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ELECTROMAGNETIC WAVE TRANSIENTS INTERACTING WITH A DISSIPATIVE STRATIFIED MEDIUM

by

Thales Michael Papazoglou

A Dissertation Submitted to the Faculty of the DEPARTMENT OF NUCLEAR ENGINEERING In Partial Fulfillment of the Requirements For the Degree of DOCTOR OF PHILOSOPHY In the Graduate College THE UNIVERSITY OF ARIZONA

1974
I hereby recommend that this dissertation prepared under my direction by Thales Michael Papazoglou, entitled "Electromagnetic Wave Transients Interacting with a Dissipative Stratified Medium," be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy.

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Date

After inspection of the final copy of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:

Robert E. White  
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ABSTRACT

A time domain solution to the problem of a transient electromagnetic plane wave and a dissipative stratified semi-infinite medium is presented. The problem is solved in the frequency domain; the solution is multiplied by the frequency spectrum of the incident plane wave and the inverse Fourier transform is taken. The transform is treated analytically by suitable complex plane contour selection which deals with the branch cuts of the Fresnel coefficients. The solution to the two layer problem contains both the reflective and transmittive results. The three layer problem is solved for the reflective results. Realistic frequency dependence models, for the constitutive parameters, are incorporated in the two layer problem.

The result of analysis is a sum of algebraic terms plus real definite integrals which may be easily handled numerically, offering promise for more efficient computer codes than those previously used. The additional information thus obtained offers also a greater insight into the physics of the problem itself.

Field curves are included for both incident polarizations, for the constant parameters case as well as the frequency dependent parameters case. The Brewster's
angle phenomenon is evident for the H-field parallel to the surface of the dissipative medium.

Possible applications of the problem include the interaction of natural or man-made transients with the earth's surface, geophysical probing and subsurface communication.
CHAPTER 1

INTRODUCTION

1.1 Works Related to the Subject Matter

The present study deals with the interaction of plane electromagnetic transients with a dissipative stratified medium. The Fourier transform method is employed, and a detailed complex analysis is pursued to obtain simple analytical expressions for all the physical quantities. The question of the frequency dependence of the electrical constitutive parameters is also discussed.

The problem of a monochromatic plane wave incident at an arbitrary angle in air on an infinite imperfectly conducting half-space is a classic boundary-value problem in electromagnetic theory treated thoroughly in standard textbooks, as by Stratton (1941) and Jackson (1962). The reflection of electromagnetic waves from horizontally stratified lossy media, insofar as the frequency domain is concerned, is also analyzed thoroughly in standard references, as by Wait (1970). However, the problem of the actual reflection and transmission for frequencies that extend from zero to infinity, such as the case of electromagnetic pulses, is mathematically complex and has not been completely solved for all the cases of practical interest.
In the last quarter of a century a considerable amount of work has been done on the general subject of the propagation of electromagnetic transients. Wait (1951a) used the Laplace transformation and neglected the contribution of displacement currents (which is equivalent to neglecting the product of frequency times the dielectric permittivity compared with the conductivity, thereby limiting the validity of the treatment to frequencies that are not too high, while results for the transient response are only valid for sufficiently large times) to compute the transient electric fields for several types of step function current sources embedded in a conducting infinite medium. Wait (1953), Richards (1958), Zisk (1960), and Anderson and Moore (1960) have investigated the electromagnetic fields generated by pulsed electric and magnetic dipoles in an infinite lossy medium with emphasis on the behavior of the disturbance as a function of the distance from the point source. Bhattacharyya (1957a) considered a more realistic case of an electric dipole with a ramp function or sawtooth function current; he also neglected displacement currents. In a later paper, Bhattacharyya (1957b) considered a step-function electric pulse through a homogeneous and isotropic conducting medium but took into account the effect of the displacement currents. Wait and Hill (1971) and Mijharends (1962) studied the transient signals coming from dipoles buried in a conducting half-space; in both
studies displacement currents are neglected. Bhattacharyya (1959) has determined the transient electric and magnetic fields close to the surface of the earth as developed by a step-function current flowing in a circular loop wire; in this study he has included displacement currents. An introductory work on the subject of the transient magnetic dipole over a horizontally stratified earth is that by Wait (1951b). A general analysis for the electromagnetic response of conducting media (including infinite and half-space geometries) due to pulse excitation is presented by Wait (1961). Wait and Spies (1970a, 1970b) studied transient fields for an electric dipole in a homogeneous dissipative medium with particular emphasis on the determination of the influence of the displacement currents on the waveform. Grumet (1959) has reported on the transient electromagnetic fields produced in a semi-infinite conductor when a uniform electric field of infinite extent is abruptly applied at the surface in the form of a step function. The same problem but with accounting for the effect of the frequency dependence of the constitutive parameters (conductivity and permittivity) of rock for pulse transmission is discussed by Fuller and Wait (1972), who employed the algorithm known as the fast Fourier transform (Cochran et al., 1967) for their numerical computations. Wait (1956a, 1957a, 1957b), Levy and Keller (1958), and Keilson and Row (1959) investigated the propagation of transient ground or surface waves
over curved and plane earths including the diffusion of the disturbance into the earth. Harrison (1964), Harrison et al. (1965), and Harrison and Papas (1965) have analyzed the propagation of transient electromagnetic fields through conducting walls into plane and spherical cavities. Wait (1956b) has examined the problem of shielding an observer from a transient electromagnetic source in the form of a dipole by a conductive sheet. Wait and Froese (1955) have studied the case of a transverse magnetic wave transient for both step and rectangular pulse excitation and have given solution for the reflection of the wave at oblique incidence from the plane interface of a dissipative medium. The problem of distortion of electromagnetic pulses at totally reflecting layers is examined by Nicolis (1967, 1969) and Wait (1969). Wait (1971) and Hill and Wait (1974) examined the electromagnetic fields in a conducting half-space for a line source with a delta-function current where all displacement currents are omitted. Singh (1973) derived the electromagnetic transient response of a permeable and conducting sphere embedded in a finitely conducting infinite space. King and Harrison (1968) have considered the case of normal incidence of an electromagnetic transient on a lossy half-space and have presented results for transmission into the lossy medium when the input pulse is Gaussian. Monroe (1969) has given a three term approximate solution as a step-function response of a vertically
polarized electromagnetic plane wave reflected at an imperfectly conducting surface; this solution he states is approximately valid only for early times. Papazoglou (1972) and Dudley, Papazoglou, and White (1974) have given exact analytical solutions for the fields on and above the surface of a conducting half-space excited by a causal double-exponential plane wave incident at an arbitrary angle.

1.2 The Approach Followed in This Study

We simulate the input pulse-shapes that are occurring in natural as well as man-made transient phenomena with a double-exponential causal pulse of the form:

\[ f(t) = u(t) \cdot \frac{(e^{-\alpha t} - e^{-\beta t})}{N} \]  \hspace{1cm} (1.1)

\[ u(t) \equiv \text{unit step function} \]

\[ N = N(\alpha, \beta) \equiv \text{normalization constant.} \]

This form appears particularly attractive, since with an appropriate choice of \( \alpha \)'s and \( \beta \)'s we can simulate a wide variety of pulses ranging from a unit-step function to a delta function and the in-between pulse shapes (Dudley et al., 1974). From the mathematical standpoint, by using this form, we gain in simplicity due to the fact that the corresponding Fourier transform has only two simple poles at \( i\alpha \) and \( i\beta \) (Papazoglou, 1972; Dudley et al., 1974).
For the treatment of the electromagnetic wave response, we note that the general case of polarization is derivable as a linear combination of the two basic cases: Transverse electric (TE) and Transverse magnetic (TM) (Papazoglou, 1972; Dudley et al., 1974). We use the Maxwell's equations in the frequency domain and the boundary conditions at the plane interfaces and thereby we obtain the functional forms of the EM-waves in all regions, i.e., the atmosphere and the dissipative medium (earth). We also obtain all the reflection and transmission coefficients expressed in the frequency-domain.

To return to the time-domain we multiply the expressions for the electromagnetic fields in the frequency domain times the input pulse in the frequency-domain and we invert the Fourier transform. We follow a complete analysis on the complex plane and reduce all results into algebraic terms and real definite integrals.

In performing the numerical computations and in order to evaluate the principal value integrals contained in some of the expressions, we have made use of the idea of folding the range of integration around the singularity of the integrand (Davis and Rabinowitz, 1967; Dudley et al., 1974). This yields two regular definite integrals that are readily Computable by standard algorithms. The method reduces the computation time to only a small fraction of
the corresponding time needed for direct numerical integration.

As contrasted with the method of direct numerical Fourier inversion using Filon's integration formulas (Abramowitz and Stegun, 1964) concurrently employed at the Electrical Engineering Department of The University of Arizona, the method presented here offers substantial information given in terms of the explicit analytical form of the solution in addition to considerable savings in computation costs in many cases of practical interest, such as the complete two layer case and the early times for the many-layered problem.

In the case of seawater the model with frequency independent constitutive parameters is adequate (Wait, 1961) but in the case of soils and rocks the conductivity as well as the electric permittivity are frequency dependent. The following works are informative on the subject: Scott, Carroll, and Cunningham (1964); Fuller and Wait (1972); Collette and Katsube (1973); Katsube and Collette (1973); Ryu et al. (1973); Longmire and Longley (1973); Alvarez (1973); and Lytle (1973). Based on the observation that in the region of 10 HZ to 10 kHZ the dielectric constant varies approximately as the inverse of the frequency for many geological media while it levels off for high frequencies (Wait, 1961) we develop a simple model for the frequency
dependent constitutive parameters that approximates existing data and offers mathematical simplicity.

1.3 Related Physical Problems and Applications

The considerable current interest in the propagation of transient electromagnetic waves through the earth is due to applications in at least three problems of the real world: the geophysical probing, the subsurface communication, and the study of the effects of a wide range of phenomena ranging from lightning strokes to nuclear detonations.

In connection with the first application the following works are notable: Morrison, Phillips, and O'Brien (1969); Dey and Morrison (1973); Vanyan (1967); and Keller and Frischknecht (1970). In this connection the present work appears promising.

Possible applications of the present work lie also in the third application above. In this connection, relevant works are by Suydam (1961); Gilinsky (1965); Johler and Morgenstern (1965); Johler (1967); Sandmeier, Dupree, and Hansen (1972); a reference text by Headquarters, Department of the Army (1962); and Merewether (1969).

1.4 Outline of the Following Chapters

Chapter 2 deals with the physical problem. We discuss in detail the physics of the problem and subsequently we formulate the general N-layer case. As
particular cases we deduce the formulas for the two-layer and the three-layer cases. We then analyze the special three-layer case of air-dielectric-perfect conductor.

Chapters 3 and 4 present the two-layer case in its entirety. The former deals with the TE case, in the latter we present the TM case. We include a review of the existing results for the fields on the interface and the reflective results (Papazoglou, 1972; Dudley et al., 1974) and then we present a complete analysis of the fields within the dissipative medium.

Chapters 5 and 6 cover the three-layer problem, with the TE and TM cases exposed separately. We analyze the reflective fields.

Chapter 7 deals with the question of the frequency dependent constitutive parameters of the dissipative medium.

Chapter 8 concludes the study and presents the outlook for future work.

At the end we include three appendices. Appendix A presents a list of relevant physical constants. In Appendix B we present in detail the mathematical background for the study; as a starting basis we assume a standard textbook on complex calculus (Marsden, 1973). Appendix C contains a set of field curves referred to in the text.
CHAPTER 2

FORMULATION OF THE PHYSICAL PROBLEM

2.1 Introduction

We consider a N-layer dissipative medium, as in Figure 2.1, where layer #1 is an empty half-space, layers #2, 3, ..., (N-1) and layer #N, a half-space, are lossy media. We assume a plane wave incident on the interface between layers #1 and #2 at an arbitrary angle θ. The case of general polarization can be broken up into two subcases (Papazoglou, 1972; Dudley et al., 1974), i.e., a transverse electric case (TE) and a transverse magnetic case (TM). We define a polarization angle φ as the angle which the vector E makes with the interface between layers #1 and #2, measured in the plane perpendicular to the Poynting vector. Looking back along the Poynting vector from the interface (Figure 2.2), we project E and H through the angle φ. We now choose the x-direction as the direction of the component of E in the interface plane. The result is a decomposition of the general case into TE and TM modes as follows:

\[
\begin{align*}
\text{TE} & : E \cos \phi & H \sin \phi \\
\text{TM} & : H \cos \phi & E \sin \phi
\end{align*}
\]

The six components thus obtained constitute the field.
Figure 2.1. The n-layer problem.

Figure 2.2. The polarization angle.
2.2 TE Subcase N-Layers

We solve the wave equation (Panofsky and Phillips, 1962), in the frequency domain, for $E_x$ in every substratum:

$$(
\nabla^2 + k_n^2\right) E_{xn} = 0
$$

$$n = 1, 2, ..., N \tag{2.1}
$$

where:

$$\nabla^2 = \nabla^2_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.2}
$$

Since the results do not depend on $x$, and

$$k_n^2 = \omega^2 \mu_o (\varepsilon_n + \frac{\sigma_n}{\omega}) \tag{2.3}
$$

We use subsequently the Maxwell equation in the frequency domain:

$$\nabla \times \bar{E} = -i\omega \mu \bar{H} \tag{2.4}
$$

to obtain the following plane-wave solutions:

$$E_{xn} = A_n^+ \exp[-ik_n(y\sin \theta_n + z\cos \theta_n)]
$$

$$+ A_n^- \exp[-ik_n(y\sin \theta_n - z\cos \theta_n)]
$$

$$H_{yn} = \frac{\cos \theta_n}{n_n} [A_n^+ \exp[-ik_n(y\sin \theta_n + z\cos \theta_n)]
$$

$$- A_n^- \exp[-ik_n(y\sin \theta_n - z\cos \theta_n)]\]$$
\[ H_{zn} = -\frac{\sin \theta}{n_n} [A_n^+ \exp(-ik_n(y\sin \theta_n + z\cos \theta_n))] \]

\[ + A_n^- \exp(-ik_n(y\sin \theta_n - z\cos \theta_n))] \] (2.5)

\[ n_n = \mu_0^{1/2} \cdot (\varepsilon_n + \sigma_n/i\omega)^{-1/2} \]

n: index of the particular layer

A\(^+_n\): transmission in the nth layer

A\(^-_n\): reflection in the nth layer

In (2.5) we have used Snell's law implicitly.

The following boundary conditions apply on the arbitrarily chosen jth interface (Figure 2.3).

\[ \frac{\text{Region (j)}}{\text{Region (j+1)}} \]

\[ z = z_j; \]

\[ H_{yj} = H_{y(j+1)} \] (2.6)

Continuity of the tangential fields:

\[ E_{xj} = E_{x(j+1)} \]

\[ k_j \sin \theta_j = k_{j+1} \sin \theta_{j+1} \] (2.7)
The application of (2.6) due to (2.7) gives:

\[ A_j^+ \exp(-ik_j \cos \theta_j z_j) + A_j^- \exp(i k_j \cos \theta_j z_j) = \]

\[ = A_{j+1}^+ \exp(-i k_{j+1} \cos \theta_{j+1} z_{j+1}) + A_{j+1}^- \exp(i k_{j+1} \cos \theta_{j+1} z_{j+1}) \]

\[ \frac{\cos \theta_j}{n_j} [A_j^+ \exp(-i z_j k_j \cos \theta_j) - A_j^- \exp(i z_j k_j \cos \theta_j)] = \]

\[ = \frac{\cos \theta_j}{n_j} [A_{j+1}^+ \exp(-i z_{j+1} k_j \cos \theta_j) - A_{j+1}^- \exp(i z_{j+1} k_j \cos \theta_j)] \]

(2.8)

Equations (2.8) considered simultaneously constitute a linear 2x2 system the solution of which provides us with recursion formulas that express the reflection and transmission coefficients for one layer in terms of those of the next layer. The solution of the system (2.8) gives:

\[ A_j^+ = \frac{1}{2} \left\{ \left(1 + \frac{Z_j}{Z_{j+1}}\right) \cdot e^{iz_j (k_j \cos \theta_j - k_{j+1} \cos \theta_{j+1})} \cdot A_{j+1}^+ + \right. \]

\[ + \left. \left(1 - \frac{Z_j}{Z_{j+1}}\right) \cdot e^{iz_j (k_j \cos \theta_j + k_{j+1} \cos \theta_{j+1})} \cdot A_{j+1}^- \right\} \]

\[ A_j^- = \frac{1}{2} \left\{ \left(1 - \frac{Z_j}{Z_{j+1}}\right) \cdot e^{-iz_j (k_j \cos \theta_j + k_{j+1} \cos \theta_{j+1})} \cdot A_{j+1}^+ + \right. \]

\[ + \left. \left(1 + \frac{Z_j}{Z_{j+1}}\right) \cdot e^{-iz_j (k_j \cos \theta_j - k_{j+1} \cos \theta_{j+1})} \cdot A_{j+1}^- \right\} \]

(2.9)
where:

\[ Z_i = \frac{n_i}{\cos \theta_i} \]  \hspace{1cm} (2.10)

The problem of determining the \( A_j^\pm \) for all \( j \) such that

\[ 1 \leq j \leq N \]

is thus reduced to solving the system (2.9) starting from the boundary relation for

\[ j = N-1 \]

which is:

\[ A_{N-1}^+ = \frac{1}{2}(1 + \frac{Z_{N-1}}{Z_N}) \cdot e^{iz_{N-1}(k_{N-1}\cos \theta_{N-1} - k_N\cos \theta_N)} \cdot A_N^+ \]

\[ A_{N-1}^- = \frac{1}{2}(1 - \frac{Z_{N-1}}{Z_N}) \cdot e^{-iz_{N-1}(k_{N-1}\cos \theta_{N-1} + k_N\cos \theta_N)} \cdot A_N^+ \]  \hspace{1cm} (2.11)

Since \( A_N^- = 0 \), which follows from the fact that there are \( N-1 \) interfaces present in the \( N \)-layer problem, no reflection exists in the \( N \)th layer.

Proceeding gradually to smaller \( j \)'s we can express in the end the reflection and transmission coefficients of the first layer in terms of the transmission coefficient for the \( N \)th layer, i.e.,

\[ A_1^+ = G_N^+(Z_1, Z_2, \ldots, Z_N; k_1\cos \theta_1, \ldots, k_N\cos \theta_N) \cdot A_N^+ \]

\[ A_1^- = G_N^-(Z_1, Z_2, \ldots, Z_N; k_1\cos \theta_1, \ldots, k_N\cos \theta_N) \cdot A_N^+ \]  \hspace{1cm} (2.12)
where $G_N^+$ are known functions that arise from the solution of the system (2.9).

By normalizing the incident wave we deduce for the transmission coefficient into $N$:

$$A_N^+ = \frac{1}{G_N^+} \quad (2.13)$$

Using (2.13) we substitute into (2.11) and proceeding step by step we express all $A_j^\pm$ in terms of known quantities, thereby creating the complete solution of the problem in the frequency domain. In particular for the reflection coefficient in medium 1 we obtain:

$$A_1^- = \frac{G_N^-}{G_N^+} \quad (2.14)$$

### 2.3 TE Subcase 2-Layers

We apply results of the previous paragraph and obtain:

$$N = 2, \ z_1 = 0 \ \text{hence}$$

$$A_1^+ = 1 = \frac{1}{2} \left( 1 + \frac{n_1 \cos \theta}{n_2 \cos \theta} \right) A_2^+$$

$$A_1^- = \frac{1}{2} \left( 1 - \frac{n_1 \cos \theta}{n_2 \cos \theta} \right) A_2^+ \quad (2.15)$$
therefore:

\[ A_2^+ = \frac{1}{G_2} = \frac{2n_2 \cos \theta}{n_2 \cos \theta + n_1 \cos \theta_2} \]  
(2.16)

\[ A_1^- = \frac{n_2 \cos \theta - n_1 \cos \theta_2}{n_2 \cos \theta + n_1 \cos \theta_2} \]  
(2.17)

For the \( \cos \theta_2 \) because of (2.7) we get:

\[ \cos \theta_2 = \sqrt{1 - \sin^2 \theta_2} = \sqrt{1 - \left( \frac{n_2}{n_1} \right)^2 \sin^2 \theta} \]

and hence

\[ A_1^- = \frac{\cos \theta - \sqrt{\frac{n_1}{n_2}}^2 - \sin^2 \theta}{\cos \theta + \sqrt{\frac{n_1}{n_2}}^2 - \sin^2 \theta} \]

\[ A_2^+ = \frac{2 \cos \theta}{\cos \theta + \sqrt{\frac{n_1}{n_2}}^2 - \sin^2 \theta} \]  
(2.18)

The parameter \( n_1/n_2 \) is such that:

\[ \left( \frac{n_1}{n_2} \right)^2 = k + \frac{\sigma}{\omega \varepsilon_0} \]

\[ k = \frac{\varepsilon_2}{\varepsilon_0} \]  
(2.19)
2.4 TE Subcase 3-Layers

We will refer to Figure 2.4:

Region (1)
\[ z = 0 \]
\[ \bar{d} \]
Region (2)
\[ z = d \]
Region (3)

Figure 2.4. The three-layer problem.

Here \( A_1^+ = 1, A_3^- = 0 \) and following the same procedure as previously explained we get:

\[
A_3^+ = \frac{4 \cdot e^{iDK_3 \cos \theta_3}}{(1 + \frac{Z_1}{Z_2})(1 + \frac{Z_2}{Z_3}) \cdot e^{iDK_2 \cos \theta_2} + (1 - \frac{Z_1}{Z_2})(1 - \frac{Z_2}{Z_3}) \cdot e^{-iDK_2 \cos \theta_2}}
\]

(2.20)

and

\[
A_1^- = \frac{(1 - \frac{Z_1}{Z_2})(1 + \frac{Z_2}{Z_3}) + (1 + \frac{Z_1}{Z_2})(1 - \frac{Z_2}{Z_3}) \cdot e^{-iDK_2 \cos \theta_2}}{(1 + \frac{Z_1}{Z_2})(1 + \frac{Z_2}{Z_3}) + (1 - \frac{Z_1}{Z_2})(1 - \frac{Z_2}{Z_3}) \cdot e^{-iDK_2 \cos \theta_2}}
\]

(2.21)

We now introduce two new parameters
\[ \lambda_{12} = \frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} \]
\[ \lambda_{23} = \frac{1 - \frac{Z_2}{Z_3}}{1 + \frac{Z_2}{Z_3}} \]  

by means of which (2.21) becomes:

\[ A_1 = \frac{\lambda_{12} + \lambda_{23} e^{-i2dk_2 \cos \theta_2}}{1 + \lambda_{12} \lambda_{23} e^{-i2dk_2 \cos \theta_2}} \]  

(2.23)

Eq. (2.23) can be now transformed into a physically more meaningful and mathematically more suggestive equation as follows:

We first observe that for real \( \omega \)

\[ |\lambda_{12} \lambda_{23} e^{-i2dk_2 \cos \theta_2}| < 1 \]  

(2.24)

(see Appendix B for complete proof). The condition of real \( \omega \) is not narrow as far as our problem is concerned because its implication (2.24) will be used within an integral that extends solely on the real axis, as will become clear later.

Because of (2.24) we can write (Marsden, 1973, p. 165) by expanding the fraction in a convergent series:
where

\[ A_1 = \lambda_{12} + (\lambda_{12}^2 - 1) \sum_{m=1}^{\infty} (-1)^m \lambda_{12}^m \lambda_{23} e^{-i2mdk_2 \cos \theta_2} \]

\[ = \lambda_{12} + (1 - \lambda_{12}^2) \lambda_{23} e^{-i2dk_2 \cos \theta_2} \]

\[ + (\lambda_{12}^2 - 1) \lambda_{12} \lambda_{23} e^{-i4dk_2 \cos \theta_2} + ... \]  

(2.25)

As will become clear later by considering the corresponding time-domain expressions we may interpret Eqn. (2.25) as an identity that gives \( A_1 \) as a series of the contributions (in the frequency domain) of all the partial wave transients that arise as a result of multiple reflections between the two interfaces of the three-layer problem. Accordingly the first term of the series represents the contribution of the primary wave as in the two-layer case. The second term represents the contribution of the wave that bounces back after one reflection at the interface between layers \#2 and \#3 and so forth. Note that due to the finite speed of wave propagation the resultant reflection in the time domain at any time will be given as a finite sum of non-zero terms. Similar observations in other cases have been reported by other researchers, notably by Johler and Morgenstern (1965) and Johler (1967).
2.5 TM Subcase N-Layers

For this case, we solve the wave equation for $H_x$ in every substratum and apply the Maxwell equation:

$$\nabla \times \vec{H} = i\omega \frac{\mu}{n^2} \vec{E}$$

to obtain in a similar manner as previously:

$$H_{xn} = -\frac{B^+}{n} \exp[-ik_n (ysin0_n + zcos0_n)]$$

$$+ \frac{B^-}{n} \exp[-ik_n (ysin0_n - zcos0_n)]$$

$$E_{yn} = cos0_n \{B^+ \exp[-ik_n (ysin0_n + zcos0_n)]$$

$$+ B^- \exp[-ik_n (ysin0_n - zcos0_n)]\}$$

$$E_{zn} = -sin0_n \{B^+ \exp[-ik_n (ysin0_n + zcos0_n)]$$

$$- B^- \exp[-ik_n (ysin0_n - zcos0_n)]\}$$

(2.26)

The following boundary conditions apply on an arbitrarily chosen interface (see Figure 2.2).

$$H_{xj} = H_{x(j+1)}$$

$$E_{yj} = E_{y(j+1)}$$

$$k_j sin\theta_j = k_{j+1} sin\theta_{j+1}$$

(2.27)
By means of (2.26), (2.27) are readily translated into the following 2x2 linear system:

\[
\begin{align*}
B^+_j \frac{\exp(-iz_j k_j \cos \theta_j)}{n_j} + B^-_j \frac{\exp(iz_j k_j \cos \theta_j)}{n_j} &= \\
= - \frac{B^+_{j+1}}{n_{j+1}} \exp(-iz_j k_{j+1} \cos \theta_{j+1}) + \frac{B^-_{j+1}}{n_{j+1}} \exp(iz_j k_{j+1} \cos \theta_{j+1}) \\
\cos \theta_j \{B^+_j \exp(-iz_j k_j \cos \theta_j) + B^-_j \exp(iz_j k_j \cos \theta_j)\} &= \\
= \cos \theta_{j+1} \{B^+_j \exp(-iz_j k_{j+1} \cos \theta_{j+1}) + B^-_j \exp(iz_j k_{j+1} \cos \theta_{j+1})\} \quad (2.28)
\end{align*}
\]

The solution of (2.28) yields the recursion formulas:

\[
B^+_j = \frac{1}{2} \left[ \frac{n_j}{n_{j+1}} + \frac{\cos \theta_{j+1}}{\cos \theta_j} \right] e^{iz_j (k_j \cos \theta_j - k_{j+1} \cos \theta_{j+1})} \cdot B^+_{j+1} + \\
\frac{\cos \theta_{j+1}}{\cos \theta_j} - \frac{n_j}{n_{j+1}} e^{iz_j (k_j \cos \theta_j + k_{j+1} \cos \theta_{j+1})} \cdot B^-_{j+1}
\]

\[
B^-_j = \frac{1}{2} \left[ \frac{\cos \theta_{j+1}}{\cos \theta_j} - \frac{n_j}{n_{j+1}} \right] e^{-iz_j (k_j \cos \theta_j + k_{j+1} \cos \theta_{j+1})} \cdot B^+_{j+1} + \\
\frac{\cos \theta_{j+1}}{\cos \theta_j} + \frac{n_j}{n_{j+1}} e^{-iz_j (k_j \cos \theta_j - k_{j+1} \cos \theta_{j+1})} \cdot B^-_{j+1}
\]

(2.29)

Again the solution for the system (2.29) follows the following steps: The boundary condition for the last
interface \( j = N - 1 \) is:

\[
B_{N-1}^+ = \frac{1}{2} \left( \frac{n_{N-1}}{n_N} + \frac{\cos \theta_{N-1}}{\cos \theta_{N}} \right) e^{iz_{N-1} \left( k_{N-1} \cos \theta_{N-1} - k_{N} \cos \theta_{N} \right)} \cdot B_N^+ \\
B_{N-1}^- = \frac{1}{2} \left( \frac{n_{N-1}}{\cos \theta_{N-1}} - \frac{n_{N-1}}{n_N} \right) e^{-iz_{N-1} \left( k_{N-1} \cos \theta_{N-1} - k_{N} \cos \theta_{N} \right)} \cdot B_N^+ 
\]

\[ (2.30) \]

and proceeding gradually to smaller \( j \)'s we can express in the end the reflection and transmission coefficients of the first layer in terms of the transmission coefficient for the \( N \)th layer:

\[
B_1^+ = F_N^+ (n_1, \cos \theta, \ldots, n_N, \cos \theta_N; k_1 \cos \theta, \ldots, k_N \cos \theta_N) \cdot B_N^+ \\
B_1^- = F_N^- (n_1, \cos \theta, \ldots, n_N, \cos \theta_N; k_1 \cos \theta, \ldots, k_N \cos \theta_N) \cdot B_N^+ 
\]

\[ (2.31) \]

where \( F_N^+ \) are known functions that arise from the solution of the system \((2.29)\).

Normalization of the incident wave yields:

\[
B_N^+ = \frac{1}{F_N^+} 
\]

\[ (2.32) \]

Using this value for \( B_N^+ \) we can readily obtain any \( B_j^\pm \) through the process described above. For the reflection coefficient in medium 1 we get:

\[
B_1^- = \frac{F_N^-}{F_N^+} 
\]

\[ (2.33) \]
2.6 TM Subcase 2-Layers

For \( N = 2, z_1 = 0 \) we obtain:

\[
B_1^+ = 1 = \frac{1}{2} \left( \frac{n_1}{n_2} + \frac{\cos \theta_2}{\cos \theta} \right) B_2^+
\]

\[
B_1^- = \frac{1}{2} \left( \frac{\cos \theta_2}{\cos \theta} - \frac{n_1}{n_2} \right) \cdot B_2^+
\]

and solving:

\[
B_2^+ = \frac{1}{F_2^+} = \frac{2n_2 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta_2}
\]  

(2.35)

and

\[
B_1^- = \frac{n_2 \cos \theta_2 - n_1 \cos \theta}{n_2 \cos \theta_2 + n_1 \cos \theta}
\]

(2.36)

and since

\[
\cos \theta_2 = [1 - \left( \frac{n_2}{n_1} \right)^2 \sin^2 \theta]^{1/2}
\]

(2.37)

we get:

\[
B_2^+ = \frac{2 \left( \frac{n_1}{n_2} \right) \cos \theta}{\left( \frac{n_1}{n_2} \right)^2 \cos \theta + \left[ \frac{n_1}{n_2} \right]^2 \sin^2 \theta]^{1/2}
\]

\[
B_1^- = \frac{\left[ \frac{n_1}{n_2} \right]^2 \sin^2 \theta]^{1/2} - \left( \frac{n_1}{n_2} \right)^2 \cos \theta}{\left[ \frac{n_1}{n_2} \right]^2 \sin^2 \theta]^{1/2} + \left( \frac{n_1}{n_2} \right)^2 \cos \theta
\]

(2.38)
2.7 TM Subcase 3-Layers

We will refer to the previous schematic (Figure 2.4). Here $B_1^+ = 1$, $B_3^- = 0$, and, following the same procedure as previously explained, we get: (we let $n_1 = n_o$)

$$B_3^+ = \frac{\text{idk}_3 \cos \theta_3}{4e} \frac{\cos \theta_2}{\cos \theta} \frac{\cos \theta_3}{n_o} \frac{n_2 \text{idk}_2 \cos \theta_2}{(\cos \theta + \frac{n_o}{n_2})(\cos \theta + \frac{n_2}{n_3})e}$$ (2.39)

$$B_1^- = \frac{\cos \theta_2}{\cos \theta} \frac{n_0}{n_2} \frac{\cos \theta_3}{n_3} + \frac{\cos \theta_2}{\cos \theta} + \frac{n_0}{n_2} \frac{\cos \theta_3}{n_3} - \frac{n_2}{\text{idk}_2 \cos \theta_2}$$ (2.40)

We now introduce two new parameters:

$$\mu_{12} = \frac{n_2 \cos \theta_2 - n_o \cos \theta}{n_2 \cos \theta_2 + n_o \cos \theta} \quad (= B_1^-)$$

$$\mu_{23} = \frac{n_3 \cos \theta_3 - n_2 \cos \theta_2}{n_3 \cos \theta_3 + n_2 \cos \theta_2} \quad (2.41)$$

Hence, concentrating on the reflective properties in the first interface, we can write:
We now recognize the analogy of the forms in the RHS of the equations (2.23) and (2.42) and we proceed in a similar fashion to expand this form in a series similar to (2.25) where the \( \lambda \)'s are interchanged with the \( \mu \)'s. We note here too, in the same context as in Section 2.4, that we may interpret the resulting series as a series of the contributions (in the frequency domain) of all the partial wave transients that arise as a result of multiple reflections between the two interfaces of the three-layer problem.

2.8 A Special Case

We now imagine the case of 3-layers where region #3 (Figure 2.4) is a perfect conductor. This is a case that is physically interesting and that helps to illustrate numerous mathematical points. For further simplification, we will assume that region #2 (Figure 2.4) is a dielectric.

The case is characterized by:

\[ \sigma_2 = 0 \]

\[ \sigma_3 \to \infty \]  \hspace{1cm} (2.43)

In the transverse electric subcase we have:

\[ \sigma_3 \to \infty \Rightarrow n_3 = 0 \Rightarrow \cos \theta_3 = 1, \ z_3 = 0 \]
and since $z_2 \neq 0$ we have $z_2/z_3 \rightarrow \infty$ hence we apply (2.21) and get

$$A_1 = \frac{1 - \frac{Z_1}{Z_2} - (1 + \frac{Z_1}{Z_2})e^{-i2dk_2\cos\theta_2}}{1 + \frac{Z_1}{Z_2} - (1 - \frac{Z_1}{Z_2})e^{-i2dk_2\cos\theta_2}} \quad (2.44)$$

Recalling Eqn. (2.22), letting in this case,

$$\lambda = \lambda_{12} = \frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} \quad (2.45)$$

we translate (2.44) into:

$$A_1 = \frac{\lambda - e^{-i2dk_2\cos\theta_2}}{1 - \lambda e^{-i2dk_2\cos\theta_2}} \quad (2.46)$$

In this case (let $k = \frac{e_2}{\epsilon_0}$)

$$k_2\cos\theta_2 = \frac{\omega}{C} \sqrt{k - \sin^2\theta} = \frac{\omega}{v} \quad (2.47)$$

where:

$$v = \frac{C}{\sqrt{k - \sin^2\theta}} \quad (2.48)$$

and (2.46) takes the form:

$$A_1 = \frac{\lambda - e^{-i\frac{2d\omega}{v}}}{1 - \lambda e^{-i\frac{2d\omega}{v}}} \quad (2.49)$$
Now:

\[
\frac{z_1}{z_2} = \frac{n_0 \cos \theta_2}{n_2 \cos \theta} = \sqrt{\frac{k \sin^2 \theta}{\cos \theta}}
\] (2.50)

hence:

\[
\lambda = \frac{\cos \theta - \sqrt{k \sin^2 \theta}}{\cos \theta + \sqrt{k \sin^2 \theta}}
\] (2.51)

and \[|\lambda| < 1\] (2.52)

Applying Eqn. (2.25) we obtain the expansion form:

\[
\lambda_{23} = -1
\]

\[
A_1 = \lambda + (\lambda^2 - 1) \sum_{m=1}^{\infty} \lambda^{m-1} e^{-i2m \frac{d\omega}{v}}
\]

\[= \lambda + (\lambda^2 - 1) e^{-i2 \frac{d\omega}{v}} + (\lambda^2 - 1) \lambda e^{-i4 \frac{d\omega}{v}} + ...\] (2.53)

In the transverse magnetic subcase we have:

\[\sigma_3 + \infty \Rightarrow n_3 = 0\] and, since \(n_2 \neq 0\) \(\Rightarrow n_2/n_3 \rightarrow \infty\),

from (2.40) we get:

\[
B_1 = \frac{\cos \theta_2 - \frac{n_0}{n_2} - \left(\frac{\cos \theta_2}{\cos \theta} + \frac{n_0}{n_2}\right)e^{-i2 \frac{d\omega}{v}}}{\cos \theta_2 + \frac{n_0}{n_2} - \left(\frac{\cos \theta_2}{\cos \theta} - \frac{n_0}{n_2}\right)e^{-i2 \frac{d\omega}{v}}}
\] (2.54)
Letting $\mu = \mu_{12}$, in view of (2.41) we get:

$$B_1 = \frac{\mu - e^{-i2\frac{d\omega}{v}}}{1 - \mu e^{-i2\frac{d\omega}{v}}} \tag{2.55}$$

in this case; since $n_2 \cos \theta_2 = n_0 \sqrt{k - \sin^2 \theta}$

$$\mu = \frac{\sqrt{k - \sin^2 \theta}}{\sqrt{k - \sin^2 \theta + k \cos \theta}} \tag{2.56}$$

In this case too: $|\mu| < 1$. The expansion for $B_1$ proceeds in an identical fashion as before, leading to the form of (2.53), where the $\lambda$ is replaced by $\mu$.

In connection with earlier remarks (see Sections 2.4 and 2.7) on the physical interpretation of the expansions for $A_1$ and $B_1$ we may now observe the following:

Let us consider as an input a decaying exponential pulse: $\exp(-\alpha t)$. The reflection in the time domain is found after multiplying (2.53) times the Fourier transform of the input $1/(\alpha + i\omega)$ and inverting into the time domain the result, thus:

$$e_{\text{x1 refl}} = (1+\lambda)e^{-\alpha t} + (\lambda^2 - 1)e^{-\alpha(t - \frac{2d}{v})} u(t - \frac{2d}{v}) +$$

$$+ \lambda(\lambda^2 - 1)e^{-\alpha(t - \frac{4d}{v})} u(t - \frac{4d}{v}) + \ldots, \text{ where} \tag{2.57}$$

$u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$
The first term in (2.57) is the reflection due to the presence of medium (2) as in the two-layer problem. The second term which contributes only after $t > 2d/v$ is the contribution of the transient that bounces back after one reflection at the surface of the perfect conductor. The third term which becomes non-zero after $t > 4d/v$ is the contribution of the transient that bounces back after two reflections at the surface of the perfect conductor, and so forth.

2.9 The Mathematical Analysis

The problem consists of finding the fields at the first interface for the general problem and the fields in the dissipative medium in the two-layer case, when the input plane wave is a transient pulse whose time dependence is given by the double exponential causal function:

$$f(t) = u(t) \cdot \frac{(e^{-\alpha t} - e^{-\beta t})}{N}$$

with Fourier transform:

$$F(\omega) = \frac{1}{\alpha + i\omega} - \frac{1}{\beta + i\omega}/N$$

Therefore the solution to the problem is determined by finding the inverse Fourier transform of the fields in the frequency domain as exposed previously in detail, multiplied by $F(\omega)$. For example:
Let $E(\omega)$ be the field in the frequency domain. Then the field $e(t)$ in the time domain is:

$$e(t) = \left[ e^{(\alpha)}(t) - e^{(\beta)}(t) \right]/N$$

$$e^{(\alpha)}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E(\omega)}{\alpha + i\omega} e^{i\omega t} d\omega$$

$$e^{(\beta)}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E(\omega)}{\beta + i\omega} e^{i\omega t} d\omega$$

(2.60)

We should note that the problem reduces to that of determining $e^{(\alpha)}$ since $e^{(\beta)}$ can be obtained then by substituting $\beta$ for $\alpha$ in $e^{(\alpha)}$. 
3.1 The Reflective Results

The author in a previous study (Papazoglou, 1972) has dealt thoroughly with the solution of the two-layer problem for the fields on the interface. Dudley et al. (1974) gave a complete account of the reflective results.

For an input pulse given by the causal double-exponential function:

\[ f(t) = u(t) \left[ \exp(-\alpha t) - \exp(-\beta t) \right]/N \] (3.1)

\[ N = \exp(-\alpha T) - \exp(-\beta T) \] (3.2)

\[ T = \frac{\ln(\beta/\alpha)}{\beta - \alpha} \] (3.3)

it is found that the field for \( Z \leq 0 \) (Figure 3.1) is:

\[ e_{xl}(Z,t) = f(t - \frac{Z}{C}\cos\theta) + a_1^-(t + \frac{Z}{C}\cos\theta) \] (3.4)

where: \( C = \) the speed of light in free space

\[ a_1^- = \frac{(I_1^{(0)} - I_2^{(0)})}{N} \] (3.5)
and:

\[ I_1^{(0)} = \frac{1}{2\pi} \cdot \]

\[ \int_{-\infty}^{+\infty} \frac{(K+\cos \theta)\omega - i(\sigma/\varepsilon_o) - 2\cos \theta [(\omega)^2 (K - \sin^2 \theta) - i(\sigma/\varepsilon_o)\omega]}{(\alpha + i\omega)[(1-K)\omega + i(\sigma/\varepsilon_o)]} e^{i\omega t} d\omega \]

(3.6)

and \( I_2^{(0)} \) is obtained from (3.6) by substituting \( \beta \) for \( \alpha \).

We have further shown that:

\[ I_1^{(0)} = \frac{(\sigma/\varepsilon_o) - \alpha(K+\cos \theta) + 2\cos \theta [\alpha^2(K - \sin^2 \theta) - \sigma/\varepsilon_o]}{(K-1)\sigma/\varepsilon_o} \]

\[ \exp(-at) - \frac{2\cos \theta}{\pi} \int_0^\infty \frac{[\sigma y/\varepsilon_o - y^2(K - \sin^2 \theta)]^{1/2}}{(\alpha - y)[(1-K)y + \sigma/\varepsilon_o]} \exp(-yt) dy \]

(3.7)
for $\alpha > \xi$ where $\xi = \sigma / (\varepsilon_0 (K - \sin^2 \theta))$

and for $\alpha \leq \xi = \sigma / (\varepsilon_0 (K - \sin^2 \theta))$

$$I_1(0) = \frac{\sigma / \varepsilon_0 - \alpha (K + \cos 2\theta)}{(K - 1) \alpha - \sigma / \varepsilon_0} \exp(-\alpha t)$$

$$= \frac{2 \cos \theta}{\pi} \xi \frac{\sigma y / \varepsilon_0 - y^2 (K - \sin^2 \theta)}{[(\alpha - y) [(1 - K) y + \sigma / \varepsilon_0]} \exp(-yt) dy$$

(3.8)

where the double parentheses above indicate the singularity of the integrand in the principal value integral (see also Appendix B). On the interface ($Z = 0$) the effect of the dissipative half-space on the tangential electric field is that the fields tend to cancel with the result that the electric fields are much less in peak magnitude than the input field. This is illustrated clearly by the curves for the principal field component ($e_{x1}$) for the TE case at the interface (see Figure 1, Appendix C); the data given therein are sufficient to determine the fields at an arbitrary height above the interface; indeed we can apply (3.4) and add the input pulse to the reflection coefficient, each taken at the appropriate time. The parameters for the illustration are those of a lossy earth: $\sigma = 2 \times 10^{-2}$ and $K = 15$. Curves are given for four arrival angles: $\theta = 0^\circ$, $28^\circ$, $56^\circ$, and $84^\circ$ and the input pulse is included for reference. The input pulse has a rise time of 10 ns and fall time of 350 ns.
3.2 Fields Within the Dissipative Medium

Within the half-space #2 the fields are:

\[ E_{x2} = A_2^+ \exp[-ik_2\cos\theta_2 z] \]

\[ H_{y2} = A_2^+ \frac{\cos\theta_2}{n_2} \exp(-izk_2\cos\theta_2) \]

\[ H_{z2} = -A_2^+ \frac{\sin\theta_2}{n_2} \exp(-izk_2\cos\theta_2) \] \hspace{1cm} (3.9)

\[ A_2^+ = \frac{2\cos\theta}{\cos\theta + [\frac{n_1}{n_2} - \sin^2\theta]^{1/2}} \]

\[ = \frac{2\omega\cos^2\theta - 2\cos\theta \sqrt{\omega^2(K-\sin^2\theta) - i\sigma}}{(1-K)\omega + i\sigma} \] \hspace{1cm} (3.10)

and for the time-domain solutions:

\[ e_{x2}^{(a)}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{x2}(\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \]

\[ h_{y2}^{(a)}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_{y2}(\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \]

\[ h_{z2}^{(a)}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_{z2}(\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \] \hspace{1cm} (3.11)

We now proceed to the analytical reduction of \( e_{x2}^{(a)} \); we have:
\[ k_2 \cos \theta_2 = \frac{\omega}{c} \sqrt{K \sin^2 \theta + \frac{\sigma}{\omega \varepsilon_0}} \quad (3.12) \]

and hence:

\[ e^{(\alpha)}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A^+(\omega)}{\alpha + i\omega} e^{-\omega t - i\omega} \frac{z}{\omega^2 (K \sin^2 \theta) - i\omega} \frac{1}{\varepsilon_0} d\omega \quad (3.13) \]

The integral in the RHS of (3.13) is an improper integral on the real axis with a complex integrand. Based on the residue theorem we may reduce (3.13) further (Figure 3.2). We examine the integrand of (3.13) on the complex \( \omega \)-plane. It has a simple pole at \( \alpha \) (Figure 3.2) and a removable singularity (see Appendix B) at

\[ \omega = i \frac{\sigma}{\varepsilon_0 (k-1)} \quad (3.14) \]

This is because the numerator in (3.10) becomes zero when (3.14) is used for \( \omega \). The integrand possesses a branch cut \((0, i\xi)\) as shown in Figure 3.2, where as before:

\[ \xi \equiv \sigma/(\varepsilon_0 (K \sin^2 \theta)) \quad (3.15) \]

Next we take the integrand of (3.13) integrated along \( C_2 \) (the infinite upper semicircle); we let:

\[ \omega = R e^{i\phi} \quad \text{where:} \]

\[ 0 \leq \phi \leq \pi \]

\[ R \to \infty \]  

\[ \text{(3.16)} \]
We first observe that:

\[
\lim_{\omega \to \infty} A_2^+(\omega) = \frac{2(\cos \theta - \sqrt{K \sin^2 \theta}) \cos \theta}{1 - K} \tag{3.17}
\]

hence the limit of \( A_2^+ \) is finite. The rest of the integrand has a limit:

\[
\lim_{\omega \to \infty} \left( e^{i\omega t - \frac{Z}{C} \sqrt{\omega^2 (K \sin^2 \theta) - i \omega \varepsilon}} \right) / (\alpha + i\omega)
\]

\[
= \lim_{\omega \to \infty} e^{i\omega (t - \frac{Z}{C} \sqrt{K \sin^2 \theta})} / i\omega \tag{3.18}
\]

and \( i\omega = i\text{Re}i\phi = i\text{Re} \cos \phi - i\text{Re} \sin \phi \)

\[
\frac{d\omega}{i\omega} = \frac{i\text{Re}i\phi}{i\text{Re}i\phi} = d\phi
\]
hence the integral in question becomes:

\[
\frac{(\cos \theta - \sqrt{K \sin^2 \theta}) \cos \theta}{\pi (1-K)} \int_0^\pi (i \cos \phi - R \sin \phi) (t - \frac{Z}{C} \sqrt{K \sin^2 \theta}) \cos \theta \, d\phi
\]

It is readily seen from this form that the contour integration along \( C_2 \) yields zero when:

\[
t > \frac{Z}{v}
\]  

(3.19)

where:

\[
v = \frac{C}{\sqrt{K \sin^2 \theta}}
\]  

(3.20)

As will become clearer subsequently, the condition \( z < vt \) is physically equivalent to the fact that the speed of propagation of the EM pulse in the dissipative medium is finite; hence, as one may have anticipated, the solution will be identically zero for depths greater than the product of the speed of propagation by the time.

Under the condition (3.19) the integral in (3.13) will be equal to \( 2\pi i \) times the residue at \( i\alpha \) minus the integral along the branch cut as indicated in Figure 3.2. Hence we can write (3.13) as:
\[ e^{(\alpha)}_{x2}(z,t) = \]
\[ = \frac{2\cos\theta (\cos\theta - \sqrt{\alpha^2 (K\sin^2 \theta) - \frac{\sigma \alpha}{\varepsilon_0}})}{(1-K)\alpha + \frac{\sigma}{\varepsilon_0}} - \frac{at + \frac{z}{C} \sqrt{\alpha^2 (K\sin^2 \theta) - \frac{\sigma \alpha}{\varepsilon_0}}}{(1-K)\alpha + \frac{\sigma}{\varepsilon_0}} \]
\[ - \frac{1}{2\pi} \int \frac{2y\cos^2 \theta}{(a-y) [(1-K)y + \frac{\sigma}{\varepsilon_0}]} \cdot e^{-yt} \sin \left( \frac{z}{C} \sqrt{\frac{\sigma y - y^2 (K\sin^2 \theta)}{\varepsilon_0}} \right) \, dy \]
\[ + \frac{1}{\pi} \int \frac{2y\cos^2 \theta}{(a-y) [(1-K)y + \frac{\sigma}{\varepsilon_0}]} \cdot e^{-yt} \sin \left( \frac{z}{C} \sqrt{\frac{\sigma y - y^2 (K\sin^2 \theta)}{\varepsilon_0}} \right) \, dy \]
\[ - \frac{1}{\pi} \int \frac{2\cos\theta \sqrt{\frac{\sigma y - y^2 (K\sin^2 \theta)}{\varepsilon_0}}}{(a-y) [(1-K)y + \frac{\sigma}{\varepsilon_0}]} \cdot e^{-yt} \sin \left( \frac{z}{C} \sqrt{\frac{\sigma y - y^2 (K\sin^2 \theta)}{\varepsilon_0}} \right) \, dy \]
\[ \cdot \cos \left( \frac{z}{C} \sqrt{\frac{\sigma y - y^2 (K\sin^2 \theta)}{\varepsilon_0}} \right) \, dy \]

(3.21)
Equation (3.21) gives the field $e^{(\alpha)}_{x2}$ in the non-singular case, when

$$\alpha > \xi$$

(Figure 3.2); on the other hand when $\alpha \leq \xi$ we have:

$$e^{(\alpha)}_{x2}(z,t) = \frac{1}{\pi^2} \int_\alpha^\xi \frac{2y\cos^2 \theta}{(\alpha-y)[(1-K)y + \frac{\sigma}{\varepsilon_0}]} e^{-yt} \cdot$$

$$\cdot \sin\left(\frac{z}{\sqrt{\frac{\sigma y - y^2(1-K^2)\sin^2 \theta)}}\right) dy$$

$$+ \frac{2a\cos \theta}{[(1-K)a + \frac{\sigma}{\varepsilon_0}]} e^{-at} \cos\left(\frac{z}{\sqrt{\frac{\sigma a}{\varepsilon_0} - a^2(1-K^2)\sin^2 \theta)}}\right) -$$

$$- \frac{1}{\pi^2} \int_\alpha^\xi \frac{2\cos \theta}{\sqrt{\frac{\sigma y - y^2(1-K^2)\sin^2 \theta)}} e^{-yt} \cdot$$

$$\cdot \cos\left(\frac{z}{\sqrt{\frac{\sigma y - y^2(1-K^2)\sin^2 \theta)}}\right) dy$$

$$+ \frac{2\cos \theta}{\sqrt{\frac{\sigma a}{\varepsilon_0} - a^2(1-K^2)\sin^2 \theta)}} e^{-at} \sin\left(\frac{z}{\sqrt{\frac{\sigma a}{\varepsilon_0} - a^2(1-K^2)\sin^2 \theta)}}\right)$$

(3.22)

Next we compute the field $h^{(\alpha)}_{y2}$. We begin by noting that:

$$\frac{\cos^2 \theta}{n_2^2} = \frac{1}{n_0} \sqrt{k\sin^2 \theta} + \frac{\sigma}{i\omega \varepsilon_0}$$

(3.23)
hence we get:

\[ h^{(\alpha)}_{y2}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{A^+(\omega)}{\alpha + i\omega} \frac{1}{n_o} \left[ \sqrt{K \sin^2 \theta + \frac{\sigma}{i\omega \varepsilon_0}} \right] \left[ \frac{i\omega \sin \frac{Z}{C}}{\omega^2 (K \sin^2 \theta) - \frac{i\omega}{\varepsilon_0}} \right] e^{-\frac{i\omega t}{\varepsilon_0}} d\omega \]  

(3.24)

The strategy for analysis is here similar to the aforementioned. Again the integration along the infinite upper semicircle on the complex plane (Figure 3.2) vanishes when the proper condition is valid (3.19). Consequently we will have: \((\alpha > \xi)\)

\[ h^{(\alpha)}_{y2}(z,t) = \frac{2\cos \theta \left[ \frac{\sigma}{\varepsilon_0} - \alpha (K \sin^2 \theta) + \cos \theta \sqrt{\alpha^2 (K \sin^2 \theta) - \frac{\sigma^2}{\varepsilon_0}} \right]}{n_o \left[ (1-K)\alpha + \frac{\sigma}{\varepsilon_0} \right]} \]

\[ + \frac{\cos \theta}{n_o \pi} \int_{BC}^{+} \frac{i}{[(1-K)\alpha + \frac{\sigma}{\varepsilon_0}] (\alpha - y)} \left[ \sqrt{\frac{\sigma y - y^2 (K \sin^2 \theta)}{\varepsilon_0}} - \frac{\sigma y - y^2 (K \sin^2 \theta)}{\varepsilon_0} \right] e^{\frac{iy}{\varepsilon_0}} dy \]

\[ - \frac{\cos^2 \theta}{n_o \pi} \int_{BC}^{+} \frac{1}{(\alpha - y) [(1-K)y + \frac{\sigma}{\varepsilon_0}]} \left[ \sqrt{\frac{\sigma y - y^2 (K \sin^2 \theta)}{\varepsilon_0}} - \frac{\sigma y - y^2 (K \sin^2 \theta)}{\varepsilon_0} \right] e^{\frac{iy}{\varepsilon_0}} dy \]
\[
\frac{2\cos \theta \left[ \frac{\sigma}{\varepsilon_0} - \alpha (K \sin^2 \theta) + \cos \theta \sqrt{\alpha^2 (K \sin^2 \theta) - \frac{\sigma \alpha}{\varepsilon_0}} \right]}{n_0 \left[ (1-K) + \frac{\sigma}{\varepsilon_0} \right]} \\
- \alpha t + \frac{2}{C} \sqrt{\alpha^2 (K \sin^2 \theta) - \frac{\sigma \alpha}{\varepsilon_0}} e^{-yt} \sin \left( \frac{zy}{C \varepsilon_0} - y^2 (K \sin^2 \theta) \right) dy \\
+ \frac{2 \cos^2 \theta}{n_0} \xi \int_0^{\pi n_0} \frac{\sqrt{\alpha y - y^2 (K \sin^2 \theta)}}{\varepsilon_0} e^{-yt} \cos \left( \frac{zy}{C \varepsilon_0} - y^2 (K \sin^2 \theta) \right) dy
\]

(3.25)

The singular case $\alpha \leq \xi$ is:

\[
h_{y^2}(z,t) = - \frac{2 \cos \theta}{\pi n_0} \xi \int_0^{(\alpha-y)[(1-K) + \frac{\sigma}{\varepsilon_0}] } \frac{(K \sin^2 \theta) y - \frac{\sigma}{\varepsilon_0}}{(\alpha-y)[(1-K)y + \frac{\sigma}{\varepsilon_0}] } e^{-yt} \\
\sin \left( \frac{zy}{C \varepsilon_0} - y^2 (K \sin^2 \theta) \right) dy \\
- \frac{2 \cos \theta}{n_0} \frac{(K \sin^2 \theta) \alpha - \frac{\sigma}{\varepsilon_0}}{(1-K) \alpha + \frac{\sigma}{\varepsilon_0}} e^{-at} \cos \left( \frac{zy}{C \varepsilon_0} - \alpha^2 (K \sin^2 \theta) \right) \\
+ \frac{2 \cos^2 \theta}{n_0} \xi \int_0^{(\alpha-y)[(1-K)y + \frac{\sigma}{\varepsilon_0}] } \frac{\sqrt{\alpha y - y^2 (K \sin^2 \theta)}}{\varepsilon_0} e^{-yt} \right) .
\]
\[
- \cos \left( \sqrt{\frac{\sigma y}{\varepsilon_0} - y^2 (K \sin^2 \theta)} \right) \ dy \\
- \frac{2 \cos^2 \theta}{n_0} \left( \sqrt{\frac{\sigma a}{\varepsilon_0} - a^2 (K \sin^2 \theta)} \right) e^{-at} \sin \left( \sqrt{\frac{\sigma a}{\varepsilon_0} - a^2 (K \sin^2 \theta)} \right) 
\]

Finally we have that:

\[
h_z^{(a)} (z, t) = - \frac{\sin \theta}{n_0} e_{x2}^{(a)} (z, t) 
\]

Hence this last case is reduced to the relations (3.21) and (3.22) multiplied by the constant \(-\sin \theta / n_0\).

As an important check, we note that for \( z = 0 \) the fields for the dissipative medium as given above reduce to the expressions for the fields on the interface in agreement with previous studies (Papazoglou, 1972; Dudley et al., 1974).

The principal electric field \( e_{x2} \) is presented (see Figures 3 and 4 in Appendix C) for two cases. The first case (I) is that of wet earth with parameters \( \sigma = 3 \times 10^{-2} \), \( K = 30 \) and with an input pulse represented by Eqn. (1.1) with \( \alpha = 2.5 \times 10^8 \) and \( \beta = 3 \times 10^8 \). In the second case (II) we have considered moist earth with parameters \( \sigma = 2 \times 10^{-2} \) and \( K = 10 \) and as an input pulse one with \( \alpha = 5.9 \times 10^7 \) and \( \beta = 2.1 \times 10^8 \). The angle of incidence for the input has been chosen as 45°.
It is clear that the wetter earth attenuates the electric fields more rapidly; however, substantial signals (36% of the peak at the surface) penetrate below one meter of wet earth. The output exhibits a narrow positive lobe of large magnitude and a wide negative lobe of small magnitude such that the time average of the field is always zero. This is also true for the electric fields on the interface (the relatively small magnitude of the negative lobe makes it disappear under the scale in the corresponding Figure 1 in Appendix C). This has a straightforward theoretical proof and has been observed by other researchers in previous studies (King and Harrison, 1968).

The speed of propagation of the wave-front within the dissipative medium can be gathered by observing the moment of arrival of the transient at a particular depth.

3.3 The Effectiveness of the Analytical Results

The value of the results obtained thus far, through detailed analysis on the complex plane, is due to at least two important novel features: (1) the pulse distortion (branch cut contribution) is separated from the undistorted output (residue contribution) and (2) the results are promising in achieving higher efficiencies in the evaluation of the fields by means of computers, as compared with the method based on direct numerical Fourier inversion. In this connection, we may note that a direct numerical Fourier
inversion using a standard numerical integration scheme is extremely costly, because of the highly oscillatory nature of the integrands for high frequencies. In addition, the computations are performed on the field of complex numbers. An important improvement is achieved by using the Filon's integration formulas (Abramowitz and Stegun, 1964) for the evaluation of integrals involving a rapidly oscillating trigonometric type integrand-factor. Using the analytical results of our method, we obtain comparable answers with only a small fraction of the integration steps required in the Filon integration scheme (less than one-third and as low as one-eighth). These savings are attributable to the fact that the integrands involved in the analytical method are real and do not oscillate within their respective ranges of integration. As an illustration we present (see Figure 5, Appendix C), for the case of $\sigma = 0.02$ and $K = 10$ the integrand for the E-field on the interface (TE-case) with $\alpha = 2.5 \times 10^8$. An anomalous behavior is observed in the case of singular integrands, which we illustrate in the following figure (for $\alpha = 5.9 \times 10^7$; see Figure 6, Appendix C) but this is readily rectified using the scheme of folding the range of integration around the singularity (Figure 7, Appendix C). This method, known conceptually before (Davis and Rabinowitz, 1967), is explained in great detail in a recent study (Dudley et al., 1974). As we may last note, the treatment
of the singular integrands with the aforementioned technique is so effective that actually the integration steps required are then fewer than in the case of regular integrands for comparable accuracy. This is to be expected after a comparison of the graphs of the regular integrands and those of the rectified (singular) integrands (Appendix C, Figures 5 and 7).
TM-CASE TWO LAYERS

4.1 The Reflective Results

In a previous study (Papazoglou, 1972) we have dealt thoroughly with the solution of the two-layer problem for the fields on the interface; in another study (Dudley et al., 1974) we give a complete account of the reflective results and in addition discuss the concept of Brewster's angle in the time-domain.

For an input pulse represented by the causal double-exponential function of Eqns. (3.1) through (3.3) it is found that the field for $Z \leq 0$ (Figure 3.1) is:

$$h_{x1}(z,t) = -(1/n_0)f(t - \frac{Z}{C} \cos \theta) - b_1^-(t + \frac{Z}{C} \cos \theta))$$

(4.1)

where $C$ is the speed of light in free space, $\theta$ the angle of incidence (Figure 3.1) and:

$$b_1^- = (K_1^{(0)} - K_2^{(0)})/N$$

(4.2)

where for $\alpha > \xi \equiv \sigma/(\varepsilon_0 K \sin^2 \theta)$
\[ K_1^{(0)} = - \frac{\alpha^2(K \sin^2 \theta + K^2 \cos^2 \theta) + \frac{\sigma^2}{\varepsilon_o} \cos^2 \theta - \frac{\sigma \alpha}{\varepsilon_o} (l + 2K \cos^2 \theta)}{\left[ (K-1)\alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \right] \left[ (K+1-\sec^2 \theta) \alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \cos^2 \theta \right]} + \]

\[ + \frac{-2 \left( 2 \alpha \left( \sqrt{\frac{\sigma y}{\varepsilon_o} - y \left( K \sin^2 \theta \right)} - \frac{\sigma \alpha}{\varepsilon_o} \right) \right)}{\left[ (K-1)\alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \right] \left[ (K+1-\sec^2 \theta) \alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \cos^2 \theta \right]} e^{-\alpha t} + \]

\[ + \frac{2 \cos \theta}{\pi (K-1) \left[ (K+1) \cos^2 \theta - 1 \right]} \int_0^\xi \frac{(Ky - \frac{\sigma}{\varepsilon_o}) \sqrt{\frac{\sigma y}{\varepsilon_o} - y \left( K \sin^2 \theta \right)} \cdot e^{-yt}}{(\alpha-y) \left[ y - \frac{\sigma}{\varepsilon_o} \right] \left[ y - \frac{\sigma}{\varepsilon_o (K+1-\sec^2 \theta)} \right]} \cdot dy \quad (4.3) \]

and for \( \alpha \leq \xi \equiv \sigma/(\varepsilon_o (K \sin^2 \theta)) \)

\[ K_1^{(0)} = - \frac{\alpha^2(K \sin^2 \theta + K^2 \cos^2 \theta) + \frac{\sigma^2}{\varepsilon_o} \cos^2 \theta - \frac{\sigma \alpha}{\varepsilon_o} (l + 2K \cos^2 \theta)}{\left[ (K-1)\alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \right] \left[ (K+1-\sec^2 \theta) \alpha - \left(\frac{\sigma}{\varepsilon_o}\right) \cos^2 \theta \right]} \cdot e^{-\alpha t} + \]

\[ + \frac{2 \cos \theta}{\pi (K-1) \left[ (K+1) \cos^2 \theta - 1 \right]} \int_0^\xi \frac{(Ky - \frac{\sigma}{\varepsilon_o}) \sqrt{\frac{\sigma y}{\varepsilon_o} - y \left( K \sin^2 \theta \right)} \cdot e^{-yt}}{(\alpha-y) \left[ y - \frac{\sigma}{\varepsilon_o} \right] \left[ y - \frac{\sigma}{\varepsilon_o (K+1-\sec^2 \theta)} \right]} \cdot dy \quad (4.4) \]
$k_2^{(0)}$ is obtained from (4.3) and (4.4), accordingly, when $\beta$ is substituted for $\alpha$. As in the case of the TE wave (see Section 3.1) the transform is reduced to an algebraic term plus a real definite integral.

In Appendix C (Figure 8) curves are presented for the principal field $(h_x^1)$, for the TM case, at the interface. We have assumed the same parameters as in the TE case (see Section 3.1). Again we note that these curves are sufficient data to determine the fields at an arbitrary height above the interface ($Z < 0$); indeed we can apply (4.1) and add the input pulse to the reflection coefficient, each taken at the appropriate time according to the same relation. Note that, in contrast to the TE case, on the interface ($Z = 0$) the incident and reflected fields tend to add and produce resultant magnetic fields with peak magnitudes greater than unity. Some peculiar behavior is noted at $\theta = 84^\circ$. For this case, the reflection coefficient changes sign once, as a function of time. Thus the resulting total magnetic field $(h_x^1)$ shows an accentuated late-time behavior and a depressed early-time. In all, the result is a marked loss in rise time from 10 ns to 80 ns. This phenomenon is related to the phenomenon of Brewster's angle and is discussed by Dudley et al. (1974).
4.2 Fields Within the Dissipative Medium

Within the lossy half-space #2 (Z > 0) the fields are:

\[ H_{x2}(Z, \omega) = - \frac{B^+_2}{n_2} \exp(-izk_2 \cos \theta_2) \]

\[ E_{y2}(Z, \omega) = B^+_2 \cos \theta_2 \exp(-izk_2 \cos \theta_2) \]

\[ E_{z2}(Z, \omega) = -B^+_2 \sin \theta_2 \exp(-izk_2 \cos \theta_2) \] (4.5)

where, according to the first of (2.38), we have:

\[ B^+_2 = \frac{2}{n_2} \left( \frac{n_1}{n_2} \right) \cos \theta = \frac{2}{n_2} \left( \frac{n_1}{n_2} \right)^{1/2} \cos \theta + \left[ \frac{n_1}{n_2} \right] \sin^2 \theta \]

\[ = \frac{2 \cos \theta \sqrt{K + \frac{\sigma}{\omega \varepsilon_0}} \cdot \omega \cdot \left[ (Kw + \frac{\sigma}{\varepsilon_0}) \cos \theta - \sqrt{\omega^2 (K - \sin^2 \theta) - \frac{\sigma^2 \omega}{\varepsilon_0}} \right]}{\left[ ((K+1) \cos^2 \theta - 1) \omega - \frac{\sigma}{\varepsilon_0} \cos^2 \theta \right] \left[ (K-1) \omega - \frac{\sigma}{\varepsilon_0} \right]} \] (4.6)

and for the time-domain solutions:

\[ h_{x2}^{(\alpha)}(Z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_{x2}(Z,\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \]

\[ e_{y2}^{(\alpha)}(Z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{y2}(Z,\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \]

\[ e_{z2}^{(\alpha)}(Z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{z2}(Z,\omega) \frac{e^{i\omega t}}{\alpha + i\omega} d\omega \] (4.7)
Next, we proceed to the computation of $h_{x2}^{(\alpha)}$, as follows:

$$\frac{1}{n_2} = \frac{1}{n_o} \sqrt{K + \frac{\sigma}{i\omega \varepsilon_o}}$$

(4.8)

hence:

$$\frac{B_2^+}{n_2} = \frac{2\cos\theta(K\omega-i\varepsilon_o^\sigma)[(K\omega-i\varepsilon_o^\sigma)\cos\theta-\sqrt{\omega^2(K-\sin^2\theta)-i\varepsilon_o^\sigma\omega}]}{n_o[((K+1)\cos^2\theta-1)\omega-i\varepsilon_o^\sigma-\cos^2\theta][(K-1)\omega-i\varepsilon_o^\sigma]}$$

(4.9)

and:

$$h_{x2}^{(\alpha)} = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{B_{x2}^+}{\alpha+i\omega} e^{i\omega t-\frac{iZ}{C} \sqrt{\omega^2(K-\sin^2\theta)-i\varepsilon_o^\sigma\omega}} d\omega$$

(4.10)

An examination of the integrand of (4.10) on the complex $\omega$-plane reveals that it has a simple pole at $i\alpha$ (Figure 3.2) and two removable singularities at:

$$\omega = i\sigma/\varepsilon_o(K-1)$$

and

$$\omega = i\sigma/\varepsilon_o(K+1-\sec^2\theta)$$

The procedure that follows in integrating on the complex $\omega$-plane is analogous to the analysis for the TE case (see Section 3.2).

The condition for a vanishing line integral (of the integrand under consideration) along $C_2$ (Figure 3.2) is:

$$t > \frac{Z}{v}$$

(4.11)

where:

$$v = C/\sqrt{K-\sin^2\theta}$$

(4.12)
Eqns. (4.11) and (4.12) express the condition for non-zero fields within the dissipative medium. Under this condition the integral in (4.10) will be equal to $2\pi i$ times the residue at $ia$ minus the corresponding integral along the branch cut as indicated in Figure 3.2; hence, we can write (4.10) as:

$$h_{x2}^{(a)}(z,t) = -\frac{2\cos^2(K\alpha - \frac{g}{\varepsilon_o})[(K\alpha - \frac{g}{\varepsilon_o})\cos^2 - \sqrt{\alpha^2(K\sin^2 \theta) - \frac{g\alpha}{\varepsilon_o}}]}{n_o[((K+1)\cos^2 \theta - 1)\alpha - \frac{g}{\varepsilon_o}\cos^2 \theta][(K-1)\alpha - \frac{g}{\varepsilon_o}]} \cdot \exp(-at + \frac{z}{C}\sqrt{\alpha^2(K\sin^2 \theta) - \frac{g\alpha}{\varepsilon_o}}) +$$

$$+ \frac{1}{2\pi} \int_{BC+} \frac{2\cos^2 \theta(Ky - \frac{g}{\varepsilon_o})^2}{n_o(\alpha-y)[((K+1)\cos^2 \theta - 1)y - \frac{g}{\varepsilon_o}\cos^2 \theta][(K-1)y - \frac{g}{\varepsilon_o}]} \cdot \exp(-at + \frac{z}{C}\sqrt{\alpha^2(K\sin^2 \theta) - \frac{g\alpha}{\varepsilon_o}}) \cdot dy = -$$

$$\frac{2\cos^2(K\alpha - \frac{g}{\varepsilon_o})[(K\alpha - \frac{g}{\varepsilon_o})\cos^2 - \sqrt{\alpha^2(K\sin^2 \theta) - \frac{g\alpha}{\varepsilon_o}}]}{n_o[((K+1)\cos^2 \theta - 1)\alpha - \frac{g}{\varepsilon_o}\cos^2 \theta][(K-1)\alpha - \frac{g}{\varepsilon_o}]} \cdot \exp(-at + \frac{z}{C}\sqrt{\alpha^2(K\sin^2 \theta) - \frac{g\alpha}{\varepsilon_o}})$$
Equation (4.13) expresses the field $h_{x2}^{(a)}$ as a function of time and depth and is valid for the non-singular case, when

$$a > \xi$$

(4.14)

In case:

$$a \leq \xi \quad \text{(singular)}$$

(4.15)

we have:

$$h_{x2}^{(a)}(z,t) =$$

$$- \frac{2\cos^2 \theta}{\pi \epsilon_o} \int_0^\xi \frac{(Ky - \frac{\sigma}{\epsilon_o})^2 e^{-yt} \sin\left(\frac{z}{C} \sqrt{\frac{\sigma v}{\epsilon_o} - y^2 (K-\sin^2 \theta)}\right)}{(\alpha - y) [((K+1)\cos^2 \theta - 1)y - \frac{\sigma}{\epsilon_o} \cos \theta] [K(y - \frac{\sigma}{\epsilon_o} - y^2 (K-\sin^2 \theta))]} \, dy$$

$$+ \frac{2\cos \theta}{\pi \epsilon_o} \int_0^\xi \frac{(Ky - \frac{\sigma}{\epsilon_o})^2 e^{-yt} \cos\left(\frac{z}{C} \sqrt{\frac{\sigma v}{\epsilon_o} - y^2 (K-\sin^2 \theta)}\right)}{(\alpha - y) [((K+1)\cos^2 \theta - 1)y - \frac{\sigma}{\epsilon_o} \cos \theta] [K(y - \frac{\sigma}{\epsilon_o} - y^2 (K-\sin^2 \theta))]} \, dy$$

(4.13)
Next, we compute the field $e_y^{(\alpha)}(z,t)$; we note that:

$$\cos \theta_2 = \sqrt{K - \sin^2 \theta + \frac{\sigma}{\omega \varepsilon_0}} / \sqrt{K + \frac{\sigma}{\omega \varepsilon_0}}$$  (4.17)

therefore:

$$B_2^+ \cos \theta_2 =$$

$$2 \cos \theta \sqrt{\omega^2 (K - \sin^2 \theta) - i \frac{\sigma \omega}{\varepsilon_0}} [(K \omega - i \frac{\sigma}{\varepsilon_0}) \cos \theta - \sqrt{\omega^2 (K - \sin^2 \theta) - i \frac{\sigma \omega}{\varepsilon_0}}]$$

$$= \frac{2 \cos^2 \theta (K \omega - i \frac{\sigma}{\varepsilon_0}) \sqrt{\omega^2 (K - \sin^2 \theta) - i \frac{\sigma \omega}{\varepsilon_0}} - 2 \cos \theta (\omega^2 (K - \sin^2 \theta) - i \frac{\sigma \omega}{\varepsilon_0})}{[((K+1) \cos^2 \theta - 1) \omega - i \frac{\sigma \omega}{\varepsilon_0}][(K-1) \omega - i \frac{\sigma}{\varepsilon_0}]}$$  (4.18)
The analysis on the complex plane yields

\[
e_{y2} (z,t) = \frac{2\cos^2\theta (ka - \frac{a}{\varepsilon_0}) \sqrt{\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0}} - 2\cos\theta (\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0})}{[(k^2 + 1)\cos^2\theta - 1] \alpha - \frac{\sigma\cos^2\theta}{{\varepsilon_0}}][(k^2 - 1) \alpha - \frac{\sigma}{{\varepsilon_0}}]} \\
\cdot \exp(-at + \frac{z}{C} \sqrt{\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0}}) - \\
- \frac{1}{2\pi} \int_{BC} \\
\cdot \exp(-\pi + \frac{z}{C} \sqrt{\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0}}) \\
\cdot \frac{2\cos^2\theta (ky - \frac{a}{\varepsilon_0}) \sqrt{\frac{\sigma y}{\varepsilon_0} - y^2 (k^2 - \sin^2\theta)} e}{[(k^2 + 1)\cos^2\theta - 1] y - \frac{\sigma\cos^2\theta}{{\varepsilon_0}}][(k^2 - 1) y - \frac{\sigma}{{\varepsilon_0}}] (a-y)} \\
\cdot \frac{dy}{2\pi} \int_{BC} \\
\cdot \frac{2\cos\theta (\frac{\sigma y}{\varepsilon_0} - y^2 (k^2 - \sin^2\theta)) e}{[(k^2 + 1)\cos^2\theta - 1] y - \frac{\sigma\cos^2\theta}{{\varepsilon_0}}][(k^2 - 1) y - \frac{\sigma}{{\varepsilon_0}}] (a-y)} \\
\cdot \frac{dy}{2\pi} \int_{BC} \\
\cdot \frac{2\cos^2\theta (ka - \frac{a}{\varepsilon_0}) \sqrt{\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0}} - 2\cos\theta (\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0})}{[(k^2 + 1)\cos^2\theta - 1] \alpha - \frac{\sigma\cos^2\theta}{{\varepsilon_0}}][(k^2 - 1) \alpha - \frac{\sigma}{{\varepsilon_0}}]} \\
\cdot \exp(-at + \frac{z}{C} \sqrt{\alpha^2 (k^2 - \sin^2\theta) - \frac{\sigma\alpha}{\varepsilon_0}}) +
\[ + \frac{2\cos^2 \theta}{\pi} \int_0^\xi \frac{(Ky - \frac{\sigma}{\varepsilon_o}) \sqrt{\frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta)} e^{-yt} \cos \left( \frac{2}{C} \sqrt{\frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta)} \right)}{(\alpha-y) \left[ ((K+1)\cos^2 \theta - 1) \frac{\sigma}{\varepsilon_o} - \cos^2 \theta \right] \left[ (K-1) \frac{\sigma}{\varepsilon_o} \right]} \, dy \]

\[ + \frac{2\cos \theta}{\pi} \int_0^\xi \frac{\left[ \frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta) \right] e^{-yt} \sin \left( \frac{2}{C} \sqrt{\frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta)} \right)}{(\alpha-y) \left[ ((K+1)\cos^2 \theta - 1) \frac{\sigma}{\varepsilon_o} - \cos^2 \theta \right] \left[ (K-1) \frac{\sigma}{\varepsilon_o} \right]} \, dy \]

The case above is for \( \alpha > \xi \) (non-singular). For \( \alpha \leq \xi \) (singular), we get:

\[ e_{y2}^{(a)}(z,t) = \frac{2\cos^2 \theta}{\pi} \int_0^\xi \frac{(Ky - \frac{\sigma}{\varepsilon_o}) \sqrt{\frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta)} e^{-yt} \cos \left( \frac{2}{C} \sqrt{\frac{\sigma y}{\varepsilon_o} - y^2 (K-\sin^2 \theta)} \right)}{(\alpha-y) \left[ ((K+1)\cos^2 \theta - 1) \frac{\sigma}{\varepsilon_o} - \cos^2 \theta \right] \left[ (K-1) \frac{\sigma}{\varepsilon_o} \right]} \, dy \]

\[ - \frac{2\cos^2 \theta}{\pi} \left[ \left[ ((K+1)\cos^2 \theta - 1) \frac{\sigma}{\varepsilon_o} - \cos^2 \theta \right] \left[ (K-1) \frac{\sigma}{\varepsilon_o} \right] \right] \]

\[ \cdot \sin \left( \frac{2}{C} \sqrt{\frac{\sigma y}{\varepsilon_o} - \alpha^2 (K-\sin^2 \theta)} \right) \]

\[ (4.19) \]
\[ + \frac{2\cos \theta}{\pi} \xi \int_{0}^{\infty} e^{-y} \sin \left( \sqrt{\frac{\sigma y - y^2 (K-\sin^2 \theta)}{\epsilon_0}} \right) dy \]

\[ + \frac{2\cos \theta}{\pi} \xi \int_{0}^{\infty} e^{-y} \cos \left( \sqrt{\frac{\sigma y - y^2 (K-\sin^2 \theta)}{\epsilon_0}} \right) dy \]

\[ \frac{[\alpha - \alpha^2 (K-\sin^2 \theta)] e^{-\alpha t} \cos \left( \sqrt{\frac{\sigma y - y^2 (K-\sin^2 \theta)}{\epsilon_0}} \right) }{\left[ ((K+1)\cos^2 \theta - 1)\alpha - \frac{\sigma}{\epsilon_0} \cos^2 \theta \right] \left[ (K-1)\alpha - \frac{\sigma}{\epsilon_0} \right] } \]

(4.20)

Last, we compute the field \( e_{z2}^{(\alpha)}(z,t) \); we note that:

\[ \sin \theta_2 = \frac{k_1}{k_2} \sin \theta = \frac{n_2}{n_0} \sin \theta = \frac{\sin \theta}{\sqrt{K + \frac{\sigma}{\omega \epsilon_0}}} \]

(4.21)

therefore:

\[ B_2^+ \sin \theta_2 = \frac{\omega \sin \theta \left[ (K\omega - i\sigma) \cos \theta - \sqrt{\omega^2 (K-\sin^2 \theta) - i\sigma \omega} \right] }{\left[ ((K+1)\cos^2 \theta - 1)\omega - i\sigma \epsilon_0 \cos^2 \theta \right] \left[ (K-1)\omega - i\sigma \epsilon_0 \right] } \]

(4.22)

Consequently: for \( \alpha > \xi \) (non-singular)

\[ e_{z2}^{(\alpha)}(z,t) = \]

\[ \frac{\alpha \sin 2\theta \left[ (K\alpha - \frac{\sigma}{\epsilon_0}) \cos \theta - \sqrt{\alpha^2 (K-\sin^2 \theta) - \frac{\sigma \alpha}{\epsilon_0}} \right] }{\left[ ((K+1)\cos^2 \theta - 1)\alpha - \frac{\sigma}{\epsilon_0} \cos^2 \theta \right] \left[ (K-1)\alpha - \frac{\sigma}{\epsilon_0} \right] } \]

\[ -at + \frac{Z}{C} \sqrt{\alpha^2 (K-\sin^2 \theta) - \frac{\sigma \alpha}{\epsilon_0}} \]

\[ e^\ldots \]
\[ y(Ky - \frac{\sigma}{\varepsilon_0})\cos\theta \sin\theta \exp\left[-yt - \frac{\beta}{C} \sqrt{\frac{\sigma_y}{\varepsilon_0}} - y^2(K\sin^2\theta)\right] \]

\[ \frac{1}{2\pi} \int \frac{1}{i} \frac{\sqrt{\sigma_y} - y^2(K\sin^2\theta) \exp\left[-yt - \frac{\beta}{C} \sqrt{\frac{\sigma_y}{\varepsilon_0}} - y^2(K\sin^2\theta)\right]}{(\alpha - y) [(K+1)\cos^2\theta - 1] y - \frac{\sigma}{\varepsilon_0} \cos^2\theta] [(K-1) y - \frac{\sigma}{\varepsilon_0}] (\alpha - y) } \, dy \]

\[ \frac{\sin \theta \sqrt{\sigma_y} - y^2(K\sin^2\theta) \exp\left[-yt - \frac{\beta}{C} \sqrt{\frac{\sigma_y}{\varepsilon_0}} - y^2(K\sin^2\theta)\right]}{(\alpha - y) [(K+1)\cos^2\theta - 1] y - \frac{\sigma}{\varepsilon_0} \cos^2\theta] [(K-1) y - \frac{\sigma}{\varepsilon_0}] } \, dy \]

\[ \frac{\cos \theta \sin \theta}{\pi} \int \frac{y(Ky - \frac{\sigma}{\varepsilon_0}) e^{-yt} \sin \left(\frac{\beta}{C} \sqrt{\frac{\sigma_y}{\varepsilon_0}} - y^2(K\sin^2\theta)\right)}{(\alpha - y) [(K+1)\cos^2\theta - 1] y - \frac{\sigma}{\varepsilon_0} \cos^2\theta] [(K-1) y - \frac{\sigma}{\varepsilon_0}] } \, dy \]

\[ \frac{\sin \theta}{\pi} \int \frac{y \sqrt{\sigma_y} - y^2(K\sin^2\theta) \exp\left[-yt \cos \left(\frac{\beta}{C} \sqrt{\frac{\sigma_y}{\varepsilon_0}} - y^2(K\sin^2\theta)\right)\right]}{(\alpha - y) [(K+1)\cos^2\theta - 1] y - \frac{\sigma}{\varepsilon_0} \cos^2\theta] [(K-1) y - \frac{\sigma}{\varepsilon_0}] } \, dy \]

(4.23)

and for \( \alpha \leq \xi \) (singular), we get:
In this case, as in the TE case (see Section 3.2), we let \( Z = 0 \) and the results for medium \#2 reduce to the expressions for the fields on the interface as derived earlier (Papazoglou, 1972; Dudley et al., 1974).

At the end (see Figures 9 and 10, Appendix C) the principal magnetic field \( (h_{x2}) \) is presented for the two cases already considered in the TE case (Chapter 3). Again we may observe that significant signals (40% of the peak at the surface) penetrate below one meter of earth.
Note that in this case the magnetic field has negative values throughout. This again has a theoretical foundation with a proof analogous to that for the electric field. This is clearly seen in the case of fields on the interface (see Figure 8, Appendix C).

The times of arrival of the transients at various depths coincide with those for the TE case, as does the speed of propagation of the wave-front.

In Appendix C (Figure 11) we present the principal magnetic field at depths 10 cm and 1 m where we have taken \( \theta = 84^\circ \) and all other quantities as in case II. The Brewster's angle \( \theta_B \) is given by the relation

\[
\cos\theta_B = \left( K + \sigma/i\omega\varepsilon_0 + 1 \right)^{-1/2}
\]

and for high frequencies such that

\[
\omega \gg \omega_{cr}
\]

\[
\omega_{cr} \equiv \frac{\sigma}{K\varepsilon_0}
\]

we can neglect the contribution of the conductivity currents and get:

\[
\cos\theta_B \bigg|_{\omega \gg \omega_{cr}} \approx (K + 1)^{-1/2}
\]

which in this case (\( K = 10 \)) gives: \( \theta_B = 72.5^\circ \). We now note that the input pulse (case II) has a rise time, measured
from the 10% to the 90% point, of 4.5 ns and a fall time, measured from the origin to the 1/e-fall point, of 31 ns. In comparison for a 45° angle of incidence the field at 10 cm within the lossy space exhibits a 5.5-ns rise time and a 34-ns fall time and the field at 1 m exhibits a 7.5-ns rise time and a 60-ns fall time. Last, for an 84° angle of incidence the field at 10 cm within the dissipative medium has a 7.5 ns rise time and a 60 ns fall time, and the field at 1 m shows an 11 ns rise time and an 100 ns fall time. This behavior is attributable to two distinct phenomena: (1) the broadening and flattening of the field curves with depth into the dissipative medium (for Gaussian type inputs and the case of electric fields, this trend has been observed by King and Harrison [1968]); and (2) the Brewster's angle effect according to which there is a selective absorption of the high frequencies and consequently a depressed early time behavior for large angles of incidence. This concept was discussed in detail for the case of the fields on the interface by Dudley et al. (1974).
CHAPTER 5

TE CASE THREE LAYERS

5.1 Introduction

The reflection coefficient $A_1^-$ is given in a series representation by:

$$A_1^- = \lambda_{12} + (\lambda_{12}^2 - 1) \sum_{m=1}^{\infty} (-1)^m \lambda_{12}^{m-1} \lambda_{23}^m e^{-i2mdk_2\cos \theta_2}$$

(5.1)

For an input pulse given by Eqns. (3.1), (3.2), and (3.3) the principal electric field ($e_{x1}$) on the surface of the layered dissipative medium is:

$$e_{x1}(t) = f(t) + a_1^-(t)$$

(5.2)

while the field within the empty space is given by (3.4).

$$a_1^- = (I_1 - I_2)/N$$

(5.3)

where:

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_1^-}{a+i\omega} e^{i\omega t} d\omega$$

(5.4)

and $I_2$ is obtained from (5.4) by substituting $\beta$ for $\alpha$. In (5.4) we then substitute the RHS of (5.1) for $A_1^-$. The Fourier inversion in (5.4) is then carried out term by term. The first term reproduces the reflective results in the two layer case (see Section 3.1). The
general term contains (in the time domain) as a factor a unit step function delayed by an integer multiple of the time for the wave front to travel from interface #1 to interface #2 and back; this gives rise to a straightforward physical interpretation of the series (5.1) (see Section 2.4). Because of these unit step functions the representation of the field \( e_{x1} \) in the time domain contains only a finite number of terms.

5.2 Inversion of the mth Term of the Series

For convenience of notation we write (5.4) as:

\[
I_1^{(m)} = \sum_{m=0}^{\infty} I_1^{(m)}
\]

(5.5)

where \( I_1^{(m)} \) represents the Fourier-inverted mth term of (5.4). Thus \( I_1^{(0)} \) will be:

\[
I_1^{(0)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\lambda_{12}}{\alpha+i\omega} e^{i\omega t} d\omega
\]

(5.6)

and the mth term will be:

\[
I_1^{(m)} = \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\alpha+i\omega} \left( \frac{\lambda_{12}}{\lambda_{12}-1} \right)^m \lambda_{12}^{m-1} \lambda_{23}^{-m} e^{-i2mdk_2\cos\theta} d\omega
\]

(5.7)

where the explicit frequency dependences are:

\[
\lambda_{12}(\omega) = \frac{(K_{r2}+\cos2\theta)\omega - i\frac{\sigma_{2}}{\varepsilon_0} - 2\cos\theta \sqrt{\omega^2 (K_{r2} - \sin^2\theta) - i\frac{\sigma_{2}\omega}{\varepsilon_0}}}{(1-K_{r2})\omega + i\frac{\sigma_{2}}{\varepsilon_0}}
\]

(5.8)
\[ \lambda_{23}(\omega) = \frac{\omega^2(K_{r2} + K_{r3} - 2\sin^2\theta)}{\omega^2(K_{r2} - K_{r3}) - i\frac{\sigma_2 - \sigma_3}{\varepsilon_o} \omega} - i\frac{\sigma_2 + \sigma_3}{\varepsilon_o} \frac{\omega - 2\sqrt{\omega^2(K_{r2} - \sin^2\theta) - i\frac{\sigma_2}{\varepsilon_o} \omega}}{\sqrt{\omega^2(K_{r3} - \sin^2\theta) - i\frac{\sigma_3}{\varepsilon_o} \omega}} \]

where:

\[ K_{r2} \equiv \frac{\varepsilon_2}{\varepsilon_o}, \quad K_{r3} \equiv \frac{\varepsilon_3}{\varepsilon_o} \quad (5.10) \]

In (5.7) the only non-removable singularity is at \( \omega = i\alpha \), since both \( \omega = i\sigma_2/(\varepsilon_o(K_{r2} - 1)) \) and \( \omega = i(\sigma_2 - \sigma_3)/(\varepsilon_o(K_{r2} - K_{r3})) \) are removable singularities.

For simplicity of notation we now define the following auxiliary functions:

\[ S_2(\omega) \equiv \sqrt{\omega^2(K_{r2} - \sin^2\theta) - i\frac{\sigma_2}{\varepsilon_o} \omega} \quad (5.11) \]

\[ S_3(\omega) \equiv \sqrt{\omega^2(K_{r3} - \sin^2\theta) - i\frac{\sigma_3}{\varepsilon_o} \omega} \quad (5.12) \]

\[ A_2(\omega) \equiv (K_{r2} + \cos2\theta)\omega - i\frac{\sigma_2}{\varepsilon_o} \quad (5.13) \]

\[ B_2(\omega) \equiv (1 - K_{r2})\omega + i\frac{\sigma_2}{\varepsilon_o} \quad (5.14) \]
\[ A_{23}(\omega) = \omega^2 (K_{r2} - K_{r3}) - 2\sin^2 \theta - \frac{i\sigma_2 + \sigma_3}{\varepsilon_0} \omega \] (5.15)

\[ B_{23}(\omega) = \omega^2 (K_{r2} + K_{r3}) - \frac{i\sigma_2 - \sigma_3}{\varepsilon_0} \omega \] (5.16)

then we may rewrite (5.8) and (5.9) as:

\[ \lambda_{12}(\omega) = \frac{A_2(\omega) - 2\cos\theta S_2(\omega)}{B_2(\omega)} \] (5.17)

\[ \lambda_{23}(\omega) = \frac{A_{23}(\omega) - 2S_2(\omega)S_3(\omega)}{B_{23}(\omega)} \] (5.18)

We note that the functions defined by (5.11) and (5.12) possess branch cuts while the functions defined by (5.13) through (5.16) are analytic functions in the whole complex \( \omega \)-plane. Further we note that any even power of \( S_2(\omega) \) and/or \( S_3(\omega) \) constitutes an analytic function while an odd power will possess a branch cut. Thus, in analyzing the integral in (5.7), by using the binomial expansion, we separate the integrand into four distinct terms: one which is an analytic function and three others that are products of analytic functions multiplied by, respectively, the functions \( S_2(\omega) \), \( S_3(\omega) \) and \( S_2 \cdot S_3 \) that possess branch cuts. In this fashion the result can be expressed (as will be seen next) in terms of only real quantities and known real functions.
Accordingly we find that:

\[
(\lambda_{12}^2 - 1)\lambda_{12}^{m-1}\lambda_{23} = \Sigma_0^{(m)}(\omega) + \Sigma_2^{(m)}(\omega)S_2(\omega) + \Sigma_3^{(m)}(\omega)S_3(\omega) + \\
+ \Sigma_{23}^{(m)}(\omega)S_2(\omega)S_3(\omega)
\]

where \(\Sigma_0^{(m)}(\omega)\), \(\Sigma_2^{(m)}(\omega)\), \(\Sigma_3^{(m)}(\omega)\) and \(\Sigma_{23}^{(m)}(\omega)\) are analytic functions given explicitly in terms of the frequency \(\omega\) as follows:

\[
\Sigma_0^{(m)}(\omega) = \frac{X_1^{(m)}(\omega)Y_1^{(m)}(\omega)}{B_2^{m+1}B_2^{m+1}}
\]

\[
\Sigma_2^{(m)}(\omega) = -2\cos\theta \frac{X_2^{(m)}(\omega)Y_1^{(m)}(\omega)}{B_2^{m+1}B_2^{m+1}}
\]

\[
\Sigma_3^{(m)}(\omega) = 4\cos\theta S_2^{(m)} \frac{X_2^{(m)}(\omega)Y_2^{(m)}(\omega)}{B_2^{m+1}B_2^{m+1}}
\]

\[
\Sigma_{23}^{(m)}(\omega) = -2 \frac{X_1^{(m)}(\omega)Y_2^{(m)}(\omega)}{B_2^{m+1}B_2^{m+1}}
\]

The auxiliary \(X\) and \(Y\)-functions are:

\[
X_1^{(m)}(\omega) = \sum_{p=0}^{2[m+1]} (m+1)\lambda_2^{m+1-p}2^p\cos^p\theta S_2^p - \\
- B_2^{m+1} \sum_{p=0}^{2[m-1]} (m-1)\lambda_2^{m-1-p}2^p\cos^p\theta S_2^p
\]
\[ x_2^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m}{2}\right]} (m+1) A_2^{m-p} p_2^p \cos \theta S_2^p - \]
\[ - B_2^2 \sum_{p=0}^{2\left[\frac{m-2}{2}\right]} (m-1) A_2^{m-2-p} p_2^p \cos \theta S_2^p \]
\[ y_1^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m}{2}\right]} (m) A_2^{m-p} p_2^p S_2^p S_3 \]
\[ y_2^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m-1}{2}\right]} (m) A_2^{m-1-p} p_2^p S_2^p S_3 \]

where the summation index \( p \) takes only even non-negative values \((p = 0, 2, 4, \ldots)\) and \( \lfloor x \rfloor \equiv \text{integer part of } x \).

We now combine Eqns. (5.7) and (5.19) to obtain:

\[ I_1^{(m)} = \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \sum_{0}^{\Sigma^{(m)}(\omega)} i\omega t - 2mdk_2 \cos \theta_2 \] 
\[ e \] 
\[ + \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \sum_{2}^{\Sigma^{(m)}(\omega)} i\omega t - 2mdk_2 \cos \theta_2 
S_2(\omega) e \] 
\[ + \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \sum_{3}^{\Sigma^{(m)}(\omega)} i\omega t - 2mdk_2 \cos \theta_2 
S_3(\omega) e \] 
\[ + \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \sum_{23}^{\Sigma^{(m)}(\omega)} i\omega t - 2mdk_2 \cos \theta_2 
S_2(\omega) S_3(\omega) e \] 
\[ 2 \] 
\( 5.22 \)
In the above expressions the only non-removable singularity is at \( \omega = i\alpha \), because the singularities carried by the functions \( B_2, B_{23} \), as noted before, are removable. Equation (5.22) permits a straightforward calculation for the general term \( I^{(m)}_1 \). Indeed we close the contour in the complex \( \omega \)-plane with a semicircle of infinite radius \( (0 \leq \theta \leq \pi) \) and require that:

\[
\tau > 2m \frac{d}{v_2}, \quad v_2 = \frac{C}{\sqrt{K_{r2} \sin^2 \theta}}
\]

(5.23)

Then the integrals along path \( C_2 \) (Figure 3.2) vanish, therefore the result appears as the algebraic sum of a residue contribution (for the non-singular case) and a branch cut contribution. The branch cut contribution is given for all the possible cases in a systematized form in Appendix B. Here we will illustrate the final result for the non-singular case:

\[
\alpha > \max(\xi_2, \xi_3)
\]

(5.24)

where:

\[
\xi_2 \equiv \sigma_2 / (\varepsilon_0 (K_{r2} \sin^2 \theta))
\]

\[
\xi_3 \equiv \sigma_3 / (\varepsilon_0 (K_{r3} \sin^2 \theta))
\]

(5.25)
Then:

\[ I_1^{(m)} = (-1)^m \left[ \Sigma_0^{(m)} (i\alpha) + i \Sigma_2^{(m)} (i\alpha) \right] \sqrt{\alpha^2 (K_{r2} - \sin^2 \theta) - \frac{\sigma_2^2}{\varepsilon_0}} \]

\[ + i \Sigma_3^{(m)} (i\alpha) \cdot \sqrt{\alpha^2 (K_{r3} - \sin^2 \theta) - \frac{\sigma_3^2}{\varepsilon_0}} \cdot e \]

\[ - \Sigma_{23}^{(m)} (i\alpha) \sqrt{\alpha^2 (K_{r2} - \sin^2 \theta) - \frac{\sigma_2^2}{\varepsilon_0}} \cdot \sqrt{\alpha^2 (K_{r3} - \sin^2 \theta) - \frac{\sigma_3^2}{\varepsilon_0}} \]

\[ - \alpha t + \frac{2md}{C} \sqrt{\alpha^2 (K_{r2} - \sin^2 \theta) - \frac{\sigma_2^2}{\varepsilon_0}} \cdot e \]

\[ + \frac{(-1)^{m+1}}{2\pi} \int_{\alpha-y}^{\alpha+y} \sigma_0^{(m)} (iy) \, dy \quad \text{(BC)} \]

\[ + \frac{(-1)^{m+1}}{2\pi} \int_{\alpha-y}^{\alpha+y} i \Sigma_2^{(m)} (iy) \sqrt{\frac{\sigma_2 y}{\varepsilon_0} - y^2 (K_{r2} - \sin^2 \theta)} \, dy \quad \text{(BC)} \]

\[ + \frac{(-1)^{m+1}}{2\pi} \int_{\alpha-y}^{\alpha+y} \Sigma_3^{(m)} (iy) \sqrt{\frac{\sigma_3 y}{\varepsilon_0} - y^2 (K_{r3} - \sin^2 \theta)} \, dy \quad \text{(BC)} \]

\[ - \alpha t - \frac{2md}{C} \sqrt{\frac{\sigma_2 y}{\varepsilon_0} - y^2 (K_{r2} - \sin^2 \theta)} \cdot e \]

\[ + \frac{(-1)^{m+1}}{2\pi} \int_{\alpha-y}^{\alpha+y} i \Sigma_3^{(m)} (iy) \sqrt{\frac{\sigma_3 y}{\varepsilon_0} - y^2 (K_{r3} - \sin^2 \theta)} \, dy \quad \text{(BC)} \]
\[ + \frac{(-1)^{m+1}}{2\pi} \int_{BC^\dagger} \Sigma_0^{(m)}(iy) \frac{\sqrt{\frac{\sigma_2Y}{\epsilon_0} - Y^2(K_{r2} - \sin^2\theta)}}{\alpha - y} \sqrt{\frac{\sigma_3Y}{\epsilon_0} - y^2(K_{r3} - \sin^2\theta)} e^{-yt} \frac{2md}{C} \sqrt{\frac{\sigma_2Y}{\epsilon_0} - y^2(K_{r2} - \sin^2\theta)} \, dy \]

We now call, for simplicity:

\[ \phi_2(y) \equiv \sqrt{\frac{\sigma_2Y}{\epsilon_0} - y^2(K_{r2} - \sin^2\theta)}; \quad \phi_2^*(y) \equiv \sqrt{y^2(K_{r2} - \sin^2\theta) - \frac{\sigma_2Y}{\epsilon_0}} \]

\[ \phi_3(y) \equiv \sqrt{\frac{\sigma_3Y}{\epsilon_0} - y^2(K_{r3} - \sin^2\theta)}; \quad \phi_3^*(y) \equiv \sqrt{y^2(K_{r3} - \sin^2\theta) - \frac{\sigma_3Y}{\epsilon_0}} \]

and applying the relations in Appendix B, we get for \( \xi_2 > \xi_3 \):

\[ I_1^{(m)} = (-1)^m \left[ \Sigma_0^{(m)}(i\alpha) + i\Sigma_2^{(m)}(i\alpha)\phi_2^*(\alpha) + i\Sigma_3^{(m)}(i\alpha)\phi_3^*(\alpha) \right] \exp(-\alpha t + \frac{2md}{C}\phi_2^*(\alpha)) \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_2} \frac{\Sigma_0^{(m)}(iy)}{\alpha - y} e^{-yt}\sin\left(\frac{2md}{C}\phi_2(y)\right) \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_2} \frac{i\Sigma_2^{(m)}(iy)}{\alpha - y} \phi_2(y) e^{-yt}\cos\left(\frac{2md}{C}\phi_2(y)\right) \, dy \]
\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_3} \frac{i \Sigma_3(iy)}{\alpha - y} \phi_3(y) e^{-yt \cos \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_{\xi_3}^{\xi_2} \frac{i \Sigma_3(iy) \phi_2^*(y)}{\alpha - y} e^{-yt \sin \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_3} \frac{\Sigma_2(iy)}{\alpha - y} \phi_2(y) \phi_3(y) e^{-yt \sin \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ - \frac{(-1)^m}{\pi} \int_{\xi_3}^{\xi_2} \frac{\Sigma_2(iy) \phi_2^*(y) \phi_3^*(y)}{\alpha - y} e^{-yt \cos \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \quad (5.28) \]

For \( \xi_3 > \xi_2 \), we get:

\[ I_1^{(m)} = (-1)^m [\Sigma_0^{(m)}(ia) + i \Sigma_2^{(m)}(ia) \phi_2^*(\alpha) + i \Sigma_3^{(m)}(ia) \phi_3^*(\alpha) \] 

\[ - \Sigma_2^{(m)}(ia) \phi_3^*(\alpha) \phi_3^*(\alpha)] \exp(-\alpha t + \frac{2m\theta}{C} \phi_2^*(\alpha)) \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_2} \frac{\Sigma_0^{(m)}(iy)}{\alpha - y} e^{-yt \sin \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_2} \frac{i \Sigma_2(iy) \phi_2(y)}{\alpha - y} e^{-yt \cos \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_0^{\xi_2} \frac{i \Sigma_3(iy) \phi_3(y)}{\alpha - y} e^{-yt \cos \left( \frac{2m\theta}{C} \phi_2(y) \right)} \, dy \]

\[ + \frac{(-1)^m}{\pi} \int_{\xi_2}^{\xi_3} \frac{i \Sigma_3(iy) \phi_3(y)}{\alpha - y} e^{-yt + \frac{2m\theta}{C} \phi_2^*(y)} \, dy \]
\begin{align*}
+ \frac{(-1)^m}{\pi} & \int_0^\xi_2 \Sigma_{23}^{(m)}(i\gamma) \frac{\phi_2(y) \phi_3(y) e^{-yt} \sin(\frac{2md}{C}\phi_2(y))}{\alpha-y} \ dy \\
- \frac{(-1)^m}{\pi} & \int_\xi_2^{\xi_3} \Sigma_{23}^{(m)}(i\gamma)^* \frac{\phi_2(y) \phi_3(y) e^{-yt} + \frac{2md}{C}\phi_2(y)^*}{\alpha-y} \ dy \quad (5.29)
\end{align*}

All the above expressions (integrands, explicit terms, etc.) are real.

5.3 Application to the Term for \( m = 1 \)

For the initial times, such that:

\[ t < \frac{2d}{v_2} \quad (5.30) \]

when the only non-zero term in \( I_1^{(m)} \) is the one for \( m = 0 \), the response of the dissipative medium is that due to only layer \#2. The first reflection of the transient from the surface of medium \#3 that is detected on the interface \#1 after

\[ t \geq \frac{2d}{v_2} \quad (5.31) \]

bears the pertinent information about medium \#3, that may be used for remote sensing of medium \#3, as, for example, in geophysical probing.

We now begin the computation of \( I_1^{(1)} \) by first estimating the \( X \) and \( Y \)-functions (Section 5.2).
\[ x_1^{(1)} = A_2^2 - B_2^2 - 4S_2^2 \cos^2 \theta \]  
(5.32)

\[ x_2^{(1)} = 2A_2 \]  
(5.33)

\[ y_1^{(1)} = A_{23} \]  
(5.34)

\[ y_2^{(1)} = 1 \]  
(5.35)

Hence we obtain for the \( \Sigma \)-functions:

\[ \Sigma_0^{(1)} = \frac{A_{23} (A_2^2 - B_2^2 + 4S_2^2 \cos^2 \theta)}{B_2^2 B_{23}} \]  
(5.36)

\[ \Sigma_2^{(1)} = - \frac{4A_2 A_{23} \cos \theta}{B_2^2 B_{23}} \]  
(5.37)

\[ \Sigma_3^{(1)} = \frac{8A_2 S_2^2 \cos \theta}{B_2^2 B_{23}} \]  
(5.38)

\[ \Sigma_{23}^{(1)} = - \frac{2(A_2^2 - B_2^2 + 4S_2^2 \cos^2 \theta)}{B_2^2 B_{23}} \]  
(5.39)

In order now to apply the formulas (5.28) and (5.29) (Section 5.2), we estimate the \( \Sigma \)-functions at \( ix \), with \( x \) real, by using the formulas (5.36) through (5.39):

\[ A_2(ix) = i[(K r_2 + \cos \theta)x - \frac{\sigma_2}{\varepsilon_0}] \]  
(5.40)
\[ B_2(ix) = i[(1-Kr_2)x + \frac{\sigma_2}{\epsilon_0}] \] (5.41)

\[ A_{23}(ix) = \frac{\sigma_2+\sigma_3}{\epsilon_0}x-x^2(Kr_2+Kr_3-2\sin^2\theta) \] (5.42)

\[ B_{23}(ix) = \frac{\sigma_2-\sigma_3}{\epsilon_0}x-x^2(Kr_2-Br_3) \] (5.43)

\[ S_2^2(ix) = \frac{\sigma_2}{\epsilon_0}x-x^2(Kr_2-\sin^2\theta) \] (5.44)

Therefore:

\[ \Sigma_0^{(1)}(ix) = \]

\[ \frac{\frac{\sigma_2+\sigma_3}{\epsilon_0} - x(Kr_2+Kr_3-2\sin^2\theta)}{\frac{[(1-Kr_2)x + \frac{\sigma_2}{\epsilon_0}]^2}{\epsilon_0}} \]

\[ \cdot \frac{[(1-Kr_2)x + \frac{\sigma_2}{\epsilon_0}] \cdot [x(Kr_2-Kr_3) - \frac{\sigma_2-\sigma_3}{\epsilon_0}]}{\epsilon_0} \]

\[ - \frac{[(Kr_2+\cos\theta)x - \frac{\sigma_2}{\epsilon_0}]^2 + 4[\frac{\sigma_2}{\epsilon_0} - x^2(Kr_2-\sin^2\theta)]\cos^2\theta}{\epsilon_0} \] (5.45)

\[ iE_2^{(1)}(ix) = \]

\[ \frac{4[(Kr_2+\cos\theta)x - \frac{\sigma_2}{\epsilon_0}] \cdot [\frac{\sigma_2+\sigma_3}{\epsilon_0} - x(Kr_2+Kr_3-2\sin^2\theta)]\cos\theta}{\epsilon_0} \]

\[ = \frac{\frac{\sigma_2}{\epsilon_0}x-x^2(Kr_2-Kr_3) - \frac{\sigma_2-\sigma_3}{\epsilon_0}}{\epsilon_0} \] (5.46)
\[ i\Sigma_3^{(1)}(ix) = \frac{8 \left[ \frac{\sigma_2}{\varepsilon_o} - x(K_{r2} \cos \theta) \right] \left[ \frac{\sigma_2}{\varepsilon_o} - x(K_{r2} \sin^2 \theta) \right] \cos \theta}{\left[ (1 - K_{r2})x + \frac{\sigma_2}{\varepsilon_o} \right] \cdot \left[ x(K_{r2} - K_3) - \frac{\sigma_2 - \sigma_3}{\varepsilon_o} \right]} \]

\[ \Sigma_{23}^{(1)}(ix) = \]

\[ 2 \cdot \left[ \frac{\sigma_2^2}{\varepsilon_o^2} - \left[ (1 - K_{r2})x + \frac{\sigma_2}{\varepsilon_o} \right]^2 \right] \]

\[ \left[ (1 - K_{r2})x + \frac{\sigma_2}{\varepsilon_o} \right] \cdot \left[ x(K_{r2} - K_3) - \frac{\sigma_2 - \sigma_3}{\varepsilon_o} \right] \cdot x \]

\[ \frac{-4 \left[ \frac{\sigma_2^2}{\varepsilon_o^2} - x^2(K_{r2} \sin^2 \theta) \cos^2 \theta \right]}{\varepsilon_o} \]

(5.47)

Then we will have:

Then we will have:

for \( \xi_2 < \xi_3 \):

\[ I_1^{(1)} = -[\Sigma_0^{(1)}(i\alpha) + i\Sigma_2^{(1)}(i\alpha)\phi^*_2(\alpha) + i\Sigma_3^{(1)}(i\alpha)\phi^*_3(\alpha)] \exp(-\alpha t + \frac{2d}{C}\phi^*_2(\alpha)) \]

\[ - \Sigma_{23}^{(1)}(i\alpha)\phi^*_2(\alpha)\phi^*_3(\alpha) \exp(-\alpha t + \frac{2d}{C}\phi^*_2(\alpha)) \]

\[ + \frac{1}{\pi} \left[ \int_0^{\xi_2} \frac{\Sigma_0^{(1)}(iy)}{\alpha - y} e^{-yt \sin(\frac{2d}{C}\phi_2(y))} dy + \right. \]

\[ \left. + \int_0^{\xi_2} \frac{i\Sigma_2^{(1)}(iy)}{\alpha - y} \phi_2(y) e^{-yt \cos(\frac{2d}{C}\phi_2(y))} dy \right] \]
\[
\xi_2 + \int_0^\xi_2 \frac{i \Sigma_3^{(1)}(iy)}{a-y} \phi_3(y) e^{-yt \cos\left(\frac{2d}{C} \phi_2(y)\right)} dy
\]

\[
\xi_3 + \int_\xi_2^{\xi_2} \frac{i \Sigma_3^{(1)}(iy)}{a-y} \phi_3(y) e^{-yt + \frac{2d}{C} \phi_2(y)} dy
\]

\[
\xi_2 \Sigma_2^{(1)}(iy) + \int_0^{\xi_2} \frac{\Sigma_2^{(1)}(iy)}{a-y} \phi_2(y) \phi_3(y) e^{-yt \sin\left(\frac{2d}{C} \phi_2(y)\right)} dy
\]

\[
\xi_3 \Sigma_3^{(1)}(iy) + \int_\xi_2^{\xi_2} \frac{\Sigma_3^{(1)}(iy)}{a-y} \phi_2(y) \phi_3(y) e^{-yt + \frac{2d}{C} \phi_2(y)} dy
\]

Equation (5.49) is seen to contain only real expressions (after the substitution of the \(\Sigma\)-functions) and real definite integrals. For \(\xi_3 < \xi_2\) the corresponding expression becomes:

\[
I_1^{(1)} = - \left[ \Sigma_0^{(1)}(i\alpha) + i \Sigma_2^{(1)}(i\alpha) \phi_2^*(\alpha) + i \Sigma_3^{(1)}(i\alpha) \phi_3^*(\alpha) \right]
\]

\[
- \Sigma_2^{(1)}(i\alpha) \phi_2^*(\alpha) \phi_3^*(\alpha) \exp\left(-\alpha t + \frac{2d}{C} \phi_2^*(\alpha)\right)
\]

\[
- \frac{1}{\pi} \left[ \int_0^{\xi_2} \frac{\xi_2 \Sigma_0^{(1)}(iy)}{a-y} e^{-yt \sin\left(\frac{2d}{C} \phi_2(y)\right)} dy 
\right]
\]

\[
\xi_2 + \int_0^{\xi_2} \frac{i \Sigma_2^{(1)}(iy)}{a-y} \phi_2(y) e^{-yt \cos\left(\frac{2d}{C} \phi_2(y)\right)} dy
\]

\[
\xi_3 + \int_0^{\xi_3} \frac{i \Sigma_3^{(1)}(iy)}{a-y} \phi_3(y) e^{-yt \cos\left(\frac{2d}{C} \phi_2(y)\right)} dy
\]
\[
\begin{align*}
&+ \xi_2 \left( i \Sigma_{3}^{(1)}(iy) \right) \phi_3^*(y) e^{-yt} \sin \left( \frac{2d}{C} \phi_2(y) \right) dy \\
&+ \int_{\xi_3}^{\xi_3} \Sigma_{23}^{(1)}(iy) \\
&+ \int_{0}^{\xi_3} \frac{\phi_2(y) \phi_3(y) e^{-yt} \sin \left( \frac{2d}{C} \phi_2(y) \right) dy}{\alpha-y} \\
&- \xi_2 \left( i \Sigma_{23}^{(1)}(iy) \right) \phi_2(y) \phi_3(y) e^{-yt} \cos \left( \frac{2d}{C} \phi_2(y) \right) dy \\
\end{align*}
\] (5.50)

Again we observe that, after substitution of the $\Sigma$-functions as given by Eqns. (5.45) through (5.48), Eqn. (5.50) gives $I_1^{(1)}$ as an expression involving only real terms and operations.

### 5.4 Another Method; a Special Case

An alternative expression for $A_1^-$ [see Eqn. (2.21), Section 2.4] is:

\[
A_1^- = \frac{(1 - \frac{Z_1}{Z_3}) \cos (dk_2 \cos \theta_2) - i \left( \frac{Z_1}{Z_2} - \frac{Z_2}{Z_3} \right) \sin (dk_2 \cos \theta_2)}{(1 + \frac{Z_1}{Z_3}) \cos (dk_2 \cos \theta_2) + i \left( \frac{Z_1}{Z_2} + \frac{Z_2}{Z_3} \right) \sin (dk_2 \cos \theta_2)}
\] (5.51)

Consequently, an alternative way to analyze $I_1$ as given by (5.4) (Section 5.1) is to substitute $A_1^-$ as expressed by (5.51) into (5.4). The resulting complex improper integral can then be analyzed using the same contour integrations (Figure 3.2), where now the branch cut has upper branch point $i\xi$, with:
\[ \xi = \frac{\sigma_3}{\varepsilon_0(K_{r3} - \sin^2 \theta)} \]  

(5.52)

This is due to the fact that in (5.51) the RHS possesses no branch cut other than \((0, i\xi)\). There arises a difficulty, however, in determining the non-removable singularities of the integrand, other than \(\omega = i\alpha\), which are found as solutions to the equation:

\[
(1 + \frac{Z_1}{Z_3}) \cos(\alpha k_2 \cos \theta) + i(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_3}) \sin(\alpha k_2 \cos \theta) = 0 \tag{5.53}
\]

There has been found no explicit general analytical solution to Eqn. (5.53). The asymptotic solution of (5.53) can be expressed analytically with relative ease, we thus find for \(|\omega| \to \infty\) the following solution:

for \(K_{r3} > K_{r2}\):

\[
\omega_\nu = \frac{\nu v_2}{2d} (\nu \pi - i \ln \mu), \quad \nu \to \infty \text{ odd} \tag{5.54}
\]

and for \(K_{r2} > K_{r3}\):

\[
\omega_\nu = \frac{\nu v_2}{2d} (\nu \pi - i \ln |\mu|), \quad \nu \to \pm \infty \text{ even} \tag{5.55}
\]

where

\[
\mu \equiv \frac{\cos \theta - \sqrt{K_{r2} - \sin^2 \theta}}{\cos \theta + \sqrt{K_{r2} - \sin^2 \theta}}, \quad \frac{\sqrt{K_{r2} - \sin^2 \theta - \sqrt{K_{r3} - \sin^2 \theta}}}{\sqrt{K_{r2} - \sin^2 \theta + \sqrt{K_{r3} - \sin^2 \theta}}} \tag{5.56}
\]
The only remaining way to determine the complete solution to (5.53), that is numerically, appears to be extremely tedious. We will now illustrate this method in the case where region #3 (Figure 2.4) is a perfect conductor (\(\sigma_3 \to \infty\)) and region #2 is a lossless dielectric. In that case:

\[
A_1^{-} = \frac{-i\frac{2dw}{v}}{(\lambda-e^{-\frac{2dw}{v}})/(1-\lambda-e^{-\frac{2dw}{v}})}
\]  

(5.57)

where (Section 2.8):

\[
-1 < \lambda = \frac{\cos \theta - \sqrt{K \sin^2 \theta}}{\cos \theta + \sqrt{K \sin^2 \theta}} < 0
\]  

(5.58)

Hence:

\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\alpha + i\omega} \cdot \lambda - e^{-\frac{2dw}{v}} \frac{-i\frac{2dw}{v}}{1-\lambda-e^{-\frac{2dw}{v}}} \, d\omega
\]  

(5.59)

The non-removable singularities of the integrand are then:

\[
\omega = ia \text{ and } \omega_v = \frac{v}{2d}(v\pi - i\log|\lambda|)
\]  

(5.60)

where \(v = \pm1, \pm3, \pm5, \ldots\).

We have thus a case of damped oscillations. The oscillations are dictated by the geometry of the problem and the speed of the wave, i.e., by the real part of the singularities \(\omega_v\). The damping is caused by the escaping
radiation back into layer #1 and it depends on the frequency 
\( (v/2d) \) of travel of the wave back and forth between the two 
interfaces (imaginary \( \omega \)). Note that all the singularities 
lie in the upper complex semiplane. Now, since the 
integrand possesses no branch cuts, the integral will be 
given (for \( t > 0 \)) by the sum (or in this case series) of 
all the residue contributions, i.e.:

\[
I_1 = \frac{2da}{\lambda - e^{v/2d}} \cdot e^{-\alpha t} +
\]

\[
+ \frac{1-\lambda}{(d/v)} \left( \frac{1}{\lambda} \right) \sum_{v=1, v \text{ odd}}^\infty \frac{1 - \frac{v t}{2d}}{\frac{2d}{\lambda}} \left( \frac{\alpha + \frac{v}{2d} \log |\lambda|}{\frac{2d}{\lambda}} \right) \cos \left(\frac{\pi v t}{2d}\right) + \frac{v}{2d} \sin \left(\frac{\pi v t}{2d}\right)
\]

\[
= Ae^{\lambda} + (A-1)e^{\frac{\lambda^2}{2d} + \alpha \left(\lambda^2 - 1\right) e^{\frac{2d}{\lambda}} u(t - \frac{2d}{\lambda}) + \sum_{v=1}^\infty \lambda^{v-1} e^{-\alpha \left(\lambda^2 - 1\right) e^{\frac{2d}{\lambda}} u(t - \frac{2vd}{\lambda})}
\]

Had we followed the expansion method for \( A^-_1 \), the result 
would have been: \( t \geq 0 \)

\[
I_1 = \lambda e^{\lambda^2 t} + (\lambda^2 - 1)e^{\frac{-\alpha \left(\frac{2d}{\lambda^2} - 1\right) e^{\frac{-\alpha \left(\frac{2d}{\lambda^2} - 1\right) u(t - \frac{2d}{\lambda})}}
\]

\[
+ \lambda (\lambda^2 - 1)e^{\frac{-\alpha \left(\frac{4d}{\lambda} - 1\right) e^{\frac{-\alpha \left(\frac{4d}{\lambda} - 1\right) u(t - \frac{4d}{\lambda})}} + \ldots
\]

\[
= \lambda e^{\lambda^2 t} + (\lambda^2 - 1) \sum_{v=1}^\infty \lambda^{v-1} e^{-\alpha \left(\lambda^2 - 1\right) e^{\frac{-\alpha \left(\frac{2vd}{\lambda} - 1\right) e^{\frac{-\alpha \left(\frac{2vd}{\lambda} - 1\right) u(t - \frac{2vd}{\lambda})}}}
\]

It is easy to see that the two expressions for \( I_1 \) above are 
identical except at the points of discontinuity:
\[ t = \frac{2d}{v}, \frac{4d}{v}, \ldots, \frac{2vd}{v}, \ldots \]

at which times the sine-cosine series expression takes the average values between the left-approach and the right-approach limits. Hence the equivalence of the two methods.

The expansion method (for \( A_l \)) appears to offer an advantage in that by separating the contributions to the final result of the successively reflected transients it gives a clearer physical picture of the phenomenon. Furthermore there is no indication that the expansion method is numerically more expensive than the method of the residues (explained in this paragraph). Actually in the simple case of air-dielectric-perfect conductor, the opposite is true.

In the case of air-dielectric-perfect conductor for an input pulse given by Eqns. (3.1), (3.2), and (3.3) the principal electric field (\( e_{x1} \)), according to the expansion method, is:

\[
e_{x1} = \frac{1+\lambda}{N} \cdot (e^{-\alpha t} - e^{-\beta t}) + \frac{\lambda^2-1}{N} \cdot \]

\[
\cdot \left[ e^{-\alpha \left( t - \frac{2d}{v} \right)} - e^{-\beta \left( t - \frac{2d}{v} \right)} \right] u(t - \frac{2d}{v}) + \frac{\lambda(\lambda^2-1)}{N} \cdot \]

\[
\cdot \left[ e^{-\alpha \left( t - \frac{4d}{v} \right)} - e^{-\beta \left( t - \frac{4d}{v} \right)} \right] u(t - \frac{4d}{v}) + \ldots \]  

(5.63)
It is seen from (5.63) that, as expected, the field function is continuous for \( t \geq 0 \) with discontinuous first derivative (slope). The change in slope (discontinuous jumps of the first derivative) occur for finite times at the points:

\[
t = \frac{2d}{v}, \frac{4d}{v}, \ldots, \frac{2vd}{v}, 
\]

and correspondingly the derivative jumps are:

\[
\Delta \frac{de_{x1}}{dt} = \frac{\lambda^{2-1}}{N}(\beta-\alpha), \frac{\lambda(\lambda^{2-1})}{N}(\beta-\alpha), \ldots, \frac{\lambda^{n-1}(\lambda^{2-1})}{n}(\beta-\alpha), \ldots
\]

(5.65)

In the graph representing \((e_{x1})\) the distance (on the time scale) between two successive derivative jumps is the quantity \(2d/v\) (Figure 9, Appendix C).

5.5 Numerical Applications

At the end (Appendix C) we present four field curves for the principal field \(e_{x1}\). The case of air-dielectric-perfect conductor is considered for both \(K = 7\) and \(K = 30\) (Figure 12, Appendix C). As input pulse we have taken a normalized causal double exponential with \(\alpha = 2.5 \times 10^8\) and \(\beta = 3 \times 10^8\) with angle of incidence \(\theta = 45^\circ\) (curve I, Figure 2, Appendix C). The discontinuities in the slope are illustrated clearly with these curves.

Theory suggests that in practice information may be obtained on the thickness of the dielectric and the permittivity of the dielectric by observing the response
curves to known inputs. To illustrate the point, we consider a dielectric with permittivity $K$ and thickness $d$ as noted previously (Section 5.4). The response to a double exponential pulse incident at an angle $\theta_1$ exhibits jumps of slope every $\Delta t_1$ seconds, where $\Delta t_1 = 2d/v_1$; 

$v_1 = C/\sqrt{K\sin^2\theta_1}$. If we change the angle of incidence to $\theta_2$ the jumps of slope appear every $\Delta t_2$ seconds where $\Delta t_2 = 2d/v_2$; $v_2 = C/\sqrt{K\sin^2\theta_2}$. A combination of these results in:

$$K = [(\Delta t_2)^2\sin^2\theta_1 - (\Delta t_1)^2\sin^2\theta_2]/[(\Delta t_2)^2 - (\Delta t_1)^2] \quad (5.66)$$

$$d = C\Delta t_1/(2\sqrt{K\sin^2\theta_1}) \quad (5.67)$$

Numerical errors associated with (5.66) applied in conjunction with a graph are expected to become overwhelming for large $K$'s and/or not too different $\theta_1$, $\theta_2$.

As an example we will consider a simple criterion of a measurable difference between $\Delta t_1$ and $\Delta t_2$:

$$(\Delta t_1 - \Delta t_2)/\Delta t_1 \geq 2\%$$

But assuming $K$ large we can write:

$$\frac{\Delta t_1 - \Delta t_2}{\Delta t_1} = \frac{\sqrt{K\sin^2\theta_1} - \sqrt{K\sin^2\theta_2}}{\sqrt{K\sin^2\theta_1}} \approx \frac{\sin^2\theta_2 - \sin^2\theta_1}{2K} \frac{\sin^2\theta_1}{1 - \frac{\sin^2\theta_1}{2K}}$$
Hence:

\[
\frac{\sin^2 \theta_2 - \sin^2 \theta_1}{2K - \sin^2 \theta_1} \geq 0.02
\]

which gives:

\[(2K - \sin^2 \theta_1) \cdot 0.02 \leq \sin^2 \theta_2 - \sin^2 \theta_1\]

Since \(2K >> \sin^2 \theta_1\) we get

\[K \leq \frac{100}{4} = 25\]

hence (5.66) will be of value for small \(K\) (less than 25).

Figure 13 (Appendix C) presents the case of 1 meter of dry earth on top of wet earth (for dry earth \(K = 7\) and \(\sigma = 10^{-3}\) and for wet earth \(K = 30\) and \(\sigma = 3 \times 10^{-2}\)). The input pulse is here the same (curve I, Figure 2, Appendix C) as in the previous case. Figure 14 (Appendix C) presents the case of 1 meter of wet earth on top of dry earth. As expected the fields in the latter case (wet earth on top) are smaller in magnitude (less than half in peak) for both the positive and the negative lobes. This is due to the fact that the higher conductivity of the wet earth tends to attenuate more the electric field on the interface and also the fact that the dry earth substratum has poor reflectivity.

Again Eqns. (5,66) and (5,67) are applicable in this case (where \(K\) is \(K_{r2}\)), while a comparison of experimentally obtained field curves with an appropriate set of
theoretically constructed curves will give information about the other constitutive parameters (conductivities, etc.).

On the effectiveness of the analytical results similar conclusions can be drawn as previously (see Section 3.3) for $t < 4d/v_2$. However, for large times and many terms of the series (5.1), the analytical method becomes eventually more expensive than the Filon direct-inversion. The analytical method provides a unique way for the computation of the contribution of the multi-reflected transients between the two interfaces. It thus provides a clearer physical understanding. Last, note that for great $t$, the contribution to the fields on the interface of the first terms of (5.1) becomes increasingly smaller relative to that of subsequent terms; hence one may reduce the overall cost of computation by neglecting the initial terms of (5.1) as the desired accuracy will permit. Under such circumstances the method may again become competitive costwise with the Filon direct-inversion for large times.
CHAPTER 6

TM CASE THREE LAYERS

6.1 Introduction

Following previous observations (see Section 2.7) we can express the reflection coefficient $B_{1}^{-}$ as a series:

$$
B_{1}^{-} = \mu_{12}^{2} - (\mu_{12}^{2} - 1) \sum_{m=1}^{\infty} (-1)^{m \cdot m - 1} \mu_{12}^{m - 1} \mu_{23}^{m} e^{-2\cdot m \cdot k_{2} \cos^{2} \theta_{2}}
$$

(6.1)

For an input given by Eqns. (3.1), (3.2), and (3.3) the principal magnetic field ($h_{11}$) on the surface of the layered dissipative medium is:

$$
h_{11}(t) = -(1/\eta_{o}) (f(t) - b_{1}^{-}(t))
$$

(6.2)

The field within the empty space (layer #1) is given by Eqn. (4.1)

$$
b_{1}^{-} = (K_{1} - K_{2})/N
$$

(6.3)

$$
K_{1} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{B_{1}^{-}}{\alpha + i\omega} e^{i\omega t} d\omega
$$

(6.4)

and $K_{2}$ is obtained from (6.4) by substituting $\beta$ for $\alpha$.

The Fourier inversion in (6.4) is carried out, as in the TE case, term by term, with the first term reproducing the reflective results of the two layer case and subsequent
terms yielding the contribution of the reflected waves from
the interface #2. For any given finite time the series in
the frequency domain yields a finite sum of terms in the
time domain (as in Section 5.1).

6.2 Inversion of the mth Term of the Series

We rewrite Eqn. (6.4) in the form:

\[ K_1 = \sum_{m=0}^{\infty} K_1^{(m)} \]  
\[ (6.5) \]

where:

\[ K_1^{(0)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mu_{12} e^{i\omega t}}{\alpha + i\omega} d\omega \]  
\[ (6.6) \]

and

\[ K_1^{(m)} = \frac{(-1)^m}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\alpha + i\omega} (\mu_{12} - 1) \mu_{12}^{m-1} \mu_{12} \mu_{23} e^{-i2mdk_2 \cos \theta} d\omega \]  
\[ (6.7) \]

In order to give the explicit frequency expressions for \( \mu_{12}, \mu_{23} \) we will adopt the following auxiliary functions:

\[ C_2(\omega) = \omega^2 (K_{r2} - \sin^2 \theta + K_{r2}^2 \cos^2 \theta) - \frac{\sigma_2^2}{\varepsilon_0} (1+2K_{r2} \cos^2 \theta) \omega - \frac{\sigma_2^2}{\varepsilon_0} \cos^2 \theta \]  
\[ (6.8) \]

\[ C_{23}(\omega) = (K_{r2} \omega - \frac{\sigma_{23}^2}{\varepsilon_0}) \left[ \omega^2 (K_{r3} - \sin^2 \theta) - \frac{\sigma_{23}^2}{\varepsilon_0} \right] + \]  
\[ + (K_{r3} \omega - \frac{\sigma_{23}^2}{\varepsilon_0}) \left[ \omega^2 (K_{r2} - \sin^2 \theta) - \frac{\sigma_{23}^2}{\varepsilon_0} \omega \right] \]  
\[ (6.9) \]
\[
D_2(\omega) \equiv k_{r2} \omega - \frac{i \sigma_2}{\varepsilon_o} \tag{6.10}
\]

\[
D_3(\omega) \equiv k_{r3} \omega - \frac{i \sigma_3}{\varepsilon_o} \tag{6.11}
\]

\[
P_2(\omega) \equiv -[\omega((k_{r2}+1)\cos^2\theta-1)-i\frac{\sigma_2}{\varepsilon_o}\cos^2\theta][\omega(k_{r2}-1)-i\frac{\sigma_2}{\varepsilon_o}] \tag{6.12}
\]

\[
P_{23}(\omega) \equiv \omega(k_{r2}-k_{r3})\cdot \left[\frac{\sigma_2 - \sigma_3}{\varepsilon_o}\right] \cdot \left[\omega(k_{r2} - k_{r3})\sin^2\theta\right] \omega^2 - i\omega(k_{r2} - \sin^2\theta)\sigma_3 + (k_{r3} - \sin^2\theta)\sigma_2} \right)/\varepsilon_o - \sigma_2 \sigma_3/\varepsilon_o^2 \right]\tag{6.13}
\]

We obtain:

\[
\mu_{12} = \frac{\sqrt{\omega^2(k_{r2} - \sin^2\theta) - i\frac{\sigma_2}{\varepsilon_o} - (k_{r2} - \omega - i\frac{\sigma_2}{\varepsilon_o})\cos\theta}}{\sqrt{\omega^2(k_{r2} - \sin^2\theta) - i\frac{\sigma_2}{\varepsilon_o} + (k_{r2} - \omega - i\frac{\sigma_2}{\varepsilon_o})\cos\theta}} = \frac{C_2(\omega) - 2\cos\theta D_2(\omega) S_2(\omega)}{P_2(\omega)} \tag{6.14}
\]

\[
\mu_{23} = \left[(k_{r2} - \omega - i\frac{\sigma_2}{\varepsilon_o})\sqrt{\omega^2(k_{r3} - \sin^2\theta) - i\frac{\sigma_3}{\varepsilon_o} - (k_{r3} - \omega - i\frac{\sigma_3}{\varepsilon_o})}\right]/\left[(k_{r2} - \omega - i\frac{\sigma_2}{\varepsilon_o})\sqrt{\omega^2(k_{r3} - \sin^2\theta) - i\frac{\sigma_3}{\varepsilon_o}}\right]
+ (k_{r3} - \omega - i\frac{\sigma_3}{\varepsilon_o})\sqrt{\omega^2(k_{r2} - \sin^2\theta) - i\frac{\sigma_2}{\varepsilon_o}} = \]

\[
C_{23}(\omega) - 2D_2(\omega)D_3(\omega)S_2(\omega)S_3(\omega)
\]
\[
P_{23}(\omega)
\]
(6.15)

where the S-functions are the same ones defined in Section 5.2 by Eqns. (5.11) and (5.12). We note that the S-functions are the only ones possessing branch cuts on the complex \(\omega\)-plane.

The similarity of forms between Eqns. (6.14), (6.15) and (5.17), (5.18) suggests an analogous treatment, thus we let:

\[
(\mu_{12}^2 - 1)\mu_{12}^{m-1}\mu_{23}^{m} = M_0^{(m)}(\omega) + M_2^{(m)}(\omega)S_2(\omega) + M_3^{(m)}(\omega)S_3(\omega) + \]
\[
+ M_{23}^{(m)}(\omega)S_2(\omega)S_3(\omega)
\]
(6.16)

where the M's are analytic functions given explicitly in terms of the frequency \(\omega\) as follows:

\[
M_0^{(m)}(\omega) = \frac{W_1^{(m)}(\omega)Z_1^{(m)}(\omega)}{P_2^{m+1}P_{23}^m}
\]

\[
M_2^{(m)}(\omega) = -2\cos\theta D_2(\omega)\frac{W_2^{(m)}(\omega)Z_1^{(m)}(\omega)}{P_2^{m+1}P_{23}^m}
\]

\[
M_3^{(m)}(\omega) = 4\cos\theta D_3(\omega)D_2^2(\omega)S_2^2\frac{W_2^{(m)}(\omega)Z_1^{(m)}(\omega)}{P_2^{m+1}P_{23}^m}
\]

\[
M_{23}^{(m)}(\omega) = -2D_2(\omega)D_3(\omega)\frac{W_1^{(m)}(\omega)Z_2^{(m)}(\omega)}{P_2^{m+1}P_{23}^m}
\]
(6.17)
The auxiliary functions \( W \) and \( Z \) are:

\[
W_1^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m+1}{2}\right]} (m+1)C_{2}^{m+1-p}P_{2}^{p}\cos P_{2}D_{2}^{p}S_{2}^{p}
\]

\[
W_2^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m}{2}\right]} (m+1)C_{2}^{m-p}P_{2}^{p}\cos P_{2}D_{2}^{p}S_{2}^{p}
\]

\[
Z_1^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m-1}{2}\right]} (m-1)C_{2}^{m-2+p}P_{2}^{p}\cos P_{2}D_{2}^{p}S_{2}^{p}
\]

\[
Z_2^{(m)}(\omega) = \sum_{p=0}^{2\left[\frac{m-1}{2}\right]} (m-1)C_{2}^{m-1+p}P_{2}^{p}\cos P_{2}D_{2}^{p}S_{2}^{p}
\]

\[\text{where the summation index } p \text{ takes even, non-negative values } (p = 0, 2, 4, \ldots) \text{ and:}\]

\[\lfloor x \rfloor \equiv \text{integer part of } x \text{ for } x \geq 0\]
Combination of (6.7) and (6.16) gives:

\[
K_1^{(m)} = \frac{(-1)^m}{2\pi} \left( \int_{-\infty}^{\infty} \frac{M_0^{(m)}(\omega)}{\alpha+i\omega} e^{-i2mk_2\cos\theta} d\omega \right)
\]

\[
+ \int_{-\infty}^{\infty} \frac{M_2^{(m)}(\omega)}{\alpha+i\omega} e^{-i2mk_2\cos\theta} d\omega
\]

\[
+ \int_{-\infty}^{\infty} \frac{M_3^{(m)}(\omega)}{\alpha+i\omega} e^{-i2mk_2\cos\theta} d\omega
\]

\[
+ \int_{-\infty}^{\infty} \frac{M_{23}^{(m)}(\omega)}{\alpha+i\omega} e^{-i2mk_2\cos\theta} d\omega \}
\]

(6.22)

It is now seen that, by substituting the \(M\)-functions for the \(\Sigma\)-functions in the formulas for the \(I_1^{(m)}\) of the TE case, we obtain the corresponding formulas for \(K_1^{(m)}\) of the TM case.

Thus Eqns. (5.28) and (5.29) apply here for \(K_1^{(m)}\), by substituting the \(M\)-functions in the place of \(\Sigma\)-functions.

We note that all resulting expressions are real and no imaginary quantities are involved in the final results.

6.3 Application to the Term for \(m = 1\)

For the initial times, such that \(t < 2d/v_2\), when the only non-zero term in (6.5) is the \(K_1^{(0)}\), the response of the dissipative medium is due to layer \#2 alone. At the time \(t = 2d/v_2\) the reflection of the transient coming from the interface \#2 surfaces on the first interface. This reflection, and specifically its contribution to the fields
on interface #1, is given by $k_1^{(1)}$. We now proceed to analyze $k_1^{(1)}$:

The auxiliary functions $W$ and $Z$ are:

$$
W_1^{(1)} = C_2^2 - P_2^2 + 4D_2^2 S_2^2 \cos^2 \theta \\
W_2^{(1)} = 2C_2 \\
Z_1^{(1)} = C_{23} \\
Z_2^{(1)} = 1
$$

hence, we obtain for the $M$-functions:

$$
M_0^{(1)} = C_{23} (C_2^2 - P_2^2 + 4D_2^2 S_2^2 \cos^2 \theta) / (P_2^2 P_{23}) \\
M_2^{(1)} = -4C_2 C_{23} \cos \theta D_2 / (P_2^2 P_{23}) \\
M_3^{(1)} = 8C_2 D_3 D_2^2 S_2^2 \cos \theta / (P_2^2 P_{23}) \\
M_{23}^{(1)} = -2D_2 D_3 (C_2^2 - P_2^2 + 4D_2^2 S_2^2 \cos^2 \theta) / (P_2^2 P_{23})
$$

Next, in order to apply the integral formulas for $I_1^{(1)}$ we need the $M$-functions estimated at $ix$, where $x$ is real, and in view of (6.24) this will be accomplished when the auxiliary functions $C_2$, $C_{23}$, $D_2$, $D_3$, $P_2$, $P_{23}$ have been expressed as follows:
\[ C_2(ix) = \frac{\sigma_2^2 x}{\varepsilon_o} (1+2K_{r2}\cos^2 \theta) - x^2(K_{r2}^2 - \sin^2 \theta + K_{r2}^2 \cos^2 \theta) - \frac{\sigma_2^2}{\varepsilon_o} \cos^2 \theta \]  

(6.25)

\[ C_{23}(ix) = (K_{r2}^2 - \frac{\sigma_2^2}{\varepsilon_o}) [x^2(K_{r3} - \sin^2 \theta) - \frac{\sigma_2^2 x}{\varepsilon_o}] + (K_{r3}^2 - \frac{\sigma_3^2}{\varepsilon_o}) [x^2(K_{r2} - \sin^2 \theta) - \frac{\sigma_2^2 x}{\varepsilon_o}] \]  

(6.26)

\[ D_2(ix) = i(K_{r2}^2 - \frac{\sigma_2^2}{\varepsilon_o}) \]  

(6.27)

\[ D_3(ix) = i(K_{r3}^2 - \frac{\sigma_3^2}{\varepsilon_o}) \]  

(6.28)

\[ P_2(ix) = [x((K_{r2} + 1)\cos^2 \theta - 1) - \frac{\sigma_2^2 \cos^2 \theta}{\varepsilon_o}] [x(K_{r2} - 1) - \frac{\sigma_2^2}{\varepsilon_o}] \]  

(6.29)

\[ P_{23}(ix) = x[x(K_{r2} - K_{r3}) - \frac{\sigma_2^2 - \sigma_3^2}{\varepsilon_o}] [K_{r2}K_{r3} - (K_{r2} + K_{r3})\sin^2 \theta]x^2 - x(K_{r2} - \sin^2 \theta)\sigma_3 + (K_{r3} - \sin^2 \theta)\sigma_2] / \varepsilon_o + \sigma_2 \sigma_3 / \varepsilon_o \]  

(6.30)

Accordingly, we find for the non-singular case:

for \( \xi_2 < \xi_3 \):

\[ K_1^{(1)} = -[M_0^{(1)}(ia) + iM_2^{(1)}(ia)\phi_2^{*}(a) + iM_3^{(1)}(ia)\phi_3^{*}(a) - M_{23}^{(1)}(ia)\phi_2^{*}(a)\phi_3^{*}(a)] \exp(-\alpha t + \frac{2d}{C} \phi_2^{*}(a)) \]
For $\xi_3 < \xi_2$ the corresponding expression is:

\[
K_{1}^{(1)} = -[M_{0}^{(1)}(ia)+iM_{2}^{(1)}(ia)\phi_{2}^{*}(a)+iM_{3}^{(1)}(ia)\phi_{3}^{*}(a)]
- M_{23}^{(1)}(ia)\phi_{2}^{*}(a)\phi_{3}^{*}(a) \exp(-at + \frac{2d}{c}\phi_{2}^{*}(a))
\]

\[
- \frac{1}{\pi} \left\{ \xi_2 \frac{M_{0}^{(1)}(iy)}{\alpha-y} e^{-yt\sin\left(\frac{2d}{c}\phi_{2}(y)\right)} dy +
\xi_2 \frac{iM_{2}^{(1)}(iy)}{\alpha-y} \phi_{2}(y) e^{-yt\cos\left(\frac{2d}{c}\phi_{2}(y)\right)} dy
\right. 
\]

\[
\left. + \xi_2 \frac{iM_{3}^{(1)}(iy)}{\alpha-y} \phi_{3}(y) e^{-yt\cos\left(\frac{2d}{c}\phi_{2}(y)\right)} dy \right\} \quad (6.31)
\]
\[ \xi_3 \frac{iM(1)_{3\alpha}}{y} \int_0^\infty \phi_3(y) e^{-\frac{2d}{C} \phi_2(y)} dy \]

\[ + \xi_2 \frac{iM(1)_{3\alpha}}{y} \int_0^\infty \phi_2(y) \phi_3(y) e^{-\frac{2d}{C} \phi_2(y)} dy \]

\[ + \xi_3 \frac{M(1)_{3\alpha}}{y} \int_0^\infty \phi_2(y) \phi_3(y) e^{-\frac{2d}{C} \phi_2(y)} dy \]

\[ - \xi_2 \frac{M(1)_{3\alpha}}{y} \int_0^\infty \phi_2(y) \phi_3(y) e^{-\frac{2d}{C} \phi_2(y)} dy \] (6.32)

The usual statement on the reality of the quantities and operations involved is true here too.

6.4 A Special Case

We will present a different method, as we did in Section 5.4, by assuming the case of air-dielectric-perfect conductor. In this case \( \sigma_2 = 0, \sigma_3 \to \infty \) we have

\[ B_1 = (\mu - e^{-i2d\omega/v})/(1 - \mu e^{-i2d\omega/v}) \] (6.33)

where

\[ \mu = \frac{\sqrt{K\sin^2 \theta - K\cos \theta}}{\sqrt{K\sin^2 \theta + K\cos \theta}} \] (6.34)
We see here that $|\mu| < 1$, but that the sign of $\mu$ is not fixed. This is the major difference between the TM case and the TE case where the sign of $\lambda$ was known in the outset to be negative [Eqn. (5.58), Section 5.4].

Clearly for fixed $K$ the sign is expected to undergo change as the angle of incidence crosses a critical value. This value suggested physically and produced mathematically, is the Brewster's angle such that

$$\tan^2\theta_B = K \quad (6.35)$$

We thus find:

$$1 > \mu > 0 \quad \text{for} \quad \theta > \theta_B$$

$$-1 < \mu < 0 \quad \text{for} \quad \theta < \theta_B$$

$$\mu = 0 \quad \text{for} \quad \theta = \theta_B \quad (6.36)$$

In the time domain we have:

$$K_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\alpha + i\omega} \cdot \frac{-i2d\omega}{\mu - e^{\alpha}} \, d\omega \quad (6.37)$$

The non-removable singularities of the integrand except the usual at $\omega = i\alpha$, are according to the case:

When the incidence is at exactly Brewster's angle:

$$\theta = \theta_B \quad (6.38)$$
No singularities of $B_1^-$ exist and the solution in this case is:

\[ K_1(t) = -u(t - \frac{2d}{v}) e^{-\alpha(t - \frac{2d}{v})} \]  \hspace{1cm} (6.39)

This is to be expected physically since no wave is reflected from the first interface. The whole transient penetrates into the lossless dielectric, undergoes total reflection, and reappears at the interface #1 with a delay of $2d/v$.

When the incidence is at less than the Brewster's angle

\[ \theta < \theta_B \]  \hspace{1cm} (6.40)

then the singularities of $B_1^-$ are:

\[ \omega_v = \frac{v}{2d}(\nu \pi - \text{ilog}|\mu|) \]  \hspace{1cm} (6.41)

where $\nu = \pm 1, \pm 3, \pm 5, \ldots$

The case is similar to the TE case; we have thus a case of damped oscillations. The oscillations are dictated by the geometry of the problem and the speed of the wave, i.e., by the real part of the singularities $\omega_v$. The damping is caused by the escaping radiation back in the air and it depends on the frequency ($\nu/2d$) of travel of the wave back and forth between the two interfaces ($\text{Im}\omega_v$).
The solution is accordingly found to be:

\[
K_1 = \frac{2\alpha a}{v} e^{-\alpha t} + \frac{1 - \mu^2}{(d/v)(1/\mu)} \left( 1 - \frac{v t}{2d} \right) \sum_{\nu=1, \text{odd}}^{\infty} \frac{(\alpha + \frac{v}{2d} \ln |\mu|) \cos (\nu \pi \frac{v t}{2d}) + \nu \pi \frac{v}{2d} \sin (\nu \pi \frac{v t}{2d})}{2^n (\alpha + \nu \pi \frac{v}{2d} \ln |\mu| + \nu^2 \pi \frac{v^2}{4d^2})}
\]

(6.42)

The corresponding expression, according to the expansion method is:

\[
K_1 = \mu e^{-\alpha t} + (\mu^2 - 1)e^{-\alpha (t - \frac{2d}{v})} u(t - \frac{2d}{v}) + \mu (\mu^2 - 1)e^{-\alpha (t - \frac{4d}{v})} u(t - \frac{4d}{v}) + \ldots + \mu^{\nu-1} (\mu^2 - 1)e^{-\alpha (t - \frac{2\nu d}{v})} u(t - \frac{2\nu d}{v}) + \ldots
\]

(6.43)

The two expressions above are equal except at the points of discontinuity:

\[
t = \frac{2d}{v}, \frac{4d}{v}, \ldots, \frac{2\nu d}{v}, \ldots \quad \text{(for } t > 0)\]

at which times the sine-cosine series expression takes the average values between the left-approach and right-approach limits of the step function series. Hence the two results are indeed equivalent.
When the incidence is at an angle greater than
Brewster's angle, i.e.:

\[ \theta > \theta_B \tag{6.44} \]

the non-removable singularities of \( B_1 \) are:

\[ \omega_v = \frac{v}{2d} (\nu \pi - i \ln \mu) \tag{6.45} \]

where \( v = 0, \pm 2, \pm 4, \ldots \)

Thus, the solution is:

\[
K_1 = \frac{2 \alpha}{\nu} \frac{e^{-\alpha t}}{1 - \frac{2 \alpha}{\nu} e^{-\alpha t}} + \frac{1 - \frac{\nu t}{2d}}{2d} \left( \frac{\mu^2 - 1}{\nu^2} \right) \left( \frac{1}{\nu^2} \right) \left( \frac{1 - \frac{\nu t}{2d}}{2d} \right) + \frac{\mu^2 - 1}{(d/\nu)} \left( \frac{d}{\nu} \right)^{-1} \left( \frac{\nu^2 v^2}{4d^2} \right) \tag{6.46}
\]

The corresponding expression, according to the expansion method remains unchanged from that given by Eqn. (6.43).

We note that the expansion method, followed here throughout the 3 layer problem, offers a clearer physical picture (see also remarks in Section 5.4), and in the case of air-dielectric-perfect conductor proves to be much more economic than the method of the residues. Indeed, to achieve third decimal accuracy with the sine-cosine series (method of the residues) of (5.61), (6.42), or (6.46), an
enormous number of terms is necessary for the time range of practical interest (number of terms from 5,000 to 10,000) which in turn makes the computation prohibitively expensive. However, in the general problem this comparison is expected to change significantly.

For the principal electric field \( h_{x1} \), in the case of air-dielectric-perfect conductor for an input pulse given by Eqns. (3.1), (3.2), and (3.3), according to the expansion method, we have:

\[
\begin{align*}
    n_0 h_{x1} &= \frac{\mu-1}{N}(e^{-\alpha t} - e^{-\beta t}) + \frac{\mu^2-1}{N}[e^{-\alpha(t-\frac{2d}{v})} - e^{-\beta(t-\frac{2d}{v})}]u(t-\frac{2d}{v}) + \\
    &+ \frac{\mu(\mu^2-1)}{N}[e^{-\alpha(t-\frac{4d}{v})} - e^{-\beta(t-\frac{4d}{v})}]u(t-\frac{4d}{v}) + \ldots \quad (6.47)
\end{align*}
\]

As expected, the field function is continuous for \( t \geq 0 \), with discontinuities in the first derivative. The change in slope occurs for finite times at the points:

\[
t = \frac{2d}{v}, \frac{4d}{v}, \ldots, \frac{2vd}{v}, \ldots \quad (6.48)
\]

and respectively, the derivative jumps are:

\[
\begin{align*}
    \Delta^d_{t} n_0 h_{x1} &= \frac{\mu^2-1}{N}(\beta-\alpha), \frac{\mu(\mu^2-1)}{N}(\beta-\alpha), \ldots, \\
    &\frac{\mu^{v-1}(\mu^2-1)}{N}(\beta-\alpha), \ldots \quad (6.49)
\end{align*}
\]
6.5 Numerical Applications

In Appendix C we present field curves for the principal magnetic field $h_{x1}$. The case of air-dielectric-perfect conductor is considered for $K = 7$ and $K = 30$ (Figures 15 and 16, Appendix C) with an input pulse as the pulse shape I (Figure 2, Appendix C) and with incidence at $45^\circ$ and $84^\circ$. The discontinuities in slope occurring every $2d/v$ seconds are clearly illustrated with these curves. For $84^\circ$ incidence (larger than the Brewster's angles for both $K = 7$ and $K = 30$) the second lobes of the field curves are larger in magnitude than the first lobes. This is because in that case the larger portion of the transient penetrates the first interface and is subsequently reflected back to reappear with a time lag in the interface #1. (Figure 16, Appendix C). Eventually the fields decay due to losses by escaping radiation into layer #1.

Figure 17 (Appendix C) presents the principal magnetic field for the case of dry earth ($K = 7$ and $\sigma = 10^{-3}$) on top of wet earth ($K = 30$ and $\sigma = 3 \times 10^{-2}$). The input pulse is again pulse shape I (Figure 2, Appendix C). The angle of incidence was given the values $45^\circ$, $69.4^\circ$, and $84^\circ$ respectively. For $t < 4d/v$ the field curves consist of two negative lobes, with the second one being smaller, even for large angles of incidence (here $\theta_B \approx 69.4^\circ$) due to the losses in both layers #2 and #3. Note that for $\theta = 69.4^\circ$ the second lobe is on top of the
corresponding one for $\theta = 45^\circ$. This is due to the earlier arrival of the reflected transient in the former case and the Brewster's angle effect. Note also that because of the relatively low critical frequency ($\omega_{cr} \equiv \sigma/(ke_o)$) and the small width of the input pulse, no significant depression of the early time behavior for large angles of incidence are observed here.

Figure 18 (Appendix C) shows the principal magnetic field for the case of wet earth on top of dry earth. The input pulse and the various angles of incidence are here the same as for Figure 17 (Appendix C). For $t < 4d/v$ there is only one negative lobe, this being due to the relatively poor reflective properties of the dry-earth substratum. For large angles of incidence ($\theta > \theta_B = 79.67^\circ$) we observed an accentuated late time behavior due to the Brewster's angle effect. For the early time behavior the Brewster's angle effect is relatively small in this case due to the small width of the input pulse.

Regarding the effectiveness of the analytical results in achieving efficient computer codes, similar conclusions can be drawn as in previous sections (Section 3.3 and Section 5.5).
CHAPTER 7

FREQUENCY DEPENDENT ELECTRICAL CONSTITUTIVE PARAMETERS

7.1 Introduction

The frequency dependence of the complex relative dielectric constant for a compound Debye medium is described by the relation:

\[ K^*(i\omega) = \frac{\sigma}{i\omega\varepsilon_0} + \sum_{m=1}^{M} \left[ a_m + \frac{b_m - a_m}{1 + i\omega\tau_m} \right] \] (7.1)

\( \varepsilon_0 \) is, as usual, the free space permittivity, \( \sigma \) the dc conductivity and the Debye parameters \( a_m, b_m, \tau_m \) are determined to fit experimental data (Scott et al., 1964; Longmire and Longley, 1973). The data are usually available in the form of a real effective dielectric constant and an effective conductivity, which according to the model (7.1) are fitted into the forms:

\[ \frac{\varepsilon_{\text{eff}}(\omega)}{\varepsilon_0} = \sum_{m=1}^{M} \left[ a_m + \frac{b_m - a_m}{1 + (\omega\tau_m)^2} \right] \] (7.2)

and

\[ \sigma_{\text{eff}}(\omega) = \sigma + \sum_{m=1}^{M} \frac{\omega^2 \varepsilon_0 \tau_m (b_m - a_m)}{1 + (\omega\tau_m)^2} \] (7.3)
where we have considered as valid the familiar relationship:

\[ K^*(i\omega) = \frac{\varepsilon_{\text{eff}}(\omega)}{\varepsilon_0} - i \frac{\sigma_{\text{eff}}(\omega)}{\omega \varepsilon_0} \]  
(7.4)

It is interesting to note the following limits:

\[ \sum_{m=1}^{M} a_m = \lim_{\omega \to \infty} \frac{\varepsilon_{\text{eff}}(\omega)}{\varepsilon_0} \]  
(7.5)

\[ \sum_{m=1}^{M} b_m = \lim_{\omega \to 0} \frac{\varepsilon_{\text{eff}}(\omega)}{\varepsilon_0} \]  
(7.6)

\[ \sigma = \lim_{\omega \to 0} \sigma_{\text{eff}}(\omega) \]  
(7.7)

\[ \sum_{m=1}^{M} \frac{b_m - a_m}{\tau_m} = \lim_{\omega \to \infty} \frac{\sigma_{\text{eff}}(\omega)}{\varepsilon_0} - \frac{\sigma}{\varepsilon_0} \]  
(7.8)

Physically, the Debye model describes several possible mechanisms for material polarization in the presence of an electric field, as for example damped motion of free charge and damped reorientation of polar molecules (Fuller and Wait, 1972).

### 7.2 The M = 1 Model

The Debye model, taking one term of the summation becomes:

\[ K^*(i\omega) = a + \frac{\sigma}{i\omega \varepsilon_0} + \frac{b-a}{1+i\omega \tau} \]  
(7.9)

We will now adopt this for the two layer problem to compute the principal electric field \( e_{x1} \) on the interface.
For convenience of notation we let:

\[ K \equiv a \] (7.10)

\[ \Delta K \equiv b-a \] (7.11)

These are in accordance with the meaning of the parameters in the Debye model, where:

\[ a = \lim_{\omega \to \infty} \varepsilon_2(\omega)/\varepsilon_0 \quad \text{and} \quad b = \lim_{\omega \to 0} \varepsilon_2(\omega)/\varepsilon_0. \]

We substitute the expression for the complex \( K^* \) into \( A_1^-(2.18) \) and the resulting expression is:

\[
A_{1f}^-(\omega) = \frac{\cos\theta - \sqrt{K^2 \sin^2 \theta + \frac{\sigma}{1+i\omega} + \frac{\Delta K}{1+i\omega}}}{\cos\theta + \sqrt{K^2 \sin^2 \theta + \frac{\sigma}{1+i\omega} + \frac{\Delta K}{1+i\omega}}} = \\
\frac{\omega \Delta K + (1+i\omega) \left[ \omega (K+\cos 2\theta) - i \frac{\sigma}{\varepsilon_0} - 2\cos \theta \sqrt{(1+i\omega)[(1+i\omega)]} \right]}{(1+i\omega)[(1-K) \omega + i \frac{\sigma}{\varepsilon_0}] - \omega \Delta K} = \\
\frac{\{\omega^2 (K^2 \sin^2 \theta) - i \frac{\sigma \omega}{\varepsilon_0}\} + \Delta k \omega^2}{(1-K) \omega + i \frac{\sigma}{\varepsilon_0}}
\]

This form possesses two removable singularities, found as solutions when the complex quadratic trinomial in the denominator is set equal to zero:

\[
\omega = i \frac{\Delta K + K - 1 + \frac{\sigma T}{\varepsilon_0} \pm \sqrt{\Delta K^2 + 2\Delta K (K - 1 + \frac{\sigma T}{\varepsilon_0}) + (K - 1 - \frac{\sigma T}{\varepsilon_0})^2}}{2\tau (K-1)}
\] (7.13)
For these values of \( \omega \) it is easily seen that the numerator of \( A^f_{1f}(\omega) \) (7.12) vanishes, hence the singularities are indeed removable.

In the subsequent analysis we will need the branch points of the square-root function in (7.12). We let the argument-function be equal to zero and obtain four roots, which constitute the branch points:

\[
\omega = 0, \, \omega = i/\tau
\]

and:

\[
\omega = i \frac{\Delta K + K \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}}{2\tau (K - \sin^2 \theta)} \pm \sqrt{\Delta K^2 + 2\Delta K (K - \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}) + (K - \sin^2 \theta - \frac{\sigma_T}{\epsilon_O})^2}
\]

(7.14)

which (except for \( \omega = 0 \)) lie all in the upper-half imaginary axis on the complex \( \omega \)-plane.

Letting, for neatness' sake:

\[
\rho_1 \equiv \frac{\Delta K + K \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}}{2\tau (K - \sin^2 \theta)} + \sqrt{\Delta K^2 + 2\Delta K (K - \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}) + (K - \sin^2 \theta - \frac{\sigma_T}{\epsilon_O})^2}
\]

(7.15)

\[
\rho_2 \equiv \frac{\Delta K + K \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}}{2\tau (K - \sin^2 \theta)} - \sqrt{\Delta K^2 + 2\Delta K (K - \sin^2 \theta + \frac{\sigma_T}{\epsilon_O}) + (K - \sin^2 \theta - \frac{\sigma_T}{\epsilon_O})^2}
\]

(7.16)
where:

\[ \rho_1 > \rho_2 \quad (7.17) \]

we then have the following four branch points:

\[ \omega = 0, \ i\rho_2, \ i/\tau, \ i\rho_1 \quad (7.18) \]

Now there is an interesting relationship between \( \rho_1, 1/\tau, \rho_2 \), found as follows:

The trinomial

\[ T(p) = -\tau(K - \sin^2 \theta)p^2 + (K - \sin^2 \theta + \Delta K + \frac{\sigma T}{\varepsilon_0})p - \frac{\sigma}{\varepsilon_0} \quad (7.19) \]

can be written, in terms of its roots, as:

\[ T(p) = -\tau(K - \sin^2 \theta)(p - \rho_1)(p - \rho_2) \quad (7.20) \]

Now substituting in the former, \( p = 1/\tau \), we obtain:

\[ T(1/\tau) = \frac{\Delta K}{\tau} > 0 \quad (7.21) \]

Hence because of (7.20) it must be that

\[ \rho_1 > \frac{1}{\tau} > \rho_2 \quad (7.22) \]

Consequently, the order of the branch points is:

\[ 0 < \rho_2 < \frac{1}{\tau} < \rho_1 \quad (7.23) \]

It is worth noting, at this point, that the frequency-independent model can be obtained as a limit when \( 1/\tau \to 0 \). Indeed, by means of the last inequality, we get:
\[ \frac{1}{\tau} \to 0 \Rightarrow \rho_2 = 0 \]  
\[ (7.24) \]

and from the definition of \( \rho_1 \) (7.15), we obtain:

\[ \frac{1}{\tau} \to 0 \Rightarrow \rho_1 = \frac{\sigma}{\varepsilon_o (K - \sin^2 \theta)} = \xi \]

the frequency-independent parameters case, where the number of branch points reduces to two.

Now we evaluate the integral function \( I_1^{(0)} \), as was defined by (3.6) (Section 3.1), with the Debye model:

\[
I_1^{(0)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\omega \Delta K \{ (1+i\omega t) \frac{\sigma}{\varepsilon_o} \}}{\alpha + i\omega} \left\{ (1+i\omega t) \{ (1-K) \omega + i\frac{\sigma}{\varepsilon_o} \} - \omega \Delta K \right\} \]

\[ \times -2\cos \theta \sqrt{\frac{(1+i\omega t) \{ (1+i\omega t) \{ \omega^2 (K - \sin^2 \theta) - i\frac{\sigma \omega}{\varepsilon_o} \} + \Delta K \omega^2 \}}{\text{e}^{i\omega t} \text{d}\omega}} \]
\[ (7.25) \]

This integration can be separated into two parts. First we integrate the analytic part of the function, which, by closing the contour on the upper complex semiplane and including the non-removable singularity at \( \omega = i\alpha \), is given by its residue contribution (Figure 7.1). Second, we integrate the part containing the square-root function. This will be equal to its residue-contribution term (non-singular case) minus the branch-cut contribution computed along the branch-cut segments as shown in Figure 7.1. We thus obtain:
Figure 7.1. The complex plane for the simple Debye model.

\[ I_1(0) = \frac{\alpha \Delta K + (1-\alpha) [\alpha (K+\cos \theta) - \frac{\sigma}{\varepsilon_0}] - 2\cos \theta}{[(1-\alpha) [(1-K) \alpha + \frac{\sigma}{\varepsilon_0}] - \alpha \Delta K] e^{\alpha t}} \]

\[ \cdot \, \text{sign}(1-\alpha) \sqrt{(1-\alpha) [(1-\alpha) (\alpha^2 (K-\sin^2 \theta) - \frac{\sigma \alpha}{\varepsilon_0}) + \alpha^2 \Delta K]} \]

\[ \rho_2 \int_{0}^{\infty} \frac{(1-\gamma) [(1-\gamma) \{\frac{\sigma \gamma}{\varepsilon_0} - \gamma^2 (K-\sin^2 \theta)] - \Delta K \gamma^2]}{(\alpha-\gamma) [(1-\gamma) [(1-K) \gamma + \frac{\sigma}{\varepsilon_0}] - \gamma \Delta K]} e^{-\gamma t} d\gamma + \]

\[ \rho_1 \int_{1/\tau}^{\infty} \frac{(1-\gamma) [(1-\gamma) \{\frac{\sigma \gamma}{\varepsilon_0} - \gamma^2 (K-\sin^2 \theta)] - \Delta K \gamma^2]}{(\alpha-\gamma) [(1-\gamma) [(1-K) \gamma + \frac{\sigma}{\varepsilon_0}] - \gamma \Delta K]} e^{-\gamma t} d\gamma \]

(7.26)
As noted before in this section, in the limit $1/\tau \to 0$ we would expect to reduce the frequency-dependent model to the frequency-independent case. Indeed, because of (7.24) and $\lim_{t \to \infty} \rho_1 = \xi$, for $1/\tau \to 0$, (7.26) yields (3.7). Equivalently we may obtain the constant-parameters model by letting $\Delta K \to 0$.

There are two singular cases, occurring when the singularity of excitation ($\omega = \imath \alpha$) enters upon either branch cut (Figure 7.1); the corresponding results are as follows:

For $\rho_1 > \alpha > 1/\tau$:

$$I_1^{(0)} = \frac{\alpha \Delta K + (1-\alpha \tau) [\alpha (K+\cos \theta) - \frac{\alpha}{\xi_o}]}{(1-\alpha \tau) [(1-K)\alpha + \frac{\alpha}{\xi_o} - \alpha \Delta K]} \cdot e^{-\alpha t} -$$

$$\rho_1 \sqrt{(1-\tau y) [(1-\tau y) \left( \frac{\alpha y}{\xi_o} - y^2 (K-\sin^2 \theta) \right) - y^2 \Delta K]} e^{-y \tau dy}$$

$$+ \frac{2 \cos \theta}{\pi} \int_{\pi}^{1/\tau} \frac{\rho_2 \sqrt{(1-\tau y) [(1-\tau y) \left( \frac{\alpha y}{\xi_o} - y^2 (K-\sin^2 \theta) \right) - y^2 \Delta K]} e^{-y \tau dy}}{((\alpha-y)) [(1-\tau y) [(1-K) y + \frac{\alpha}{\xi_o} - y \Delta K]}$$

and for $\rho_2 > \alpha > 0$, we get

$$I_1^{(0)} = \frac{\alpha \Delta K + (1-\alpha \tau) [\alpha (K+\cos \theta) - \frac{\alpha}{\xi_o}]}{(1-\alpha \tau) [(1-K)\alpha + \frac{\alpha}{\xi_o} - \alpha \Delta K]} \cdot e^{-\alpha t} -$$

$$\rho_1 \sqrt{(1-\tau y) [(1-\tau y) \left( \frac{\alpha y}{\xi_o} - y^2 (K-\sin^2 \theta) \right) - y^2 \Delta K]} e^{-y \tau dy}$$

$$+ \frac{2 \cos \theta}{\pi} \int_{\pi}^{1/\tau} \frac{\rho_2 \sqrt{(1-\tau y) [(1-\tau y) \left( \frac{\alpha y}{\xi_o} - y^2 (K-\sin^2 \theta) \right) - y^2 \Delta K]} e^{-y \tau dy}}{((\alpha-y)) [(1-\tau y) [(1-K) y + \frac{\alpha}{\xi_o} - y \Delta K]}$$

(7.27)
\[ -\frac{2\cos\theta}{\pi} \int_{0}^{\rho_2} \sqrt{(1-\tau y)\left\{ (1-\tau y)\left[ \frac{\sigma y}{\varepsilon_0} - y^2(K - \sin^2\theta) \right] - y^2\Delta K \right\} } \cdot e^{-yt} dy \]

\[ + \frac{2\cos\theta}{\pi} \int_{1/\tau}^{\rho_1} \sqrt{(1-\tau y)\left\{ (1-\tau y)\left[ \frac{\sigma y}{\varepsilon_0} - y^2(K - \sin^2\theta) \right] - y^2\Delta K \right\} } \cdot e^{-yt} dy \]

(7.28)

The expressions above are all real and separate the undis­torted pulse (residue contribution) from the pulse distor­tion (branch-cut contribution).

For the TM-case the procedure is analogous. There are four removable singularities for \( B_{1}^{+} \). The integration is performed using the same contours (Figure 7.1). The results are:

For the non-singular case:

\[ K_1^{(0)} = \frac{(1-\alpha\tau)^2\left[ \frac{\sigma}{\varepsilon_0} - \alpha^2(K - \sin^2\theta) \right] - \alpha^2(1-\alpha\tau)\Delta K - }{\left( (1-\alpha\tau)\left[ (K\alpha - \frac{\sigma}{\varepsilon_0})\cos^2\theta - \alpha\sin^2\theta \right] + \alpha\Delta K\cos^2\theta \right) } \cdot \\
\]

\[ -\frac{2}{\left( (1-\alpha\tau)\left[ (K\alpha - \frac{\sigma}{\varepsilon_0})\cos^2\theta - \alpha\sin^2\theta \right] + \alpha\Delta K \right) } \cdot e^{-\alpha\tau} \]

\[ \cdots \]
\[ + \frac{2\cos \theta}{\pi} \int_0^\rho_2 \]

\[
\frac{\sqrt{(1-\tau y) (K_y - \frac{\sigma}{\varepsilon_o} + \Delta K_y) (1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o}) (K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \cos^2 \theta \} - y^2 \Delta K}}{(1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o}) (K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \cos^2 \theta \}} \cdot e^{-yt} dy
\]

\[ \cdot [ (1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \]} (\alpha - y) \]

\[ - \frac{2\cos \theta}{\pi} \int_0^{\rho_1} \]

\[
\frac{\sqrt{(1-\tau y) (K_y - \frac{\sigma}{\varepsilon_o} + \Delta K_y) (1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o}) (K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \cos^2 \theta \} - y^2 \Delta K}}{(1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o}) (K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \cos^2 \theta \}} \cdot e^{-yt} dy \]

\[ \cdot [ (1-\tau y) \{(K_y - \frac{\sigma}{\varepsilon_o})\cos^2 \theta - y \sin^2 \theta \} + y \Delta K \]} (\alpha - y) \]

(7.29)

The singular cases are formed as in the TE case; the residue is then:

\[
(1-\alpha \tau)^2 \left[ \frac{\sigma \alpha}{\varepsilon_o} - \alpha^2 (K \sin^2 \theta) \right] - \alpha^2 (1-\alpha \tau) \Delta K - (1-\alpha \tau) (K \alpha - \frac{\sigma}{\varepsilon_o}) + \]

\[
\frac{\left[ (1-\alpha \tau) \{(K \alpha - \frac{\sigma}{\varepsilon_o}) \cos^2 \theta - \alpha \sin^2 \theta \} + \alpha \Delta K \cos^2 \theta \}\right] (1-\alpha \tau) \cdot \]

\[
\frac{+ \alpha \Delta K} {\{(K \alpha - \frac{\sigma}{\varepsilon_o}) \cos^2 \theta - \alpha \sin^2 \theta \} + \alpha \Delta K} \cdot e^{-\alpha \tau} \]

(7.30)
The integral terms are formed with the integrand in (7.29) in the same fashion as the integrals in (7.27) and (7.28).

### 7.3 Numerical Applications

A set of parameters \((a_m, b_m, \tau_m)\) for a Debye model selected to fit one set of the U. S. Geological Survey data for a subsurface radio propagation site at Raymond, Colorado is reported by Fuller and Wait (1972) as:

<table>
<thead>
<tr>
<th>(m)</th>
<th>(a_m)</th>
<th>(b_m)</th>
<th>(\tau_m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0</td>
<td>1555.0</td>
<td>2.93 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>200.0</td>
<td>6.74 \times 10^{-5}</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>38.0</td>
<td>4.48 \times 10^{-6}</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>25.2</td>
<td>9.05 \times 10^{-7}</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>1.8</td>
<td>2.40 \times 10^{-8}</td>
</tr>
</tbody>
</table>

where \(\sigma = 5 \times 10^{-4}\) mhos/m. We now take the parameters for each \(m\) separately and form a one-term Debye model; the branch points \(i\rho_1, i\rho_2\) are then as follows:

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\rho_1(\theta=0))</th>
<th>(\rho_1(\theta=90^\circ))</th>
<th>(\rho_2(\theta=0))</th>
<th>(\rho_2(\theta=90^\circ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2210.06/\tau_1</td>
<td>2516.99/\tau_1</td>
<td>.9143/\tau_1</td>
<td>.9143/\tau_1</td>
</tr>
<tr>
<td>2</td>
<td>489.00/\tau_2</td>
<td>556.90/\tau_2</td>
<td>.9505/\tau_2</td>
<td>.9505/\tau_2</td>
</tr>
<tr>
<td>3</td>
<td>35.41/\tau_3</td>
<td>40.31/\tau_3</td>
<td>.8724/\tau_3</td>
<td>.8728/\tau_3</td>
</tr>
<tr>
<td>4</td>
<td>9.41/\tau_4</td>
<td>10.66/\tau_4</td>
<td>.6634/\tau_4</td>
<td>.6665/\tau_4</td>
</tr>
<tr>
<td>5</td>
<td>1.22/\tau_5</td>
<td>1.26/\tau_5</td>
<td>.1350/\tau_5</td>
<td>.1494/\tau_5</td>
</tr>
</tbody>
</table>

The results of this table confirm earlier observations on the relative location of the branch points [Section 7.2, (7.23)]. The model with \(M = 1\) [Section 7.2, (7.9)] has
one relaxation. An approximation of the aforementioned five term Debye model ($M = 5$) with an $M = 1$ model is obtained as follows:

We choose:

\[ K = \Sigma a_m = 8.2 \]

\[ \Delta K = \Sigma b_m - \Sigma a_m = 1811.8 \] \hspace{1cm} (7.31)

Thus we retain the same permittivities for zero frequency and infinite frequency, \( \sigma = 5 \times 10^{-4} \) mhos/m as in the full Debye model. And we choose \( \tau \) so as to retain the same conductivity for infinite frequency:

\[ \frac{\Delta K}{\tau} = \frac{5}{\Sigma} \frac{b_m - a_m}{\tau_m} \] \hspace{1cm} (7.32)

from which we solve for \( \tau \):

\[ \tau = 1.668 \times 10^{-5} \] \hspace{1cm} (7.33)

This model allows for significant changes in both the real permittivity (which varies between 1820 and 8.2) and the conductivity (which varies between \( 5 \times 10^{-4} \) and \( 14.62 \times 10^{-4} \)) of the dispersive medium. Figure 19 (Appendix C) presents the real permittivity ($K_{\text{real}}$) and conductivity ($\sigma$) as functions of frequency.

Figure 20 (Appendix C) presents the normalized principal electric field ($e_{x1}$) for the case with frequency dependent parameters ($M = 1$ model) as compared with that for
the case of constant parameters. For both models, so that they are comparable, we have taken $K = 8.2$ and $\sigma = 5 \times 10^{-4}$. As input we have chosen pulse #II (Figure 2, Appendix C) at an angle of incidence $45^\circ$. We see that the early-time behavior for the two models is, for practical purposes, identical; this comes as no surprise, since in the frequency dependent case the permittivity levels off, to the value of the constant parameters model, for large frequencies (corresponding to the early-time behavior). The frequency dependent model exhibits a depressed late-time behavior with a narrow and sharply peaked negative lobe. The cross-over point occurs, for the frequency-dependent model, approximately $15$ ns earlier than the corresponding one for the constant-parameters model.

Figure 21 (Appendix C) presents the same comparison for the normalized principal magnetic field (TM-case). Again the early-time behavior differs very little from one model to the other, while there is a pronounced late time behavior for the frequency dependent model.

Another illustration is presented in Figures 22, 23, and 24 (Appendix C), where the same computations are repeated for the $(M = 1)$ model: $K = 14$, $\Delta K = 60$, $\tau = 1.989 \times 10^{-6}$ and $\sigma = 10^{-5}$ mhos/m. A broader incident pulse is chosen for this case (pulse #III, Figure 2, Appendix C) while the angle of incidence is kept at $45^\circ$. For the TE-case (Figure 23, Appendix C) the cross-over point occurs,
for the frequency dependent model, approximately 700 ns earlier than the corresponding one for the constant-parameters model. This larger time difference is because of the wider incident pulse (as compared to the previous illustration).

7.4 The General M Model

The general case of the Debye model for $K^*(i\omega)$ is to be treated along the same lines as the case $M = 1$ (Section 7.2). In the case of general $M$ we find, proceeding as previously, that branch points are at $\omega = 0$, $\omega = i/\tau_1$, $\lambda = 1, 2, \ldots, M$ and the roots of the equation $K^*(i\omega) = 0$. We will now prove an important theorem that assures a straightforward extension of the theory for the $M = 1$ model (Section 7.2) to the general model with $M > 1$.

**Theorem**: The branch points for the general Debye model lie all on the non-negative imaginary axis on the complex $\omega$-plane.

**Proof**: It suffices to prove the statement for the roots of the equation:

$$K^*(i\omega) = K + \frac{\sigma}{i\omega \tau_0} + \sum_{i=1}^{M} \frac{\Delta K_i}{1+i\omega \tau_i} = 0 \quad (7.34)$$

It is an alternative way to (7.1) for $K^*(i\omega)$, where:

$$K \geq 1 \quad (7.35)$$
\[ \Delta K_i, \tau_i \geq 0 \quad (7.36) \]

Now let the solutions to (7.34) be expressed in the form:
\[ \omega = \omega_1 + i\omega_2 \quad (7.37) \]
where \( \omega_1 \) and \( \omega_2 \) are real.

By substitution of (7.37) into (7.34), letting the imaginary part of the expression on the LHS be equal to zero, we obtain:
\[ \omega \left[ \frac{\sigma/\varepsilon_0}{\omega_1^2 + \omega_2^2} + \frac{M}{i} \frac{\Delta K_i \tau_i}{(1-\tau_i \omega_2)^2 + \tau_i^2 \omega_1^2} \right] = 0 \quad (7.38) \]

Since the quantity within the brackets is positive definite it turns out:
\[ \omega_1 = 0 \quad \text{q.e.d.} \quad (7.39) \]

The branch-cut integrations will thus be performed in the case of the general model on the imaginary axis, as was done in the case of \( M = 1 \) model (Figure 7.1).

To determine the location of the branch points we then substitute (7.39) and (7.37) into (7.34) and obtain:
\[ K \sin^2 \theta = \frac{\sigma}{\omega_2 \varepsilon_0} + \frac{M}{i} \frac{\Delta K_i}{1 - \omega_2^2 \tau_i} = 0 \quad (7.40) \]

Equation (7.40) is analogous to the familiar inhour equation of point reactor dynamics. Accordingly, and since
\[ K - \sin^2 \theta > 0 \]  \hspace{1cm} (7.41)

(7.40) possesses \( M + 1 \) positive roots (Hetrick, 1971), their qualitative behavior for \( M = 5 \) is shown in Figure 7.2. Consequently the branch cut in the case \( M = 5 \) will be as shown in Figure 7.3. The branch cut segments have as lower limits the points \( i/\tau_\ell \) (\( \ell = 1, 2, \ldots, 5 \)) and the origin and as upper limits the \( i\omega_{2m}'s \) where \( \omega_{2m} \) takes the six different values as depicted in Figure 7.2. The contour-integration paths surrounding the branch cut are also shown in Figure 7.3. Here, we have chosen the order of the constants \( \tau_m \) to be:

\[ \tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 \]  \hspace{1cm} (7.42)

Figure 7.2. Real branch points (qualitative plot) for the \( M = 5 \) Debye model.
Figure 7.3. The complex plane for the full Debye model ($M = 5$).
CHAPTER 8

OUTLOOK

We have examined in detail and have given analytical solutions to the problem of the interaction of transient EM pulses with a stratified dissipative medium, for the cases of two layers and three layers with a double exponential causal pulse input. We have examined also the effects of realistic frequency-dependent constitutive parameters on the results.

Actual extensions of this work in the case of four layers and more are possible, if need arises.

We have also left, for future work, numerical calculations incorporating the general Debye model for frequency dependent constitutive parameters. In light of our work (Chapter 7) this field seems now open.

Last, we think that the method of the residues for the multilayer case (Sections 5.4 and 6.4) deserves a closer look in the future. It may add new and useful information from the theoretical standpoint, and perhaps it may prove to be economically competitive with the existing numerical inversion method.
APPENDIX A

PHYSICAL CONSTANTS

\[ 1 \leq k \equiv \frac{\varepsilon_2}{\varepsilon_0} < 100 \] (frequency independent)

\[ \varepsilon_0^{-1} = 36\pi 10^9 \, \text{[farad/m]} \]

\[ 10^{-5} \, \text{[mhos/m]} \leq \sigma < 10^7 \, \text{[mhos/m]} \text{ at } 20^\circ \text{C} \]

\[ 10^3 \, \text{secs}^{-1} < \alpha, \beta < 10^9 \, \text{secs}^{-1} \] (pulse constants)

\[ 0 \leq \theta < \frac{\pi}{2} \] (angle of incidence)

\[ c = 2.9979 \times 10^8 \, \text{[m/sec]} \]
APPENDIX B

MATHEMATICAL CONCEPTS AND FUNDAMENTALS

B.1 Definitions

Definition 1. For \( a < x_1 < b \) and \( f(x) \) bounded on \([a,b]\), we define:

\[
\int_{a}^{b} \frac{f(x)}{(x_1-x)} \, dx = \lim_{\varepsilon \to 0} \left\{ \int_{a}^{x_1-\varepsilon} \frac{f(x)}{x_1-x} \, dx + \int_{x_1+\varepsilon}^{b} \frac{f(x)}{x_1-x} \, dx \right\}
\]

as the principal value integral of the singular integrand \( f(x)/(x_1-x) \) over \((a,b)\), when the RHS of the defining Eqn. exists. The double parentheses in the LHS integral are part of the special notation of the principal value integration which we use to clearly indicate the location of the singularity (Dudley et al., 1974).

Definition 2. We define as a removable singularity \( x_0 \), of a function \( f(x) \) the value of the independent variable \( x = x_0 \), for which \( f(x_0) \) does not exist and \( \lim_{x \to x_0} f(x) \) exists.

It is readily seen (Papazoglou, 1972) that removable singularities, encountered in this work, do not contribute residues.
B.2 Relations on Contour Integrals

We develop here integral relationships used in the text. We begin by defining the following functions:

\[ \phi_i(y) = \sqrt{\frac{\sigma_i y}{\epsilon_0}} - y^2 (K_{ri} \sin^2 \theta); \quad \phi_i^*(y) = \sqrt{y^2 (K_{ri} \sin^2 \theta) - \frac{\sigma_i y}{\epsilon_0}} \]

where: \( i = 2, 3 \), we have:

\[ \phi_i^2 + \phi_i^{*2} = 0 \]

\( \sigma_i, K_{ri} \) and \( \epsilon_0, \theta \) are all positive physical constants (Appendix A), such that:

\[ 1 < K_{ri}, \sigma > 0; \quad 0 \leq \theta < \frac{\pi}{2} \]

For our purposes, the \( \phi_i, \phi_i^* \) functions are generated from another set of functions \( (S_i(\omega)) \) of a complex variable \( \omega \), when in the latter we substitute \( \omega = iy \) or \( \omega = y \) respectively [Section 5.2, Eqns. (5.11) and (5.12)]. Thus we have:

Relations involving \( \phi_2(y) \)

Let \( \xi_2 \equiv \sigma_2/\epsilon_0 (K_{r2} \sin^2 \theta) > 0 \)

we observe that:

\[ \phi_2(0) = \phi_2(\xi_2) = 0 \]

and hence a branch cut for \( \phi_2 \) is:
Next we consider a real function of $y$, $f(y)$, non-singular on $[0, \xi_2]$, such that it does not possess any branch cuts. The integral along both sides of the branch cut is

$$\int_{BC} f(y) \phi_2(y) \, dy = -2 \int_{0}^{\xi_2} f(y) \phi_2(y) \, dy$$

The proof of this relation is simple, following the convention in Figure B.1, where the negative branch of $\phi_2$ is assumed on the left side of the branch cut and the positive branch of $\phi_2$ is on the right of the branch cut.

When a singularity exists within the range of integration (on the branch cut) we have:

$$\int_{BC} \frac{f(y) \phi_2(y)}{\alpha - y} \, dy = -2 \int_{0}^{\xi_2} \frac{f(y) \phi_2(y)}{((\alpha - y))} \, dy$$
where $0 < \alpha < \xi_2$. The configuration on the complex plane is here:

![Figure B.2. Singular case.](image)

In a similar fashion, we obtain the following relations:

\[
\begin{align*}
&i \int_{BC^{++}} f(y) \exp(-i \frac{Z}{C} \phi_2(y)) \, dy = -2 \int_{0}^{\xi_2} f(y) \sin(\frac{Z}{C} \phi_2(y)) \, dy \\
&\int_{BC^{++}} f(y) \phi_2(y) \exp(-i \frac{Z}{C} \phi_2(y)) \, dy = -2 \int_{0}^{\xi_2} f(y) \phi_2(y) \cos(\frac{Z}{C} \phi_2(y)) \, dy
\end{align*}
\]

where $Z$, $C$ are positive constants, and the following singular relations:
\[
\int_{BC++} f(y) \exp(-i\frac{Z_{\phi_2}(y)}{\alpha-y}) \, dy = -2 \int_{0}^{\xi_2} f(y) \sin(\frac{Z_{\phi_2}(y)}{C_{\phi_2}(y)}) \, dy - 2\pi f(\alpha) \cos(\frac{Z_{\phi_2}(\alpha)}{C_{\phi_2}(\alpha)})
\]

\[
\int_{BC++} f(y) \phi_2(y) \exp(-i\frac{Z_{\phi_2}(y)}{\alpha-y}) \, dy =
\]

\[
= -2 \int_{0}^{\xi_2} f(y) \phi_2(y) \cos(\frac{Z_{\phi_2}(y)}{C_{\phi_2}(y)}) \, dy + 2\pi f(\alpha) \phi_2(\alpha) \sin(\frac{Z_{\phi_2}(\alpha)}{C_{\phi_2}(\alpha)})
\]

Relations involving \( \phi_2 \) and \( \phi_3 \)

We let: \( \xi_3 = \sigma_3/(\varepsilon_0 (K_{r3} - \sin^2 \theta)) \) and observe that:

\( \phi_3(0) = \phi_3(\xi_3) = 0 \)

For the integral formulas containing both \( \phi_2 \) and \( \phi_3 \) it is imperative to state the relative position of the branch points \( i\xi_2 \) and \( i\xi_3 \). Thus we recognize the following two subcases:

Subcase for \( \xi_2 < \xi_3 \)

We draw the following schematic on the complex \( \omega \)-plane:
We integrate along the branch cut and obtain:

\[
\int_{BC^{++}} f(y) \phi_3(y) \exp(-i\frac{Z}{C} \phi_2(y)) \, dy = \xi_2
\]

\[
= -2 \int_0^{\xi_2} f(y) \phi_3(y) \cos\left(\frac{Z}{C} \phi_2(y)\right) \, dy - \xi_3
\]

\[
- 2 \int_{\xi_2}^{\xi_3} f(y) \phi_3(y) \exp\left(\frac{Z}{C} \phi_2^*(y)\right) \, dy
\]

\[
i \int_{BC^{++}} f(y) \phi_2(y) \phi_3(y) \exp(-i\frac{Z}{C} \phi_2(y)) \, dy = \xi_2
\]

\[
= -2 \int_0^{\xi_2} f(y) \phi_2(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \, dy + \xi_3
\]

\[
+ 2 \int_{\xi_2}^{\xi_3} f(y) \phi_2^*(y) \phi_3(y) \exp\left(\frac{Z}{C} \phi_2^*(y)\right) \, dy
\]

Figure B.3. The $\xi_2 < \xi_3$ case.
The singular formulas are as follows:

For $\xi_2 < a < \xi_3$:

$$
\int_{\text{BC}^{++}} \frac{f(y)\phi_3(y)\exp(-i\frac{Z}{C}\phi_2(y))}{a-y} \, dy = \frac{\xi_2}{2} f(y)\phi_3(y)\cos(\frac{Z}{C}\phi_2(y)) - \frac{\xi_3}{2} f(y)\phi_3(y)\exp(\frac{Z}{C}\phi_2(y))
$$

$$
= -2 \int_{0}^{\xi_2} \frac{f(y)\phi_3(y)\cos(\frac{Z}{C}\phi_2(y))}{a-y} \, dy
$$

$$
- 2 \int_{\xi_2}^{\xi_3} \frac{f(y)\phi_3(y)\exp(\frac{Z}{C}\phi_2(y))}{((a-y))} \, dy
$$

and for $a < \xi_2 < \xi_3$:

$$
\int_{\text{BC}^{++}} \frac{f(y)\phi_3(y)\exp(-i\frac{Z}{C}\phi_2(y))}{a-y} \, dy = \frac{\xi_2}{2} f(y)\phi_3(y)\cos(\frac{Z}{C}\phi_2(y)) + 2\pi f(a)\phi_3(a)\sin(\frac{Z}{C}\phi_2(a))
$$

$$
= -2 \int_{0}^{\xi_2} \frac{f(y)\phi_3(y)\cos(\frac{Z}{C}\phi_2(y))}{((a-y))} \, dy
$$
\[ \xi_3 \int \frac{f(y) \phi_3(y) \exp \left( \frac{i}{C_2} \phi_2(y) \right)}{a-y} \, dy \]

\[ - 2 \int \frac{f(y) \phi_2(y) \phi_3(y) \exp \left( -i \frac{Z}{C_2} \phi_2(y) \right)}{a-y} \, dy = \]

\[ \xi_2 \int \frac{f(y) \phi_2(y) \phi_3(y) \sin \left( \frac{Z}{C_2} \phi_2(y) \right)}{(a-y)} \, dy - \]

\[ - 2\pi \int \frac{f(a) \phi_2(a) \phi_3(a) \cos \left( \frac{Z}{C_2} \phi_2(a) \right)}{a-y} \, dy + \]

\[ + 2 \int \frac{f(y) \phi_2^*(y) \phi_3(y) \exp \left( -i \frac{Z}{C_2} \phi_2(y) \right)}{a-y} \, dy \]

Subcase \( \xi_3 < \xi_2 \)

The configuration on the complex plane is, in this case:

\[ \begin{array}{c}
\xi_2 \\
\xi_3 \\
0 \\
\end{array} \]

\( (-) \quad \begin{array}{c}
\xi_2 \\
\xi_3 \\
0 \\
\end{array} \quad \begin{array}{c}
i\xi_2 \\
i\xi_3 \\
+ \end{array} \]

Figure B.4. The \( \xi_3 < \xi_2 \) case.
The following identities are obtained:

\[ \int_{BC^{+\pm}} f(y)\phi_3(y)\exp(-i\frac{Z}{C}\phi_2(y))\,dy = \]

\[ \xi_3 \]

\[ = -2 \int_0^\xi_2 f(y)\phi_3(y)\cos\left(\frac{Z}{C}\phi_2(y)\right)\,dy - \]

\[ \xi_3 \]

\[ - 2 \int_{\xi_3}^{\xi_2} f(y)\phi_3^*(y)\sin\left(\frac{Z}{C}\phi_2(y)\right)\,dy \]

\[ i \int_{BC^{+\pm}} f(y)\phi_2(y)\phi_3(y)\exp(-i\frac{Z}{C}\phi_2(y))\,dy = \]

\[ \xi_3 \]

\[ = -2 \int_0^\xi_2 f(y)\phi_2(y)\phi_3(y)\sin\left(\frac{Z}{C}\phi_2(y)\right)\,dy \]

\[ \xi_3 \]

\[ + 2 \int_{\xi_3}^{\xi_2} f(y)\phi_2(y)\phi_3^*(y)\cos\left(\frac{Z}{C}\phi_2(y)\right)\,dy \]

The singular cases are as follows:

For \( \xi_3 < \alpha < \xi_2 \):

\[ \int_{BC^{+\pm}} \frac{f(y)\phi_3(y)\exp(-i\frac{Z}{C}\phi_2(y))}{\alpha-y}\,dy = \]

\[ \xi_3 \]

\[ = -2 \int_0^\xi_2 \frac{f(y)\phi_3(y)\cos\left(\frac{Z}{C}\phi_2(y)\right)}{\alpha-y}\,dy - \]

\[ \xi_3 \]

\[ - 2 \int_{\xi_3}^{\xi_2} \frac{f(y)\phi_3^*(y)\sin\left(\frac{Z}{C}\phi_2(y)\right)}{(\alpha-y)}\,dy - 2\pi f(\alpha)\phi_3^*(\alpha)\cos\left(\frac{Z}{C}\phi_2(\alpha)\right) \]
\[\begin{align*}
&i \int_{BC^{+\downarrow}} f(y) \phi_2(y) \phi_3(y) \exp\left(-i\frac{Z}{C} \phi_2(y)\right) \frac{dy}{\alpha-y} = \\
&\quad = \frac{\xi_3}{2} f(y) \phi_2(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad - 2 f(y) \phi_2(y) \phi_3(y) \cos\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad + 2 f(y) \phi_2(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad \cdot - 2 \pi f(\alpha) \phi_2(\alpha) \phi_3(\alpha) \sin\left(\frac{Z}{C} \phi_2(\alpha)\right)
\end{align*}\]

For \(\alpha < \xi_3 < \xi_2\)

\[\begin{align*}
&i \int_{BC^{+\downarrow}} f(y) \phi_3(y) \exp\left(-i\frac{Z}{C} \phi_2(y)\right) \frac{dy}{\alpha-y} = \\
&\quad = \frac{\xi_3}{2} f(y) \phi_3(y) \cos\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad - 2 f(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad + 2 f(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad \cdot - 2 \pi f(\alpha) \phi_3(\alpha) \sin\left(\frac{Z}{C} \phi_2(\alpha)\right)
\end{align*}\]

and:

\[\begin{align*}
&i \int_{BC^{+\downarrow}} f(y) \phi_2(y) \phi_3(y) \exp\left(-i\frac{Z}{C} \phi_2(y)\right) \frac{dy}{\alpha-y} = \\
&\quad = \frac{\xi_3}{2} f(y) \phi_2(y) \phi_3(y) \sin\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad - 2 f(y) \phi_2(y) \phi_3(y) \cos\left(\frac{Z}{C} \phi_2(y)\right) \\
&\quad + 2 f(y) \phi_2(y) \phi_3(y) \cos\left(\frac{Z}{C} \phi_2(y)\right)
\end{align*}\]
It is worth noting that all the aforementioned identities have real RHS, involving only real functions and real numbers. Thus the usefulness of these relationships, in that through them we are able to reduce all expressions in our analytical results to real quantities.

B.3. Evaluation of the Principal Value Integrals

For a real principal value integral, such as those encountered herein, we have shown (Dudley et al., 1974), that:

\[
\int_{\xi}^{y} \frac{F(y)}{(\alpha-y)} dy = \int_{0}^{\lambda} G(y) dy + \int_{0}^{\mu} f(vy) dy
\]

where: \(0 < \alpha < \xi\), \(F(y)\) is a continuous function possessing a derivative at \(\alpha\) and:

\[
G(y) \equiv \begin{cases} 
\frac{F(\alpha-y) - F(\alpha+y)}{y}; & y \neq 0 \\
-2F'(\alpha); & y = 0
\end{cases}
\]

\[
f(y) = \frac{F(\alpha+y)}{y}
\]
This formula is the direct result of the simple idea of folding the range of integration around the point of singularity, so that the combination of the unbounded left and right branches of the integrand yields a bounded integrand and consequently a regular definite integration. Due to its simplicity this technique is applicable in cases where any number of singularities may be present within the range of integration; in such a case one applies the method for each and every one of the singularities in succession. Under extreme conditions we may be required, at worst, to apply the formula repeatedly to obtain the desired result.

B.4 Proof of $|\lambda_{12}\lambda_{23}e^{-i2dk_2\cos\theta}| < 1$ for $\omega$ real

Let

\[ z_2 = x_2 + iy_2 = \sqrt{K_{\lambda_{23}}\sin^2\theta + \frac{\sigma_2}{i\omega\varepsilon_0}} \]

\[ z_3 = x_3 + iy_3 = \sqrt{K_{\lambda_{23}}\sin^2\theta + \frac{\sigma_3}{i\omega\varepsilon_0}} \]
The plus "+" subscript in front of each square root indicates the principal branch having positive real part.

We note that

\[ K_{r2} - \sin^2 \theta > 0 \]
\[ K_{r3} - \sin^2 \theta > 0 \]
and of course \( x_2, x_3 > 0 \)

Now:

\[ x_2 y_2 = -\frac{\sigma_2}{2\omega \varepsilon_0} \]
\[ x_3 y_3 = -\frac{\sigma_3}{2\omega \varepsilon_0} \]

hence we conclude that

\[ \text{sign}(y_2) = \text{sign}(y_3) = -\text{sign}(\omega) \]

Now

\[ \lambda_{12} = \frac{\cos \theta - x_2 - iy_2}{\cos \theta + x_2 + iy_2} \]

hence

\[ |\lambda_{12}| = \left| \frac{\cos \theta - x_2 - iy_2}{\cos \theta + x_2 + iy_2} \right| \]

and

\[ |\lambda_{12}| < 1 \quad \text{since } x_2 > 0 \]

Furthermore

\[ \lambda_{23} = \frac{x_2 - x_3 + i(y_2 - y_3)}{x_2 + x_3 + i(y_2 + y_3)} \]

and

\[ |\lambda_{23}| < 1 \quad \text{since } x_2, x_3 > 0 \text{ and } y_2 \cdot y_3 > 0 \]
Finally

\[ e^{-i2dk_2 \cos \theta_2} = e^{-i \frac{2d\omega}{c}(x_2 + iy_2)} \]

and

\[ e^{-i2dk_2 \cos \theta_2} \quad \text{and} \quad \frac{2d\omega}{c} |y_2| < 1 \]

since

\[ \text{sign} \left( \frac{2d\omega}{c} \right) = \text{sign} (\omega y_2) = \text{sign} (\omega) \text{sign} (y_2) = \]

\[ = - \text{sign}^2 (\omega) = -1 \]

Therefore:

\[ \text{Real } \omega \Rightarrow |\lambda_{12,23} e^{-i2dk_2 \cos \theta_2}| < 1 \quad \text{q.e.d.} \]
Figure 1. TE-case, interface fields.
Figure 2. Normalized input pulses.

Normalized input pulses:

I: $\alpha = 2.5 \times 10^8$, $\beta = 3 \times 10^8$

II: $\alpha = 5.9 \times 10^7$, $\beta = 2.1 \times 10^8$

III: $\alpha = 3.1 \times 10^6$, $\beta = 1.1 \times 10^8$
Figure 3. TE-output, for input I.
Figure 4. TE-output, for input II.
Figure 5. TE non-singular integrand.
Figure 6. TE-singular integrand.
Figure 7. TE rectified integrands (singular case).
Figure 8. TM-case interface fields.
Figure 9. TM-output, for input I.
Figure 10. TM-output, for input II.

\[ \sigma = 2 \times 10^{-2} \text{ mhos/m} \]
\[ \varepsilon_{r2} = 10. \]
\( \sigma = 2 \times 10^{-2} \, \text{m}^2/\text{m} \)
\( \epsilon_r = 10. \)

Figure 11. TM-output for input II, at 84°.
Figure 12. TE special case, three layers, input I.
Figure 13. TE three layers for input I.
Figure 14. TE three layers for input I (a).

\[ \begin{align*}
K_0 &= 1 \\
K_2 &= 30 \\
\sigma_2 &= 3 \times 10^{-2} \\
K_3 &= 7 \\
\sigma_3 &= 1 \times 10^{-3} \text{ mhos/m}
\end{align*} \]
Figure 15. TM special case, three layers input I.
Figure 16. TM special case, three layers input I at 84°.
Figure 17. TM three layers for input I.
Figure 18. TM three layers for input I (a).
Figure 19. Frequency dependent parameters.
Figure 20. TE-output for frequency dependent parameters and input II.
Figure 21. TM-output for frequency dependent parameters and input II.
Figure 22. Frequency dependent parameters (a).
Figure 23. TE output for frequency dependent parameters and input III.
Figure 24. TM output for frequency dependent parameters and input III.
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