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DIFFRACTION THROUGH A SQUARE APERTURE

by

Clifford William Prettie

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I hereby recommend that this dissertation prepared under my direction by Clifford William Prettie entitled DIFFRACTION THROUGH A SQUARE APERTURE be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

Dissertation Director   Date 5/23/74

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The author would also like to dedicate this work to his wife, Carol Ann Flyer.

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ABSTRACT

The dissertation develops the problem of vector diffraction through a square aperture through generalizations of a method developed for the thin wire scattering problem. In the latter problem the harmonic operator of the Pocklington integral equation is brought through the integration sign onto the unknown current with integration by parts. The solution of the integral equation by moment methods gives the forcing function of a differential equation for the current which can be solved by green's function techniques. Care is taken to remove the singularity in the derivative of the current. The behavior of this removable singularity is shown to be that of the inverse square root of the distance to the nearest edge. Computational results are given that show the superiority of this green's function moment method over conventional methods. The method is next applied to the narrow rectangular aperture problem which is shown to be identical to the thin wire problem with some reasonable approximations. An iterative technique for the narrow aperture is reviewed with a discussion of the approximations involved. Its results and the green's function method are then favorably compared. The green's function moment method is then applied to the vector diffraction problem of the square

aperture. The analysis carries the integral equations up to a point where a programmer could take over. Care is taken to make sure the fields satisfy edge conditions. Numerical results good for an aperture size of up to about a wavelength square are expected; up to about two wavelengths square if modifications are made to take advantage of symmetries.

CHAPTER 1

INTRODUCTION

In this dissertation the problem of diffraction through a square aperture is addressed with the purpose of obtaining an approximate solution. The method developed is based on a moment method solution to the integral equations for the aperture electric field. For an excellent discussion of moment methods the reader is directed to Harrington (1968). These integral equations involve harmonic operators. The method developed carefully brings this operator through integration signs and onto the unknown. The harmonic operator acting on the unknown is solved for via moment methods. The unknown is then solved for with green's function techniques motivating the name "green's function moment method."

In Chapter 2 motivation for the method is given by applying it to the relatively simple problem of thin wire scattering. The basic idea of moving the harmonic operator onto the unknown is set forth. Unfortunately the method presented gives poor answers.

In Chapter 3 the failure of the method of Chapter 2 is analyzed. With some analytical techniques involving inverse finite Hilbert transforms the reason for the failure of the method is laid to an invalid assumption regarding integration by parts. The derivative of the current on the thin wire scatterer is shown to go singular with the inverse square root of the distance to the ends of the wire. This singularity is shown to be removable.

In Chapter 4 the results of Chapter 3 are incorporated into the method of Chapter 2. The result is the green's function moment method for the wire. Computational results are given showing the method's superiority over conventional moment methods.

In Chapter 5 results from the green's function method are compared with results from an iterative type solution. The problem involved is diffraction through a narrow aperture. The approximations made in this problem reduce it to a scalar diffraction problem similar to the thin wire problem. The errors involved in the approximations made in the iterative solution (Suzuki, 1956) are discussed in detail to develop confidence in its numerical results. Fairly good agreement with the green's function moment method is then shown.

In Chapter 6 the green's function moment method is applied to the square aperture problem. The integral equations are developed until a programmable set of linear equations is obtained. Care is taken with the analysis and the solutions obtained from this formulation meet edge conditions.

Chapter 7 concludes the dissertation with some ideas for further applications of green's function moment methods.

It should be noted that the heart of this dissertation is the method developed, the green's function moment method. The emphasis is on this method throughout the text. Its applicability to the diffraction problem is used basically as an example of its versatility. It should also be noted that no computational results for the square aperture are given. Such results were incompatible with the monetary resources available to the author.

CHAPTER 2

THE THIN WIRE (INTRODUCTION)

The mathematical problem of solving for the current on a thin straight wire scatterer has long been a testing ground for approximate solutions by moment methods as well as other methods. For a review of the latest conventional moment methods applied to this problem the reader is directed to Miller et al. (1974). The relative simplicity of the differentio-integral equations involved has probably been the reason for this fact. In this chapter a new method is applied to this problem. Although the results of this application are not good, they motivate further modifications which lead to excellent results.

The Green's Function Moment Method for Solving the Thin Wire Scattering Problem

The integral equation for the current has the following form:

$$\frac{\partial}{\partial x} \int_{-1}^{1} \frac{\partial I(z)}{\partial z} K(z-x)\, dz + k^2 \int_{-1}^{1} I(z)\, K(z-x)\, dz = f(x) \qquad (2.1)$$

The current at $\pm$ 1 is forced to vanish. The kernel of the equation is called the "exact" kernel and is defined as follows for a wire of radius a:

$$K(x) = \int_0^{2\pi} \frac{e^{-ikR}}{R} \, d\phi,$$

where

$$R^2 = x^2 + 4a^2\sin^2(\phi/2) \tag{2.2}$$

and where k is the wavenumber. Note that because of the boundary condition on I(z) and the fact that

$$\frac{\partial}{\partial x} K(z-x) = - \frac{\partial}{\partial z} K(z-x), \tag{2.3}$$

integral equation (2.1) is basically no different from the following:

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) \int_{-1}^{1} I(z) \, K(z-x) \, dz = f(x) \tag{2.4}$$

The latter is called the Pocklington integral equation. Equation (2.1) is called the potential integrodifferential equation or, briefly, the potential integral equation (Miller et al. 1974).

The derivation of this equation is widely known and only sketched here. Basically it follows an argument that the tangential electric field must go to zero on the wire. The electric field is given in terms of the currents and charges on the wire through the use of potential functions. Conservation of charge is used to express the charge density in terms of the current. With several approximations (2.1) appears readily. For a complete derivation of (2.1) see Miller et al. (1974) or King (1956).

Arriving at Equation (2.4) from Equation (2.1) is an easy matter. The differentiation on the unknown of the integral equation I(z) glides through the integration to the outside of the integral without problems. The Pocklington equation, (2.4), can be treated as a differential equation, the unknown being the integral over the current. Straightforward solution of that equation leads to Hallèn's integral equation for the current, namely to the following integral equation:

$$\int_{-1}^{1} I(z)\ K(z-x)\ dz = -\frac{1}{k}\int_{-1}^{x} f(t)\ \sin(k(t-x))dt$$

$$+ A \cos ky + B \sin ky \qquad (2.5)$$

A and B are unknown constants that are determined by the boundary condition on I. Suppose now that the differentiations are taken into the integral onto I(z). An ordinary integral equation of the first kind with the harmonic operator operating on the current is obtained. A moment method solution of this integral equation would give the forcing function for a harmonic differential equation for I(z). The boundary conditions on I(z) are known and green's function techniques can subsequently be employed to find I(z). Unknown constants do not enter the picture.

This method has been explored by Mayes (1974) as early as 1972.

Following this process in more detail, an integration by parts of (2.1) gives:

$$\int_{-1}^{1} \{\frac{\partial^2}{\partial z^2} + k^2\} I(z) K(z-x) dz - \left(\frac{\partial I(z)}{\partial z} K(z-x)\right)\Bigg|_{z=-1}^{z=+1} = f(x) \tag{2.6}$$

Note that the assumption that the differentiation can be interchanged with the integration and the assumption that an integration by parts can be performed have been made. Letting U(z) be defined as follows:

$$\{\frac{\partial^2}{\partial z^2} + k^2\} I(z) = U(z) \tag{2.7}$$

with the boundary conditions on I one can obtain I(x) in terms of U as:

$$I(x) = \frac{\sin(k(x+1))}{k \sin(2k)} \int_{x}^{1} U(z) \sin(k(z-1)) dz$$

$$+ \frac{\sin(k(x-1))}{k \sin(2k)} \int_{-1}^{x} U(z) \sin(k(z+1)) dz \tag{2.8}$$

This expression is a straightforward application of green's function techniques. With (2.8) the derivative of I can be represented in terms of U. The second term on the left hand side of (2.6) becomes an integral expression of U.

This expression when combined with the first integral gives the following integral equation for U.

$$\int_{-1}^{1} U(z)\, \tilde{K}(z,x)\, dz = f(x) \tag{2.9}$$

where the kernel is defined as follows:

$$\tilde{K}(z,x) = K(z-x) + \{K(x+1)\sin(k(z-1)) - K(x-1)\, \sin(k(z+1))\}/\sin(2k) \tag{2.10}$$

Thus the desired integral equation has been found. Solving (2.9) by moment methods I(z) can be found.

Advantages and Disadvantages of the Green's Function Moment Method

The advantages of the above method are at least twofold. First, as noted before, there are no undetermined coefficients to bother with as in the Hallèn integral equation (2.5). The toil of determining these constants has been done, in a sense, by the green's function of the harmonic operator with Dirichlet boundary conditions [as in (2.8)]. Second, the resulting current functions have continuous derivatives even when U is expanded in crude pulse functions. This fact is in contrast with standard moment methods where the function that is used to solve the integral equation is generally the function that describes the current. If pulse functions are used in such methods, I has a derivative that is very non-physical.

Disadvantages of the method exist. First calculating $\tilde{K}$ may require more effort than calculating K. This objection is not that important since the major contributor to computer storage and run time in moment methods is the process of solving the linear equations for the unknowns. Second, upon programming the problem into the computer and calculating some currents, poor convergence is noted for almost any geometry (convergence is a subjective measure of how many unknown expansion functions must be used before the current function is accurately enough described so that use of more expansion functions will not significantly alter its value). This disadvantage cannot be taken lightly. Improved convergence over standard methods seemed to be a motivation for use of this method. This reversal is very discouraging. If this method with its nice quality of giving smooth solutions is to be saved, some modifications must be made.

CHAPTER 3

THE THIN WIRE (ANALYSIS)

The poor convergence of the method of Mayes casts suspicion on the validity of the assumptions underlying its formulation. It will be recalled that those assumptions were first, that the differentiation and integration operations could be interchanged to bring the differentiation onto the kernel, and second, that an integration by parts could be performed in order to create an integral equation of the first kind, the unknown being the harmonic operator operating on the current function. A closer look at that integral equation reveals further difficulties. If integral equation (2.9) of Chapter 2 is reshaped by grouping appropriate terms one obtains the following:

$$\int_{-1}^{1} U(z)\ K(z-x)\ dz - a^{+}\ K(1-x) + a^{-}\ K(x+1) = f(x) \qquad (3.1)$$

where: $a^{+} = \int_{-1}^{1} U(p)\ \sin(k(p+1))\ dp/\sin(2k)$

$$a^{-} = \int_{-1}^{1} U(p)\ \sin(k(p-1))\ dp/\sin(2k) \qquad (3.2)$$

It is well known that K(a-z) has a logarithmic singularity at z = a (Butler, 1972). This fact may be recognized by considering the integration over the region where the integrand of the integral expression (2.2) for the kernel is singular:

$$\frac{1}{2\pi}\int_{\varepsilon/a}^{\varepsilon/a}\frac{1}{r}\,d\phi \simeq \frac{1}{2\pi a}\int_{\varepsilon/a}^{\varepsilon/a}\frac{d(a\phi)}{(x^2+(a\phi)^2)^{1/2}} = \frac{1}{2\pi a}\ln\left(\frac{R+\varepsilon}{R-\varepsilon}\right)$$

where ε/a is small;

$$R^2 = x^2 + \varepsilon^2;\text{ and where } x \text{ has replaced } z - x \tag{3.3}$$

Now as $x \to 0$ it is easily seen that $R + \varepsilon \to 2\varepsilon$. Expanding $R = \varepsilon(1+x^2/\varepsilon^2)^{1/2}$ in a binomial series gives $R - \varepsilon =$ $[\varepsilon + \frac{1}{2}x^2/\varepsilon + 0(x^4)] - \varepsilon \simeq \frac{1}{2}x^2/\varepsilon$ and the desired result:

$$K(x) = -\frac{\ln|x|}{\pi a} + O(1) \text{ as } x \to 0 \tag{3.4}$$

Looking at (3.1) with (3.4) in mind a distressing fact can be seen. The right-hand side of (3.1) is finite at $x = \pm 1$. If a^+ or a^- is non-zero, however, the left-hand side has singular terms at $x = +1$ or at $x = -1$. To effect cancellation of these terms the integrand $U(z)\ K(z-x)$ must be non-integrable when $x = +1$ or when $x = -1$. Since $K(z\pm 1)$ is integrable, $U(z)$ must therefore be singular at at least one of the end points. A moment method solution for such a function would converge slowly near the ends. This poor end convergence of U would seem to affect the function I over the entire interval. Such behavior is indeed witnessed in the computational results of this method. The notion arises that perhaps these two underlying assumptions are valid and that all that is needed is either a conclusive proof that a^+ and a^- are, in all cases, zero, or knowledge of the asymtotic behavior of U near the ends. This notion dissipates, however, in the next section where it is shown that neither alternative is easily acceptable. In the rest of the chapter the two assumptions are looked at and a rigorous development of the problem is performed with the help of Tricomi's (1957) writings on the airfoil equation. The assumption that integration by parts can be performed is seen to be invalid in this development; on the other hand, the assumption regarding the interchange of differentiation and integration is seen to be valid, with certain restrictions.

Arguments for Reexamination of Basic Assumptions

In the previous section it was seen that the method of Chapter 2 could be saved only if the expression in (3.1) as $x \to \pm 1$ could be made finite. First consider the possibility of having a^+ and a^- go to zero for any forcing function of the integral equation. Inspection of Equation (2.8) or of Equations (2.6) and (3.2) reveals a^+ and a^- as the derivatives of the current function evaluated at each of the end points: $a^+ = \partial I(z)/\partial z|_{z=1}$ $a^- = \partial I(z)/\partial z|_{z=-1}$. Requiring a^+ and a^- to be zero in all cases is equivalent to requiring that the derivative of the current vanish at the ends. This restriction must be considered in addition to the basic restriction of the problem that the current itself vanish at the ends. Physically this added restriction is equivalent to requiring the charge to vanish near the end-points of the wire--an idea which disagrees with intuition, since in the static problem the charge density would be singular at these points. Furthermore, a simple moment method solution to this problem with the new restrictions acting as constraints on the solution vector becomes over-determined. Getting stable numerical results to this over-determined problem would be very difficult. Although the possibility of both a^+ and a^- being zero for any forcing function has not been ruled completely out, the weight of these intuitive arguments seems to make the possibility improbable.

The other possibility for saving the method of Chapter 2 is to make $U(z)$ go singular at $z = \pm 1$ in such a fashion that $\int U(z)\, K(x-z)\, dz$ will cancel the singularities of the logarithmic functions. Looking at (3.1) shown again here,

$$\int_{-1}^{1} U(z)\, K(z-x)\, dz - a^+ K(1-x) + a^- K(1+x) \qquad (3.5)$$

it can be seen that by setting $U(z) = u(z) + a^+ \delta(z-1) - a^- \delta(z+1)$, the undesirable singularities are cancelled leaving

$$\int_{-1}^{1} u(z)\, K(z-x)\, dz = f(x) \qquad (3.6)$$

Upon putting this assumed form of U into the inversion formula (2.8) for the current gives the startling result

$$I(x) = \frac{\sin(k(x+1))}{k \sin(2k)} \int_x^1 u(z) \sin(k(z-1))\, dz$$

$$+ \frac{\sin(k(x-1))}{k \sin(2k)} \int_{-1}^x u(z) \sin(k(z+1))\, dz \qquad (3.7)$$

The terms $a^+\delta(z-1)$ and $a^-\delta(z+1)$ have no effect on the current I. This fact says in effect, that if the singular terms of integral equation (3.1) are ignored the current function I is the same. It is not surprising then that computational results based on this idea converge poorly, in general, and are in poor agreement with published results. One fundamental objection to this idea is the meaninglessness of terms like $\delta(z-1)$ in the inversion formula. Indeed the interpretation of a jump discontinuity in the derivative at the end points of the interval of definition of a function is difficult. Further, the problem

$$u''(x) + k^2u(x) = \delta(1-x);\ x \in [-1,1],\ u(-1) = u(1) = 0 \qquad (3.8)$$

is like asking what happens to the immovable object when struck by an irresistible force, and Equation (3.7) tried to answer it.

The failure to put the method of Chapter 2 on sturdier ground can only lead to its abandonment and a return to basics. Although the above attempts to resolve the problem of the singularities have been fruitless, the remote possibility that the method might yield valid results exists; however, to proceed further without a reexamination of the method and its assumptions would be reckless.

Analysis of the Interchange of Differentiation with Integration

Heuristically, the integral equation

$$Qu(x) \overset{df}{=} \frac{\partial}{\partial x} \int_{-1}^{1} u(z) \ln|x-z| \, dz = f(x) \tag{3.9}$$

can be considered instead of the more complicated potential equation with exact kernel (2.1). The problems of Equation (3.1) only reflect the singular nature of the exact kernel which in (3.3) and (3.4) is seen to be of logarithmic nature. The same problems arise in (3.9) if u(z) is considered as playing the role of the first derivative of the current function in the potential equation problem. Indeed, interchange of differentiation and integration followed by an integration by parts gives

$$\int_{-1}^{1} u'(z) \ln|x-z| \, dz - u(1) \ln|1-x| + u(-1) \ln|1+x| = f(x) \tag{3.10}$$

The left-hand side is potentially singular again as in Equation (3.1). The simplicity of the log function, however, makes it an easier matter to examine these manipulations.

Consider the interchange of the differentiation and the integration in this light. All the purist need note is that $1/(x-z)$ is a non-integrable function in the sense of Lebesque in order to answer the question of its validity. However, it is possible to redefine the sense of the integral to avoid this embarrassment. Suppose a function $g(x)$ is singular at $x = x_o \in (a,b)$ and is integrable elsewhere in $[a,b]$. The Cauchy principal value of the integral of $g(x)$ is defined as (Whittaker and Watson, 1927):

$$\text{CPV} \int_a^b g(x)\ dx = \lim_{\delta \downarrow 0} \left[\int_a^{x_o-\delta} g(x)\ dx + \int_{x_o+\delta}^{b} g(x)\ dx \right]$$

$$\equiv \int_a^{*b} g(x)\ dx \tag{3.11}$$

Following the notation of Tricomi (1957), the asterisk over the integral sign shall denote the fact that the Cauchy principal value is meant. The Cauchy principal value of $\int_{-1}^{1} u(z)/(x-z)\ dz$ exists, a.e., if $u(z)$ is merely integrable (Titchmarsh, 1948, p. 132). Noting that

$$\int_{-1}^{*1} u(z)\ \ln|x-z|\ dz = \int_{-1}^{1} u(z)\ \ln|x-z|\ dz \tag{3.12}$$

if $u(z)$ is bounded over some closed interval around x, no matter how small, redefinition of the integration in the integral equation (3.9) to a CPV integration does not change the diffraction problem. Now the question of the validity of interchanging the differentiation with the integration cannot be answered so hastily in the negative.

The first step in the interchange is to bring the derivative inside the CPV limit process, that is, to interchange the derivative with the limit in (3.11). For this step to be valid the following conditions must occur:

$$\int_{-1}^{x-\delta} + \int_{x+\delta}^{1} u(z) \ln|x-z| \, dz \to \int_{-1}^{*1} u(z) \ln|x-z| \, dz$$

uniformly almost everywhere (3.13a)

$$\lim_{\delta \downarrow 0} \frac{\partial}{\partial x} \int_{-1}^{x-\delta} + \int_{x+\delta}^{1} u(z) \ln|x-z| \, dz$$

exists almost everywhere (3.13b)

From a theorem on Lebesque integration (McShane, 1944, p. 217) since over $(-1, x-\delta)$ and over $(x+\delta, 1)$ $u(z)/(x-z)$ is absolutely integrable in z, the integration and differentiation in condition (3.13b) may be interchanged. With the help of Liebniz's rule condition (3.13b) becomes

$$\lim_{\delta \downarrow 0} \left[\int_{-1}^{x-\delta} + \int_{x+\delta}^{1} \frac{u(z)}{x-z} \, dz \right] + \lim_{\delta \downarrow 0} \left[u(x-\delta) \ln\delta - (x+\delta) \ln\delta \right]$$

exists almost everywhere (3.14)

As noted before the first term of (3.14) exists almost everywhere if u(z) is integrable. The second term exists almost everywhere if for almost any $x \varepsilon (-1,1)$ a number K large enough and numbers p and δ_o small enough can be found so that $|u(x-\delta) - u(x+\delta)| < K\delta^p$ for all $0 < \delta < \delta_o$. This fact is seen by noting that $x^p \ln|x| \to 0$ as $x \to 0$ so long as $p > 0$. The condition on this second term in (3.14) is definitely the stronger of the two and (3.13b) is met if u satisfies it. Moreover, the interchange has also been justified under these conditions on u if it can be shown that (3.13a) is true under the same, since it has been shown that

$$\lim_{\delta \to 0} \left[\frac{\partial}{\partial x} \int_{-1}^{x-\delta} u(z) \ln|x-z| \, dz + \frac{\partial}{\partial x} \int_{x+\delta}^{1} u(z) \ln|x-z| \, dz\right]$$

$$= \int_{-1}^{*1} \frac{u(z)}{x-z} \, dz \qquad (3.15)$$

The uniform convergence of the limit of (3.13a) can be looked at in a different manner. With the fact that the integral and its CPV are identical the uniform convergence to the principal value is seen to be equivalent to the uniform convergence of

$$\left| \int_{-1}^{x-\delta} + \int_{x+\delta}^{1} u(z) \ln|x-z| \, dz - \int_{-1}^{*1} u(z) \ln|x-z| \, dz \right|$$

$$= \left| \int_{x-\delta}^{x+\delta} u(z) \ln|x-z| \, dz \right| = \left| \int_{-\delta}^{\delta} u(x+z) \ln|z| \, dz \right| \qquad (3.16)$$

to zero. A uniform bound for u would guarantee the uniform convergence of (3.16) everywhere. Suppose u is restricted by the condition that was found for the existence of (3.13b) everywhere instead of almost everywhere, that is, suppose that for any $x \varepsilon (-1,1)$ there exists numbers p and δ_o small enough and a number K large enough such that $|u(x) - u(x_o)|$ can be made strictly smaller than $K|x-x_o|^p$ for all $0 < |x-x_o| < \delta_o$. u(z) is continuous on (-1,1) under this condition everywhere and therefore bounded on any closed interval contained in (-1,1). This condition guarantees therefore uniform convergence of (3.16) everywhere on any closed interval contained in (-1,1). This condition can also be shown to be sufficient for the existence of the CPV of the integral everywhere in (-1,1). These results lead to the following theorem.

Theorem 3-1: If u(z) is such that there exist positive numbers p, K, δ_o such that for any $x \varepsilon (-1,1)$ $|u(x) - u(x_o)| < K|x-x_o|^p$ for all $0 < |x_o-x| < \delta_o$, then an interchange of the differentiation and integration may be performed in Equation (3.9) provided the integration is understood as the CPV of the integral. That is for all $x \varepsilon (-1,1)$ $Qu(x) = \frac{\partial}{\partial x} \int_{-1}^{1} u(z) \ln|x-z|\, dz = \int_{-1}^{*1} \frac{u(z)}{x-z}\, dz$. This theorem follows from the fact that any $x \varepsilon (-1,1)$ is a member of a closed interval $[-1 + \frac{x+1}{2}, 1 - \frac{1-x}{2}]$ contained in (-1,1) and from the above arguments the interchange is valid within this closed interval.

The condition on u(z) in the theorem is similar to the Hölder condition. A function u(z) is said to satisfy the Hölder condition on the interval I if for any two points t_1 and t_2 in I, $|u(t_1) - u(t_2)| < K|t_1-t_2|^p$ for some K and p which exist independently of t_1 and t_2. Using some liscense, let the following terminology be adopted: u(z) is said to satisfy a Hölder condition at x if there exist K, p, and δ such that $|u(z) - u(x)| < K|z-x|^p$ whenever $0 < |x-z| < \delta$. Physically the assumption on u in the preceding theorem does not seem unreasonable. The charge on the wire is represented by u, which can be expected to be continuous and differentiable at all but the ends of the wire. As a consequence of the differentiability of u(z), u can be expected to satisfy a Hölder condition at all x on the open interval (-1,1). Thus any function u(z) that is of physical interest and is a solution of the integral equation (3.9) is also a solution of the integral equation

$$Qu(x) = \int_{-1}^{*1} \frac{u(z)}{x-z}\, dz = f(x) \tag{3.17}$$

The Airfoil Equation

Consider this left-hand side of Equation (3.17). The integral operator operating on u(z), if multiplied by the constant $-1/\pi$, is known as the finite Hilbert transform. The equation, itself, occurs in the study of the motion of lifting surfaces and is known as the "airfoil equation." This equation is nice in that it has a closed form solution. It is not hard to show that the finite Hilbert transform of $1/\sqrt{(1-x^2)}$ is zero (Tricomi, 1957), thus, the solution (3.17) is determined only up to finite constant times $1/\sqrt{(1-x^2)}$. Tricomi (1957) gives this solution in closed form as

$$u(x) = \frac{1}{\sqrt{(1-x^2)}} \left[\int_{-1}^{*1} \frac{\sqrt{(1-y^2)}\ f(y)}{x-y}\, dy + C \right] \qquad (3.18)$$

The factor of $1/\sqrt{(1-x^2)}$ in u(z) brings about an interesting consequence. If the finite Hilbert transform of $\sqrt{(1-y^2)}f(y)$ can be shown to be bounded and continuous near the endpoints, u(x) would go singular as $1/\sqrt{(1\pm x)}$ at the appropriate endpoint. This behavior would strictly rule out performing an integration by parts as in Equation (3.10) since the derivative of u would be a non-integrable function. If f(x) is bounded and has a bounded derivative on the open interval (-1,1) it would be fairly easy to show that

$$Q\ \{\sqrt{(1-x^2)}\ f(x)\} = \int_{-1}^{*1} \frac{\sqrt{(1-x^2)}\ f(x)}{x-y}\, dx \qquad (3.19)$$

is bounded and continuous. Another tact, however, gives still more explicit results.

If f(x) is bounded and has a bounded derivative then $|f(x)|^{2+\varepsilon}$ is integrable for an arbitrary $\varepsilon > 0$ and an alternate form for u(x) in terms of f(x) can be given (Tricomi, 1957) as follows:

$$u(x) = Q_1 f(x) = - \int_{-1}^{*1} \frac{\sqrt{(1-x^2)}}{\sqrt{(1-y^2)}} \frac{f(y)}{y-x} dy + \frac{Ax + B}{\sqrt{(1-x^2)}} \tag{3.20}$$

If

$$\int_{-1}^{*1} \frac{f(x)\ dx}{\sqrt{(1-x^2)}(x-y)} \tag{3.21}$$

can be shown to be bounded, then, the singularity at the ends is not only established but also established as removable. The proof of the boundedness of (3.21) is not hard. First,

$$\int_{-1}^{*1} \frac{f(x)\ dx}{\sqrt{(1-x^2)}(x-y)} = \int_{-1}^{*1} \frac{f(x) - f(y)}{\sqrt{(1-x^2)}(x-y)} dx \tag{3.22}$$

from the fact that $Q\{1/\sqrt{(1-x^2)}\} = 0$. By the mean value theorem $(f(x)-f(y))/(x-y)$ is equal to the derivative of f at some value between x and y. Thus since the derivative is bounded on (-1,1), so too must be this function. Calling this bound K_m it is seen that (3.21) is bounded by πK_m

$$\left| \int_{-1}^{*1} \frac{f(x)\ dx}{\sqrt{(1-x^2)}(x-y)} \right| \leq \int_{-1}^{*1} \frac{1}{\sqrt{(1-x^2)}} \left| \frac{f(x) - f(y)}{x - y} \right| dx$$

$$\leq K_m \int_{-1}^{*1} \frac{dx}{\sqrt{(1-x^2)}} = \pi K_m \tag{3.23}$$

The restrictions on f(x) could be weakened and the theorem would still hold; however, as they stand they are not hard to meet in diffraction problems. In most diffraction problems the forcing function easily meets this requirement because it is usually an infinitely differentiable function.

The singularity in the homogeneous term gives the result that integration by parts in (3.10) is definitely not valid in the general case. However, this singularity can be removed with addition of appropriate terms. That is, b_1 and b_2 can be found so that,

$$u(x) = b_1/\sqrt{(1+x)} + w(x) + b_2/\sqrt{(1-x)}$$

$$\text{where:} \quad w(\pm 1) = 0 \tag{3.24}$$

The importance of this fact cannot be overstressed for it makes it possible to avoid the singular behavior of u. Integration by parts is, in general, invalid for u; however, under not too stringent conditions an integration by parts will be valid for w.

Invalidity of an Integration by Parts of the Potential Equation

The results of the previous section seem to point to the fact that $\partial I(z)/\partial z$ has a removable singularity that behaves like the inverse square root of the distance to the nearest endpoint. To show this fact, the logical first step would be to isolate the logarithmic singularity of the kernel. The leftover kernel of this isolation process in the potential integral equation is a function, say M(x), which can be shown to be bounded. More explicitly, the potential integral equation is remembered as,

$$\frac{\partial}{\partial x} \int_{-1}^{1} \frac{\partial I(z)}{\partial z} K(z-x)\, dz + k^2 \int_{-1}^{1} I(z)\, K(z-x)\, dz = f(x) \quad (3.25)$$

An integration by parts of the second integral is permissible. Defining $u(z) = \partial I(z)/\partial z$ gives

$$\frac{\partial}{\partial x} \int_{-1}^{1} u(z)\, K(z-x)\, dz - k^2 \int_{-1}^{1} u(z) \int_{x}^{z} K(t-x)\, dt dz = f(x) \quad (3.26)$$

Defining $M(x)$ with

$$M(z-x) = \frac{\partial}{\partial x} \left[K(z-x) + \frac{\ln|z-x|}{\pi a}\right] - k^2 \int_{x}^{z} K(t-x)\, dt \quad (3.27)$$

the potential integral equation can be given in the following form:

$$\frac{-1}{\pi a} \int_{-1}^{*1} \frac{u(z)}{x-z}\, dz + \int_{-1}^{1} u(z)\, M(z-x)\, dz = f(x) \quad (3.28)$$

or in operator notation with $Ru(x)$ defined as the integral of $M(z-x)\, u(z)$ with respect to z, ignoring constants:

$$Qu + Ru = f \quad (3.29)$$

If both sides of the equation are operated on with the inversion operator Q_1 defined in (3.20), the following results:

$$Q_1\ Qu + Q_1\ Ru = Q_1\ f$$

$$u + \frac{Ax + B}{\sqrt{(1-x^2)}} + Q_1\ Ru = Q_1\ f \qquad (3.30)$$

where A and B are constant functions.

$Q_1\ f(x)$ is known to be bounded from the previous section. If a bound could be illustrated for Q_1 Ru the conclusions of the previous section would extend to the case here. u(x) would thus have a removable singularity. To show a bound for Q_1 Ru, Ru is shown to have the property of having a bounded derivative on the open interval (-1,1). Arguments can then be made for its boundedness identical with those in the previous section for Q_1 f.

In Appendix A M(x) is shown to be continuous on (-2,2) and differentiable everywhere except x = 0. Its derivative is shown to be a function of the form ln(x) plus some bounded function. From Lebesque integration theory (McShane, 1944, p. 217) consideration of the boundedness of the derivative of Ru(x) can be made by investigating the integral of the product of u(x) and the derivative of M(z-x). The integral of u(x) with a bounded kernel is a bounded function making the key question whether the integral of u with the logarithmic kernel is bounded.

The <u>derivative</u> of the integral of u with the logarithmic kernel is seen to be bounded everywhere in the interval (-1,1) in Equation (3.28). From this fact it is possible to conclude that the integral itself of u with the logarithmic kernel is a bounded function implying the same conclusion for the derivative of Ru(x). Thus Q_1 Ru(x) is bounded. u(x) in the general case suffers a removable singularity at the ends, that singularity going as the inverse square root of the distance to the nearest end-point. Hence the assumption of Chapter 2 regarding an integration by parts is in general invalid.

Note that cases may exist where Ax+B = 0 in (3.30). Integration by parts becomes valid in the method of Chapter 2 and this method should work. These cases can be seen to correspond to the cases where u(x) is zero at the ends. This is because Q_1 f and Q_1 Ru(x) are not only bounded, but also go to zero as x $\pm$ 1. When the wire length is an integral number of wavelengths and the incidence is broad-side u(x) does indeed go to zero at the ends. Early numerical results at these values of wire length were very good and were the basic motivation for this further work. Note also that u(x) going to zero at x = $\pm$ 1 is equivalent with the case considered in a previous section of this chapter where $a^+ = a^- = 0$.

The next chapter develops a method with very good convergence rate from the results given in this chapter. The singularity in u(x) is removed and a similar approach to the one in Chapter 2 is taken.

CHAPTER 4

THE THIN WIRE AND THE REFINED GREEN'S FUNCTION MOMENT METHOD

In Chapter 2, the method presented was found to be based on an erroneous assumption that an integration by parts could be performed in the potential integral equation. In this chapter that error is remedied. From the preceding chapter the nature of the singularity that causes the integration by parts to go awry is known. Since there are only two singularities in the derivative of the current, one at each endpoint, it would seem profitable to remove them. This chapter develops this idea and presents results of a moment method based on it.

The Green's Function Moment Method

The integral equation that is to be solved is remembered from Chapter 2 as:

$$\frac{\partial}{\partial x} \int_{-1}^{1} \frac{\partial I(z)}{\partial z} K(x-z)\, dz + k^2 \int_{-1}^{1} I(z)\, K(z-x)\, dz = f(x) \qquad (4.1)$$

where:

$$K(x) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{-ikR}}{R}\, d\phi$$

$$R^2 = x^2 + 4a^2 \sin^2(\phi/2) \qquad (4.2)$$

The current has a boundary value restriction on it in that it must be zero at $x = \pm 1$. From the results of Chapter 3 it is known that the derivative of the current goes as the inverse square root of the distance from the ends. Also from Chapter 3, b_1 and b_2 can be assigned with

$$\frac{\partial I(x)}{\partial x} - \frac{b_1}{\sqrt{(1-x)}} - \frac{b_2}{\sqrt{(1+x)}} \to 0 \text{ as } x \to \pm 1 \tag{4.3}$$

Thus I(x) can be assumed to be of the form

$$I(x) = i(x) + 2b_1\sqrt{(1+x)} + 2b_2\sqrt{(1-x)} \tag{4.4}$$

with

$$\frac{\partial i(x)}{\partial x} \to 0 \text{ as } x \to \pm 1 \tag{4.5}$$

Letting

$$G(x) = \frac{\partial}{\partial x}\int_0^2 \frac{K(x-z)}{\sqrt{(z)}}\, dz + 2k^2 \int_0^2 \sqrt{(z)}\; K(x-z)\, dz \tag{4.6}$$

the integral equation (4.1) becomes:

$$- \int_{-1}^{1} \frac{\partial i(z)}{\partial z} \frac{\partial K(z-x)}{\partial z}\, dz + k^2 \int_{-1}^{1} i(z)\; K(z-x)\, dz$$

$$+ b_1\; G(1+x) + b_2\; G(1-x) \tag{4.7}$$

Since the first partial of i(z) vanishes at the endpoints and since it can be shown with some effort that $\partial^2 i(z)/\partial z^2$ is an integrable function, an integration by parts can be performed to yield

$$\int_{-1}^{1} [\{\frac{\partial^2}{\partial z^2} + k^2\} i(z)] K(z-x) dz + b_1 G(1+x) + b_2 G(1-x)$$

$$= f(x) \qquad (4.8)$$

Letting $U(z) = \{\partial^2/\partial z^2 + k^2\}i(z)$ the following set of equations, amenable to moment method solution, is obtained:

$$\int_{-1}^{1} U(z) K(z-x) dz + b_1 G(1+x) + b_2 G(1-x) = f(x)$$

$$I(z=+1) = \frac{1}{k \sin(2k)} \int_{-1}^{1} U(z) \cos(k(z+1)) dz + b_1 2\sqrt{2} = 0$$

$$I(z=-1) = \frac{1}{k \sin(2k)} \int_{-1}^{1} U(z) \cos(k(z-1)) dz + b_2 2\sqrt{2} = 0 \qquad (4.9)$$

The current I(z) is then given by the following relation:

$$I(y) = \frac{1}{k \sin(2k)} [\cos(k(y+1)) \int_{y}^{1} U(z) \cos(k(z-1)) dz$$

$$+ \cos(k(y-1)) \int_{-1}^{y} U(z) \cos(k(z+1)) dz] \qquad (4.10)$$

The above relation between i(z) and U(z) is found by treating U(z) as a forcing function for the differential equation

$$[\frac{d^2}{dz^2} + k^2] i(z) = U(z); \frac{d\, i(z)}{dz}\Big|_{z=\pm 1} = 0 \qquad (4.11)$$

The inversion may be performed using classical green's function techniques.

The Case of Resonance

The above method fails at the resonances of the wire because sin(2k) goes to zero at such points. When there exists an integer N such that $|N\pi-2k|$ is small, another method for calculating U must be utilized. The method, which is based on the use of a modified green's function (Stakgold, 1968, p. 86) instead of the classical green's function, gives the following set of equations for U:

$$\int_{-1}^{1} U(z)\ K(z-x)\ dz + b_1\ G(1+x) + b_2\ G(1-x) = f(x)$$

$$I(z=+1) = -\frac{\cos(2k)}{k} \int_{-1}^{1} U(z)\ \sin(k(z+1))\ dz + 2b_1\sqrt{2}$$

$$+ A\cos(2k) = 0$$

$$I(z=-1) = b_2 2\sqrt{2} + A = 0$$

$$\int_{-1}^{1} U(z)\ \cos(k(z+1))\ dz = 0 \qquad (4.12)$$

The last equation assures orthogonality of U with the homogeneous solution. After the above equations have been solved by moment methods, the current may be found with the following relations:

$$I(z) = i(z) + 2b_1\sqrt{(1+z)} + 2b_2\sqrt{(1-z)} \qquad (4.13)$$

$$i(z) = -\frac{1}{k}\,[\cos(k(z+l)) \int_{-l}^{z} U(p)\,\sin(k(p+l))\,dp$$

$$+ \sin(k(z+l)) \int_{z}^{l} U(p)\,\cos(k(p+l))\,dp] + A\cos(k(z+l)) \quad (4.14)$$

Computational Results for the Thin Wire Scatterer

In the above analysis K(x-z) was defined to be the exact kernel. The analysis depended on the exact kernel only to justify the assumption on the shape of the current near the ends of the wire. Once the assumption is made, it is irrelevant in the following analysis whether the exact kernel or Waterman's (1965) extended kernel is used. The computer analysis can thus be simplified by using the latter kernel, that is, by setting

$$K(x) = e^{-ikr}/r \quad \text{where } r^2 = x^2 + a^2.$$

In the computation of all the following results, the extended kernel has been used.

To calculate U(z) in the green's function method, moment methods were used with pulse expansion functions and delta function testing. Even with such crude expansion functions, the resulting current has continuous first derivatives on the open interval (-l,l) which is not the case with the piecewise sinusoidal method or the triangular pulse method.

Integration of the kernel function times the pulse functions of the expansion was accomplished by splitting the integral into two parts:

$$\int_a^b \frac{e^{-ikr}}{r} dz' = \int_a^b \frac{e^{-ikr}-1}{r} dz' + \int_a^b \frac{1}{r} dz'$$

The second integral on the right hand side was evaluated analytically. The first integral was done using 24-point Gaussian integration. Care must be taken in the evaluation of the function G(z) as defined in (4.6). In the following results this integral was performed as follows:

$$\int_0^{2H} \frac{1}{\sqrt{z'}} \frac{\partial}{\partial z} \frac{e^{-ikr}}{r} dz' + 2k^2 \int_0^{2H} \frac{e^{-ikr}}{r} \sqrt{z'}\, dz' =$$

$$\frac{1}{.02\ a} \left(\int_{\ln w}^{\ln a} \frac{-e^u\ (F(-e^u)-C)}{\sqrt{(w-e^u)}} du + \int_{-a}^{a} \frac{F(u)\ du}{\sqrt{(w+u)}} + \right.$$

$$\left. \int_{\ln a}^{\ln(2H-w)} \frac{e^u\ F(e^u)}{\sqrt{(w+e^u)}} du \right) \Bigg|_{w\,=\,z\,-\,.01\,a}^{w\,=\,z\,+\,.01\,a}$$

$$+ 2k^2 \left(\int_{\ln z}^{\ln a} (-e^u)\ F(-e^u)\ \sqrt{(z-e^u)}\ du + \int_{-a}^{a} F(u)\ \sqrt{(z+u)}\ du \right.$$

$$\left. + \int_{\ln a}^{\ln(2H-z)} e^u\ F(e^u)\ \sqrt{(e^u+z)}\ du + 2C\ [\sqrt{(z+.01a)}-\sqrt{(z-.01a)}] \right.$$

where $F(x) = e^{-ikr}/r$; $r = x + a$, and $C = F(-z)$. The purpose of the exponential substitutions is to make the integrand more slowly varying. The above integrals were also evaluated using 24-point Gaussian integration.

Plotted are results by the above two methods compared with a standard moment method solution (Butler, 1972) of the Pocklington equation that uses piecewise sinusoidal expansion functions and delta function testing. The forcing functions are $A\exp(-ikz\cos(\theta))$ where θ is the angle of incidence and A is a physical constant. Also plotted is a check against the results of Harrington (1968) who uses 32 triangular pulses as the expansion set and point matching. In all the results the total length of the wire was 74.2 times the diameter, thus fixing the value of a. To try to give a measure of convergence rates we consider the absolute value of the difference between the converged curve and the curve in question.

A statement that the method has converged to within, say, 1% at a certain sample rate should be understood to mean that this difference curve--between the converged curve and this certain sample rate curve--divided by the peak value of the converged curve is less than, in this case, .01.

Figure 1 shows plots that compare the non-resonant green's function method and the piecewise sinusoidal method for a $1/\pi$-wavelength wire. The former method has converged essentially when sampled 16 times along its total length. No symmetry has been invoked in either method. The green's function method has converged to within 1 per cent at 8 samples while at 79 samples the piecewise sinusoidal method has yet to converge to within 2 per cent. The ratio of

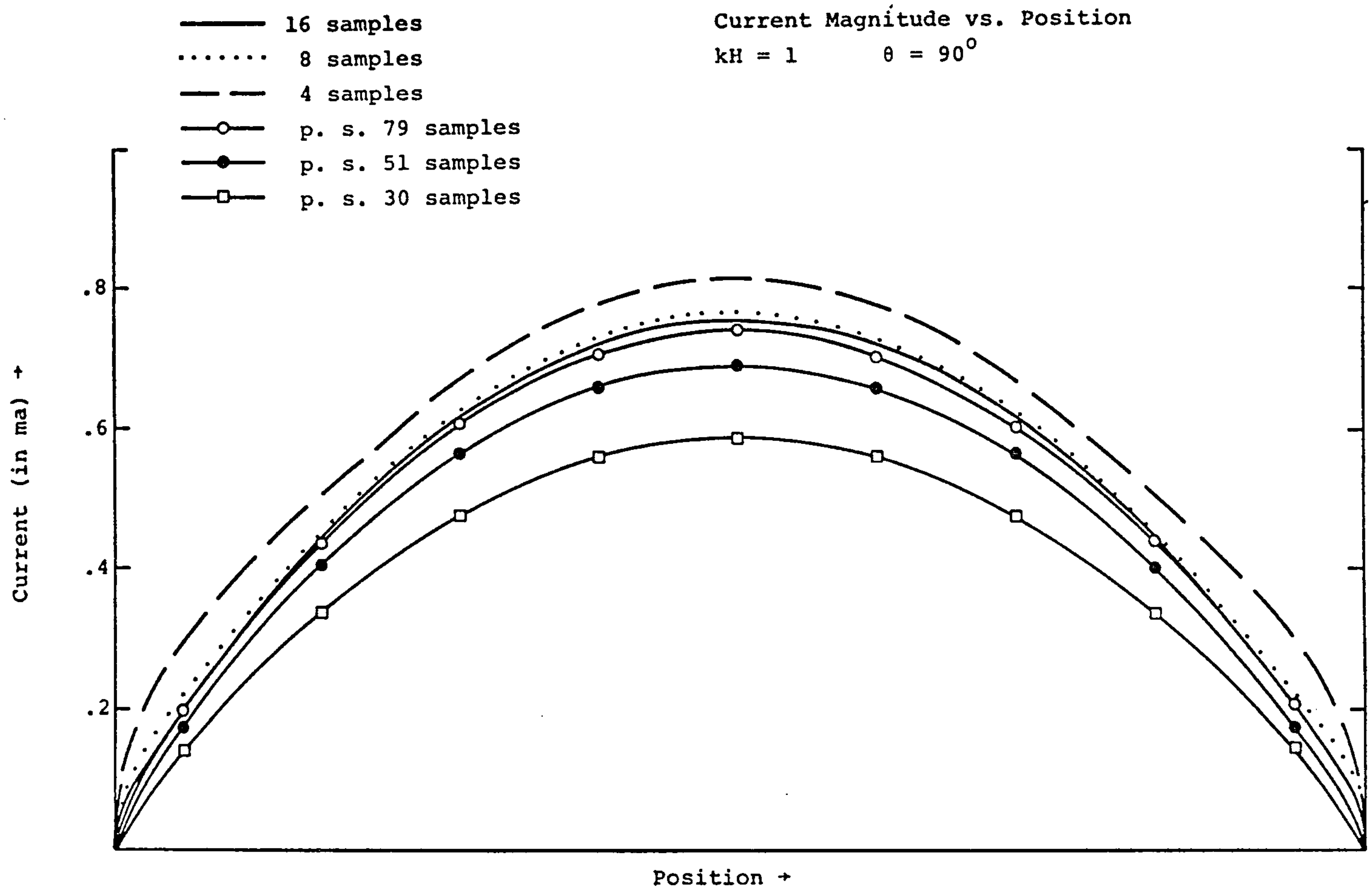

Fig. 1. Plot showing the faster convergence rate of the non-resonant method over the piecewise sinusoidal (p. s.) method -- Incident field strength is 1 volt/wavelength.

improvement in convergence seems to be at least 10-to-1 in this case.

Figure 2 shows a comparison between the resonant green's function method and the piecewise sinusoidal method for a wavelength and a half wire and oblique angle of incidence. The green's function method seems to have converged to within less than 1 per cent at 16 samples and to within 2 per cent at 10 samples. The convergence for the piecewise sinusoidal method does not seem as good even at 60 samples.

Figures 3 and 4 show comparisons between the resonant green's function method and the results published in Harrington (1968) for a two wavelength wire at two different oblique angles of incidence. The green's function method has essentially converged at 14 sample points for an incident angle of 60°. For the 15° case the method was very close to converging at 16 samples. It was within 5% of converging at 10 samples. The results of Harrington using 30 triangular pulses is within 2% of the 15° case and within 4% of the 45° case. It should be noted that the Harrington curves have been redrawn from numbers obtained by digitizing enlarged copies of the curves in his book and may, therefore, be subject to the errors of that process.

Figure 5 illustrates the fact that although the non-resonant green's function method cannot be applied to resonant cases it can be used to get good results for these

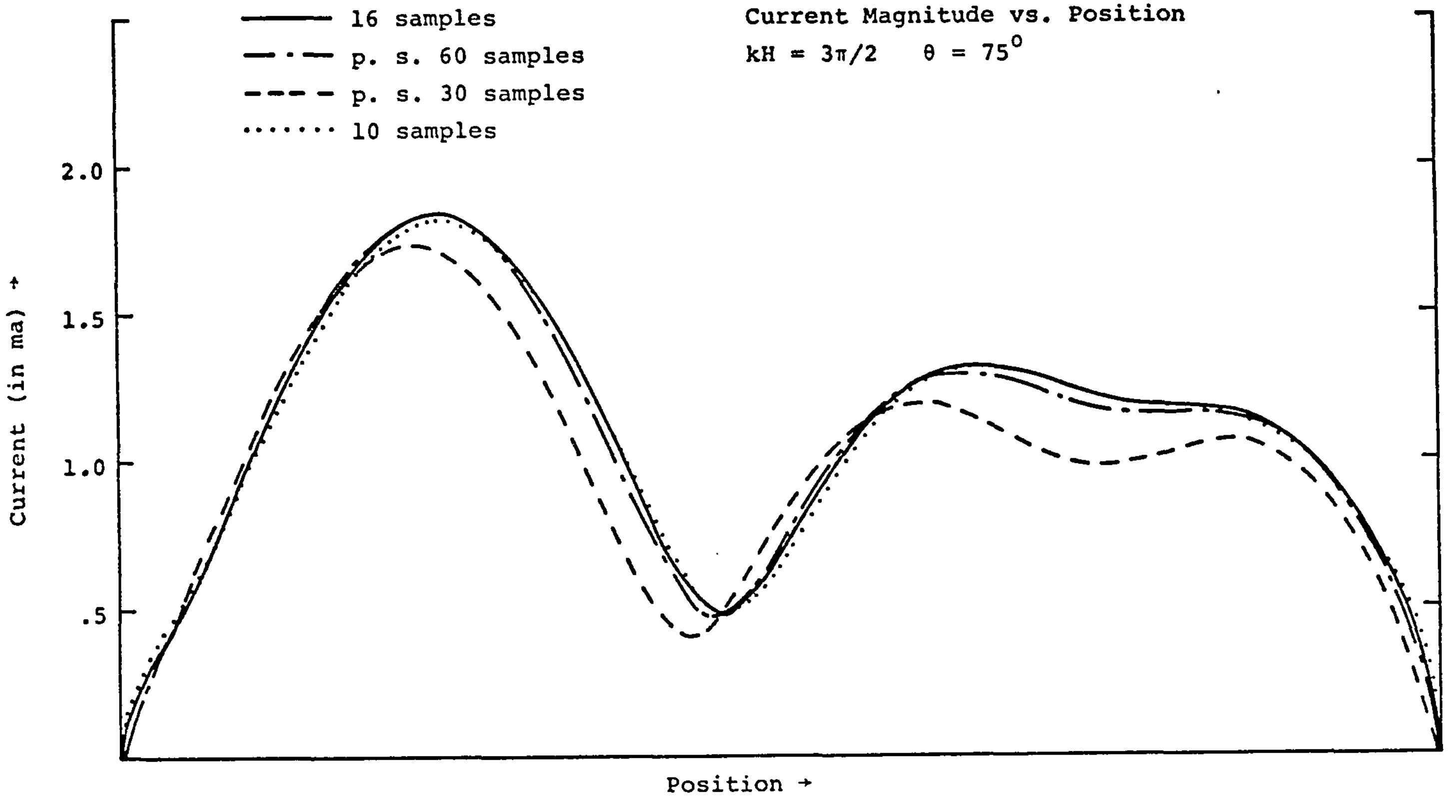

Fig. 2. Plot of the resonant method for two different sample rates against the piecewise sinusoidal (p. s.) method -- Incident field strength is 1 volt/wavelength.

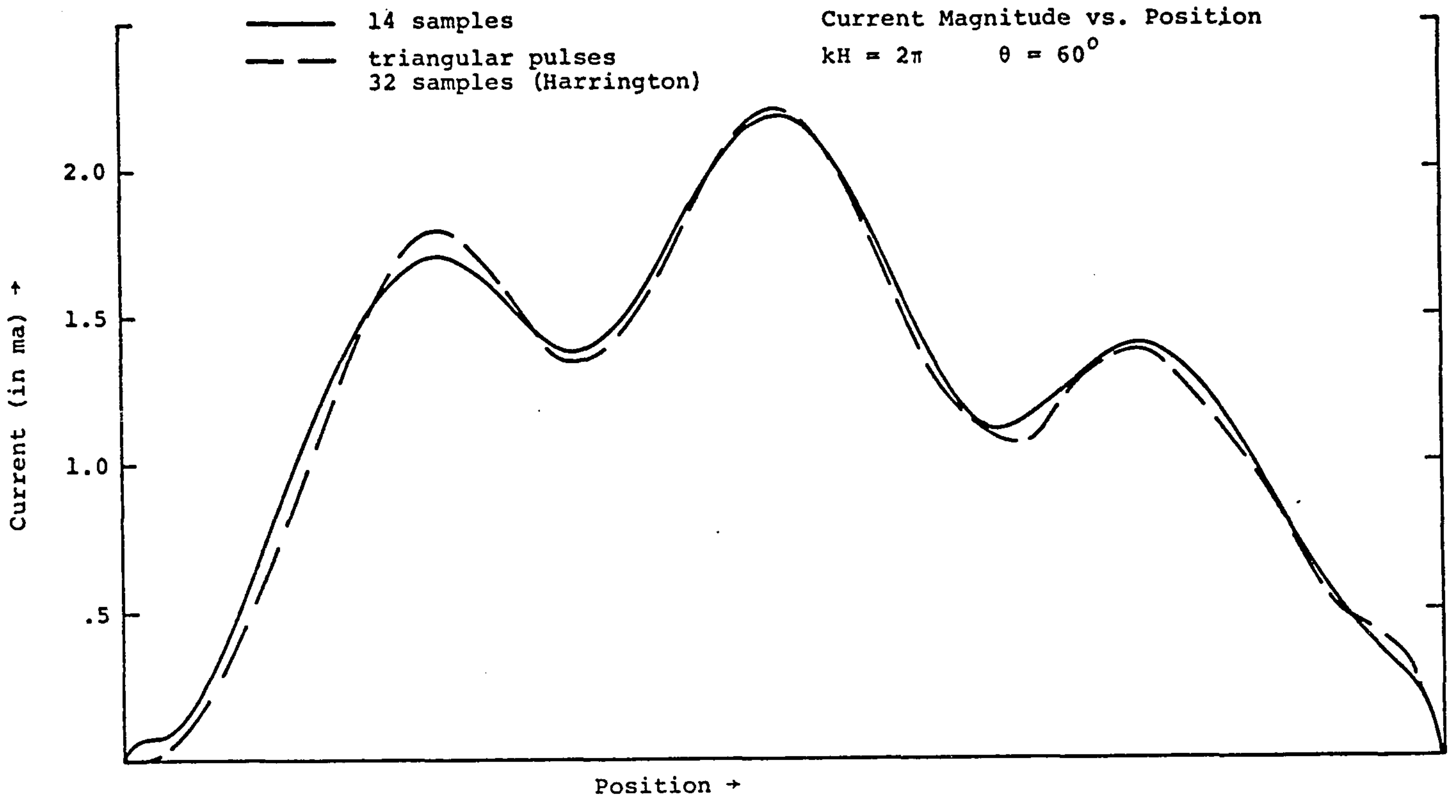

Fig. 3. Plot of the resonant method against published results -- Incident field strength is 1 volt/wavelength.

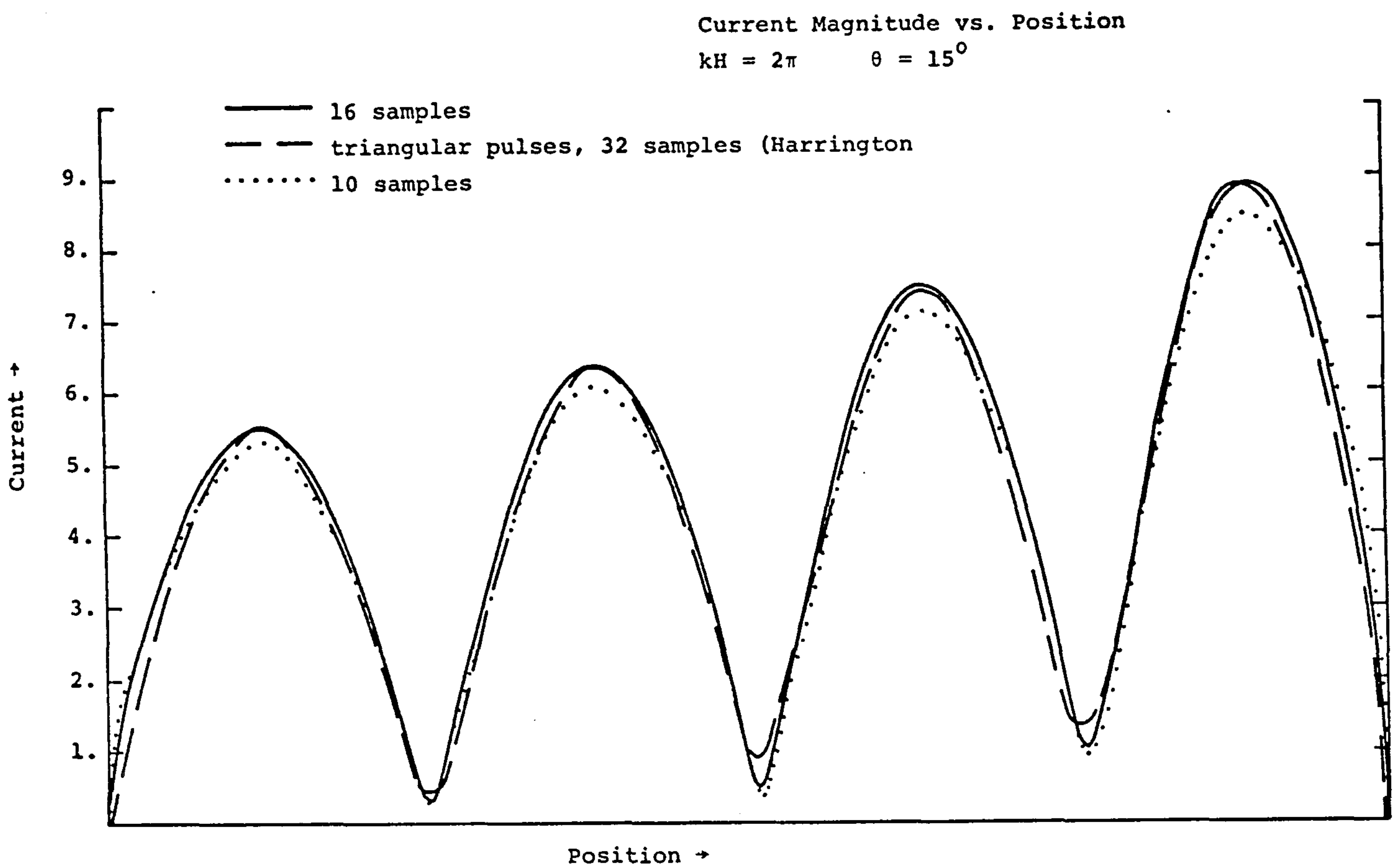

Fig. 4. Plot of the resonant method for the two different samples rates against published results -- Incident field strength is 1 volt/wavelength.

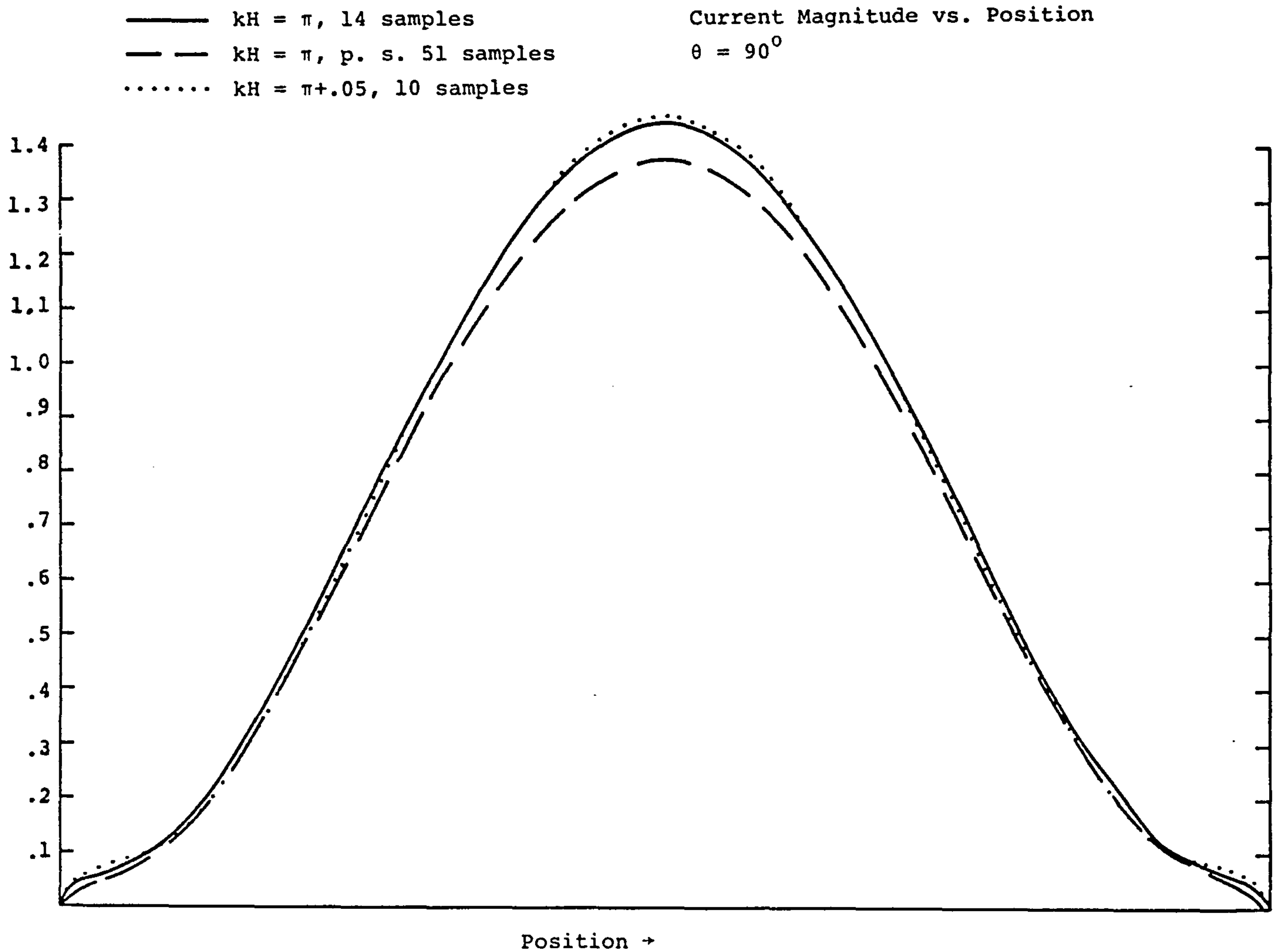

Fig. 5. Plot showing that the non-resonant method can be used near resonance to approximate the current at a resonance with less sampling than the piecewise sinusoidal method -- Incident field strength is 1 volt/wavelength.

cases by making them slightly off-resonant. The figure shows the resonant green's function method and the piecewise sinusoidal method for the wavelength wire. It also has a plot of the results of the non-resonant green's function method for the $1 + .05/\pi$ wavelength wire. Both the resonant green's function method and the non-resonant green's function method have converged, the former at 12 samples and the latter at 14 samples. The piecewise sinusoidal method has yet to converge to within 8 per cent at 51 samples. The non-resonant method is within only 2 per cent of the resonant method at only 14 samples showing its superiority over the piecewise-sinusoidal method by a ratio of at least 4 to 1, in this case.

Conclusions About the Method Applied to the Thin Wire Problem

Using the green's function method as presented herein, one can achieve significantly faster convergence rates over conventional methods. Since the major contribution to computer run time in moment method solutions is matrix inversion time, it can be seen that this method not only saves large amounts of storage, but also runs faster for large problems.

The method seemed to work best near resonances where $\cos(2k) = 1$ and at near normal incidence angles. It converged less quickly near resonances where $\cos(2k) = -1$

and at small incidence angles. Larger coefficients of the singularities were observed in the latter cases.

The overall fast rate of convergence of the method is probably due to two factors. The first factor is the smoothness of the assumed current which has a continuous first derivative. The second factor is that the solution anticipates the correct edge singularity. It is not known if the first factor contributes as heavily as the second. Perhaps by using a piecewise sinusoidal method or any other conventional method modified to anticipate this singularity, similar convergence rates might be obtained.

CHAPTER 5

SCALAR DIFFRACTION THROUGH A NARROW APERTURE

The problem of diffraction of a monochromatic plane wave by an aperture in a plane conducting screen is a classic problem treated extensively by many authors. Approximate methods for an aperture small compared with the wavelength are available in the literature. An early review of these techniques is afforded by Bouwkamp (1954). In addition, the infinite length slit (Sieger, 1908; Morse and Rubenstein, 1938) and the circular aperture (Meixner and Andrejewski, 1950; Flammer, 1953) have been solved exactly.

Although the rectangular aperture problem has not been solved exactly, there are several approximate solutions. Suzuki (1956), for one, has presented a method which is an adaptation of Hallèn's method of solution of the integral equation for the wire as modified by Gray (1944). Suzuki's method can only be applied to apertures narrow in one dimension. The green's function method of Chapter 4 can be easily applied to such a geometry with a few assumptions about the nature of the fields. In this chapter the natural comparison of these two methods is made.

The Suzuki (1956) method is an iterative method. An integral equation of the second kind that describes the unknown aperture field is iterated once analytically. The result is an approximation for this field. The extraction of this integral equation of the second kind from the original differentio-integral equation for the aperture field is an involved process. The approximations made in this process are numerous though none seem severe. An analysis of these approximations is presented preceded by a review of Suzuki's method.

The green's function moment method solution requires for its formulation approximations no more severe than the iterative formulation. Once the approximations are made, the problem becomes similar to the wire problem of Chapter 4. In fact the problems are the same except for a few constants. The application of the green's function moment method to this problem, then, requires only minor changes from the one discussed for the wire.

The comparison between the iterative method and the green's function moment method offers an interesting test for these numerical methods. The approaches are different enough so that agreement in solutions might give one a sense of confidence in both. Numerical results of this comparison are given after the discussion of the approximations made in the iterative approach.

Review of Suzuki's Iterative Formulation

Consider a narrow rectangular slot in an infinite screen located in the z = 0 plane and oriented with the long dimension along the y-axis (Fig. 6). Assume plane wave incidence with the Poynting vector normal to the screen and with the electric field in the direction of the x-axis. With the narrow dimension of the slot small enough, the magnitude of the y-directed electric field in the aperture may be neglected with respect to the x-directed electric field. The result is that the entire problem may be formulated from the y-directed magnetic Hertz potential as follows:

$$E_x = i\omega\mu_o \frac{\partial \Pi}{\partial z} \qquad H_x = \frac{\partial^2 \Pi}{\partial x \partial y}$$

$$E_y = 0 \qquad H_y = \frac{\partial^2 \Pi}{\partial^2 y} + k^2 \Pi$$

$$E_z = -i\omega\mu_o \frac{\partial \Pi}{\partial x} \qquad H_z = \frac{\partial^2 \Pi}{\partial y \partial z} \tag{5.1}$$

where in the region $z < 0, \Pi$ is given by:

$$\Pi = \frac{E^i}{\omega\mu_o k}\left(e^{-ikz} + e^{ikz}\right) + \frac{1}{2\pi i\omega\mu_o} \iint_{\text{Aperture}} \varepsilon(\xi,\eta) \frac{e^{-ikr}}{r} dA \tag{5.2}$$

and in the region $z > 0$ Π is given by:

$$\Pi = -\frac{1}{2\pi i\omega\mu_o} \iint_{\text{Aperture}} \varepsilon(\xi,\eta) \frac{e^{-ikr}}{r} dA \tag{5.3}$$

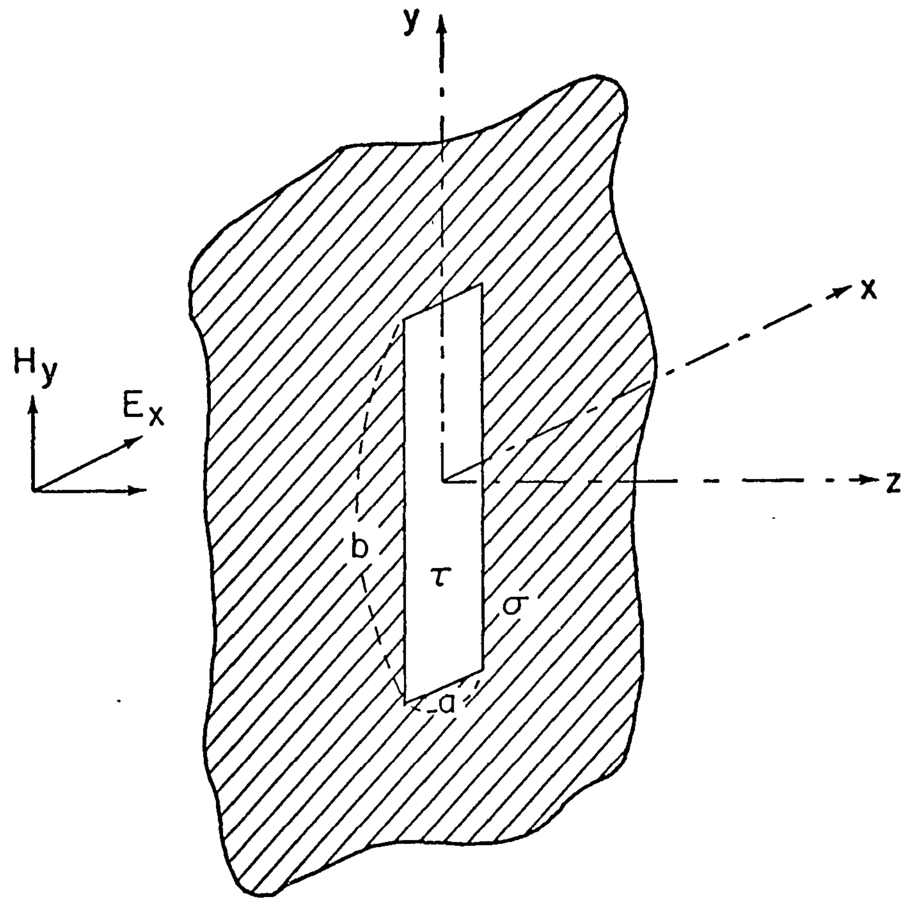

Fig. 6. The geometry of the narrow rectangular aperture problem.

where $\varepsilon(x,y)$ is the unknown aperture electric field (in the x-direction), E^i is the incident electric field, and

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2 \tag{5.4}$$

All continuity requirements across the aperture and all boundary conditions on the screen are met if $\varepsilon(x,y)$ satisfies two conditions:

Condition I:

$$\iint_\tau \varepsilon(\xi,\eta) \frac{e^{-ikr_o}}{r_o} dA = -\frac{2\pi i}{k} (E^i + B\cos(ky)) \tag{5.5}$$

for all $(x,y)\varepsilon\tau$ where τ is the set of all points on the area of the aperture and where B is an arbitrary constant, and where $r_o^2 = (x-x')^2 + (y-y')^2$.

Condition II:

$$\varepsilon(x,b/2) = \varepsilon(x,-b/2) = 0 \tag{5,6}$$

for all $x\varepsilon(-z/2,\ a/2)$.

By adding and subtracting

$$\iint_\tau \varepsilon(\xi,y) \frac{e^{-ikr_o}}{r_o} dA$$

one obtains

$$I_1(x,y) + I_2(x,y) = \frac{-2\pi i}{k} (E^i + B\cos(ky)) \tag{5.7}$$

where I_1 and I_2 are the integrals defined as follows:

$$I_1(x,y) = \iint_\tau \varepsilon(\xi,y)\ \frac{e^{-ikr_o}}{r_o}\ dA \tag{5.8}$$

$$I_2(x,y) =$$

$$\iint_\tau \frac{\{\varepsilon(\xi,\eta)-\varepsilon(\xi,y)\}\cos(kr_o) - i\varepsilon(\xi,\eta)\sin(kr_o)}{r_o}\ dA \tag{5.9}$$

To make the integration over η possible in I_1, I_1 is split

$$I_1(x,y) = I_1'(x,y) + I_1''(x,y) \tag{5.10}$$

where

$$I_1'(x,y) = \iint_\tau \varepsilon(\xi,y)\ \frac{[\cos(kr_o) - 1]}{r_o}\ dA \tag{5.11}$$

$$I_1''(x,y) = \iint_\tau \frac{\varepsilon(\xi,y)}{r_o}\ dA \tag{5.12}$$

and r_o is approximated by $|y-\eta|$ in I_1'. The result after integration over η is:

$$I_1(x,y) = \int_{-a/2}^{a/2} \varepsilon(\xi,y)\ K(x,\xi,y)\ d\xi \tag{5.13}$$

where

$$K(x,\xi,y) = \log\left(\frac{\sqrt{(b/2-y)^2 + (x-\xi)^2} + (b/2-y)}{\sqrt{(b/2+y)^2 + (x-\xi)^2} - (b/2+y)}\right)$$

$$- C_{in}\{k(b/2+y)\} - C_{in}\{k(b/2-y)\} \tag{5.14}$$

and where

$$C_{in}(x) = \log x + 0.5772 - C_i(x)$$

$$C_i(x) = \text{the cosine integral of } x \tag{5.15}$$

Suzuki next approximates $\int \varepsilon(\xi,y)\, K(x,\xi,y)\, d\xi$ by $\int \varepsilon(\xi,y)\, K(x,\xi)\, d\xi$, where

$$K(x,\xi) = \frac{1}{b} \int_{-b/2}^{b/2} K(x,\xi,y)\, dy \tag{5.16}$$

$$\simeq 2\{\log(2b/|x-\xi|) - C_{in}(kb) - \sin(kb)/kb\} \tag{5.17}$$

and then makes the substitutions:

$$x = a/2 \sin(\theta) \tag{5.18}$$

$$\xi = a/2 \sin(\theta') \tag{5.19}$$

$$\sin(\theta)\ \varepsilon(x,y) = \sum_{m=0}^{\infty} C_m(y) \cos(m\theta) \tag{5.20}$$

If r_o is approximated by $|y-\eta|$ in $I_2(x,y)$, Equation (5.5) becomes

$$-\frac{\pi a}{2} \left[\Omega_G C_o(y) + 2 \sum_{1}^{\infty} \frac{C_m(y)\ \cos(m\theta)}{m} \right]$$

$$= \frac{2\pi i}{k} (E^i + B \cos(ky))$$

$$+ \frac{\pi a}{2} \int_{-b/2}^{b/2} \frac{\{C_o(\eta) - C_o(y)\} \cos(k|y-\eta|)}{|y-\eta|} d\eta$$

$$- \frac{\pi a}{2} \int_{-b/2}^{b/2} \frac{i\ C_o(\eta)\ \sin(k|y-\eta|)}{|y-\eta|} d\eta \tag{5.21}$$

where:

$$\Omega_G = 2 \left[\log\left(\frac{4b}{a}\right) - C_{in}(kb) - \frac{\sin(kb)}{kb} + \log(2) \right] \tag{5.22}$$

In the above calculations the integration over θ' has been performed, and use of the following identity has been made to further simplify $K(x,\xi)$:

$$\sum_{1}^{\infty} \frac{\cos(m\theta)\ \cos(m\theta')}{m} = -\frac{1}{2}\log(2|\cos(\theta) - \cos(\theta')|) \tag{5.23}$$

with the result

$$K(x,\xi) = \Omega_G + 4\sum_{1}^{\infty} \frac{\cos(m\theta)\ \cos(m\theta')}{m} \tag{5.24}$$

With the next step the motivation for all the approximations and substitutions becomes clear. Equating coefficients of $\cos(m\theta)$, Suzuki obtains an integral equation of the second kind for $C_o(y)$. Ω_G being large, the iteration of this equation converges very rapidly. Suzuki performs one iteration and then solves for B using Equation (5.6) to give $\varepsilon(x,y)$. In more detail, the equating of coefficients gives:

$$0 = C_1(y) = C_2(y) = C_3(y) = \ldots \tag{5.25}$$

$$C_o(y) = \frac{f}{\Omega_G} (E^i + B \cos(kY))$$

$$- \frac{1}{\Omega_G} \int_{-b/2}^{b/2} \frac{\{C_o(\eta) - C_o(y)\} \cos(k|y-\eta|)}{|y-\eta|} d\eta$$

$$+ \frac{1}{\Omega_G} \int_{-b/2}^{b/2} \frac{iC_o(\eta) \sin(k|y-\eta|)}{|y-\eta|} d\eta \qquad (5.26)$$

where

$$f = - \frac{4i}{ka} \qquad (5.27)$$

If $C_o(y)$ is set equal to $\frac{f}{\Omega_G} (E^i + B \cos(ky))$ and one iteration is performed, the following results:

$$C_o(y) = \frac{f}{\Omega_G} \left[E^i + B \cos(ky) + \frac{1}{\Omega_G} \left[iE^i L(y) - \frac{B}{2} M(y) + i\frac{B}{2} N(y)\right]\right] \qquad (5.28)$$

where S_i is the integral sine and:

$$L(y) = S_i(v^+) + S_i(v^-) \qquad (5.29)$$

$$M(y) = \sin(ky) \left[S_i(2v^+) - S_i(2v^-)\right] - \cos(ky) \left[C_{in}(2v^+) + C_{in}(2v^-) - 2 \left[C_{in}(v^+) + C_{in}(v^-)\right]\right] \qquad (5.30)$$

$$N(y) = \cos(ky) \left[S_i(2v^+) - S_i(2v^-)\right] + \sin(ky) \left[C_{in}(2v^+) - C_{in}(2v^-)\right] \qquad (5.31)$$

$$v^{+} = k(b/2+y) \tag{5.32}$$

$$v^{-} = k(b/2-y) \tag{5.33}$$

Note that we find M(y) as presented by Suzuki (1956) to be in error and the form given above to be correct. M(y) represents

$$2 \int_{-b/2}^{b/2} \frac{\{\cos(k\eta) - \cos(ky)\} \cos(k|y-\eta|)}{|y-\eta|} d\eta$$

If one performs the integration, the result will be as given herein. B is now solved using Equation (5.6) to give

$$B = -E^{i}\delta \tag{5.34}$$

where

$$\delta = \frac{1 + \frac{1}{\Omega_G}\left(i\ L(b/2)\right)}{\cos(kb/2) + \frac{1}{2\Omega_G}\left[-M(b/2) + iN(b/2)\right]} \tag{5.35}$$

The end product is Suzuki's result for the aperture field

$$\varepsilon(x,y) = \frac{C_o(y)}{\sin(\theta)}$$

$$= -\frac{4iE^{i}}{ka\Omega_G} \frac{1}{\sqrt{1-(2x/a)^2}} \left(1 - \delta\cos(ky) + \frac{1}{\Omega_G}\left(iL(y)\right.\right.$$

$$\left.\left. + \frac{\delta}{2} M(y) - i\frac{\delta}{2}N(y)\right)\right) \tag{5.36}$$

which completes the review of the Suzuki formulation with the noted corrections.

Analysis of Suzuki's Approximations

The Suzuki paper and, thus, this work is limited to narrow apertures. There are several approximations that rely crucially on the fact that a, the narrow side dimension is very much less than the long dimension, b. One of these approximations is the approximation of r_o by $|y-\eta|$ in two integrals: in I_1' of Equation (5.11) to give (5.13), and in I_2 of Equation (5.9) to give (5.21). In both integrals this approximation has two effects. It makes $1/r_o \simeq 1/|y-\eta|$ and it makes $\cos(kr_o) \simeq \cos(k|y-\eta|)$. When $|y-\eta|$ is small the errors are serious at best showing that on a pointwise basis the approximation may be completely without meaning. The consideration that it occurs within an integral and that the true measure of the goodness of the approximation might lie in the value of this integral leads to the examination of the extent and location of the area where the approximation is bad. Rather than examine the above approximation in this light, the simpler approximation of $\sqrt{(\eta^2+\xi^2)}$ by $|\eta|$ where $-a/2 < \eta < a/2$ and $-b/2 < \eta < b/2$ can be looked at as a worst possible case. This approximation is a worst possible case because it guarantees maximum inclusion of the small area around $|y-\eta| = 0$ where the approximation is worst. The shaded area of Fig. 7 shows the region where $1/|\eta|$ is at least 40% bigger than $1/\sqrt{(\xi^2+\eta^2)}$. The area of this region can be seen to be small. The value of the absolute error becoming

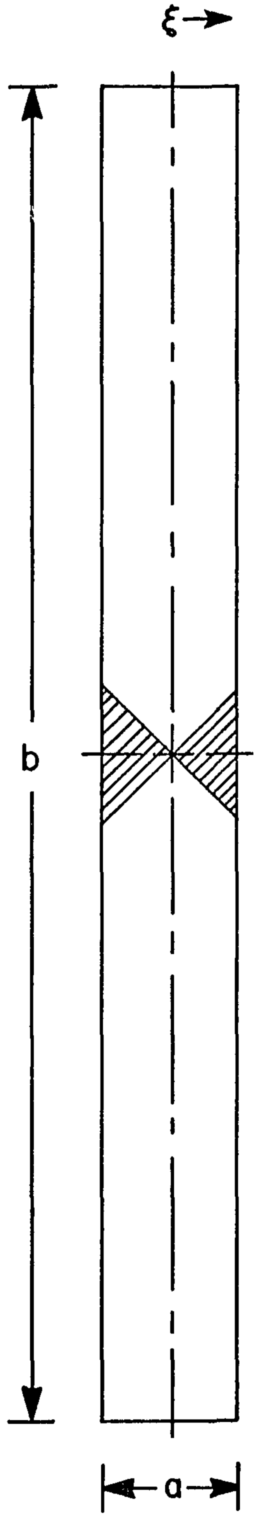

Fig. 7. Area of 40% or greater error in the approximation of r by $|y-\eta|$.

systematically larger in entering the region, however, could seriously affect the value of the integral if it were not for the saving fact that it is in this region that the numerators of both of the integrals go to zero.

Consider now the error in the approximation of $k\sqrt{(\eta^2+\xi^2)}$ by $k|\eta|$ which appears as the argument of a sinusoidal function. As k becomes larger and larger the extent of the region of error becomes larger. With larger and larger ratios of b to a the overall effect of this region becomes less significant. As an arbitrary but reasonable criterion let us say that the approximation becomes invalid when the area of phase error greater than $\pi/8$ is one-quarter of the area of the aperture. The borderline case will be as in Fig. 8, where the phase error on the dividing line is exactly $\pi/8$. The length to width ratio necessary to preserve validity of the approximation for a given value of λ $(= 2\pi/k)$ can be found as follows: Taylor's expansion formula gives the error of the approximation at $\xi = a$ as about $k|\eta|(a/\eta)^2/2$.

$$(k\eta/2)(a/\eta)^2 = \pi/8 \text{ implies}$$

$$\eta = 8a^2/\lambda$$

Choosing $\eta = b/2$ to give the borderline case as in Fig. 8, and letting $\gamma = b/a$ gives the following relation:

$$\gamma = 4\sqrt{b/\lambda}$$

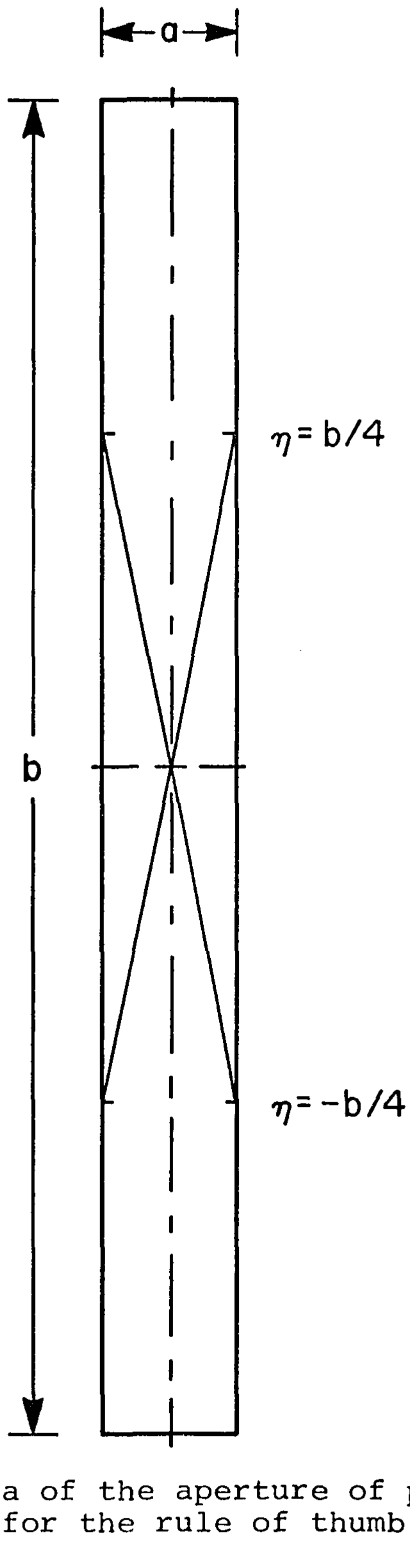

Fig. 8. Quarter area of the aperture of phase error $\pi/8$ or greater for the rule of thumb developed in the text.

As an example, the length to width ratio should be no less than 8 if the shortest wavelength of the incoming signal is one-quarter of the length of the longest edge.

Another situation where the approximation $a \ll b$ enters is in the approximation made in evaluating $K(x,\xi)$ of Equation (5.17) from Equation (5.16). Following are the exact and approximated values of $\frac{1}{b}\int K(x,\xi,y)\,dy$:

$$\frac{1}{b}\int_{-b/2}^{b/2} K(x,\xi,y)\,dy$$

$$= \frac{1}{b}\int_{-b/2}^{b/2} \left\{\log \frac{\sqrt{(b/2-y)^2+(x-\xi)^2} + (b/2-y)}{\sqrt{(b/2+y)^2+(x-\xi)^2} - (b/2+y)}\right.$$

$$\left. - C_{in}\{k(b/2+y)\} - C_{in}\{k(b/2-y)\}\right\} dy$$

$$\text{(exactly)} \quad = \log \left\{\frac{2b^2 + (x-\xi)^2 + 2b\sqrt{b^2+(x-\xi)^2}}{(x-\xi)^2}\right.$$

$$\left. - 2\,\frac{\sqrt{b^2+(x-\xi)^2}}{b} + 2\,\frac{|x-\xi|}{b} + 2\left(1-C_{in}(kb) - \frac{\sin(kb)}{kb}\right)\right.$$

$$\text{(approximately)} = 2\left\{\log\left(\frac{2b}{|x-\xi|}\right) - C_{in}(kb) - \frac{\sin(kb)}{kb}\right\}$$

$$= K(x,\xi)$$

The approximation of $b^2 + (x-\xi)^2$ by b^2 is used three times. Even on a point-wise basis this approximation seems to be good for an aperture length to width ratio of ten to one or greater. Consider now the approximation of $2|x-\xi|/b$ by 0 for the same length to width ratio. At small values of kb the smallest value of $\log(2b/|x-\xi|)$ is about 4.6 and the value of $C_{in}(kb) + \sin(kb)/kb$ is about 1. Thus the smallest value of $K(x,\xi)$ is about 7.2 (= 2 x (4.6-1)). Considering now the values of $2|x-\xi|/b$, the largest being only .2, it can be seen that on a point wise basis this approximation is off at most by .2/7.2 ≃ 3%. Now let kb become large. As an example let the aperture be four wavelengths long. kb is 8π in this case and $C_{in}(kb) + \sin(kb)/kb$ evaluated from tables is found to be approximately 3.8. The smallest value of $K(x,\xi)$ becomes 1.6 (= 2 x (4.6-3.8)) and on a point-wise basis the approximation is off by a maximum of (.2/1.6) ≃ 13%. If this term were not ignored it would be found that $C_1(y)$, $C_2(y)$ and the other Fourier coefficients would no longer be necessarily zero.

Indeed the approximations discussed above bring about necessary and sufficient conditions for the vanishing of these coefficients. These approximations are again:

1. The approximation of r_o by $|y-\eta|$ in I_1' and I_2 discussed previously.
2. The approximation of $\int \varepsilon(\xi,y)\, K(\xi,x,y)\, d\xi$ by

$$\int \varepsilon(\xi,y) \frac{1}{b} \int_{-b/2}^{b/2} K(\xi,x,y)\, dy\, d\xi.$$

3. The approximation of $1/b \int K(\xi,x,y)\, dy$ by $K(\xi,x)$ discussed above.

The vanishing of these coefficients implies that $\varepsilon(x,y)$ behaves as a simple edge singularity in the narrow dimension and that

$$\iint \varepsilon(\xi,\eta) \frac{e^{-ikr_o}}{r_o}\, d\xi\, d\eta$$

is a constant function in the variable x.

Suppose that at the start the following assumption is made

$$\varepsilon(x,y) = C_o(y)/\sqrt{(1-4x^2/a^2)}$$

Approximation 2 can be reexamined in the light of this "known" dependence; that is, the following test can be made:

$$\int_{-a/2}^{a/2} \frac{C_o(y)}{\sqrt{(1-4\xi^2/a^2)}} K(\xi,x,y)\, d\xi$$

$$= \int_{-a/2}^{a/2} \frac{C_o(y)}{\sqrt{(1-4\xi^2/a^2)}} K(\xi,x)\, d\xi.$$

Using the substitutions given in (5.18) and (5.19) together with Equation (5.24) the right-hand side of the above equation can be found to be $\pi\, \Omega_G\, C_o(y)$ for all $x \varepsilon (-a/2,a/2)$. Test the approximation with $C_o(y) = 1$ identically. The integral on the left-hand side is not easy to evaluate and must be done numerically. The integrand has singularities at $\xi = \pm\, a/2$ and at $\xi = x$. Consequently in the results given the integral has been evaluated by splitting it into two parts,

$$\int_{-a/2}^{a/2} \ldots d\xi = \int_{-a/2}^{x} \ldots d\xi + \int_{x}^{a/2} \ldots d\xi$$

An appropriate trigonometric substitution was then used in each of the integrals to eliminate these singularities. In the first integral

$$\xi = \frac{x+a/2}{2} \cos(u) + \frac{x-a/2}{2}; \quad u\varepsilon|-\pi,0|$$

In the second integral

$$\xi = -\frac{x-a/2}{2} \cos(u) + \frac{x+a/2}{2}; \quad u\varepsilon|-\pi,0|$$

The two integrations were then performed with respect to u numerically using 40-point Gaussian integration. The results are seen in Figs. 9, 10, and 11. Fig. 9 shows the result for an aperture 2 wavelengths long with a length to width ratio of ten to one. The constant curve is the value of $\pi\Omega_G$. Also plotted in this figure are the values of the integral for two widely separated x values with respect to the narrow dimension. The variation in the x-direction is seen to be small. The variation in the y-direction, however, is relatively large near the ends of the slot but small near the middle. Figures 10 and 11 show the same integral plotted for an aperture with a 120 to 1 length to width ratio. In these cases the variation in the x-direction

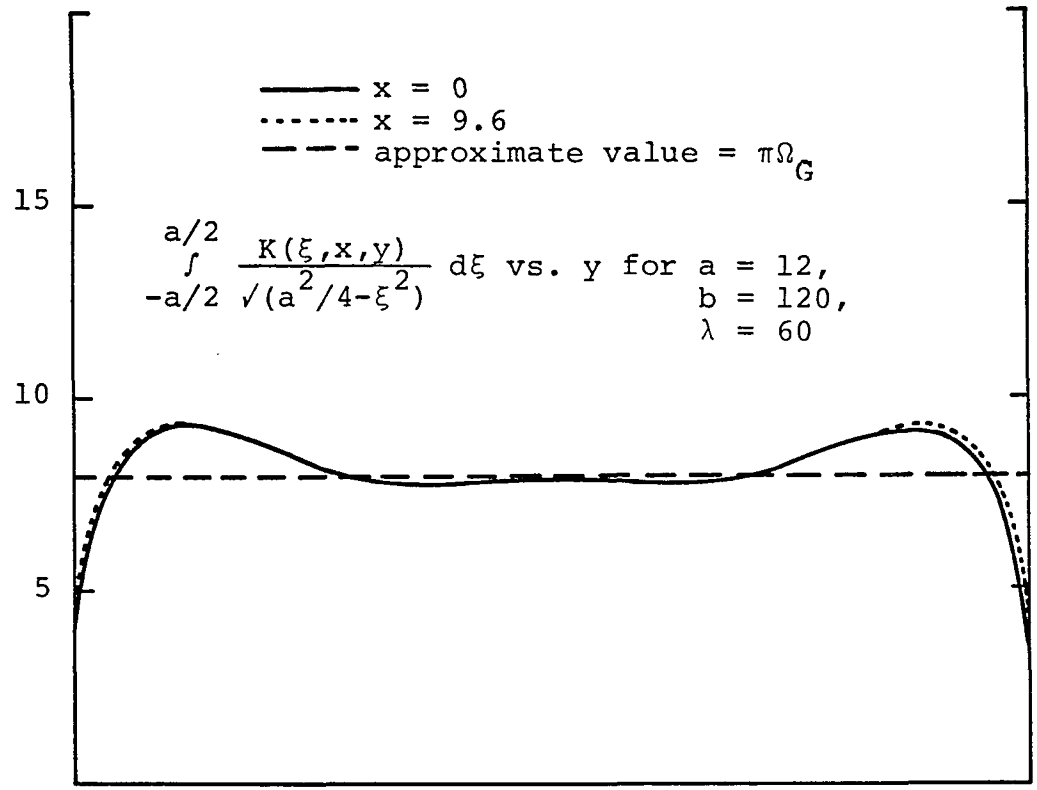

Fig. 9. Approximate vs. true value of an important integral for a length to width ratio of 10 to 1.

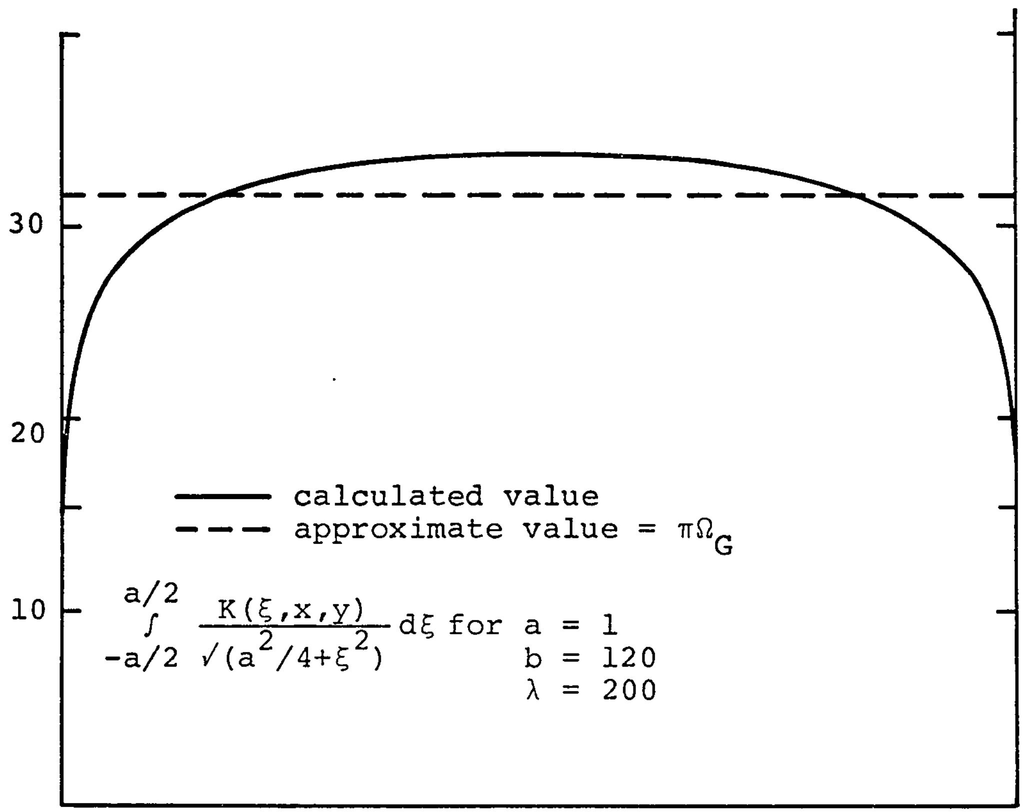

Fig. 10. Approximate vs. true value of an important integral for an electrically long slot with a length to width ratio of 120 to 1.

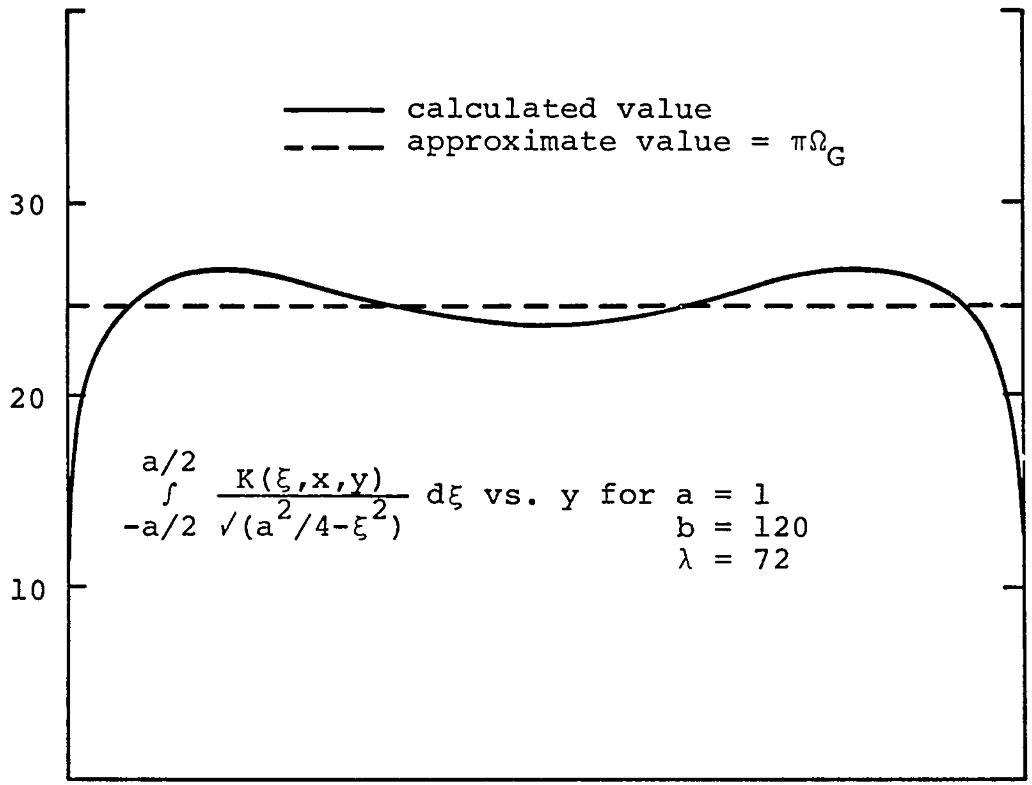

Fig. 11. Approximate vs. true value of an important integral for a shorter (electrically) slot with a length to width ratio of 120 to 1.

is not noticeable, again however, the y-variation is significantly, but not overwhelmingly, not constant.

Green's Function Moment Method Applied to Narrow Aperture Diffraction

With all the approximations that result in $\varepsilon(x,y) = C_o(y)/\sqrt{(1-4x^2/a^2)}$ the problem is brought into a form similar to the wire scatterer problem. The application of the Hallèn iterative approach as developed by Gray (1944) brings out this similarity. Suppose that at the beginning of the analysis one had assumed such a form for $\varepsilon(x,y)$ instead of following Suzuki's route. When the substitution for $\varepsilon(x,y)$ is made into Equation (5.5) with the trigonometric substitution (5.19), the resulting integral equation is:

$$\left(\frac{\partial^2}{\partial y^2} + k^2\right) \int_{-b/2}^{b/2} C_o(\eta)\ T(\eta,y)\ d\eta = -\frac{4ik}{a} E^i$$

where

$$T(\eta,y) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ik\rho}}{\rho}\, d\phi;\ \rho^2 = (\eta-y)^2 + \left(\frac{a}{2}\sin\left(\frac{\phi}{2}\right)\right)^2$$

This equation is essentially Pocklington's equation for the current on a wire scatterer with a radius of a/4. The kernel of the equation is the exact kernel. Only the forcing function is different in the problem. $C_o(y)$ can be and has been calculated using the green's function moment method. A comparison of magnitudes between moment methods solutions and iterative solutions is shown in Figs. 12, 13, 14, 15, and 16. The moment method values were calculated using the extended kernel of Waterman (1965) and green's function methods. The figures show fairly good agreement. For the cases λ = 200, 90, 80, 72, and 60 with a = 1 and b = 120 the range of maximum absolute difference over the maximum value of the moment method curve was 7-33%. In four of the five cases the iterative method gave larger magnitudes for $C_o(y)$.

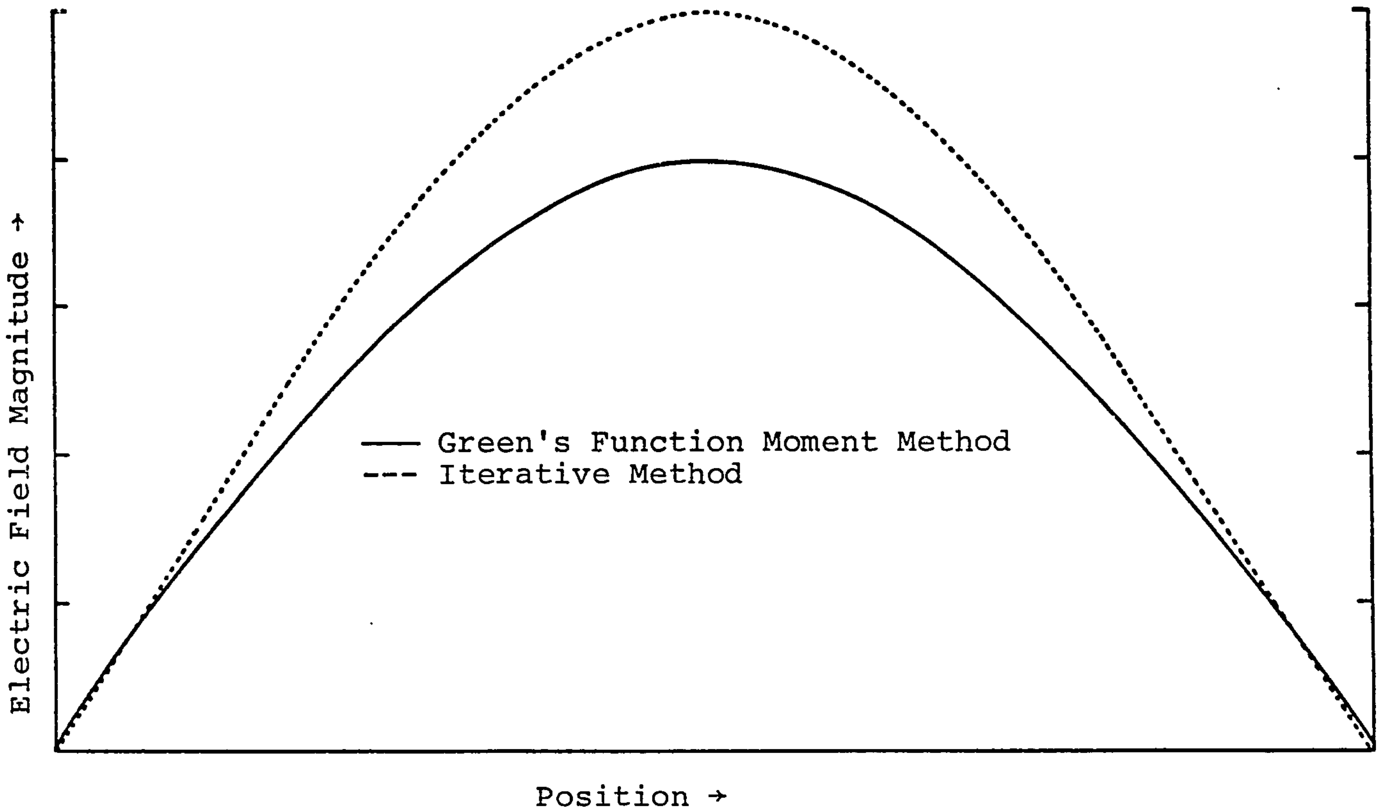

Fig. 12. Comparison of the field for an aperture with length to width ratio of 120 to 1 for λ = 200.

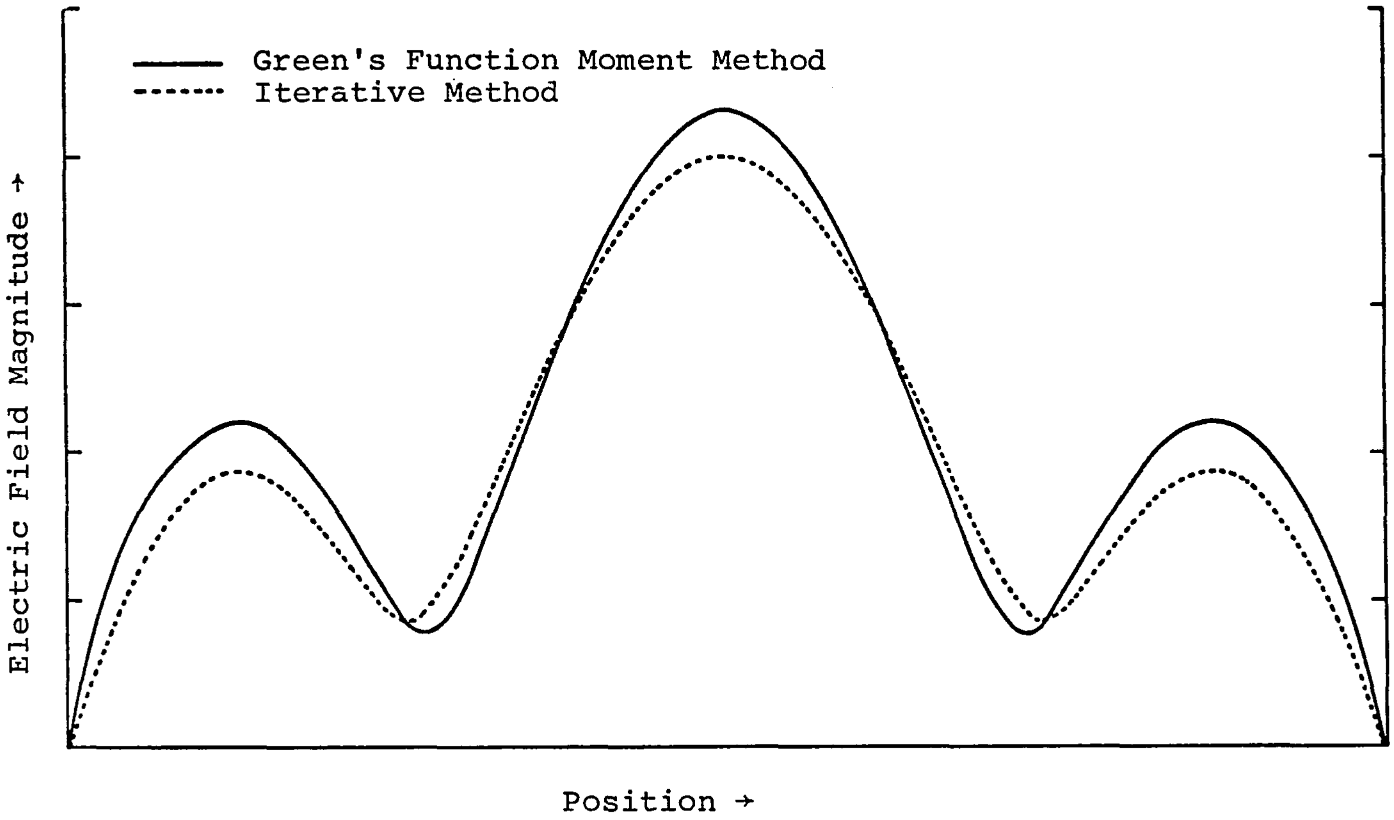

Fig. 13. Comparison of the field for an aperture with length to width ratio of 120 to 1 for $\lambda = 90$.

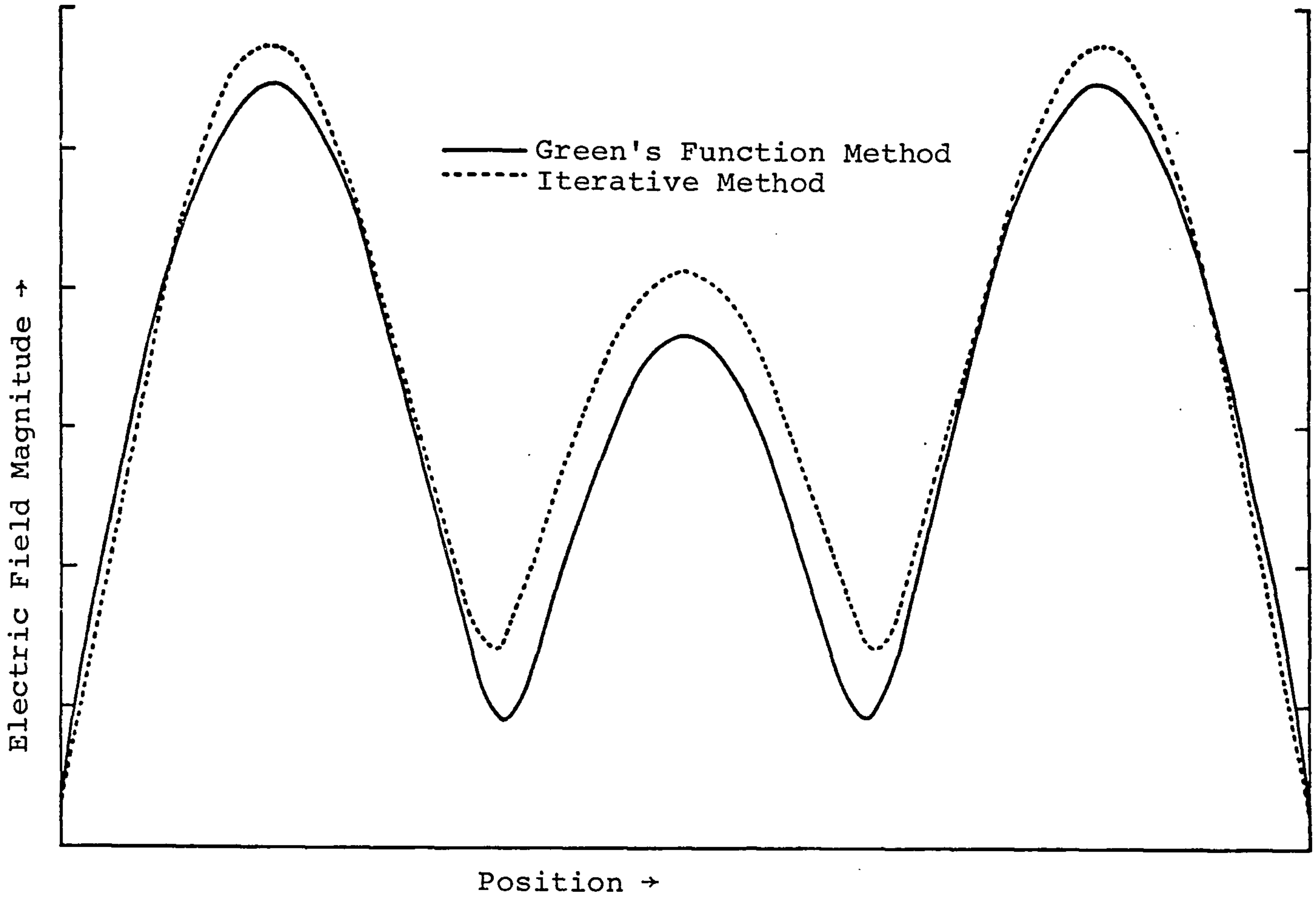

Fig. 14. Comparison of the field for an aperture with length to width ratio of 120 to 1 for $\lambda = 80$.

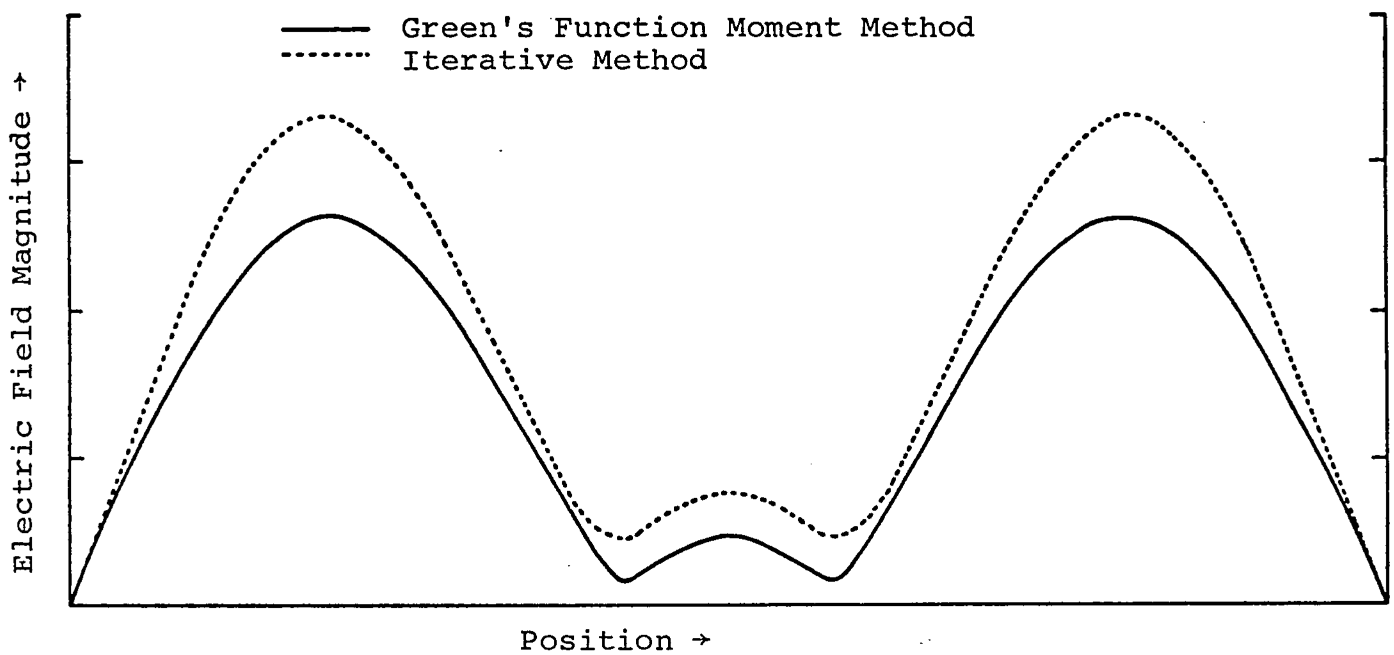

Fig. 15. Comparison of the field for an aperture with length to width ratio of 120 to 1 for λ = 72.

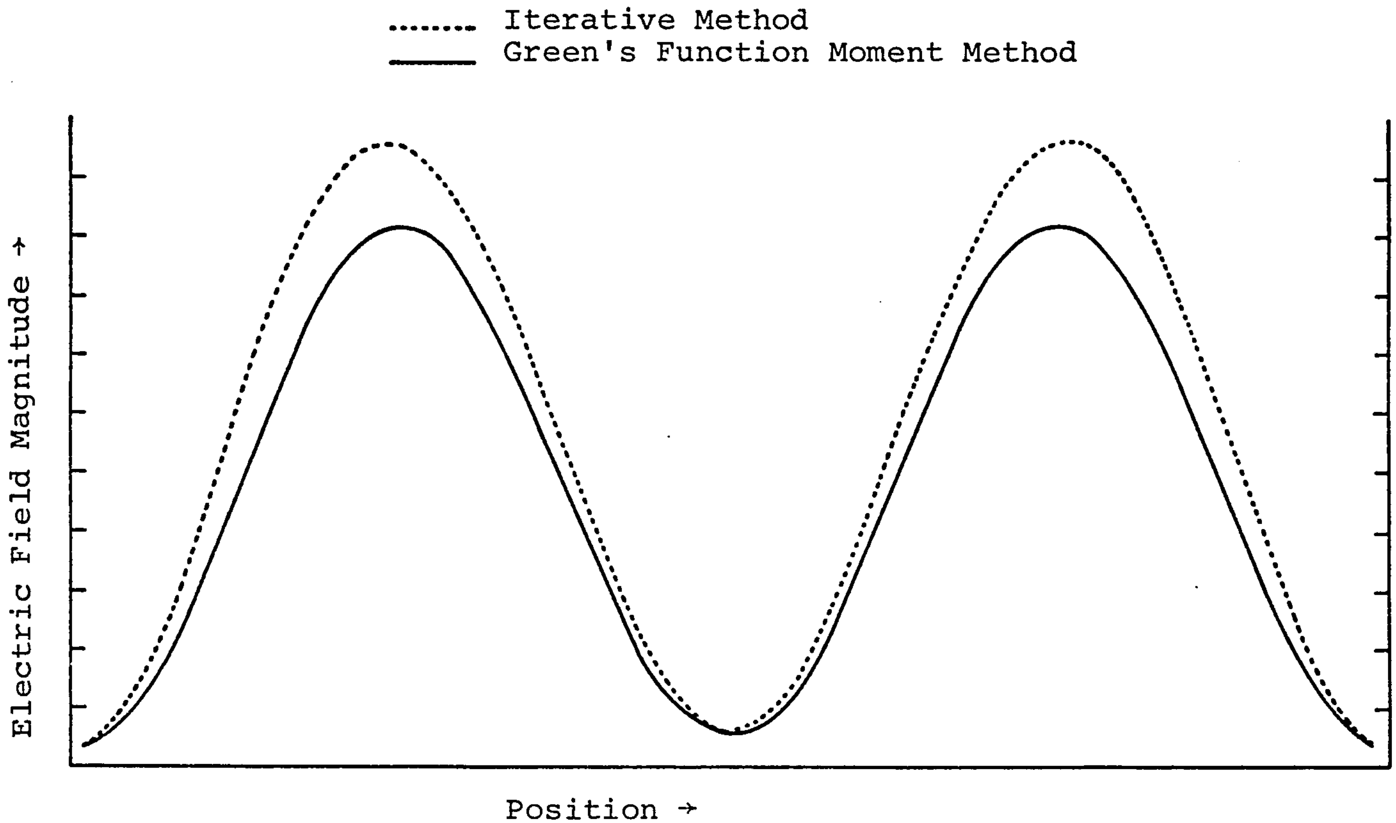

Fig. 16. Comparison of the field for an aperture with length to width ratio of 120 to 1 for $\lambda = 60$.

CHAPTER 6

THE SQUARE APERTURE

The treatments of the wire scatterer problem and the narrow aperture problem in the previous chapters were characterized by the fact that only one one-dimensional unknown field had to be solved for. The general problem of diffraction through apertures or of scattering of perfect conductors does not have this nice property, and in general, two two-dimensional unknowns have to be solved for. In the special case of diffraction through rectangular apertures or scattering from conducting rectangular plates, the integral equations for these two unknowns are very similar to the integral equation for the unknown in the previous one-dimensional problems.

In this chapter the extension of the green's function moment method to these special two-dimensional problems is illustrated by considering the problem of diffraction through a square aperture. The geometry is very similar to the geometry for the diffraction problem in Chapter 5. Instead of a rectangular aperture of length b and width a in the x-y plane, the aperture is square and has a length and width of 2. The incident electric field polarization is arbitrary as well as the angle of incidence in the present problem. Thus, the approximation regarding the small magnitude of the aperture electric field component in the y-direction made in Chapter 5 cannot be reasonably applied to this situation. A more complex solution can be expected.

The Field Expansions and Integral Equations for the Square Aperture Problem

For this problem determination of the aperture electric field components in the x- and y-directions is sufficient for solution of the problem. Once these field components are known the electric and magnetic fields at any point can be easily found. The expressions for the fields everywhere in terms of the aperture components of the electric field are well known and are given as follows:

$$E_x = \frac{\partial u}{\partial z} \qquad H_x = -\frac{1}{ik}\left[\left(\frac{\partial^2}{\partial x^2} + k^2\right)v - \frac{\partial^2 u}{\partial x \partial y}\right]$$

$$E_y = \frac{\partial v}{\partial z} \qquad H_y = +\frac{1}{ik}\left[\left(\frac{\partial^2}{\partial y^2} + k^2\right)u - \frac{\partial^2 v}{\partial y \partial x}\right]$$

$$E_z = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \qquad H_z = +\frac{1}{ik}\left[\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 v}{\partial x \partial z}\right] \qquad (6.1)$$

where each component is a function of x, y, and z. For instance,

$$E_y = E_y(x,y,z)$$

And where

$$(u,v) = \frac{1}{2\pi} \int_{-1}^{1} \int_{-1}^{1} (E_x^A, E_y^A) \phi \, dx' \, dy'$$

with $\quad E_x^A = E_x^A\ (x',y') = E_x(x',y',0)$

and $\quad E_y^A = E_y^A\ (x',y') = E_y(x',y',0)$

ϕ is defined as:

$$\phi = \begin{cases} + \dfrac{e^{-ikr}}{r}; & \text{if } z < 0 \\ \\ - \dfrac{e^{-ikr}}{r}; & \text{if } z > 0 \end{cases}$$

where $\quad r^2 = (x-x')^2 + (y-y')^2 + z^2$

The notation is Copson's (1950). For the rest of the chapter the concern will be the finding of these fields.

The pertinent equations that describe E_x^A and E_y^A are the continuity requirements on H_x and H_y.

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) \iint E_y^A \frac{e^{-ikr}}{r} dA - \frac{\partial^2}{\partial x \partial y} \iint E_x^A \frac{e^{-ikr}}{r} dA$$

$$= - \frac{ik}{2} H_x^i \tag{6.2a}$$

$$\left(\frac{\partial^2}{\partial y^2} + k^2\right) \iint E_x^A \frac{e^{-ikr}}{r} dA - \frac{\partial^2}{\partial y \partial x} \iint E_y^A \frac{e^{-ikr}}{r} dA$$

$$= \frac{ik}{2} H_y^i \tag{6.2b}$$

for all (x,y) in the aperture region, A. Or from Stratton (1941),

$$-\hat{y} \iint E_y^A \frac{e^{-ikr}}{r} dA + \frac{\partial}{\partial x} \iint H_z^A \frac{e^{-ikr}}{r} dA = \frac{\hat{y}}{k^2} \frac{ik}{2} H_x^i \qquad (6.2c)$$

$$-\hat{y} \iint E_x^A \frac{e^{-ikr}}{r} dA + \frac{\partial}{\partial y} \iint H_z^A \frac{e^{-ikr}}{r} dA = -\frac{\hat{y}}{k^2} \frac{ik}{2} H_y^i \qquad (6.2d)$$

again for all (x,y) in A. In the above and in the future, the following conventions are used:

H_x^i and H_y^i are the incident fields at $z = 0$

$\hat{y} = i\omega\varepsilon$

$r^2 = (x-x')^2 + (y-y')^2$

$$\iint (\text{Term})\, dA \overset{\text{df}}{=} \int_{-1}^{1} \int_{-1}^{1} (\text{Term})\, dx'\, dy'$$

and the latter type integral is assumed to hold for all (x,y) in the aperture region, A.

As the equations are written in (6.1), E_x, E_y, and H_z are already continuous across the aperture. If E_x^A and E_y^A satisfy (6.2), perhaps since H_x and H_y are continuous, it might seem that the continuity of E_z would follow. This is not the case, however, for although the incident field satisfies Ampere's equation H_x and H_y do not satisfy it necessarily. Straightforward differentiation of the forms given in (6.1) for H_x and H_y and substitution into Ampere's equation does not give E_z as one would hope. Performing that operation

$$\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = -\frac{1}{ik}\left[\frac{\partial^3 v}{\partial y \partial x^2} + k^2 \frac{\partial v}{\partial y} - \frac{\partial^3 u}{\partial y \partial x \partial y}\right]$$

$$-\frac{1}{ik}\left[\frac{\partial^3 u}{\partial x \partial y^2} + k^2 \frac{\partial u}{\partial x} - \frac{\partial^3 v}{\partial x \partial y \partial x}\right]$$

$$= ik\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] - \frac{1}{ik}\left[\frac{\partial^3 v}{\partial y \partial x^2} - \frac{\partial^3 v}{\partial x \partial y \partial x} + \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y \partial x \partial y}\right] \tag{6.3}$$

It can be seen from (6.1) that a necessary and sufficient condition for getting E_z and thereby assuring continuity of E_z across the aperture (if the incident field satisfies Maxwell's equations) is:

$$\left[\frac{\partial^3 v}{\partial y \partial x^2} - \frac{\partial^3 v}{\partial x \partial y \partial x} + \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y \partial x \partial y}\right] = 0 \tag{6.4}$$

There are further conditions that E_x^A and E_y^A must meet. On the plane of the aperture E_x is zero everywhere except on the aperture. Continuity of tangential electric fields must be enforced everywhere and particularly at the aperture-metallic screen boundary. Thus E_x^A must go to zero as x goes to plus or minus one.

The treatment of the integrals in this problem is similar to the treatment of the singular integral in Chapter 3. E_x^A, E_y^A, and H_z^A are assumed to satisfy certain smoothness criteria so that interchange of differentiation and integration may be performed. $E_x^A(x,y)$, for instance, is assumed to have the following properties. For any y_o there exists an integrable $M(x) > 0$ and a $\delta > 0$ such that $|E_x^A(x,y) - E_x^A(x,y_o)| < M(x)\ |y-y_o|$ whenever $0 < |y-y_o| < \delta$. Also for any x_o there exists an integrable $M(y) > 0$ and $\delta > 0$ (not necessarily the same M and δ as before) with $|E_x^A(x,y) - E_x^A(x_o,y)| < M(y)\ |x-x_o|$ whenever $0 < |x-x_o| < \delta$. Let the phrase "ψ is smooth" where ψ is a field defined on the aperture imply that ψ has these same two properties. The criteria on aperture fields E_y^A and H_z^A are the same as on E_x^A, that is, E_y^A and H_z^A are assumed to be smooth in the above sense.

The smoothness condition as defined here is perhaps a bit strong. It says indeed more than what might be expected from the common usage of the word smooth. Considering E_x^A again, smoothness is met at (x_o,y_o) if

$$\int_{-1}^{1} \frac{\partial E_x^A(x',y_o)}{\partial y'}\, dx' \quad \text{and} \quad \int_{-1}^{1} \frac{\partial E_x^A(x_o,y')}{\partial x'}\, dy'$$

both exist. The integrability of the partial derivatives of the field over lines perpendicular to the direction of the partial is intrinsically tied into the definition. A function like

$$\frac{\partial E_x^A}{\partial x}\ (x,y)$$

under these considerations would not be expected to be smooth since from the behavior of this function near $x = \pm 1$ might be expected to be like $1/(\sqrt{(1-x^2)})^3$, a non-integrable function. To meet smoothness criteria, then E_x^A, E_y^A, and H_z^A must not go infinite as fast as or faster than the inverse distance to the nearest edge of the aperture. With physical reasoning about the radiation of energy from edges, Jones (1964) and Bouwkamp (1954) among others have argued that these functions go infinite no faster than the inverse square root of the distance to the nearest edge. If this be the case then E_x^A, E_y^A, and H_z^A surely can be expected to meet the desired smoothness conditions.

Similarity with Potential Integral

Looking at the Stratton Chu formulation of the integral equations in (6.2) (c,d), a situation is found similar to the problem in Chapters 2 and 3. If the order of differentiation and integration is changed, say, in the first equation on the term involving H_z without regard to rigor, and an integration by parts is performed, the resulting boundary term goes infinite. It will be shown that the singularity in this term is logarithmic as $x \to \pm 1$. The likeness of this situation with Chapter 3 motivates the question of whether the fields behave like the current or perhaps like its derivative. This question has an affirmative answer.

It is now shown that the z-component of the magnetic field must go infinite as the inverse square root of the distance to the edge. Because the terms are more involved, the arguments presented will be briefer and perhaps not quite as rigorous as those in Chapter 3.

The continuity equations of H_z and H_y can be cast into a one-dimensional form to illustrate the logarithmic singularity of the kernel and subsequent behavior of H_z. These operations are now performed with only the continuity equation of H_x, the results for the H_y case would be obtained in an analogous manner. Let y be fixed and $y = y_o$. Equation (6.2c) becomes with some rearranging of terms

$$-\hat{y} \iint E_y^A \frac{e^{-ikr}}{r} dA + \frac{\partial}{\partial x} \iint H_z^A \frac{e^{-ikr}-1}{r} dA$$

$$+ \frac{\partial}{\partial x} \iint \frac{\{H_z^A(x',y') - H_z^A(x',y_o)\}}{r} dA + \frac{\partial}{\partial x} \iint \frac{H_z^A(x',y_o)}{r} dx'\, dy'$$

$$= \frac{\hat{y}}{k^2} \frac{ik}{2} H_x^i \quad \text{where } r^2 = (x-x')^2 + (y_o-y')^2 \tag{6.5}$$

The right-most term on the left-hand side can be integrated in the y' variable to give

$$\frac{\partial}{\partial x} \int_{-1}^{1} \int_{-1}^{1} \frac{H_z^A(x',y_o)}{r} dx'\, dy'$$

$$= \frac{\partial}{\partial x} \int_{-1}^{1} H_z^A(x',y_o) \ln\left(\frac{r^+ + (1-y_o)}{r^- - (1+y_o)}\right) dx' \tag{6.6}$$

where $$(r^+)^2 = (x-x')^2 + (1-y_o)^2$$

$$(r^-)^2 = (x-x')^2 + (1+y_o)^2$$

The logarithmic function in the right-hand integrand of (6.6) is similar to the logarithmic function found in Chapter 3 when investigating the behavior of the exact kernel K(x) near x = 0. Indeed as $x \to x'$ this function goes as

$$\ln\left(\frac{4(1-y_o)(1+y_o)}{(x-x')^2}\right)$$

Thus Equation (6.5) can be written as:

$$-2\frac{\partial}{\partial x}\int_{-1}^{1} H_z^A(x',y_o)\ \ln|x-x'|dx' + \int_{-1}^{1} H_z(x',y_o)\ M(x-x')\ dx'$$

$$-\ \hat{y}\iint E_y^A \frac{e^{-ikr}}{r}\ dA + \frac{\partial}{\partial x}\iint H_z^A \frac{e^{-ikr}-1}{r}\ dA$$

$$+\iint \frac{\{H_z^A(x,y)-H_z^A(x,y_o)\}}{r}\ dx' = f(x,y_o) \qquad (6.7)$$

where $$M(x) = \ln\left(\frac{(r^+ + (1-y_o))x^2}{r^- - (1+y_o)}\right)$$

The first term is the operator Q of Chapter 3, the finite Hilbert transform operator, the other terms are all bounded and continuous and have bounded derivatives in the x-variable. The implication of these facts is that H_z goes singular as the inverse square root of the distance from the points $x = \pm 1$ for any y_o in (-1,1). It does this to preserve the finiteness of the equation and of the first term. From Chapter 3 it is known that this singularity is removable. Thus

$$H_z^A(x,y) = \frac{f_1(y)}{\sqrt{(1-x)}} + \frac{f_2(y)}{\sqrt{(1+x)}} + \frac{g_1(x)}{\sqrt{(1-y)}} + \frac{g_2(x)}{\sqrt{(1+y)}} + h_z(x,y) \tag{6.8}$$

where $h_z(x,y) \to 0$ as $x \to \pm 1$ or as $y \to \pm 1$

is true, the g_1 and g_2 terms coming from symmetric arguments about Equation (6.2d).

With motivation from the above arguments and the results of Chapter 3, the following rule for dealing with derivatives of integrals and integration by parts can be formulated. Let $\psi^A(x,y)$ be a field component or any derivative of a field component restricted to the aperture. Suppose $\frac{\partial}{\partial x} \iint \psi^A(x',y') \frac{e^{-ikr}}{r}\, dA$ appears in an equation for the continuity of fields. Then either $\psi^A(x,y) = 0$ at $x = \pm 1$ or $\psi^A(x,y) = f_1(y)/\sqrt{(1-x)} + \phi(x,y) + f_2(y)/\sqrt{(1+x)}$ where $\phi(x,y) = 0$ at $x = \pm 1$.

The same rule can easily be formulated with x switched with y everywhere and vice versa. Application of this rule for finiteness saves the effort of going through the long argument just applied to H_z^A for every similar instance.

Returning now to the continuity equations for H_x and H_y, this rule for finiteness can be applied to give the singular nature of E_x^A and E_y^A and their derivatives. Looking at (6.2a), by the boundary conditions on E_x^A and E_y^A at the edges of the slot, integrations by parts may be performed to give

$$\frac{\partial}{\partial x} \iint \frac{\partial E_y^A}{\partial x} \frac{e^{-ikr}}{r} dA + k^2 \iint E_y^A \frac{e^{-ikr}}{r} dA - \frac{\partial}{\partial x} \iint \frac{\partial E_x}{\partial y} \frac{e^{-ikr}}{r} dA = - \frac{ik}{2} H_x^i \qquad (6.9)$$

Applying the rule gives the singular behavior of

$$\frac{\partial E_y^A}{\partial x} (x,y)$$

and

$$\frac{\partial E_x^A}{\partial y}$$

in x. The behavior of

$$\frac{\partial E_x^A}{\partial y}$$

may be assumed to be that of E_x^A, that is, E_x^A is expected to go infinite as the inverse square root of $x \pm 1$ since the rule gives this behavior for

$$\frac{\partial E_x^A}{\partial y}$$

Applying similar arguments to (6.2b) gives the following form for E_x^A and E_y^A

$$E_x^A(x,y) = e_x(x,y) + \hat{a}_1(y)/\surd(1-x) + \hat{a}_2(y)/\surd(1+x)$$

$$+ \hat{a}_3(y)\surd(1-x) + \hat{a}_4(y)\surd(1+x) + \hat{a}_5(x)\surd(1-y) + \hat{a}_6(x)\surd(1+y)$$

$$E_y^A(x,y) = e_y(x,y) + \hat{b}_1(x)/\surd(1-y) + \hat{b}_2(x)/\surd(1+y)$$

$$+ \hat{b}_3(x)\surd(1-y) + \hat{b}_4(x)\surd(1+y) + \hat{b}_5(y)\surd(1-x) + \hat{b}_6(y)\surd(1+x) \qquad (6.10)$$

where for all x and y, $e_x(\pm 1,y) = e_y(x,\pm 1) = 0$ and where $\frac{\partial e_x}{\partial n} = \frac{\partial e_y}{\partial n} = 0$ on the boundary of A, $\frac{\partial}{\partial n}$ denoting the derivative in the direction normal to the boundary of A.

This decomposition can be taken still further. The terms $\hat{a}_1, \ldots, \hat{a}_6, \hat{b}_1, \ldots, \hat{b}_6$ can be decomposed. Consider $\hat{a}_5(x)$. The behavior of $E_x(x,y)$ is known as $x \to \pm 1$ and $\hat{a}_5(x)$ would be expected to have the same behavior; namely $\hat{a}_5(x) = a_{51}/\surd(1-x) + a_5(x) + a_{52}/\surd(1+x)$ where $a_5(\pm 1) = 0$. The following form for E_x^A and E_y^A, then, proceeds from the rule for finiteness.

$$E_x^A(x,y) = e_x(x,y) + a_1(y)/\surd(1-x) + a_2(y)/\surd(1+x)$$

$$+ a_3(y)\surd(1-x) + a_4(y)\surd(1+x) + a_5(x)\surd(1-y) + a_6(x)\surd(1+y)$$

$$+ a_{11}\surd(1-y)/\surd(1-x) + a_{21}\surd(1-y)/\surd(1+x) + a_{31}\surd(1-y)\surd(1-x)$$

$$+ a_{41}\sqrt{(1-y)}\sqrt{(1+x)} + a_{12}\sqrt{(1+y)}/\sqrt{(1-x)} + a_{22}\sqrt{(1+y)}/\sqrt{(1+x)}$$

$$+ a_{32}\sqrt{(1+y)}\sqrt{(1-x)} + a_{42}\sqrt{(1+y)}\sqrt{(1+x)}$$

$$E_y^A(x,y) = e_y(x,y) + b_1(x)/\sqrt{(1-y)} + b_2(x)/\sqrt{(1+y)}$$

$$+ b_3(x)\sqrt{(1-y)} + b_4(x)\sqrt{(1+y)} + b_5(y)\sqrt{(1-x)} + b_6(y)\sqrt{(1+x)}$$

$$+ b_{11}\sqrt{(1-x)}/\sqrt{(1-y)} + b_{21}\sqrt{(1-x)}/\sqrt{(1+y)} + b_{31}\sqrt{(1-x)}\sqrt{(1-y)}$$

$$+ b_{41}\sqrt{(1-x)}\sqrt{(1+y)} + b_{12}\sqrt{(1+x)}/\sqrt{(1-y)} + b_{22}\sqrt{(1+x)}/\sqrt{(1+y)}$$

$$+ b_{32}\sqrt{(1+x)}\sqrt{(1-y)} + b_{42}\sqrt{(1+x)}\sqrt{(1+y)} \qquad (6.11)$$

where $e_x(\pm 1,y) = e_y(x,\pm 1) = 0$

$\frac{\partial e_x}{\partial n} = \frac{\partial e_y}{\partial n} = 0$ on the boundary of A where $\frac{\partial}{\partial n}$ is the derivative in the direction normal to the boundary, and where

$$a_1(\pm 1) = a_2(\pm 1) = a_3(\pm 1) = a_4(\pm 1) = a_5(\pm 1) = a_6(\pm 1) = 0$$

$$b_1(\pm 1) = b_2(\pm 1) = b_3(\pm 1) = b_4(\pm 1) = b_5(\pm 1) = b_6(\pm 1) = 0$$

$$a_1'(\pm 1) = a_2'(\pm 1) = a_3'(\pm 1) = a_4'(\pm 1) = a_5'(\pm 1) = a_6'(\pm 1) = 0$$

$$b_1'(\pm 1) = b_2'(\pm 1) = b_3'(\pm 1) = b_4'(\pm 1) = b_5'(\pm 1) = b_6'(\pm 1) = 0$$

the primes denoting the derivative.

Note that like terms have been combined so that a_{51} and a_{52} in the above argument are represented in terms a_{11} and a_{21} of (6.11).

This detailed expansion for E_x^A and E_y^A is equivalent to the expansion for I(z) in Chapters 3 and 4. All singular terms that would affect integration by parts in (6.2a) or (6.2b) are isolated in (6.11) as they were in I(z). The behavior of E_x^A and E_y^A near the edges is not surprising since the Sommerfeld solution of the half plane problem exhibits the same behavior. Bouwkamp (1954) has noted that this type of singularity seems to occur in all known solutions of diffraction problems edges as though it were the result of a "natural boundary condition." In this light the only significant fact brought out in these expansions is that these singularities are removable. This fact is not overly surprising, when it is kept in mind that fields are generally analytic in sourceless regions away from points and edges.

Matching of Some Edge Coefficients

Straightforward attempts to insert the forms for E_x^A and E_y^A of (6.11) into (6.2a,b) and to give a moment method solution of these equations run into a major difficulty. There are too many unknowns for the number of constraints, these constraints being those listed in (6.11) and the boundary conditions. Further constraints must be lurking in the problem because physical problems generally have unique solutions. These constraints are found in the condition (6.4); (6.4) is remembered to constrain the solution to be a solution of Maxwell's equations.

Heuristically, the facts

$$\frac{\partial^3 v}{\partial y \partial x^2} = \frac{\partial^3 v}{\partial x \partial y \partial z}, \quad \frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^3 u}{\partial y \partial x \partial y} \tag{6.12}$$

where $\quad (u,v) = \iint (E_x^A, E_y^A) \frac{e^{-ikr}}{r} dA$

can be shown to be false. If they were true the two continuity equations, (6.2a) and (6.2b), could be shown to be sufficient conditions for the continuity of E_z. Thus, by merely assuring continuity of H_x and H_y, the E_z continuity would be assured. If the fields were expanded in terms of the two magnetic vector potentials, A_x and A_y (where $\vec{H} = \vec{\nabla} \times \vec{A}$) continuity of either field, H_x or H_y, could be assured with only one potential. The resulting field would be decoupled in $TM_{\text{to } x}$ and $TM_{\text{to } y}$ modes. This fact disagrees with the intuitive notion that the scattered field of a polarized wave need not be polarized in the same direction.

Expression (6.4) must go to zero, then, not solely because of (6.12), but rather because some of the terms of the derivative of u cancel some of those of the derivative of v. To investigate this cancellation the expansions for E_x^A and E_y^A are used. In this investigation care is taken so that the order of the differentiation is not changed on the outside of the integrand. To perform interchanges integration by parts is used only when permissible. As an example $-a_1'(x)$ is shown to be equal to $b_5(x)/2$

$$\frac{\partial^3}{\partial y \partial x^2} \iint \frac{a_1(x')}{\sqrt{(1-y')}} \frac{e^{-ikr}}{r} dA - \frac{\partial^3}{\partial y \partial x \partial y} \iint b_5(x)\sqrt{(1-y)} \frac{e^{-ikr}}{r} dA$$

$$= 0$$

$$\frac{\partial^2}{\partial y \partial x} \iint \frac{a_1'(x')}{\sqrt{(1-y')}} \frac{e^{-ikr}}{r} dA + \frac{\partial^2}{\partial y \partial x} \iint \frac{b_5(x')}{2\sqrt{(1-y')}} \frac{e^{-ikr}}{r} dA = 0$$

$$\Longrightarrow \; -a_1'(x) = b_5(x)/2 \qquad (6.13)$$

Underlying this argument, of course, is the assumption that the solution, $u(x)$, to the integral equation

$$\frac{\partial^2}{\partial y \partial x} \int_{-1}^{1} u(x') \int_{-1}^{1} \frac{e^{-ikr}}{r\sqrt{(1-y')}} dy'\, dx' = f(x,y) \qquad (6.14)$$

for all $y \in (-1,1)$ where $r^2 = (x-x')^2 + (y-y')^2$

with $u(\pm 1) = 0$

is unique if it exists. This fact shall not be proved. Instead appeal is made to the fact that consistent use of this argument gives a system of numerical equations which is neither over-determined nor under-determined. The resulting constraints then are as follows:

$$-2a_1'(x) = b_5(x)$$

$$+2a_2'(x) = b_6(x)$$

$$-2b_1'(y) = a_5(y)$$

$$+2b_2'(y) = a_6(y) \qquad (6.15)$$

When these conditions are met (6.4) applies. Moreover, with these conditions the problem is no longer under-determined.

The constraints of (6.15) could also have been derived by requiring $\frac{\partial H_x}{\partial z}$ and $\frac{\partial H_y}{\partial z}$ to be integrable over the aperture, or by requiring that they go singular no faster than the inverse square root of the distance to the nearest edge. If (6.15) are not met, $\frac{\partial H_x}{\partial z}$ and $\frac{\partial H_y}{\partial z}$ are found to go singular with the inverse three halves root of the distance to an edge. This type of argument has been used by Clemmow (1951) and Copson (1950) in the problem of diffraction by a half-plane. In the Copson (1950) paper instead of using (6.4) he required continuity of E_z^i. He found that the solution was under-determined so he used this added condition. It seems that he ignored the obvious constraint that the incident field must satisfy Maxwell's equations. Both this constraint on the incident field and the continuity of E_z are implicit in (6.4) indicating that perhaps it might be a more natural choice for numerical work than mere continuity of E_z. Using it one would not be subject to the problems Copson ran into.

A Numerical Method for the Aperture Problem

A method is now presented for finding the aperture fields with moment methods. There are a large number of ways of attacking the problem numerically. Finding one that gives reasonably smooth fields at a minimum of storage and within reasonable time restrictions is not so easy. The method presented here gives results that have continuous first derivatives in one direction and is discontinuous in the other. It is like other two-dimensional moment method problems in that it has the disadvantage of requiring large amounts of storage. Since moment method solutions generally require storage of the order of the square of the number of unknowns, the storage requirement will be of the order of N^4 for a wavelength square aperture sampled N times per wavelength. The algorithm presented requires solution of equations with $2(N^2 + 4)$ unknowns. Six or seven samples along the length of the square is its practical limit since it will involve storage capabilities of a $(2N^2 + 8)$ by $(2N^2 + 8)$ matrix.

There are ways of reducing the requirement so that about twice this sampling rate can be achieved for the same amount of storage. These ways make use of the symmetries of the square to reduce an arbitrary problem into eight symmetric problems. The discussion of these methods shall be omitted since they are, basically, a problem of bookkeeping. By other methods storage requirements can be further reduced with a pseudo-decompling of E_x^A and E_y^A. These methods, however, seem to require inversion of the two-dimensional harmonic operator with Neumann boundary conditions. Such inversions require considerable amounts of computer time. They do have the advantage, however, of giving solutions with continuous derivatives in any direction.

Returning now to the method to be presented, solution of (6.2a,b) is required. E_x^A and E_y^A are assumed to be of the following form:

$$
\begin{aligned}
E_x^A(x,y) = {} & L_y^{-1}\,[U(x,y) + u_1(y)\omega'(1-x) + u_2(y)\omega'(1+x)] \\
& + \frac{\partial}{\partial x} L_x^{-1}\,[-\,v_1(x)\omega(1-y) + v_2(x)\omega(1+y)] \\
& + a_{11}\omega(1-y)\omega'(1-x) + a_{21}\omega(1-y)\omega'(1+x) \\
& + a_{21}\omega(1+y)\omega'(1-x) + a_{22}\omega(1+y)\omega'(1+x)
\end{aligned}
$$

$$
\begin{aligned}
E_y^A(x,y) = {} & L_x^{-1}\,[V(x,y) + v_1(x)\omega'(1-y) + v_2(x)\omega'(1+y)] \\
& + \frac{\partial}{\partial y} L_y^{-1}\,[-\,u_1(y)\omega(1-x) + u_2(y)\omega(1+x)] \\
& + b_{11}\omega(1-x)\omega'(1-y) + b_{21}\omega(1-x)\omega'(1+y) \\
& + b_{12}\omega(1+x)\omega'(1-y) + b_{22}\omega(1+x)\omega'(1+y) \qquad (6.16)
\end{aligned}
$$

where $L_x\, g(x,y) = (\frac{\partial^2}{\partial x^2} + k^2) g(x,y); \quad \frac{\partial g(x,y)}{\partial x}\Big|_{x=\pm 1} = 0$

$$L_y\, g(x,y) = (\frac{\partial^2}{\partial y^2} + k^2) g(x,y); \quad \frac{\partial g(x,y)}{\partial y}\Big|_{y=\pm 1} = 0$$

and L_x^{-1} and L_y^{-1} define the appropriate inverse operators. Furthermore, N being the sampling rate per side length

$$U(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} U_{nm}\, P_n(x)\, P_m(y)$$

$$V(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} V_{nm}\, P_n(x)\, P_m(y)$$

$$u_1(y) = \sum_{n=1}^{N} u_{1n}\, P_n(y)$$

$$u_2(y) = \sum_{n=1}^{N} u_{2n}\, P_n(y)$$

$$v_1(y) = \sum_{n=1}^{N} v_{1n}\, P_n(y)$$

$$v_2(y) = \sum_{n=1}^{N} v_{2n}\, P_n(y)$$

where P_n are pulse functions defined as follows

$$P_n(x) = 1 \text{ if } -1 + \frac{2(n-1)}{N} \le x \le -1 + \frac{2n}{N}$$

and $P_n(x) = 0$ otherwise. $\omega(x)$ is defined so that it is analogous to the square root function

$$\omega(x) = (2-x)\sqrt{x}$$

$$\omega'(x) = \frac{d\omega(x)}{dx} = \frac{(2-x)}{2\sqrt{x}} - \sqrt{x}.$$

Note that although the form of E_x^A and E_y^A as given by (6.11) has not been duplicated, the important singularities of these fields have been represented. The exception in (6.16) of the terms whose derivatives are singular at the $x = \pm 1$ edge of E_x^A and at the $y = \pm 1$ edge of E_y^A should be noted. No difficulties arise since the method being presented does not make use of an integration by parts that brings the derivative onto $\frac{\partial}{\partial x} L_y^{-1} U(x,y)$ or onto $\frac{\partial}{\partial y} L_x^{-1} V(x,y)$. Indeed these terms do not appear in this method.

Note also that in (6.16) the square root functions have been replaced by terms like $\omega(1-y)$ and $\omega'(1-y)$. The term $\omega(1-y)$ behaves like $2\sqrt{(1-y)}$ near $y = 1$ and $\omega'(1-y)$ behaves like $1/\sqrt{(1-y)}$. The reason for replacing a square root function say $\sqrt{(1-y)}$ with $\omega(1-y)$ is the fact that the square root function does not go to zero at $y = -1$ whereas $\omega(1-y)$ does. Although this advantage of $\omega(1-y)$ does not seem very striking at first, it is indeed of great importance for matching the boundary conditions on E_x^A and E_y^A. With E_x^A and E_y^A as given in (6.16), these boundary conditions become $L_y^{-1} U(x,\pm 1) = L_x^{-1} V(\pm 1,y) = 0$. Had square root functions been used the problem of matching a series of pulse functions to a continuous function would arise when the boundary conditions are enforced.

Finally note that in (6.16) the fields have been prepared for moment method solution. They have been expressed in terms of a finite number of weights U_{nm}, V_{nm}, u_{1n}, u_{2n}, v_{1n}, v_{2n}, a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, and b_{22}.

The number of unknowns is thus, strictly, $2N^2 + 4N + 8$, however, the relation $L_y^{-1} U(\pm 1,y) = L_x^{-1} V(x,\pm 1) = 0$ that arises from removal of the inverse square singularities from E_x^A and E_y^A gives $U_{1m} = U_{Nm} = V_{n1} = V_{nN} = 0$. This relation sets 4N unknowns to zero and there is no need to retain them. Thus, there are $2N^2 + 8$ unknowns of importance.

The task is to now specify $2N^2 + 8$ equations for these unknowns. $2N^2$ of these equations in this method are point matching equations. The remaining 8 are

$$L_x^{-1} v_1(x) = L_x^{-1} v_2(x) = 0 \quad \text{at } x = \pm 1$$

$$L_y^{-1} u_1(y) = L_y^{-1} u_2(y) = 0 \quad \text{at } y = \pm 1 \tag{6.17}$$

Looking more closely at the $2N^2$ point matching equations Equation (6.2a) is rewritten here in terms of the unknowns. Note that integrations by parts, wherever permissible have been performed.

$$- \frac{ik}{2} H_x = \iint \{U(x',y') + u_1(y')\omega'(1-x') + u_2(y')\omega'(1+x')\} \frac{e^{-ikr}}{r} dA$$

$$+ \frac{\partial}{\partial y} \iint \frac{\partial}{\partial x'} L_{x'}^{-1} \{v_1(x')\omega'(1-y') + v_2(x') \omega'(1+y')\} \frac{e^{-ikr}}{r} dA$$

$$+ k^2 \iint \frac{\partial}{\partial x'} L_{x'}^{-1} \{- v_1(x')\omega(1-y') + v_2(x')\omega(1+y')\} \frac{e^{-ikr}}{r} dA$$

$$+ a_{11} G_1((1-y),(1-x)) + a_{21} G_1((1-y),(1+x))$$

$$+ a_{12} G_1((1+y),(1-x)) + a_{22} G_1((1+y),(1+x))$$

$$- \frac{\partial}{\partial y} \iint \frac{\partial}{\partial x'} L_{x'}^{-1} \{V(x',y') + v_1(x')\omega'(1-y') + v_2(x')\omega'(1-y')\} \frac{e^{-ikr}}{r} dA$$

$$- \iint \frac{\partial^2}{\partial y'^2} L_{y'}^{-1} \{u_1(y')\omega'(1-x') + u_2(y')\omega'(1+x')\} \frac{e^{-ikr}}{r} dA$$

$$+ b_{11} G_2((1-x),(1-y)) + b_{21} G_2((1-x),(1+y))$$

$$+ b_{12} G_2((1+x),(1-y)) + b_{22} G_2((1+x),(1+y)) \qquad (6.18)$$

where $G_1(x,y) = (\frac{\partial^2}{\partial y^2} + k^2) \iint \omega(x')\omega'(y') \frac{e^{-ikr}}{r} dA$

$$G_2(x,y) = - \frac{\partial^2}{\partial x \partial y} \iint \omega(x')\omega'(y') \frac{e^{-ikr}}{r} dA$$

The equation as written in (6.18) mimics the form of (6.2a). Note that some of the terms can be cancelled. Also the second derivative of the inverse harmonic operator in y can be replaced since

$$\frac{\partial^2}{\partial y^2} L_y^{-1} f(x,y) = f(x,y) - k^2 L_y^{-1} f(x,y) \tag{6.19}$$

This replacement gives terms that cancel with still others. The final result is:

$$- \frac{ik}{2} H_x(x,y) = \iint U(x',y') \frac{e^{-ikr}}{r} dA$$

$$+ k^2 \iint L_{y'}^{-1} \{u_1(y')\omega'(1-x') + u_2(y')\omega'(1+x')\} \frac{e^{-ikr}}{r} dA$$

$$+ a_{11} G_1((1-y),(1-x)) + a_{21} G_1((1-y),(1+x))$$

$$+ a_{21} G_1((1+y),(1-x)) + a_{22} G_1((1+y),(1+x))$$

$$- \frac{\partial}{\partial y} \iint \frac{\partial}{\partial x'} L_{x'}^{-1} V(x',y') \frac{e^{-ikr}}{r} dA$$

$$+ k^2 \iint \frac{\partial}{\partial x'} L_{x'}^{-1} \{- v_1(x')\omega(1-y') + v_2(x')\omega(1+y')\}$$

$$\frac{e^{-ikr}}{r} dA$$

$$+ b_{11}\, G_2((1-x),(1-y)) + b_{21}\, G_2((1-x),(1+y))$$

$$+ b_{21}\, G_2((1+x),(1-y)) + b_{22}\, G_2((1+x),(1+y)) \qquad (6.20)$$

This relation gives N^2 equations when it is satisfied at all the N^2 centerpoints of $P_n(x)$ and $P_m(y)$, $n = 1$ to N, $m = 1$ to N. The corresponding result for Equation (6.2b) that expresses the continuity of H_y gives an additional N^2 equations. Thus, the problem is now virtually ready for the programmer to take over.

This dissertation will not treat the programming of the square aperture problem. The programming is straightforward but tedious for the above method; there should be no difficulties. Considering this moment method formulation, the first integral in (6.20) involves calculation of the integral of the product of $P_n(x)\ P_m(y)$ with e^{-ikr}/r. The procedure for its calculation is well known for those who have programmed solutions of diffraction problems with moment methods. The second integral involves integration of $L_y^{-1}\ P_n(y)\omega'(1\pm x)$ over e^{-ikr}/r and produces no great difficulties since $L_y^{-1}\ P_n(y)$ can be given with a simple expression in terms of sines and cosines. The evaluations of G_1 and G_2 are straightforward although they may require some care in the same manner as the G function of Chapter 4 did. The next integral

$$-\frac{\partial}{\partial y} \iint \frac{\partial}{\partial x'}\, L_{x'}^{-1}\ P_n(x')\ P_m(y')\ \frac{e^{-ikr}}{r}\, dA \qquad (6.21)$$

looks more formidable than it is. The inverse harmonic operator in x merely gives a trigonometric term inside the integrand. The product of this term with the y-derivative of e^{-ikr}/r is an expression which is integrable in the x-variable in closed form (Jordan and Balmain 1968, p. 338). The fourth integral involves integration of $\frac{\partial}{\partial x} L_x^{-1} P_n(x)$ times e^{-ikr}/r. As with $L_y^{-1} P_n(y)$, $\frac{\partial}{\partial x} L_x^{-1} P_n(x)$ is a simple expression involving only trigonometric terms and should not be difficult to evaluate. Thus the equations are in suitable form for solution.

A method has therefore been devised to take advantage of the known behavior of E_x^A and E_y^A at the edge. Knowledge of this behavior has allowed integrations by parts which in turn have allowed for finding smooth solutions with little effort. The method developed shows promise of handling square apertures up to two wavelengths long with suitable modifications.

CHAPTER 7

CONCLUSIONS

The green's function moment method seems adequately powerful to give good numerical results for square apertures the order of a wavelength on a side. Its application to this problem seems almost natural because of the presence of harmonic operators in the integrodifferential equation formulation of the problem. This conclusion is based on the power of the method when applied to smaller problems, such as the narrow aperture and the thin wire. Programming the method in Chapter 6 would, therefore, seem to be a fruitful pursuit, an assurance that is nice to have before embarking on such a large scale project.

By following the formulation for the square aperture, simple generalizations of the method presented in Chapter 6 give a method for an arbitrary rectangular aperture. It would be interesting to investigate the narrow aperture scalar methods with such a vector solution. Also, integral equations of the same form as those of Chapter 6, but with modifications of the kernel function of Chapter 6 that do not affect its singular portion, could be solved. Such equations could give solutions for diffraction into cavities or into dielectric mediums. With the use of Babinet's principle, scattering from rectangular plates may also be treated.

This method is not only powerful because of the fast convergence rate found in the numerical results, but also because it says something definite about the nature of the edge singularities. Without such beforehand knowledge the reduction of the integral equations for the square aperture problem to moment methods would have been difficult because of the under-determined nature of the problem. The method, therefore, seems to show promise, not only as a numerical method but also as an analytical tool.

APPENDIX A

M(x), THE LEFTOVER OF THE EXTRACTION OF THE AIRFOIL KERNEL FROM THE EXACT KERNEL

Chapter 3 appeals to this appendix for some facts concerning the exact kernel and its derivative. The illustration of these facts is probably a bit tedious to include in Chapter 3, yet it is important enough to be shown in some detail here. These facts are that M(x) to be defined in a moment, is bounded and has a derivative with a logarithmic singularity. This singularity is to be shown to be removable. M(x) is defined in Chapter 3 as follows:

$$2\pi\, M(x) = \frac{\partial}{\partial x}\,[K(x) + \frac{2\ln|x|}{a}] - k^2 \int_0^x K(t)\, dt \qquad (A.1)$$

where $K(x) = \int_0^{2\pi} \frac{e^{-ikR}}{R}\, d\phi,\ R^2 = x^2 + 4a^2 \sin^2(\phi/2)$

The approach taken in this appendix is brute force. If the brutality of long involved equations should shock the reader he is advised to skip this appendix.

One of the first steps to take is to characterize the exact kernel in terms of complete elliptic integrals. The complete elliptic integral of the first and second kinds are defined as follows:

$$\overline{K}(x) = \int_0^{\pi/2} \frac{dy}{(1-x^2 \sin^2(y))^{1/2}}, \quad \text{elliptic integral of the first kind}$$

$$\overline{E}(x) = \int_0^{\pi/2} (1-x^2 \sin^2(y))^{1/2}\, dy; \quad \text{elliptic integral of the second kind} \tag{A.2}$$

The definitions follow Gradshteyn and Ryzhik (1965). Care is advised when using tables. Abramowitz and Stegun (1964) define $\overline{K}(x)$ and $\overline{E}(x)$ slightly differently.

The elliptic integrals are related to each other through differentiations. These relations from Gradshteyn and Ryzhik (1965) are:

$$x\overline{K}'(x) = \overline{E}(x)/(1-x^2) - \overline{K}(x)$$

$$x\overline{E}'(x) = \overline{E}(x) - \overline{K}(x) \tag{A.3}$$

Note that in this appendix primes are used exclusively to denote derivatives.

The behavior of the elliptic integrals as their arguments go to one is of importance since it will be shown that this is where M(x) has a singular derivative. These behaviors are now given and should be kept in mind.

$$\lim_{x\to 1} \{\overline{K}(x) - \ln(4/\surd(1-x^2))\} = 0$$

$$\lim_{x\to 1} \{\overline{E}(x) - 1\} = 0 \tag{A.4}$$

M(x) is broken down three ways in this appendix. The three components are as follows:

$$2\pi\, M(x) = 2M_1(x)/a + M_2(x) + M_3(x)$$

where
$$M_1(x) = \frac{\partial}{\partial x}\left\{\frac{a}{2}\int_0^{2\pi} \frac{1}{R}\, d\phi + \ln|x|\right\}$$

$$- k^2 \int_0^x \frac{a}{2}\int_0^{2\pi} \frac{1}{R_t}\, d\phi\, dt$$

$$M_2(x) = \frac{\partial}{\partial x}\left\{\int_0^{2\pi} \frac{e^{-ikR}-1}{R}\, d\phi\right\}$$

$$M_3(x) = k^2 \int_0^x \int_0^{2\pi} \frac{e^{-ikR_t}-1}{R_t}\, d\phi\, dt$$

and where $R^2 = x^2 + 4a^2 \sin^2(\phi/2)$,

$$R_t^2 = t^2 + 4a^2 \sin^2(\phi/2) \qquad \text{(A.5)}$$

For each component boundedness is illustrated and the nature of its derivative at x = 0 is analyzed. The terms are arranged in decreasing difficulty of argument. $M_1(x)$ requires the use of the whole gamut of elliptic integral properties. $M_2(x)$ requires a few machination with elliptic integrals and work with derivatives of e^{-ikr}/r. There may be some argument as to whether this function might not be more difficult to analyze than $M_1(x)$. $M_3(x)$ requires very little argument. It can be considered very briefly right now.

The integrand of the double integration that defines $M_3(x)$ is bounded for any t in the interval [0,2]. This interval should be noted now as the interval under consideration for x in M_1 and M_2 as well as M_3. Thus $M_3(x)$ is bounded. Furthermore the derivative with respect to x is given by

$$M_3'(x) = k^2 \int_0^{2\pi} \frac{e^{-ikR}-1}{R}\, d \tag{A.6}$$

where R^2 is as before in (A.5)

The integrand is again bounded and therefore $M_3(x)$ is bounded as $x \to 0$. $M_3(x)$ can thus be neglected from further considerations of M(x) and M'(x). The remainder of the appendix concentrates on $M_1(x)$ and $M_2(x)$.

Treatment of the "Nasty" Term of M(x)

In this section $M_1(x)$ is given the consideration promised. First, however, the following fact is shown.

$$\frac{a}{2} \int_0^{2\pi} \frac{1}{R}\, d\phi = \beta\overline{K}(\beta) \tag{A.7}$$

where $\beta = 2a/r_a$ and $r_a^2 = 4a^2 + x^2$

With a simple change of integration variable and trigonometric identity the following is obtained

$$\int_0^{2\pi} \frac{1}{R}\, d\phi = 2 \int_0^{\pi} \frac{d\theta}{((x^2+4a^2) - 4a^2\cos^2(\theta))^{1/2}}$$

$$= 2 \int_{-\pi/2}^{\pi/2} \frac{d\theta}{(r_a(1-(\frac{4a^2}{r_a^2})\ \sin^2\theta)^{1/2})} \tag{A.8}$$

The second equality follows from a translation of variables and another trigonometric identity. Note that r_a is independent of θ and may be removed from the integrand. Symmetry of the $\sin^2$ function gives the result:

$$\frac{a}{2} \int_0^{2\pi} \frac{1}{R}\, d\phi = \frac{2a}{r_a} \int_0^{\pi/2} \frac{d\theta}{(1 - (\frac{2a}{r_a})^2\ \sin^2(\theta))^{1/2}} \tag{A.9}$$

which is identical with (A.7) if the definition for $\overline{K}(x)$ in (A.2) is remembered. Thus, M(x) can be expressed in terms of elliptic integrals as follows:

$$M_1(x) = \frac{\partial}{\partial x} \{\beta\overline{K}(\beta) + \ln|x|\} - k^2 \int_0^x \beta_t\ \overline{K}(\beta_t)\ dt \tag{A.10}$$

where $\beta = 2a/(4a^2+x^2)^{1/2}$, $\beta_t = 2a/(4a^2+t^2)^{1/2}$

Looking more closely at (A.7), the behavior of the integral is seen to be singular as $x \to 0$. $\beta \to 1$ when $x \to 0$ causing $\overline{K}(\beta) \to +\infty$ from (A.3). Indeed $\overline{K}(\beta)$ goes as $\ln(4/\sqrt{(1-\beta^2)}) = \ln(4r_a/|x|)$. Note that the singular behavior of the exact kernel is embodied in this term. This derivation of the logarithmic singularity could be used as an alternative to the one given in Chapter 3. Note also that for any x the relation $\beta \leq 1$ holds. This relation is important for application of series expansions and numerical approximations to the elliptic integrals.

$M_1(x)$ is now shown to be bounded. Of the two terms up for consideration the integral term is the easiest prey. Since K(x) is of a logarithmic nature in t near t = 0,

$$k^2 \int_0^x \beta_t \, \overline{K}(\beta_t) \, dt \tag{A.11}$$

where $\beta_t = 2a/(4a^2 + t^2)^{1/2}$

is of the nature of xlnx near x = 0 and is therefore bounded for x in the desired interval. It can be seen now that this integral term has a derivative whose behavior is logarithmic as $x \to 0$. More precisely

$$\lim_{x \to 0} [\frac{\partial}{\partial x} k^2 \int_0^x \beta_t \, \overline{K}(\beta_t) \, dt + k^2 \ln|x|]$$

$$= \lim_{x \to 0} k^2 \, [\beta\overline{K}(\beta) + \ln|x|] = k^2 \ln(8a) \tag{A.12}$$

Thus this term of $M_1(x)$ has a removable logarithmic singularity.

Consider now the term

$$\frac{\partial}{\partial x}\ [\beta\overline{K}(\beta) + \ln|x|];\ \text{where } \beta = 2a/r_a \tag{A.13}$$

and r_a is remembered as $\surd(4a^2 + x^2)$

Performing the differentiation gives:

$$\beta'[\overline{K}(\beta) + \beta\overline{K}'(\beta)] + \frac{1}{x};\ \text{where } \beta' = -x\beta/r_a^2 \tag{A.14}$$

As $x \rightarrow 0$ note that β' goes as $-x$ and $\beta \rightarrow 1$. It would not be surprising to see the $1/x$ term cancel with the $\beta'\overline{K}'$ term as $x \rightarrow 0$. Since $\overline{K}$ goes as the logarithm its derivative would be expected to go as $1/x$. What is surprising is the relative simplicity of the term of obtained when (A.3) is applied to $\beta\overline{K}'(\beta)$ to give:

$$\frac{\partial}{\partial x}\ [\beta\overline{K}(\beta) + \ln|x|] = \frac{\beta\overline{E}(\beta)-1}{x} \tag{A.15}$$

The numerator and denominator of this term both go to zero as x goes to zero. L'Hopital's rule must be applied to show that this term is bounded. This application gives

$$\begin{aligned}\lim_{x\rightarrow 0}\ \left[\frac{\beta\overline{E}(\beta)-1}{x}\right] &= \lim_{x\rightarrow 0}\ [\overline{E}(\beta) + \beta\overline{E}'(\beta)]\ \beta' \\ &= \lim_{x\rightarrow 0}\ [2\overline{E}(\beta) - \overline{K}(\beta)]\ \beta' \\ &= 0 \end{aligned} \tag{A.16}$$

The last equality follows from the fact that β' goes to zero as x and $\overline{K}$ goes infinite only as the logarithm of x.

The consequence of the boundedness of the limit is that (A.13) is bounded. Now the derivative of this term is looked at near $x = 0$. Using (A.15) the derivative is found to be

$$\frac{\partial}{\partial x}\,\frac{\beta\overline{E}(\beta)-1}{x} = \frac{\beta'\{2\overline{E}(\beta)-\overline{K}(\beta)\}}{x} - \frac{\beta\overline{E}(\beta)-1}{x^2} \tag{A.17}$$

Both of these terms can be shown to have a removable logarithmic singularity. Remembering $\beta' = -x\beta/r_a^2$ the first term on the right-hand side is seen to go as

$$\frac{\ln(4/\sqrt{(1-\beta^2)})}{4a^2} = \frac{\ln(4r_a/|x|)}{4a^2} \tag{A.18}$$

This singularity is removable, that is,

$$\lim_{x\to 0}\,[\beta'(2\overline{E}(\beta) - \overline{K}(\beta))/x + \ln|x|/4a^2] = \frac{\ln(8a)-2}{4a^2} \tag{A.19}$$

To analyze the second term of (A.17), L'Hopital's rule must be used. The denominator will become $2x$. The numerator will become $\beta'[2\overline{E}(\beta)-\overline{K}(\beta)]$. From (A.19) the entire term might be expected to have a removable logarithmic singularity of $-\ln|x|/8a^2$. Formally this hypothesis must be tested with L'Hopital as follows:

$$\lim_{x\to 0} [(\beta\overline{E}(\beta)-1)/x^2 + \ln|x|/8a^2]$$

$$= \lim_{x\to 0} [(\beta\overline{E}(\beta)-1 + x^2\ln|x|/8a^2)/x^2]$$

$$= \lim_{x\to 0} [(\beta'\{2\overline{E}(\beta)-\overline{K}(\beta)\} + 2x\ln|x|/8a^2 + x/8a^2)/2x]$$

$$= \lim_{x\to 0} [\frac{\overline{K}(\beta)-2\overline{E}(\beta)}{2r_a^2} + \frac{\ln|x|}{8a^2} + \frac{1}{16a^2}]$$

$$= (2\ln(8a) - 3)/16a^2 \qquad \text{(A.20)}$$

Each of the terms in (A.17) has a removable logarithmic singularity. These singularities do not cancel. Thus (A.13) has a removable logarithmic singularity. With (A.20) and (A.19) this singularity can be deduced as $-\ln|x|/8a^2$. The singularity of the integral term of $M_1(x)$ may be combined with this one to give the removable logarithmic singularity of $M_1(x)$ as $-\ln|x|(1/8a^2+k^2)$.

Treatment of the "Not so Nasty" Term of M(x)

The term

$$M_2(x) = \frac{\partial}{\partial x}\int_0^{2\pi} \frac{e^{-ikR}-1}{R}\, d\phi; \text{ where } R^2 = x^2 + 4_a^2\sin^2(\phi/2) \qquad \text{(A.21)}$$

is now treated. At all x except x = 0 it is permissible to interchange the order of differentiation and integration. This interchange gives

$$M_2(x) = - x \int_0^{2\pi} \frac{e^{-ikR}(1+ikR)-1}{R^3} d\phi \tag{A.22}$$

The numerator of the integrand may be expanded in a power series in R. The cancellations leave a 1/R term, the sum of a constant and a term in R, and higher powers of R. With this expansion boundedness is fairly easy to show. The nature of the derivative also follows with a little effort.

Performing the expansion then gives

$$M_2(x) = + x \int_0^{2\pi} \frac{\sum_{n=2}^{\infty} \frac{(-ik)^n R^n (n-1)}{n!}}{R^3} d\phi$$

$$= x \int_0^{2\pi} \frac{k^2}{2R} d\phi + x \int_0^{2\pi} \frac{-ik^3}{3} - \frac{k^4 R}{8} d\phi$$

$$+ x \int_0^{2\pi} O(R^2) \; d\phi \tag{A.23}$$

The integral of the terms of order R^2 is easy to characterize.

Its integrand is bounded in both x and ϕ. The derivative of its integrand is also bounded in x and ϕ as seen when term by term differentiation is performed (which is valid). The result that this term has a bounded derivative for $x \varepsilon [0,2]$ is obvious.

Consider now the 1/R term. From previous results (A.7) this term may be expressed in terms of the complete elliptic integral of the first kind. Thus

$$x \int_0^{2\pi} \frac{k^2}{2R}\, d\phi = \frac{k^2 x}{2}\, \beta\overline{K}(\beta) \text{ where } \beta = 2a/(4a^2+x^2)^{1/2} \qquad (A.24)$$

Since $\overline{K}(\beta)$ has the removable singularity + $\ln(8a/|x|)$ as $x \rightarrow 0$ and since $x \ln x \rightarrow 0$ as $x \rightarrow 0$ the $1/R$ term is seen to be bounded. The derivative of this term is

$$\frac{\partial}{\partial x}\, x \int_0^{2\pi} \frac{k^2}{2R}\, d\phi = \frac{k^2}{2}\, [\beta\overline{K}(\beta) + x\beta'\{\overline{K}(\beta) + \overline{K}'(\beta)\}]$$

$$= \frac{k^2}{2}\, [\beta\overline{K}(\beta) - \frac{x^2}{r_a^2}\, \{\overline{E}(\beta)/(1-\beta^2)\}]$$

$$= \frac{k^2}{2}\, [\beta\overline{K}(\beta) - \overline{E}(\beta)] \qquad (A.25)$$

Since $\beta\overline{K}(\beta)$ has the removable logarithmic singularity $\ln(8z/|x|)$ as $x \rightarrow 0$, this term has the removable logarithmic singularity of $k^2 \ln(8a/|x|)/2$.

Consider the remaining term of M_2,

$$-x \int_0^{2\pi} [\frac{ik^3}{3} + \frac{k^4 R}{8}]\, d\phi \qquad (A.26)$$

The integrand of this term is bounded in x and ϕ, thus the integral is bounded in x, and, therefore, the whole term is bounded. The derivative of this term is found to be for $x = 0$

$$\frac{\partial}{\partial x}\, x \int_0^{2\pi} - \frac{ik^3}{3} - \frac{k^4 R}{8}\, d\phi = \int_0^{2\pi} - \frac{ik^3}{3} - \frac{k^4 R}{8}\, d\phi$$

$$+ x \int_0^{2\pi} - \frac{k^4 x}{8R}\, d\phi \qquad \text{(A.27)}$$

The first integral has been seen to be bounded. The second integral gives $-k^4\, x^2\, \beta\overline{K}(\beta)/8$. From the logarithmic nature of $\overline{K}(\beta)$ in x as $x \to 0$ and the fact $x^2 \ln x \to 0$ as $x \to 0$ the derivative of this term is seen to be bounded in x.

In conclusion $M_2(x)$ is seen to be bounded. Its derivative is also seen to have the removable logarithmic singularity $k^2 \ln (8a/|x|)/2$. From the previous section $M_1(x)$ is bounded and has a removable logarithmic singularity. The conclusion must be, then, that M(x) has these properties also.

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