LOGISTICS WITH COMPETING USERS

by

Ling Shen

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UMI Number: 9901678

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Ling Shen

entitled Logistics with Competing Users

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Phylosophy

P.B. Mirchandani 7/22/98
Date

7/22/98
Date

Frank U. Granallo 7/22/98
Date

Date

Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

P.B. Mirchandani 7/22/98
Dissertation Director Date

STATEMENT BY AUTHOR

SIGNED: Ling Shen

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DEDICATION

To my parents, Ruyu Liu and Weigong Shen

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ABSTRACT

This dissertation addresses a class of fundamental logistical problems where two or more potential users (or players) compete for a common set of resources. Each user has a criterion (cost or performance requirement) that he/she wishes to optimize. The users' criteria are often in conflict, that is, choosing a decision that optimizes one user's criterion may not also optimize the criteria of others. How should the resources be utilized to satisfy the user demands?

In this dissertation, optimization and game theoretical models are employed to examine the equilibrium points and efficiently find the frontier of non-dominated solutions to three logistics problems with competing users: (1) single machine scheduling, (2) network resource allocation and (3) assignment to multiple servers (queues). New cooperative game theoretic methods are developed to negotiate on the Pareto frontier.

In addition, a Stackelberg leader-follower game framework is introduced in a queueing system which includes both competitive users and competitive servers. The existence of a unique equilibrium is shown.

The models and methodologies developed in the dissertation can be applied in many areas, such as Internet pricing, scheduling resources among competitors, network routing of users' requirements, analysis of competitive market, etc.

Chapter 1

INTRODUCTION

This dissertation addresses a class of fundamental logistical problems where two or more potential users (or players) compete for a common set of resources. Each user has a criterion (cost or performance requirement) that he/she wishes to optimize. Users' criteria are often in conflict, that is, choosing a decision that optimizes one user's criterion may not also optimize the criteria of others. How should the resources be utilized to satisfy the user demands?

Due to the vastness and complexity of such logistical problems, we will only address some typical problem scenarios, and some aspects of these problems. In particular, we will study primarily the two-user case, but provide some extensions to N-user case when the extensions hold in a straight forward manner.

Three general themes in addressing such competing user problems are as follows:

1. Can we find a user Nash equilibrium[1]? Is such a Nash equilibrium unique?
2. Can we efficiently generate a nondominated (or Pareto optimal) solution[2] set over which the two users can negotiate?

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[1] A Nash equilibrium point is defined as a vector consisting of a decision alternative for each user with the property that no user is able to change decision unilaterally to increase his/her own pay-off (Nash, 1953).

[2] Nondominated solutions are the simultaneous decisions from which no user can obtain better pay-off without worsening the pay-off of another user. The set of all nondominated solutions are referred to as the Pareto frontier.

3. Can we characterize a negotiation process for the two users?

The general concepts of equilibria, Pareto solutions and the overview of the methodology will be discussed in section 1.1 and in the Appendix. We will explore the ideas of the first two themes in three typical logistic problems. For the last theme, we will present a new negotiation concept between the two conflicting parties where the Pareto frontier is discrete. In this chapter, the three problems and the new solution concept will be briefly introduced in section 1.2, 1.3, 1.4 and 1.5, respectively. The outline of the dissertation will be given in section 1.6.

1.1 Concepts of Game, Equilibria and Pareto Frontier

Many real-life decisions often require the resolution of conflicts among multiple objectives or multiple decision makers. For a single decision maker problem, which we will refer to as the single-user case, the decision maker makes some type of trade-off to come up with a decision or solution. Solutions of the single-user problems are most often found by multiple objective programming. In the literature, the method of sequential optimization, the ε-constraint method, the weighting method, distance-based and direction-based methods are most frequently applied to find sets of efficient solutions by reformulating and solving the problem as a single objective optimization problem (e.g., see Szidarovszky et al., 1986).

In the case of multiple decision makers or the multiple-user case, game theoretic models have been employed. Game theory can be grouped into two major classes, noncooperative and cooperative games. In a noncooperative game, where no cooperation is assumed among the players, and an equilibrium is often considered as the solution of the game. The concept of equilibrium was introduced by game theorists and others. In equilibrium, conflicts may be resolved so that none of the users can improve his/her payoff by unilaterally changing strategies. Since equilibria are inherently inefficient, it is the interests of all players to seek a nondominated solution to resolve the conflict. In a cooperative game, some kind of cooperation or interaction among the players is assumed. Rational players are able to improve their payoffs simultaneously by negotiating on the Pareto frontier. Hence the solution of a cooperative game is usually a Pareto optimal solution.

Harsanyi (1977) has divided the theory of rational behavior into two categories: game theory and ethics. Game theory is concerned with individuals who are pursuing their own interests and personal values against other individuals. Ethics, on the other hand, involves rational pursuit of the interests of the whole group. Noncooperative game theory belongs to the first category and almost all other models fall into the ethical category. Noncooperative games are often employed to model situation with conflicting players (i.e., competitive servers or users) where each player has an assumed objective. The Nash equilibrium point is usually considered as the solution. One practical way to find an "equilibrium" solution for two users is by a *virtual dynamic negotiation process* over the feasible set, as reported by Beroggi and Mirchandani (1996). Here a compromise or a solution is obtained when the process terminates or converges. Such solutions depend on the strategies being employed by the users of this process. In many cases, the solution reached by this process is not on the Pareto frontier. On the other hand, in a cooperative game, it is assumed that competitors decide on a solution from the Pareto frontier.

Negotiating methodologies over a Pareto set are usually divided into two classes: axiomatic and bargaining methods (Roth, 1979). The most important concepts and solution techniques will be reviewed in section 5.1. In this dissertation, a new fictitious negotiation method will be developed to select a compromise Pareto solution.

Most negotiation methods reported in the literature assume that the Pareto frontier can be either derived in a closed form or obtained as a set of discrete points. However it may be difficult to find the Pareto solutions. In this case, as Roth (1985) has demonstrated, if two parties negotiate over many alternatives and issues, then the likelihood of reaching a Pareto-optimal settlement is small. If the settlement is not Pareto-optimal, there is room for improvement. This is called *post-settlement* by Raiffa (1982,1985). Teich et al. (1996) proposed a heuristic post-settlement technique to improve a given settlement. Post-settlement issues are beyond the scope of the dissertation. We will focus on applications where the Pareto frontier is obtainable.

1.2 Single machine scheduling problem with two competitive users

The first competitive logistics problem that this dissertation will address is the scheduling of a single machine for two users. Here each user (or player) has a set of jobs to be processed on the machine. The machine can process only one job at a time. Each user would like to minimize a function related to the job completion times, where early completion times are preferred. For example, a user may have the objective of minimizing the total flow time.

In the analysis of the two-user machine scheduling problem on a single machine, a branch-and-bound algorithm will be presented to obtain the Pareto frontier. This problem is further illustrated in chapter 2.

1.3 Competing users for two queues/servers

Consider now the problem where two users compete for two non-identical substitutable servers, each with a separate queue of infinite capacity, utilizing a first-come-first-serve discipline. Each user generates jobs at a given rate. The cost of performing each job depends on the service prices and congestion level at the facilities. A user is able to split the demand to the servers to optimize its cost function. Given two servers and their service prices, it is assumed that users will operate at a Nash equilibrium. On the other hand, the two service providers compete for the market. Each provider may seek to maximize its net profit by selecting appropriate prices. This may be referred to as a Stackelberg leader-follower game (see, e.g., Stackelberg, 1934) in which service providers behave as leaders and impose their prices on the users who behave as followers.

There are many applications which involve many users, and each user selects a server to optimize his/her objective. These users can be Internet users, airline passengers, etc. In an Internet application, packets could be routed through a packet switched network to optimize the individual performance at minimum cost. On the other hand, network service providers, by providing incentives to the users through a proper pricing scheme, may compete for the market and maximize their expected net profits.

This problem is further illustrated in chapter 3. We will demonstrate the existence of a unique user Nash equilibrium in section 3.3. The Pareto frontier between the two providers will be characterized in section 3.4.

1.4 Network capacity allocation problem among competitive users

Consider a resource allocation problem where a number of users share a common set of resources, which can be modeled as a set of parallel links. Each user has a throughput demand. There is a usage cost and a capacity limit associated with each link. Users compete for the links with lowest costs. For example, several shippers may want to rent capacities from a number of ships (or aircrafts, trucks, etc.). Conflicts occur when the capacities demanded by the shippers exceed available ship capacities. In this dissertation a polynomial algorithm to obtain the Pareto frontier is developed for the case where two users compete for n parallel links. This problem is analyzed in Chapter 4.

1.5 A new negotiation concept in the case of discrete Pareto frontier

A fictitious negotiation process is usually applied for two conflicting players to select a compromise solution from Pareto frontier. Each player could either breakdown the negotiation process so that both players get disagreement payoff[1], or make a further compromise with the other player, or terminate the process by accepting the offer. In this type of dynamic negotiations, the main problem is to determine the order in which offers

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[1] Disagreement payoff is defined as the payoff for both players when the negotiation process breaks down.

are made. The *alternating offer* method of Osborne and Rubinstein (1990) is one of the most popular bargaining models. The process is complicated by the introduction of *breakdown probabilities*. In the model of Osborne and Rubinstein, the probability that a player breaks down the negotiation process is a constant. However, the breakdown probability of a player depends on both the rigidity[1] of the player and the goodness of the other player's offer. Moreover, the solution does not depend on the initial positions of the players. These drawbacks were overcome by the one-step solution concept of Szidarovszky and Shen (1997) in which offer-dependent breakdown probabilities were introduced. The related literature and this one-step solution concept will be reviewed in section 5.1 and 5.2.

In chapter 5, we will develop a one-step solution algorithm that can be applied on a discrete Pareto frontier to select a Pareto solution which maximizes the expect payoff of each individual decision maker. Users' schedules on a single machine will be used as an example to illustrate this solution concept. In addition, the idea of the one-step solution concept can also be applied to select a compromise solution from a piecewise linear Pareto frontier.

1.6 Outline of the dissertation

Given the current popularity of deregulation, it is not surprising that most of our resource allocation problems can be solved in the context of competitive economic market. Researcher in operations research, systems engineering and communication

[1] The rigidity of a player is a subjective figure to indicate the level of inflexibility of a player from the offer the other player has just made.

systems have begun to consider economics (specifically pricing) as a means of ensuring the realization of the potential profit. It is believed that the goal of providing an adequate level of performance and customer satisfaction lies beyond technology – in the realm of resource allocation mechanism based on economic theory. In this dissertation, the relationships between the resource (e.g., network bandwidth, machine capacity, service, etc.) providers and resource users (e.g., network subscriber, jobs, customers, etc.) will be characterized, and negotiation approaches will be studied.

The main contributions of this dissertation are (1) the study of logistical problems of competing users having individual criteria associated with single machine scheduling, queueing systems, and network resource allocation, (2) the derivation of results from optimization and game theoretic points of view, (3) the development of efficient methods to obtain nondominated solutions for two competing users and (4) the proposal of an approach to negotiate over the nondominated solution set.

The dissertation will be organized as follows. Fundamental concepts, results in decision making and game theory are briefly summarized in the Appendix. In chapter 2, nondominated solutions on a single machine scheduling problem will be analyzed. Then, a conflict arising in two-user two-server queueing systems will be introduced and investigated in chapter 3. A network resource allocation problem for multiple users will be presented in chapter 4. In chapter 5, a concept for negotiating over the Pareto frontier will be introduced. An example of conflicts on a single machine scheduling problem will be used to illustrate the methodology. Future research areas will be discussed in chapter 6.

Chapter 2

NONDOMINATED SOLUTIONS FOR THE TWO-USER SINGLE MACHINE SCHEDULING PROBLEM

2.1 Problem Scenarios and Determination of Pareto Frontier

In this chapter we consider the following single machine problem. Two users compete for processing time on a machine. Let the jobs of the two users be denoted by A and B. Each user has a performance criteria, which could be, for example, minimize total delay cost, minimize maximum tardiness, or minimize total flowtime, etc. In other words, both users prefer schedules with early completion times. Hence conflicts occur.

Previous reported work in multi-criteria scheduling has dealt primarily with problems where a single decision maker (or a single user) faces conflicting criteria. Here researchers have developed methods to come up an "optimum" decision. The most commonly used techniques are the sequential method (Smith 1956, Chen and Bulfin 1990, John and Sadowski 1984) and the weighting method (Peha, 1995). In the sequential method, the most important criterion is optimized first; the solution is considered as one of the constraints when the less important criteria are to be optimized. In the weighting method, each criterion is assigned a weight and the weighted summation of all criteria is optimized.

Recently game theoretic approaches have been utilized to model user choice, competition among users, and user equilibrium. From the game theoretic perspective, each user is considered as a player, who competes for some time on the machine. The game ends when each player has chosen a schedule for its job. Typically, for such a game, either the existence of a unique Nash equilibrium is proven, which is considered as the solution of a noncooperative game, or Pareto frontier is obtained on which two conflicting users can negotiate and select a nondominated solution. In this later case, the description of the Pareto frontier is critical because it provides the basis for further negotiation. However, this is not a trivial task even for the simple single machine scheduling setting. For example, see Agnetis et al. (1996) who have studied a job shop with two competing jobs with due dates and developed a polynomial algorithm to find the set of nondominated schedules.

Since Nash equilibrium is inherently inefficient, we assume that the two competing users are able to negotiate over the Pareto frontier. Our conflict resolution approach consists of two stages. In the first stage, all Pareto solutions are generated by a branch-and-bound procedure, which is the focus of this chapter. In the second stage, a negotiation concept is applied to find a "solution" on the Pareto frontier; this concept is discussed in chapter 5.

We assume in this single-machine problem that

(1) jobs are non-preemptive;

(2) jobs are ready for processing at $t=0$; and

(3) processing time, t_j, of each job j is known.

We assume a penalty $P_j(C_j)$ for job j in terms of its completion time C_j. We will consider two types of objectives for the two users: either minimize the total penalty, that is,

$$\text{minimize } z_A = \sum_{j \in A} P_j(C_j)$$

$$\text{minimize } z_B = \sum_{j \in B} P_j(C_j),$$

or minimize the maximum penalty, that is,

$$\text{minimize } z_A = \max_{j \in A}\{P_j(C_j)\}$$

$$\text{minimize } z_B = \max_{j \in B}\{P_j(C_j)\}.$$

In general, a branch-and-bound search procedure can be utilized to find the Pareto solutions. Each vertex corresponds to a sequence of some jobs of both types. Any one of the remaining jobs can be the next in the sequence. Therefore the number of out-going arcs from a vertex equals the number of jobs yet to be sequenced. The root vertex of the branch-and-bound tree corresponds to the empty set. At each vertex, the corresponding objective values can be easily updated. In the search process the following deletion rules can applied to reduce the size of the tree.

(a) If any vertex dominates another vertex with the same jobs, then cut (from further consideration) all branches from the dominated vertex and the branch to the vertex;

(b) If two vertices have the same objective values and have the same jobs but in different orders, then the two permutations are equivalent. Merge the two vertices as one vertex;

(c) If a vertex is dominated by a point on the Pareto frontier, then cut all branches from this vertex and the branch to the vertex; and

(d) If a new nondominated end-point (leaf vertex) is found, all points from the Pareto frontier that are dominated by the new end-point are dropped.

The above branch-and-bound procedure is computationally intensive. Its complexity is non-polynomial. However, as we show below, the branch-and-bound tree can be reduced to a binary tree when both types of jobs are in SPT (shortest processing time) sequence at any Pareto point. The next two theorems give sufficient conditions to simplify the branch-and-bound procedure.

Theorem 2.1 Assume that the following two conditions hold for all jobs:

(i) For every job, j, P_j is a regular measure, that is, $C_j < C_j^{'}$ implies that $P_j(C_j) \le P_j(C_j^{'})$.

(ii) If jobs $i, j \in A$ (or B) and $t_i < t_j$, then $P_i(t) - P_j(t)$ is non-decreasing in t.

When the objective of each user is to minimize the total penalty, then, for any nondominated objective pair (or any point on the Pareto frontier), there is a corresponding job sequence in which both user A jobs and user B jobs are in SPT order.

Proof. Assume one kind of job (say a user A job) is not in SPT sequencing. Then there are two adjacent jobs of user A which are in opposite order, $t_j > t_i$ (see Figure 2.1). Let t_B be the start of job j and T the completion time of job i in sequence S.

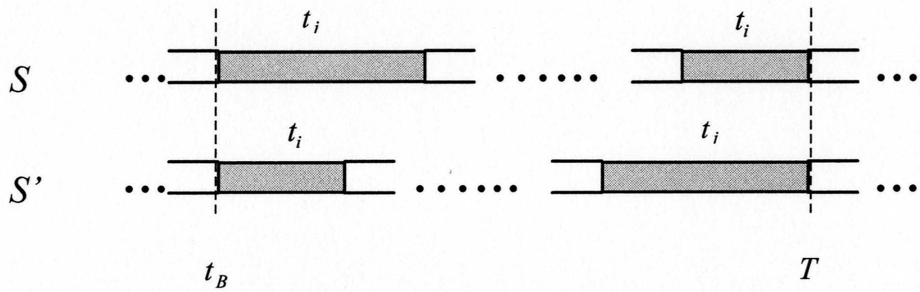

Figure 2.1 Interchanging adjacent jobs

We will show that by interchanging t_j and t_i, the resulting new sequencing, S', is equivalent to or dominates the original sequencing *S*.

Let *K* denote the set of jobs between t_j and t_i. Notice that they all belong to user *B*. Then $t_i < t_j$ implies that for all $k \in K$, C_k decreases after the interchange, and for all $k \in B$, $k \notin K$, C_k remains the same. Therefore with the interchange of t_j and t_i, $z_B(S') \leq z_B(S)$.

Notice also that if $l \in A$ and $l \neq i$, $l \neq j$, then C_l remains the same. Therefore

$$z_A(S) - z_A(S') = \left[P_j(t_B + t_j) + P_i(T)\right]$$

$$-\left[P_i(t_B + t_i) + P_j(T)\right]$$

$$= \left[P_i(T) - P_j(T)\right] - \left[P_i(t_B + t_i) - P_j(t_B + t_j)\right] \tag{2.1}$$

$$\geq \left[P_i(T) - P_j(T)\right] - \left[P_i(t_B + t_i) - P_j(t_B + t_i)\right] \geq 0,$$

since $T > t_B + t_i$,

which completes the proof. ■

Remarks. Condition (ii) in theorem 2.1 is a quite strong assumption. Assume that two jobs of the same user have the same completion time. If the shorter job has the larger penalty, then the later the completion time, the larger the incurred difference between the penalties of the two jobs. However, if the longer job has the larger penalty, then the later the completion time, the smaller the incurred difference between the penalties of the two jobs.

Theorem 2.2 When the objective of each user is to minimize the maximum job penalty, if condition (i) of Theorem 2.1 and the following condition holds, then, for each point on the Pareto frontier, there is a corresponding job sequence in which both user A jobs and user B jobs are in SPT order:

(iii) If jobs i and j$\in A$ (or B) and $t_i < t_j$, then for all $t \geq 0$, $P_i(t) \geq P_j(t)$.

Proof. This theorem can also be proven by pair-wise job interchanges.

Remarks. Condition (iii) is a much weaker assumption. If two jobs of the same user have the same completion times, then the one with shorter processing time incurs larger penalty.

2.2 Computational Experiments

One approach to demonstrate the efficiency of the deletion rules and SPT rules is to conduct some computational experiments (see Table 2.1). A number of problems were randomly generated. These are in columns 2 and 3, which give processing times for users A and B, respectively. The total number of sequences to be examined for complete enumeration is given in column 4. In these problems, suppose each user wishes to minimize the total flowtime, i.e., $\sum_{j\in A} C_j$ for user A and $\sum_{j\in B} C_j$ for user B.

Table 2.1 Deletion Rule Experiments

Proble m #	Job Processing Times (A)	Job Processing Times (B)	Number of Permutations	Number of nodes visited	Reduction (%)
1	2,4,8,10,16,20	5,8,10,15	10!	10871	99.7%
2	4,6,8,10,16,20	4,8,10,15	10!	11743	99.68%
3	4,6,16,20	2,10,12,15	8!	1964	95.13%
4	4,6,16,20	12,18,25,35	8!	1725	95.72%
5	14,16,16,20	2,8,25,35	8!	1764	95.62%
6	14,16,33,44	2,8,55,66	8!	1848	95.42%
7	23,24,26,44	12,18,55,66	8!	1822	95.48%
8	23,24,26,104	12,18,55,70	8!	1798	95.54%
9	2,24,26,104	12,18,35,70	8!	1777	95.59%
10	2,24,26,104	12,18,135,17	8!	1818	95.49%
11	2,20,80,140	150,160,165,	8!	1596	96.04%
12	2,20,80,120	150,160,180,	8!	1593	96.05%
13	2,20,80,120	15,160,180,2	8!	1767	95.62%
14	2,4,8,10,16	5,5,10	8!	1402	96.52%
15	2,4,8,8,16	5,15,25	8!	1804	95.53%
16	2,4,8,16,18	5,15,25	8!	1649	95.91%
17	12,14,18,36,4	5,15,25	8!	2070	94.87%
18	52,54,66,68,8	5,15,25	8!	1868	95.37%
19	52,54,66,68,8	5,100,200	8!	1623	95.97%
20	52,54,66,68,8	50,100,200	8!	1623	95.97%
21	52,54,66,68,1	50,100,200	8!	1601	96.03%
22	22,54,66,68,1	50,180,200	8!	1612	96%
23	2,4,66,88,188	50,180,200	8!	1588	96.06%
24	2,4,66,88,188,	50,180,200	9!	3936	98.92%
25	2,40,66,88,18	5,80,200	9!	4307	98.81%
26	2,14,16,28,35,	60,80,200	9!	4088	98.87%

Table 2.1 (continued)

Proble m #	Job Processing Times (A)	Job Processing Times (B)	Number of Permutations	Number of nodes visited	Reduction (%)
27	2,14,56,58,95,	60,80,200	9!	4046	98.89%
28	2,14,56,58,95,	6,8,100	9!	4171	98.85%
29	2,14,56,58,95,	6,80,100	9!	4170	98.85%
30	2,14,56,58,95,	600,800,1000	9!	3774	98.96%
31	22,44,55,66,8	6,8,10	9!	3963	98.91%
32	4,77,78,150,2	66,88,150	9!	3801	98.95%
33	50,60,70,80,9	71,81,91	9!	4301	98.81%
34	50,60,70,80,9	61,71,81,91	9!	5158	98.58%
35	5,60,80,100,1	6,71,81,91	9!	4956	98.73%
36	5,6,8,10,15	6,71,81,91	9!	4844	98.73%
37	2,66,88,101,1	600,700,800,	9!	4565	98.74%
38	26,62,88,111,	6,7,8,9	9!	4199	98.84%
39	26,62,88,111,	6,27,38,49	9!	4671	98.71%
40	2,8,10,16,24,3	10,15,20	9!	3716	98.98%
41	2,8,10,16,24,3	100,150,200	9!	3716	98.98%
42	2,8,10,16,24,3	10,150,200	9!	4128	98.86%
43	2,8,10,16,240,	10,15,200	9!	4153	98.86%
44	2,88,100,160,	10,15,200	9!	4172	98.85%
45	12,18,20,26,5	10,15,200	9!	4151	98.86%
46	12,18,20,26,5	15,200	9!	2961	99.18%
47	2,8,20,26,54,8	5,200	9!	2971	99.18%
48	2,8,22,66,88,9	5,200	9!	2963	99.18%
49	2,8,22,66,88,9	500,1000	9!	2691	99.26%

When we apply the four deletion rules for the branch-and-bound search procedure, the number of nodes (sequences) that were actually evaluated is shown in column 5. This is a reduction of over 95%. The experimental data showed that the reduction in terms of percentage tend to be larger in problem instants with more jobs. For example, the computational reduction is around 95% in eight-job problems, but ten-job problems achieved 99% of reduction.

In the above 49 problem instants, conditions (i) and (ii) hold. Therefore, we can also user SPT (shortest processing time) rule to generate all nondominated solutions efficiently. In this case, the branch-and-bound search is simplified to a binary tree. From each vertex, the two branches considered correspond to the next shortest user-A job and the next shortest user-B, respectively.

Table 2.2 Binary Tree Search

N_A	N_B	Number of permutations	Complexity of binary search	Reduction (%)
5	5	10!	$\binom{10}{5}$	99.9931%
5	4	9!	$\binom{9}{5}$	99.9653%
5	3	8!	$\binom{8}{5}$	99.86%
5	2	7!	$\binom{7}{5}$	99.58%
4	4	8!	$\binom{8}{4}$	99.83%
4	3	7!	$\binom{7}{4}$	99.31%

Denote the number of user A jobs by N_A and number of user B jobs by N_B. In general the complexity of the binary branch-and-bound procedure is $\binom{N_A + N_B}{N_A}$. Though it is not a polynomial number, the binary branch-and-bound procedure is an efficient algorithm to compute all nondominated sequences. Table 2.2 shows that the reduction achieved was more than 99% in the binary tree search.

Now we will show an example where Theorems 2.1 or 2.2 can be applied to reduce the tree to a binary branch-and-bound tree. Again, assume that minimizing the total flow-time is the objective for each user. User A has 4 jobs and user B has 2 jobs. The processing times are summarized in Table 2.3. From Theorem 2.1, we need only to examine the sequences where both type A and type B jobs are in SPT order. These sequences and corresponding objective values are shown in Table 2.4. We exclude any pair (z_A, z_B) that are dominated by others and obtain the set of nondominated solutions. In this small problem, the number of sequence that need to be evaluated is 15.

After the Pareto frontier is obtained, the two users can select a compromise solution through a negotiation process. We will address this issue in chapter 5.

Table 2.3 Job Processing Times

Job #	Type	Process Times
1	A	8
2	A	16
3	A	20
4	A	30
5	B	15
6	B	21

Table 2.4 Pareto Points

Sequence	z_A	z_B	Pareto
123456	150	199	(150,199)
123546	165	216	dominated
123564	186	139	(186,139)
125346	180	149	(180,149)
125364	201	119	(201,119)
125634	222	99	(222,99)
152346	195	133	(195,133)
152364	216	103	(216,103)
152634	237	83	(237,83)
156234	258	67	(258,67)
512346	257	128	dominated
512364	231	95	(231,95)
512634	252	75	(252,75)
516234	273	59	(273,59)
561234	294	51	(294,51)

Chapter 3

COMPETING USERS FOR TWO QUEUES/SERVERS

3.1 Introduction

This chapter addresses a scenario where multiple users compete for processing resources provided by multiple servers. In the meantime, the servers compete with each other to maximize their profit. Each user's cost is characterized by both a performance criterion and a usage cost. The performance criterion could be minimizing the average delay, or maximizing the throughput. Users' jobs arrive randomly. Upon arrival, each user selects a server to minimize the cost. On the other hand, servers may have different pricing scheme to achieve the maximum profit.

The *incentive-compatible pricing* mechanism is a well known and widely used to determine an efficient allocation of service capacities among competing (self-interested) users. Low prices of a server provide incentives for users to be more likely to select it. However, due to the limited capacity, the server could got congested when there are too many users. In spite of the low prices, users may then choose other servers in order to achieve better performance (e.g. delay, throughput, etc.).

A number of studies have been conducted recently to examine the user allocation problem in queueing systems via appropriate pricing scheme.

Lee and Cohen (1985) studied competitive allocation of customers to servers to minimize customers' expected waiting times. Reitman (1991) examined competitive capacity and pricing decisions when customers' demands depend on price and delay, and numerically computed a Nash equilibrium with respect to capacities and prices for several examples. Li and Lee (1994) and Loch (1991) studied duopolistic pricing with time-sensitive customers when delay is modeled by queueing relations. Loch assumed that customers' choices are based on long-run average behavior. The model of Li and Lee analyzes the case when customers observe queue lengths before choosing a queue. Mendelson and Whang (1990) considered a system that is modeled as an M/M/1 queueing system with multiple user classes. They derived an optimal *incentive-compatible* price-setting scheme when service requirements are stochastic and heterogeneous, meaning that the arrival rates and execution priorities jointly maximize the expected net value of the system while being determined on a decentralized basis. That is, while each user makes an individual decision on whether or not to join the system and at what priority level, these decisions maximize the objective of the system as a whole. Mendelson (1985) analyzed a special case where all jobs are homogeneous in their time values and expected service requirement. Korilis et al. (1997) studied decentralized routing control schemes in parallel telecommunication networks. They showed that Nash equilibria are generically inefficient and lead to suboptimal network performance. In particular, they investigated *incentive-compatible* pricing schemes that drive the system to a target operating point that is considered efficient from the network's point of view.

The purpose of this chapter is to study the behavior of the competitive users and servers in the scenario described above. The Stackelberg (1934) leader-follower game is the appropriate framework to model such a scenario, where competitive servers, facilities or providers are modeled as leaders and users are the followers. Two interactive games are involved: the users' game and the servers' game.

Given service quality and prices, each user seeks to optimize its own cost and performance. Since no cooperation is assumed among the users, the user (Nash) equilibrium is usually considered as the solution of the users' game. The existence of a unique Nash equilibrium has been proven under the condition of diagonal strict convexity (Rosen, 1965). Such specific requirements are not generally satisfied. Orda et al. (1993) have shown the existence of the unique Nash equilibrium under more general assumptions. Unfortunately, none of these results provide any efficient method to obtain a Nash equilibrium. This chapter presents a constructive proof to a two user Nash equilibrium. Furthermore, monotonocity properties of the user equilibrium is analyzed in terms of server decision variables, hence the servers' Pareto frontier can be characterized.

When the servers compete for the users, the problem becomes a leader-follower game. The main challenge in this leader-follower game framework is to examine the behavior of the servers. The payoff functions are complicated by the assumption of user equilibrium. The price and service quality will affect user equilibrium, as reflected in the payoffs of the servers. We will investigate the Pareto frontier of the two-server case and characterize the solutions of the game. It will be interesting to note that the best choice for the servers is to play a cooperative game, where a large variety of conflict resolution methodology can be employed.

There is little literature on user/server behavior in the leader-follower game. In most early works on this type of game, it is assumed that homogeneous servers are to be shared by competitive customers. Only a few recent papers have considered competitive heterogeneous customers with competitive heterogeneous servers. This chapter serves as an attempt to further investigate the latter case.

The chapter is organized as follows. In section 3.2 the user Nash equilibrium will be analyzed, and the existence of a unique equilibrium will be proved. Moreover, the algorithm for finding such an equilibrium will be presented. Profit function of the servers will be studied, and the server Pareto solutions will be characterized in section 3.3. In section 3.4 the two users are considered as one player (assuming the possibility of transfer of payment) and their joint optimum (or the "system" optimal) will be determined. The joint optimum of servers will be discussed in section 3.5. Conclusions will be drawn in section 3.6.

3.2 Modeling the Competing Users/Servers

Suppose the two nonidentical servers (providers or stations) are denoted by A and B, each with a separate queue of infinite capacity utilizing a first-come-first-serve (*FCFS*) discipline. Jobs (e.g., packets in a communication network or customers in a service system) can be processed by either of the two servers. Suppose there are two types of jobs, each type belonging to a different user. Jobs of user i will also be called type-i jobs. Each user seeks to optimize a cost or other performance measure by controlling the routing of its demand. This gives rise to a noncooperative game.

Each server charges an inventory holding cost for each job in the queue and service cost for each job processed. We will assume that it takes a random amount of time to process a job. In our model, both offered prices and service time distributions are considered to be independent of job type.

Assume that jobs arrive according to a Poisson process. Let the arrival rate of type-i jobs be λ_i. Each user allocates its demand by splitting it into two streams for the two servers. Upon arrival, each job is allocated to a server based on the allocation percentages. With, say, probability p_i, a type-i job is sent to server A, and with probability $1-p_i$ it is sent to server B.

3.3 Equilibrium Solution for Users

As mentioned earlier, we assume that users consider only their own interest and do not cooperate with each other in managing the servers. This situation can be mathematically modeled as a noncooperative game.

The model for the above scenario consists of two *FCFS* servers with exponentially distributed service time and class independent service rates, μ_j (where $j = A$ or B). Let b_j be the service charge and h_j be the unit time inventory holding cost. In this model, we assume each user-i will select p_i to minimize the expected costs per job given by

$$S_i(\mathbf{p}) = h_A p_i W_A + h_B(1-p_i)W_B + b_A p_i + b_B(1-p_i) \qquad (3.1)$$

where W_A and W_B are the average waiting time for server A and server B, respectively. Notice that every server can be modeled as an M/G/1 queue. Since no user will select a server with unstable queue, it is reasonable to assume that all queues are stable (i.e., having utilization less than 1). Therefore, W_A and W_B can be derived as

$$W_A = \frac{\sum_{i=1}^{2} \lambda_i p_i}{\mu_A(\mu_A - \sum_{i=1}^{2} \lambda_i p_i)} \qquad (3.2)$$

and

$$W_B = \frac{\sum_{i=1}^{2} \lambda_i (1-p_i)}{\mu_B(\mu_B - \sum_{i=1}^{2} \lambda_i (1-p_i))}. \qquad (3.3)$$

The strategy set of each user is the unit interval, $p_i \in [0,1]$, which is closed, convex and bounded. Therefore the objective functions are continuous. Moreover, it is convex in its own strategy. Therefore, the well known result of Nikaido and Isoda (1955) implies the existence of at least one user Nash equilibrium. However, the proof of this theorem is based on fixed point theorems. It is not constructive and does not provide a practical method to find the equilibrium.

The uniqueness of an equilibrium in a N-person game is guaranteed when the objective functions are strictly diagonally convex (Rosen, 1965). In our problem, this condition is not necessarily satisfied. In a similar network problem, Orda et al. (1993) have shown the existence and uniqueness of the equilibrium among the users. However the special conditions required in their proof do not hold in our case. Therefore, in our analysis, we cannot use these earlier results. Instead, a new constructive proof is presented below which guarantees the existence of a unique equilibrium and, in addition, provides an efficient algorithm to find it. For the sake of simplicity, we consider first the case with only two users (or players).

Theorem 3.1 When two users compete for two servers, there is a unique user equilibrium.

Proof. From (3.2) and (3.3), the average waiting times in the two queues are

$$W_A = \frac{\lambda_1 p_1 + \lambda_2 p_2}{\mu_A(\mu_A - \lambda_1 p_1 - \lambda_2 p_2)} \tag{3.4}$$

and

$$W_B = \frac{\lambda_1(1-p_1) + \lambda_2(1-p_2)}{\mu_B(\mu_B - \lambda_1(1-p_1) - \lambda_2(1-p_2))}, \tag{3.5}$$

respectively. Hence the expected system cost of type-1 and type-2 jobs are

$$S_1(p_1, p_2) = h_A p_1 W_A + h_B(1-p_1)W_B + b_A p_1 + b_B(1-p_1) \tag{3.6}$$

and

$$S_2(p_1, p_2) = h_A p_2 W_A + h_B(1-p_2)W_B + b_A p_2 + b_B(1-p_2) \tag{3.7}$$

respectively.

Simple algebra shows that

$$\frac{\partial S_1(p_1,p_2)}{\partial p_1}=\frac{h_A(\lambda_1 p_1+\lambda_2 p_2)}{\mu_A(\mu_A-\lambda_1 p_1-\lambda_2 p_2)}+\frac{h_A\lambda_1 p_1}{(\mu_A-\lambda_1 p_1-\lambda_2 p_2)^2}$$
$$-\frac{h_B(\lambda_1(1-p_1)+\lambda_2(1-p_2))}{\mu_B(\mu_B-\lambda_1(1-p_1)-\lambda_2(1-p_2))}-\frac{h_B\lambda_1(1-p_1)}{(\mu_B-\lambda_1(1-p_1)-\lambda_2(1-p_2))^2}+b_A-b_B.$$

(3 8)

and

$$\frac{\partial S_2(p_1,p_2)}{\partial p_2}=\frac{h_A(\lambda_1 p_1+\lambda_2 p_2)}{\mu_A(\mu_A-\lambda_1 p_1-\lambda_2 p_2)}+\frac{h_A\lambda_2 p_2}{(\mu_A-\lambda_1 p_1-\lambda_2 p_2)^2}$$
$$-\frac{h_B(\lambda_1(1-p_1)+\lambda_2(1-p_2))}{\mu_B(\mu_B-\lambda_1(1-p_1)-\lambda_2(1-p_2))}-\frac{h_B\lambda_2(1-p_2)}{(\mu_B-\lambda_1(1-p_1)-\lambda_2(1-p_2))^2}+$$
$$b_A-b_B. \qquad (3.9)$$

Note that $\frac{\partial S_i(p_1,p_2)}{\partial p_i}$ is strictly monotone increasing in both p_1 and p_2. If

$$\left.\frac{\partial S_i(p_1,p_2)}{\partial p_i}\right|_{p_1=0,p_2=0}>0,\quad i=1,2,$$

then, at equilibrium, either $p_1=0$ or $p_2=0$. If

$$\left.\frac{\partial S_i(p_1,p_2)}{\partial p_i}\right|_{p_1=1,p_2=1}>0,\quad i=1,2,$$

then, at equilibrium, either $p_1=1$ or $p_2=1$. In both cases, the equilibrium at a boundary point is called a corner equilibrium. When corner equilibria occur, the Nash equilibrium cannot be an interior point.

Without loss of generality, let $\lambda_1>\lambda_2$. We now examine the Nash equilibrium for the following five cases, in which the range for (b_A-b_B) covers the entire real line. In the first four cases, we have boundary equilibria, and in the last case, we have a unique interior equilibrium.

Case 1:

$$b_A - b_B \geq \frac{h_B(\lambda_1+\lambda_2)}{\mu_B(\mu_B-\lambda_1-\lambda_2)} + \frac{h_B\lambda_1}{(\mu_B-\lambda_1-\lambda_2)^2}. \tag{3.10}$$

Substituting in (3.8) and (3.9), at $p_1 = p_2 = 0$, this implies

$$\left.\frac{\partial S_i(p_1,p_2)}{\partial p_i}\right|_{p_1=0,p_2=0} \geq 0,\ i=1,2.$$

These relations hold if and only if $p_1 = 0$ and $p_2 = 0$ is the unique Nash equilibrium.

Case 2:

$$\frac{h_B(\lambda_1+\lambda_2)}{\mu_B(\mu_B-\lambda_1-\lambda_2)} + \frac{h_B\lambda_1}{(\mu_B-\lambda_1-\lambda_2)^2} > b_A - b_B \geq \frac{h_B(\lambda_1+\lambda_2)}{\mu_B(\mu_B-\lambda_1-\lambda_2)} + \frac{h_B\lambda_2}{(\mu_B-\lambda_1-\lambda_2)^2}. \tag{3.11}$$

Substituting in (3.8) and (3.9), at $p_1 = p_2 = 0$, this implies

$$\left.\frac{\partial S_1(p_1,p_2)}{\partial p_1}\right|_{p_1=0,p_2=0} < 0 \text{ and } \left.\frac{\partial S_2(p_1,p_2)}{\partial p_2}\right|_{p_1=0,p_2=0} \geq 0.$$

Hence, at equilibrium, $p_2 = 0$.

Since

$$\left.\frac{\partial S_1(p_1,p_2)}{\partial p_1}\right|_{p_1=1,p_2=0} > 0,$$

there is a unique root, $p_1^* \in (0,1)$, to the equation

$$\left.\frac{\partial \pi_1(p_1,p_2)}{\partial p_1}\right|_{p_2=0} = 0.$$

Therefore, (p_1^*,0) is the unique Nash equilibrium. It is also easy to see that if there is an equilibrium (p_1^*,0) with some $p_1^* \in (0,1)$, then case 2 occurs.

Case 3:

$$b_B - b_A \geq \frac{h_A(\lambda_1+\lambda_2)}{\mu_A(\mu_A-\lambda_1-\lambda_2)} + \frac{h_A\lambda_1}{(\mu_A-\lambda_1-\lambda_2)^2}. \tag{3.12}$$

Substituting in (3.8) and (3.9), at $p_1 = p_2 = 1$, this implies

$$\left.\frac{\partial \pi_i(p_1,p_2)}{\partial p_i}\right|_{p_1=1,p_2=1} \leq 0, \ i=1,2.$$

This is the sufficient and necessary condition that $p_1 = 1$ and $p_2 = 1$ is the unique Nash equilibrium.

Case 4:

$$\frac{h_A(\lambda_1+\lambda_2)}{\mu_A(\mu_A-\lambda_1-\lambda_2)} + \frac{h_A\lambda_1}{(\mu_A-\lambda_1-\lambda_2)^2} > b_B - b_A \geq \frac{h_A(\lambda_1+\lambda_2)}{\mu_A(\mu_A-\lambda_1-\lambda_2)} + \frac{h_A\lambda_2}{(\mu_A-\lambda_1-\lambda_2)^2}. \tag{3.13}$$

Substituting in (3.8) and (3.9), at $p_1 = p_2 = 1$, this implies

$$\left.\frac{\partial \pi_1(p_1,p_2)}{\partial p_1}\right|_{p_1=1,p_2=1} > 0 \text{ and } \left.\frac{\partial \pi_2(p_1,p_2)}{\partial p_2}\right|_{p_1=1,p_2=1} \leq 0.$$

Hence, at equilibrium, $p_2 = 1$.

Since

$$\left.\frac{\partial \pi_1(p_1,p_2)}{\partial p_1}\right|_{p_1=0,p_2=1} < 0,$$

there is a unique root, $p_1^{**} \in (0,1)$, to the equation

$$\left.\frac{\partial \pi_1(p_1,p_2)}{\partial p_1}\right|_{p_2=1} = 0.$$

Therefore, $(p_1^{**},1)$ is the unique Nash equilibrium. It is also easy to see that if there is an equilibrium $(p_1^{**},1)$ with some $p_1^{**} \in (0,1)$, then case 4 occurs.

Notice that corner equilibria of the form $(0, p_2)$ and $(1, p_2)$ are not possible since the corresponding inequalities are contradictory.

Case 5:

$$\frac{h_B(\lambda_1+\lambda_2)}{\mu_B(\mu_B-\lambda_1-\lambda_2)}+\frac{h_B\lambda_2}{(\mu_B-\lambda_1-\lambda_2)^2} > b_A - b_B > -\frac{h_A(\lambda_1+\lambda_2)}{\mu_A(\mu_A-\lambda_1-\lambda_2)}-\frac{h_A\lambda_1}{(\mu_A-\lambda_1-\lambda_2)^2}. \tag{3.14}$$

In this case, we may have only an equilibrium only in the interior decision space. It is easy to verify that

$$\left.\frac{\partial \pi_i(p_1,p_2)}{\partial p_i}\right|_{p_1=p_1^{*},p_2=0} < 0 \text{ or } \left.\frac{\partial \pi_i(p_1,p_2)}{\partial p_i}\right|_{p_1=p_1^{**},p_2=1} > 0,\ i=1,2,$$

where p_1^{*} is the root of

$$\left.\frac{\partial \pi_1(p_1,p_2)}{\partial p_1}\right|_{p_2=0} = 0,$$

when

$$b_A - b_B = \frac{h_B(\lambda_1 + \lambda_2)}{\mu_B(\mu_B - \lambda_1 - \lambda_2)} + \frac{h_B \lambda_2}{(\mu_B - \lambda_1 - \lambda_2)^2},$$

and p_1^{**} is the root of

$$\left.\frac{\partial S_1(p_1, p_2)}{\partial p_1}\right|_{p_2=1} = 0$$

when

$$b_B - b_A = \frac{h_A(\lambda_1 + \lambda_2)}{\mu_A(\mu_A - \lambda_1 - \lambda_2)} + \frac{h_A \lambda_2}{(\mu_A - \lambda_1 - \lambda_2)^2}.$$

Any interior equilibrium can be found by solving the following two equations:

$$\frac{\partial S_1(p_1, p_2)}{\partial p_1} = 0$$

and

$$\frac{\partial S_2(p_1, p_2)}{\partial p_2} = 0.$$

Let

$$K = \lambda_1 p_1 + \lambda_2 p_2.$$

and

$$F(K) = \left(\frac{\partial S_1(p_1, p_2)}{\partial p_1} + \frac{\partial S_2(p_1, p_2)}{\partial p_2}\right)/2$$

Simple algebra shows that

$$F(K) = \frac{h_A K}{\mu_A(\mu_A - K)} + \frac{h_A K}{2(\mu_A - K)^2} - \frac{h_B(\lambda_1 + \lambda_2 - K)}{\mu_B(\mu_B - \lambda_1 - \lambda_2 + K)} - \frac{h_B(\lambda_1 + \lambda_2 - K)}{2(\mu_B - \lambda_1 - \lambda_2 + K)^2} +$$

$$b_A - b_B. \tag{3.15}$$

As K increases, every term increases, so $F(K)$ is strictly monotone increasing in K. From the previous derivation,

$$F(\lambda_1 p_1^*) < 0 \text{ and } F(\lambda_1 + \lambda_2) > 0.$$

Therefore, there is a unique root, K^*, of $F(K)$. Substitution into (3.8) and (3.9), yields the unique interior Nash equilibrium (p_1 , p_2):

$$p_1 = \left(\frac{h_B(\lambda_1 + \lambda_2 - K)}{\mu_B(\mu_B - \lambda_1 - \lambda_2 + K)} + \frac{h_B \lambda_1}{(\mu_B - \lambda_1 - \lambda_2 + K)^2} - \frac{h_A K}{\mu_A(\mu_A - K)} - b_A + b_B \right) \Big/ \left(\frac{h_A \lambda_1}{(\mu_A - K)^2} + \frac{h_B \lambda_1}{(\mu_B - \lambda_1 - \lambda_2 + K)^2} \right)$$

$$p_2 = \left(\frac{h_B(\lambda_1 + \lambda_2 - K)}{\mu_B(\mu_B - \lambda_1 - \lambda_2 + K)} + \frac{h_B \lambda_2}{(\mu_B - \lambda_1 - \lambda_2 + K)^2} - \frac{h_A K}{\mu_A(\mu_A - K)} - b_A + b_B \right) \Big/ \left(\frac{h_A \lambda_2}{(\mu_A - K)^2} + \frac{h_B \lambda_2}{(\mu_B - \lambda_1 - \lambda_2 + K)^2} \right)$$

Hence the two users always have a unique equilibrium. ∎

From the proof of Theorem 3.1, some properties of the user Nash equilibrium can be summarized in the following corollaries.

Corollary 3.1 The value of K^* is non-decreasing and continuous in h_B.

Proof. Notice that in each of the 5 cases, the equilibrium values of p_i are either constants (0 or 1) or the solution of continuous, strictly monotonic equations. Also, in each case, the implicit function theorem implies that the equilibrium values of p_i are continuous in the model parameters. In addition, at the boundary between two cases, the same solution is obtained regardless of the case assumed. Specifically as h_B increases, p_i (i = 1,2) remains unchanged in cases 1 and 3, p_1 increases and p_2 remains constant in cases 2 and 4. In the case of interior equilibrium (case 5), the value of K^* is obtained by finding the root of the monotonic $F(K)$, which also implies that K^* is non-decreasing in h_B. ■

Corollary 3.2 The value of K^* is non-increasing and continuously in h_A.

Remark. Since the user cost function (3.1) complies with the assumptions of the "*Type-A functions*" that Orda et al. (1993) defined in the N-user case, their results directly imply the existence of a unique Nash equilibrium as well as corollary 3.1 and 3.2. However they did not give a method to compute this equilibrium, which we do above.

3.4 Pareto Solutions and Nash Equilibria for the Competing Servers

As we have shown in section 3.2, at the user Nash equilibrium, more customers would select server A when $(b_A - b_B)$ decreases. For the sake of simplicity, we assume that each server has only one decision variable, the unit holding charge. Assume that h_A ranges from 0 to $\overline{H}_A$, h_B ranges from 0 to $\overline{H}_B$. With any fixed pair (h_A, h_B), it is assumed that the customers always select the equilibrium solution (p_1, p_2). Since the payoff of each server depends on only the aggregated customer arrival rate to that server at user equilibrium, the expected aggregated customer arrival rate to server A at user equilibrium will be denoted by K throughout this section to simplify the notation. Every formula is expressed in terms of K. Nonetheless, the expected total arrival rate to server B is $\lambda_1 + \lambda_2 - K$. Then the corresponding payoffs of the two servers are as follows:

$$J_A(b_A, h_A) = \lambda_1 p_1 (W_A h_A + b_A) + \lambda_2 p_2 (W_A h_A + b_A) = K W_A h_A + K b_A \tag{3.16}$$

and

$$J_B(b_B,h_B) = \lambda_1(1-p_1)(W_B h_B + b_B) + \lambda_2(1-p_2)(W_B h_B + b_B)$$
$$= (\lambda_1 + \lambda_2 - K)W_B h_B + (\lambda_1 + \lambda_2 - K)b_B. \tag{3.17}$$

Based on the monotone properties of the user equilibrium, we may examine the partial derivatives, $\partial J_A / \partial h_A$ and $\partial J_B / \partial h_B$, and hence characterize the Pareto frontier of the servers. In the next several pages, we will develop six lemmas to show that the Pareto frontier is continuous, monotone in h_A, h_B, and either h_A, h_B, or both are at the maximum.

Lemma 3.1. For any fixed pair $(h_A, h_B) \in [0, \overline{H}_A] \times [0, \overline{H}_B]$, the corresponding server payoffs form a continuous curve in the (J_A, J_B) plane.

Proof. In corollaries 3.1 and 3.2, we have shown that K is continuous in h_A and h_B. Furthermore, the expected waiting times, W_A and W_B are continuous in K. Therefore, (J_A, J_B) given by (3.16) and (3.17) is a continuous curve. ■

Lemma 3.2 Assume that $0 \le h_A^{(1)} < h_A^{(2)} \le \overline{H}_A$ and $0 \le h_B^{(1)} < h_B^{(2)} \le \overline{H}_B$. Then, for $i, j = 1,2$,

(i) $J_A(h_A^{(i)}, h_B^{(2)}) \ge J_A(h_A^{(i)}, h_B^{(1)})$,

(ii) $J_B(h_A^{(2)}, h_B^{(j)}) \ge J_B(h_A^{(1)}, h_B^{(j)})$, and

(iii) equality holds at a corner user-equilibrium.

That is, J_A increases in h_B and J_B increases in h_A.

Proof. From corollary 3.1, as h_B increases, K is non-decreasing. Therefore, since $K = \lambda_1 p_1 + \lambda_2 p_2$, from (3.4), W_A is also non-decreasing in h_B. Hence, J_A is non-decreasing in h_B. Similarly, as h_A increases, both ($\lambda_1 + \lambda_2 - K$) and W_B are also non-decreasing. Therefore, J_B is non-decreasing. So J_A increases in h_B and J_B increases in h_A. ■

Corollary 3.3 $\partial J_A / \partial h_B \geq 0$, $\partial J_B / \partial h_A \geq 0$.

Lemmas 3.1 and 3.2 are illustrated in Figures 3.1-3.4. These figures demonstrate four typical examples of the feasible set for the two servers. For each distinct pair $(h_A, h_B) \in [0, \overline{H}_A] \times [0, \overline{H}_B]$, the corresponding point (J_A, J_B) has been plotted. The curves on the figure are constructed by fixing the value of h_A (or h_B) and varying h_B (or h_A). Figures 3.1 and 3.2 show the feasible set of the two symmetric servers, whereas Figures 3.3 and 3.4 illustrate the non-symmetric situation.

Lemma 3.3 Assume that at user equilibrium, K_1 and K_2 are the expected numbers of jobs selecting Server A at $(h_A^{(1)}, h_B^{(1)})$ and at $(h_A^{(2)}, h_B^{(2)})$, respectively. If $K_1 = K_2$ and $h_A^{(1)} > h_A^{(2)}$, then $h_B^{(1)} > h_B^{(2)}$.

Proof. Assume $h_B^{(1)} \leq h_B^{(2)}$. Increase the value of $h_B^{(1)}$ until $h_B^{(1)} = h_B^{(2)}$. Notice that K_1 increases. Decrease next the value of $h_A^{(1)}$ until $h_A^{(1)} = h_A^{(2)}$. Then K_1 also increases. Therefore, the new value of K_1 becomes greater than K_2, which contradicts the assumption that at $(h_A^{(2)}, h_B^{(2)})$, K_2 is the number of jobs selecting Server A. Therefore, $h_B^{(1)} > h_B^{(2)}$. ■

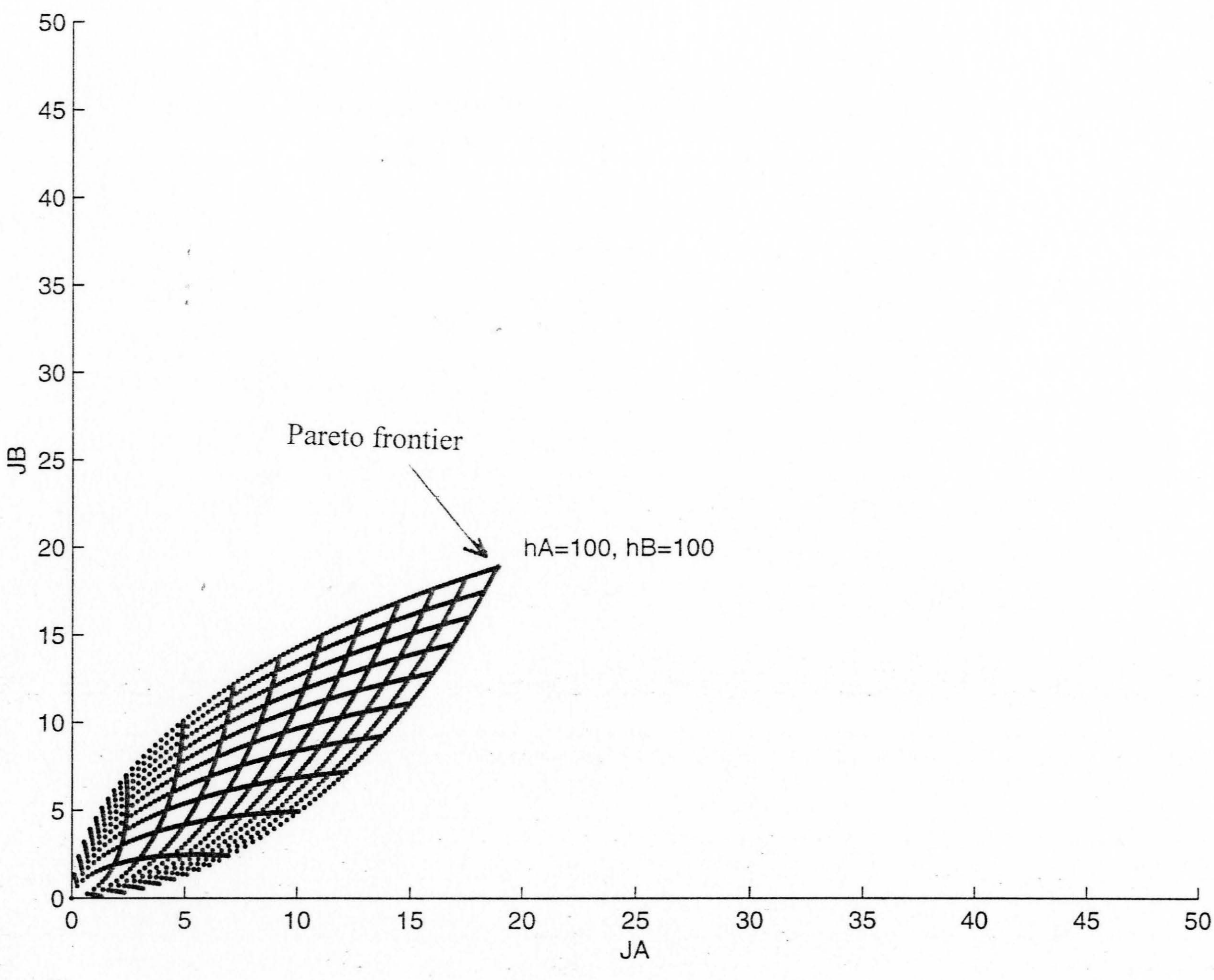

Figure 3.1 User-equilibrium trajectories (in terms of provider revenues) by varying h_A and h_B ($\lambda_1 = 4$, $\lambda_2 = 10$, $\mu_A = 10$, $\mu_B = 20$, $b_A = 0$, $b_B = 0$)

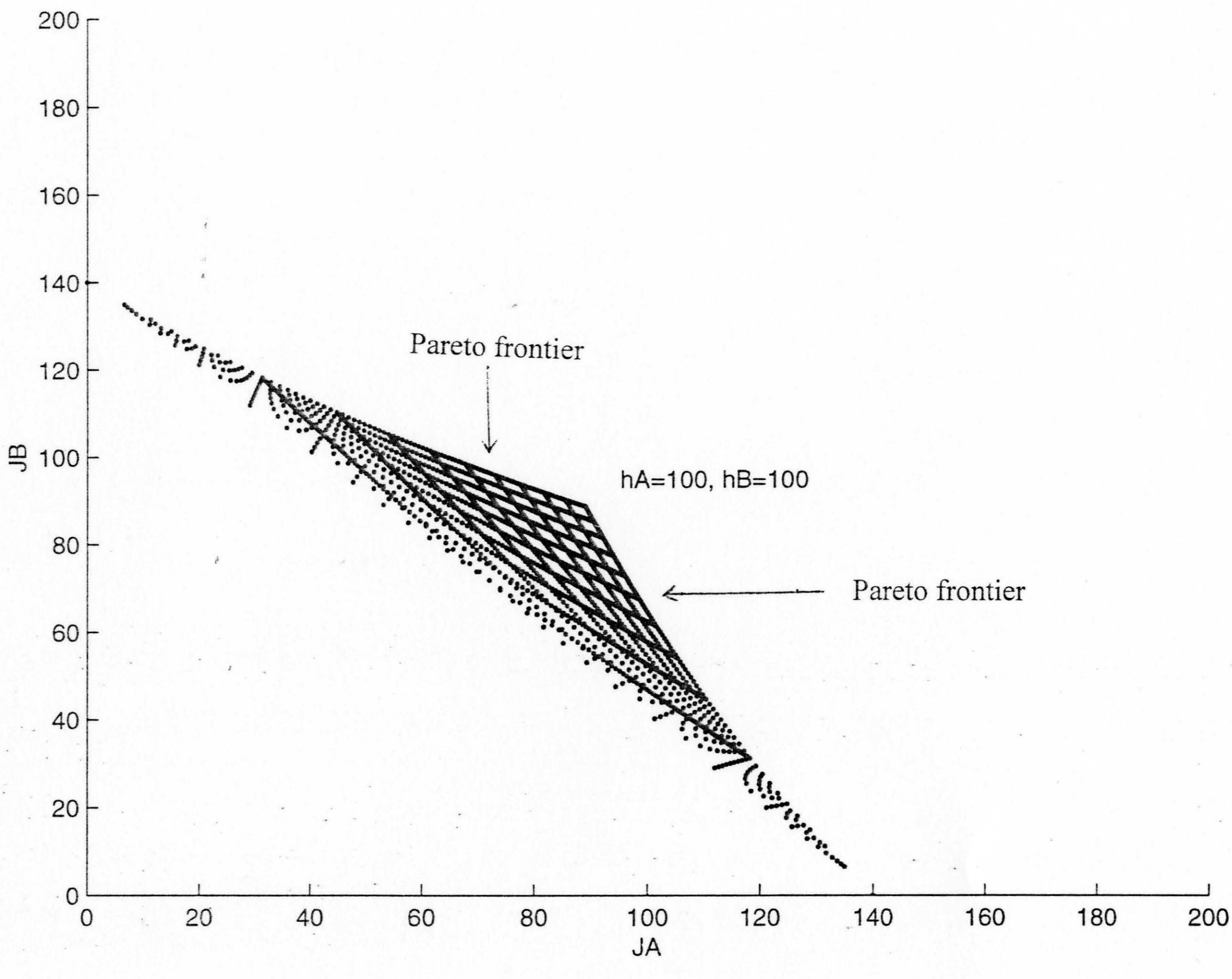

Figure 3.2 User-equilibrium trajectories (in terms of provider revenues) by varying h_A and h_B ($\lambda_1 = 4$, $\lambda_2 = 10$, $\mu_A = 20$, $\mu_B = 20$, $b_A = 10$, $b_B = 10$)

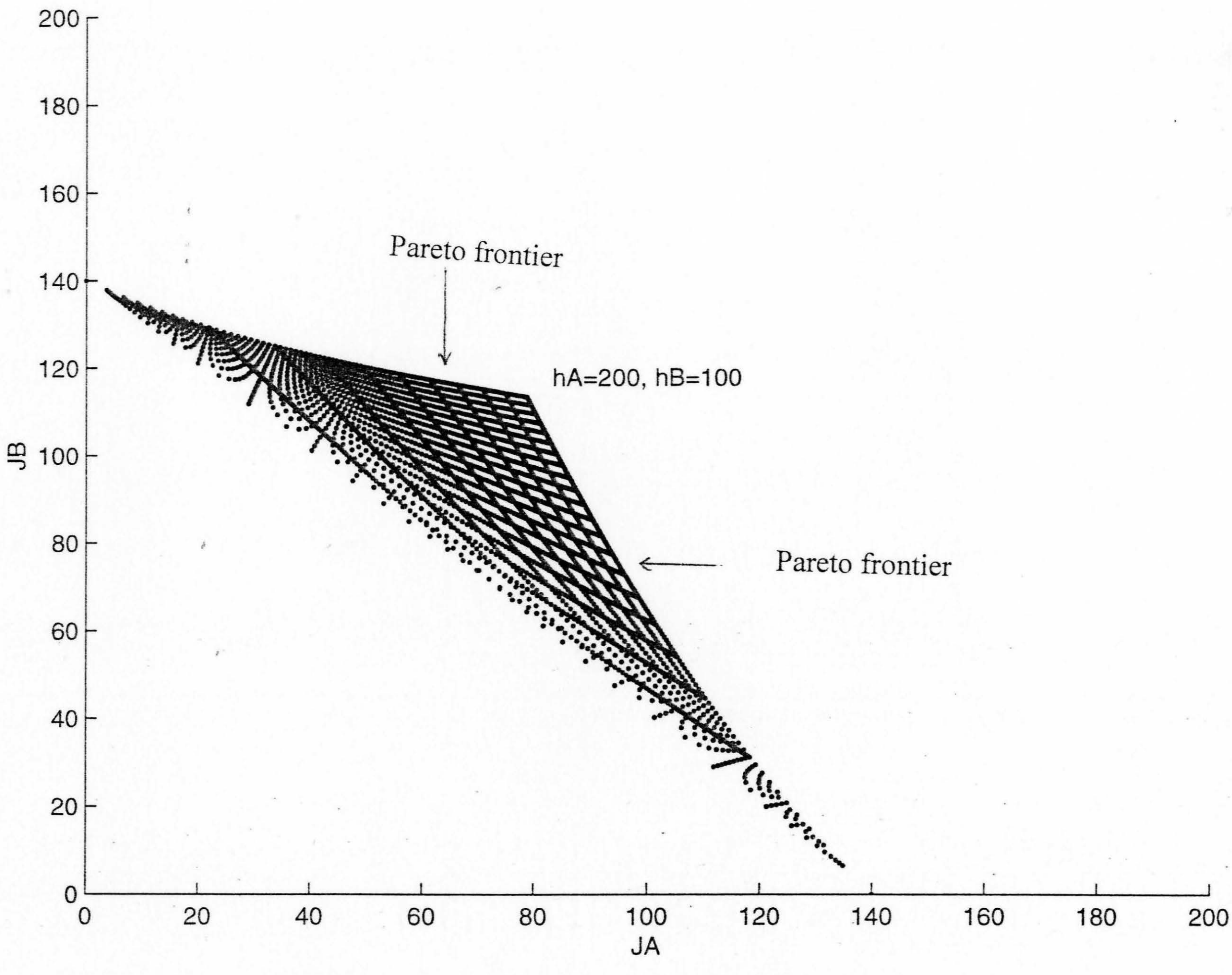

Figure 3.3 User-equilibrium trajectories (in terms of provider revenues) by varying h_A **and** h_B ($\lambda_1 = 4$, $\lambda_2 = 10$, $\mu_A = 20$, $\mu_B = 20$, $b_A = 10$, $b_B = 10$)

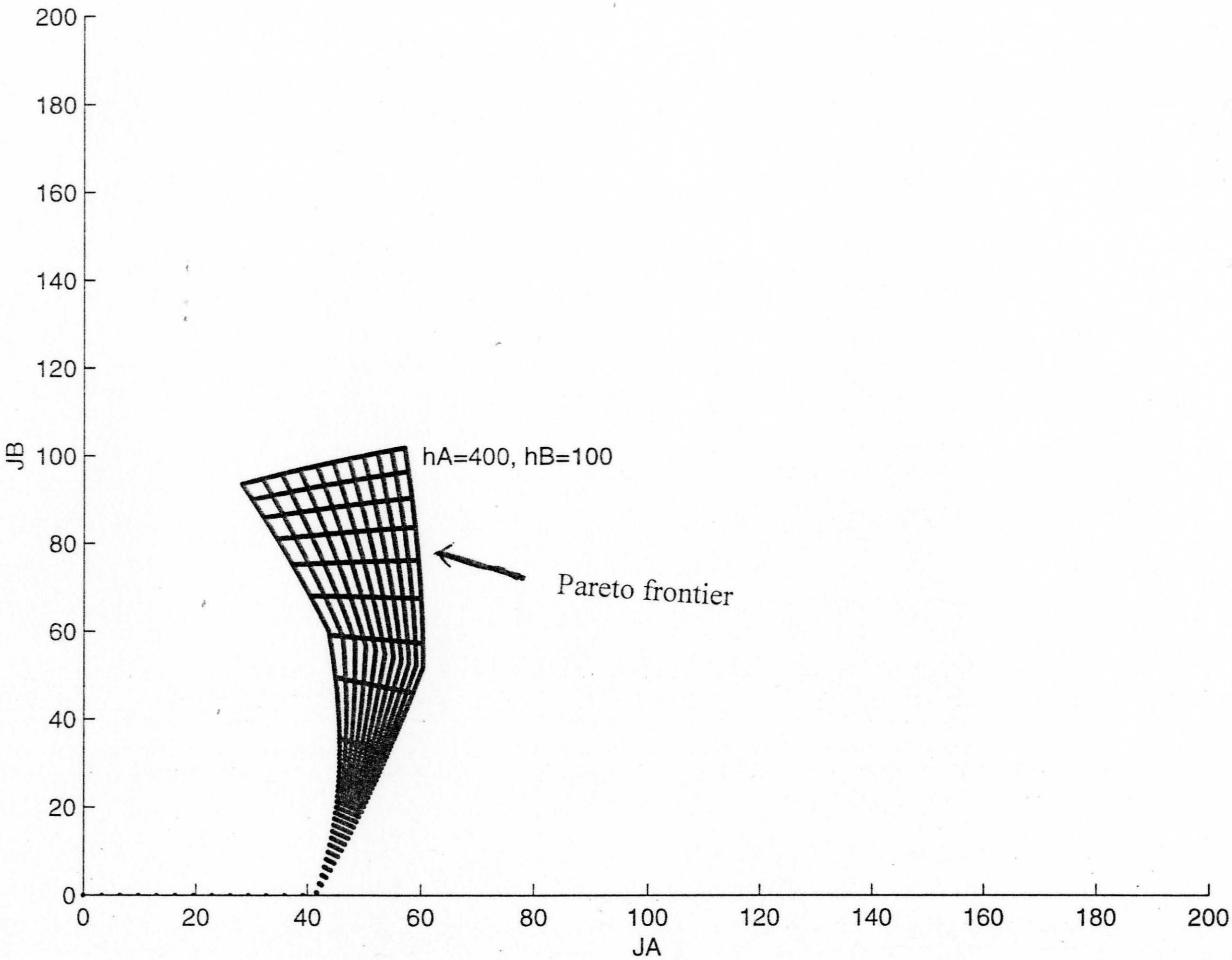

Figure 3.4 User-equilibrium trajectories (in terms of provider revenues) by varying h_A and h_B ($\lambda_1 = 4$, $\lambda_2 = 10$, $\mu_A = 20$, $\mu_B = 20$, $b_A = 0$, $b_B = 10$)

In order to find the servers' Pareto frontier and examine the existence of Nash equilibrium between the two servers, the derivatives $\partial J_A/\partial h_A$ and $\partial J_B/\partial h_B$ will be investigated. The sign of $\partial J_A/\partial h_A$ depends on both service prices and relationships between K and μ_A, μ_B. Given service prices, say, $b_A = b_B = 0$, theorem 3.2 gives a sufficient condition for $\partial J_A/\partial h_A$ to be positive.

Theorem 3.2 Assume $b_A = b_B = 0$. If $K \le \frac{3-\sqrt{3}}{2}\mu_A$, then $\partial J_A/\partial h_A > 0$.

Proof. Substituting (3.4) into (3.16) and simplifying, we have

$$J_A = \frac{K^2 h_A}{\mu_A(\mu_A - K)} + Kb_A.$$

Assume that K is a function of h_A, $K = K(h_A)$. Then, by differentiation,

$$\frac{\partial J_A}{\partial h_A} = \frac{(K^2 + 2\dot{K}Kh_A)(\mu_A - K) + K^2 h_A \dot{K}}{\mu_A(\mu_A - K)^2} + \dot{K}b_A,$$

where $\dot{K}$ is $\partial J_A/\partial h_A$. Therefore $\partial J_A/\partial h_A$ is negative if and only if

$$\dot{K} \le \frac{K(K-\mu_A)}{h_A(2\mu_A - K)}. \tag{3.18}$$

Since $b_A = b_B = 0$, condition (3.14) holds and, hence, case 5 occurs. At user equilibrium, K can be obtained by finding the root of function (3.15). From (3.15), we can compute

$$\frac{\partial F}{\partial h_A} = \frac{K}{\mu_A(\mu_A - K)} + \frac{K}{2(\mu_A - K)^2} = \frac{3\mu_A K - 2K^2}{2\mu_A(\mu_A - K)^2},$$

$$\frac{\partial F}{\partial K} = \frac{h_A(3\mu_A - K)}{2(\mu_A - K)^3} + \frac{h_B(K - \lambda_1 - \lambda_2 + 3\mu_B)}{2(\mu_B - \lambda_1 - \lambda_2 + K)^3} < \frac{h_A(3\mu_A - K)}{2(\mu_A - K)^3}.$$

Then

$$\dot{K} = \frac{\partial K}{\partial h_A} < -\frac{(3\mu_A K - 2K^2)(\mu_A - K)}{\mu_A h_A (3\mu_A - K)}.$$

The inequality (3.18) necessarily holds if

$$\dot{K} = \frac{\partial K}{\partial h_A} < -\frac{(3\mu_A K - 2K^2)(\mu_A - K)}{\mu_A h_A (3\mu_A - K)} \le \frac{K(K - \mu_A)}{h_A (2\mu_A - K)}.$$

Simple algebra shows that the second inequality holds if and only if

$$2K^2 - 6\mu_A K + 3\mu_A^2 \ge 0. \qquad (3.19)$$

The roots of the left hand side of inequality (3.19) are

$$K_{1,2} = \frac{3 \pm \sqrt{3}}{2}\mu_A,$$

therefore (3.19) holds if and only if

$$K \le \frac{3 - \sqrt{3}}{2}\mu_A$$

which proves the theorem. •

The analysis can be carried out in the same manner as shown in theorem 3.2 for a nonzero value of b_A. Similar conditions can be found to guarantee that $\partial J_A / \partial h_A > 0$. However, b_A cannot be arbitrarily large, otherwise $\partial J_A / \partial h_A$ would become negative (since $\dot{K}$<0).

Lemma 3.4. If $\partial J_A / \partial h_A |_{h_B = \overline{H}_B} > 0$, then $(h_A, \overline{H}_B)$ cannot be Pareto optimal for $h_A \in [0, \overline{H}_A)$, but $(\overline{H}_A, \overline{H}_B)$ always gives a Pareto solution.

Proof. By assumption and lemma 3.2,

$$J_A(h_A, H_B) < J_A(H_A, H_B) \quad \text{and} \quad J_B(h_A, H_B) \le J_B(H_A, H_B)$$

where $h_A \in [0, \overline{H}_A)$. That is, $(h_A, \overline{H}_B)$ is always dominated by $(\overline{H}_A, \overline{H}_B)$ and therefore cannot be a Pareto solution.

Again by lemma 3.2, we have

$$J_A(\overline{H}_A, h_B) \le J_A(\overline{H}_A, \overline{H}_B)$$

where $h_B \in [0, \overline{H}_B)$. Equality holds only at a corner user equilibrium. Hence $(\overline{H}_A, h_B)$ cannot dominate $(\overline{H}_A, \overline{H}_B)$.

Next we will show that no interior point (h_A, h_B) can dominate $(\overline{H}_A, \overline{H}_B)$. Suppose there exists such (h_A, h_B) which dominates $(\overline{H}_A, \overline{H}_B)$, that is,

$$J_A(h_A, h_B) \ge J_A(\overline{H}_A, \overline{H}_B) \quad \text{and} \quad J_B(h_A, h_B) \ge J_B(\overline{H}_A, \overline{H}_B),$$

and at least one of the inequalities is strict.

At the user equilibrium with (h_A, h_B), let K_1 be the expected number of jobs selecting server A, and with $(\overline{H}_A, \overline{H}_B)$, let K_2 be the expected number of jobs selecting server A. If $K_1 = K_2$, since $\overline{H}_A > h_A$ and $\overline{H}_B > h_B$, then by lemma 3.2,

$$J_A(h_A, h_B) < J_A(\overline{H}_A, \overline{H}_B) \quad \text{and} \quad J_B(h_A, h_B) < J_B(\overline{H}_A, \overline{H}_B),$$

which contradicts the assumption. Without loss of generality, assume that

$$K_1 > K_2.$$

Decrease the value of $\overline{H}_A$ to h_A' until $K_1 = K_2$. By lemma 3.2, $h_A < h_A' < \overline{H}_A$, so

$$J_A(h_A, h_B) < J_A(h_A', \overline{H}_B) < J_A(\overline{H}_A, \overline{H}_B),$$

which contradicts the assumption. The case of $K_1 < K_2$ leads to a similar contradiction. The above analysis shows that ($\overline{H}_A$, $\overline{H}_B$) is a Pareto solution. •

Lemma 3.5 If $\partial J_A / \partial h_A |_{h_B = \overline{H}_B} < 0$ for $h_A \in [0, \overline{H}_A]$, then any point $(h_A, \overline{H}_B)$ is Pareto optimal.

Proof. By assumption and lemma 3.2,

$$J_A(\overline{H}_A, h_B) \le J_A(\overline{H}_A, \overline{H}_B) \le J_A(h_A, \overline{H}_B).$$

The first becomes an equality only at user corner equilibrium and the second becomes an equality at $h_A = \overline{H}_A$. Therefore, the point $(h_A, \overline{H}_B)$ cannot be dominated by ($\overline{H}_A$, h_B). Now we will show that it cannot be dominated by any other point (h_A', h_B'), where $h_A' < \overline{H}_A$ and $h_B' < \overline{H}_B$.

Suppose there exists such (h_A', h_B') that can dominate $(h_A, \overline{H}_B)$. Assume $h_A' \ge h_A$. Then lemma 3.2 and the assumption imply that

$$J_A(h_A', h_B) \le J_A(h_A', \overline{H}_B) \le J_A(h_A, \overline{H}_B).$$

The first equality holds only at user corner equilibrium and the second equality holds at $h_A' = h_A$. The above inequality contradicts assumption that (h_A', h_B') dominates $(h_A, \overline{H}_B)$.

Next assume that $h_A' < h_A$. Given (h_A', h_B'), let K_1 be the expected number of jobs selecting server A at the equilibrium. Given (h_A , $\overline{H}_B$), let K_2 be the expected number of jobs selecting server A at the corresponding equilibrium. Without loss of generality, assume $K_1 > K_2$. By lemma 3.2, there exists an $h_A'' < h_A$ such that $(h_A'', \overline{H}_B)$ dominates (h_A', h_B'). Then

$$J_B(h_A', h_B) < J_B(h_A'', \overline{H}_B) < J_B(h_A, \overline{H}_B)$$

which contradicts the assumption. Similar contradictions can be derived when $K_1 < K_2$ and $K_1 = K_2$. Therefore, any point $(h_A, \overline{H}_B)$ gives a Pareto solution. •

Lemma 3.6 At any Pareto solution, either $h_A = \overline{H}_A$ or $h_B = \overline{H}_B$.

Proof. For any pair (h_A, h_B), where $h_A < \overline{H}_A$, $h_B < \overline{H}_B$, K is the corresponding aggregated customer arrival rate to server A at user equilibrium. From the continuity of the revenue functions of the servers, there always exists a point $(h_A + \delta_A, h_B + \delta_B)$ with small positive values of δ_A and δ_B, such that at the user equilibrium, the corresponding aggregated customer arrival rate to server A remains the same. Lemma 3.3 implies that any interior point (h_A, h_B) is always dominated by such $(h_A + \delta_A, h_B + \delta_B)$. Therefore, at any Pareto solution, either $h_A = \overline{H}_A$ or $h_B = \overline{H}_B$. •

The assertion of lemma 3.6 becomes significant if a conflict resolution methodology or multi-objective programming is used to find the most appropriate compromise pricing strategies of the servers. Both methodologies seek the solution on the Pareto frontier, therefore the complete description of the Pareto frontier makes the application of such methods efficient.

In Figure 3.1, both $\partial J_A/\partial h_A$ and $\partial J_B/\partial h_B$ are positive. Only $h_A = \overline{H}_A$ and $h_B = \overline{H}_B$ gives the Pareto solution. In this case, Nash equilibrium coincides with the Pareto solution.

Figures 3.2 and 3.3 illustrate the case where both $\partial J_A/\partial h_A$ and $\partial J_B/\partial h_B$ are negative. Either $h_A = \overline{H}_A$ or $h_B = \overline{H}_B$ gives the Pareto solutions.

Sufficient conditions for $\partial J_A/\partial h_A$ and $\partial J_B/\partial h_B$ to be positive are shown in theorem 3.2. If $\partial J_A/\partial h_A > 0$, server A can make more profit by increasing the unit holding price. However the signs of the partial derivatives of $\partial J_A/\partial h_A$ and $\partial J_B/\partial h_B$ may change. Assume that together both servers can handle all jobs, that is, $\mu_A, \mu_B > \lambda_1 + \lambda_2$. If h_A can be increased to an arbitrarily large value, then the value of J_A will eventually converge to zero because no customer will select server A. The above capacity assumption is crucial in our analysis. If one server cannot handle all demands and demands cannot be dismissed or taken care of by external servers, then the other server can always make large profits by arbitrarily raising prices. By the same token, it is important to enforce the limit on the prices that servers can offer. Otherwise, the two servers can make infinitely large profit by raising the prices together.

Based on our analysis, if Nash equilibrium exists, it could coincide with the ideal point, as shown in Figure 3.1, or it might be an interior point where $\partial J_A/\partial h_A = 0$ and $\partial J_B/\partial h_B = 0$, which is less likely to occur.

The Pareto frontier can be obtained by examining the boundary at either $h_A = \overline{H}_A$ or $h_B = \overline{H}_B$. Regardless of the sign of $\partial J_A / \partial h_A$ and $\partial J_B / \partial h_B$, we can always apply lemma 3.4 and 3.5 to find the dominant relationships. Figure 3.4 illustrates the case where the sign of $\partial J_A / \partial h_A$ changes at a boundary. The curve at $h_B = \overline{H}_B$ can be divided into two segments: $\partial J_A / \partial h_A \le 0$ and $\partial J_A / \partial h_A > 0$. The nondominated solutions can be determined by using the results given above. We conclude that the best strategy of the servers is to cooperate with each other and find a compromise solution on the Pareto frontier (Roth, 1979).

3.5 System Optimum for Users

Assume that users can transfer payments to reach a compromise. Then, it is logical to minimize the overall costs per job of the two users. An optimal point, referred to as "system optimum" as opposed to "user equilibrium", is found by solving the following two equations for the two unknowns p_1 and p_2:

$$\frac{\partial\,(\lambda_1 S_1(p_1, p_2) + \lambda_2 S_2(p_1, p_2))}{\partial\, p_1} = 0$$

$$\frac{\partial\,(\lambda_1 S_1(p_1, p_2) + \lambda_2 S_2(p_1, p_2))}{\partial\, p_2} = 0.$$

Since

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} S_1(p_1, p_2) + \frac{\lambda_2}{\lambda_1 + \lambda_2} S_2(p_1, p_2) = (h_A W_A + b_A)\frac{K}{\lambda_1 + \lambda_2} + (h_B W_B + b_B)\frac{\lambda_1 + \lambda_2 - K}{\lambda_1 + \lambda_2},$$

the system optimum depends only on K, the expected number of jobs selecting server A. Therefore, under the same assumptions of section 3.2 that all service prices and service time distributions are user-independent, the problem is equivalent to the single user case where the job arrival rate is $\lambda = \lambda_1 + \lambda_2$, probability of selecting server A is $p = K/\lambda = K/(\lambda_1 + \lambda_2)$, probability of selecting server B is $1 - p = (\lambda_1 + \lambda_2 - K)/(\lambda_1 + \lambda_2)$, and the objection function is

$$S(p) = pW_A h_A + (1-p)W_B h_B + pb_A + (1-p)b_B .$$

Following a similar analysis as shown in section 3.2, we may compute

$$\frac{\partial S(p)}{\partial p} = \frac{\lambda p h_A}{\mu_A(\mu_A - \lambda p)} + \frac{\lambda p h_A}{(\mu_A - \lambda p)^2} - \frac{\lambda(1-p)h_B}{\mu_B(\mu_B - \lambda(1-p))} + \frac{\lambda(1-p)h_B}{(\mu_B - \lambda(1-p))^2} + b_A - b_B . \tag{3.20}$$

Since the partial derivative is a monotone increasing convex function in p, exactly one of the following three cases will occur:

(i) When $\left.\frac{\partial S}{\partial p}\right|_{K=\max(0,\lambda-\mu_B)} \geq 0$, $K = \max(0,\ \lambda - \mu_B)$ is the optimal solution;

(ii) When $\left.\frac{\partial S}{\partial p}\right|_{K=\min(\lambda,\mu_A)} \leq 0$, $K = \min(\lambda, \mu_A)$ is the optimal solution;

(iii) When $\left.\frac{\partial S}{\partial p}\right|_{K=\max(0,\lambda-\mu_B)} < 0$ and $\left.\frac{\partial S}{\partial p}\right|_{K=\min(\lambda,\mu_A)} > 0$, the system optimal solution can be found by finding the unique root of function (3.20) via common techniques, such as the bisection or Newton method.

Notice that as h_A (or h_B) increases, the same monotone properties hold as stated in corollary 3.1 and 3.2. In this case, the Pareto Frontier and Nash equilibria (if any exists) of the servers can be analyzed in the same manner as in section 3.3.

3.6 System Optimum for Servers

If payment transfer is allowed between the two servers, the objective of the servers can be written as

$$J(h_A, h_B) = KW_A h_A + Kb_A + (\lambda_1 + \lambda_2 - K)W_B h_B + (\lambda_1 + \lambda_2 - K)b_B.$$

The system optimal for the servers must be also Pareto optimal. From our analysis given in section 3.3, if $\partial J_A / \partial h_A > 0$ and $\partial J_B / \partial h_B > 0$, then ($\overline{H}_A, \overline{H}_B$) gives the system optimal, which coincides with Nash equilibrium.

3.7 Conclusions

This chapter investigated the competitive scenario of multiple users and multiple servers (or providers, facilities, etc.) where users' jobs are queued for service from the servers. The leader-follower game framework was adopted to analyze the game between users and servers. Given service prices, a constructive proof was presented to obtain the unique user Nash equilibrium. Based on the monotone properties of the user Nash equilibrium, the servers' Pareto frontier was studied and characterized. We mention here that no theory known to the author can be directly applied in this scenario. In addition, in this chapter we show that the system optimum for users and the system optimum of servers can be investigated in a similar manner. Then the two users behave as one player, the two servers behave as the competitive player, and the system can be modeled as the classical Stackelberg (1934) leader-follower game, where both users and servers achieve their optimum at equilibrium.

This research presented a critical step to understanding the game of multiple competitive users queueing up at competitive servers. We will extend these results to the N-user two-server case in a future study. There are still several open questions. For example, what is the situation if the service rates are user-dependent? What would be the pricing scheme where there are N servers (N>2)? How does one model and solve the situation where user demands gradually diminish as service prices increase? The results in this chapter certainly encourage further investigation in these directions.

Chapter 4

USERS COMPETING FOR PARALLEL RESOURCES

4.1 Problem Introduction and Formulation

Consider a system of m users competing for a set of parallel resources, for example, when the users wish to send their goods through a transport network on a set of parallel routes. Let us denote the set of parallel resources as links $\mathcal{L} = \{l_1, l_2, \ldots, l_n\}$. Let the processing (or transporting) capacity for link l_i be denoted by $\mathcal{C}_i$. Let the demand for user j be $D^{(j)}$ and let the unit processing (or transportation) cost of user j on link l_i is assumed to be $c_i^{(j)}$.

Suppose every user wishes to minimize its cost of processing (or transporting) its demand on the available links. We will assume that the total demand is no more than the processing capacity given by the sum of the link capacities. Let $x_i^{(j)}$ be the allocated flow that user-j sends on link l_i. Conflict occurs when two or more users schedule to send amounts to link l_i whose total exceeds the capacity of link l_i. Then the optimization problem for each user can be formulated as a linear program (LP):

$$\text{minimize} \qquad \sum_{i=1}^{n} c_i^{(j)} x_i^{(j)}$$

$$\text{subject to} \quad \sum_j x_i^{(j)} \le \mathcal{C}_i \tag{4.1}$$

$$\sum_i x_i^{(j)} = D^{(j)}$$

$$x_i^{(j)} \ge 0, \qquad \text{for all } i.$$

Since each user will be confronted with an LP, there may exist preferred link order over which the user may compete.

Note that this formulation differs from the multicommodity flow problem with one objective function. In multicommodity flow problems, the total cost of the flow is to be minimized, and it is assumed that the total cost will be borne by a single decision maker. Researchers have developed several methods to solve the multicommodity flow problem, including pricing-directive decomposition (e.g., Lagrangian relaxation, Dantzig-Wolfe), resource-directive decomposition and partitioning methods. These methods have been summarized in Ahuja et al. (1993).

On the other hand, another class of problems, which we refer to as heterogeneous criteria resource allocation problems, arise when several criteria are controlled by either a single decision maker, or each user is considered as an independent decision maker. In the first case, the problems are usually approached by multiobjective programming. In the second case, the problems are modeled and studied within the framework of game theory.

The most commonly used techniques in multiobjective programming problems were the sequential method (Smith, 1956; Chen and Bulfin, 1990; John and Sadowski, 1984) and the weighting method (Peha, 1995). These techniques are summarized, for example, in Szidarovszky et al. (1986). Game theoretic approach has not been utilized until only recently to model customer behavior, competition among firms, and to determine market equilibrium.

Recall that there are two approaches to resolving such competitive situations. When no cooperation is assumed among the users, Nash equilibria are suggested as the "solutions" for the problem. However, Nash equilibria are usually inefficient as demonstrated by Korilis(1995, 1997). In the second approach, the users are assumed to be cooperative. They would then negotiate over the Pareto frontier and settle at a compromising Pareto solution. A key point in this approach is to identify the Pareto frontier.

In this chapter, we will first show that the payoff set is convex and linear. We will also prove that in the case of two users, the Pareto frontier is a piecewise linear function. A polynomial algorithm will be presented to compute the Pareto frontier for a special two-user case when both users have the same link preference order. In addition, a new concept to get a compromise solution will be proposed, which does not require the computation of the Pareto frontier.

The chapter will be organized as follows. In section 4.2, we will prove that the feasible payoff set is a linear convex set. A polynomial algorithm to obtain the Pareto frontier will be presented in section 4.3. A new concept for solving n-user conflicts will be introduced in section 4.4, and conclusions will be drawn in section 4.5.

4.2 LP Analysis and Payoff Space

Observe that each user solves an LP problem (4.1), where each of them has the same convex feasible decision set. In this section, we will prove that the resultant feasible payoff set is a linear convex set. In the case of two users, the Pareto frontier is a convex piece-wise linear function.

Define vector $\mathbf{w}$ as

$$\mathbf{w} = \left(x_1^{(1)},\cdots,x_n^{(1)},x_1^{(2)},\cdots,x_n^{(2)},\cdots\cdots,x_1^{(m)},\cdots,x_n^{(m)}\right)^T.$$

Consider the feasible decision set

$$F = \{\mathbf{w} | \mathbf{w} \in \mathbf{R}^{n\times m}, \mathbf{A}\mathbf{w} \le \mathbf{b}, \mathbf{w} \ge \mathbf{0}\}$$

with $\mathbf{A}$ being a real matrix and $\mathbf{b}$ being a real vector. The vector of payoffs of the m users for each $\mathbf{w} \in F$ can be written as

$$\varphi = \begin{pmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_n^T \end{pmatrix} \mathbf{w} = \mathbf{C}\mathbf{w},$$

where $\mathbf{c}_j^T\mathbf{w}$ corresponds to the objective function of user-j, and

$$\mathbf{C} = \begin{pmatrix} c_1^{(1)}\cdots c_n^{(1)} & 0\cdots 0 & \cdots\cdots & 0 \cdots 0 \\ 0\cdots 0 & c_1^{(2)}\cdots c_n^{(2)} & \cdots\cdots & 0\cdots 0 \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 0 \cdots 0 & \cdots\cdots & 0\cdots 0 & c_1^{(m)}\cdots c_n^{(m)} \end{pmatrix}.$$

Without loss of generality, we may assume that $\mathbf{C} \neq \mathbf{0}$.

Let the payoff set be denoted by

$$S = \{\varphi \mid \varphi = \mathbf{Cw},\ \mathbf{w} \in F\}.$$

This is obviously convex and closed, since mapping $\mathbf{w} \mapsto \mathbf{Cw}$ is linear and F is bounded and convex. Since F is bounded, all elements of F can be expressed as a convex linear combination of its vertices, that is,

$$\mathbf{w} = a_1 v_1 + \cdots + a_r v_r$$

with $a_i \geq 0$, and $\sum_{i=1}^{r} a_i = 1$, where vectors $v_1, \ldots, v_r$ are the vertices of F.

Now consider the following set:

$$S^* = \{\varphi \mid \varphi = \sum_{i=1}^{r} a_i \mathbf{C}\, v_i, a_i \geq 0 \text{ for all } i, \sum_{i=1}^{r} a_i = 1\},$$

which is the simplex generated by vertices $\mathbf{C}_i\, v_i (i = 1,2,\ldots,r)$. The following is a simple description of the payoff set.

Lemma 4.1 $S = S^*$.

Proof. Let $\varphi \in S$, then with some $\mathbf{w} \in F$,

$$\varphi = \mathbf{Cw} = \mathbf{C} \sum_{i=1}^{r} a_i v_i = \sum_{i=1}^{r} a_i \mathbf{C}\, v_i \in S^*.$$

Assume next that $\varphi \in S^*$, then

$$\varphi = \sum_{i=1}^{r} a_i \mathbf{C}\, v_i = \mathbf{C} \sum_{i=1}^{r} a_i v_i = \mathbf{Cz} \in S,$$

with $\mathbf{z} = \sum_{i=1}^{r} a_i v_i \in F$. ■

Corollary 4.1 The payoff space is also a simplex. If only two users are present, then the Pareto frontier is a convex piecewise linear function, where the line segments connect the Pareto vertices.

Corollary 4.2 The set of vertices of S is a subset of $\{\mathbf{C}\, v_1,\ldots,\mathbf{C}\, v_r\}$, where $v_1,\ldots, v_r$ are the vertices of F.

One way to obtain the Pareto frontier is to compute the set of all vertices of the payoff set S. Corollary 4.2 implies that the set of vertices of S is a subset of the set of the images of the vertices of F by mapping $\mathbf{w} \mapsto \mathbf{Cw}$. Such vertices correspond to the extreme points of the LP problem (1). In this problem, there are $n \times m + m$ variables (including slack variables), and $m + n$ constraints. The number of extreme points is therefore $\binom{n \times m + m}{m+n}$, which is non-polynomial. Hence, using the LP model to obtain all extreme points to determine the Pareto frontier has non-polynomial complexity.

4.3 Computation of Pareto Frontier

It has been shown above that in the case of two users, the Pareto frontier is a convex piecewise linear function. We will present now a polynomial algorithm to characterize the Pareto frontier. First note the following simple observation.

Lemma 4.2 If $c_1^{(j)} < c_2^{(j)} < \cdots < c_n^{(j)}$, then the best strategy for user-j (j =1, 2) to allocate $D^{(j)}$ units is the following:

(i) If $D^{(j)} \geq \sum_{k=1}^{i} \mathcal{C}_k$, then $\mathcal{C}_k$ amount is allocated to link l_k for all $k \leq i$;

(ii) If $\sum_{k=1}^{i-1} \mathcal{C}_k \leq D^{(j)} \leq \sum_{k=1}^{i} \mathcal{C}_k$, then allocate amount $D^{(j)} - \sum_{k=1}^{i-1} \mathcal{C}_k^{(j)}$ to link l_i.

(iii) otherwise, zero amount is allocated to link l_i.

4.3.1 Preliminary analysis for the case with two parallel links

Lemma 4.3 Suppose there are two parallel links, l_1 and l_2, and two users, user-1 and user-2.

(i) If $c_1^{(1)} \leq c_2^{(1)}$ and $c_1^{(2)} \geq c_2^{(2)}$, and at least one of the inequalities is strict, then an allocation is Pareto if and only if $x_1^{(1)} = \min\{D^{(1)}, \mathcal{C}_1\}$ and $x_2^{(2)} = \min\{D^{(2)}, \mathcal{C}_2\}$.

(ii) If $c_1^{(1)} \leq c_2^{(1)}$, $c_1^{(2)} < c_2^{(2)}$, and link l_1 is unable to allocate all demands of both users, then the allocation is Pareto if and only if $x_1^{(1)} + x_1^{(2)} = \mathcal{C}_1$. In this case, the Pareto frontier is a linear line,

$$\phi^{(2)} = -\frac{c_1^{(2)} - c_2^{(2)}}{c_1^{(1)} - c_2^{(1)}}(\phi^{(1)} - \phi_{\min}^{(1)}) + \phi_{\min}^{(2)},$$

where $\phi_{\min}^{(1)}$ and $\phi_{\min}^{(2)}$ can be obtained as follows:

$$\phi_{\min}^{(j)} = c_1^{(j)} \min\{D^{(j)}, \mathcal{C}_1\} + c_2^{(j)} \max\{0, D^{(j)} - \mathcal{C}_1\}, j = 1,2.$$

Proof.

(i) Assume that there exists a Pareto solution such that $x_1^{(1)} < \min\{D^{(1)}, \mathcal{C}_1\}$ or $x_2^{(2)} < \min\{D^{(2)}, \mathcal{C}_2\}$. Then $x_2^{(1)} > 0$ and $x_1^{(2)} > 0$. Both objectives can be improved by interchanging a small allocation of user-1 on link l_2 with the same allocation of user-2 on link l_1. This contradicts the definition of Pareto solutions.

(ii) Assume that there exists a Pareto solution such that $x_1^{(1)} + x_1^{(2)} < \mathcal{C}_1$. Since link l_1 is unable to process all the demand of both users, at least one of the users has to allocate some demand to link l_2, say, $x_2^{(1)} > 0$. Without worsening the cost objective of user-2, we can improve the cost objective of user-1 by sending some of this $x_2^{(1)}$ to link l_1. This also contradicts the definition of Pareto optimality.

On the other hand, when $x_1^{(1)} + x_1^{(2)} = \mathcal{C}_1$, it is impossible to improve any one of the objectives without worsening the other. Hence such allocation is Pareto optimal. In this case,

$$\phi^{(1)} = c_1^{(1)} x_1^{(1)} + c_2^{(1)} (D^{(1)} - x_2^{(1)})$$

and

$$\phi^{(2)} = c_1^{(2)} (\mathcal{C}_1 - x_1^{(1)}) + c_2^{(2)} (D^{(2)} - (\mathcal{C}_1 - x_1^{(1)})).$$

Then we have

$$\frac{\partial \phi^{(1)}}{\partial x_1^{(1)}} = c_1^{(1)} - c_2^{(1)} \text{ and } \frac{\partial \phi^{(2)}}{\partial x_1^{(1)}} = -c_1^{(2)} + c_2^{(2)}.$$

Hence

$$\frac{\partial \phi^{(2)}}{\partial \phi^{(1)}} = -\frac{c_1^{(2)} - c_2^{(2)}}{c_1^{(1)} - c_2^{(1)}}.$$

The minimum of $\phi^{(1)}$ is obtained at the maximum of $\phi^{(2)}$ and vice versa, so

$$\phi_{\min}^{(j)} = c_1^{(j)} \min\{D^{(j)}, \mathcal{C}_1\} + c_2^{(j)} \max\{0, D^{(j)} - \mathcal{C}_1\}, j = 1,2.$$

Therefore, the Pareto frontier is a linear segment between points ($\phi_{\min}^{(1)}$, $\phi_{\max}^{(2)}$) and ($\phi_{\max}^{(1)}$, $\phi_{\min}^{(2)}$) with slope $-\dfrac{c_1^{(2)} - c_2^{(2)}}{c_1^{(1)} - c_2^{(1)}}$. ■

Lemma 4.3 illustrates a simple conflict when two users are competing for only two links. When the two users have the same preference order of the links, the vertices of the payoff set are obtained at the minimum cost of either user-1 or user-2 (see Figure 4.1). This provides the insight to the case when n, n>2, links are available.

4.3.2 The *N*-link two-user case with the same preference order

Consider the case of two users, user-1, user-2 and n (≥ 2) links. Without loss of generality, suppose the links be indexed so that $c_1^{(1)} < c_2^{(1)} < \cdots < c_n^{(1)}$. If user-2 has the same preference order, that is, $c_1^{(2)} < c_2^{(2)} < \cdots < c_n^{(2)}$, then a flow is a Pareto solution if and only if $x_i^{(1)} + x_i^{(2)} = \min\{D^{(1)} + D^{(2)} - \sum_{k=1}^{i-1} (x_k^{(1)} + x_k^{(2)}), \mathcal{C}_i\}$, because a more preferred link is allocated first before a lesser preferred one.

Given the Pareto frontier of two users competing for n-1 links, we can determine by induction the Pareto frontier for the case with one more link (i.e. n links). At any Pareto solution, $x_1^{(1)} + x_1^{(2)} = \min\{D^{(1)} + D^{(2)}, \mathcal{C}_1\}$. After the entire capacity of the first link has been allocated to two users, the problem becomes that of allocating demand on remaining (n-1) links, where the demand of user-j becomes $D^{(j)} - x_1^{(j)}, j = 1,2$.

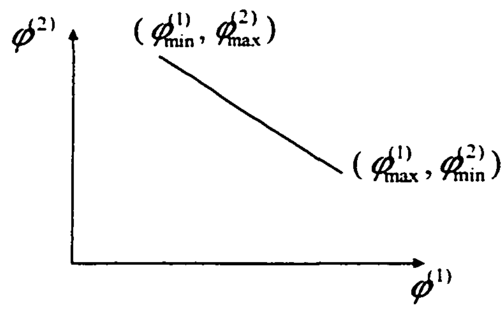

Figure 4.1 Pareto frontier of two users sharing two links

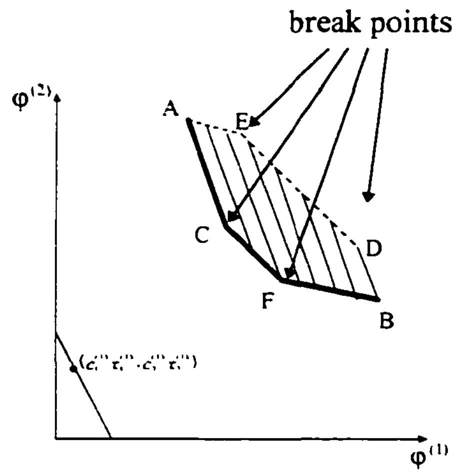

Figure 4.2 Pareto frontier of two users sharing three links

Based on Lemma 4.3, the Pareto frontier can be computed in the 3-link competition case (see Figure 4.2). The allocation costs for both users on link l_1 are $c_1^{(1)}x_1^{(1)}$ and $c_1^{(2)}x_1^{(2)}$, respectively. Corresponding to any capacity allocation on link l_1, there is a Pareto frontier for the two users on the link l_2 and l_3. For example, if user-1 occupies the entire link l_1, line AC is the Pareto frontier of the two users on link l_2 and l_3. If user-2 occupies the entire link l_1, line *BD* is the Pareto frontier of the two users on link l_2 and l_3. For any other allocation of link l_1, the Pareto frontier is a parallel line segment somewhere between *AC* and *BD*. Several such lines are shown in Figure 4.2. There are at most six possible vertices in the space of these Pareto frontiers. Connecting four of them constitutes the convex piecewise Pareto frontier for the 3-link case.

Following Lemma 4.3, we have the Pareto frontier of the case when two users compete for the other two links:

$$\phi^{(2)} = -\frac{c_2^{(2)} - c_3^{(2)}}{c_2^{(1)} - c_3^{(1)}}(\phi^{(1)} - \phi_{\min}^{(1)} - c_1^{(1)}x_1^{(1)}) + \phi_{\min}^{(2)} + c_1^{(2)}x_1^{(2)}$$

with

$$\phi_{\min}^{(j)} = c_2^{(j)} \min\{D^{(j)} - x_1^{(j)},\ \mathcal{C}_2\} + c_3^{(j)} \max\{0,\ D^{(j)} - x_1^{(j)} - \mathcal{C}_2\}, j = 1,2.$$

The Pareto frontier consists of these parallel line segments obtained by the above formula. Notice that $\phi_{\min}^{(1)}$ on links l_2 and l_3 is obtained at

$$x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\} \text{ and } x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)}.$$

We need to consider the value of $x_2^{(1)}$ in the following two cases to compute the points ($\phi_{\min}^{(1)}$, $\phi_{\max}^{(2)}$). In the first case, $x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\} = D^{(1)} - x_1^{(1)}$, and in the second case, $x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\} = \mathcal{C}_2$.

Case 1. $x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\} = D^{(1)} - x_1^{(1)}$.

Then user-1 can send all its demand to link l_2, therefore

$$\phi_{\min}^{(1)} = c_1^{(1)}x_1^{(1)} + c_2^{(1)}(D^{(1)} - x_1^{(1)}).$$

User-2 optimizes its allocation in the network:

$$x_2^{(2)} = \min\{D^{(2)} - x_1^{(2)}, \mathcal{C}_2 - (D^{(1)} - x_1^{(1)})\}\ .$$

If user-2 can also send all its demand to along link l_2, then

$$\varphi_{\max}^{(2)} = c_1^{(2)} x_1^{(2)} + c_2^{(2)} (D^{(2)} - x_1^{(2)})\ ,$$

otherwise,

$$\varphi_{\max}^{(2)} = c_1^{(2)} x_1^{(2)} + c_2^{(2)} (C_2 - (D^{(1)} - x_1^{(1)}) + c_3^{(2)} (D^{(2)} - x_1^{(2)} - x_2^{(2)})\ .$$

It can be shown that in either case,

$$\frac{\partial \varphi_{\max}^{(2)}}{\partial x_1^{(1)}} = -c_1^{(2)} + c_2^{(2)}\ .$$

Therefore, the slope of this line segment is

$$\frac{\partial \varphi_{\max}^{(2)}}{\partial \varphi_{\min}^{(1)}} = \frac{\partial \varphi_{\max}^{(2)} / \partial x_1^{(1)}}{\partial \varphi_{\min}^{(1)} / \partial x_1^{(1)}} = -\frac{c_1^{(2)} - c_2^{(2)}}{c_1^{(1)} - c_2^{(1)}}\ .$$

Case 2. $x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\} = \mathcal{C}_2$.

Then link l_2 is fully utilized by user-1. Both users send all remaining demands along link l_3, therefore,

$$\varphi_{\min}^{(1)} = c_1^{(1)} x_1^{(1)} + c_2^{(1)} \mathcal{C}_2 + c_3^{(1)} (D^{(1)} - x_1^{(1)} - \mathcal{C}_2)\ ,$$

and

$$\varphi_{\max}^{(2)} = c_1^{(2)} x_1^{(2)} + c_3^{(2)} (D^{(2)} - x_1^{(2)})\ .$$

The slope of this line segment can be similarly computed as

$$\frac{\partial \varphi_{\max}^{(2)}}{\partial \varphi_{\min}^{(1)}} = \frac{\partial \varphi_{\max}^{(2)} / \partial x_1^{(1)}}{\partial \varphi_{\min}^{(1)} / \partial x_1^{(1)}} = -\frac{c_1^{(2)} - c_3^{(2)}}{c_1^{(1)} - c_3^{(1)}}.$$

The linear contour of ($\varphi_{\max}^{(1)}$, $\varphi_{\min}^{(2)}$) can be also determined in the same way. Refer to the point that connects two line segments as the *breakpoint*. It is easy to see that the breakpoint is at $D^{(j)} - x_1^{(j)} = \mathcal{C}_2$, which is the border line in the above two cases. In other words, after the two users share the first link, the remaining demand of user-j equals the capacity on link l_2. In general, vertices of the user cost space can be obtained only in the following cases:

Vertex A: User-1 achieves his/her minimum cost:

$$x_1^{(1)} = \min\{D^{(1)}, \mathcal{C}_1\},$$

$$x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\},$$

$$x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)},$$

$$x_1^{(2)} = \min\{D^{(2)}, \mathcal{C}_1 - x_1^{(1)}\},$$

$$x_2^{(2)} = \min\{D^{(2)} - x_1^{(2)}, \mathcal{C}_2 - x_2^{(1)}\},$$

$$x_3^{(2)} = D^{(2)} - x_1^{(2)} - x_2^{(2)}.$$

Vertex B: User-2 achieves his/her minimum cost:

$$x_1^{(2)} = \min\{D^{(2)}, \mathcal{C}_1\},$$

$$x_2^{(2)} = \min\{D^{(2)} - x_1^{(2)}, \mathcal{C}_2\},$$

$$x_3^{(2)} = D^{(2)} - x_1^{(2)} - x_2^{(2)},$$

$$x_1^{(1)} = \min\{D^{(1)}, \mathcal{C}_1 - x_1^{(2)}\},$$

$$x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2 - x_2^{(2)}\},$$

$$x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)}.$$

Vertex C: User-1 takes the capacity of link l_1 as much as possible, while user-2 optimizes on links l_2 and l_3:

$$x_1^{(1)} = \min\{D^{(1)}, \mathcal{C}_1\},$$

$$x_1^{(2)} = \min\{D^{(2)}, \mathcal{C}_1 - x_1^{(1)}\},$$

$$x_2^{(2)} = \min\{D^{(2)} - x_1^{(2)}, \mathcal{C}_2\},$$

$$x_3^{(2)} = D^{(2)} - x_1^{(2)} - x_2^{(2)},$$

$$x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2 - x_2^{(2)}\},$$

$$x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)}.$$

Vertex D: User-2 takes the capacity of link l_1 as much as possible, while user-1 optimizes on links l_2 and l_3:

$$x_1^{(2)} = \min\{D^{(2)}, \mathcal{C}_1\},$$

$$x_1^{(1)} = \min\{D^{(1)}, \mathcal{C}_1 - x_1^{(2)}\},$$

$$x_2^{(1)} = \min\{D^{(1)} - x_1^{(1)}, \mathcal{C}_2\},$$

$$x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)},$$

$$x_2^{(2)} = \min\{D^{(2)} - x_1^{(2)}, \mathcal{C}_2 - x_2^{(1)}\},$$

$$x_3^{(2)} = D^{(2)} - x_1^{(2)} - x_2^{(2)}.$$

Vertex E: If $D^{(1)} > \mathcal{C}_2$ and $D^{(2)} > D^{(1)} - \mathcal{C}_2$, a vertex exists for $D^{(1)} - x_1^{(1)} = \mathcal{C}_2$, when user-1 optimizes on links l_2 and l_3:

$x_1^{(1)} = D^{(1)} - \mathcal{C}_2$,

$x_2^{(1)} = D^{(1)} - x_1^{(1)}$,

$x_3^{(1)} = 0$,

$x_1^{(2)} = D^{(2)} - x_1^{(1)}$,

$x_2^{(2)} = 0$,

$x_3^{(2)} = D^{(2)} - x_1^{(2)} - x_2^{(2)}$.

Vertex F: If $D^{(2)} > \mathcal{C}_2$ and $D^{(1)} > D^{(2)} - \mathcal{C}_2$, a vertex exists for $D^{(2)} - x_1^{(2)} = \mathcal{C}_2$, when user-2 optimizes on links l_2 and l_3:

$x_1^{(2)} = D^{(2)} - \mathcal{C}_2$,

$x_2^{(2)} = D^{(2)} - x_1^{(2)}$,

$x_3^{(2)} = 0$,,

$x_1^{(1)} = \mathcal{C}_1 - x_1^{(2)}$,

$x_2^{(1)} = 0$,

$x_3^{(1)} = D^{(1)} - x_1^{(1)} - x_2^{(1)}$.

It has been shown in the previous section that the Pareto frontier is a convex piecewise linear function. Thus , when we connect vertices A, C, F, B, with line segments, and also vertices A, E, D, B, the line segments that dominate the other line segments constitute the

Pareto frontier. In Figure 4.2, $ACFB$ are the dominating line segments that make up the Pareto frontier. The following numerical example illustrates this result.

Example 4.1 In a connectionless oriented network, suppose two users wish to send data through three parallel routes. User-1 wishes to send 12 gigabits and user-2 15 gigabits. The capacity of each route is 10 gigabits. The costs for user-1 on the three routes are \$100/gigabit, \$200/gigabit and \$300/gigabit, respectively. The costs for user-2 on the three routes are \$100/gigabit, \$200/gigabit and \$600/gigabit, respectively. The resultant Pareto frontier is shown in Figure 4.3. ■

In this example, it is easy to verify that the Pareto frontier is a convex piecewise linear function by connecting vertices A, C, F, B. The line segment AE is parallel to the line segment FB with slope $-(c_1^{(2)}-c_2^{(2)})/(c_1^{(1)}-c_2^{(1)})$ and the line segment ED is parallel to the line segment CF with slope $-(c_1^{(2)}-c_3^{(2)})/(c_1^{(1)}-c_3^{(1)})$.

The above analysis provides an insight to develop a polynomial algorithm to obtain the Pareto frontier in the special case where the two users have the same link preference order. Suppose the Pareto frontier in the n-1 link case consists of a chain of vertices starting from vertex A and terminating at vertex B. Vertex A is obtained at ($\phi_{\min}^{(1)}$, $\phi_{\max}^{(2)}$) and vertex B is obtained at ($\phi_{\max}^{(1)}$, $\phi_{\min}^{(2)}$). We can compute the Pareto frontier, AB', when user-1 occupies link l_1 totally and the Pareto frontier, A'B, when user-2 occupies link l_1 totally, as depicted in Figure 4.4. By examining

$$\frac{\partial \varphi^{(2)}_{\max}}{\partial \varphi^{(1)}_{\min}} = \frac{\partial \varphi^{(2)}_{\max} \big/ \partial x^{(1)}_1}{\partial \varphi^{(1)}_{\min} \big/ \partial x^{(1)}_1}$$

and

$$\frac{\partial \varphi^{(2)}_{\min}}{\partial \varphi^{(1)}_{\max}} = \frac{\partial \varphi^{(2)}_{\min} \big/ \partial x^{(1)}_1}{\partial \varphi^{(1)}_{\max} \big/ \partial x^{(1)}_1},$$

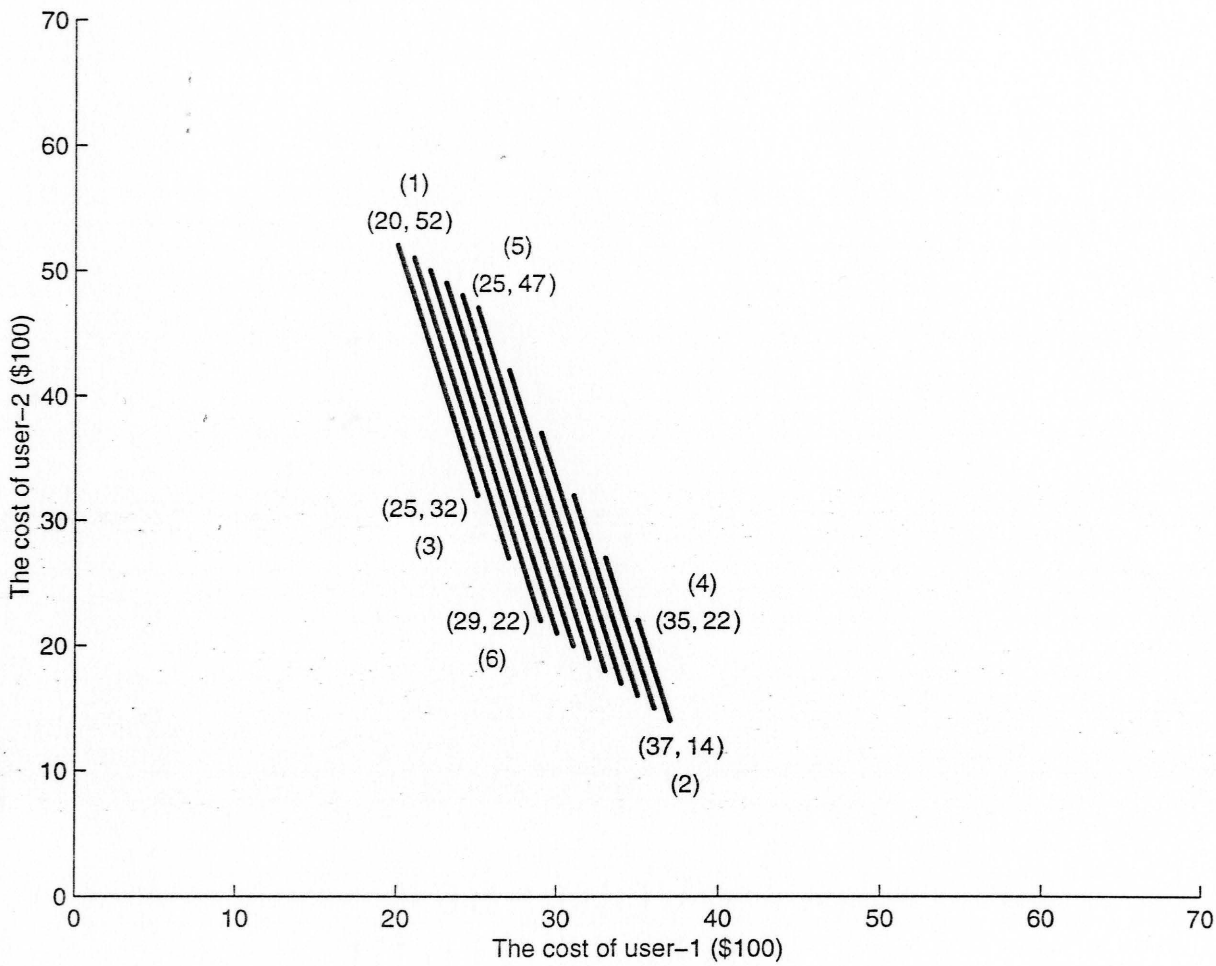

Figure 4.3 Pareto frontier of two users with three parallel links

(numerical example)

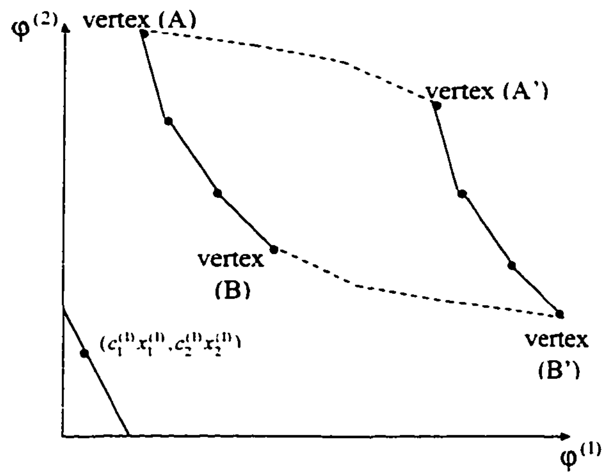

Figure 4.4 Pareto frontier of two users with *n* parallel links

we can find the breakpoints on the connecting line segments between *AA*' and *BB*' at $x_1^{(j)} = D^{(j)} - \sum_{l=2}^{k} C_l$ with k = 2, 3, ..., n-1. Theoretically, there are at most 2(n-2) new breakpoints when we examine the above derivatives. In other words, the breakpoints could only possibly appear when the remaining demand of user-j (j =1,2) equals total capacity on link l_2 through l_k (k = 2, 3, ..., n-1) after the first link is shared by the two users. As shown in Figure 4.4, the Pareto frontier are the linear segment connecting the chain of vertices (A,..., B , ..., B') or (A,..., A',...,B'), whichever dominates the other.

Theorem 4.1 The complexity of the above algorithm is $O(n^3)$.

Proof. In order to find the Pareto frontier in the n link case (n > 2), we will have to examine at most 2(n-2) new breakpoints due to the additional nth link, in addition to the number of points computed so far in the n-1 link case. Therefore, the total number of breakpoints to be examined for n links is at most

$$2+\sum_{k=3}^{n} 2(k-2) = 2+(n-2)(n-1).$$

where the first term is due to the two-link case. It takes $O(n)$ operations to compute each point. Hence the overall complexity of the algorithm is $O(n^3)$. ■

Based on the knowledge of the Pareto frontier any known method (such as the Nash solution, the non-symmetric Nash solution, Rubinstein's alternating offer method, or any of its extensions) can be applied for finding an appropriate solution on the frontier. We will address negotiation methods over the Pareto frontier in chapter 5.

When the assumption, $c_1^{(j)} < c_2^{(j)} < \cdots < c_n^{(j)}, j = 1, 2$, does not hold, the finite induction can be also employed in the same spirit to obtain the Pareto frontier. Each user allocates his demand to most preferred link. If conflict occurs, a set of line segment Pareto frontiers for that link can be determined. This constitutes a new Pareto frontier. Starting from a distinct point of the Pareto frontier, continue this process until all demands are satisfied.

Suppose that the users have shared capacity of n links, among which $n^{(1)}$ links are utilized by user-1 without conflict with user-2 and $n^{(2)}$ links are used by user-2 without conflict with user-1. The capacities on the rest of (n - $n^{(1)}$ - $n^{(2)}$) links are allocated between the two users. During the process, assume that we have determined the Pareto frontier over the k links and user-1 wishes to send its remaining demand to the next preferred link. If the next preferred link of user 1 is not in conflict with user 2, then assign the link to user 1 and shift the Pareto frontier by an amount equal to the user-1 cost on that link. Conflict arises when the next preferred link of user 1 has already been allocated to user-2 demand. In order to divide the capacity of this link between the two users, user-2's demand needs to be withdrawn from the link. Then shift the Pareto frontier down by an amount equal to user-2 cost on that link. The same technique discussed above can be used now for the search of the extreme points on the Pareto frontier.

4.4 A "Solution" Concept for m Users

In Section 4.3, we developed a polynomial algorithm to compute the entire Pareto frontier for the two-user case. In this section, we discuss a new "solution" concept to resolve allocation conflicts in a more general situation where m users compete for n links.

In this conceptual system, we assume that there is a "user's club". All users who join the club will operate in a system optimum manner; that is, demands will be allocated on available links to minimize total cost of the users in the club. However, the cost to each user will be "fairly" re-allocated. In other words, the club will decide a fair price for each user. In this conceptual system, we assume users who are not in the club can only use the links after the users in the club have achieved their system optimum.

The worst case for each individual user is when he/she is the only one who is not in the club, hence he/she would be the last one to use the remaining capacity on the links. Denote such cost for each user-j by $\phi^{(j)}_{worst}$. Ideally a user can send all its demands to the most preferred links. Denote the corresponding minimum cost for user-j by $\phi^{(j)}_{ideal}$. In this conceptual system, club users pay the fair amount $\phi^{(j)}$ between $\phi^{(j)}_{worst}$ and $\phi^{(j)}_{ideal}$:

$$\phi^{(j)} = \phi^{(j)}_{ideal} + \alpha(\phi^{(j)}_{worst} - \phi^{(j)}_{ideal}), (j = 1,2,\ldots, m)$$

where α is the common factor to all users in the club. The value of α will be specified later, when the system optimum has been achieved.

The system optimum can be computed by solving a standard LP problem as follows:

Minimize $$\sum_j \sum_i c_i^{(j)} x_i^{(j)}$$

subject to :

$$\sum_j x_i^{(j)} \le C_i$$

$$\sum_i x_i^{(j)} = D^{(j)}$$

$$x_i^{(j)} \ge 0, \qquad \text{for all link } l_i.$$

Suppose the optimal solution of the above problem has the minimum cost S. The values of $\phi^{(j)}_{worst}$ and $\phi^{(j)}_{ideal}$ can also be obtained through a similar LP problem. Then α can be obtained by solving the following equation:

$$S = \sum_j (\phi^{(j)}_{ideal} + \alpha(\phi^{(j)}_{worst} - \phi^{(j)}_{ideal})) .$$

That is,

$$\alpha = \frac{S - \sum_j \phi^{(j)}_{ideal}}{\sum_j (\phi^{(j)}_{worst} - \phi^{(j)}_{ideal})} .$$

This concept can be easily applied when several users are involved in a conflict situation. The solution coincides with one of the Pareto solutions and it directly suggests a compromising solution.

4.5 Conclusions

In this chapter, we have examined a fundamental network resource allocation problem with two competing users and n substitutable links. The feasible set was proven to be convex and linear. In the case of two competing users, the Pareto frontier was shown to be a convex piecewise linear function.

We have also demonstrated a polynomial algorithm to compute this Pareto frontier for the special cases of (1) two links and (2) n links with users having the same link preference order. In addition, a new "solution" concept was suggested to obtain a possible compromising position for the n users.

Several questions for further research stem from the current study. At this point it is not clear what *pricing scheme* network providers could adopt to enforce an efficient capacity allocation. It is also the subject of future research to extend the results of this chapter to general networks. However, the results presented in this chapter certainly can be considered as an initial step to the understanding of the nature of multiuser *routing games* on networks.

Chapter 5

NEGOTIATION ON THE PARETO FRONTIER

In chapter 2 and 4, we have introduced and studied logistic problems with competing users; there we developed methods to obtain the Pareto frontier over which the users may negotiate. In the literature, it is usually assumed that the negotiation takes place over a concave, continuously differentiable Pareto frontier. However, the Pareto frontier in the two-user single machine scheduling problem (chapter 2) consists of only discrete points. In this chapter, we will extend the one-step negotiation concept that is briefly introduced in section 5.2 to the case with a discrete Pareto frontier. Before we report some new results in this arena, we will briefly review some concepts and procedures which may be used to negotiate on the Pareto frontier.

5.1 Literature Review

Over the past half century, a large variety of concepts and methodologies have been devised for resolving two-person conflicts. There are three major approaches for conflict resolution. The first group of methods is based on certain axioms that the solution has to satisfy. The second group of approaches is based on a *fictitious dynamic bargaining* process which usually takes place over the Pareto frontier. The third approach is called *virtual* *negotiation process*, which is similar to the second group, except that the negotiation takes place over the entire feasible space. The last two groups of approaches can also be classified into one broad category.

The classic Nash's bargaining model (1953) has been a well-studied axiomatic model. Since then, his original axioms have been generalized, new solution concepts have been introduced. Most of the common solution concepts are summarized by Roth (1979).

An interesting extension of the Nash solution is given by Mariotti (1996), where the solution is not necessarily Pareto optimal. He showed that the solution is a point on the linear segment connecting the Nash solution and the disagreement point. A non-convex solution concept was introduced by Conley (1996); here the solution is defined as the intercept of the Pareto frontier of the original problem and the Nash solution based on the convex hull of the non-convex feasible payoff set.

In the *fictitious dynamic bargaining process* (Shapiro, 1958), either an equilibrium is considered to be the solution if it exists, or the limit of the process is accepted as the solution if the process converges. If bargaining is considered as a two-person noncooperative game where players' strategy pair equals their payoff pair, then Nash (1953) claimed that the equilibrium set coincides with the set of Pareto solutions. Bargaining can also be considered as a single-player decision problem, when the strategy selection of the other player is considered to be random. Each player then maximizes his/her expected profit. If uniform distribution is assumed, then the resulting solution is equivalent to the *Nash-product maximizing alternative* (e.g. Roth, 1979).

In dynamic negotiations, players take turns to make decisions among the following three alternatives: (1) accept the last offer of the competitor, (2) provide a counter offer (or a concession), or (3) terminate negotiation, which results in the *breakdown* of negotiations. The main problem is to determine the order in which offers are made. If the principle of Zeuthen (1930) is assumed, then the limit of the dynamic process coincides with the Nash solution. The *alternating offer* method (Rubinstein, 1982 and 1990) is a popular model in this regard. There are two major difficulties in applying this method. The process is complicated by the introduction of *breakdown probabilities*. In the model of Osborne and Rubinstein, the probability that a player breaks down the negotiation process is a constant. However, the breakdown probability of a player depends on both the *rigidity* of the player and the goodness of the other player's offer. Moreover, the solution does not depend on the initial positions of the players.

In the recent paper of Szidarovszky and Shen (1997), offer-dependent breakdown probabilities were introduced in the negotiation process. Furthermore, a one-step common optimal choice for both players was proposed as a new solution concept based on their relative *rigidities*. Such one-step two-person game has several advantages: (1) the solution no longer depends on the order in which concessions are made; and (2) the *breakdown probabilities* depend on the current offers. The mathematical properties of this procedure have been examined by Szidarovszky (1997). This concept can be easily extended to bargaining over a concave piecewise linear Pareto frontier. This concept, as well as this extension is briefly discussed in the next section.

However, this model cannot be applied when the decision sets are finite, since continuity and convexity may not hold. This model and solution concept need to be extended to the discrete case. For example, in multi-objective scheduling and assignment problems, the feasible solution set is given by all permutation sequences, which implies that the Pareto frontier is necessarily discrete. In section 5.3, we provide an extension to the Szidarovszky-Shen study to the conflict situations with discrete feasible payoff sets. We will show the existence of a solution and that the number of solutions will be one or two. An algorithm will be introduced to find an appropriate compromise solution. The single-machine scheduling problem will be used as an example to illustrate the algorithm of section 5.4. Some conclusions on this negotiation process will be presented in section 5.5.

5.2 Dynamic Negotiation Process on a Continuous Pareto Frontier of Two-User (Review of Szidarovszky and Shen, 1997)

In a conventional two-person bargaining game (***S*, d**), it is assumed that ***S*** is compact, convex and comprehensive in R^2. We may assume without loss of generality (if necessary, after a suitable positive affine transformation of utility scales) that the disagreement point is the origin. Assume that Pareto frontier is a twice continuously differentiable function, denoted by $z_2 = g(z_1)$. The initial offers of the two players are (a_1, a_2) and (b_1, b_2). If player 1 makes a concession, then he/she offers (z_1, z_2), which decreases his/her payoff and at the same time, increases the payoff of player 2. Player 2 will either continue the negotiaion with probability $P_2(z_2)$ by proposing a further compromising offer, or break up the negotiation process with probability $(1 - P_2(z_2))$, resulting in zero payoffs for both players. Assume that

$$P_2(z_2) = (\frac{z_2 - a_2}{b_2 - a_2})^{r_2},$$

where $r_2 > 0$ is a given constant *rigidity index* of player 2.

Under these assumptions, player 1 will choose a point that maximizes his/her expected payoff:

$$\text{maximize } z_1(\frac{z_2 - a_2}{b_2 - a_2})^{r_2}$$

$$\text{subject to } a_1 \geq z_1 \geq b_1.$$

Similarly player 2 will also select a point that maximizes his/her expected payoff.

$$\text{maximize } z_2(\frac{z_1 - b_1}{a_1 - b_1})^{r_1}$$

$$\text{subject to } b_2 \geq z_2 \geq a_2.$$

A point (z_1^*, z_2^*) is optimal for both players if and only if it is the solution of equations

$$(z_1(\frac{z_2 - a_2}{b_2 - a_2})^{r_2})' = 0$$

and

$$(z_2(\frac{z_1 - b_1}{a_1 - b_1})^{r_1})' = 0.$$

Since $z_2 = g(z_1)$ is a monotone decreasing function, there exists its inverse function $z_1 = G(z_2)$. Furthermore, note that at optimum, $z_1 \neq b_1$ and $z_2 \neq a_2$, and, therefore, the above equations can be simplified to

$$z_1 r_2 g'(z_1) + g(z_1) - a_2 = 0$$

and

$$z_2 r_1 G'(z_2) + G(z_2) - b_1 = 0.$$

Since $z_2 G'(z_2) = g(z_1) / g'(z_1)$, these equations are equivalent to a single equation for unknown z_1:

$$\frac{a_2 - g(z_1)}{z_1 g'(z_1)} - \alpha \frac{b_1 - z_1}{\frac{g(z_1)}{g'(z_1)}} = 0, \tag{5.1}$$

where $\alpha = r_2 / r_1$ is the ratio of the rigidity indices. Assume that g is twice continuously differentiable, $g'(z_1) < 0$, and the following relation holds,

$$g''(z_1) \leq \min\{\frac{g'(z_1)^2}{g(z_1)}, -\frac{g'(z_1)}{z_1}\}.$$

One can easily verify that the left hand side of equation (5.1) is strictly decreasing, is positive at b_1 and negative at a_1. Consequently, there is a unique solution (z_1^*, z_2^*) on the Pareto frontier between the two initial offers, which is the optimal choice for both players with the given ratio of the *rigidity* indices $\alpha = r_2 / r_1$. In other words, at (z_1^*, z_2^*), the partial derivative of the objective function of each player is zero.

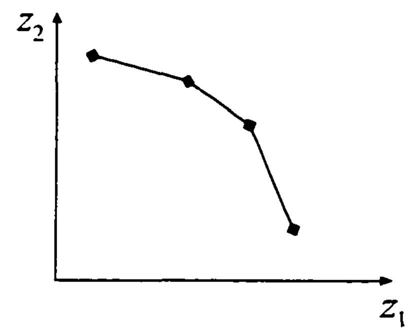

Figure 5.1 Concave piecewise linear Pareto frontier

In the case of concave piecewise linear Pareto frontier (see Fig. 5.1), one can also verify that the same result holds. Therefore, this one-step solution concept can also be applied to a piecewise linear Pareto frontier, for example, to the one that we derived in chapter 4.

5.3 Dynamic Negotiation Process on a Discrete Pareto Frontier for Two-Users

Consider a conflict (***S***, **d**), where ***S*** is a finite feasible payoff set and **d** is the disagreement payoff vector. Assume, without loss of generality, that **d** = **0**. Let

$$\{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\},$$

denote the set of discrete Pareto payoffs, where $x_1 > x_2 > \cdots > x_N \geq 0$. The Pareto property implies that $0 \leq y_1 < y_2 < \cdots < y_N$. For the sake of simplicity, the initial positions of the two players are assumed to be (x_1, y_1) and (x_N, y_N). Let r_1 and r_2 denote the rigidity indices of the two players, respectively. Applying the offer dependent breakdown probability introduced earlier, if player 1 makes a concession to a point (x_k, y_k), then player 2 will continue the bargaining process with probability

$$(\frac{y_k - y_1}{y_N - y_1})^{r_2}.$$

Hence the expected payoff of player 1 is

$$x_k(\frac{y_k - y_1}{y_N - y_1})^{r_2} \geq 0.$$

If $k = 1$, player 1 receives zero expected payoff. He/she always gets positive payoff when $k \neq 1$. It has to be the first player's choice to move to a position other than (x_1, y_1). Offer (x_k, y_k) is the optimal choice for player 1 if and only if for all $l \neq k$,

$$x_k(y_k - y_1)^{r_2} \geq x_l(y_l - y_1)^{r_2}.$$

After taking logarithms, this relationship becomes

$$r_2 \log\frac{y_k - y_1}{y_l - y_1} \geq \log\frac{x_l}{x_k}.$$

That is,

$$r_2 \begin{cases} \geq \dfrac{\log\dfrac{x_l}{x_k}}{\log\dfrac{y_k - y_1}{y_l - y_1}} & \text{if } l < k, \\ \leq \dfrac{\log\dfrac{x_l}{x_k}}{\log\dfrac{y_k - y_1}{y_l - y_1}} & \text{if } l > k. \end{cases}$$

Therefore,

$$r_2 \begin{cases} \geq \max\{P_{1k}, \cdots, P_{k-1,k}\} \\ \leq \min\{P_{k+1,k}, \cdots, P_{Nk}\} \end{cases} \tag{5.2}$$

where

$$P_{lk} = \begin{cases} 0, & \text{if } l = 1, \\ \dfrac{\log \dfrac{x_l}{x_k}}{\log \dfrac{y_k - y_1}{y_l - y_1}}, & \text{if } l > 1. \end{cases} \qquad (5.3)$$

Denote by I_k the interval $[\max\{P_{1k},\cdots,P_{k-1,k}\},\min\{P_{k+1,k},\cdots,P_{Nk}\}]$ for $2 \le k \le N$.

Notice that (x_k, y_k) will be an optimal solution for player 1 if and only if $r_2 \in I_k$. Moreover, for any rigidity index of player 2, there always exists an optimal selection for player 1. Hence, the union of those intervals therefore covers the positive axis. That is,

$$\bigcup_{k=2}^{N} I_k = (0,\infty).$$

Similarly, when player 2 has to make a concession, (x_k, y_k) is his/her optimal selection if and only if for all $l \neq k$,

$$y_k(x_k - x_N)^{r_1} \ge y_l(x_l - x_N)^{r_1}.$$

This relationship can be rewritten as

$$r_1 \begin{cases} \ge \max\{Q_{Nk},\cdots,Q_{k+1,k}\} \\ \le \min\{P_{k-1,k},\cdots,Q_{1k}\} \end{cases} \qquad (5.4)$$

where

$$Q_{lk} = \begin{cases} 0, & \text{if } l = N, \\ \dfrac{\log \dfrac{y_l}{y_k}}{\log \dfrac{x_k - x_N}{x_l - x_N}}, & \text{if } l < N. \end{cases} \quad (5.5)$$

Denote by J_k the interval $[\max\{Q_{Nk}, \cdots, Q_{k+1,k}\}, \min\{Q_{k-1,k}, \cdots, Q_{1k}\}]$ for $2 \le k \le N$.

Notice that (x_k, y_k) will be optimal solution for player 1 if and only if $r_1 \in J_k$. Moreover, for any rigidity index of player 2, there always exists an optimal selection for player 1. Hence, the union of those intervals therefore covers the positive axis. That is,

$$\bigcup_{k=2}^{N} J_k = (0, \infty).$$

This alternating process is finite since at each stage, either a player will choose a point between the current positions or the game ends up with one player accepting the offer of the other player. The terminating solution depends on the initial offers as well as the order in which concessions are made.

A one-step solution concept can be defined which is the simultaneous optimal selection for both players with a given value of the rigidity index ratio $\alpha = r_2 / r_1$. Consider a point (x_k, y_k) such that both intervals I_k and J_k are non-empty. Then interval $L_k = [\dfrac{\min\, I_k}{\max\, J_k}, \dfrac{\max\, I_k}{\min\, J_k}]$ is non-empty, and for any value $\alpha \in L_k$, (x_k, y_k) is an equilibrium point. However, for some α, there is no interval L_k such that $\alpha \in L_k$. In such cases we define the solution of the game as follows: the interval which contains the closest real value to α is chosen and the corresponding point is accepted as the solution. Because of the existence of possibly empty intervals I_k and J_k, as well as the fact that some intervals L_k may overlap, multiple solutions might exist. Theorem 5.1 below shows that intervals I_k and J_k are non-empty and intervals L_k can overlap only at the end points if all Pareto points are located on the curve of a continuous concave function.

Theorem 5.1 Assume that all Pareto points are located on a graph of a twice differentiable function $y = g(x)$,such that $g^{'}(x) < 0$ and $g^{''}(x) \leq 0$ for all x. Then

1. I_k is non-empty for $k = 2,\cdots,N$;
2. J_k is non-empty for $k = 1,\cdots,N-1$;
3. Intervals L_k may overlap only at the end points;
4. There is at least one equilibrium and the number of equilibria is at most two.

Proof. Define function

$$h(u) = \log(g(e^u) - y_1).$$

Simple differentiation yields

$$h'(u) = \frac{g'(e^u)e^u}{g(e^u) - y_1} < 0$$

and

$$h''(u) = -\frac{(g'(e^u))^2 e^{2u}}{(g(e^u) - y_1)^2} + \frac{g''(e^u)e^{2u}}{g(e^u) - y_1} + \frac{g'(e^u)e^u}{g(e^u) - y_1} < 0.$$

Therefore, function $h(u)$ is strictly concave. Hence, function

$$\frac{h(u_k)-h(u_l)}{u_l-u_k}$$

increases in u_l. Denote the above function by $H(u_l,u_k)$. By introducing new variables $x_k = e^{u_k}$ and $x_l = e^{u_l}$, and by definition (5.3),

$$P_{lk} = \frac{1}{H(u_l,u_k)}.$$

Since $H(u_l,u_k)$ is also increasing in x_l, it is implied that P_{lk} is decreasing in x_l. Consequently,

$$\max\{P_{2k},\dots,P_{k-1,k}\} = P_{k-1,k}$$

and

$$\max\{P_{k+1,k},\dots,P_{N,k}\} = P_{k+1,k}.$$

Hence $I_k = [P_{k-1,k}, P_{k+1,k}]$. The monotonicity of P_{lk} also implies that $P_{k-1,k} < P_{k+1,k}$ and interval I_k is therefore non-empty.

Similarly, interval J_k can be shown to be $[Q_{k-1,k}, Q_{k-1,k}]$ and it is also non-empty. This observation implies that intervals L_k become

$$L_k = \begin{cases} (0,\dfrac{P_{23}}{Q_{23}}], & \text{if } k=2 \\ [\dfrac{P_{k-1,k}}{Q_{k-1,k}},\dfrac{P_{k+1,k}}{Q_{k+1,k}}], & \text{if } k=3,\dots,N-2 \\ [\dfrac{P_{N-2,N-1}}{Q_{N-2,N-1}},\infty), & \text{if } k=N-1. \end{cases} \tag{5.6}$$

If for a given α, $\alpha \in (P_{k-1,k}/Q_{k-1,k}, P_{k+1,k}/Q_{k+1,k})$, then (x_k, y_k) is the unique equilibrium. If $\alpha = P_{k-1,k}/Q_{k-1,k}$ with some $k \in \{3,\ldots, N-1\}$, then both points (x_{k-1}, y_{k-1}) and (x_k, y_k) are the equivalent equilibrium solutions. ■

5.4 Application to the Competing Single-Machine Problem

An interesting application of the above methodology stems from resolving conflicts in scheduling problems. In such cases, the Pareto frontier is sometimes finite and discrete. We will next elaborate this conflict resolution methodology on single machine scheduling problem studied in chapter 2.

Suppose that there are two users, each with a set of jobs, namely A and B, requiring processing on a single machine. Suppose each user seeks to minimize the total flowtime. Therefore, both users prefer schedules with early completion times and, hence, conflicts occur.

Assume that if no agreement is reached, jobs will be processed in a random order. If every possible sequence has the same probability, then, in case of disagreement, the expected payoffs are the averages of total flowtimes of all possible sequences. For example, as discussed in chapter 2, assume that A = {1,2,3,4} and B = {5,6}. The processing times are summarized in Table 2.3. The average total flowtimes are d_1=257 and d_2=128; these may be considered as disagreement payoff values. The set of Pareto solutions are given in Tables 2.4 and 5.1.

Table 5.1 Pareto Points

Sequence	z_A	z_B	Pareto Solutions	Standard Pareto Solutions $(x,y)^1$
123456	150	199	(150,199)	—
123546	165	216	X	—
123564	186	139	(186,139)	—
125346	180	149	(180,149)	—
125364	201	119	(201,119)	(56,9)
125634	222	99	(222,99)	(35,29)
152346	195	133	(195,133)	—
152364	216	103	(216,103)	(41,25)
152634	237	83	(237,83)	(20,45)
156234	258	67	(258,67)	—
512346	257	128	X	—
512364	231	95	(231,95)	(26,33)
512634	252	75	(252,75)	(5,53)
516234	273	59	(273,59)	—
561234	294	51	(294,51)	—

It is realistic to assume that the two users negotiate over the Pareto solutions that dominate the disagreement payoff. Thus, introduce linear transformations as follows:

$$x = d_1 - z_A \qquad (d_1 \geq z_A)$$

and

$$y = d_2 - z_B \qquad (d_2 \geq z_B)$$

to the conventional bargaining model with zero disagreement payoff (see Table 5.1). Now the proposed solution concept described in section 5.2 can be applied to find a negotiated solution. The values of P_{kl} and Q_{kl} are calculated from (5.3) and (5.5) respectively. Hence I_k and J_k can be computed by (5.2) and (5.4) (Some of the intervals could be empty). Finally, we determine $L_k = [\frac{\min I_k}{\max J_k}, \frac{\max I_k}{\min J_k}]$, where both I_k and J_k are non-empty. The following algorithm summarizes the bargaining procedure.

Bargaining Algorithm

1. Compute the values P_{kl} and Q_{kl} from (5.3) and (5.5), respectively.
2. For $k = 2, \cdots, N-1$, compute the range of rigidity index r_2, $I_k = [\max\{P_{1k}, \cdots, P_{k-1,k}\}, \min\{P_{k+1,k}, \cdots, P_{Nk}\}]$ where the kth Pareto solution is the best choice for player 1, and the range of rigidity index r_1, $J_k = [\max\{Q_{Nk}, \cdots, Q_{k+1,k}\}, \min\{Q_{k-1,k}, \cdots, Q_{1k}\}]$ where the kth Pareto solution is the best choice for player 2.
3. Compute intervals $L_k = [\frac{\min I_k}{\max J_k}, \frac{\max I_k}{\min J_k}]$, where both I_k and J_k are non-empty.
4. For any $\alpha \in L_k$, (x_k, y_k) is the equilibrium solution. In the case when no interval L_k contains α, select (x_k, y_k) as the solution where L_k has the closest real value to α.

[1] By standard, it is meant that users are maximizing with the disagreement payoff of (0,0).

If there are N discrete Pareto points, it takes $\binom{N}{2}$ to compute P_{lk} and Q_{lk}, and it takes $O(N)$ to compute L_k and locate an interval for α. Therefore, the computational complexity of the bargaining algorithm is $O(N^2(N-1))$.

Table 5.2 L_k Intervals, Solutions, and Optimal Schedules

L_k	(x,y)	Sequence
0	(56,9)	125364
(0,0.8710]	(41,25)	152364
[0.8710,1.5020]	(35,29)	125634
[1.5020, ∞)	(20,45)	152634
$r_2 = 0$	(5,53)	512634

The results for the illustrative single-machine scheduling problem are summarized in Table 5.2. The first column shows the different intervals L_k, the second column presents the payoff values at the solutions for all $\alpha \in L_k$, and the last column shows the corresponding job schedules. Notice that point (26,33) is deeply inside the convex hull, the corresponding interval L_k is empty. Therefore it is not listed as a possible solution in Table 5.2.

5.5 Conclusions

This chapter summarized major two-user negotiation methodologies. A one-step solution concept by Szidarovszky and Shen (1997) was reviewed, and the concept was extended to two-user dynamic negotiation over a concave piecewise linear Pareto frontier and a Pareto frontier consisting of discrete points. An example of a two-user competing single machine scheduling problem was used to illustrate this one-step negotiation procedure, where the Pareto frontier is discrete. Similarly, the procedure can be applied to piecewise linear frontiers such as those obtained in problems where users compete for parallel resources (see chapter 4).

Chapter 6

CONCLUSIONS AND FUTURE RESEARCH

This dissertation has studied three fundamental logistic problems where two potential users (or players) compete for a common set of resources. The three problems are discussed in chapter 2, 3 and 4, respectively. The concepts of games, equilibria and Pareto frontier are addressed in these problems with competing users. In chapter 5, two-user negotiation methodologies are discussed.

In the two-user single machine scheduling problem, the set of nondominated solutions for the two users are efficiently generated by a branch-and-bound search. Sufficient conditions are provided to use the SPT (shortest processing times) rule to simplify the general branch-and-bound search to binary search tree.

In the problem of competitive users of two servers/queues, the leader-follower game framework is adopted to analyze the game between users and servers. Given service prices, a constructive proof is provided to obtain the unique user Nash equilibrium. Based on the monotone properties of the user Nash equilibrium, the servers' Pareto frontier is studied and characterized.

We have examined a network resource allocation problem with two competing users and n substitutable links. The feasible set is proven to be convex and linear. In the case of two competing users, the Pareto frontier is shown to be a convex piecewise linear function. A polynomial algorithm is presented to compute this Pareto frontier for the special case of n links where the users have the same preference order for the links.

Some two-user negotiation methodologies are summarized in chapter 5. We review the concept of the one-step dynamic negotiation process over a continuous twice differentiable Pareto frontier. This one-step solution concept has been extended to the case of concave piecewise linear Pareto frontier and to a Pareto frontier consisting of discrete points.

This dissertation has opened the door to much future research. For one, such types of models can be developed for other logistics problems with competing users, for example, related to inventory control and network routing. For another, the three problems studied in this dissertation can be generalized to the case of N users with $N>2$. Also, pricing schemes for service providers (or network providers) can be investigated to enforce an efficient capacity allocation (or desired network performance).

Appendix A

BASIC CONCEPTS AND DEFINITIONS OF DECISION MODELS AND GAMES

This appendix introduces some basic concepts for modeling decision problems and games. Models for decision-making depend on the number of decision makers and their interactions. In a broad sense, they can be divided into three major categories. If there is a single decision maker and only one objective function to optimize, then single-objective optimization models are used. If there is a single decision maker facing multiple (often conflicting) objectives, then multiobjective programming models are used. In cases when there are n decision makers, and each wanting to optimize his/her objective function, then game theoretic approaches are most appropriate. In the next three sections, we will briefly introduce some fundamental concepts and definitions for these three categories.

A.1 Single Objective Optimization Problems

Typically, optimization models such as linear programming, nonlinear programming, dynamic programming, and integer programming are commonly used in single objective optimization problems. Such problems may be stated as

$$\text{Maximize} \quad f(\mathbf{x}),$$
$$\text{subject to} \quad \mathbf{x} \in X,$$

where $\mathbf{x}$ is the decision variable, usually a vector. X is the **feasible set** of decisions, and $f(\mathbf{x})$ is the objective function, or the performance function, or the utility function that needs to be maximized in the optimization problem.

The optimal objective (function) value is always greater than or equal to the objective value of any non-optimal solution. Or, equivalently, there is no higher objective value than that of the optimal solution.

A.2 Multiobjective Programming Problems

The solutions of multiobjective programming problems are the "Pareto" optimal solutions. Mathematically, a multiobjective programming problem may be formulated as follows:

$$\text{Maximize} \quad f_k(\mathbf{x}), \qquad k = 1, \ldots n,$$

$$\text{subject to} \quad \mathbf{x} \in X.$$

Here $f_k : X \mapsto R$ is the kth objective function. The **ideal point** of a multiobjective programming problem is an n-element vector where the kth component is the maximum value of the kth objective. If the ideal point is feasible, then it satisfies the optimality condition given above, and it is the solution of the problem. However in most cases, the ideal point is infeasible, and therefore "Pareto" optimal solutions are accepted as a set of solutions over which the decision maker must choose.

Definition 1. A feasible solution $\mathbf{x}^*$ is **weakly Pareto optimal** if there is no feasible solution $\mathbf{x}$ such that $f_k(\mathbf{x}) > f_k(\mathbf{x}^*)$ for all k.

In other words, $\mathbf{x}^*$ is weakly Pareto optimal if it is impossible to increase all objective functions simultaneously.

Definition 2. A feasible solution $\mathbf{x}^*$ is **strongly Pareto optimal** if there is no feasible solution $\mathbf{x}$ such that $f_k(\mathbf{x}) \geq f_k(\mathbf{x}^*)$ for all k, with strict inequality for at least one k.

In the sequel, if we state that a solution is **Pareto optimal**, we will mean that it is strongly Pareto optimal. This definition implies that if $\mathbf{x}$ is strongly Pareto optimal, then no objective can be improved without decreasing at least one of the other objectives. In the multiobjective programming literature, Pareto optimal solutions are often called **efficient**, or **nondominated solutions**.

In a multiobjective programming problem, there are usually multiple Pareto optimal solutions. Additional preference information of the decision maker is needed to select a particular one as the solution of the problem. A large variety of methods are available. For example, sequential optimization, ε-constraint method, weighting, distance-based, and direction-based algorithms are the most popular algorithms (Szidarovszky, et al., 1986).

The existence of Pareto optimal solutions are guaranteed by the following theorem.

Theorem 1. If X is a finite set, then there is at least one strongly nondominated solution (or Pareto optimal solution).

Figures A.1 and A.2 show examples of unique and multiple Pareto solutions.

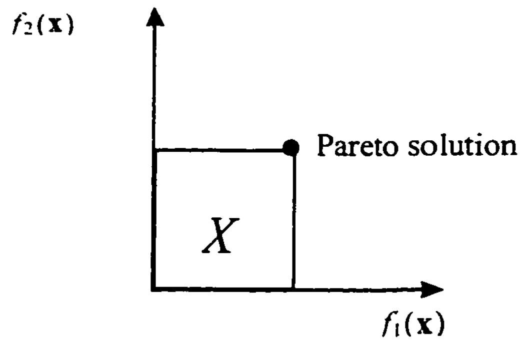

Figure A.1 Unique Pareto solution

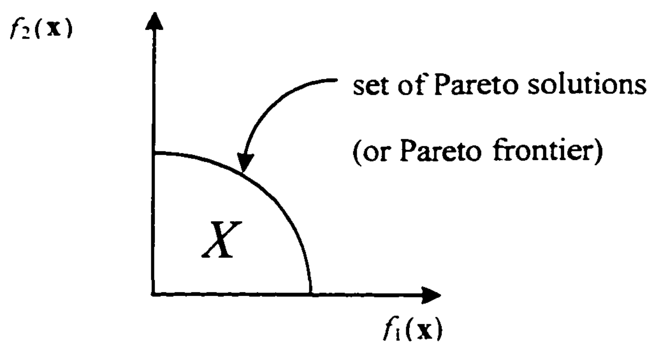

Figure A.2 Multiple Pareto solutions

A.3 *n*-Person Games and Conflicts

In an n-person game, the decision makers are usually called **players**. In the dissertation, users and players are used interchangeably when we study the competitive user behavior. The objectives are called the **payoff functions**.

Game theoretic models are divided into two groups: noncooperative and cooperative games. If no cooperation is assumed among the players, then the problem is modeled as a **noncooperative game**. If there is cooperation among the players of an n-person game, then the problem is modeled as **cooperative game**.

A.3.1 Noncooperative Games

The classical book of Neumann and Morgenstern (1944) is the oldest reference on noncooperative games. A more concise and technical treatment of noncooperative game theory is provided in Fudenberg and Tirole (1991) and van Damme (1987).

An n-person noncooperative game is defined as

$$\{n; X_1,\ldots, X_n, X; \varphi_1,\ldots, \varphi_n\},$$

where n is the number of players, X_k is the **feasible decision set of player *k***, $X \subseteq X_1 \times X_2 \times \cdots \times X_n$ is the feasible **simultaneous decisions set**, and $\varphi_k : X \mapsto R$ is the **payoff function** of player k. In the game theory literature, the decision alternatives of the players are called **strategies**; set X_k is called **strategy set** of player k, and X is called **simultaneous strategy set**. The interrelationship among the strategy selection of the different players is modeled by set X.

The existence of multiple Pareto optimal solutions implies a *conflict* among the players in choosing a solution point in the simultaneous strategy set. A solution concept for noncooperative games can be defined as a possible "equilibrium" among players' decisions. The most common one of these equilibria is the Nash equilibrium defined as follows.

Definition 3. A vector $\mathbf{x}^* = (x_1^*,\ldots,x_n^*) \in X$ is a **Nash-equilibrium**, if for all k, and $\mathbf{x} = (x_1^*,\ldots,x_{k-1}^*, x_k, x_{k+1}^*,\ldots x_n^*) \in X$, $\varphi_k(\mathbf{x}^*) \geq \varphi_k(\mathbf{x})$.

In other words, a simultaneous strategy vector is a Nash equilibrium if no player can unilaterally increase his/her payoff value by changing strategy.

If the strategy set X_k is discrete, then the game is called **discrete**, otherwise, the game is called **continuous**. An n-person game is **finite** if all strategy sets X_k, $k = 1,2\ldots n$, are finite sets.

The **two-person game**, that is when $n = 2$, has been studied extensively. In the special case, when $\varphi_2 = -\varphi_1$, then the game is called **two-person, zero-sum game.** A **two-person finite game** can be described by two **payoff matrices** for both players. The rows correspond to the strategies of the first player, the columns are associated with the strategies of the second player. For any strategy pair, the corresponding element in a payoff matrix gives the payoff values of the players. In a two-person zero-sum finite game, the negative of the first player's payoff gives the payoff of the second player. In this case, an element of the payoff matrix gives an equilibrium point if it is the largest in its column and it is the smallest in its row. Therefore, the equilibrium points of zero-sum games are usually referred to as "saddle points" in the space of objective values.

In a two-person zero-sum finite game, there is no guarantee for the existence of an equilibrium. Also, if an equilibrium point exists it may not necessarily be unique. To illustrate this point, three payoff matrices are given in Figure A.3. In the first case (Figure A.3(a)), the corresponding two-person zero-sum game has no equilibrium, in the second case (Figure A.3(b)), every strategy pair is an equilibrium. In the third case (Figure A.3(c)), the strategy pair corresponding to the element (1,2) of the payoff matrix is the unique equilibrium.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

(a) (b) (c)

Figure A.3 Examples of payoff matrices

The most general existence theorem for Nash equilibria, the Nikaido-Isoda theorem (1955), can be stated as follows.

Theorem 2. Assume that, for all k,

(1) X_k is a convex, closed, bounded subset of a finite dimensional Euclidean space,

(2) $X = X_1 \times X_2 \times \cdots \times X_n$,

(3) $\varphi_k(\mathbf{x})$ is continuous as an n variable function, and

(4) $\varphi_k(\mathbf{x})$ is concave in x_k,

then there is at least one Nash equilibrium point.

This result does not guarantee the uniqueness of the equilibrium point. For example, if the payoff functions are constants, then every simultaneous strategy vector is a Nash equilibrium point.

The uniqueness of the Nash equilibrium was examined by Rosen (1965). In addition to the condition of Theorem 5, assume that for all k,

(5) $\varphi_k(\mathbf{x})$ is continuously differentiable with respect to x_k.

Introduce the following vector valued function,

$$\mathbf{g}(\mathbf{x},\mathbf{r}) = \begin{pmatrix} r_1\nabla_1 \varphi_1(\mathbf{x}) \\ r_2\nabla_2 \varphi_2(\mathbf{x}) \\ \vdots \\ r_n\nabla_n \varphi_n(\mathbf{x}) \end{pmatrix},$$

where $\mathbf{r} = (r_1,\ldots,r_n)$ is a nonzero, nonnegative vector and $\nabla_k \varphi_k$ is the gradient of the φ_k with respect to x_k.

Definition 4. An n-person game is called diagonally strictly concave, if for all different $\mathbf{x}$ and $\mathbf{z} \in X$,

$$(\mathbf{x}-\mathbf{z})^T(\mathbf{g}(\mathbf{x},\mathbf{r})-\mathbf{g}(\mathbf{z},\mathbf{r})) > 0$$

with some constant $\mathbf{r}$.

Theorem 3. If an n-person game satisfies conditions (1) - (5), and is diagonally strictly concave, then it has a unique Nash equilibrium point.

A.3.2 Cooperative Games

Nash equilibria are inherently inefficient. For example, the game known as "the Prisoner's dilemma" (see, for example, Osborne and Rubinstein, 1994) illustrates that the Nash equilibrium is not necessarily nondominated. Cooperative games may remedy this problem by assuming some cooperation or interaction among the players.

In the literature on cooperative games, it is usually assumed that the players seek a Pareto optimal solution. Since there are may be many Pareto solutions for a given cooperative game, either an axiomatic approach or a negotiation procedure has to be used in order to find a compromise (Pareto optimal) solution. The most popular axiomatic models are summarized by Roth (1979).

The major literature on the negotiation process has been reviewed in Chapter 5. In a recent paper by Szidarovszky and Shen (1997), a one-step negotiation process was proposed. Here, each player has an initial position on the Pareto frontier. If a player presents a compromise position, the other player will either terminate the negotiation process with an assumed "*breakdown*" probability, or accept the offer. This probability was assumed to be dependent on the compromise offerred. Under general conditions, it was proved that there exists a unique position on the Pareto frontier which maximizes the expected payoffs of both players simultaneously. This is a fair solution because it does not depend on the concession order (the order in which the offers are made by both players).

This dissertation, the referred one-step negotiation process will be extended to the situations where the payoff sets are discrete. The solution concept of this one-step negotiation process is developed as follows.

A **two-person cooperative game** (or a **conflict situation**) is defined as a pair (*S*, **d**),where $S \subseteq R^2$ is the feasible payoff set:

$$S = \{(\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}) | \mathbf{x} \in X\}.$$

It is usually assumed that *S* is convex, closed, and comprehensive (that is, $\mathbf{u} \in S$ and $\mathbf{u} \geq \mathbf{v}$ imply that $\mathbf{v} \in S$), and both payoff functions are bounded from above. Vector $\mathbf{d} = (d_1, d_2)$ is called the **disagreement payoff vector**. By this we mean that, in the case of a disagreement, player *k* receives the payoff value d_k. To rule out trivial bargaining problem, we assume that there is a $\mathbf{u} \in S$ such that $\mathbf{u} \leq \mathbf{d}$.

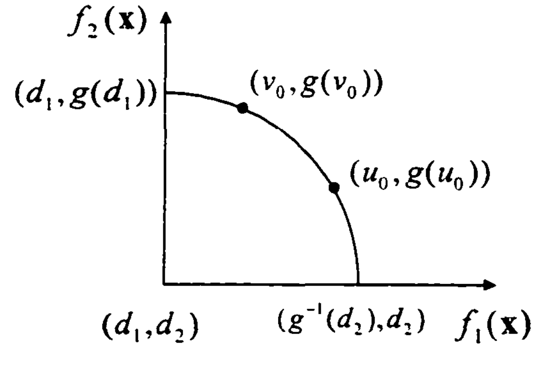

Figure A.4 Initial positions of both players in a negotiation process

For mathematical convenience, assume that the Pareto frontier is characterized by a strictly decreasing function $z_2 = g(z_1)$. At the beginning of the negotiation process (see Figure A.4), both players have their initial positions, denoted by, say, $(u^0, g(u^0))$ and $(v^0, g(v^0))$. It is realistic to assume that the initial positions are better than the disagreement payoff values, since no player will accept worse negotiated outcome than the disagreement payoff, that is

$$d_1 \le v^0 < u^0 \le g^{-1}(d_2) \text{ and } d_2 \le g(u^0) < g(v^0) \le g(d_1).$$

We define the following negotiation process in this conflict situation. If any player presents a position (a possible compromise), then it is either accepted or rejected by the other player. In the case of acceptance, the players receive the agreed payoff values. In the case of rejection, the players receive the disagreement payoff values. In this negotiation process, we assume that the probability of accepting a position depends on both the competitor's value of the position and some preference parameters r_1, r_2 of the two players. We model players' acceptance of a position as follows: If player 1 presents a candidate position, (z, $g(z)$), then player 2 will accept it with probability

$$P_2(z) = (\frac{g(z)-g(x^0)}{g(y^0)-g(x^0)})^{r_2}, \quad (p_2>0) \qquad \text{(A.1)}$$

where r_2 is the preference parameter for player 2.

This expression models the following behavior. If $z = u^0$, that is, player 1 makes no concession, then player 2 will not accept it, that is, the acceptance probability of player 2 is zero. If $z = v^0$, that is, player 1 accepts the initial position of the other player, then the acceptance probability of player 2 is one. A larger value of r_2 decreases the acceptance probability for the same offers. Therefore paremeter r_2 can be interpreted as the **rigidity** of player 2's position.

Given this acceptance probability, the expected payoff of player 1 is

$$zP_2(z) + d_1(1 - P_2(z)). \qquad \text{(A.2)}$$

A similar expression can be formulated when player 2 presents an initial position:

$$g(z)P_1(g(z)) + d_2(1 - P_1(g(z))). \qquad \text{(A.3)}$$

In conflict resolution literature, $1 - P_2(z)$ (or $1 - P_1(g(z))$) is called the **breakdown probability** (Rubinstein, 1982). To model the breakdwon of a negotiation process over a negotiation period, this probability is assumed to be constant in the literature. It can be shown (Obsborne and Rubinstein, 1990) that with a fixed amortization rate and infinite time horizon, the expected payoffs of both players can be written in a similar form as (A.2). However, the constant breakdown probability assumption does not reflect the patience levels of the players which should depend on the goodness of the position offered by the other player.

The offer (z, $g(z)$), offer dependent acceptance probability (A.1), and the expected payoff (A.2) can be interpreted as a negotiation process where each player considers each other's utility. Player 1 believes that the normalized utility function of player 2 is given as $P_2(z)$ and by maximizing expected payoff (A.2) she wishes to maximize the product of her gain ($z - d_1$) and the utility value $P_2(z)$ of the other player. This product is analogous to the multiplicative multiattribute utility function (Keeney and Raiffa, 1976) for a single decision maker. In this case, player 1's multiplicative utility function is a product of a function of her payoff (attribute 1) and a function of player 2's payoff (attribute 2).

In our one-step solution concept, we assume that the relative rigidity index $\alpha = r_2 / r_1$ is given. A point (z, $g(z)$) is the one-step solution of the conflict if this point maximizes the expected payoff of both players for some values of r_1 and r_2 with $\alpha = r_2 / r_1$.

It can be shown that under some general conditions, a continuous conflict has exactly one solution. In the discrete case, there are at most two solutions (as we show in chapter 5). In the continuous case, a monotonic nonlinear equation has to be solved (by using the bisection or secant search methods) to obtain the negotiated compromise (Szidarovszky and Shen, 1997). For the discrete case, a special numerical algorithm is introduced in chapter 5.

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IMAGE EVALUATION TEST TARGET (QA-3)

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