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I hereby recommend that this dissertation prepared under my direction by Stillman Eugene Sims

entitled Convergence Properties of Spline Functions

be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

M. S. Cheema
Dissertation Director

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After inspection of the final copy of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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SIGNED: S Eugene Sims

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ABSTRACT

The primary purpose of this paper is to prove some convergence properties for spline functions in various function spaces. However, in the more general setting of a normed linear space with strictly convex norm, it is first shown that the operator which assigns to each element in the space its best approximant from a given subspace is continuous. Next in the function space L_p, $p > 1$, some convergence theorems for best approximation by spline functions are shown to follow from known results. The main conclusions, which are convergence properties for the type I polynomial spline are then obtained. In addition, similar convergence properties for spline functions which resemble the type I polynomial spline are cited. Finally, equations for obtaining the best least square approximant by a spline function are developed, and it is noted that these equations may also be used to find the type I polynomial spline.

CHAPTER 1

INTRODUCTION

A polynomial spline function of degree m is a function whose $(m - 1)^{th}$ derivative is a step function. The name spline function is derived from a mechanical spline which is a device used by draftsmen to draw a smooth curve between specified points. In 1946, Schoenberg first introduced the concept of a mathematical spline function, but the study of these functions lay somewhat dormant until 1957, when Holladay proved the minimum curvature property for cubic splines [1]. Using this property as a guide Ahlberg, Nilson and Walsh introduced in 1964, a function space aspect to the theory of spline functions which has led to a considerable amount of interest in this subject in recent years. Those mentioned above have been interested in spline functions which interpolate to a given set of points. On the other hand, Rice and Schumaker have characterized best approximation by spline functions in uniform norm, while Marsden and Schoenberg have introduced spline functions which generalize the Bernstein polynomials, thus displaying nice graphical behaviour.

Spline functions have found wide application in both theory and practice. Indeed, the polynomial spline usually provides a smoother interpolation to a given set of data, and a smoother and closer approximation to a given function than a polynomial. Moreover, spline functions can be used in numerical integration and differentiation, and have been applied in space technology.

From a theoretical point of view various kinds of spline functions have been defined, and have been shown to satisfy a variety of properties many of which are stated in subsequent chapters. In particular, if the spline function is confined to a closed interval on which a sequence of meshes is given, then convergence theorems which depend upon the convergence of a corresponding sequence of mesh norms have been proved [1]. In this paper some convergence properties for polynomial spline functions called type I spline functions as well as for other spline functions which are closely associated with the type I spline are proved. These convergence properties rely on the convergence of a sequence of functions on a fixed mesh defined on a closed interval rather than the convergence of a sequence of mesh norms. Chapters 4 and 5 are devoted to these topics.

The motivation for this study was an interest in determining whether the best approximant operator is continuous in a normed linear space. To be more precise let X be a normed linear space, and let V be a subspace of X. A best approximant from the subspace V to an element y in X is an element x^* in V which satisfies

$$||y - x^*|| = \inf_{x \in V} ||y - x||.$$

It happens that the operator which maps any element y in X to its best approximant x^* in V is a continuous operator in the topology induced by the norm provided that X has a strictly convex norm. The proof of this statement is the subject of Chapter 2.

In Chapter 3 a basis for the subspace of polynomial spline functions in a given function space is exhibited, and is used together with known results to obtain convergence properties for the best spline approximant in L_p-norm, $p > 1$. The final chapter is devoted to developing equations which yield the best spline approximant in L_2-norm, and ultimately the type I polynomial spline.

In this paper a mesh Δ on the closed interval $[a,b]$ is defined to be a division of $[a,b]$ into a finite number of subintervals by the points $x_0, x_1, \ldots, x_N$ satisfying

$$a = x_0 < x_1 < \ldots < x_N = b.$$

The mesh norm is defined by

$$||\Delta|| = \max_{0 \leqq i \leqq N} |x_i - x_{i-1}|.$$

Furthermore, several different notations are used for the norm of a given linear space, and the various kinds of spline functions are denoted according to their type, degree, or dependence on a particular function or mesh. In both instances it should be clear from the context how the notation is to be interpreted.

CHAPTER 2

CONTINUITY OF THE BEST APPROXIMANT OPERATOR

Throughout this chapter it is assumed that X is a normed linear space with strictly convex norm, and that V is a subspace of X spanned by the linearly independent elements x_1, x_2, ..., x_n. Then for each element y in X there exists a unique element x^* in V which is the best approximant to y in the sense that

$$||y - x^*|| = \inf_{x \in V} ||y - x||.$$

Let F be the operator which assigns to each element in X its best approximant in V, that is, $F(y) = x^*$ for each y in X. In this chapter it will be shown that this operator is continuous in the topology induced by the norm.

Before attempting a proof it should be pointed out that a number of results have been obtained in this direction. It is easy to prove that if the unique best approximant to any element in a metric space is taken from a compact subset of the space, then the best approximant operator is continuous [4]. Furthermore, it has been shown that the best approximant operator is continuous in C[a,b] with uniform norm provided that the subspace from which the best approximant is taken satisfies a Haar condition [4]. But this proof depends on the equioscillation property of the best uniform approximant and does not extend, for example, to L_p space. However, Smith proved [14] that the best approximant operator is continuous in $L_p[a,b]$, and the proof given here, with slight modifications, extends his proof to the more general setting.

In order to obtain the proof several lemmas will be needed.

Lemma 2.1. If $\lim_{k\to\infty} ||y - y_k|| = 0$, and if $x_k^*(k = 1, 2 \ldots)$ and x^* are the best approximants for $y_k(k = 1, 2 \ldots)$ and y respectively, then

$$\lim_{k\to\infty} ||y_k - x_k^*|| = ||y - x^*||.$$

Proof. Since x_k^* is the best approximant to y_k and x^* is the best approximant to y

$$||y_k - x_k^*|| \leqq ||y_k - x^*||,$$

and

$$||y - x^*|| \leqq ||y - x_k^*||.$$

These two inequalities yield

$$||y_k - x_k^*|| \leqq ||y - y_k|| + ||y - x^*||, \qquad (2.1)$$

and

$$||y - x^*|| \leq ||y - y_k|| + ||y_k - x_k^*||,$$

or, rewriting this last inequality

$$||y - x^*|| - ||y - y_k|| \leq ||y_k - x_k^*||. \tag{2.2}$$

Now inequalities (2.1) and (2.2) can be combined to give

$$||y - x^*|| - ||y - y_k|| \leq ||y_k - x_k^*|| \leq ||y - y_k|| + ||y - x^*||.$$

Taking the limit of both sides of this last inequality yields

$$||y - x^*|| \leq \lim_{k\to\infty} ||y_k - x_k^*|| \leq ||y - x^*||,$$

or

$$\lim_{k\to\infty} ||y_k - x_k^*|| = ||y - x^*||.$$

Lemma 2.2. Given the same hypothesis as in the previous lemma

$$\lim_{k\to\infty} ||y - x_k^*|| = ||y - x^*||.$$

<u>Proof.</u> As in the above proof

$$||y - x^*|| \leqq ||y - x_k^*|| \leqq ||y - y_k|| + ||y_k - x_k^*||.$$

In the limit this becomes

$$||y - x^*|| \leqq \lim_{k\to\infty} ||y - x_k^*|| \leqq \lim_{k\to\infty} ||y - y_k|| + \lim_{k\to\infty} ||y_k - x_k^*||.$$

Now an application of the hypothesis and Lemma 2.1 results in the inequality

$$||y - x^*|| \leqq \lim_{k\to\infty} ||y - x_k^*|| \leqq ||y - x^*||.$$

Finally, then

$$\lim_{k\to\infty} ||y - x_k^*|| = ||y - x^*||.$$

At this point it becomes necessary to define the subspace

$$U = \{by + \sum_{i=1}^{n} a_i x_i\}$$

where y is a fixed element in the derived set of X, that is, such that $\lim_{k\to\infty} ||y - y_k|| = 0$ for some sequence $\{y_k\}$ in X. Clearly U is closed in X since it is a finite dimensional subspace of a normed linear space. Furthermore, the subset W of U will be defined by

$$W = \{y - x_k^*\} \quad (k = 1, 2, \ldots)$$

where the x_k^* are the best approximants to the y_k.

<u>Lemma 2.3</u>. W is a bounded subset of U.

<u>Proof</u>. From Lemma 2.2

$$\lim_{k \to \infty} ||y - x_k^*|| = ||y - x^*||.$$

Hence for any positive number ε there exists a positive integer k_0 such that

$$||y - x^*|| - \varepsilon \leqq ||y - x_k^*|| \leqq ||y - x^*|| + \varepsilon$$

for all $k > k_0$. Thus $\{||y - x_k^*||\}$ is bounded by

$$\max \{||y - x_1^*||, \ldots, ||y - x_{k_0}^*||, ||y - x^*|| + \varepsilon\} .$$

The proof of the final lemma needed before proceeding to the main theorem can be found in [5].

<u>Lemma 2.4</u>. Let W be a bounded subset of a finite dimensional normed linear space U. If $\{z_k\}$ is an arbitrary sequence of elements

in W, then there exists a subsequence $\{z_{k_j}\}$ that converges to an element z in U.

Theorem 2.1. Let X be a normed linear space with strictly convex norm, and let $\{y_k\}$ be a sequence in X such that

$$\lim_{k\to\infty} ||y - y_k|| = 0.$$

If $x_k^*(k = 1, 2 \ldots)$ and x^* are the best approximants to y_k and y, respectively, from the subspace V, then

$$\lim_{k\to\infty} ||x^* - x_k^*|| = 0.$$

In other words, the best approximant operator is continuous.

Proof. Assume that the best approximant operator is not continuous, that is

$$\lim_{k\to\infty} ||x^* - x_k^*|| \neq 0.$$

This implies that

$$\lim_{k\to\infty} ||(y - x_k^*) - (y - x^*)|| \neq 0.$$

Hence there exists a positive number ε_0 and a subsequence $\{y - x_{k_j}^*\}$

of $\{y - x_k^*\}$ such that

$$||(y - x_{k_j}^*) - (y - x^*)|| > \varepsilon_0.$$

Now $\{y - x_{k_j}^*\}$ is a sequence in the bounded subset W of U as defined above. Thus Lemma 2.4 implies that $\{y - x_{k_j}^*\}$ has a convergent subsequence $\{y - x_{k_{j_m}}^*\}$ in U. So $\{y - x_{k_{j_m}}^*\}$ is a Cauchy sequence in U which implies that for each positive number ε there exists a positive integer m_0 such that

$$||(y - x_{k_{j_m}}^*) - (y - x_{k_{j_n}}^*)|| < \varepsilon$$

for all m, $n > m_0$. But this reduces to

$$||x_{k_{j_m}}^* - x_{k_{j_n}}^*|| < \varepsilon$$

for each ε and all m, $n > m_0$. Therefore, $\{x_{k_{j_m}}^*\}$ is a Cauchy sequence in U since V is a subset of U. Thus $\{x_{k_{j_m}}^*\}$ converges to some element z in U since any finite dimensional normed linear space is complete, and U is closed in X. Furthermore, each $x_{k_{j_m}}^*$ is in V, and V is closed in U implies that z is in V.

Now, suppressing the subscripts,

$$\lim_{k \to \infty} ||x_k^* - z|| = 0$$

implies that

$$\lim_{k\to\infty} ||(y - x_k^*) - (y - z)|| = 0.$$

The norm, $||\cdot||$, is a continuous functional, thus

$$\lim_{k\to\infty} ||y - x_k^*|| = ||y - z||. \tag{2.3}$$

On the other hand, Lemma 2.2 gives

$$\lim_{k\to\infty} ||y - x_k^*|| = ||y - x^*||. \tag{2.4}$$

Taken together (2.3) and (2.4) yield

$$||y - z|| = ||y - x^*||. \tag{2.5}$$

But $z \neq x^*$ since

$$||(y - x_k^*) - (y - x^*)|| = ||x_k^* - x|| > \varepsilon_0$$

in the limit becomes

$$\lim_{k\to\infty} ||x_k^* - x^*|| = ||z - x^*|| > \varepsilon_0.$$

Thus (2.5) is a contradiction to the uniqueness of the best approximant. Hence the initial assumption was false, and the best approximant operator is continuous.

It is worthy of note that in the above proof the strict convexity of the norm was used only to insure that the best approximant was unique. Therefore, this requirement could be replaced by the less stringent condition that the best approximant from a given subspace be unique. For example, this type argument could be used as an alternate proof of the continuity of the best approximant operator in $C[a,b]$ with uniform norm when the subspace in question satisfies a Haar condition.

CHAPTER 3

THE BEST SPLINE APPROXIMANT IN L_p SPACE

There seems to be a lack of information concerning approximation by spline functions in L_p space, $p > 1$. Perhaps this is due to the ease with which information can be obtained in L_p space from known results in a more general framework. However, it will become evident in the subsequent chapters that the study of various types of spline functions can be reduced to a study of spline approximation in L_2 space. Therefore, it seems worthwhile to write down some of these theorems, and show how they follow from the known results.

Definition 3.1. Let $\Delta: a = x_0 < x_1 < \dots < X_N = b$ be a mesh on the interval $[a,b]$. A polynomial spline of degree m with respect to the mesh Δ is defined to be a function on $[a,b]$ which is in the class $C^{m-1}[a,b]$, and which reduces to a polynomial of degree m in each subinterval (x_{i-1}, x_i): $i = 1, \dots, N$.

In order to consider the approximation of a given function in L_p by a polynomial spline of degree m it is necessary to show that any spline function can be written as a linear combination of a set of linearly independent functions in L_p.

Definition 3.2. The truncated power function is defined by

$$x_+^m = \begin{cases} 0 & \text{if } x < 0 \\ x^m & \text{if } x \geqq 0. \end{cases}$$

The following theorem gives a representation of a polynomial spline of degree m as a sum of a polynomial of degree m and a linear combination of truncated power functions. The proof given here can be found in [6].

Theorem 3.1. Every polynomial spline function of degree m on the mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ has a unique representation in the form

$$S_m(x) = P_m(x) + \sum_{i=1}^{N-1} b_i (x - x_i)_+^m \tag{3.1}$$

where $P_m(x)$ is a polynomial of degree m.

Proof. For $i = 0, 1, \ldots, N - 1$, let $P_{mi}(x)$ be the polynomial that gives $S_m(x)$ in the interval (x_i, x_{i+1}). Then according to Definition 3.1 $P_{mi}(x) - P_{m,i-1}(x)$ is a polynomial of degree m having an m-fold zero at $x = x_i$. That is, for some b_i this polynomial reduces to

$$P_{mi}(x) - P_{m,i-1}(x) = b_i(x - x_i)^m. \tag{3.2}$$

But

$$P_{mk}(x) = P_{m0}(x) + (P_{m1}(x) - P_{m0}(x)) + \ldots + (P_{mk}(x) - P_{m,k-1}(x)).$$

Thus

$$P_{mk}(x) - P_{m0}(x) = \sum_{i=1}^{k} b_i(x - x_i)^m.$$

Taking $P_m(x) = P_{m0}(x)$ this is equivalent to (3.1) since for x in (x_i, x_{i+1}) the truncated power function $(x - x_i)_+^m = 0$ for $i = k + 1, k + 2, \ldots, N - 1$.

All that remains is to show that this representation is unique. By definition, $S_m(x) = P_{m0}(x)$ for $x < x_1$. But for $x < x_1$ (3.1) reduces to $S_m(x) = P_m(x)$. Hence $P_m(x) \equiv P_{m0}(x)$. Furthermore, m-fold differentiation of (3.2) yields

$$P_{mi}^{(m)}(x) - P_{m,i-1}^{(m)}(x) = b_i m!.$$

Letting $x = x_i$, this may be written as

$$b_i = \frac{1}{m!}[S_m^{(m)}(x_i + 0) - S_m^{(m)}(x_i - 0)]$$

where $S_m^{(m)}(x_i + 0)$ and $S_m^{(m)}(x_i - 0)$ are the right and left hand m^{th}-derivatives of $S_m(x)$ at x_i. Hence the representation is unique.

Thus, the set of all polynomial spline functions is precisely the subspace of the linear space L_p which is spanned by the linearly independent set of functions

$$\{1, x, \ldots, x^m, (x - x_1)_+^m, \ldots, (x - x_{N-1})_+^m\}. \tag{3.3}$$

The following theorem is a consequence of the fact that L_p, $p > 1$ is a normed linear space with strictly convex norm.

<u>Theorem 3.2</u>. Let f in L_p, $p > 1$, and a mesh Δ be given on $[a,b]$. Then there exists a unique best polynomial spline approximant to the function f in L_p-norm.

If S and S^* are used to denote an arbitrary polynomial spline and the best approximant to f by a spline function in L_p-norm respectively, then

$$||f - S^*||_p \leqq ||f - S||_p.$$

The next theorem follows from Theorem 2.1 provided that the best spline approximant in L_p-norm is considered as an operator mapping any function f in L_p into the subspace of polynomial splines.

Theorem 3.3. The best L_p spline approximant operator is a continuous operator.

There is a classical result in the literature [10] known as Polya's algorithm, which connects best L_p-approximants, $p > 1$, with best approximants in the uniform norm. This latter approximant is often referred to as the best Tchebycheff approximant.

Theorem 3.4. Let $\{\phi_i : i = 1, \ldots, n\}$ be a linearly independent set of elements in the space $C[a,b]$, and let $A_p = (a_1^p, \ldots, a_n^p)$ be the set of coefficients of the best L_p-approximant, $p > 1$, from the subspace spanned by the $\phi_i (i=1, \ldots, n)$ to a given function f in $C[a,b]$. Then the sequence of points $\{A_p : p = 1, 2, \ldots\}$ in E_n contains a convergent subsequence. Further, the limit A^* of any convergent subsequence of $\{A_p\}$ is the set of coefficients of a best Tchebycheff approximant to f on $[a,b]$.

Corollary 3.1. With the hypothesis of Theorem 3.4, if there is a unique best Tchebycheff approximant to f on $[a,b]$ with coefficient set $A^* = (a_1^*, \ldots, a_n^*)$, then

$$\lim_{p \to \infty} A_p = A^*.$$

Obviously, the above theorem holds for approximation of a given continuous function by a polynomial spline. Unfortunately, the best spline approximant in uniform norm is not necessarily unique since the set of functions (3.3) do not satisfy a Haar condition [12]. However, Rice [11] and Schumaker [12] have characterized the best spline approximant in uniform norm. Indeed, a combination of Theorem 4 of [11], and Corollary 3.1 above yields the analog of this Corollary for polynomial splines.

Theorem 3.5. Let f in $C[a,b]$ and a mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ be given. Further, let $S(A_p,\Delta,x)$ and $S(A^*,\Delta,x)$ represent the best L_p-approximant to f and a best Tchebycheff approximant to f, respectively, by a polynomial spline of degree m. If $f(x) - S(A^*,\Delta,x)$ alternates $m + r + 1$ times $(r \leqq N - 1)$ on $[x_i, x_{i+r+1}]$, but does not alternate $m + q + 1$ times on any subinterval $[x, x_{i+q+1}]$ of $[x_i, x_{i+r+1}]$, then

$$\lim_{p\to\infty} A_p = A^*.$$

In the next theorem the convergence of the best L_p spline approximant as the degree of the polynomial spline tends to infinity will be investigated.

Theorem 3.5. Let f in $C[a,b]$ and a mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ be given. Furthermore, let $S^*_m(x)$ denote the best L_p - approximant to f by a polynomial spline of degree m. Then

$$\lim_{m\to\infty} ||f - S^*_m||_p = 0.$$

Proof. Since f is in $C[a,b]$ it is known that

$$\lim_{m\to\infty} ||f - P_m^*||_p = 0 \tag{3.4}$$

where P_m^* represents the best L_p - approximant to f by a polynomial of degree m [4]. Now the subspace generated by the set $\{x^j : j = 0, 1, \ldots, m\}$ is a subspace of the space spanned by the set (3.3). Hence,

$$||f - S_m^*||_p \leqq ||f - P_m^*||_p$$

for all m. This along with (3.4) implies

$$\lim_{m\to\infty} ||f - S_m^*||_p = 0.$$

In order to prove that the sequence $\{P_m^*(x) : m = 1, 2 \ldots\}$ of best L_p polynomial approximants to a function f converges to the function f in L_p-norm as m tends to infinity it is sufficient to know that there exists a sequence of polynomials which converges to f as the degree of the polynomials increases. This is guaranteed by the Weierstrass Approximation Theorem. Recently Marsden [8] has proved a similar approximation theorem for polynomial spline functions. Without getting sidetracked by details it suffices to remark that the sequence of splines used in Marsden's proof are the variation diminishing splines introduced by Marsden and Schoenberg [9]. It is not necessary at present to know what this sequence of splines look like, but merely to know that such a sequence exists. Indeed Marsden's Theorem implies the following theorem as a special case.

Theorem 3.6. If f is in $C[a,b]$ and if $\{\Delta_N\}$ is a sequence of meshes such that

$$\lim_{N\to\infty} ||\Delta_N|| = 0$$

then there exists a sequence of splines $\{\overline{S}_{\Delta_N}\}$ of fixed but arbitrary degree m such that

$$\lim_{N\to\infty} ||f - \overline{S}_{\Delta_N}||_\infty = 0.$$

As an immediate corollary of this result the next theorem follows.

Theorem 3.7. If f is in $C[a,b]$ and $\{\Delta_N\}$ is a sequence meshes such that

$$\lim_{N\to\infty} ||\Delta_N|| = 0$$

then the sequence of best L_p spline approximants $\{S^*_{\Delta_N}\}$ of degree m converges in L_p-norm to the function f as N tends to infinity.

That is,

$$\lim_{N\to\infty} ||S^*_{\Delta_N} - f||_p = 0.$$

Proof. Since $S^*_{\Delta_N}$ is the best L_p spline approximant to f on the mesh Δ_N

$$||f - S^*_{\Delta_N}||_p \leqq ||f - \overline{S}_{\Delta_N}||_p \tag{3.5}$$

where $\overline{S}_{\Delta_N}$ is the spline whose existence was guaranteed in Theorem 3.6. Now (3.5) implies

$$||f - S^*_{\Delta_N}||_p \leqq ||f - \overline{S}_{\Delta_N}||_\infty (b - a)^{1/2}.$$

Therefore,

$$\lim_{N\to\infty} ||f - S^*_{\Delta_N}||_p = 0.$$

CHAPTER 4

CONVERGENCE PROPERTIES OF SPLINE FUNCTIONS

Numerous convergence properties have been proved for different types of spline functions. However there seems to have been no exploration of the convergence properties of a sequence of spline functions which are defined in terms of a convergent sequence of functions in a given normed linear space. It will become clear how the sequence of spline functions is determined as the chapter progresses. The following definitions and results from the theory of spline functions will be essential to the development. In stating these definitions, and henceforth in this paper Δ will denote a mesh $\Delta: a = x_0 < x_1 < \dots < x_N = b$ on the interval $[a,b]$.

Definition 4.1. Let f be an arbitrary function defined on $[a,b]$. The polynomial spline of degree m which interpolates to the function f on the mesh Δ is called a spline of interpolation to f on Δ and is denoted by $S_m(f,x)$. That is,

$$S_m(f,x_i) = f(x_i), \qquad (i = 0, 1, \dots, N).$$

Definition 4.2. Let f be a function defined on $[a,b]$ such that $f^{(\alpha)}(a)$ and $f^{(\alpha)}(b)$ exist for $\alpha = 0, 1, \dots, n-1$, and let

$S_{2n-1}(f,x)$ be a polynomial spline of interpolation to f on the mesh Δ of degree $2n-1$. Then $S_{2n-1}(f,x)$ is called a type I spline of interpolation provided

$$S_{2n-1}^{(\alpha)}(f,x_i) = f^{(\alpha)}(x_i) \ (i = 0, N) \ (\alpha = 1, \ldots, n - 1).$$

The preceding definition is given only for polynomial splines of odd degree because the polynomial spline of interpolation of even degree may fail to exist. On the other hand, the type I polynomial spline of odd degree exists and is unique for a rather broad class of functions.

Definition 4.3. The class of all functions defined on $[a,b]$ which possess an absolutely continuous $(n - 1)^{th}$ derivative and whose n^{th} derivative is in $L_2[a,b]$ will be denoted by $K^n(a,b)$.

It is for this class of functions that the existence and uniqueness theorem will be needed in this presentation. Although the existence and uniqueness theorems and their respective proofs are given separately in [1], here the two will be combined into a single statement for the sake of brevity.

Theorem 4.1. Let f be in $K^n(a,b)$ and let Δ be a mesh on the interval $[a,b]$. Then there exists a unique type I polynomial spline of interpolation to f on the mesh Δ of degree $2n-1$.

Several properties of the type I polynomial spline function whose utility will soon become apparent will now be listed. For proofs the reader is again referred to [1].

First Integral Relation. If f is in $K^n(a,b)$ and $S_{2n-1}(f,x)$ is the type I spline of interpolation to f on Δ, then

$$\int_a^b [f^{(n)}(x)]^2 dx = \int_a^b [S_{2n-1}^{(n)}(f,x)]^2 dx + \int_a^b [f^{(n)}(x) - S_{2n-1}^{(n)}(f,x)]^2 dx.$$

Minimum Norm Property. Let a mesh Δ on $[a,b]$ and a function f having $n - 1$ derivatives at both $x = a$ and $x = b$ be given. Then, of all functions g in $K^n(a,b)$ which interpolate to f on Δ, and such that $g^{(\alpha)}(x_i) = f^{(\alpha)}(x_i)$ $(i = 0, N)$ $(\alpha = 1, \ldots, n - 1)$, the type I spline of interpolation to f on Δ of degree $2n-1$ is the unique admissible function which minimizes

$$\int_a^b [g^{(n)}(x)]^2 dx.$$

Best Approximation Property. Let a mesh Δ on $[a,b]$ and f in $K^n(a,b)$ be given. Let $S_{2n-1}(x)$ be any polynomial spline on Δ of degree $2n-1$, and let $S_{2n-1}(f,x)$ be the type I spline of interpolation to f on Δ of degree $2n-1$. Then

$$\int_a^b [f^{(n)}(x) - S_{2n-1}^{(n)}(x)]^2 dx \geqq \int_a^b [f^{(n)}(x) - S_{2n-1}^{(n)}(f,x)]^2 dx, \qquad (4.1)$$

and equality holds if, and only if

$$S_{2n-1}(f,x) = S_{2n-1}(x) + p_{n-1}(x)$$

where $p_{n-1}(x)$ is a polynomial of degree $n - 1$.

A psuedo inner product can be defined on the linear space $K^n(a,b)$ by

$$(f,g) = \{\int_a^b f^{(n)}(x)g^{(n)}(x)dx\}^{1/2}. \tag{4.2}$$

The associated psuedo norm is, of course, defined by

$$||f||_n^2 = \int_a^b [f^{(n)}(x)]^2 dx. \tag{4.3}$$

Now if an equivalence relation is defined on $K^n(a,b)$ by; f is equivalent to g if, and only if, f and g differ by a polynomial of degree less than or equal to $n - 1$, then (4.2) and (4.3) lose their psuedo-character on the resulting equivalence classes of functions. Thus if functions differing by a polynomial of degree less than or equal to $n - 1$ are identified then $K^n(a,b)$ becomes an inner product space. Furthermore, it can be shown that this inner product space is a Hilbert space [1].

Given a fixed mesh Δ on $[a,b]$ an operator Ψ which will be called the type I spline operator can be defined on the linear space

$K^n(a,b)$ by assigning to each function in this space its unique type I polynomial spline of interpolation of degree $2n-1$. Moreover, if the identification mentioned above is made, then this operator can be considered as an operator on the Hilbert space $K^n(a,b)$. In order to interpret the operator in this manner it must be shown that the operator is well-defined [13].

Lemma 4.1. If f and g are any two functions which are in the same equivalence class in the Hilbert space $K^n(a,b)$, and if Ψ is the type I spline operator, then $\Psi(f)$ and $\Psi(g)$ are in the same equivalence class.

Proof. Since f and g are in the same equivalence class,

$$f - g = P_{n-1}$$

where P_{n-1} is a polynomial of degree less than or equal to $n - 1$. It can be shown that the type I polynomial spline is linear [1]; hence

$$S_{2n-1}(f - g) = S_{2n-1}(f) - S_{2n-1}(g).$$

But

$$S_{2n-1}(f - g) = S_{2n-1}(P_{n-1}) = P_{n-1}$$

where the last equality resulted from the uniqueness of the type I polynomial spline. Thus

$$\Psi(f) - \Psi(g) = S_{2n-1}(f - g) = P_{n-1}.$$

That is, $\Psi(f)$ and $\Psi(g)$ are in the same equivalence class.

Theorem 4.2. The type I spline operator is a continuous operator on the Hilbert space $K^n(a,b)$.

Proof. The best approximation property (4.1) implies that given a function f in the linear space $K^n(a,b)$ the equivalence class containing the type I polynomial spline of interpolation to f on a mesh Δ is the best spline approximant to the equivalence class containing the function f. The proof now follows from Theorem 2.1 since an inner product space is a normed linear space with strictly convex norm.

However, it is interesting to note that in this case the continuity of the operator can be arrived at using the minimum norm property. Let f be in $K^n(a,b)$, and let $\{f_k\}$ be a sequence in $K^n(a,b)$ such that $\lim_{k\to\infty} ||f_k - f||_n = 0$. Now the type I polynomial spline is linear on $K^n(a,b)$. [1] which implies that

$$S_{2n-1}(f - f_k) = S_{2n-1}(f) - S_{2n-1}(f_k).$$

Obviously, $f - f_k$ is a function which satisfies the hypothesis of the minimum norm property for each k. Thus

$$||\Psi(f) - \Psi(f_k)||_n = ||S_{2n-1}(f) - S_{2n-1}(f_k)||_n$$

$$= ||S_{2n-1}(f - f_k)||_n \leqq ||f - f_k||_n.$$

Hence,

$$\lim_{k\to\infty} ||\Psi(f) - \Psi(f_k)||_n = 0,$$

which implies that type I spline operator is a continuous operator on the Hilbert space $K^n(a,b)$.

The question that now arises is whether these convergence properties carry over to the space $C^n[a,b]$ with uniform norm. A partial answer is given in the following theorem.

<u>Theorem 4.3</u>. Let a function f and a sequence of functions $\{f_k\}$ be given in $C^n[a,b]$ such that

$$\lim_{k\to\infty} f_k^{(\alpha)}(x) = f^{(\alpha)}(x) \quad (\alpha = 0, 1, \ldots, n)$$

uniformly for all x in $[a,b]$. Then

$$\lim_{k\to\infty} S_{2n-1}^{(\alpha)}(f_k,x) = S_{2n-1}^{(\alpha)}(f,x) \quad (\alpha = 0, 1, \ldots, n - 1)$$

uniformly for all x in $[a,b]$ where the $S_{2n-1}(f_k,x)(k = 1, 2, \ldots)$ and $S_{2n-1}(f,x)$ are the type I polynomial splines of interpolation to the $f_k(k = 1, 2, \ldots)$ and f, respectively, on a given mesh Δ.

Proof. Let $||f||_\infty = \max\limits_{a \leq x \leq b} |f(x)|$. Then

$$\left\{\int_a^b [f^{(n)}(x) - f_k^{(n)}(x)]^2 dx\right\}^{1/2} \leq \max_{a \leq x \leq b} |f^{(n)}(x) - f_k^{(n)}(x)| \left\{\int_a^b dx\right\}^{1/2}$$

$$= ||f^{(n)} - f_k^{(n)}||_\infty (b - a)^{1/2}. \tag{4.4}$$

Thus,

$$\lim_{k\to\infty} ||f^{(n)} - f_k^{(n)}||_\infty = 0$$

implies that

$$\lim_{k\to\infty} \left\{\int_a^b [f^{(n)}(x) - f_k^{(n)}(x)]^2 dx\right\}^{1/2} = 0.$$

Now since the spline functions in discussion are the type I polynomial splines

$$S_{2n-1}^{(n-1)}(f,x) - S_{2n-1}^{(n-1)}(f_k,x)$$

$$= \int_a^x [S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)]du + S_{2n-1}^{(n-1)}(f,a) - S_{2n-1}^{(n-1)}(f_k,a)$$

$$= \int_a^x [S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)]du + f^{(n-1)}(a) - f_k^{(n-1)}(a).$$

Hence, for any x in $[a,b]$

$$|S_{2n-1}^{(n-1)}(f,x) - S_{2n-1}^{(n-1)}(f_k,x)|$$

$$\leqq \int_a^x |S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)|du + |f^{(n-1)}(a) - f_k^{(n-1)}(a)|$$

$$\leqq \int_a^b |S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)|du + |f^{(n-1)}(a) - f_k^{(n-1)}(a)|. \quad (4.5)$$

By Schwarz's inequality applied to the function $S_{2n-1}^{(n)}(f - f_k,x)$ and the constant function $1(x) \equiv 1$,

$$\int_a^b |S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)|du$$

$$\leqq \{\int_a^b [S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)]^2 du\}^{1/2}(b - a)^{1/2}. \quad (4.6)$$

The inequalities (4.5) and (4.6) combine to give for x in $[a,b]$

$$|S_{2n-1}^{(n-1)}(f,x) - S_{2n-1}^{(n-1)}(f_k,x)|$$

$$\leqq \{\int_a^b [S_{2n-1}^{(n)}(f,u) - S_{2n-1}^{(n)}(f_k,u)]^2 du\}^{1/2}(b - a)^{1/2}$$

$$+ |f^{(n-1)}(a) - f_k^{(n-1)}(a)|$$

$$\leq \{\int_a^b [f^{(n)}(u) - f_k^{(n)}(u)]^2 \, du\}^{1/2}(b - a)^{1/2}$$

$$+ |f^{(n-1)}(a) - f_k^{(n-1)}(a)|, \tag{4.7}$$

where the last inequality resulted from an application of the minimum norm property, and the linearity of the type I spline operator. Taking the supremum over the interval $[a,b]$ and using (4.4) yields

$$||S_{2n-1}^{(n-1)}(f) - S_{2n-1}^{(n-1)}(f_k)||_\infty$$

$$\leq ||f^{(n)} - f_k^{(n)}||_\infty(b - a) + |f^{(n-1)}(a) - f_k^{(n-1)}(a)|. \tag{4.8}$$

Therefore,

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-1)}(f) - S_{2n-1}^{(n-1)}(f_k)||_\infty = 0. \tag{4.9}$$

Again since the splines are of type I

$$S_{2n-1}^{(n-2)}(f,x) - S_{2n-1}^{(n-2)}(f_k,x)$$

$$= \int_a^x [S_{2n-1}^{(n-1)}(f,u) - S_{2n-1}^{(n-1)}(f_k,u)]du + f^{(n-2)}(a) - f_k^{(n-2)}(a).$$

Hence, for any x in $[a,b]$

$$|S_{2n-1}^{(n-2)}(f,x) - S_{2n-1}^{(n-2)}(f_k,x)|$$

$$\leqq \int_a^b |S_{2n-1}^{(n-1)}(f,u) - S_{2n-1}^{(n-1)}(f_k,u)|du + |f^{(n-2)}(a) - f_k^{(n-2)}(a)|$$

$$\leqq ||S_{2n-1}^{(n-1)}(f) - S_{2n-1}^{(n-1)}(f_k)||_\infty(b - a) + |f^{(n-2)}(a) - f_k^{(n-2)}(a)|.$$

After taking the supremum over $[a,b]$ of both sides of this inequality, the estimate (4.8) can be used to acquire

$$||S_{2n-1}^{(n-2)}(f) - S_{2n-1}^{(n-2)}(f_k)||_\infty \leqq ||f^{(n)} - f_k^{(n)}||_\infty(b - a)^2$$

$$+ |f^{(n-1)}(a) - f_k^{(n-1)}(a)|(b - a) + |f^{(n-2)}(a) - f_k^{(n-2)}(a)|, \quad (4.10)$$

which implies that

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-2)}(f) - S_{2n-1}^{(n-2)}(f_k)||_\infty = 0.$$

Proceeding by induction, suppose at the j^{th} step $(0 \leqq j \leqq n - 1)$

$$||S_{2n-1}^{(n-j)}(f) - S_{2n-1}^{(n-j)}(f_k)||_\infty \leqq ||f^{(n)} - f_k^{(n)}||_\infty(b - a)^j$$

$$+ \sum_{i=1}^{j} |f^{(n-i)}(a) - f_k^{(n-i)}(a)|(b - a)^{j-i}, \quad (4.11)$$

so that

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-j)}(f) - S_{2n-1}^{(n-j)}(f_k)||_\infty = 0. \tag{4.12}$$

Then, as above

$$S_{2n-1}^{(n-j-1)}(f,x) - S_{2n-1}^{(n-j-1)}(f_k,x)$$

$$= \int_a^x [S_{2n-1}^{(n-j)}(f,u) - S_{2n-1}^{(n-j)}(f_k,u)]du + f^{(n-j-1)}(a) - f_k^{(n-j-1)}(a),$$

and for x in $[a,b]$

$$|S_{2n-1}^{(n-j-1)}(f,x) - S_{2n-1}^{(n-j-1)}(f_k,x)|$$

$$\leqq \int_a^b |S_{2n-1}^{(n-j)}(f,u) - S_{2n-1}^{(n-j)}(f_k,u)|du + |f^{(n-j-1)}(a) - f_k^{(n-j-1)}(a)|$$

$$\leqq ||S_{2n-1}^{(n-j)}(f) - S_{2n-1}^{(n-j)}(f_k)||_\infty(b - a) + |f^{(n-j-1)}(a) - f_k^{(n-j-1)}(a)|.$$

Now taking the maximum over $[a,b]$ of both sides of this inequality, and applying (4.11) gives

$$||S_{2n-1}^{(n-j-1)}(f) - S_{2n-1}^{(n-j-1)}(f_k)||_\infty \leqq \{||f^{(n)} - f_k^{(n)}||_\infty(b - a)^j$$

$$+ \sum_{i=1}^{j} |f^{(n-i)}(a) - f_k^{(n-i)}(a)|(b - a)^{j-i}\}(b - a)$$

$$+ |f^{(n-j-1)}(a) - f_k^{(n-j-1)}(a)| = ||f^{(n)} - f_k^{(n)}||_\infty (b - a)^{j+1}$$

$$+ \sum_{i=1}^{j+1} |f^{(n-i)}(a) - f_k^{(n-i)}(a)|(b - a)^{j+1-i}.$$

This inequality and the hypothesis imply that

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-j-1)}(f) - S_{2n-1}^{(n-j-1)}(f_k)||_\infty = 0.$$

Hence the induction argument is complete, and (4.11) and (4.12) hold for all α such that $0 \leqq \alpha \leqq n - 1$.

Although the type I polynomial spline is of primary interest in this paper, it is worthwhile to note that the above theorems hold for periodic polynomial splines.

<u>Definition 4.4</u>. Let f be a periodic function of period $(b - a)$, and let a mesh Δ be given on $[a,b]$. Then a periodic polynomial spline of interpolation to f on Δ of degree $2n-1$ is a polynomial spline of interpolation to f on Δ such that

$$S_p^{(\alpha)}(f,a+) - S_p^{(\alpha)}(f,b-) \quad (\alpha = 1, \ldots, n - 1).$$

The periodic polynomial spline of interpolation to a function f on a given mesh Δ satisfies conditions that are sufficient to obtain the aforementioned theorems provided that f is in the subclass $K_p^n(a,b)$ of $K^n(a,b)$ consisting of those functions which, together with their first $(n-1)^{th}$ derivatives, have continuous periodic extensions to $(-\infty,\infty)$ of period $(b-a)$. That is if f is in $K_p^n(a,b)$ then the periodic polynomial spline of interpolation to f not only exists and is unique, but also satisfies a minimum norm property, and a best approximation property. Moreover, if any two functions which differ by a constant are identified then $K_p^n(a,b)$ is an inner product space.

Now if a periodic spline operator is defined on $K_p(a,b)$ by corresponding with each function in $K_p(a,b)$ its unique periodic polynomial spline of interpolation, on a given mesh Δ, then a proof similar to that of Theorem 4.2 suffices to yield the following theorem.

Theorem 4.4. The periodic spline operator is a continuous operator on the inner product space $K_p^n(a,b)$.

Furthermore, a theorem analogous to Theorem 4.3 can be proved for periodic spline functions.

Theorem 4.5. Let a function f along with a sequence of functions $\{f_k\}$ be given in $C_p^n[a,b]$ such that

$$\lim_{k\to\infty} ||f_k^{(\alpha)} - f^{(\alpha)}||_\infty = 0 \ (\alpha = 0, 1, \ldots, n).$$

Then,

$$\lim_{k\to\infty} ||S_p^{(\alpha)}(f) - S_p^{(\alpha)}(f_k)||_\infty = 0 \ (\alpha = 0, 1, \ldots, n-1).$$

For the sake of completeness the foregoing theorems will be stated for a wider class of polynomial splines which include the type I and periodic polynomial splines defined above. To accomplish this task the following definitions are essential.

Definition 4.5. Let a mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ be given, and let $S_{2n-1}(x)$ be a function in $K^{2n-t}(a,b)$ with the property that $S_{2n-1}^{(2n)}(x)$ vanishes identically in each open mesh interval (x_{i-1}, x_i) of Δ. Then $S_{2n-1}(x)$ is called a polynomial spline of degree $2n-1$ with deficiency t where t is restricted by $0 \leqq t \leqq n$.

In this terminology, the polynomial spline of degree $2n-1$ in Definition 4.1 has deficiency one, while an ordinary polynomial of degree $2n-1$ has deficiency zero. The polynomial spline of deficiency one is sometimes called a simple spline.

Definitions analogous to those given for the type I and periodic simple splines can be made for the type I and periodic polynomial splines with deficiency t.

Definition 4.6. Let f be in the class $K^{n-t}(a,b)$ $(K_P^{n-t}(a,b))$, and let a mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ on $[a,b]$ be given. Then the type I (periodic) polynomial spline of interpolation to f on Δ of degree $2n-1$ with deficiency t is defined by the equations

$$S^{(\alpha)}_{2n-1}(f,x_i) = f^{(\alpha)}(x_i) \; (i = 0, 1, \ldots, N) \; (\alpha = 0, 1, \ldots, t - 1)$$

$$S^{(\alpha)}_{2n-1}(f,x_i) = f^{(\alpha)}(x_i) \; (i = 0, N) \; (\alpha = t, \ldots, n - 1).$$

Note that the continuity conditions are relaxed, but that this is compensated for by requiring interpolation not only to the function f, but also to its first $t - 1$ derivatives at the interior mesh points. This does not affect the nature of the proofs already given for simple splines since existence, uniqueness, a minimum norm property and a best approximation property hold for this class of deficient spline functions. Indeed, if the proper identification of functions is made, and if a type I (periodic) deficient spline operator is defined in an obvious way then the following theorems obtain.

Theorem 4.6. The type I (periodic) deficient spline operator is a continuous operator on the inner product space $K^n(a,b)$ $(K^n_p(a,b))$.

Theorem 4.7. Let f and $\{f_k\}$ be functions in $C^n[a,b]$ $(C^n_p[a,b])$ such that

$$\lim_{k\to\infty} ||f^{(\alpha)} - f^{(\alpha)}_k||_\infty = 0 \; (\alpha = 0, 1, \ldots, n).$$

If $S_{2n-1}(f,x)$ and $S_{2n-1}(f_k,x)$ represent the type I (periodic) polynomial spline of interpolation of degree $2n-1$ with deficiency t $(t \leqq n)$, then

$$\lim_{k\to\infty} ||S_{2n-1}^{(\alpha)}(f) - S_{2n-1}^{(\alpha)}(f_k)||_\infty = 0 \ (\alpha = 0, 1, \ldots, n-1).$$

Now let

$$L = a_n(x) \cdot D^n + a_{n-1}(x) \cdot D^{n-1} + \ldots + a_0(x),$$

where each $a_j(x)$ $(j = 0, 1, \ldots, n)$ is in $C^n[a,b]$, and $a_n(x)$ does not vanish on $[a,b]$, be a general linear differential operator. Furthermore, let

$$L^* = (-1)^n \cdot D^n\{a_n(x)\} + (-1)^{n-1} \cdot D^{n-1}\{a_{n-1}(x)\} + \ldots + a_0(x)$$

be the adjoint of L.

Definition 4.7. A generalized spline of deficiency t $(0 \leqq t \leqq n)$ with respect to a given mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ is a function $S_\Delta(x)$ which is in $K^{2n-t}(a,b)$, and satisfies the differential equation

$$L^* L(S_\Delta) = 0$$

on each open mesh interval of Δ. $S_\Delta(x)$ is said to have order $2n$.

A further discussion of generalized splines can be found in [1]. It will be assumed here that the operator L satisfies conditions which insure the existence, uniqueness, minimum norm, and best approximation properties of the generalized spline of interpolation. It should be clear that the special case $L = D^n$ was the one dealt with up to this point.

The definitions of the type I and periodic generalized splines of interpolation of order $2n$ with deficiency t $(t \leq n)$ can be patterned after the corresponding definitions for polynomial splines. Also definitions of the appropriate equivalence relation on $K^n(a,b)$ or $K^n_p(a,b)$ and of the respective generalized spline operators can be made in an evident manner. Furthermore, the proof of Theorem 4.2 can be used as a model to prove that each of the generalized spline operators so defined is a continuous operator on their respective inner product spaces.

To get the proof of the analog of Theorem 4.3 for generalized splines an argument similar to the one used there would be adequate. However, a slightly different treatment which avoids the minimum norm property, and instead relies upon the continuity of the best approximant operator is given.

Theorem 4.8. Let the function f together with the sequence of functions $\{f_k\}$ be given in $C^n[a,b]$ $(C^n_p[a,b])$ such that

$$\lim_{k\to\infty} ||f^{(\alpha)} - f_k^{(\alpha)}||_\infty = 0 \ (\alpha = 0, 1, \ldots, n).$$

If $S_\Delta(f,x)$ represents the type I (periodic) generalized spline of interpolation to f of order $2n$ with deficiency t ($t \leqq n$), then

$$\lim_{k\to\infty} ||S_\Delta^{(\alpha)}(f) - S_\Delta^{(\alpha)}(f_k)||_\infty = 0 \ (\alpha = 0, 1, \ldots, n).$$

Proof. The proof will be given for the type I spline. By the best approximation property, the type I generalized spline of interpolation to f of order $2n$ with deficiency t ($t \leqq n$) satisfies

$$\int_a^b [f^{(n)}(x) - S_\Delta^{(n)}(f,x)]^2 dx \leqq \int_a^b [f^{(n)}(x) - S_\Delta^{(n)}(x)]^2 dx, \qquad (4.13)$$

where in the right hand side $S_\Delta(x)$ denotes any generalized spline of order $2n$ with deficiency t. Now the set of generalized splines of order $2n$ with deficiency t form a subspace of the linear space $K^n(a,b)$. Moreover, the set of n^{th} derivatives of these spline functions is a subspace, Ω, of $K^n(a,b)$. Therefore, by (4.13) $S_\Delta^{(n)}(f)$ is the best approximant to $f^{(n)}$ in L_2-norm from the subspace Ω. A similar statement holds for each of the functions $f_k^{(n)}$ and $S_\Delta^{(n)}(f_k)$ ($k = 1, 2, \ldots$). Hence, by the continuity of the best approximant operator, Theorem 2.1, and inequality (4.4)

$$\lim_{k\to\infty} ||S_\Delta^{(n)}(f) - S_\Delta^{(n)}(f_k)||_2 = 0. \qquad (4.14)$$

But the subspace Ω is finite dimensional, which implies that any two norms on this subspace are equivalent. Thus, there exists a real number β such that

$$||S_\Delta^{(n)}(f) - S_\Delta^{(n)}(f_k)||_\infty \leqq \beta||S_\Delta^{(n)}(f) - S_\Delta^{(n)}(f_k)||_2.$$

Clearly then (4.14) implies

$$\lim_{k\to\infty} ||S_\Delta^{(n)}(f) - S_\Delta^{(n)}(f_k)||_\infty = 0.$$

Now as in the proof of Theorem 4.3

$$S_\Delta^{(n-1)}(f,x) - S_\Delta^{(n-1)}(f_k,x)$$

$$= \int_a^x [S_\Delta^{(n)}(f,u) - S_\Delta^{(n)}(f_k,u)]du + f^{(n-1)}(a) - f_k^{(n-1)}(a),$$

which finally gives

$$||S_\Delta^{(n-1)}(f) - S_\Delta^{(n-1)}(f_k)||_\infty$$

$$\leqq ||S_\Delta^{(n)}(f) - S_\Delta^{(n)}(f_k)||_\infty(b - a)^{1/2} + |f^{(n-1)}(a) - f_k^{(n-1)}(a)|.$$

This yields

$$\lim_{k\to\infty} ||S_\Delta^{(n-1)}(f) - S_\Delta^{(n-1)}(f_k)||_\infty = 0.$$

The proof can now be completed by using an inductive argument similar to that of Theorem 4.3.

Finally, a theorem similar to Theorem 4.3 can be proven for a polynomial spline which is about to be defined.

<u>Definition 4.8</u>. A polynomial spline of interpolation to a function f on a mesh Δ of degree $2n-1$ with deficiency t $(t \leqq n)$ is called a type t spline provided that

$$S_t^{(\alpha)}(f,x_i) = f^{(\alpha)}(x_i) \;\; (\alpha = 0, 1, \ldots, t-1) \;\; (i = 0, 1, \ldots, N)$$

$$S_t^{(\alpha)}(f,x_i) = 0 \;\; (\alpha = n, n+1, \ldots, 2n-t-1) \;\; (i = 0, N).$$

Once again existence and uniqueness properties, a minimum norm property, and a best approximation property are valid for this family of spline functions. More to the point of the present discussion, however, is the fact that the type t spline satisfies what might be called a "minimum norm property of order t". Some of the details of the proof will be included although they differ only slightly from those given in [1] for the minimum norm property for the type I spline.

Lemma 4.2. Let f in $K^n(a,b)$ and a mesh Δ on $[a,b]$ be given. Also let $S_t(f,x)$ be the type t polynomial spline of interpolation to f on the mesh Δ of degree $2n-1$. Then

$$\left\{\int_a^b [S_t^{(t)}(f,x)]^2 dx\right\}^{1/2} \leqq \left\{\int_a^b [f^{(t)}(x)]^2 dx\right\}^{1/2}.$$

Proof. Using the proof of the fundamental identity given in [1], an integration by parts t times ultimately yields

$$\int_a^b [f^{(t)}(x) - S_t^{(t)}(f,x)]^2 dx = \int_a^b [f^{(t)}(x)]^2 dx$$

$$- 2 \sum_{j=1}^{N} \sum_{\nu=1}^{t} (-1)^{\nu+1} \{f^{(t-\nu)}(x) - S_t^{(t-\nu)}(f,x)\} S_t^{(t+\nu-1)}(f,x) \Big|_{x_{j-1}}^{x_j}$$

$$- \int_a^b [S_t^{(t)}(f,x)]^2 \, dx.$$

An examination of the definition shows that

$$f^{(\alpha)}(x_i) - S_t^{(\alpha)}(f,x_i) = 0 \ (i = 0, 1, \ldots, N) \ (\alpha = 0, \ldots, t - 1),$$

or equivalently

$$f^{(t-\nu)}(x_i) - S_t^{(t-\nu)}(f,x_i) = 0 \ (i = 0, 1, \ldots, N) \ (\nu = 1, \ldots, t).$$

Hence,

$$\int_a^b [f^{(t)}(x) - S_t^{(t)}(f,x)]^2 dx = \int_a^b [f^{(t)}(x)]^2 dx - \int_a^b [S_t^{(t)}(f,x)]^2 dx$$

which is the "first integral relation of order t". This implies

$$\int_a^b [f^{(t)}(x)]^2 dx - \int_a^b [S_t^{(t)}(f,x)]^2 dx \geqq 0,$$

or

$$\{\int_a^b [S_t^{(t)}(f,x)]^2 dx\}^{1/2} \leqq \{\int_a^b [f^{(t)}(x)]^2 dx\}^{1/2}$$

which is the "minimum norm property of order t".

Theorem 4.9. Let f and $\{f_k\}$ $(k = 1, 2, \ldots)$ be given in $C^n(a,b)$ such that

$$\lim_{k\to\infty} ||f^{(\alpha)} - f_k^{(\alpha)}||_\infty = 0 \ (\alpha = 0, 1, \ldots, t;\ t \leqq n).$$

Furthermore, let $S_t(f,x)$ and $S_t(f_k,x)$ $(k = 1, 2, \ldots)$ be the type t polynomial splines of interpolation to f and f_k $(k = 1, 2, \ldots)$, respectively on a given mesh Δ. Then

$$\lim_{k\to\infty} ||S_t^{(\alpha)}(f) - S_t^{(\alpha)}(f_k)||_\infty = 0 \ (\alpha = 0, 1, \ldots, t - 1).$$

Proof. The proof follows exactly as in Theorem 4.3 with n replaced by t.

It is natural to wonder whether the requirement that the sequence of functions $\{f_k^{(\alpha)}\}$ converge uniformly to $f^{(\alpha)}$ for $\alpha = 0, 1, \ldots, n$ is a reasonable one. The answer is in the affirmative since for any function in $C^n[a,b]$ the Bernstein polynomials behave in precisely this manner. Unfortunately, the rate of convergence of these polynomials is quite slow which makes them poor candidates for any practical application. In the next chapter a method which does not necessitate the high order of convergence is devised.

CHAPTER 5

A CONVERGENT SEQUENCE OF SPLINE FUNCTIONS

The convergence properties given in Chapter 4 depend upon a high order of convergence of a sequence of functions in uniform norm. In this chapter a sequence of spline functions is developed which avoids the requirement for a high order of convergence. This is accomplished by considering the n^{th} derivative of the type I spline of interpolation to a given function f on a mesh Δ of degree $2n-1$ as the best approximant to the n^{th} derivative of f in L_2-norm. The best approximation property for the type I polynomial spline (4.1) allows this interpretation.

<u>Theorem 5.1</u>. Let a function f in $C^n[a,b]$ and a mesh $\Delta: a = x_0 < x_1 < \dots < x_N = b$, be given. Further let a sequence of functions $\{g_k\}$ be given in $C[a,b]$ such that

$$\lim_{k\to\infty} ||f^{(n)} - g_k||_\infty = 0.$$

Then,

$$\lim_{k\to\infty} ||S_{2n-1}^{(n)}(f) - S_k^*||_\infty = 0,$$

where S_k^* denotes the best approximant to g_k in L_2-norm from the subspace Ω with basis $\{1, x, \ldots, x^{n-1}, \ldots, (x - x_{N-1})_+^{n-1}\}$.

Proof. In Chapter 3 it was pointed out that the best spline approximant in L_2-norm, or the best least square spline approximant, exists and is unique. Now by the best approximation property for type I polynomial splines of interpolation (4.1),

$$||f^{(n)} - S_{2n-1}^{(n)}(f)||_2 \leqq ||f^{(n)} - S_{2n-1}^{(n)}||_2,$$

where $S_{2n-1}(x)$ represents any polynomial spline of degree $2n-1$. It is apparent, upon a re-examination of Theorem 3.1, that the subspace of $K^n(a,b)$ composed of the n^{th} derivatives of all polynomial splines of degree $2n-1$ is precisely the subspace Ω defined above. Hence, $S^*(x) = S_{2n-1}^{(n)}(f,x)$ is the best least square approximant to $f^{(n)}(x) = g(x)$ from the subspace Ω.

Now $g(x) = f^{(n)}(x)$ is in $C[a,b]$, and the hypothesis insures that

$$\lim_{k\to\infty} ||g - g_k||_\infty = 0.$$

But uniform convergence on a finite interval implies mean-square convergence, that is,

$$\lim_{k\to\infty} ||g - g_k||_2 = 0.$$

Letting S_k^* be the best least square approximant to g_k $(k = 1, 2, \ldots)$, and recalling that the best approximant operator is continuous in L_2-norm yields

$$\lim_{k\to\infty} ||S^* - S_k^*||_2 = 0.$$

Moreover, the finite dimensionality of the subspace Ω implies that the L_2-norm is equivalent to the uniform norm on this subspace [2]. Hence, there exists a real number β such that

$$||S^* - S_k^*||_\infty \leq \beta ||S^* - S_k^*||_2.$$

This implies that

$$\lim_{k\to\infty} ||S^* - S_k^*||_\infty = 0$$

or,

$$\lim_{k\to\infty} ||S_{2n-1}^{(n)}(f) - S_k^*||_\infty = 0. \tag{5.1}$$

A sequence of polynomial splines of degree $2n-1$ whose convergence to $S_{2n-1}(f,x)$ will depend upon the convergence of the sequence $\{g_k\}$ to $f^{(n)}$ can now be developed.

Theorem 5.2. Let f be in $C^n[a,b]$, and let a mesh Δ be given on $[a,b]$. Also let a sequence of functions $\{g_k\}$ be given in $C[a,b]$ such that

$$\lim_{k\to\infty} ||f^{(n)} - g_k||_\infty = 0.$$

Then there exists a sequence $\{\sigma_k\}$ of polynomial splines of degree $2n-1$ such that

$$\lim_{k\to\infty} ||S_{2n-1}(f) - \sigma_k||_\infty = 0.$$

Proof. The proof will be given for the interval $[0,1]$ for the sake of simplicity. The results can then be easily transferred to the interval $[a,b]$ by means of the linear transformation

$$T(x) = \frac{x - a}{b - a}.$$

The proof proceeds by observing that

$$S_{2n-1}^{(n-1)}(f,x) - \int_0^x S_k^*(u)du - f^{(n-1)}(0) = \int_0^x [S_{2n-1}^{(n)}(f,u) - S_k^*(u)]du,$$

and that for any x in $[0,1]$

$$|S_{2n-1}^{(n-1)}(f,x) - \sigma_{k,1}(x)| \leq \int_0^1 |S_{2n-1}^{(n)}(f,u) - S_k^*(u)|du,$$

where

$$\sigma_{k,1}(x) = \int_0^x S_k^*(u)du + f^{(n-1)}(0).$$

Now (5.1) can be combined with this last inequality to acquire

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-1)}(f) - \sigma_{k,1}||_\infty = 0.$$

To complete the proof by induction let

$$\sigma_{k,j}(x) = \int_0^x \overset{(j)}{\cdots} \int_0^x S_k^*(u)du + \sum_{i=1}^{j} \frac{f^{(n-i)}(0)}{(j-i)!} x^{j-i} \quad (j = 1, \ldots, n),$$

and suppose that

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-j)}(f) - \sigma_{k,j}||_\infty = 0. \tag{5.2}$$

Then for $j \leqq n$

$$\int_0^x [S_{2n-1}^{(n-j)}(f,u) - \sigma_{k,j}(u)]du$$

$$= S_{2n-1}^{(n-j-1)}(f,x) - S_{2n-1}^{(n-j-1)}(f,0) - \int_0^x \sigma_{k,j}(u)du$$

$$= S_{2n-1}^{(n-j-1)}(f,x) - f^{(n-j-1)}(0) - \int_0^x \overset{(j+1)}{\cdots} \int_0^x S_k^*(u)du$$

$$- \int_0^x [\sum_{i=j}^{j} \frac{f^{(n-i)}(0)}{(j-i)!} u^{j-i}] du$$

$$= S_{2n-1}^{(n-j-1)}(f,x) - \int_0^x (j+1) \int_0^x S_k^*(u)du - \sum_{i=1}^{j+1} \frac{f^{(n-i)}(0)}{(j+1-i)!} x^{j+1-i}$$

$$= S_{2n-1}^{(n-j-1)}(f,x) - \sigma_{k,j+1}(x).$$

This gives, for x in $[0,1]$

$$|S_{2n-1}^{(n-j-1)}(f,x) - \sigma_{k,j+1}(x)| \leqq \int_0^x |S_{2n-1}^{(n-j)}(f,u) - \sigma_{k,j}(u)| du$$

$$\leqq \int_0^1 |S_{2n-1}^{(n-j)}(f,u) - \sigma_{k,j}(u)| du \leqq ||S_{2n-1}^{(n-j)}(f) - \sigma_{k,j}||_\infty.$$

Taking the supremum of both sides over $[0,1]$, and using the assumption (5.2) yields

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-j-1)}(f) - \sigma_{k,j+1}||_\infty = 0.$$

Hence, by induction

$$\lim_{k\to\infty} ||S_{2n-1}^{(n-j)}(f) - \sigma_{k,j}||_\infty = 0 \quad (j = 1, \ldots, n).$$

In particular, for $j = n$

$$\lim_{k\to\infty} ||S_{2n-1}(f) - \sigma_k||_\infty = 0$$

with

$$\sigma_k(x) = \sigma_{k,n}(x) = \int_0^x \overset{(n)}{\cdots} \int_0^x S_k^*(u)\,du + \sum_{i=1}^{n} \frac{f^{(n-i)}(0)}{(n-i)!} x^{n-i}.$$

In order to utilize this convergence theorem, an estimate on the rate of convergence is needed. Such an estimate is furnished by the following lemma and theorem.

Lemma 5.1. Let f be given in $C^n[a,b]$, and h be given in $C[a,b]$. Furthermore, let a mesh Δ be defined on $[a,b]$, and let $S_{2n-1}(f,x)$ be the type I polynomial spline of interpolation to f on Δ. Then,

$$||S_{2n-1}^{(n)}(f) - S^*||_2 \leqq ||f^{(n)} - h||_\infty (b - a)^{1/2} \tag{5.3}$$

where S^* denotes the best least-square approximant to h from the subspace Ω defined in Theorem 5.1.

Proof. Let H be an n-fold antiderivative of h, that is, let H satisfy the condition

$$H^{(n)}(x) = h(x).$$

Then H is in $C^n[a,b]$, and the type I polynomial spline of interpolation to H on the mesh Δ of degree $2n-1$ exists and is unique.

Now by the best approximation property (4.1) applied to $H(x)$

$$\left\{\int_a^b [H^{(n)}(x) - S_{2n-1}^{(n)}(H,x)]^2 dx\right\}^{1/2} \leqq \left\{\int_a^b [H^{(n)}(x) - S_{2n-1}^{(n)}(x)]^2 dx\right\}^{1/2}, \quad (5.4)$$

where $S_{2n-1}(x)$ represents any polynomial spline on Δ of degree $2n-1$. Furthermore, the subspace of n^{th} derivatives of polynomial splines of degree $2n-1$ is the subspace Ω defined above.

Moreover,

$$\left\{\int_a^b [H^{(n)}(x) - S_{2n-1}^{(n)}(H,x)]^2 dx\right\}^{1/2} = \left\{\int_a^b [h(x) - S_{2n-1}^{(n)}(H,x)]^2 dx\right\}^{1/2}$$

so that (5.4) implies

$$\left\{\int_a^b [h(x) - S_{2n-1}^{(n)}(H,x)]^2 dx\right\}^{1/2} \leqq \left\{\int_a^b [h(x) - S_{n-1}(x)]^2 dx\right\}^{1/2}$$

where $S_{n-1}(x)$ represents any polynomial spline of degree $n-1$ from the subspace Ω. But the best least square approximant to $h(x)$ from the subspace Ω is unique, which implies

$$S_{2n-1}^{(n)}(H,x) = S^*(x).$$

Therefore,

$$||S_{2n-1}^{(n)}(f) - S^*||_2 = ||S_{2n-1}^{(n)}(f) - S_{2n-1}^{(n)}(H)||_2$$

$$= ||S^{(n)}_{2n-1}(f - H)||_2 \tag{5.5}$$

where the last equality follows from the linearity of the type I polynomial spline of interpolation. The minimum norm property can now be used to obtain

$$||S^{(n)}_{2n-1}(f - H)||_2 \leqq ||f^{(n)} - H^{(n)}||_2 = ||f^{(n)} - h||_2. \tag{5.6}$$

Combining (5.5) and (5.6) gives

$$||S^{(n)}_{2n-1}(f) - S^*||_2 \leqq ||f^{(n)} - h||_2.$$

The proof is finished by observing that

$$||f^{(n)} - h||_2 \leqq ||f^{(n)} - h||_\infty (b - a)^{1/2}.$$

<u>Theorem 5.3</u>. Let f in $C^n[0,1]$ and a sequence $\{g_k\}$ in $C[0,1]$ be given such that

$$\lim_{k\to\infty} ||f^{(n)} - g_k||_\infty = 0.$$

Then, for $\alpha = 0, 1, \ldots, n-1$ and $k = 1, 2, \ldots$

$$||S^{(\alpha)}_{2n-1}(f) - \sigma^{(\alpha)}_k||_\infty \leqq ||f^{(n)} - g_k||_\infty.$$

Proof. The lemma asserts that for each k,

$$||S_{2n-1}^{(n)}(f) - S_k^*||_2 \leqq ||f^{(n)} - g_k||_\infty. \tag{5.7}$$

As in the proof of Theorem 5.2, for any x in $[0,1]$

$$|S_{2n-1}^{(n-1)}(f,x) - \sigma_{k,1}(x)| \leqq \int_0^1 |S_{2n-1}^{(n)}(f,u) - S_k^*(u)|du. \tag{5.8}$$

But an application of the Schwarz inequality gives

$$\int_0^1 |S_{2n-1}^{(n)}(f,u) - S_k^*(u)|du \leqq \{\int |S_{2n-1}^{(n)}(f,u) - S_k^*(u)|^2 du\}^{1/2}. \tag{5.9}$$

Now (5.7), (5.8), and (5.9) together imply that for x in $[0,1]$

$$|S_{2n-1}^{(n-1)}(f,x) - \sigma_{k,1}(x)| \leqq ||f^{(n)} - g_k||_\infty,$$

which further implies

$$||S_{2n-1}^{(n-1)}(f) - \sigma_{k,1}||_\infty \leqq ||f^{(n)} - g_k||_\infty.$$

Noting that

$$\sigma_k^{(j)}(x) = \sigma_{k,n-j}(x) \qquad (j = 0, 1, \ldots, n-1),$$

the proof can be completed by induction as in Theorem 5.2. The end result is that for $\alpha = 0, 1, \ldots, n-1$ and $k = 1, 2, \ldots$

$$||S_{2n-1}^{(\alpha)}(f) - \sigma_k^{(\alpha)}||_\infty \leqq ||f^{(n)} - g_k||_\infty. \qquad (5.10)$$

These theorems can be combined with convergence properties of the type I polynomial spline, the proofs of which are given in [1], to obtain yet another convergence theorem. First a theorem and corollary from [1] will be given.

<u>Theorem 5.4</u>. Let a function f in $K^n(a,b)$ and a sequence of meshes $\{\Delta_N\}$ $(N = 1, 2 \ldots)$ be given such that $\lim_{N\to\infty} ||\Delta_N|| = 0$, that is, such that the sequence of mesh norms converges to zero. Then,

$$\lim_{N\to\infty} ||f^{(\alpha)} - S_{\Delta_N}^{(\alpha)}(f)||_\infty = 0 \qquad (\alpha = 0, 1, \ldots, n-1) \qquad (5.11)$$

where $S_{\Delta_N}(f)$ is the type I polynomial spline of interpolation to f on the mesh Δ_N of degree $2n-1$.

<u>Corollary 5.1</u>. Under the conditions given in Theorem 5.4,

$$||f^{(\alpha)} - S_{\Delta_N}^{(\alpha)}(f)||_\infty \leqq \eta_\alpha \qquad (\alpha = 0, 1, \ldots, n-1) \qquad (5.12)$$

where

$$\eta_\alpha = (n)^{1/2}(n-1)\ \ldots\ (\alpha+1)\cdot(||\Delta_N||/2)^{(2n-2\alpha-1)/2}$$

$$\cdot\ 2\{\int_a^b [f^{(n)}(x)]^2 dx\ \}^{1/2}.$$

Theorem 5.5. Let f be in $C^n[0,1]$ and let a sequence of functions $\{g_k\}$ be given in $C[0,1]$ such that

$$\lim_{k\to\infty} ||f^{(n)} - g_k||_\infty = 0. \tag{5.13}$$

Furthermore, let a sequence $\{\Delta_N\}$ of meshes be given on $[0,1]$ such that

$$\lim_{N\to\infty} ||\Delta_N|| = 0. \tag{5.14}$$

Then,

$$\lim_{\substack{k\to\infty \\ N\to\infty}} ||f^{(\alpha)} - \sigma_{N,k}^{(\alpha)}||_\infty = 0 \qquad (\alpha = 0, 1, \ldots, n-1)$$

where $\sigma_{N,k}$ $(k = 1, 2, \ldots)$ is the polynomial spline of degree $2n-1$ on the mesh Δ_N $(N = 1, 2, \ldots)$ which was introduced in Theorem 5.2.

Proof. It is apparent that for each α, $(\alpha = 0, 1, \ldots, n-1)$,

$$||f^{(\alpha)} - \sigma_{N,k}^{(\alpha)}||_\infty \leqq ||f^{(\alpha)} - S_{\Delta_N}^{(\alpha)}(f)||_\infty + ||S_{\Delta_N}^{(\alpha)}(f) - \sigma_{N,k}^{(\alpha)}||_\infty.$$

A use of the estimates (5.10) and (5.12) produces the inequality

$$||f^{(\alpha)} - \sigma_{N,k}^{(\alpha)}||_\infty \leqq \eta_\alpha + ||f^{(n)} - g_k||_\infty.$$

Taking the limit as both N and k tend to infinity proves the theorem.

CHAPTER 6

APPROXIMATING POLYNOMIAL SPLINES

In this final chapter the continuity of the best approximant operator is used as an aid in finding the best least square spline approximant. The equations developed can then be used to obtain the type I polynomial spline. Examples which illustrate these techniques are included. For the sake of simplicity it is assumed throughout this chapter that $[a,b] = [0,1]$.

By Theorem 3.1 any polynomial spline of degree m can be written

$$S_m(x) = \sum_{\ell=0}^{m} a_\ell x^\ell + \sum_{i=1}^{N-1} b_i (x - x_i)_+^m,$$

or

$$S_m(x) = \sum_{\ell=0}^{m} a_\ell \phi_\ell(x) + \sum_{i=1}^{N-1} b_i \psi_i(x)$$

where ϕ_ℓ and ψ_i are defined by

$$\phi_\ell(x) = x^\ell \qquad (\ell = 0, 1, \ldots, m)$$

$$\psi_i(x) = (x - x_i)_+^m \qquad (i = 1, \ldots, N-1).$$

Therefore, if g is a continuous function on $[0,1]$, then the best approximant to g in L_2-norm from the subspace with basis

$$\{\phi_0, \phi_1, \ldots \phi_m, \psi_1, \ldots, \psi_{N-1}\}$$

can be obtained by solving the normal equations

$$\sum_{\ell=0}^{m} a_\ell(\phi_\ell,\phi_q) + \sum_{i=1}^{N-1} b_i(\psi_i,\phi_q) = (g,\phi_q) \qquad (q = 0, 1, \ldots, m) \tag{6.1}$$

$$\sum_{\ell=0}^{m} a_\ell(\phi_\ell,\psi_j) + \sum_{i=1}^{N-1} b_i(\psi_i,\psi_j) = (g,\psi_j) \qquad (j = 1, \ldots, N-1)$$

where the inner product is given by

$$(f,g) = \int_0^1 f(x)g(x)dx.$$

Now if the right hand members of the normal equations can be readily obtained, then a solution of these equations will produce the coefficients of the best least square spline approximant to g of degree m. In general, however, the inner products on the right hand side of (6.1) must be approximated by numerical integration. The intent of the method to be developed here is to circumvent the necessity for numerical integration by making use of the continuity of the best approximant operator in L_2-norm. Putting these matters aside for the moment, the inner products on the left hand side of (6.1) will first be evaluated.

It is easy to see that

$$(\phi_\ell, \phi_q) = \frac{1}{\ell+q+1} \quad \text{for} \quad 0 \leqq \ell,\ q \leqq m. \tag{6.2}$$

In order to obtain (ϕ_q, ψ_i) it is convenient to prove the following lemmas.

<u>Lemma 6.1.</u> $(\phi_q, \psi_i) = (1 - x_i)^{m+1} \sum_{r=0}^{q} \frac{\binom{q}{r}(1-x_i)^{q-r}}{q+m+1-r} x_i^r.$

<u>Proof.</u> $(\phi_q, \psi_i) = \int_0^1 x^q (x - x_i)_+^m dx = \int_{x_i}^1 x^q (x - x_i)^m dx.$

Make the substitution $y = x - x_i$ to obtain

$$\begin{aligned}
(\phi_q, \psi_i) &= \int_0^{1-x_i} (y + x_i)^q y^m dy \\
&= \int_0^{1-x_i} [\sum_{r=0}^{q} \binom{q}{r} y^{q-r} x_i^r] y^m dy \\
&= \sum_{r=0}^{q} \binom{q}{r} \int_0^{1-x_i} y^{q+m-r} x_i^r dy \\
&= (1 - x_i)^{m+1} \sum_{r=0}^{q} \frac{\binom{q}{r}(1-x_i)^{q-r} x_i^r}{q+m+1-r}.
\end{aligned}$$

Lemma 6.2. $(\phi_q, \psi_i) = (1 - x_i)^{m+1} \sum_{r=0}^{q} \Delta^{(r)} h(0) \binom{q}{r} x_i^r$

where h is a function which interpolates to $\frac{1}{q+m+1-r}$ at the points $r/q (r = 0, 1, \ldots, q)$, and $\Delta^{(r)} h(0)$ is the r^{th} divided difference of h on this set of points.

Proof. The Bernstein polynomial for h of degree q can be written as

$$B_q(h,x) = \sum_{r=0}^{q} \binom{q}{r} h(r/q)(1 - x)^{q-r} x^r.$$

But it is known [5] that this reduces to

$$B_q(h,x) = \sum_{r=0}^{q} \Delta^{(r)} h(0) \binom{q}{r} x^r.$$

Hence at $x = x_i$

$$\sum_{r=0}^{q} \frac{\binom{q}{r}(1-x_i)^{q-r} x_i^r}{q+m+1-r} = \sum_{r=0}^{q} \Delta^{(r)} h(0) \binom{q}{r} x_i^r.$$

Now a multiplication of both sides of this equation by $(1 - x_i)^{m+1}$ yields the lemma.

Lemma 6.3. $P(x) = \sum_{\nu=0}^{r} \frac{x(x-1)(x-2)\ldots(x-r)}{(x-\nu)} \binom{r}{\nu} (-1)^{r-\nu} = r!$

Proof. If $x = 0$, then

$$P(0) = (-1)(-2) \ \ldots \ (-r)\binom{r}{0}(-1)^r = (-1)^r r!\,(-1)^r = r!$$

If $x = 1$,

$$P(1) = 1 \cdot (1 - 2)(1 - 3) \cdot \ldots \cdot (1 - r)\binom{r}{1}(-1)^{r-1}$$

$$= 1(-1)(-2) \cdot \ldots \cdot [-(r - 1)]\binom{r}{1}(-1)^{r-1}$$

$$= (-1)^{r-1} \cdot (-1)^{r-1}(r - 1)!\binom{r}{1} = r!$$

If $x = \nu$, $0 \leqq \nu \leqq r$

$$P(\nu) = \nu(\nu - 1) \cdot \ldots \cdot [\nu - (\nu - 1)] \cdot [\nu - (\nu + 1)]$$

$$\cdot \ldots \cdot (\nu - r)\binom{r}{\nu}(-1)^{r-\nu}$$

$$= \nu(\nu - 1) \ \ldots \ 1 \cdot (-1) \cdot \ldots \cdot [-(r - \nu)]\binom{r}{\nu}(-1)^{r-\nu}$$

$$= (-1)^{r-\nu}(-1)^{r-\nu}\nu!\binom{r}{\nu} = r!$$

So $P(x)$ is a polynomial of degree r which is equal to $r!$ at the $r + 1$ points $0, 1, \ldots, r$. Hence,

$$P(x) = r!$$

Lemma 6.4. $\Delta^{(r)}h(0) = \frac{r!(q+m-r)!}{(q+m+1)!}$.

Proof. In [5] it is shown that

$$\Delta^{(r)}h(0) = \sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu}h(\nu/r).$$

Hence

$$\Delta^{(r)}h(0) = \sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu}\frac{1}{q+m+1-\nu}$$

$$= \sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu}\frac{(q+m+1)(q+m)\dots(q+m+1-r)}{(q+m+1)(q+m)\dots(q+m+1-r)(q+m+1-\nu)}$$

$$= \frac{(q+m-r)!}{(q+m+1)!}\sum_{\nu=0}^{r} (-1)^{r-\nu}\binom{r}{\nu}\frac{(q+m+1)(q+m)\dots(q+m+1-r)}{(q+m+1-\nu)}$$

$$= \frac{(q+m-r)!}{(q+m+1)!}\, r!$$

where the final equality results from Lemma 6.3 with $x = q + m + 1$.

Now a combination of Lemmas 6.2 and 6.4 can be used to obtain

$$(\phi_q,\psi_i) = (1 - x_i)^{m+1}\sum_{r=0}^{q}\frac{r!(q+m-r)!}{(q+m+1)!}\binom{q}{r}x_i^r$$

$$= \frac{(1-x_i)^{m+1}}{q+m+1}\sum_{r=0}^{q}\frac{\binom{q}{r}}{\binom{q+m}{r}}x_i^r. \tag{6.3}$$

If $x_i = x_j$ a straightforward evaluation gives

$$(\psi_i,\psi_i) = \int_0^1 (x - x_i)_+^m (x - x_i)_+^m dx = \frac{(x-x_i)^{2m+1}}{2m+1}. \tag{6.4}$$

On the other hand if $x_i < x_j$ then,

$$(\psi_i,\psi_j) = \int_0^1 (x - x_i)_+^m dx = \int_{x_j}^1 (x - x_i)^m (x - x_j)^m dx.$$

Now integrations by parts yields

$$(\psi_i,\psi_j) = \frac{(1-x_i)^m}{m+1}(1 - x_j)^{m+1} - \frac{m}{m+1}\int_{x_j}^1 (x - x_i)^{m-1}(x - x_j)^{m+1}dx$$

$$= \frac{(1-x_i)^m}{m+1}(1 - x_j)^{m+1} - \frac{m}{(m+1)(m+2)}(1 - x_i)^{m-1}(1 - x_j)^{m+1}$$

$$+ \frac{m}{(m+1)}\frac{(m-1)}{(m+2)}\int_{x_j}^1 (x - x_i)^{m-2}(x - x_j)^{m+2}dx$$

where the last equality resulted from another integration by parts. Repeating this process m times gives

$$(\psi_i,\psi_j) = \frac{(1-x_i)^m}{m+1}(1 - x_j)^{m+1} - \frac{m}{(m+1)(m+2)}(1 - x_i)^{m-1}(1 - x_j)^{m+2}$$

$$+ \dots + (-1)^m \frac{m!}{(m+1)(m+2)\dots(2m)(2m+1)}(1 - x_j)^{2m+1}$$

$$= \frac{(1-x_j)^{m+1}}{m+1} \sum_{r=0}^{m} (-1)^m \frac{(m+1)(m)\dots(m+1-r)}{(m+1)(m+2)\dots(m+1+r)} (1 - x_i)^{m-r}(1 - x_j)^r$$

$$= \frac{(1-x_j)^{m+1}}{m+1} \sum_{r=0}^{m} (-1)^m \frac{(m+1)!m!}{(m+1+r)!(m-r)!} (1 - x_i)^{m-r}(1 - x_j)^r. \quad (6.5)$$

This completes the evaluation of all the inner products on the left hand side of (6.1). Now in the proof of Theorem 5.1 it was indicated that the continuity of the best least square spline approximant operator in L_2-norm implies the continuity of this operator in the space $C[0,1]$ with uniform norm. That is, if g and g_k $(k = 1, 2, \dots)$ are functions in $C[0,1]$ such that

$$\lim_{k\to\infty} ||g - g_k||_\infty = 0$$

then

$$\lim_{k\to\infty} ||S^* - S_k^*||_\infty = 0$$

where S^* and S_k^* $(k = 1, 2 \dots)$ are the best least square spline approximants to g and g_k $(k = 1, 2 \dots)$, respectively. Thus if a sequence of functions $\{g_k\}$ which converge uniformly to g in $[0,1]$ can be found then S_k^* provides an approximation for S^*.

A sequence of polynomials which converge uniformly to g in $[0,1]$ is quite useful in this respect since (6.2) and (6.3) could be used to obtain the inner products on the right hand side of (6.1) for such a sequence. Moreover, it is well known that if g is continuous on $[0,1]$ then there exists a sequence of polynomials which converge uniformly to g on $[0,1]$. The Bernstein polynomials provide such a sequence, but their rate of convergence is very slow. However, there exist sequences of polynomials which converge quite rapidly in uniform norm to a given function g provided that g is differentiable of a sufficiently high order. See [3] and [7] for a discussion of such polynomials.

The variation diminishing spline functions introduced in [9] provide yet another sequence of functions whose uniform convergence to a given continuous function is rather rapid. Furthermore, due to the nature of these spline functions, the inner products on the right side of (6.1) could be readily calculated. A detailed treatment of these spline functions is not germane to this paper, but the interested reader may see [8] and [9] for further information.

Example 6.1. Let $g(x) = e^{-x}$ and let a mesh Δ be defined on $[0,1]$ by $\Delta: 0 < 1/3 < 2/3 < 1$. Then the best least square approximant to g on $[0,1]$ by a spline function of degree one can be obtained from the normal equations

$$\sum_{\ell=0}^{1} a\ (\phi_\ell,\phi_q) + \sum_{i=1}^{2} b_i(\psi_i,\phi_q) = (g,\phi_q) \qquad (q = 0,\ 1)$$

$$\sum_{\ell=0}^{1} a_\ell(\phi_\ell,\psi_j) + \sum_{i=1}^{2} b_i(\psi_i,\psi_j) = (g,\psi_j) \qquad (j = 1,\ 2).$$

Using the equations (6.2), (6.3), (6.4) and (6.5) this reduces to

$$a_0 + 0.500000\ a_1 + 0.222222\ b_1 + 0.055556\ b_2 = 0.632120$$

$$0.500000\ a_0 + 0.333333\ a_1 + 0.172840\ b_1 + 0.049383\ b_2 = 0.264240$$

$$0.222222\ a_0 + 0.172840\ a_1 + 0.098765\ b_1 + 0.030864\ b_2 = 0.103399$$

$$0.055556\ a_0 + 0.049383\ a_1 + 0.030864\ b_1 + 0.012346\ b_2 = 0.022909.$$

Solving this system for a_0, a_1, b_1 and b_2 yields

$$S_1^*(g,x) = 0.991627 - 0.946197x + 0.244377(x - \tfrac{1}{3})_+ + 0.167135(x - \tfrac{2}{3})_+.$$

Now approximate e^{-x} by the Tchebycheff series truncated at $k = 5$, that is,

$$e^{-x} \cong \sum_{\nu=0}^{5} c_\nu T_\nu(x) = \tau_5(x)$$

where $T_\nu(x)$ is the Tchebycheff polynomial of degree ν. Then

$$\tau_5(x) = 1.000045 - 1.000022x + 0.499199x^2$$

$$- 0.166488x^3 + 0.43794x^4 - 0.008687x^5$$

and

$$||g - \tau_5||_\infty \leqq 0.000045.$$

The left hand side of the above system remains the same while the right hand side is given by

$$(\tau_5,\phi_0) = 0.632123$$

$$(\tau_5,\phi_1) = 0.264564$$

$$(\tau_5,\psi_1) = 0.103400$$

$$(\tau_5,\psi_2) = 0.021853.$$

A solution of the resultant normal equations gives

$$S_1^*(\tau_5,x) = 0.994088 - 0.920482x + 0.526893(x - \tfrac{1}{3})_+$$

$$- 0.338606(x - \tfrac{2}{3})_+.$$

Unfortunately a precise estimate of the error in approximating $S_1^*(g,x)$ by $S_1^*(\tau_5,x)$ in uniform norm is not available. However, Lemma 5.1 implies that

$$||S_1^*(g) - S_1^*(\tau_5)||_2 \leqq ||g - \tau_5||_\infty.$$

But the L_2-norm and the uniform norm are equivalent on the finite dimensional subspace of spline functions. Hence there exist a real number β such that

$$||S_1^*(g) - S_1^*(\tau_5)||_\infty \leqq \beta||S_1^*(g) - S_1^*(\tau_5)||_2.$$

This implies that

$$||S_1^*(g) - S_1^*(\tau_5)||_\infty \leqq B||g - \tau_5||_\infty \leqq B(0.000045)$$

where $B = 1/\beta$. But in this particular case

$$||S_1^*(g) - S_1^*(\tau_5)||_\infty = .05206$$

which suggest that the above estimate is not very accurate.

Example 6.2. Let $f(x) = e^{-x}$ and let Δ be as in Example 6.1. Then $g(x) = f''(x) = e^{-x}$. Now in Example 6.1 the best least-square spline approximant to g of degree 1 was found. But as was pointed out in Chapter 5 the type I spline of interpolation to f of degree three, $S_3(f,x)$, satisfies the best approximation property.

$$||f'' - S_3''(f)||_2 \leqq ||f'' - S_3''||_2$$

where $S_3(x)$ represents any cubic spline on the mesh Δ. Hence $S_3''(f,x) = S_1^*(x)$ and

$$S_3(f,x) = f(0) + f'(0)x + \int_0^x \int_0^x S_1^*(x)dx.$$

Thus

$$S_3(f,x) = 1 - x + 0.495814x^2 - 0.141033x^3$$

$$+ 0.040730(x - \tfrac{1}{3})^3 + 0.027856(x - \tfrac{2}{3})_+^3$$

and from Theorem 5.2

$$\sigma_5(x) = 1 - x + 0.497044x^2 - 0.153414x^3$$

$$+ 0.087816(x - \tfrac{1}{3})_+^3 - 0.056434(x - \tfrac{2}{3})_+^3.$$

Theorem 5.3 gives the estimate

$$||S_3(f) - \sigma_5||_\infty \leqq ||f'' - \tau_5||_\infty \leqq 0.000045.$$

Furthermore, the values of $f(x) = e^{-x}$, $S_3(f,x)$ and $\sigma_5(x)$ can be compared at the mesh points 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1.

$$f(0) = 1 \qquad S_3(f,0) = 1 \qquad \sigma_5(0) = 1$$

$$f(\tfrac{1}{3}) = 0.716532 \qquad S_3(f,\tfrac{1}{3}) = 0.716534 \qquad \sigma_5(\tfrac{1}{3}) = 0.716212$$

$$f(\tfrac{2}{3}) = 0.513416 \qquad S_3(f,\tfrac{2}{3}) = 0.513415 \qquad \sigma_5(\tfrac{2}{3}) = 0.512037$$

$$f(1) = 0.367880 \qquad S_3(f,1) = 0.367881 \qquad \sigma_5(1) = 0.367560.$$

Thus it can be seen that $\sigma_5(x)$ provides a good approximation to $S_3(f,x)$. Furthermore the values of $\sigma_5(x)$ at the mesh points of Δ are approximately equal to the values of e^{-x} at these points in this example, but it cannot be assumed that this will always be the case.

In conclusion it should be pointed out that some difficulties may arise in using the techniques described in the last two chapters. It may well be that the coefficient matrix of the normal equations is ill-conditioned. This is to be expected since the upper left-hand corner of the coefficient matrix is the Hilbert matrix (γ_{ij}) where $\gamma_{ij} = \frac{1}{i+j+1}$, and it is known that this matrix is ill-conditioned [10]. Furthermore as was indicated in the above example the spline function $\sigma_k(x)$ of Theorem 5.2 will provide a good approximation to the type I spline of interpolation to a given function f on $[0,1]$, but $\sigma_k(x)$ will no longer interpolate to f on the given mesh. Hence, this method may prove useful when approximation rather than interpolation is the primary concern.

REFERENCES

1. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, The Theory of Splines and Their Applications, Academic Press, New York, 1967.

2. G. Bachman and L. Narici, Functional Analysis, Academic Press, New York, 1966.

3. P. L. Butzer, "Linear Combinations of Bernstein Polynomials," Canadian Journal of Mathematics, 5, 1953.

4. E. W. Cheney, Introduction to Approximation Theory, McGraw-Hill, New York, 1966.

5. P. J. Davis, Interpolation and Approximation, Blaisdell, New York, 1963.

6. T. N. E. Greville, "Data Fitting by Spline Functions," M. R. C. Technical Report No. 893, 1968.

7. D. Jackson, The Theory of Approximation, American Mathematical Society Colloquium Publications, Vol. XI, New York, 1930.

8. M. Marsden, "An Identity for Spline Functions with Applications to Variation Diminishing Spline Approximation," M. R. C. Technical Report No. 897, 1968.

9. __________ and I. J. Schoenberg, "On Variation Diminishing Spline Approximation Methods," Mathematica (Cluj), 8, 1966.

10. J. R. Rice, The Approximation of Functions, Addison-Wesley, Reading, Mass., 1964.

11. __________, "Characterization of Chebyshev Approximations by Splines," S.I.A.M. Journal of Numerical Analysis, 4, 1967.

12. L. L. Schumaker, "Uniform Approximation by Tchebycheffian Spline Functions," M. R. C. Technical Report No. 768, 1967.

13. G. F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill, New York, 1963.

14. J. B. Smith, "Some Problems in the Theory of Approximation," dissertation, University of Arizona, 1967.