory, but since this is the only measurement of sufficient accuracy to differentiate between theories, it is rightfully the subject of closer examination.
Theoretical Implications of Experimental Observations

To fully appreciate the significance of observations in tests of gravitational theories, it is very enlightening to examine the outgrowth of an observation by Eddington (1924, p. 105) that different terms in the Einstein metric tensor were responsible for the three relativistic effects predicted by Einstein. Eddington's insight has led to a generalized metric in which the various terms can be parameterized in a way which admits a large class of gravitational theories (Thorne and Will 1971, Robertson 1962, Schiff 1967, Schild 1960). The following presentation of this parameterization is taken largely from the article by Robertson, although the work of Thorne and Will (see also Will 1971) is much more extensive (10 parameters).

The most general form of a metric with static spherical symmetry for the gravitational field of the sun may be written in isotropic form as

\[ ds^2 = [1 - 2\alpha \left( \frac{GM}{c^2 r} \right) + 2\beta \left( \frac{GM}{c^2 r} \right)^2 + \ldots ] dt^2 \]

\[ - \frac{1}{c^2} \left[ 1 + 2\gamma \left( \frac{GM}{c^2 r} \right) + \ldots \right] (dx^2 + dy^2 + dz^2) \]

\[ r^2 = x^2 + y^2 + z^2 \]

where \( r^2 = x^2 + y^2 + z^2 \) is the geometrical distance from the
center of the sun, $G$ is the universal gravitational constant, and $\alpha$, $\beta$, and $\gamma$ are coefficients to be determined. The red shift predicted by Einstein is given to a good approximation by

$$\frac{\nu_1 - \nu_2}{\nu_1} \approx \frac{GM}{c^2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

where $\nu_1$ is the frequency of the radiation at a distance $r_1$ from the center of a body of mass $M$. However, since the centripetal acceleration produced far from the origin is $\alpha MG/r^2$, it follows $\alpha$ is absorbed in the empirical definition of the mass $M$. Only if it were possible to determine $M$ by some independent nongravitational measurement would it make sense to ask whether in fact $\alpha$ is exactly unity (Weinberg 1972, p. 184). Therefore this red shift, which is now the most precisely measured of the three tests (Pound and Snider 1965), is of little help in determining the validity of various theories, and is generally acknowledged as a test of the principle of equivalence (Bertotti, Brill, and Krotkov 1962, p. 21).

The deflection of starlight at the solar limb is given in radians by

$$\delta = 2(\alpha + \gamma) \frac{GM}{c^2R}$$

where $M$ and $R$ refer to the solar mass and radius respectively. Experimental measurement generally favors the Einstein
value of $\gamma = 1$ although the accuracy is only 10-20% (Bertotti, Brill, and Krotkov 1962, pp. 30-40), and in obtaining values for the deflection, the inverse dependence upon $r$, the distance from the center of the sun, is generally assumed rather than calculated. At present, an effort is being made under the direction of Henry Hill to measure the $r$ dependence, as well as the magnitude of the deflection to 1% accuracy.

The perihelion precession involves all three parameters as shown in the following equation:

$$\psi = \frac{2\pi GM}{c^2(1-e^2)a} [2\alpha(\alpha + \gamma) - \beta]$$  \hspace{1cm} (1.5)

where $\psi$ is expressed in radians per revolution, $a$ is the semi-major axis of the ellipse and $e$ the eccentricity. The immediately obvious result of the parameterization is to show that, in addition to being the only test where the interaction is strictly between two masses and does not involve electromagnetic phenomena, the precession is the most sensitive test of relativity theories because it measures $\beta$ which involves second order terms in the metric. Table 1 gives a comparison of the predictions of the Newton, Einstein, and Brans-Dicke (1961) gravitational theories. The latter which is also called the Scalar-Tensor theory is included since it is at present one of the more highly
developed alternative relativistic theories and predicts only slightly different observable effects than those predicted by General Relativity. This small difference in predictions makes precise experimental results highly desirable, especially since the theories are based upon significantly different philosophies.

In the Brans-Dicke Theory, \( \omega \) is an arbitrary coupling constant related to the fraction \( s \) of a particle's mass due to the postulated scalar interaction, by the formula \( s = \frac{1}{4+2\omega} \). The Scalar-Tensor theory becomes identical to the Einstein theory as \( \omega \to \infty \), but on various grounds, it has been estimated that \( \omega \approx 5 \) (Dicke and Peebles 1965). Insertion of the values appearing in Table 1 into equations 1.4 and 1.5 indicates that if 7% of a particle's mass is due to scalar coupling with the rest of the mass in the universe then the starlight deflection should be approximately 13/14 of the value predicted by Einstein and the precession would

<table>
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<th>Parameter</th>
<th>Newton</th>
<th>Einstein</th>
<th>Brans-Dicke</th>
</tr>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1 + \omega}{2 + \omega} )</td>
</tr>
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be 10% less than the perihelion precession predicted by General Relativity.

Additional experiments to those proposed by Einstein have been suggested by Schiff (1967) and Shapiro (1964). Schiff proposed measuring the precession $d\Omega$ of an orbiting gyroscope:

$$\frac{d\Omega}{\Omega} = \frac{1}{6}(\alpha + 2\gamma)\frac{m}{r}$$

where $m$ is the mass of the body about which the satellite orbits at a distance $r$. However, the experiment has not yet been performed. Shapiro's work is twofold. Using radar to track the planets, he hopes to reduce the uncertainties in their orbits and thus reduce the size of the errors in observation (see, for example, Shapiro et al. 1972); and he has suggested that when one reflects radar off a distant planet in such a way that the rays pass close to the sun, there should be a time delay. The time delay effect has been called the "fourth test of general relativity" and involves $\gamma$ (Shapiro et al. 1971); present experimental results yield an accuracy of 20%.

Radio delay measurements using the Mariner 7 satellite indicate that $\gamma$ is $0.994 \pm 0.08$ (Anderson et al. 1971) although there are large systematic errors which are not yet fully documented. Attempts have been made to measure the deflection of starlight using long base line radio
interferometry (Muhleman, Ekers, and Fomalont 1970; Seielstad, Sramek, and Weiler 1970). The most accurate results were recently reported by Sramek (1972) and indicate that the deflection is $0.94 \pm 0.06$ the deflection predicted by General Relativity. This number favors the Brans-Dicke theory although it is not definitive.

Thus at the present time, only the perihelion precession is known with sufficient accuracy to offer the possibility of experimental verification of the correct relativistic theory of gravitation.

Perihelion Precession

The problem of determining the size of the relativistic effect in planetary perihelion precession is not so much a matter of observational accuracy as the painstaking efforts required to include all possible sources of perturbation in the theoretical calculations. The magnitude of the relativistic effect is simply the difference between the observed precession and the theoretical perturbation predictions based on classical mechanics. Any error in these classical calculations will be present when one attempts to compare the relativistic residual with the theoretical predictions of various theories of relativity. The sources of error will be made easily apparent by an
examination of the procedure involved in obtaining the observed and predicted values for the precession.

Measured Precession

In examining the manner in which one obtains values for the observed precession, it is instructive to consult a general review article by G. M. Clemence (1947). Observations are made using meridian coordinates of the planetary positions as seen from the Earth and transits of the inner planets across the disk of the sun. The difficulties encountered in reducing these observations are due to the fact that a) Mercury’s proximity to the sun makes it one of the most difficult planets to observe, b) the planetary orbits are not ellipses because of perturbations, and c) the observations are not made in a stationary frame, but are referred to the general precession of the equinoxes. This precession of the equinoxes is determined by observation of distant stars which are all in our galaxy. Considerations involving galactic rotation are thought to show that measurements of the general precession of the equinoxes is accurate to within 0.1 second of arc per century (Bertotti, Brill, and Krotkov 1962, p. 44).

Clemence describes how a least-squares fit with 12 variables, one of them being the perihelion precession, is made on the observations in order to determine the elements
of the planetary orbit. Errors in the results are due largely to observational systematic errors and must be estimated.

Theoretical Values for the Precession

Theoretical motions of perihelia are obtained using Newtonian Mechanics and perturbation theory. To gain an appreciation for the exhaustiveness of these calculations it is helpful to examine Table 2 (Clemence 1947) which shows the calculated theoretical precession for a wide variety of different causes.

Close examination of Table 2 shows that the largest errors were due to the fact that the masses of Mercury and Venus were not well known in 1947. A more recent calculation using new values of planetary masses was performed by Duncombe (1958) with the results shown in table 3.

As seen in Table 3, the precession of Mercury's perihelion certainly indicates that, barring any errors or omissions in Table 2, the Einstein formulation of relativity is the most correct description of gravitation. Following the lead of Professor Dicke, it is wise to reexamine carefully the effects listed in Table 2.

To get some feeling for possible omissions or discrepancies, it is extremely helpful to review the
Table 2. Contributions to the Motion of the Perihelia of Mercury and the Earth

The units of $m^{-1}$ and the perihelion motion are reciprocal solar masses and arcs seconds/century, respectively.

<table>
<thead>
<tr>
<th>Cause</th>
<th>$m^{-1}$</th>
<th>Motion of perihelion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>6,000,000 ± 1,000,000</td>
<td>0.025 ± 0.00</td>
</tr>
<tr>
<td>Venus</td>
<td>408,000 ± 1,000</td>
<td>277.856 ± 0.68</td>
</tr>
<tr>
<td>Earth</td>
<td>329,390 ± 300</td>
<td>90.038 ± 0.08</td>
</tr>
<tr>
<td>Mars</td>
<td>3,088,000 ± 3,000</td>
<td>2.536 ± 0.00</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1,047.39 ± 0.03</td>
<td>153.584 ± 0.00</td>
</tr>
<tr>
<td>Saturn</td>
<td>3,499 ± 4</td>
<td>7.302 ± 0.01</td>
</tr>
<tr>
<td>Uranus</td>
<td>22,800 ± 300</td>
<td>0.141 ± 0.00</td>
</tr>
<tr>
<td>Neptune</td>
<td>19,500 ± 300</td>
<td>0.042 ± 0.00</td>
</tr>
<tr>
<td>Solar oblateness</td>
<td></td>
<td>0.010 ± 0.02</td>
</tr>
<tr>
<td>Moon</td>
<td></td>
<td>7.68 ± 0.0</td>
</tr>
<tr>
<td>General precession (Julian century, 1850)</td>
<td>5025.645 ± 0.50</td>
<td>5025.65 ± 0.5</td>
</tr>
<tr>
<td>Sum</td>
<td>5557.18 ± 0.85</td>
<td>6179.1 ± 2.5</td>
</tr>
<tr>
<td>Observed motion</td>
<td>5599.74 ± 0.41</td>
<td>6183.7 ± 1.1</td>
</tr>
<tr>
<td>Difference</td>
<td>42.56 ± 0.94</td>
<td>4.6 ± 2.7</td>
</tr>
<tr>
<td>General Relativity effect</td>
<td>43.03 ± 0.03</td>
<td>3.8 ± 0.0</td>
</tr>
</tbody>
</table>
Table 3. Differences in Observed Minus Classical Theoretical Perihelion Precession (O-T) Compared to Relativistic Effects Predicted by Einstein

The units of the perihelion motion are arc seconds/century.

<table>
<thead>
<tr>
<th></th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess Precession, (O-T)</td>
<td>43.10 ± 0.44</td>
<td>8.06 ± 5.28</td>
<td>5.01 ± 1.79</td>
</tr>
<tr>
<td>General Relativity, (Eq. 1.1)</td>
<td>43.03</td>
<td>8.63</td>
<td>3.84</td>
</tr>
</tbody>
</table>

classical attempts to explain the complete excess precession before General Relativity was postulated.

Newcomb (1895, pp. 109) examined the various hypotheses which might explain the discordance between theory and observation. He noted the possibility of unknown masses or arrangements of matter and the suggestion by Hall (1894) that Newton's laws should be modified.

The first arrangement of matter considered by Newcomb is an oblate sun, i.e., a distribution of solar matter slightly flattened at the poles and bulging at the equator. He conjectures that this non-spherical distribution could be caused by a rotating core, and concludes that to account for the observed precession, the equatorial diameter should exceed the polar diameter by 0.5 seconds of arc. Although there was some disagreement as to the exact shape
of the sun, there were strong conclusions reached by Airy (1862) and Auwers (1891) that the difference between pole and equator was less than 0''.100 and 0''.050, respectively.

A ring of mass close to the surface of the sun sufficiently large to cause the observed precession would also distort the solar surface to a visible extent. Newcomb examined proposals made originally by Leverrier (1859) who had predicted the existence of an intramercurial planet. This hypothesis offered much promise since Leverrier and Adams had, as mentioned previously, been able to calculate the position of theretofore undiscovered Neptune to within 1°. Despite diligent efforts, no one has been able to observe the postulated planet Vulcan. Newcomb showed that the large angle the plane of revolution of Vulcan would make with the ecliptic, the large required mass, and the fact that such a planet would not account for a smaller but similar precession of Venus' orbit, coupled with the absence of the expected albedo or a dark spot on the sun, made the existence of Vulcan extremely doubtful. Scattered dust and planetoids were also rejected on grounds that anything large enough to cause the measured precession should be observable.

An examination of Hall's contention that the gravitational force between two masses did not vary exactly as the inverse square of the distance between them (this is also what relativity theory asserts) revealed that to account
for the observed precession and not violate other observations the attractive force would be proportional to \((1/r^{2+\delta})\) where \(\delta = 1.57 \times 10^{-7}\).

Now that relativity theory can account for a major portion if not all of the excess precession, these original proposals pertaining to unknown arrangements of matter cannot be dismissed as easily as was the case when they were expected to account for the entire effect. It should be noted that if such an arrangement were found which accounted for only 10% of the perihelion motion, the excellent agreement between Einstein's theory and the presently accepted excess would be invalidated.

An immediately apparent area of concern is the gravitational quadrupole moment of the sun. Clemence in his 1947 article (Table 2) indicated his value for the theoretical precession due to an oblate sun was based on assumptions about the interior constitution of the sun and says "The probable error is very uncertain." The original reasons for disregarding this effect were simply that observations of the visual surface precluded a quadrupole moment large enough to account for the entire discrepancy and that an oblateness this large would cause other secular variations in the elements of planetary orbits which were larger than observations would allow. The mechanism which would cause a sufficiently large oblateness was also difficult to explain.
It is important to examine possible ways, other than the precession of Mercury's perihelion, in which the existence of a solar quadrupole moment would be made manifest. The most fundamental approach to the problem is to compare the observed time variations of planetary orbital elements with those which would occur if there existed a quadrupole large enough to account for Mercury's precession. The formula for the precession caused by an oblate body is given by Danby (1962, p. 261). The total perihelion precession \( \psi_G \), in radians per revolution, is the sum of the nodal regression \( d\omega \), plus the advance of the line of apsides, \( d\Omega \). Assuming the inclination of the planetary orbit with respect to the solar equator to be small,

\[
d\psi_G = \left( \frac{3nJ_2}{a^2(1-e^2)^2} - \frac{3nJ_2}{2a^2(1-e^2)^2} \right) dt \quad \text{I.7}
\]

where \( a \) and \( e \) are the semi-major axis and eccentricity as in Equation (I.1), \( n = (2\pi/T) \) is the mean motion, and \( J_2 \) the magnitude of the quadrupole moment. Equation (I.7) can then be written

\[
\psi_G = \frac{3\pi J_2}{a^2(1-e^2)^2}. \quad \text{I.8}
\]

The magnitude of the quadrupole moment, \( J_2 \), is defined in terms of Legendre polynomials, \( P_2(\cos \theta) \), by the expression for the gravitational potential of an axially symmetric body:
\[ \phi_0 = -\frac{mG}{r}[1 - \frac{1}{r^2}J_2 P_2(\cos \theta) - \frac{1}{r^3}J_3 P_3(\cos \theta) + \ldots] \]

The fact that this precession produced by \(J_2\) varies as the inverse power of \(a\) from the relativistic precession (I.1) offers immediate hope that by looking at Venus and the Earth, as well as Mercury, it will be apparent which effect best agrees with observation. Information for making this comparison is given in Fig. 1. Of special interest in Fig. 1 is the indicated precession for the Earth and Venus if approximately 10% or 4.7 seconds of arc per century, of the total precession of Mercury were caused by a quadrupole moment and the remainder were relativistic precession predicted by the scalar tensor theory with \(\omega \approx 5\). It is obvious that the errors in the residual precession of the Earth and Venus must be further reduced before one is able to differentiate between relativistic theories of gravitation on this basis. Other attempts to put an upper limit on \(J_2\) involve the expected secular change in the inclination of Mercury's orbit. Dicke (1964b) and Shapiro (1965) concluded that a quadrupole moment large enough to account for 20% of the excess precession of Mercury would not be at variance with available observations.
Fig. 1. Comparison of various predictions for planetary precession.
Visual Solar Oblateness as Evidence of a Quadrupole Moment

Professor Dicke (1964a) pointed out that observations of the shape of solar surface would presently offer the best hope of determining $J_2$ with sufficient accuracy to allow a distinction between relativistic theories. The main theoretical problem encountered using this approach concerns the relationship between the visual photosphere and a surface of equal gravitational potential. This relationship is not well understood because of the great uncertainty about conditions in the solar interior. It is generally assumed (see Eddington 1959, p. 282, for example) that if the temperature in the photosphere is constant, the visual surface is an equipotential surface of the total gravitational potential, $\phi$, given by

$$\phi = \phi_0 + \frac{1}{2}\omega^2(x^2 + y^2)$$

where $\omega$ is the rate of the equatorial solar rotation and $x$ and $y$ are coordinates in the plane of the solar equator. This relationship between the visible surface and surfaces of equipotential is based upon the assumption that, at the surface, the matter is a perfect, stress-free fluid. This relationship is assumed by Synge (1962) in his calculations relating the observed shape of the sun to the gravitational potential on the basis of the differential rotation with latitude which is exhibited by the sun. Dicke (1970b) maintains that any deviation of the surface from equipotential
due to localized, surface-field induced stresses must be
negligible because of the highly uniform photospheric
brightness. There are differing opinions concerning this
assertion (Roxburgh 1967, Cocke 1967, and Ingersoll and
Spiegel 1971). An effort to measure anisotropy in the solar
limb profile between the poles and equator is being made in
the Physics Department of The University of Arizona by
Douglas Patz. The results should be very useful in under­
standing the relationship between the shape of the photo­
sphere and the gravitational potential.

The problem may be examined from a different aspect,
however. If a meaningful quadrupole moment does exist, it
is extremely doubtful that the $J_2$ contribution to \( \phi_0 \)
in Equation (1.10) could be canceled by any other mechanisms.
In other words, if a gravitational quadrupole moment does
exist, it would certainly show in the visible surface; how­
ever, a measured visual oblateness does not indicate with
the same certainty the existence of a quadrupole moment.
Thus, an oblateness measurement would at least put an upper
limit on $J_2$ and, pending the outcome of theoretical consid­
erations, give some indication of the magnitude. Using
Equations (1.9) and (1.10) with the assumption that the
visual surface is an equipotential surface, one can obtain
the relationship between $J_2$ and the difference between
equatorial radius, $r_e$, and the polar radius, $r_p$. Keeping
only first order terms and letting \( g_s = (GM/r_o^2) \) one obtains

\[
J_2 = \left( \frac{r_e - r_p}{r_o} - \frac{\omega^2 r_o^2}{2g_s} \right) 2r_o^2. \tag{I.11}
\]

The formula for the precession then becomes

\[
\psi_G = \frac{2\pi}{T} \left( \frac{r_o}{a} \right)^2 \left( \frac{\Delta - 1.0 \times 10^{-5}}{(1 - e^2)^2} \right), \tag{I.12}
\]

where \( \Delta = (r_e - r_p)/r_o \). A quadrupole moment large enough to cause a precession of 4.7 seconds of arc per century for Mercury as in Fig. 1 would then require that

\[
\frac{r_e - r_p}{r_o} = 5.0 \times 10^{-5}.
\]

When measuring the difference between the equatorial and polar diameters \( D_e - D_p \), one should expect to see a difference of 20 milliseconds of arc due to the rotational term and a total of 100 milliseconds if there is a quadrupole of sufficient magnitude to cause 10% of Mercury's precession. The absence of a diameter difference greater than 20 milliseconds would definitely indicate that the present General Relativistic explanation accounting for the complete residual precession is valid.

At this point it seems appropriate to mention that one reason people have been prone to accept Einstein's precession without demanding a more detailed search for the
quadrupole moment, is that it is difficult to formulate a solar model which incorporates any mechanism for causing an oblate distribution of matter. This problem has occupied a prominent place in the literature (Dicke 1970b; Howard, Moore, and Spiegel 1967; and Goldreich and Schubert 1967a,-b, 1968) since Dicke's original suggestions concerning the need for a better experimental basis of gravitational theory were made. If a visual oblateness greater than 20 milliseconds does actually exist, the ultimate explanation will require some mechanism which is not now well understood or accepted. This will be the case whether the oblateness is the result of an oblate distribution of matter and/or the result of some surface phenomenon.

Examination of Previous Solar Oblateness Measurements

In view of the situation discussed in the preceding section, one should now take great interest in the examination of previous solar oblateness measurements. Historically, the measurements fall into two categories: classical and modern. The classical methods involve the use of meridian circle transits, heliometers and photographs; all require the use of human vision and judgment. The modern methods use electronic phase sensitive detection and are completely automated. A brief discussion of the methods and results for each type of measurement will indicate why there
is still some uncertainty regarding the shape of the sun despite a classic statement made by the Astronomer Royal, G. B. Airy Esq. in 1862.

It has been proposed lately to prepare an apparatus for the purpose of examining whether the Sun's disk is really circular, and, in particular, for ascertaining whether the diameters nearly perpendicular to the ecliptic are equal to those nearly parallel to the ecliptic. I would not by any means discourage the trial of such apparatus, but I would unhesitatingly express my opinion that the result of the trial would be to show whether the apparatus is or is not trustworthy, and not to give any new information regarding the measure of the Sun's diameters in any degree comparable to that which we already possess.

Meridian Circle Transits

The mechanical details of a meridian circle transit are given in Chauvenet (1960) and Smart (1965). Basically the instrument allows an observer to record the instant in time when a star or a solar limb passes through the local meridian. This is accomplished by having the axis about which the telescope rotates as it is raised and lowered lie along a horizontal line running precisely from east to west. The field of view as the telescope is rotated sweeps along the local meridian. A series of spider webs or "wires" are installed to be parallel to the meridian in the focal plane and one or more threads is perpendicular to the meridian.

To measure the horizontal diameter of the sun, the observer records the time the limb passes over, i.e., is tangential to each of seven wires, three being on either
side of the meridian. From the mean of these observations the instant that the limb was coincident with the meridian may be obtained, and a knowledge of the Earth's rate of rotation gives the diameter of the sun. Recording the time of coincidence has been mechanized (Gething 1955, Cullen 1926) but an individual observer must still determine when the wire is tangent to the limb. The vertical diameter is measured by raising or lowering the telescope so that the thread perpendicular to the meridian is tangent to the limb. The declination of each limb is then determined using a micrometer which gives fine adjustment to the thread in connection with an accurately calibrated circle attached at the axis or rotation.

Meridian circle measurements were made at Greenwich as early as 1750 and have never given satisfactorily an accurate value for the mean diameter of the sun or a measured difference between the pole and equator. Interesting developments in the analysis of meridian circle observations have included: many claims that the diameter varies with time (see Gething 1955 and Poor 1908 for references), a highly developed system for finding the "personal equation" of an individual observer (Auwers 1886, 1887, 1889; Newcomb and Holden 1874; Cullen 1926; and Gething 1955), the realization that the measured diameter changes in the summer because of increased irradiance on the instrument (Cullen
1926), and the apparent increase in diameter when "seeing" is poor or there is cloud cover (Newcomb and Holden 1874). The works by Auwers are definitive in indicating that little knowledge about oblateness is obtained from these measurements. Poor (1908) traces these developments in detail.

Photographic Methods

Hayn (1924), Weber (1933), and Poor (1908) all sought to measure a solar image on photographic plates. The difficulties involved centering of plates in the measuring machine, bending of the plates during exposure and a poorly defined limb. Hayn accounted for "seeing" by taking many short exposures on the same plates. He remarks that the harder one tries to increase the precision, the greater the problems caused by systematic errors become, and says that "seeing" prevents more accurate results than he obtained. He concludes that, pending further events, one is justified in assuming the sun is a perfect sphere, the difference in diameters being between 30 and 10 milliseconds. Poor had a probable error (0.250 sec.), too large to contribute detailed information. Weber, although thoroughly examining the problem, does not present any meaningful results.

Schaub (1938) reanalyzes Hayn's measurements, making slight corrections but agreeing that any difference in diameters was less than 20 milliseconds. Ashbrook (1967) reviews the earlier works and indicates that the full possibilities
of the photographic technique have never been fully ex-
ploited. The problems here are much the same as determining
the deflection of starlight, however, and the photographic
method has not given completely satisfactory results in this
area either.

Heliometer Measurements

Aside from the efforts of Hayn, it is only through
the use of the heliometer that one has obtained classical
results which inspire any degree of confidence. A heliom-
eter is a refracting telescope with the objective cut in
half along a diameter. Each half of the objective forms its
own image; and by sliding the two half lenses past each
other with accurately calibrated micrometers until the
opposing solar limbs are tangent, one can measure the angu-
lar separation. The presently accepted value for the mean
diameter of the sun of 1919.26 seconds of arc was arrived at
in 1891 (Auwers) using this method and the fact that this
value has not been revised since that time is a remarkable
tribute to the painstaking attention to detail involved in
the heliometer observations and the accompanying analysis of
data.

The efforts of Schur and Ambronn (1895) and Ambronn
(1905) to measure the mean solar diameter and the difference
between equatorial and polar diameters over a period of 22
years gave extremely trustworthy results that oblateness
must be less than 10 milliseconds of arc. In fact, the
difference in diameters in both independent cases was less
than the mean error. Later analyses of their data (Hayn
1924, Poor 1908, and Ambronn 1906) showed that the differ­
ence in diameters could be at most 20 milliseconds of arc,
and that any variation of the sun's diameter with time must
have an amplitude of less than 100 milliseconds. Examina­
tion of the extreme attention given to sources of systematic
errors and of the rigorous care in making observations
manifested by Schur and Ambronn elicits widespread accep­
tance and admiration for their work (see, for example,
Ashbrook 1967, Hayn 1924, and Poor 1908). The only criti­
cism one could possibly make would involve the fact that
these observations relied on human judgment although the
length of the project and the smallness of the mean error
certainly seem to compensate for such sources of error.

In summary, the best classical measurement of Schur
and Ambronn completely eclipses other efforts and indicates
that any oblateness of the sun would at most be in accor­
dance with the expected value due to rotation and would thus
preclude the existence of a quadrupole moment large enough
to have significance in the perihelion precession of Mercury.

Modern Measurements

The increased interest in experimental relativity
and the development of new technology led to an effort to
look for a visual oblateness using phase sensitive detection (Dicke and Goldenberg 1967). This method was the first measurement which used electronic instrumentation and thus eliminated "personal equations" and commands attention because it reflected a whole new philosophy of measurement and produced unexpected results. The basic apparatus consists of a slotted disk spinning parallel to the focal plane of a coelostat in such a way that two diametrically opposite slots allow the intensity of light from the solar limb to be measured by phototubes. One uses the information from the two phototubes to keep the image centered and to look for a correlation between the relative intensity and the slot position. A correlation would be expected if the shape of the image were not spherical. Dicke and Goldenberg indicated that according to their analysis the polar diameter of the sun was 0.096 ± 0.013 seconds of arc smaller than the equatorial diameter.

The deviation of this result from the well accepted German values, plus the attempt to show that this visual oblateness implied a quadrupole moment large enough to cause a precession of 4.3 seconds of arc per century in Mercury's perihelion have generated much controversy as mentioned in the section titled "Theoretical Implications of Experimental Observations" which begins on p. 6. The fact that the only observation which supported Dicke's theoretical views was
made by Dicke himself did not help the situation. This situa-
tion is discussed in the current literature and will not be repeated here (Dicke 1970a, -b).

In the interest of generating independent tests of relativity, Carl Zanoni and Henry Hill of Wesleyan University designed and constructed in cooperation with The University of Arizona a special telescope to make astro-
metric measurements, principally of the deflection of star-
light by the Sun (Zanoni 1966). Zanoni and Hill proposed to extend the use of phase sensitive detection to the problem of oblateness in a direct measurement of diameter differ-
ences by locking oscillating slits on the inflection point of the solar limb and measuring the slit separation. By rotating the slits from the equatorial to the polar diameter and noting the change in separation, one could obtain the first direct measurement of diameter differences not depend-
ing on human judgment and an alternative check on the Dicke-
Goldenberg results. Although this project was not carried to fruition as originally planned by Zanoni, it paved the way for the presently described effort and gave valuable in-
sight into the problem of defining the solar limb using phase sensitive detection.

At present, there is considerable question as to the shape of visible surface of the sun. One must choose be-
tween Schur and Ambronn's (Ambronn 1905) spherical
conclusions and Dicke and Goldenberg's (1967) oblateness without a valid reason to explain the discrepancy. If Dicke is correct in assuming that the observed oblateness means that 4.3 seconds of Mercury's perihelion precession is due to a quadrupole moment, then the remaining precession can be accounted for in the Brans-Dicke theory and Einstein's theory is ruled out. If Schur and Ambronn's result is correct, then it is improbable that the sun has a quadrupole moment and the perihelion precession of Mercury strongly suggests that General Relativity is correct.
CHAPTER II

THEORETICAL CONSIDERATIONS INVOLVED IN AN OBLATENESS MEASUREMENT

As demonstrated in the preceding chapter there is a need for additional measurements to determine the magnitude of the visual solar oblateness. It is the purpose of this chapter to describe a reliable method of measuring the oblateness by using phase sensitive detection to lock oscillating slits onto opposing edges of the solar image and by monitoring the changes in separation between the slits when the detector is rotated between orthogonal diameters. It is extremely important to control possible sources of systematic error so that their effect on the results of the measurement will be negligible.

There are three basic areas where systematic errors could influence the accuracy of the experiment. First, there is the relationship between the shape of the observed solar image and the intrinsic dimensions of the sun. Second is the problem pertaining to the definition of the solar edge. And third is the actual relative measurement of solar diameters. In the following sections each of these problems will be treated in detail.
Relationship of the Observed Solar Image to the Intrinsic Shape of the Sun

The solar oblateness measurements have been made at the Santa Catalina Laboratory for Experimental Relativity by Astrometry (SCLERA). The facilities at SCLERA, located in the Catalina Mountains northeast of Tucson, Arizona, are under the direction of Professor Henry A. Hill, and have been supported by the National Science Foundation, The University of Arizona, and Wesleyan University.

The astrometrical telescope at SCLERA is particularly suited for a solar oblateness measurement because it was designed and built with the intent of eliminating the effects of systematic errors in astronomical measurements. The basic telescope was designed by Dr. Hill and a Princeton University graduate student, Carl A. Zanoni. Dr. Zanoni's thesis (1966) concerns the telescope design and will be the main reference for the remainder of this section.

A unique feature of the astrometrical telescope is the use of the solar diameter to determine the scale of the field. Normalizing distances in the field to the magnitude of a simultaneously measured solar diameter compensates for all first order optical aberrations. Development of an apparatus which measures changes in the observed solar diameter is of general importance for the program of daytime astrometry, as well as providing a measurement of solar oblateness.
As used in the solar oblateness experiment (see Figures 2 and 3) the telescope consists of a flat window whose normal points toward the observed object, two flat mirrors, a five inch, f/100 singlet objective lens, and an intricate array of electronics which causes the telescope to automatically track the sun and keep the solar image fixed in the field. The mirrors are used in an elevation-azimuth configuration so that the principle optic axis of the telescope remains vertical.

After passing through the window, light strikes the first mirror (Mirror I) which is mounted in a cylinder at a 45° angle with respect to the center-line of that cylinder. This center-line always lies in a plane parallel to the surface of the Earth, although the angle it makes with local north-south line is varied so that the cylinder is always perpendicular to the Azimuthal plane of the sun. The elevation is changed by rotating Mirror I and the cylinder about the center-line; this means that the normal to the first mirror lies in the plane determined by a line pointing toward the observed object and the cylinder center-line. The reflected light from Mirror I is parallel to the center-line and is reflected vertically downward through the objective by Mirror II which is also mounted at 45° to the center-line of the cylinder. The normal to Mirror II lies in a vertical
Fig. 2. View of telescope at SCLERA.
Fig. 3. Cross sectional view of telescope.
plane determined by the center-line of the cylinder and the vertical line corresponding to the principal optic axis. The position of both mirrors is controlled by feedback so that the image of the sun is always projected vertically down the telescope.

Zanoni (1966) deals with the various reasons for choosing a telescope with this configuration, but a most valuable feature, in addition to the smallness of the aberrations, is the fact that the pole of the solar image in the field rotates through approximately 270° during the day in a well determined manner with respect to projections of the various optical elements. This allows one to easily separate in a day's data the instrumental aberrations from the intrinsic shape of the sun. This capability is much more satisfactory for this purpose than reliance upon the motion of the position angle of the solar axis of rotation as required, for example, in a heliostat-fed system. The actual motion of the solar pole with respect to various types of aberrations will be derived following a discussion of the major sources of error.

Atmospheric Refraction

Atmospheric refraction is potentially the largest source of error and is not peculiar to the astrometric
telescope of SCLERA. When light from any celestial object passes from the vacuum of space into the earth's atmosphere, it is refracted according to Snell's law. The apparent zenith angle, \( z_a \), is always smaller than the actual angle \( z \). The difference \( z - z_a = (n - 1) \tan z \) (Smart 1965, p. 62) where \( n \) is the index of refraction of air at the telescope. Because different segments of the solar disk have different zenith angles, they are refracted by different amounts and the image of the sun is distorted.

Zanoni (1966, p. 87) shows that for the sun, the difference between the actual angular diameter and the observed diameter, \( \Delta \theta_v \), for a diameter in the plane of an observer's local meridian is

\[
\Delta \theta_v = (n - 1) \theta_\odot \sec^2 z
\]

where \( \theta_\odot \) is the angular diameter of the sun and \( z \) is the colatitude. For the diameter orthogonal to the meridian, the change in angular diameter is

\[
\Delta \theta_h = (n - 1) \theta_\odot .
\]

In both cases, the observed diameter is smaller than the intrinsic diameter, and for \( z \) equal to 45°, and \( n-1 \approx 43 \) seconds of arc at an altitude of 8400 ft., an oblateness of nearly 1/2 second of arc results.

Although this effect is large, past experience has shown that the agreement between observation and theory for
the refraction is excellent, and one must simply make the theoretical correction to the observed diameter. The apparent angular diameter of the sun, \( \theta_0 \), varies annually by \( \pm 2\% \) due to the earth's elliptical orbit, but the daily values are available in a solar ephemeris. The index of refraction of air varies with temperature and pressure and is given by Penndorf (1957) as

\[
n - 1 = (n_s - 1) \left( \frac{1 + \alpha T_s}{1 + \alpha T} \right) \frac{P}{P_s}
\]

where \( n \) is the index of refraction at temperature \( T \) in °C and at pressure \( P \), \( n_s \) the index of refraction at temperature \( T_s \) in °C and at pressure \( P_s \) and \( \alpha^{-1} = 273° \). Values of \( n_s \) for various wavelengths are tabulated at \( T_s = 15°C \) and \( P_s = 760 \) mm. of Hg, so if one measures the value of temperature and pressure to 0.1%, the correction for atmospheric refraction will be accurate to at least one millisecond of arc. All of the above arguments assume the existence of a plane, laminar atmosphere.

Another major problem arises because of the inhomogeneity and fluctuations found in the earth's atmosphere. This effect, which is the reason that stars appear to twinkle, blurs the image and is called "seeing." The full width at half maximum (FWHM) of the image which results as the light from a point source is observed through the atmosphere is typically between 1 and 10 seconds of arc.
"Seeing" also smears out the edge of the sun and becomes crucial in the problem of edge definition but does not distort the general shape of the solar image unless the "seeing" is anisotropic. These problems will be considered fully in the section which deals with the edge definition.

Aberrations Produced by the Optical System of the Telescope

In addition to the aberrations in the shape of the solar image caused by the passage of light through the earth's atmosphere, one must consider those aberrations due to the passage of light through the appropriate optical system. One has somewhat more control over the optical elements than the atmosphere and by judicious design and careful fabrication, it is possible to greatly reduce the size of systematic aberrations from that which one might ordinarily expect. The basic design of the telescope with a long focal length and a vertical optic axis reflects these considerations, and the small remaining aberrations are tabulated by Zanoni (1966). A discussion of the magnitude and effects of these residual intrinsic aberrations follows.

Even though the atmosphere, the window, and the lens cause dispersion (i.e., light of different wavelengths focuses at different distances from the objective), one can avoid problems in this area by choosing to make observations in a narrow spectral region and then placing the detector at
the appropriate focal length. As shown by Zanoni (p. 44), the depth of focus of ± 1 cm for a 40 foot focal length objective greatly reduces the precision to which the focal length for a given wavelength must be known. This distance is easily determined by measuring the focal length of the lens using laser light and calculating from the known relationship between the index of refraction and wavelength of light, the proper focal length for any desired wavelength.

Making measurements at a particular wavelength has additional important advantages. The fact that the correction for the oblateness caused by atmospheric refraction depends on wavelength has already been discussed. The visible photosphere of the sun emits radiation whose continuous spectrum is similar to black body radiation. This continuum is broken by the Fraunhofer absorption lines of particular elements located in and above the photosphere, and by emission lines from the corona and chromosphere. To reduce possible effects due to the contribution of one particular element, one can locate the filter in such a place as to transmit light predominantly in the continuum. It is fortuitous that a series of Mercury lines (4047Å, 4358Å, 5461Å, 5770Å, and 6907Å) lie in the continuum and that high quality but inexpensive 100Å bandwidth interference filters for these lines are readily available. The solar oblateness at different wavelengths may then be measured and compared.
The capability to probe the shape of the sun in spectral regions where absorption lines exist is an exciting possibility to be explored in future work (SCLERA Progress Report 1972, p. 55).

Thus by making observations at a particular wavelength, problems normally associated with chromatic effects not only become minimal, but the narrow band approach actually produces greater information. The remaining types of third order aberrations are spherical aberration, coma, astigmatism, distortion, and field curvature. For the small 3° field in the astrometric telescope only third order aberrations need be considered and the previously mentioned depth of focus eliminates problems due to curvature of the field, especially when the image of the sun in the field and the detector are stationary with respect to the optic axis of the telescope.

Spherical aberration is due to the fact that light rays traveling through different sections of the spherical lens focus at different depths. This effectively smears out the image at a given focal plane, but the image is symmetric when the lens is properly centered with respect to the aperture. Zanoni (1966, p. 47) has shown that for the astrometric telescope the contribution of third order spherical aberration is 0.026 seconds of arc. Since the aberration is symmetrical and much smaller than the most favorable seeing
or the 1 arc sec diffraction width of the objective, this effect becomes negligible. The window and mirrors contribute to the spherical aberration only if they deviate from their intrinsic flatness, and this will be discussed later.

Coma is the distortion of an image due to light rays which enter the optical system "off axis" and Zanoni (1966, p. 47) has shown that by choosing the proper shape factor for the objective, the coma at 5000Å can be made zero and that coma changes by less than 0.004 seconds of arc per 1000Å at 3° off axis. Since the lens is tipped, to center the field and to remove secondary reflections, the image of the sun is located one and one half degrees from the optic axis, and the maximum change in the solar oblateness resulting from the third order coma of the objective would be less than 0.0007 seconds of arc per 1000Å at 5000Å. Again, as in the case of spherical aberration, the ideally flat mirrors and window do not contribute to coma.

Astigmatism is caused by the fact that light rays passing through an objective in different planes do not focus at the same spot. For an image off the principal optic axis the resulting image will be asymmetrical. Since the objective is tipped 1.5° from the vertical axis of the telescope, the astigmatism can be as large as 1.1 seconds of arc. The worst case difference in diameters could be as large as 0.6 seconds of arc (Zanoni 1966, p. 47), or it
could be zero depending on the placement of the detector. Fortunately there are two ways in which this problem may be resolved. The first insight comes with the realization that each point source is smeared into an ellipse whose major axis runs normal to the line about which the objective is tipped. Orthogonal diameters of the sun will then have slightly different observed limb profiles due to differential smearing. This would typically have the effect of changing the effective "seeing" from one diameter to another by less than 1%. As will be shown in the section titled Unique Definition of the Solar Edge, the method of edge definition will reduce errors of this sort.

As verification of the validity of the preceding argument and to determine the actual magnitude of the effect, one can make use of the fact that the projection of the solar pole in the field rotates in a diurnal manner with respect to the astigmatism which is fixed to the projection of the objective into the field. A least squares analysis of the measured solar oblateness will indicate the magnitude of any component which rotates with the projection of the lens and will thus separate instrumental contributions from any intrinsic solar oblateness.

In practice one has additional aberrations because the optics are not perfect and because thermal gradients alter the flatness of the mirrors and window. Zanoni (1966,
p. 48) shows that the coma from the objective in the worst case consistent with the requirement that the surface figure be good to \((\lambda/20)\), is 1 second of arc. One may again argue that the corresponding effect on the definition of the solar edge will be negligible, especially since one does not expect the lens to have the worst possible figure. Since this aberration is fixed to the lens, the least squares fit technique is an adequate (final) solution. On page 54, Zanoni (1966) states that for the same \((\lambda/20)\) tolerances on the window, the worst case aberrations are similar to those for the objective. As will be shown, these aberrations also rotate in an independent manner with respect to the solar pole.

The window has a wedge of 15 seconds of arc to move the secondary reflection image so that it does not coincide with the primary image. The resulting secondary image is shifted 46 seconds of arc and has an intensity 0.0016 times that of the primary image. The resulting effect on the definition of the edge will be discussed in the section on the definition of the solar edge.

For the mirrors the worst case coma consistent with optical tolerances will be 2.8 seconds of arc for Mirror I and 1.9 seconds of arc for Mirror II. Previous arguments in this section indicate that any resultant effect on an oblateness measurement will be small and is easily removed.
A far more serious problem arises if the mirror has an intrinsic curvature. Because the aperture stop is displaced from the tipped mirror, the principal rays for the opposing limbs in the plane determined by the 45° rotation of the mirror will not have the same angle of incidence. This effect is discussed by Zanoni (1966, p. 61). Using his formalism, one can see that an instrumental oblateness as large as 100 milliseconds could possibly result if the surface of Mirror I had a dishing of (λ/10). The effect would be much less for Mirror II because it is closer to the aperture. Interferograms of the finished mirrors indicate that one should not expect effects substantially larger than 100 milliseconds, and once again the resulting aberrations rotate in the field fixed to the projection of the mirrors.

One might expect that the aberrations caused by thermal effects are likely to be the major source of error in an oblateness measurement, largely because these effects vary with time and are not necessarily fixed with respect to the optical elements. This circumstance makes it more difficult to obtain an analytic fit as in the case of all other aberrations which are constant in magnitude and fixed to the projection of the pertinent optical element in the field. The temperature effects which cause the scale of the field itself to change slowly with time will not be harmful if relative diameter measurements are made in a short period of
time. Zanoni (1966, p. 55) has shown that the aberrations caused by thermal distortion of the window will be negligible if the temperature around the perimeter is uniform to 0.1°C.

It is the aluminum coated mirrors which are most sensitive to thermal problems, especially since they absorb 10% of the incident energy. Zanoni (1966, p. 58) shows that if the mirrors are in an evacuated environment, a well defined thermal gradient between the front and back surfaces results when the back is kept at constant temperature. A constant gradient of 2°C causes the mirrors to assume a radius of curvature equal to $3.5 \times 10^6$ cm. Using formula V.15 (Zanoni 1966, p. 61), one can show that the resulting aberration which depends linearly on the distance of the curved mirror from the aperture stop, should cause an apparent oblateness of 0.072 seconds of arc. (Zanoni's numerical results are not in agreement with his formula.)

For the present measurement, the mirrors are not in an evacuated environment and are cooled by convection. The resulting air currents do cause additional seeing, but the front of the mirrors will only rise a few degrees above ambient temperature. One can calculate the expected temperature difference between the front and back of the mirror. The calculation is greatly simplified if one can show that
the time constant with which the temperature at the back of
the mirror responds to temperature changes at the front is
relatively small. Then the transient terms become negligi­
ble and one can apply certain constraints to the form of the
final solution.

The order of magnitude of the thermal time constant
is given in Braddick (1963, p. 85). Consider a slab whose
faces are initially at the temperature of the surroundings
at time \( t = 0 \), with a temperature excess inside the slab
whose magnitude is a sinusoidal function of distance from
the edge. Braddick shows that in this instance the solution
to the heat conduction equation is

\[
\theta = \theta_0 e^{-t/\tau} \sin \frac{\pi z}{\lambda}
\]

where \( \theta \) is the temperature at time \( t \) and location \( z \), \( \tau \) is
\( (\ell^2 \rho / \pi^2 k) \), \( \ell \) is the thickness of the slab, \( \rho \) is the specif­
ic heat, \( \rho \) the density and \( k \) the thermal conductivity. This
calculation can be made even more appropriate for the mirror
problem by regarding the mirror as being 1/2 the thickness
of the slab. Then one has a gradient from front to back.
The resulting time constant using \( k = 0.0033 \text{ cal/cm sec } ^\circ \text{C} \),
\( c = 0.19 \text{ cal/gm } ^\circ \text{C} \), and \( \rho = 2.1 \text{ gm/cm}^3 \) for the 2 inch thick
fused silica mirrors is \( \sim 10^3 \) seconds or 17 minutes.
Braddick points out that since any initial temperature
distribution can be expressed in a Fourier series, the time constant for the sinusoidal term calculated above would be the most significant effect in every case.

Since the response of the back of the mirror to temperature changes at the front is so rapid, one can assume that under normal conditions the time rate of change of the temperature is the same at all points even though the magnitude may be different. Measurements indicate that the temperature of the air in the mirror housing rises throughout the day. An appropriate solution to the heat flow equation

\[ \frac{\partial^2 \theta}{\partial z^2} = \frac{c \rho}{k} \frac{\partial \theta}{\partial t}. \quad \text{II.5} \]

is then

\[ \theta = R t - g(z) \quad \text{II.6} \]

where \( R \) is the rate at which the ambient temperature increases and \( g(z) \) describes the spatial temperature gradient from front to back. Solving for \( g(z) \) one obtains

\[ g(z) = \frac{c \rho R z^2}{2k} + Az + B. \quad \text{II.7} \]

By applying the boundary condition \( g(0) = 0 \) for the front and \( (dg/dz)_{z=l} = 0 \) at the back, the final solution

\[ \theta(z,t) = R t + \frac{c \rho R}{k} \left( l z - \frac{z^2}{2} \right) \quad \text{II.8} \]

is obtained. The temperature difference between the front
and back is

$$T_F - T_B = \frac{cR}{2k}.$$  \hspace{1cm} \text{II.9}$$

For $R$ equal to 1°C per hour, the difference is 0.4°C. Comparing this gradient with the 2\(\frac{1}{2}\)°C expected for vacuum operation (Zanoni 1966, p. 62), it is concluded that the magnitude of the systematic oblateness due to thermal distortion of the mirrors will be about 0.012 seconds of arc. In the afternoon the rate of increase in the ambient temperature becomes small and the effect is even less.

Another area of concern when the mirrors are not in an evacuated environment is the possible distortion due to the fact that the light must pass from the outside air with the index of refraction $n_0$, through the window and into air with a different index of refraction $n_1$. One can intuitively argue that this effect will be small because the sunlight strikes the window almost normally and because the inside air looks like a homogeneous parallel layer normal to the light as it travels from the window to the objective. An expression for the solar oblateness produced by this effect can be obtained by subtracting formulas II.1 and II.2 and changing $n-1$ in these equations to $(n_0 - n_1)/n_1$ where $n_0$ and $n_1$ are the refractive indices of air outside and inside the window, respectively. The resulting difference in solar diameter, $\Delta$, is
where $\theta_0$ is the angle between the normal to the window and the principle solar ray. This angle is typically less than a degree. For a 1% difference in $n-1$ from outside to inside, and $\theta_0 = 1^\circ$, the resulting solar oblateness by formula II.10 is $\Delta = 1.2 \times 10^{-4}$ seconds of arc which is certainly negligible.

Mirror I and the window are fixed with respect to each other, and the projections of Mirror II and the objective into the field also move together. Therefore in making a fit, one must assume a magnitude and orientation for a superimposed oblateness due to the Mirror I-window effects and a separate magnitude and orientation for the total Mirror II-lens contribution. In the following section, the relative rotation of this instrumental oblateness with respect to the solar pole will be derived.

Orientation of the Polar Axis of the Sun in the Field

In the preceeding sections, it was heavily emphasized that even though the aberrations are extremely small, one can still check for systematic errors because the polar axis of the sun rotates in the field with respect to these aberrations. This rotation not only reduces the systematic
error by reliance upon averaging, as previous experiments have done, but enables one to assume an instrumental oblateness of arbitrary magnitude and orientation with respect to a particular combination of optic elements in addition to the intrinsic oblateness of the sun. One can then perform a least squares fit which will separate any superposition of systematic effects.

To show the matter in which the pole of sun and the instrumental oblateness rotate with respect to each other in the field, it is profitable to use an approach worked out by Dr. James R. Oleson which gives the projection in the field of any vector passing through all or part of the optical system. For the geocentric celestial sphere shown in Fig. 4, the Azimuthal angle A and the Zenith angle Z of the sun located at S are defined with respect to the local coordinate system of the observer. The unit vector \( \hat{n} \) is vertical, \( \hat{j} \) is along the local north and the line ON is the polar axis of the Earth, N being the North Pole. The parallactic angle is \( \eta \) and \( \hat{\psi} \) is a vector originating at the Sun tangent to that meridian which passes through the Sun and is perpendicular to the equator of the Earth. \( \hat{\nu} \) is a vector originating at the Sun and tangent to the local azimuthal meridian, and \( \hat{h} \) a vector originating at the Sun, tangent to the sphere and orthogonal to the local meridian. The unit vector \( \hat{h} \) is normal to the window and points toward the Sun, the unit vector
As described in the text, this coordinate system is used to determine the projection of the solar pole in the field. The astronomical triangle is SNZ and the parallactic angle is given by $\eta$. 

Fig. 4. Celestial coordinate system.
\( \hat{p} \) is normal to Mirror I, \( \hat{q} \) is the direction of the center-line of the cylinder (see Chapter II, the section titled Relationship of the Observed Solar Image to the Intrinsic Shape of the Sun), and \( \hat{q} \) is normal to Mirror II. The vertical axis of the telescope is in the same direction as \( \hat{r} \).

The transfer function for a plane mirror is the dyadic

\[
\hat{\Omega}_p = \hat{1} - 2\hat{p}\hat{p}
\]

II.11

where \( \hat{p} \) is the normal to the mirror and in the case of Mirror I can be written as \( \hat{p} = (\hat{\alpha} + \hat{\omega})/\sqrt{2} \). Similarly, \( \hat{q} = (-\hat{\alpha} - \hat{\omega})/\sqrt{2} \) for Mirror II, and the transfer function of the mirrors operating on a vector \( \hat{A} \) gives

\[
\hat{A}' = \hat{\Omega}_q \cdot \hat{\Omega}_p \cdot \hat{A}
\]

II.12

or

\[
\hat{A}' = \hat{A} - 2\hat{q}(\hat{q} \cdot \hat{A}) - 2\hat{p}(\hat{p} \cdot \hat{A}) + 4\hat{q}(\hat{q} \cdot \hat{p})(\hat{p} \cdot \hat{A})
\]

II.13

where \( \hat{A}' \) is the transformed vector. In terms of \( \hat{1}, \hat{2} \), and \( \hat{3} \) the following vectors may be expressed:

\[
\hat{H} = H_0[(\cos A)\hat{1} - (\sin A)\hat{2}]
\]

II.14a

\[
\hat{V} = V_0[(-\sin A \cos Z)\hat{1} - (\cos Z \cos A)\hat{2} + (\sin Z)\hat{3}]
\]

II.14b

\[
\hat{V} = \psi_0(\hat{V} \cos \eta + \hat{3} \sin \eta)
\]

II.14c

where \( H_0 = |\hat{H}| \), \( \hat{H} = \hat{H}/H_0 \), \( V_0 = |\hat{V}| \), \( \hat{V} = \hat{V}/V_0 \), \( \psi_0 = |\hat{V}| \), and \( \hat{V} = \psi/\psi_0 \). By applying II.13 and inverting the \( \hat{1} \) and \( \hat{2} \)
components because of the passage through the objective, the following results are obtained:

\[ H' = H_0 [-\hat{1} \cos(A + Z) + \hat{j} \sin(A + Z)] \quad \text{II.15a} \]
\[ V' = V_0 [\hat{1} \sin(A + Z) + \hat{j} \cos(A + Z)] \quad \text{II.15b} \]
\[ \psi' = \psi_0 [\hat{1} \sin(A + Z + \eta) + \hat{j} \cos(A + Z + \eta)] \quad \text{II.15c} \]

where the primed vectors are the projection in the field.

The vertical component of the atmospheric refraction \( \Delta D_V \) (Eq. II.1) is projected along the vector \( \hat{V} \) which makes an angle of \( A + Z \) with North, and the horizontal component of atmospheric refraction, \( \Delta D_H \), (Eq. II.2) is projected along \( \hat{H} \) which makes an angle \( A + Z + \pi/2 \) with North. Mirror I and the window are projected into the field in the same manner as \( \hat{H} \) and \( \hat{V} \). Thus aberrations from these two elements will rotate with some constant angle, \( \alpha_1 \), plus \( A + Z \). The angle \( \alpha_1 \) is an arbitrary angle that the pole of the superimposed instrumental solar oblateness makes with respect to the semi-minor axis of Mirror I. Even though the field rotation is the same, one can still distinguish between a Mirror I-window caused solar oblateness and one caused by the atmosphere because of the \( \sec^2 z \) dependence of the atmospheric solar oblateness. The minor axis of any Mirror II-lens caused solar oblateness rotates with an angle \( \alpha_2 + A \) in the field where the angle \( \alpha_2 \) is an arbitrary angle the
pole of the superimposed instrumental solar oblateness makes with respect to the semi-minor axis of Mirror II. Finally, as defined in the Explanatory Supplement to the American Ephemeris and Nautical Almanac (1961), the pole of the Sun makes an angle $\Psi$ with the vector $\Psi$ and is therefore at an angle $A + Z + \eta - P$ measured clockwise from the North and looking in the $-$ direction. Daily values for $P$ are tabulated in the ephemeris. Table 4 summarizes the results of this section.

Table 4. Field Orientation of Various Contributions to a Measured Solar Oblateness

<table>
<thead>
<tr>
<th>Source</th>
<th>Angle of Minor Axis with respect to North</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intrinsic Solar Oblateness</td>
<td>$A + Z + \eta - P$</td>
<td>To be determined (P is tabulated)</td>
</tr>
<tr>
<td>$\Delta_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mirror I--window aberrations</td>
<td>$A + Z + \alpha_1$</td>
<td>To be determined (constant except for temperature effects)</td>
</tr>
<tr>
<td>$\Delta_M^1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mirror II--lens aberrations</td>
<td>$A + \alpha_2$</td>
<td>To be determined (constant except for temperature effects)</td>
</tr>
<tr>
<td>$\Delta_M^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atmospheric refraction</td>
<td>$A + Z$</td>
<td>$(n-1) \tan^2 z$</td>
</tr>
<tr>
<td>$\Delta_{AR}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Thus one can make a fit using the five parameters $\Delta_0$, $\Delta_{M1}$, $\Delta_{M2}$, $a_1$, and $a_2$ as defined in Table 4 and be assured that systematic errors due to a distorted image will have little effect on the resultant value of the intrinsic solar oblateness.

**Unique Definition of the Solar Edge**

The main contribution of modern solar oblateness measurements is the departure from visual and personal judgments of what point on the solar limb is defined as the edge of the sun. The Princeton experiment (Dicke and Goldenberg 1967) ingeniously circumvented this problem by integrating with a slot the light from an area extending well away from the extreme limb.

It is the purpose of this section to trace the development of a method of phase sensitive detection which allows one to electronically define the edge of the sun in a unique and consistently reproducible manner. Using a sinusoidally oscillating slit to measure the intensity of the limb profile as a function of position, it is possible in the limit of large amplitude oscillations to relate the defined edge of the observed sun to the intrinsic profile. The analysis which allows this comparison is given by Stebbins (1970) and is called the Fourier Coefficient Technique (FCT). A major achievement of this analysis is
the capability to analytically remove "seeing" dependence from the edge definition.

Before describing this approach, it is necessary to examine the characteristics of the intrinsic solar limb and the Earth's atmosphere which make the problem of edge definition so critical. This examination will show that a proper edge definition requires a high degree of sophistication in order to avoid systematic effect.

Influence of the Earth's Atmosphere and the Telescope on the Solar Edge Definition

The main problem encountered in defining a point on the intrinsic solar limb which may be called the edge of the sun is the lack of knowledge concerning the limb profile itself. This lack of knowledge is mainly due to the fact that the sun is observed through the atmosphere.

The transition between the photosphere or visual surface of the sun and the chromosphere is not well defined due to the gaseous nature of the sun, and the problem becomes even more complicated due to the phenomenon known as limb darkening. The observed light from the extreme solar limb must traverse a geometrically longer path through the partially absorbing solar atmosphere than light from the center of the solar disk which travels radially and thus comes from deeper within the photosphere. Therefore the middle of the solar disk is brighter than the limb and since
there is not sufficient knowledge of the solar atmosphere to accurately calculate the opacity of the atmosphere, the profile must be determined empirically. Minnaert (1953, pp. 99 and 129) shows, using simplified assumptions about the density and corresponding opacity of the solar atmosphere, that the actual intensity at the limb does drop extremely rapidly. According to him the intensity at the extreme limb changes exponentially and falls off by a factor of $e$ in 0.2 seconds of arc. Due to "seeing," which even for excellent daytime observations is at least 1 second (Bray and Loughhead 1965), empirical knowledge of the profile which could lead to a uniquely defined point on the edge is presently unavailable.

Daytime or solar "seeing" is much worse than in the nighttime because of the atmospheric instabilities caused by convection due to the heating effect of the sun. A point source image has brightness variations with a frequency of 1.0 to 500 Hertz and both rapid and long term motion of the image (Bray and Loughhead 1965). The slow image motions of up to 1/2 second period are caused by atmospheric inhomogeneities in the immediate neighborhood of the telescope. The index of refraction of the air changes with temperature and the whole image moves. The telescope at SCLERA is so constructed that Mirror II compensates for motion of the whole solar image up to frequencies of 50 Hertz. The high
frequency variations localized on the image come from conditions approximately 10 km in the atmosphere (Bray and Loughhead 1965) and cause the image of a point source to have a Gaussian-like distribution of intensity as a function of position. The full width of this disk of scattered light at half-maximum intensity is called the "seeing" width and varies rapidly. Bray and Loughhead (1965, p. 14) report that even on days of bad "seeing," there were abrupt changes in which the "seeing" became excellent in less than a second of time.

Thus one must closely examine edge definitions which rely on the observed shape of the sun. Initial attempts to define the solar edge made at SCLERA under the direction of Dr. Carl Zanoni confirmed the seriousness of these problems, and indicated the need for an extremely reliable approach.

A Definition of the Solar Edge Using Large Amplitude Phase Sensitive Detection and the Fourier Coefficient Technique (FCT)

In contemplation of the measurement, there was much group discussion at SCLERA about the merits and weaknesses of various definitions. Finally it was concluded that one must look at a section of the limb profile that extended over a width larger than the "seeing" in order to adequately define the solar edge. By sinusoidally oscillating a slit back and forth across the limb in the radial direction at a
frequency \((\omega/2)\) and measuring the time dependent intensity \(G(t)\), one could servo the equilibrium position of the slit so that the integral

\[
B_1(q) = \frac{2}{T} \int_{-T/2}^{T/2} G(q + a \sin \frac{\omega}{2} t) \cos \omega t \, dt \tag{II.16}
\]

would be zero. One could call this equilibrium point \(q\) the edge of the sun and by monitoring "seeing," look for empirical relationships between measured oblateness and different "seeing" conditions.

It was at this point that following suggestions made by Professor Henry Hill, R. T. Stebbins (1970) was able to show theoretically that this method was actually much stronger than initially anticipated. Stebbins was able to calculate, using simple models, the "seeing" dependence of diameter measurements using various edge definitions. He concluded that all past measurements were susceptible to error if there existed anisotropic atmospheric or instrumental seeing which could not be or was not removed by long term averaging of the data.

Stebbins, using a general approach, was able to show that the integral \(B_1\) as defined in Eq. II.16 above was related in a well defined and uniquely reproducible manner to the Fourier coefficient \(b_1\) in the Fourier series describing
the intrinsic profile of the sun. This is the basis for
the name: Fourier Coefficient Technique (FCT). This rela-
tionship involves the amplitude of the slit oscillation and
the seeing width, but both of these parameters can be readi-
ly obtained.

Since this insight has been invaluable in establish-
ing direction in the present measurement, it is appropriate
to illustrate the general theoretical approach of FCT and to
show the relationship between a point on the intrinsic solar
profile and the equilibrium point of the oscillating slit
locked onto the observed image in such a manner that the
integral \( B_1(q) \) is zero.

A unique point \( q \) on the intrinsic solar limb profile
\( g(u) \) (\( g \) is the intensity as a function of position \( u \)) may be
defined (Stebbins 1970, p. 20) as that point for which the
Fourier coefficient \( b_1 \) is zero. This coefficient may be ex-
pressed mathematically in the case of an oscillating slit
for which \( g(t) = g[q \pm a \sin(\omega t/2)] \) as follows

\[
b_1(q) = \frac{2}{T} \int_{-T/2}^{T/2} g(t) \cos \omega t dt
\]

In the preceding equation, \( a \) is the amplitude and \( (\omega/2) \) is
the frequency of oscillation. If one assumes a transfer
function, \( f \), for the atmosphere and the telescope, the
observed intensity of the solar profile \( G(x) \) as a function of the position \( x \) referred to the image may be given by

\[
G(x) = \int_{-\infty}^{+\infty} g(u)f(x - u)du.
\]  

II.18

The coefficient for the Fourier series representing \( G(x) \) expanded about a point \( q \) on the observed limb can be written as

\[
B_1(q) = \frac{2}{T} \int_{-T/2}^{T/2} \left( \int_{-\infty}^{\infty} g(u)f(x-u)du \right) \cos \omega t dt
\]  

II.19

where \( x = q + a \sin (\omega/2)t \). One can see that if \( f(x-u) \) were ideal then \( G(t) = g(t) \) and \( B_1(q) \) would equal the Fourier coefficient \( b_1(q) \) expanded about the corresponding point \( q \) on the intrinsic limb. If one assumes that the transfer function \( f(x-u) \) can be characterized by a single width parameter \( \alpha \), where \( \alpha < a \), then one can use a Taylor series expansion to set \( B_1(q) \) equal to \( b_1(q) \) plus a power series in \( \alpha \):

\[
B_1(q) = b_1(q) + \sum_{n=1}^{\infty} d_n \alpha^n
\]  

II.20

Because one does not know the analytic form of \( b_1(q) \) it must be expanded in a spatial Taylor series about a point \( q_0 \),
where \( q_0 \) is an arbitrary distance from the center of the sun chosen so that \( x = q - q_0 \) is a small quantity. The point \( q_0 \) on the intrinsic limb corresponds to a similar point on the image if there is no seeing. Keeping only the first two terms in the Taylor series one obtains for Eq. II.20:

\[
B_1(q) = b_1(q_0) + xb'_1(q_0) + \sum_{n=1}^{\infty} d_n a^n. \tag{II.21}
\]

By moving the slit until \( B_1(q) = 0 \) and assuming that \( x \) is small enough and that the spatial derivative of the intrinsic profile is well behaved, one can solve for the equilibrium point \( q = q_0 + x \) of the oscillating slit with the following result:

\[
x = -\frac{b_1(q_0)}{b'_1(q_0)} - \sum_{n=1}^{\infty} \frac{d_n}{b'_1(q_0)} a^n \tag{II.22}
\]

The quantity \( x \) is the displacement from the arbitrary but uniquely defined position \( q_0 \) to the point \( q \) which makes the integral \( B_1(q) \) equal zero. This displacement is equal to some constant term \( x_0(a) = [b_1(q_0)/b'_1(q_0)] \) plus a "seeing" dependent series.

Calculations using an empirical fit for the limb darkening function and a Gaussian transfer function were carried out by Stebbins (1970, p. 44) and he concluded that,
for amplitudes of oscillation \( a > a \), the only term important for "seeing" corrections is quadratic in \( a \). The summation in II.22 may then be replaced by a single term \( k \alpha^2 \) where 
\[
k = \left[ \frac{d_2}{b'_i(q_0)} \right].
\]
This term varies as \( a^{3/2} \) because \( b'_i(q_0) \) is amplitude dependent.

The expression for the displacement is then

\[
x = - x_0 - k \alpha^2 .
\]

The lock-on point \( x \) is determined by the amplitude dependent offset \( x_0 \), which is well defined in terms of the Fourier coefficient of the intrinsic limb, and an additional offset dependent on the "seeing" amplitude. If the Taylor series for \( b_i(q) \) expanded around \( q_0 \) does not converge as rapidly as calculations using various limb profiles indicate, the non-linear dependence of the offset upon the displacement, \( x \), does not affect the uniqueness of the lock-on point but only the interpretation relating that point to a similar point on the intrinsic sun.

In summary, the Fourier Coefficient Technique provides a qualitative analysis of phase sensitive detection in which the edge of the observed sun is defined as the equilibrium point, \( q \), of a radially oscillating slit driven to the point \( q = q_0 + x \) such that
$$B_1(q) = \frac{2}{T} \int_{-T/2}^{T/2} G(t) \cos \omega t dt$$
equals zero. This point is related to a similar point, $q_0 + x_0$, on the intrinsic solar limb which satisfies the requirement that the Fourier coefficient $b_1(q_0 + x_0)$ be zero. The displacement $x$ is related to the point $x_0$ on the intrinsic limb by the equation $x = x_0 + k\alpha^2$; and by extrapolating to the zero "seeing" limit, one can relate the position $x$ directly to the intrinsic profile.

The main assumption in the above arguments is that the transfer function can be adequately characterized by the single parameter $\alpha$. The Fourier coefficient technique greatly increases the confidence with which one may define the solar edge and experimental verification of this technique is discussed in Chapter IV.

Additional Problems—The Secondary Reflection and Localized Solar Activity

Now it is possible to discuss the effect that the secondary image of the sun will have upon the lock-on point. The secondary image is caused by internal reflections from the window and is displaced 46 seconds of arc from the primary image. Since the amount of reflected light for glass at normal incidence is 4%, the intensity of the
secondary image is 0.0016 that of the primary image. Depending on the angle between the observed diameter and the direction of displacement, the separation $d$ between the observed primary edge and observed secondary edge varies between $\pm 46$ seconds of arc. The effect on the lock-on point is best summarized by examining the three cases where the amplitude of oscillation, $a$, is respectively greater than, equal to, or less than the separation $d$.

The Fourier Coefficient $B_1(x)$ equals $b_1(x)$ plus a "seeing" dependent term. In the case where $a > d$ one can expand $B_1(x)$ in a Taylor series consisting of the sum of contributions from the primary and the secondary (denoted by the subscript $s$) images, and obtain the following

$$b_1(x) = b_1(q_0) + b_1s(q_0s) + (x+d)b_1'(q_0s) + xb_1'(q_0) .$$

II.24

where $q_0s = q_0 + d$. The offset is:

$$x = \frac{b_1(q_0)}{b'_1(q_0)} \left[ 1 + \frac{b_1s(q_0s)}{b_1(q_0)} \right] + d \frac{b_1'(q_0s)}{b_1(q_0)} \left[ 1 + \frac{b_1s(q_0s)}{b_1(q_0)} \right]$$

II.25

The ratios
add only small contributions to the zero "seeing" offset

\[ x_0 = \left\{ \frac{b_1(q_0)}{b'_1(q_0)} \right\} \]

which is calculated using limb profile data given by Allen (1963) to be approximately 1 second of arc for \( a = 10 \) seconds. Therefore,

\[ x = x_0 + 0.0016d . \]

For \( d = 3 \) or \( 4 \) seconds of arc, this offset is considerable (0.006 arc sec) but the diameter change is zero to first order because \( d \) has opposite signs at the two limbs of the diameter. This example does illustrate how sensitive one would be to anisotropic limb profiles, however.

In the case where \( d > a \), one can no longer evaluate the terms \( b_{1S} \) and \( b'_{1S} \) at \( q_{0S} \). Expanding the secondary terms around \( d \), the Fourier Coefficient can be written

\[ B_1 = b_1(q_0) + xb'_1(q_0) + b_{1S}(q_0+d) + xb'_{1S}(q_0+d) , \]

and

\[ x \approx \frac{x_0}{1 + \frac{b_{1S}(q_0+d)}{b'_1(q_0)}} + \frac{b_{1S}(q_0+d)}{b'_1(q_0)} \left[ 1 + \frac{b_{1S}(q_0+d)}{b'_1(q_0)} \right] . \]

Using Allen's (1963, p. 70) model for the intrinsic limb darkening function, the resulting integrals for \( b_1(d) \) and
b'1(d) can be evaluated using a binomial expansion in powers of a/d. The second order term is the first to contribute, and the numerical magnitude of both ratios is the same within the factor of 1/2 which comes from the derivative. There is approximately 0.05 milliseconds offset if the intensity of the second image is 0.0016 that of the first.

In the case where d ~ a one still has the same result as expressed in (II.28), but the integrals can no longer be evaluated using a binomial expansion. Here one must break the integral for b1(q0 + d) into two parts; since cos ωt is symmetric, the first part of the integral,

\[ \int_{-T/2}^{0} g(q_0 + d + a \sin \frac{\omega}{2} t) \cos \omega tdt, \]

is the same as

\[ \int_{-T/2}^{T/2} g(q_0 + a \sin \frac{\omega}{2} t) \cos \omega tdt \]

because in the second integral g(t) is 0 for all negative values of t. The ratio may then be written

\[ \frac{b_{1s}(d)}{b_1(q_0)} = (0.0016) x_0 \left[ \frac{\int_{0}^{T/2} g(q_0 + d + a \sin \frac{\omega}{2} t) \cos \omega tdt}{b_1(q_0)} \right] \]

The second term in the brackets will be much less than x0.
because the gradient of the intensity is much smaller at \( d \) than at the edge. The offset is then less than 2 milliseconds. Using the same argument, one concludes that 
\[ b \mathbf{i}_s (q_0 + d) \]  
must be smaller than \( 1.6 \times 10^{-3} \) and the offset will be less than 2 milliseconds. The secondary image rotates in the field as \( A + Z \) and is only of this magnitude for that particular diameter for which \( d = a \). Therefore, one can assume that the systematic error in an oblateness measurement introduced by the secondary reflection is less than 1 millisecond of arc.

One of the big questions concerning earlier oblateness measurements concerned sensitivity to solar structure especially possible latitude dependent variation in the limb darkening function. Dicke and Goldenberg (1967) investigated the problem of a latitude variation in brightness by using different size slots which admitted light from increasingly larger radial sections of the limb, although there is still some controversy regarding interpretation of their results (Ingersoll and Spiegel 1971).

Since \( x_0 \) is sensitive to the amplitude of oscillation, one may effectively lock onto the limb at various distances from the edge by varying the oscillation amplitude and extrapolating to the zero "seeing" offset \( x_0 \). Allen's (1963, p. 70) limb darkening function may be written
\[ g(x) = A + Bx^{\frac{1}{2}}. \]  \[ \text{II.29} \]

where \( x \) is the radial distance in from the edge of the sun and \( A \) and \( B \) are empirically determined constants. In Allen's notation \( A = 1 - u - v \) and \( B = u(2/r_o)^{\frac{1}{2}} \) where \( r_o \) is the solar radius in arc sec. The amplitude dependent offset can be written

\[ x = \frac{-0.241 \gamma a^{\frac{3}{2}}}{1 + 0.44 \gamma a^{\frac{3}{2}}} \]  \[ \text{II.30} \]

where \( \gamma = B/A \) (SCLERA Progress Report 1972, p. 19).

This amplitude dependent offset is the basis of present efforts by Douglas Patz to examine possible latitude variation in the limb darkening function. The outcome of his work will be necessary in deciding if any observed visual oblateness is the result of an intrinsic solar oblateness.

One must also be certain that diameter measurements are not affected because the limb profile is distorted by localized solar structure. The Princeton experiment was performed during a period of minimum solar activity and even though averaging also reduced the sensitivity, some have suggested that the measurement was influenced by localized phenomena (Abell 1969, p. 476; Chapman and Ingersoll 1972) because the basic apparatus scans all areas of the limb.
The present experiment, because of the capability to look at particularly selected diameters, has the potential to avoid perturbations caused by these localized effects even though the experiment must unfortunately proceed during a period 2 years removed from minimum solar activity.

A close examination of the types of solar activity and the related effect upon a particular segment of the solar limb will indicate that by relying on the available "solar patrol" observations one can either avoid taking, or discard, affected data. Sunspots at the solar limb would certainly have a most pronounced effect on an intrinsic limb profile taken at that point, but the location of sunspots is also easily obtainable. By choosing appropriate diameters and discarding data taken as a sunspot approached the limb from the back side of the sun, one can minimize this effect. Sunspots never appear at the poles and only rarely at latitudes less than 5° (Abell 1969, p. 468). During the present period of solar activity they are most likely to be found at a latitude of about 15°. Fortunately, the other effects which emit in the continuum (white light flares and faculae) are nearly always associated sunspot areas (present, past, or future) (Bray and Loughhead 1965, pp. 248 and 258) and can also be located in order that proper compensation may be made. By actually looking at a diameter when there are localized effects, one can determine the sensitivity and
estimate possible errors. (See Chapter V, the section titled Sensitivity to Localized Structure for actual results.)

Extremely few white light flares have been observed although some have increased the local intensity of the continuum by as much as 175% (Kiepenheuer 1953, p. 392; Smith and Smith 1963, p. 163). These flares last only about 10 minutes although their chromospheric counterparts (observable only in certain spectral lines) are stable for a much longer period. Faculae in the photosphere are associated with plages observed in the CaII line (Bray and Loughhead 1965, p. 248) and are usually most prominent near the limb indicating that they are formed high in the photosphere. Data regarding the position of the faculae was made available through the facilities of the Kitt Peak National Observatory. Plages are much more prominent but are only visible in emission lines. The remaining solar activity consisting of flares, prominences, spicules, etc. is all of a spectral nature which does not influence the continuum (Kiepenheuer 1953, Smith and Smith 1963).

Photospheric granulation while not associated with the cycle of solar activity also affects the intensity pattern of the photosphere. Granules are bright cells surrounded by dark boundaries thought to be caused by convection. An average granule is 1 - 2 arc seconds across and
Lasts 8-10 minutes (Abell 1969, p. 466; Bray and Loughhead 1965, pp. 60-64); however, due to their convective nature, granules are formed relatively low in the photosphere and are not visible closer than 5 seconds of arc from the limb. This fact plus the statistical nature of the phenomena makes a contribution to the limb profile negligible especially since the radially oscillating slit has a tangential length covering 100 seconds of arc. Thus by monitoring the position of sunspots and faculae as well as very rare white light flares, one insures that a particular diameter measurement will not be influenced by localized phenomena.

Relative Measurement of Solar Diameters

Now that the edge of the sun can be uniquely defined, it is important to make solar oblateness measurements in a way which makes full use of the Fourier coefficient analysis. By locking two slits onto diametrically opposite edges of the sun, the relative difference between any two solar diameters can be determined by measuring the change in separation between the equilibrium positions of the two slits as the detector is rotated from one diameter to another. If one determines that the diameter measurements are not affected by localized solar structure, then the relative diameter measurement can be corrected for "seeing" to give the visual oblateness of the observed sun.
Theoretical Predictions Relating
Observed Diameter Measurements
to "Seeing"

The separation, $D_\theta$, between slits locked on to
diametrically opposite limbs can be expressed in the light
of FCT as

$$D_\theta = 2(q_{0\theta} - x_{0\theta}) - k_A \alpha_A^2 - k_B \alpha_B^2$$  \hspace{1cm} (II.31)

where $\theta$ is the field angle with respect to north of the
measured diameter and the subscripts $A$ and $B$ refer to slit $A$
and slit $B$, respectively. The term in parentheses is called
$D_{0\theta}$ and the last two terms are due to "seeing." If $D_{0\theta}$ has
angular dependence, the intrinsic sun is not spherically
symmetric and/or has a latitude variation in the limb darkening
function. Before proceeding with this determination,
it is wise to consider the "seeing" dependent term. By
determining $k$ empirically and measuring $\alpha_A$ and $\alpha_B$, one can
eliminate the influence of atmospheric and instrumental
"seeing" upon a given diameter, even if $\alpha$ changes in time
and in position. Calculations performed by Professor Henry
Hill show that the value of $k$ is dependent only upon the
amplitude of oscillation if the value of $\alpha$ correctly refers
to the standard deviation of the transfer function. This is
generally true regardless of the shape of the transfer function, although the method used to measure $\alpha$ might not be
appropriate to a variety of transfer functions. This analysis strongly supports Stebbins' (1970) original assumption that the transfer function could be characterized by a single width parameter.

Because Mirror II oscillates rather than the slits (see Chapter II, the section titled The Solar Oblateness Detector and Primary Tracker), the image moves sinusoidally over both slits with the same amplitude and \( k_A = k_B \). The predicted value of \( k \) is 0.0102 (arc sec)\(^{-1}\) (SCLERA Progress Report 1972, p. 21) if the amplitude of oscillation is 6.8 arc seconds. The negative sign in front of \( k \) indicates that the diameter decreases as "seeing" becomes worse. This may be qualitatively verified by multiplying a step function by \( \cos \omega t \) and integrating from \( t = -(2\pi/\omega) \) to \( t = + (2\pi/\omega) \). As the step function is rounded off and light is scattered into the previously unilluminated area, the \( t = 0 \) point must be moved in from the step function to keep the value of the integral equal to zero.

To adequately correct a given diameter measurement for "seeing" dependent offsets, one must be able to measure the width parameter of the transfer function. This width parameter is almost entirely dependent upon "seeing" because the widths of the slit and the telescope transfer functions are much smaller than the "seeing" width. The resultant width is the square root of the sum of the squares of the
standard deviations, and the 1 arc second wide slit increases the width of 5 second Gaussian type seeing by 2%. Because the width change is so small, it is reasonable to expect that the shape change is also small.

Stebbins (1970, p. 51) points out that the intensity gradient at the extreme limb is the observable which is most sensitive to changes in the width parameter. He supports this choice by arguing intuitively that "... the image of a point source exactly determines the transfer function while the image of an infinitely broad source of constant intensity yields no information about the transfer function." Stebbins defines a second parameter

\[ m = \frac{G(t = 0)}{G'(t = 0)} \]  

which is the inverse logarithmic derivative (ILD) of the intensity at the limb. The parameter \( m \) is normalized by the average value of the intensity appearing in the numerator. Using a Gaussian transfer function and empirical limb darkening data from Allen (1963, p. 170), Stebbins (1970, p. 53) calculates the relationship between the ILD or "seeing" parameter \( m \) and the width parameter or variance \( \alpha \). Using this relationship and evaluating the first correction term to \( m \) for \( \alpha \) of 4 arc sec yields

\[ m = 1.19 \alpha \]  

where \( m \) and \( \alpha \) are expressed in seconds of arc.
The slit separation in Eq. (II.31) can then be written

\[ D = D_0 + K(m_A^2 + m_B^2) \]  \hspace{1cm} \text{II.34}

where \( K \) is theoretically equal to 0.0072 (arc sec\(^{-1}\)) (SCLERA Progress Report 1972, p. 22) and is obtained from \( k \) using Eq. (II.33).

The difference \( \Delta_{\text{ep}} \) between the equatorial and polar diameters may be expressed as

\[ \Delta_{\text{ep}} = D_e - D_p = (D_{oe} - D_{op}) - K(m_{Ae}^2 + m_{Be}^2 - m_{Ap}^2 - m_{Bp}^2) \]  \hspace{1cm} \text{II.35}

By setting the quantity \((m_{Ae}^2 + m_{Be}^2 - m_{Ap}^2 - m_{Bp}^2)\) equal to \( m^2 \) and plotting \( \Delta_{\text{ep}} \) versus \( m^2 \), one not only determines the zero seeing value of \( \Delta_{\text{ep}} \), but can from the slope of the line measure the magnitude of \( K \). This formulation illustrates well the fact that it is often much easier to make an accurate differential measurement than an absolute one.

If the "seeing" were the same at the pole and the equator \((m^2 = 0)\), it would not be necessary to know \( K \) at all. If the average difference between "seeing" parameters were zero, there would be no systematic contribution to a measured oblateness although the scatter in the data would increase. If there were a systematic anisotropy in the
"seeing" which on the average was one arc second worse for one diameter than for the other, a systematic error would result. For example, if $m_e = 4$ sec and $m_p = 5$ sec, a correction of 120 milliseconds to the observed oblateness would be necessary. If the value of $K$ is only accurate to 10%, the total systematic error would be 12 milliseconds.

Stebbins (1970, p. 45) studied the effect of "seeing" in FCT as well as the inflection point method and the slot technique of Dicke and Goldenberg (1967) and he concluded that anisotropic "seeing" could have been a significant factor in earlier solar oblateness measurements. Empirical verification of the "seeing" dependence is the subject of Chapter IV and it will be seen that atmospheric "seeing" effects can be suitably eliminated using FCT.

The change in separation between the slits can be measured interferometrically by attaching a beam splitter to one slit mounting and a corner cube reflector to the other. In order to show that this measurement can be made without influence from systematic errors, it is necessary to give a brief description of the solar oblateness detector.

The Solar Oblateness Detector and Primary Tracker

The solar oblateness detector (see Fig. 5) basically consists of two slits tangent to opposite edges of the sun.
Fig. 5. Oblateness detector.
whose radial movement is controlled by solenoid transducers. Below each slit is a filter and a photomultiplier to measure the intensity of the sun at a given wavelength as a function of slit position. The slits have dimensions which correspond to one second of arc in the radial direction and 100 seconds in a direction tangential to the solar edge. The slits rotate about a vertical axis which points toward the nodal point of the objective, and they lie in a plane normal to this axis of rotation. The angular position of the diameter to be examined is determined by motors and an encoder at the bottom of the axis of rotation. The radial movement of the slits is normal to the axis of rotation because of a double parallel leaf spring mounting which allows smooth motion.

The image of the sun is centered on the detector by a symmetrical arrangement of mangin tubes (see Fig. 6) which electronically drives the mirror system so that each tube receives an equal amount of illumination. The amount of light on each tube is sufficiently great that the solar position is not sensitive to "seeing" at the limb. The response of the primary tracking system will eliminate low frequency displacements of the entire image caused by local "seeing." Using computed signals for the change in the elevation and azimuth, the feedback signals from the mangin detectors keep the position of the image fixed in the field.
Fig. 6. The interferometer.
to an accuracy of better than two seconds of arc. If this remaining motion is orthogonal to the diameter being measured, the sagittal change in diameters would be at most equal to 4 milliseconds of arc. If the motion is along the diameter the slits either track it or average over it depending on the frequency.

There are two major advantages of this type of primary tracker. The first involves the fact that the mangin tubes rotate with the detector. This reduces the problems involved in centering the image of the sun on the tracker. If the electronic gains or geometrical locations of the four detectors are not perfectly matched, the line separating the two slits may be a chord rather than the diameter of the sun. This would also be the case if the slits are not perfectly centered. Since the primary tracker and the slits rotate together, the slits always remain on the same chord and again because the measurement is differential the resulting error is negligible.

The second advantage inherent in the primary tracker is due to the fact that one can change the illumination at the final image by moving the mangin mirrors. [See Zanoni (1966, pp. 78) for a detailed discussion of the mangins.] The mangin mirrors are mounted on piezo-electric bimorphs and can be driven sinusoidally. In an effort to keep the
illumination in each tracker constant, Mirror II tracks this sinusoidal oscillation and the image of the sun can be made to oscillate back and forth across the slits by driving the mangin mirrors in the proper configuration. This means that the slits must move only enough to lock on to the equilibrium point and do not need to oscillate. This considerably lightens the burden on the interferometer and enables the slits to be mounted to more massive structures with less demands on the solenoid transducer. Perhaps the best advantage is that the filter and photomultiplier now can move with the slit, and the exit pupil is fixed with respect to the photocathode.

Another potential source of error when comparing diameter readings is variation in the amplitude of oscillation. Both the $x_0$ offset and $K$ are amplitude dependent (SCLERA Progress Report 1972, p. 18; Stebbins 1970, p. 44) as is the empirical method for determining $m$. The offset for these terms goes as $a^{3/2}$ and $a^{-1/2}$, respectively. For small $a$, the "seeing" term has the largest effect. When the "seeing" correction is the maximum value of about 500 milliseconds of arc, a change in amplitude between diameters of 2% would cause an error of 15 to 20 milliseconds.

In eliminating this type of error, it is much easier to keep the amplitude constant than measure amplitude
changes. To do this, one can make use of the way Mirror II tracks the displacement of the secondary image. As described above, the mangin mirrors are driven to cause the secondary image to oscillate sinusoidally across the primary tracking detectors with frequency $\omega/2$, and Mirror II is servoed to compensate for this motion of the mangin mirrors and keep the illumination on the primary tracker constant. This means that the primary image moves sinusoidally in phase with the mangin mirrors. The mangin mirrors are driven piezo-electrically and are sensitive only to long term drifts, and since they rotate with the detector, their motion is not crucially dependent on calibration.

The error in the motion of the primary image then varies inversely as the gain of the Mirror II servo loop. This loop involves mechanical resonances which are different for the two orthogonal drive channels, but by using electronic compensation, the gains in the two channels are made equal within $10\%-15\%$. The gain at the drive frequency $\omega/2$ equal to $1.64$ Hz is about 20. So the worst case change in amplitude is less than 1% and the worst case systematic error will be less than 10 milliseconds of arc. Further reduction is possible by ascertaining the gain from the error signal in the primary tracking system at the frequency $\omega/2$ and making the appropriate correction.
The orientation of the axis of rotation is also critical in eliminating sources of systematic errors. If the axis of rotation does not point at the nodal point of the objective, the image of the sun in the plane of rotation of the slits is an ellipse and at one point the separation between the locked-on slits would correspond to a major axis, and the orthogonal diameter would be a minor axis. To contribute less than one millisecond of error, the axis of rotation must make an angle of less than $10^{-3}$ radians with the principal optic axis. The alignment may be checked by requiring the exit pupils of the mangin tubes to trace concentric circles about the bottom bearing of the rotation axis. If the axis of rotation is properly oriented, then the slits do not need to be in exactly the same plane.

Temperature problems are not important in the mechanical detector for the solar oblateness measurement because one can make relative measurements at a rate faster than temperature can change enough to have a significant effect. In keeping with the long term use of the measured solar diameter to monitor changes in scale of the field, the detector was designed so that there would be no temperature gradients. The entire detector is kept at ambient temperature by cooling the transducers, the motors and the laser housing and by installing and cooling an occulting disk.
which only allows sunlight to fall on the primary tracker and slits. This occulting disk is also necessary to reduce scattered light in the telescope.

The last problem in mechanical design of the detector concerns height changes of the detector. As mentioned previously, the depth of focus is ± 1 cm but since the angular width of the sun is constant, the diameter measurement is determined by the distance from the nodal point of the objective to the detector. To cause a systematic error of less than one millisecond difference between diameter measurements, the height of the detector must not change by more than 0.0002 inches during rotation. The detector is supported by ball bearings and the uncertainty in the vertical position due to ball bearing noise is no worse than this 0.0002 inches. However, a change of this size can be produced by only a 2°C temperature change if the detector is connected to the objective via a steel spacer. For this reason the detector is suspended from the nodal point of the lens using temperature controlled invar.

The solenoid transducers are connected to the slits using horizontal flat springs which exert minimal vertical forces. As the slits move radially there are small sagittal changes in the height. The maximum motion for differential measurements would be due to the atmospheric oblateness which could cause slit movement of 2 seconds of arc or
approximately 0.005 inches. The corresponding height change is negligible.

It is evident that it is possible to design the detector in such a manner that mechanical restraints do not produce serious systematic errors. For differential measurements, all errors are of the order of milliseconds or less. A description of the interferometer will illustrate the way in which possible systematic errors in the actual measurement process can be controlled.

The Interferometer

The design of the interferometer used in the oblateness detector was based upon an article by Sheldon Minkowitz (1968). By using circularly polarized light, it is possible to determine, using a single source of light, the change in slit separation and the direction of the change. By referring to Fig. 6, one can see the principle of operation. Right-handed circularly polarized light enters the beam splitter at point A and the reference beam $E_r$ continues through the beam splitter to point B. The measuring beam $E_m$ changes polarization at each of five reflections (3 in the corner cube) and emerges at point B with left-handed polarization. The beams recombine to form plane polarized light whose plane of rotation is given by the angle $\phi = (2\pi d/\lambda)$. By splitting the beam again and observing
the plane of polarization through polaroids at C and D which
are oriented at 45° with respect to one another, it is pos­
sible to know if the angle \( \phi \) is increasing or decreasing.
The signals from the two silicon cells used to detect the
fringes are in quadrature and enable one to detect a change
in separation of \( \lambda/8 \). In the focal plane of the solar
image at \( 5461\text{Å} \), the focal length is 486.375 inches and \( \lambda/8 \)
corresponds to 1.32 milliseconds of arc if a Helium-Neon
laser with \( \lambda = 6328\text{Å} \) is used.

As mentioned earlier, the beam splitter is fixed to
the mounting for slit B and the corner cube is fixed to slit
A. Although the laser is an ideal source of light for inter­
erometry, the laser oscillation frequency sweeps through
the Doppler half-width when the cavity length changes due to
temperature and causes slight changes in \( \lambda \). Douglas Patz
has shown that for a Doppler half-width of \( 750 \times 10^6 \) Hz,
one can expect a frequency change and a corresponding wave­
length change of \( (\Delta \lambda/\lambda) \approx 1.5 \times 10^{-6} \). This means if the
beam splitter and corner cube are kept close to each other
so that the optical path difference is less than one inch,
the maximum error because one used an unstabilized laser is
less than one millisecond of arc. Other changes of the
wavelength due to pressure and temperature changes are
negligible.
Putting the beam splitter and corner cube close to each other reduces the path which the light must travel, but the thermal expansion of the material connecting the optical elements to the slits could cause apparent diameter changes. Making the diameter rotations on a short time scale compared to the ambient temperature changes solves this problem for the solar oblateness measurements. Similar effects occur because the index of glass in the corner cube changes with temperature, thereby changing the optical path length, but this is also unimportant for the solar oblateness measurements.

**Summary of Systematic Errors**

The preceding sections of this chapter have dealt with the various types of systematic errors and have either shown that their magnitude can be made negligible by proper design or described how the effects can be removed. As a summary of this chapter, Table 5 lists the magnitude of the systematic errors and the way in which their effect upon the value for measured solar oblateness can be effectively removed or gives the design criteria necessary to reduce their size.

The major systematic error which is not eliminated prior to making the analytic fits will probably be due to the intrinsic dishing of Mirror I. If this dishing
Table 5. A Summary of Systematic Effects which Might Cause Errors in an Oblateness Measurement.

Also included is the approximate order of magnitude (in seconds of arc) of the effect before and after appropriate correction. The method used to reduce the size of the systematic error is listed.

<table>
<thead>
<tr>
<th>Type of Systematic Effect</th>
<th>Approximate Uncorrected Magnitude (Arc Second)</th>
<th>Approximate Residual Magnitude (Arc Second)</th>
<th>Method Used to Reduce the Magnitude of the Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atmospheric Refraction</td>
<td>~ 0.500</td>
<td>&lt; 0.001</td>
<td>Measurement of temperature, pressure and Z allow calculation</td>
</tr>
<tr>
<td>&quot;Seeing&quot;</td>
<td>unknown possibly zero</td>
<td>&lt; 0.010 for 1 second anisotropy</td>
<td>Measure &quot;seeing&quot; parameter m, and calculate correction using FCT</td>
</tr>
<tr>
<td>Chromatic dispersion</td>
<td>----</td>
<td>~ 0</td>
<td>narrow band observations</td>
</tr>
<tr>
<td>First order optical aberrations</td>
<td>----</td>
<td>~ 0</td>
<td>Telescope design and differential measurement</td>
</tr>
<tr>
<td>Third order optical aberrations</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Telescope design and analytic fit to data</td>
</tr>
<tr>
<td>Aberrations due to optical tolerances:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>objective</td>
<td>&lt; 0.005</td>
<td>&lt; 0.001</td>
<td>Analytical fit to data</td>
</tr>
<tr>
<td>window</td>
<td>&lt; 0.005</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
<tr>
<td>mirror</td>
<td>~ 0.100</td>
<td>&lt; 0.001</td>
<td></td>
</tr>
<tr>
<td>Type of Systematic Effect</td>
<td>Approximate Uncorrected Magnitude (Arc Second)</td>
<td>Approximate Residual Magnitude (Arc Second)</td>
<td>Method Used to Reduce the Magnitude of the Effect</td>
</tr>
<tr>
<td>--------------------------------------------------</td>
<td>-----------------------------------------------</td>
<td>------------------------------------------</td>
<td>-------------------------------------------------</td>
</tr>
<tr>
<td>Temperature scale effects</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Mechanical design and differential measurements</td>
</tr>
<tr>
<td>Mirror curvature due to temperature gradient</td>
<td>~ 0.012</td>
<td>&lt; 0.005</td>
<td>Analytical fit</td>
</tr>
<tr>
<td>Window refraction</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Keep window normal to sun</td>
</tr>
<tr>
<td>Secondary solar image</td>
<td>----</td>
<td>&lt; 1 msec</td>
<td>FCT and analytic fit</td>
</tr>
<tr>
<td>Localized solar activity</td>
<td>----</td>
<td>&lt; 0.005</td>
<td>Correlation with solar patrol data</td>
</tr>
<tr>
<td>Latitude dependent limb darkening function</td>
<td>----</td>
<td>?</td>
<td>FCT--observations at various amplitudes of oscillation</td>
</tr>
<tr>
<td>Tracking error</td>
<td>----</td>
<td>&lt; 0.004</td>
<td>Primary tracker design</td>
</tr>
<tr>
<td>Inhomogeneous photocathode</td>
<td>----</td>
<td>~ 0</td>
<td>Fixed exit pupil</td>
</tr>
<tr>
<td>Oscillation amplitude variation</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Monitor mirror servo error and make analytic fit</td>
</tr>
<tr>
<td>Axis of detector rotation</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Align axis parallel to optic axis so ( \theta ) is &lt; 10^{-3}</td>
</tr>
</tbody>
</table>

Table 5. A summary of Systematic Effects which Might Cause Errors in an Oblateness Measurement—Continued
<table>
<thead>
<tr>
<th>Type of Systematic Effect</th>
<th>Approximate Uncorrected Magnitude (Arc Second)</th>
<th>Approximate Residual Magnitude (Arc Second)</th>
<th>Method Used to Reduce the Magnitude of the Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detector height changes</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Design, differential measurement</td>
</tr>
<tr>
<td>Laser wavelength change</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Short optical path difference</td>
</tr>
<tr>
<td>Interferometer temperature change</td>
<td>----</td>
<td>&lt; 0.001</td>
<td>Design, differential measurement</td>
</tr>
</tbody>
</table>
is to first order spherical as indicated in interferograms of the mirror, the axis of the induced oblateness will lie along the principal optic axis and \( \alpha_1 \) equals zero. One can easily check this assumption by noting that the angular difference between \( \Delta_\phi \) and \( \Delta_{Ml} \) goes as \((n - P)\) (see Table 4). When \((n - P)\) is zero, the resultant solar oblateness \( \Delta_{ep} \) should be either a maximum or a minimum depending on whether \( \Delta_\phi \) and \( \Delta_{Ml} \) have the same or opposite signs. As a first step in quantitative analysis one can then fit values of the observed solar oblateness \( \Delta_{ep} \) (corrected for "seeing" and atmospheric refraction) to the equation

\[
\Delta_{ep} = \Delta_\phi + \Delta_{Ml} \cos 2(n - P). \tag{II.35}
\]

The cosine term contains the factor of two because if the minor axis of \( \Delta_{Ml} \) changes by \(90^\circ\) with respect to the minor axis of \( \Delta_\phi \), the change in values of \( \Delta_{ep} \) goes through \(180^\circ\).

The above assumptions about \( \alpha_1 \) and the magnitude of other systematic errors can be tested qualitatively by using a symmetry argument which is sensitive to all sources of error which rotate in the field in a fixed manner. If one adds the measured values of

\[
\Delta_{ep}(n_0) = \Delta_\phi + \Delta_{Ml} \cos 2(n_0 + \alpha_1 - P) \tag{II.36}
\]

and

\[
\Delta_{ep}(n_0 + 90^\circ) = \Delta_\phi + \Delta_{Ml} \cos 2(n_0 + 90^\circ + \alpha_1 - P), \tag{II.37}
\]
the sum is simply $2\Delta_\phi$ and the instrumental effects of Mirror I cancel out for any value of $\eta_0$. This cancellation is easily seen in the case where the major axes of $\Delta_\phi$ and $\Delta_{M1}$ are aligned at $\eta_0$; $\Delta_{ep} = \Delta_\phi + \Delta_{M1}$. When the relative angular orientation $\eta$ has changed by 90°, the major axis of $\Delta_{M1}$ is aligned with the minor axis of $\Delta_\phi$, and $\Delta_{ep} = \Delta_\phi - \Delta_{M1}$. The sum is just $2\Delta_\phi$ as previously indicated.

Taking any values of $\Delta_{ep}$ which were obtained for $\eta$'s with 90° difference will cause the effects of Mirror I, which are expected to be the largest errors, to cancel. The power of this approach is further increased by taking points symmetrical about noon TST where $\eta = \pm 45^\circ$. The additional help comes because the Mirror II lens caused solar oblateness, $\Delta_{M2}$, makes an angle with $\Delta_\phi$ which varies as $\eta + Z$.

While $\eta$ is antisymmetrical, $Z$ is symmetrical about noon TST; therefore $\Delta_{M2}$ effects will cancel even though $Z$ has not changed by 90° (i.e., $\cos 2(-45^\circ + Z) = - \cos 2(45^\circ + Z)$).

Even the atmospheric caused solar oblateness would cancel in adding the morning ($\eta = -45^\circ$) and afternoon ($\eta = +45^\circ$) data, if the pressure and temperature were the same. This is because its angular relation to $\Delta_\phi$ goes as $\cos 2\eta$ and its magnitude is dependent on $\sec^2 Z$.

In this chapter the sources of systematic error have been analyzed with the conclusion that the only errors which contribute substantially will rotate in the field with
respect to the pole of the sun so that one can perform an analytical fit. The Fourier analysis of phase sensitive detection using large amplitude oscillations is a particularly powerful tool for probing areas of possible systematic error ("seeing" and differential limb darkening) which may have influenced past measurements.
CHAPTER III

INSTRUMENTATION

It was shown in Chapter II that it is theoretically practical to make a solar oblateness measurement which has an ultimate accuracy of 5 parts in $10^7$, i.e., an error of less than one millisecond of arc. It is the purpose of this chapter to briefly indicate the type of instrumentation used to achieve such accuracy.

Included in the facilities at SCLERA is a small (8k memory) but extremely versatile computer which is used to control the telescope and associated experiments. The computer makes on-line calculations of the time rate of change in the azimuthal and zenith angles which are important in enabling the primary tracker system to keep the position of the solar image fixed in the field. Similarly, the computer calculates the position of the solar pole and equator and controls the rotation of the detector from one diameter to another.

Phase Sensitive Detection

Of particular interest is the way in which the computer is used to perform the integration necessary for
the phase sensitive detection which defines the edge of the sun. As indicated in Chapter II the edge of the sun is defined as the equilibrium point \( q \) of a sinusoidally oscillating slit such that the Fourier coefficient of the observed limb profile

\[
B_1(q) = \int_{-T/2}^{T/2} G(q + a \sin \frac{\omega}{2} t) \cos \omega t dt
\]

is zero. If one expands the intensity of the sun observed through the oscillating slit in the Fourier series

\[
G(t) = \frac{C}{2} + \sum (C_n \sin n \frac{\omega}{2} t + D_n \cos n \frac{\omega}{2} t)
\]

and uses the orthogonality of the harmonics, it is evident that the value of the integral \( B_1 \) is exactly equal to the amplitude \( D_2 \) of the second harmonic of \( \omega/2 \) present in the transducer signal. In effect one can perform the integration by multiplying \( G(t) \) by \( \cos \omega t \) and using a low-pass filter. The filter allows only the resulting D.C. component which is now \( D_2 \) to come through unattenuated and rolls off the other harmonics. For example after the original D.C. component \( C_0/2 \) has been multiplied by \( \cos \omega t \) the low-pass filter reduces its amplitude by \( \left\{1/[1 + (\omega T)^2]\right\}^{\frac{1}{2}} \) where \( T \) is the time constant of the filter. Using the filter to perform the integration is advantageous because by choosing a
long enough time constant, one can reduce the effect of variations in the function $G(t)$ caused by "seeing" and look only at the average value of $D_2$ over a specific number of complete cycles.

One can easily use the computer to simulate a low-pass filter according to the following algorithm:

$$\sum_1 = \sum_{i-1} + G(t_i) \cos \omega t_i - 2^{-n} \sum_{i-1}$$ \hspace{1cm} (III.1)

where $\sum_1$ is the sum at time $t_1$ and $n$ is an arbitrary number which is related to the time constant of the filter. The change in value of $\sum$ corresponding to the length of time $\Delta T$ between samples is

$$\sum_1 - \sum_{i-1} = G(t_i) \cos \omega t_i - 2^{-n} \sum_{i-1}$$ \hspace{1cm} (III.2)

and can be written in differential form as

$$\frac{d\sum}{dt} + 2^{-n}\sum = G(t) \cos \omega t$$ \hspace{1cm} (III.3)

where $f$, the sampling frequency, is the reciprocal of $\Delta T$. By letting $T$ equal $2^n/f$, the equation takes the form of a simple filter with gain.

$$(1 + Tp)\sum = 2^n G(t) \cos \omega t$$ \hspace{1cm} (III.4)

or

$$\frac{\sum}{G(t) \cos \omega t} = \frac{2^n}{1 + Tp}$$ \hspace{1cm} (III.5)
The operator $p$ is the standard symbol for differentiation with respect to time and $T$ is the time constant of the filter. By choosing the length of the time constant $T$ and the sampling frequency $f$, the gain $2^n$ is determined.

The fundamental angular frequency of oscillation $\omega/2$ was chosen as 10 radians per second (1.6 Hz) because of the need to be below the resonant frequency of Mirror II but to have a reasonable response time when averaging over several periods of oscillation. To get some insight regarding the proper time constant $T$, it is helpful to compare the magnitude of the average intensity $C_0$ and first harmonic $D_1$ with the second harmonic $D_2$ in the case where the slit equilibrium position $y_2$ is offset a distance $y_1 - y_2 = x = (\omega t_1/2)$, from the edge $y_1$. The symbol $t_1$, represents the time necessary for the image to move the distance $x$, when $D_2 \approx 0$ and $a$ is the amplitude of oscillation. For simplicity one can assume that the intensity at the edge is a step function of height $G_0$ as in Fig. 7. When the offset $x$ is small compared to the amplitude of oscillation, the intensity is

$$G(x,t) \approx G_0 \left( \frac{1}{2} - \frac{x}{\pi a} - \frac{2}{\pi} \sin \frac{\omega t}{2} - \frac{2x}{\pi a} \cos \omega t - \frac{2}{3\pi} \sin \frac{3\omega t}{2} - \frac{2x}{3\pi a} \cos 2\omega t + \ldots \right)$$

and

III.6
Fig. 7. Solar edge detection.

The time variation in image intensity \( I(t) \) at the slit is plotted for various offsets between the equilibrium position \( y_1 \) of the solar image and the slit position \( y_2 \). The value for the error signal is \( \int I(t) \cos \omega t \, dt \) and is equal to the difference between the shaded areas.
Fig. 7. Solar edge detection.
\[ G(x,t) \cos \omega t \approx \]
\[ G_0 \left( -\frac{x}{\pi a} + \frac{1}{\pi} \sin \frac{\omega t}{2} + \left( \frac{1}{2} - \frac{x}{\pi a} \right) \cos \omega t \ldots \right) \quad \text{III.7} \]

Specification of the smallest offset \( x \) which one would like to track without having the higher harmonics come through, determines that the filter must roll off the first harmonic by the ratio \( x/a \) which for a tracking accuracy of 2 milliseconds is approximately equal to \( 2 \times 10^{-4} \).

Problems arise using a digital filter because of the finite sampling rate. For example, if one samples \( G(t) \) 32 times per cycle, the computer cannot tell the difference between the 30th harmonic and the second harmonic. Normally one must sample at a frequency twice that of the frequency which contains the information, but because of this sampling error the rate was increased so that the high harmonics which come through as false signals can be identified and rolled off prior to the sampling point.

The Slit Servo System

Further boundary conditions on the digital low pass filter are imposed when one considers the entire servo system which controls the slit. The servo loop consists basically of 3 elements as shown in Fig. 8 where \( y_1 \) is the actual location of the edge and \( y_2 \) is the equilibrium position of the slit. The first operator consists of the slit,
Fig. 8. Block diagram of slit servo.

\[ \frac{\Omega}{y_2 - y_1} = \frac{k_1 G_1 G_2}{(1+T_1 p)(1+T_2 p)} \]

\[ \dot{\lambda} = k_2 \Omega + k_3 P \]
\[ (k_3 P = \dot{\lambda}_o) \]

\[ y_2 = k_4 \dot{\lambda} = k_{20} + k_4 (\dot{\lambda} - \dot{\lambda}_0) \]
a photomultiplier to measure the limb intensity as a function of the slit position, a pre-amp, prefilters, and two digital low-pass filters. The resulting transfer function is

$$\frac{\Omega}{y_1 - y_2} = \frac{G_1}{(1 + T_1 p)} \frac{G_2}{(1 + T_2 p)} k_1$$  III.8

where $G_1$ and $G_2$ are the gains of the digital filters as in Eq. III.5 and $k_1$ is a constant relating the change in $D_2$ to the displacement $x = y_1 - y_2$. A rough order of magnitude estimate for $k_1$ can be made using the first term on the right in Eq. III.7.

The second servo element provides for coarse adjustment of the slit position by adding the output of a potentiometer $P$ to $\Omega$ and by converting the resultant sum to a current which drives element three, the position transducer. As shown in the following equations the coarse adjustment greatly reduces the final position error as well as linearizing the loop equations for small displacements. The slit position depends quadratically on the transducer current, but it is possible to make a Taylor series expansion of the current in the solenoid about the value $i_0$ which corresponds to a slit position $y_{2o}$ and keep only first order terms.
The position of the slit is determined as indicated in Fig. 8:

\[ y_2 = k_4 i^2 = y_{20} + k_4 (1 - I_0) \]

\[ = y_{20} + k_4 k_2 \Omega = \frac{G_1 G_2 k_1 k_2 k_4}{(1 + T_1 p)(1 + T_2 p)} (y_1 - y_2) + y_{20}. \]

III.9

The loop is stable if \( T_1 > T_2 (G_1 G_2 k_1 k_2 k_3) \), or in other words, if the pole corresponding to \( T_2 \) is below the unity gain crossover. Rewriting Eq. III.9 to show how well \( y_2 \) tracks \( y_1 \) yields the following:

\[ \frac{y_2 - y_{20}}{y_1 - y_{20}} = \frac{1}{1 + \frac{(1 + T_1 p)(1 + T_2 p)}{G_1 G_2 k_1 k_2 k_4}}. \]

III.10

If one can adjust the potentiometer so that \( y_2 - y_{20} \) is less than 2 seconds of arc and the maximum tracking error desired is of the order of 2 milliseconds then the total D.C. open loop gain \( G = G_1 G_2 k_1 k_2 k_4 \) should be about \( 10^3 \). To insure that the tracking accuracy of the servo does not change appreciably with increased solar intensity, the voltage across the dynode chain of the photomultipliers is kept constant regardless of the anode current.

The closed loop time constant \( T_{CL} = T/G \) should be about 5 seconds. The slit position is then the result of integration over approximately 8 cycles and does not track
high frequency seeing. The open loop time constant $T_1$ is then approximately 5000 seconds. This value of $T_1$ did not completely roll-off the first harmonic in the transducer output so a second pole at $T_2$ equal to 2 seconds was inserted below the unity gain crossover which is at $\omega$ equal to $0.2 \text{ (seconds)}^{-1}$.

The interferometer count logic was developed by Douglas Patz. The up-down counter which records the fringe counts as described in Chapter II, the section titled The Solar Oblateness Detector and Primary Tracker, is sampled at 20 microsecond intervals at a time when the register is not changing. These values are fed through a digital filter with a one second time constant and the output is the averaged value of the slit separation. Any change in separation between the slits as the servo loop keeps them locked onto the edge of the sun is then easily determined. Rapid fluctuations in slit separation for a given diameter are correlated with "seeing" and will be the subject of the following chapter.
CHAPTER IV

EXPERIMENTAL VERIFICATION OF THE EDGE DEFINITION

It is the purpose of this chapter to compare experimental observation of "seeing" dependent diameter changes with the theoretical predictions of the Fourier Coefficient Technique (FCT). It will be seen that diameter measurements obtained using large amplitude phase sensitive detection to define the edge of the sun, strongly support FCT.

The relationship of the "zero-seeing" lock-on point $x_0$ to the intrinsic profile of the sun is only fundamental to a solar oblateness measurement if there is anisotropy between the polar and equatorial limb profiles. For a solar oblateness measurement in which the amplitude of oscillation of the sun's image with respect to the slit is constant, the "seeing" dependent terms become most important because the amplitude dependent $x_0$ terms will cancel out. (See Chapter II, the section titled A Definition of the Solar Edge Using Large Amplitude Phase Sensitive Detection and FCT, for a definition of $x_0$). An examination of the "seeing" dependent behavior of the observed diameter measurements shows that
PCT correctly explains the data and should be used to make corrections so that one can avoid possible systematic errors due to anisotropic transfer functions. According to Eq. 10.34, a measured diameter \( D \) should equal \( D_0 + K(m_A^2 + m_B^2) \).

Experimental verification of the above predictions involves two distinct problems. The first is the ability to empirically measure \( m \) and the second is to show that the "seeing" dependent offset in the diameter measurements does indeed vary quadratically in \( m \).

**Experimental Measurement of the "Seeing" Parameter**

To measure the ILD (inverse logarithmic derivative) \( m \), one obtains a time average of the observed solar limb profile. This is done by sampling the time dependent intensity \( G(t) \) at equally spaced time intervals, \( \Delta T \), while the slit stays locked onto the edge of the sun. The magnitude of the intensity is digitized on-line and combined with previous values which have the same phase relationship to the fundamental scanning frequency \( \omega/2 \). The ILD is then

\[
m = \frac{G(q)}{dG(q)} = \frac{G(t=0)}{dG(t=0)} \Delta T \frac{G(t=0)}{G'(t=0)}. \quad \text{IV.1}
\]

The inverse slope is computed from the profile using a least squares fit for seven points and four unknowns (the average value \( G(t=0) \) and the first three derivatives).
The same general results are obtained using a four or five point fit and the on-line computer calculations were carefully verified. The minimum value for $m$ (which corresponds to maximum profile slope) was found to be at the lock-on point for the 6.8 arc sec amplitude of oscillation. The spacing between adjacent sample points near the lock-on point was about 350 milliseconds.

The values of $m$ for a two minute run range from 4 seconds of arc to 8 seconds of arc. This corresponds to "seeing" of 8 to 16 seconds of arc (FWHM). These values of "seeing" are higher than one ordinarily encounters in daytime astronomy, but it was demonstrated that increased gain in the slit servo decreased the values of $m$ although the empirically determined value of $K$ remained the same. It was thus concluded that because the slit servo, which has a five to ten second time constant, does not follow fairly rapid movements of an entire section of the image, the profile is smeared out and the slope becomes smaller and the ILD larger.

The rapid time variation of "seeing" is shown in Fig. 9. In this case the time constant of the slit servo loop was 5 seconds and the profile was an average over 16 seconds. Values for the FWHM "seeing" are obtained by relating $m$ to $a$ via Formula (II.33). When the averaging time is increased to 128 seconds, "seeing" is worse (the average
Fig. 9. Time variation of measured "seeing" parameter $m$.

This data was taken on the afternoon of June 28, 1972.
value is ~ 6 arc seconds), but exhibits less time variation. The strong correlation between the observed values of m and the expected behavior of "seeing" is strong evidence that one can accurately parameterize "seeing" by taking the inverse logarithmic derivative at the lock-on point.

"Seeing" Dependent Diameter Corrections

Now that one has reliable values of m, one can empirically determine the correlation between diameter changes and "seeing" changes. One can check to make sure the relationship is quadratic and determine K empirically at the same time. This is done by plotting measured diameter changes as a function of $m_A^2 + m_B^2$. The straight line fit to the data in Fig. 10 shows the excellent agreement between observations and theory. The empirical value obtained from the data in Fig. 10 is $0.0072 \pm 0.0015$ which is also the theoretical value. This data was obtained June 28, 1972 at a time (afternoon) when systematic effects varied slowly, and it was corrected for atmospheric refraction.

Similar data was obtained for a much longer run on February 23, 1972 (150 points, 32 seconds per point), although systematic errors were not removed. A careful check of these data showed that m was not time correlated and that the systematic effects of the mirrors and the atmospheric
Fig. 10. The solar diameter as function of the square of the "seeing" parameter, m.

The solid curve is a linear fit to the data yielding a value for K of 0.0072 (arc second)$^{-1}$. 
refraction (which are time correlated) therefore merely increased the scatter in the data. A straight line visually fit to the February 23 data gives $K = 0.0069$ (arc seconds)$^{-1}$.

The quantitative agreement between measured values of $K$ and the theoretical predictions of FCT is very impressive. The theoretical value is based upon a quite general limb darkening model where the values in Allen's formula (1963, p. 170) were fit to empirical data. The value of $K$ is independent of the length of run or the gain of the slit servo system even though the average measured values of $m$ are affected by these parameters. It is thus well demonstrated that one can make the proper diameter corrections under all conditions.

If the "seeing" were isotropic with respect to any diameter of the sun, the errors caused by "seeing" would ultimately average to zero even though the necessary diameter corrections range from 300 to 700 milliseconds of arc. For 128 second runs, the variation was usually between 400 and 600 milliseconds. If, on the other hand, the "seeing" were somehow systematically anisotropic between the pole and the equator, an error would result when one attempted to compare diameters without the benefit of FCT. This is especially important since Stebbins (1970, p. 45) concluded that this effect may have been a significant factor in past solar oblateness experiments.
It is possible to make one more correction to a diameter measurement, although the effect is due to the particular way in which the data is obtained and is not inherent in the diameter definition itself. The average value of the diameter for a run in which the diameter is sampled \( N \) times is given by

\[
\langle D \rangle = \frac{1}{N} \sum_{i=1}^{N} D_i = \langle D_0 \rangle + \frac{K}{N} \sum_{i=1}^{N} (m_{A_i}^2 + m_{B_i}^2) \quad \text{IV.2}
\]

or

\[
\langle D \rangle = \langle D_0 \rangle + K(\langle m_A^2 \rangle + \langle m_B^2 \rangle) \quad \text{IV.3}
\]

where \( D_0, K, m_A \) and \( m_B \) were defined in Chapter II, the section titled Theoretical Predictions Relating Observed Diameter Measurements to "Seeing."

The way in which the slope is obtained allows only a determination of \( \langle m_A \rangle \) and \( \langle m_B \rangle \). However, if one assumes that the \( m_i \)'s have a Gaussian distribution around \( \langle m \rangle \), it is possible to relate \( \langle m \rangle^2 \) to \( \langle m^2 \rangle \) by using information about the difference between \( \langle D \rangle^2 \) and \( \langle D^2 \rangle \). Using the relationship \( \sigma_m^2 = 2\langle m \rangle \sigma_m \), which can be obtained from the Gaussian distribution, one can write

\[
\sigma_D^2 = 4K^2[\langle m_A \rangle \sigma_m^{m_A} + \langle m_B \rangle \sigma_m^{m_B}] \quad \text{IV.4}
\]

The proper value for the average diameter \( \langle D \rangle \) is then
\[ \langle D \rangle = \langle D_0 \rangle + K(\langle m_A \rangle^2 + \langle m_B \rangle^2) + \frac{\langle D^2 \rangle - \langle D \rangle^2}{2K(\langle m_A \rangle^2 + \langle m_B \rangle^2)} . \]  IV.5

The last term in Eq. IV.5 is important because the "seeing" parameter varies considerably in time as shown in Fig. 9, but it only contributes to a solar oblateness if the magnitude of variation is different consistently from one diameter to the other. Values for this correction range between 30 and 40 milliseconds and the correction to a 4 minute solar oblateness measurement is typically less than 10 milliseconds.

In summary, the measured diameter does vary quadratically with the inverse slope of the intensity at the extreme limb in good agreement with the predictions of FCT. The relationship is \( \Delta D = K(m_A^2 + m_B^2) \) where \( K = 0.0072 \pm 0.0015 \) (arc sec)\(^{-1}\).

This agreement increases the confidence in the experimental procedure used to implement FCT for the edge definition as well as making it possible to reduce any systematic errors due to "seeing." Scatter in the data is thus also reduced; for a single 4 minute run, the diameter measurement is good to \( \pm 0.030 \) arc seconds.
In Chapter IV, it was shown that one could accurately correct for "seeing" dependent diameter variations. The extremely close agreement between observations and the theoretical predictions of FCT is strong evidence that the method for obtaining solar oblateness measurements is reliable. FCT can also provide information about the source of a measured solar oblateness as well as removing "seeing" dependent contributions.

It is the purpose of this chapter to present the results of the present solar oblateness measurement. It will be seen that if there is no latitude variation in the limb darkening function, the measured solar oblateness is larger than classical calculations would predict. The magnitude of this solar oblateness would support the earlier observations of Dicke and Goldenberg (1967). However, there is support of explanations that the observed solar oblateness is completely the result of a latitude variation in the limb darkening function.
Before presentation of the data, it is well to examine the method for obtaining measurements. This review should be helpful in determining that sources of systematic error have been properly taken into account as explained in Chapter II.

A computer program calculates the angle in the field of the solar equator and directs the detector to rotate until the slits lie along that diameter. The reproducibility in this rotation is about 0.4° (standard deviation) although an encoder measures the actual position to 1 minute of arc. During the remainder of the day, slit A will always be associated with the same edge of the sun. The amplitude of image oscillation used to define the edge in the experiment is 6.8 arc seconds. The light from the slits is passed through a 100 Å (FWHM) interference filter with peak transmission at 5461 Å.

After a wait of 85 seconds to be sure that the slits are properly locked onto the equatorial edges, a 128 second diameter run commences. The relative diameter as given by the interferometer (see Chapter III, the section titled The Slit Servo System) is read every 1/2 second during the run. The measured diameter is the average of the 1/2 second values corrected for "seeing" as described in Chapter IV and atmospheric refraction. Halfway through a diameter run, the
temperature, pressure, solar zenith, and azimuth are read or calculated and the appropriate atmospheric refraction is obtained. The limb profile is obtained by sampling the limb intensity every 5 milliseconds (128 times per complete sinusoidal image oscillation) and adding the respective values to the sum of previous values with corresponding phase relationship.

The detector is automatically rotated to the pole for the second run of the solar oblateness cycle, and after the 85 second wait, a similar run for the polar diameter commences. The relative polar diameter is subtracted from the equatorial diameter to obtain the observed solar oblateness, $\Delta_{ep}$. The systematic errors due to "seeing" and differential atmospheric refraction have been removed from $\Delta_{ep}$ because the magnitude of these effects can be calculated. Other systematic errors of unknown magnitude as discussed in Chapter II, the section titled Summary of Systematic Errors, remain. The individual values of $\Delta_{ep}$ are obtained every 7 minutes and represent slightly more than four minutes of data. To evaluate possible "seeing" anisotropy, the difference between the "seeing" parameters of the pole and the equator is determined.

Data and Results

An extended period of unseasonably cloudy weather seriously affected the amount of data obtained during the
spring and summer of 1972, but the data which was acquired appears very reliable and reproducible.

Visual Solar Oblateness

Figure 11 displays the solar oblateness data obtained June 16 and 17, 1972. The magnitude of the observed solar oblateness $\Delta_{ep}$ is plotted vertically as a function of the parallactic angle $\eta$ (horizontal coordinate). The solid line is the best fit of the expression

$$
\Delta_{ep} = \Delta_{e} + \Delta_{M1} \cos 2(\eta - P)
$$

II.35

to the data. In making this fit, the magnitude of the intrinsic visual oblateness, $\Delta_{e}$, and the solar oblateness due to the expected curvature in Mirror I, $\Delta_{M1}$, were treated as the unknown quantities. The data was obtained for an oscillation amplitude of 6.8 arc seconds.

The least squares analysis yields $\Delta_{e} = 0.121 \pm 0.029$ arc seconds as the value for the intrinsic visual solar oblateness. The value for $\Delta_{M1}$ is $-0.095 \pm 0.054$ arc seconds.

Review of Possible Errors

The assumption implicit in Eq. II.35 that $\Delta_{M1}$ would be the major systematic error in observed values of $\Delta_{ep}$ is substantiated by applying the symmetry arguments found in Chapter II, the section titled Summary of Systematic Errors.
Fig. 11. Solar oblateness.

The observed difference $\Delta$ between the equatorial and polar diameters at $a = 6.8$ arcsec seconds as a function of the parallactic angle $\eta$. These data were obtained on June 17, 1972. The solid curve is a least squares fit of Eq. II.35 to the data. The angle $\eta$ increases monotonically during the day passing through zero at noon. The change in $\eta$ around noon is $90^\circ$ in approximately 1$\frac{1}{2}$ hours.
Adding the approximate values of $\Delta_{ep}$ for $\eta = -45^\circ$ and $\eta = +45^\circ$ and dividing by two, one obtains an approximate value of $A_0$, which agrees closely with the value obtained using Eq. II.35. This approach was used to analyze data for these days (May 22, June 24 and 28) where cloudy weather did not permit enough runs to make a quantitative fit feasible. The resulting values of $A_0$ agree with the results of June 16 and 17.

The size of the instrumental oblateness $\Delta_{M1}$ was also in qualitative agreement, on the above days. In addition, data taken at a different amplitude of oscillation by Douglas Patz on July 4, 1972 (SCLERA Progress Report 1972, p. 27) gave $\Delta_{M1} = -0.170 \pm 0.048$ arc seconds. The two quantitative values agree within error limits.

During the runs, the magnitude of the chopped error signal in the primary tracker (Chapter II, the section titled The Solar Oblateness Detector and Primary Tracker) indicated that change in oscillation amplitude was never greater than 1%. (The maximum open loop gain change was 15%.) The worst instrumental error in a single value of $\Delta_{ep}$ due to amplitude changes was less than 10 milliseconds of arc, and the effect on the value of $A_0$ obtained in this measurement is therefore not significant.

During the June 17th run, the "seeing" appeared slightly anisotropic: In two-thirds of the measurements,
the polar "seeing" was larger than that at the equator. It is not clearly indicated by the limited data presently available whether the anisotropy is atmospheric in origin, or a function of the nonlinearity of the gain in the slit servo, or a result of differing dead bands in the Mirror II servo. The difference between corrections from one diameter to the other was typically less than 50 milliseconds of arc and the maximum "seeing" correction to a value of $A_{ep}$ was 200 milliseconds. There was no correlation between either sign or magnitude of the "seeing" correction and the scatter about the fitted curve. This lack of correlation and the excellent linearity in the relationship between diameter measurements and "seeing" as discussed in Chapter IV, indicate that "seeing" is effectively eliminated as a source of systematic error in the solar oblateness measurement. The use of FCT eliminates effects that would otherwise cause increased scatter in the data, even if no anisotropy exists.

Sensitivity to Localized Solar Structure

An examination of the data obtained on June 24, 1972 shows the extreme sensitivity of the edge definition to localized solar structure. Figure 12 shows the data corrected for instrumental errors from Mirror I ($A_{M1} = -0.137$ arc seconds) (SCLERA Progress Report 1972, p. 35).
Fig. 12. Effect of sunspot on measured oblateness.

The $\Delta_\varnothing$ at $a = 6.8$ arc seconds ($\Delta_\varnothing$ corrected for the instrumental error assuming $\Delta_\varnothing = -0.137$ arc seconds) as a function of the parallactic angle $\eta$. These data were obtained on June 24th, 1972. The solid line is a smooth curve drawn through the data showing the effect of a sun spot. The angle $\eta$ increases monotonically during the day passing through zero at noon. The change in $\eta$ around noon is 90° in approximately 1½ hours.
Early in the morning the solar oblateness agrees qualitatively with the value of 0.121 arc seconds obtained from the June 16 and 17 data. Then gradually the equatorial diameter as defined by FCT became smaller and a negative oblateness resulted. The total change was approximately 0.240 arc seconds. A sunspot was visually observed two days later which would have been tangent to the limb on June 24. The affected limb would have been partially imaged on the slit at the equator. Figure 13 is a spectrogram taken on June 27 which shows the sunspot when it was no longer at the extreme limb.

A check of solar photographs taken before and after June 16 and 17 verifies that sunspots were not a factor in those solar oblateness measurements. The fact that the slits tracked the sunspots so well lends credence to the measurements not affected by sunspots.

Possible Short Term Fluctuations in the Solar Photosphere or Limb Darkening Profile

The sensitivity of the solar oblateness measurements to localized structure suggests that some of the scatter in the data which appears non-statistical may be due to fluctuations in solar structure. The scatter in successive equatorial measurements on June 16 and 17 was twice that found in the polar diameter measurements. This scatter could be due to short term oscillations of the photosphere, short
This spectrogram of June 27, 1972 (12:57 MST) was obtained with a narrow bandwidth filter centered on H$_\alpha$-4.31Å. The sunspot group near the east equatorial limb, indicated by fiducial marks, was partially intercepted by the east limb slit on the June 24, 1972 oblateness data discussed in the text. (Courtesy of Sacramento Peak Observatory, Air Force Cambridge Research Laboratories.)
term phenomena associated with convection and granulation, or actual variations in the limb darkening function. Although there is presently insufficient data to make satisfactory models for these fluctuations, continued measurements could yield exciting information about solar activity.

**Conclusions**

Measurements made during a relatively short time span, but with an accuracy not previously achieved, indicate that a visual solar oblateness of $0.121 \pm 0.029$ arc seconds exists. This value of the solar oblateness at $a = 6.8$ arc seconds which is a difference in diameters agrees quite well with the $0.050 \pm 0.007$ arc seconds of radial difference reported by Dicke and Goldenberg (1967). However, the value of $\Delta_0$ at $a = 54$ arc seconds measured by Patz (SCLERA Progress Report 1972, p. 27) is not in agreement and indicates that the equatorial and polar limb darkening functions must be different. The interpretation is not the subject of this work but is given in detail elsewhere (SCLERA Progress Report 1972, p. 27). Continuing oblateness measurements will be made (SCLERA Progress Report 1972, p. 55), which should be significant in actually determining the origin of the observed visual solar oblateness. Until more definitive measurements are made, one must conclude that although there is a visual oblateness, the relation of this to a $J_2$ is
still in question. If preliminary indications are substantiated, it will probably be concluded that the solar oblateness is due to latitude dependence in the solar limb darkening function and that $J_2$ is that produced by a uniformly rotating sun with a 25-day period. This conclusion would certainly confirm the generally accepted agreement between the observed perihelion precession and the value predicted by Einstein in General Relativity.
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