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UMI
THE OPTICS OF ELLIPSOIDAL DOMES

by

Kenneth Scott Ellis

A Dissertation Submitted to the Faculty of the
COMMITTEE ON OPTICAL SCIENCES (GRADUATE)
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

1999
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Kenneth Scott Ellis entitled The Optics of Ellipsoidal Domes and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Jose Sasian
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Jose Sasian
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SIGNED: [Signature]
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ABSTRACT

An ellipsoidal dome is a conformal optical element used to replace a hemispherical dome on a missile to enhance its performance by reducing its aerodynamic drag. Conformal optics are a general class of optical systems in which the optical elements are shaped to optimize something other than image quality, such as aerodynamics. An ellipsoidal dome has lower aerodynamic drag than a comparably sized hemispherical dome. On a missile, lower drag improves its aerodynamic performance by increasing its range and fuel efficiency but degrades the quality of the transmitted wavefront. In particular, an ellipsoidal dome introduces a varying aberration component that depends on the orientation of the aperture stop, which is pivoted about a fixed axis inside the dome. The transmitted ray bundle is incident only on a portion of the dome surface, and the included area lacks axial symmetry. To better understand the imaging characteristics of an ellipsoidal dome in this application, the first- and third-order optical properties of a constant thickness dome are investigated. Particular emphasis is placed on the geometry and symmetry of an ellipse, which impose certain constraints on the form of the aberration coefficients. The geometry is defined in terms of the aerodynamic fineness ratio, outer diameter, and center thickness of the dome. Emphasis is placed on third-order astigmatism and coma, which are shown to be the dominant aberration terms. The effects of varying the fineness ratio, thickness, and index of refraction of a dome are also investigated.
Chapter 1: Introduction

1.1 Conformal Optics

The study of conformal optics is the study of optical components or elements whose shape is governed by something other than optical performance. The field of aerodynamics is a good example of the potential usefulness of conformal optics. Aircraft, be it planes or rockets, traveling at high velocities are integrally shaped and formed to reduce drag in order to boost efficiency, range, and other factors. Smooth, continuous surfaces are desirable to maintain aerodynamic performance. Thus, for aerodynamic considerations, a conformal optical surface, be it a dome or window, is one that follows the contours of the aircraft and whose shape is dictated by the requirement for smooth aerodynamic flow.

1.1.1 Aerodynamic Properties of Conformal Domes

For a missile, the nose is often elongated so that the aerodynamic cross-section is reduced [Eggers 1958]. Airflow over the tip and along the body is smooth. Drag is reduced because air molecules do not strike the moving surface of the nose cone at normal incidence. Instead, air molecules strike the nose cone at large angles. As a result, the transfer of momentum from the air molecules to the nose cone is reduced. It is this momentum transfer that results in aerodynamic drag, which impedes the motion of the missile (or any object moving through a dense medium).
However, the introduction of an optical window, or dome, on the front of the missile has historically compromised aerodynamic performance. This is because the dome shape of choice, optically, is either a whole or a part of a hemispherical element consisting of two concentric spherical surfaces. There are a number of reasons why the latter is true. Among them is that the fabrication and testing of spherical surfaces are processes that are well understood. In addition, a hemispherical dome housing a gimbaled seeker introduces a fixed amount of aberration, regardless of the gimbal angle. Despite these optical advantages, a hemisphere is too blunt to offer optimum aerodynamic performance, particularly at the high velocities encountered by a missile.

Figure 1-1 illustrates that an elongated dome improves aerodynamic performance by reducing the air resistance or aerodynamic drag of the nose cone. It shows the normalized drag coefficients of ellipsoidal, paraboloidal, and conical nose cones, all having the same outer diameter. For comparison, an identically sized flat plate has a normalized drag coefficient of 1 [Edwards 1997]. The normalized drag coefficient is plotted against fineness ratio, which is the ratio length of length to the diameter (L/D) of the dome.

In the simple model used for the calculation, the drag coefficient is proportional to the magnitude of the component of the surface normal pointed in the direction of flight. Thus, at any point along the surface, a large angle between its normal and the direction of travel results in a small effective area for resistance, thereby reducing the aerodynamic cross section.
Figure 1-1 Aerodynamic drag vs. fineness ratio and dome shape. A hemisphere has a drag coefficient and fineness ratio of 0.5.
Both the fineness ratio and the normalized drag coefficient of a hemispherical dome are equal to 0.5. A hemisphere is simply an ellipse with equal major and minor axis lengths. As illustrated by the figure, even a modest increase in the fineness ratio results in a significant reduction in drag. The aerodynamic drag of each dome configuration is inversely proportional to the fineness ratio. This proportionality depends, to some extent, on the base (vertex) radius of the generating curve. In general, for any particular fineness ratio, the dome surface having the smallest vertex radius of curvature exhibits the least amount of aerodynamic drag.

### 1.1.2 Optical Design Considerations for Conformal Domes

However, for reasons that become more evident as this dissertation progresses, the optical properties of a dome are more ill-defined and the problem of aberration correction is more complex the further the dome shape departs from a sphere. A gimbaled imaging system inside a hemispherical dome ‘sees’ the same shape and same aberration independent of its pointing. This is the preferred optical configuration. However, as was shown, a hemispherical dome is not the preferred aerodynamic configuration. Significant reductions in drag are achieved for modest increases in the fineness ratio (elongation) of a dome. However, the transmitted optical wavefront is significantly degraded. It now depends on the orientation of the gimbal because the areas of the dome ‘seen’ by the imaging system are no longer axially symmetric for all pointing angles. As a result, the introduction of a conformal dome places strenuous requirements on the imaging system to produce a high quality image.


\[ \theta = \text{Look Angle} \]

**Figure 1-2** Conceptual layout of a conformal dome, corrector, and imaging system.
In a seeker using a conformal dome, the imaging system consists of correctors and image-forming optics that are enclosed within the dome, as shown in Figure 1-2. Either or both are mounted on a gimbai platform. The basic function of the correctors is to remove, either actively or passively, the varying aberration component of the wavefront as the gimbai is swept through a field of regard. To accomplish this, the correcting elements necessarily have an asymmetric component that counteracts the similar asymmetry introduced by the dome.

An example of an active correction technique is one in which the axial spacing between two crossed cylindrical lenses is varied to provide astigmatism and focus correction [Whalen 1997]. An example of a passive corrector is an arch corrector having variable sagittal and tangential curvatures nested inside the dome, that also corrects the astigmatic component of the wavefront error [Sparrold 1997]. Another passive correction technique uses one or more dome-like elements that are independent of the gimbal mechanism and share a common optical axis with the dome. The beam footprint moves over sub-apertures of the corrector, much as it does with the dome. Any residual bias may be removed by the imaging system. The correctors are generally afocal or low power refractive elements.

On the other hand, most of the optical power is contained by the image forming optics. That is, they are used to control the focal length and set the aperture, f/#, and field of view. These elements are mounted on the gimbal platform, but otherwise remain fixed with respect to one another. They cannot be used to correct a variable wavefront. As a
result, they should be designed to introduce little additional aberration to the system and to be capable of correcting bias errors left by the correctors. Ideally, although seldom the case, the design of the dome and correctors can proceed independently of the design of the imaging system.

That the entrance pupil of the imaging system is a significant fraction of the diameter of the dome is important. First, a large collecting aperture is beneficial from the standpoint of target detection and discrimination. So much so, that the aperture is generally as large as can be accommodated within the confines of the missile body. But a large aperture significantly increases the aberrations introduced by the dome. The effect of a large pupil diameter on image quality can be illustrated by the following example. Consider the windshield of a car or canopy of a jet. Both are conformal optical elements in accordance with the definitions set forth previously. Yet our vision through either one is scarcely affected. Why is this true?

Part of the answer is realized by considering the relative size of the pupil of the imaging system behind the glass. The human eye has a pupil diameter of about 5 mm. A projection of the pupil onto the windshield shows only a small region affecting the optical wavefront. Within this region, the windshield is nearly a plane parallel plate. Likewise, a small aperture projected onto the surface of the dome is not significantly affected by its asphericity. In addition, the human eye can rapidly accommodate focus changes introduced by the window.

In contrast, a realistic seeker assembly has an optical aperture of about half of the diameter of the missile tube. Within the aperture, the surface departures of a conformal
dome from the best-fit sphere may be tens or hundreds of waves. Such large departures limit the ability to correct the induced aberration with conventional surfaces and techniques, and instead require exotic surfaces to obtain adequate correction. Further, packaging constraints often limit the ability to actively compensate focus errors introduced by a dome.

1.2 Scope of Study

The use of nonstandard shapes and proportionately large optical apertures vastly complicates the calculation and subsequent optical correction of wavefront errors introduced by the dome [Novak 1997]. Symmetry conditions used to expand the wavefront error introduced by a spherical element may no longer apply. Mounting the imaging optics on a gimbaled platform further complicates the problem because the imaging system no longer sees constant aberration, independent of the line of sight.

In a practical sense, the surfaces required to correct the errors introduced by a dome can only be designed empirically by tracing rays in optical design software, letting the design code vary the surface shapes to achieve the desired level of correction. However, a basic understanding of the optical properties of a conformal dome can point the way to a desirable solution. The goal of this dissertation is to develop this basic understanding with regard to an ellipsoidal dome.

Basic definitions and concepts used extensively throughout the remainder of this dissertation are developed in this chapter. Also included is a brief description of a generic
missile seeker to provide some background and to put the problem of imaging through a conformal (ellipsoidal) dome in context.

In Chapter 2, mathematical expressions describing the geometry of an ellipsoidal dome are introduced. Included are geometrical aspects such as the base radius of curvature and the conic constant as well as the tangential and sagittal curvatures and the orientation of the normal with respect to the missile body axis. All are derived in terms of the fineness ratio, outer diameter of the dome, and the gimbal angle. This development provides the context for the remainder of the dissertation. The first- and third-order optical properties of a dome all depend on its geometric configuration. The first-order properties of a dome that are derived include its optical power, the location of its cardinal points, and line of sight deviation.

Chapter 3 discusses the aberration function of a dome, beginning with review of third-order aberration theory and concluding with the determination of aberration coefficients for astigmatism and coma. The emphasis of this chapter is on the influence that the symmetries of an ellipse with regard to gimbal angle have on the form of the aberration function and its coefficients.

To complete the discussion of the aberration function, Chapter 4 investigates how changing the geometry and material of a dome impacts the transmitted wavefront. Three independent parameters are chosen: the aerodynamic fineness ratio, the thickness of the dome, and its index of refraction. These three parameters are selected because they are all independent degrees of freedom that together uniquely define the dome. How each contributes to the net aberration of the dome significantly impacts the design space. Since
optical design is inherently a tradeoff between conflicting constraints, it helps to develop an understanding of how the dome moves in the solution space and, therefore, to determine where it is most appropriate to focus the design effort.

1.3 Dome Parameters and Definitions

The ellipse has been chosen for a number of reasons. It is a good compromise between aerodynamic and optical performance. Dome shapes having proportionately short vertex radii of curvature, such as the parabola and the cone, introduce significantly more optical aberration than one more closely matching a sphere. An ellipse is easily modeled in optical design software. It also has a zero slope condition at its maximum diameter. As a result, the dome can be mounted flush and without discontinuity to the missile body. Finally, the ellipse has a number of useful symmetry conditions that aid in the derivation of its optical properties.

Figure 1-3 illustrates the basic dome configuration. The outer surface of a dome is defined by the fineness ratio. As indicated previously, the fineness ratio, $F$, is an aerodynamic figure of merit indicating the elongation of a dome relative to its diameter. Specifically, it is the ratio of the length of the dome to its diameter, $L/D$. Higher values of $F$ result in a dome with lower aerodynamic drag. As a subset of this definition we have the aerodynamic fineness ratio, $F_a$, and the optical fineness ratio, $F_o$.

For the purposes of this dissertation, the determination of $F_a$ assumes an ellipsoidal outer surface. The length of the dome, by definition, equals the length of the
semi-major axis of the ellipse. It is the distance between the vertex of the outer surface to its geometric origin. The diameter of the dome is the length of the minor axis of the ellipse. For example, a hemispherical dome, has a fineness ratio \( F_a = 0.5 \) because \( L = r \) and \( D = 2r \). It is a limiting case of the more general ellipsoidal dome. In both configurations, the slope of the surface at its maximum diameter is zero. As stated earlier, this condition is imposed to ensure continuity between the outer surface of the dome and the outer diameter of the missile tube.

The latter constraint is not a true requirement for a general conformal dome element because many surface types can be smoothly integrated with the missile tube. It was chosen because the parameters defining an ellipse, its base radius of curvature and its conic constant, are uniquely defined by the fineness ratio and missile diameter. In contrast, a general aspheric dome does not necessarily have a unique radius of curvature for a given aerodynamic fineness ratio and diameter because the dome shape can be significantly altered by changing the aspheric coefficients without changing its base radius of curvature.

The optical fineness ratio is defined somewhat differently. It is the ratio, \( L/D \), of the physical extent of the dome. The difference in the two arises from the fact that the system, such as one that operates over a small scan range, may not require the dome surface extend all the way to the missile tube. In general, it has more implications with regards to the fabrication of the dome than its aerodynamic and optical properties.
Figure 1-3 Basic dome configuration. The aperture stop is at the common origin of the two ellipsoidal surfaces. It is rotated through the look angle $\theta$, which is positive as shown. $L$ is the length of the dome, measured from the vertex of the ellipse to its center. $D$ is the diameter of the ellipse. $t$ is the vertex thickness.
The aerodynamic fineness ratio is the key figure of merit because it more generally defines the dome. Unless otherwise specified, the fineness ratio $F$ and the aerodynamic fineness ratio, $F_a$, carry the same meaning.

The dome illustrated in Figure 1-3 is an example of a constant thickness dome which, as the name implies, maintains a constant separation between the outer and inner surfaces. This is the default configuration for the analysis. The inner surface is wholly constrained by the outer surface of the dome. Thus, to satisfy this constraint the inner surface must also be an ellipse, defined such that the thickness, $t$, of the dome is equal to $\frac{1}{2}(D_o - D_i)$. Once again, this is an arbitrary requirement, since the aerodynamics depend only on the shape of the outer surface. Its origins are both historical and practical.

Historically, an inner surface ellipse was selected because it lends itself to a full aperture interferometric null test having a point source at one focus and a reflecting reference sphere concentric with the other in a double-pass configuration. In a practical sense, constraining the inner surface in such a manner simplifies the discussion and analysis because it eliminates a large number of permutations of the dome. It is the ellipsoidal dome equivalent of a plane parallel plate. As such, it represents the most well defined starting point. It also allows key points about the optical characteristics of the dome to be developed without the unnecessary burden of complicated surface shapes.

Also shown in Figure 1-3 is the aperture stop, located on a gimbal platform whose center of rotation is located at the common origin of the two dome surfaces. In the optical model, the aperture stop and gimbal position are coincident so the aperture stop rotates.
The rotation angle of the stop is the look angle, $\theta$. The sign convention is defined such that a counterclockwise rotation of the stop, as shown in the figure, is a positive look angle. This is the usual convention for a positive angle in which the local z-axis of the stop must be rotated clockwise to align it with a reference axis. In this instance, the reference axis is the $z$-axis of the dome, which is itself co-linear with the missile body axis. The optical axis ray (OAR) passes through the center of the aperture stop and defines the center of the field of view of the imaging system. The field of view is distinct from the look angle. The maximum look angle is the field of regard (FOR) of the seeker. It is always a positive quantity. These terms are summarized in Table 1-1.

By definition, the aperture stop is the limiting aperture in an optical system. The chief ray, also by definition, passes through the center of the aperture stop. Pupils are simply conjugate images of the aperture stop in object and image space that necessarily rotate with the stop. The exit pupil size in this analysis is constant over look angle because this condition more accurately represents a flight configuration. This assumes the entrance pupil of an imaging system inside the dome is fixed. The entrance pupil diameter is allowed to vary over look angle so that the ray bundle always fills the aperture stop.

Since most of the emphasis of this dissertation is placed on the optical characteristics of the dome itself, the dome and aperture stop comprise the entire optical model. Where necessary, a perfect lens coincident with the aperture stop is used to illustrate some key points. A perfect lens is a mathematical construct that forms a perfect point (stigmatic) image for a single conjugate position. It is used frequently in the
analysis of afocal optical system. Departures from the ideal point image are the result of aberration and focus change introduced by the dome.

<table>
<thead>
<tr>
<th>Term</th>
<th>Symbol</th>
<th>Range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fineness Ratio</td>
<td>F or Fₐ</td>
<td>F ≥ 0.5</td>
<td>Length to diameter (L/D) of the outer surface of the dome.</td>
</tr>
<tr>
<td>Field of Regard</td>
<td>FOR</td>
<td>0 ≤ FOR ≤ π/2</td>
<td>Maximum look angle.</td>
</tr>
<tr>
<td>Look Angle</td>
<td>θ</td>
<td>-FOR ≤ θ ≤ FOR</td>
<td>Rotation of the stop with respect to the missile body axis.</td>
</tr>
<tr>
<td>Optical Axis Ray</td>
<td>OAR</td>
<td>n/a</td>
<td>Center of the field of view. The OAR is normal to and passes through the center of the aperture stop.</td>
</tr>
<tr>
<td>Line of Sight Deviation</td>
<td>LSD</td>
<td>n/a</td>
<td>Deviation of the OAR from the look angle, θ, resulting from refraction through the dome. A pointing error.</td>
</tr>
</tbody>
</table>

Table 1-1 Summary of key terms.
Chapter 2: Geometrical Aspects and First Order Characteristics

2.1 Geometry of an Ellipse

A discussion of the optical characteristics of an ellipsoidal dome must begin with a basic understanding of the geometry of the ellipse. While this may be stating the obvious, it does help to conceptualize the problem. Because the optical characteristics of the dome are critically dependent on the geometry, it is difficult to continue without developing a fundamental understanding of the ellipse and how it is represented. It will be shown that there is much about the behavior of the dome that can be intuited by the geometry.

The specification of three parameters: \( F_a, D, \) and \( t \), allow the physical dimensions of the dome to be uniquely defined. Most optical design codes require the base radius of curvature and conic constant as inputs to describe a conic surface. With this information, the sag of a conic surface is computed using Eqn. 2-1

\[
Z_s = \frac{cv \cdot R^2}{1 + \sqrt{1 - (k + 1) \cdot cv^2 \cdot R^2}} \tag{2-1}
\]

where the curvature, \( cv = 1/r_0 \), \( k = -e^2 = -(eccentricity)^2 \) is the conic constant, and the radial position from the z axis is \( R^2 = x^2 + y^2 \). The conic constant, \( k \), for an ellipse lies between \(-1 < k < 0\). It will follow that \( r \) and \( k \) depend only on the fineness ratio and the diameter.
The dome is generated using two concentric ellipsoids. In a Cartesian coordinate system, Eqn. 2-2 gives the equation for an ellipsoid.

\[
\frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1
\]  

(2-2)

This describes an ellipsoidal surface with its major axis along the z axis, and its minor axis in the x-y plane. The ellipse has a major axis of length 2a. The minor axis has a length of 2b. Since the dome has rotational symmetry, the choice of coordinate axes in the x-y plane is arbitrary. Therefore, in the discussions to follow, x = 0, and y represents a radial distance on the ellipse, measured perpendicular from the rotation axis (axis of symmetry). By definition, the tangential plane is the y-z plane.

The conic constant is easily derived because it is related to the eccentricity, or elongation, of the ellipse. An ellipse centered at the origin has two foci, \(f_1\) and \(f_2\), on the z-axis, separated from the origin by ±c, where

\[
c^2 = a^2 - b^2
\]

The eccentricity of the ellipse is

\[
\varepsilon = \frac{c}{a}
\]

Recalling that the aerodynamic fineness ratio is calculated by extending the curve of the ellipse out to its maximum diameter, we have

\[
F = \frac{L}{D} = \frac{a}{2b}
\]

(2-3)

Substituting L for a and D/2 for b and simplifying the result, the following expression for the square of the eccentricity is obtained
Finally, with $\varepsilon^2 = -k$, the conic constant is, by simple substitution

$$k = \frac{1}{4F^2} - 1 \quad (2-4)$$

The base radius $r_0$ is also related to the eccentricity and semi-major axis length [Gillett 1981]

$$r_0 = a(1 - \varepsilon^2) \quad (2-5a)$$

With the appropriate substitutions, $a = FD$, and $k = -\varepsilon^2$, we have

$$r_0 = FD(1 + k) \quad (2-5b)$$

Further substitution for $k$ yields

$$r_0 = \frac{D}{4F} \quad (2-5c)$$

Note that $r_0$ is positive for a convex surface and negative for a concave surface.

The two equations for $r$ and $k$ are an important result. The conic constant of an ellipse depends solely on the fineness ratio. The base radius of curvature is a linear scale factor to change the size of the dome without altering its basic shape.

Using the appropriate sign conventions, we can return to Eqn. 2-1 and write the surface sag in terms of the fineness ratio, diameter, and radial position.

$$Z_s = \frac{4Fy^2}{D + \sqrt{D^2 - 4y^2}}, r_0 > 0 \quad Z_s = \frac{-4Fy^2}{D + \sqrt{D^2 - 4y^2}}, r_0 < 0 \quad (2-6)$$
2.2  Geometry of the Dome

The previous development shows that the outer surface of the dome can be completely specified by $F_a$ and $D$. The inner surface also obeys the same relationships. The thickness of the dome ties the two surfaces together, allowing the inner surface to be specified in terms of the outer surface parameters.

For a constant thickness dome, the inner and outer surfaces may be considered concentric because they share the same global origin. In other words, $a_i = a_o - t$, where $a_i$ and $a_o$ refer to the semi-major axis length of the inner and outer surfaces respectively. Similarly, $b_i = b_o - t$, where the $b$ terms refer to the semi-major axis lengths of the two surfaces. With the appropriate substitutions, it can be shown that the outer and inner surface $r$ and $k$ values are

\[ k_o = \frac{1}{4F_a^2} - 1, \quad r_o = \frac{D}{4F_a}, \]  
\[ k_i = \frac{(D - 2t)^2}{4(F_a D - t)} - 1, \quad r_i = \frac{(D - 2t)^2}{F_a D - t} \]

(2-7a, b) (2-8a, b)

2.3  Surface Geometry

2.3.1  Surface Normal

For any function, $f(x,y,z) = 0$, the normal to any point $P$ on the surface is given by the gradient of $f$ evaluated at $P$,

\[ \vec{n} = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

Rewriting Eqn. 2-2 to put it in the proper form, we have
\[ f(x, y, z) = 1 - \left( \frac{x^2 + y^2}{b^2} + \frac{z^2}{a^2} \right) = 0 \]

The partial derivatives give the normal components to the surface at the point \( P_0 \)

\[ \vec{n}(x_0, y_0, z_0) = \left( \frac{-2x_0}{b^2}, \frac{-2y_0}{b^2}, \frac{-2z_0}{a^2} \right) \]

In the tangential plane, \( x = 0 \), so we are left with only an ellipse, shown in Figure 2-1. Examining the figure, it is clear that the angle between the normal and the \( z \)-axis is given by

\[ \tan(\theta) = 4F^2 \tan(\theta) \quad (2.9a,b) \]

After making the appropriate substitutions for \( a \) and \( b \) we are left with the result

\[ \tan(\theta) = 4F^2 \tan(\theta) \quad (2.10) \]

This behavior is plotted against look angle for various fineness ratios in Figure 2-2. In accordance with the established sign convention, \( \theta \) has the same sign as \( \phi \). The angle \( \phi \) between the normal to the surface and the \( z \)-axis increases rapidly with fineness ratio.
Figure 2-1 Definition of an ellipse. The foci are located at +/-c. The quantities a and b are those appearing in Eqn. 2-2. The angle between the surface normal and the z-axis is $i_n$. The angle subtended by the point $P(y_0,z_0)$ on the ellipse is $\theta$. Both are positive, as drawn.
Figure 2-2 Angle between the surface normal and the z-axis as a function of look angle for different values of the aerodynamic fineness ratio $F_a$. 
2.3.2 Curvature

To further develop the geometry of the ellipse, it is useful to define it parametrically, as a function of look angle, i.e., \( f(\theta) = (y(\theta), z(\theta)) \). Rewriting the equation of the ellipse in this fashion simplifies the calculation of the curvature of the surface as a function of look angle, as will be shown. In addition, for a particular ellipse (or dome), we are most interested in how its optical properties vary with look angle, so this provides for a more direct analysis.

As has been established, rotating the aperture stop with the gimbal platform breaks the axial symmetry of the transmitted wavefront. It instead becomes bilaterally symmetric about a centerline parallel to the \( y \)-axis, passing through the diameter of the pupil in the plane of the scan. The plane of symmetry is the tangential plane of the optical system. It is invariant for any look angle. The sagittal plane is defined with respect to the optical axis ray (OAR) and the local coordinate axes of the gimbal mechanism. Recall that the gimbal is rotated about the \( x \)-axis so that the new local \( z \)-axis \( z' \) is inclined with respect to the missile body axis \( z_0 \). The OAR is colinear with \( z' \). The sagittal plane contains the OAR, so it also rotates. It is defined by the new \( x-z' \) plane, perpendicular to the tangential plane.

Cross-sectional slices of the dome in the tangential and sagittal planes reveal a difference in curvature between the two. Although the centers of curvature are located along a common line coincident with the surface normal, the lengths \( r_T \) and \( r_S \) from a point \( P \) are different. Since the sagittal and tangential planes are orthogonal and the
centers of curvature lie along the normal line, the radius of curvature in each plane, \( r_T \) and \( r_S \) respectively, are located at the principal centers of curvature of the point \( P \). The surfaces that contain the principal centers of curvature are the caustic. Either surface of the caustic may degenerate to a line, as will prove to be the case for the sagittal centers of curvature.

The tangential centers of curvature do not degenerate to a line. Rather, the caustic surface they describe has roughly the same form as the caustic resulting from an undercorrected, spherically aberrated wavefront (Figure 2-3). The surface originates from a point \( r_0 \) on the \( z_0 \) axis for \( \theta = 0 \) and fans outward, tangent to the surface normal, away from the major axis as \( \theta \) moves away from zero until reaching a maximum at \( \theta = \pm \pi/2 \).

To calculate the tangential radius of curvature, we make use of the fact that the curvature, \( K \), at a point \( P \) on a surface is given by

\[
K = \frac{|z'y'' - y'z'|}{(z'^2 + y'^2)^{1/2}}
\]

evaluated at \( P \) [Gillett 1981]. Expressions for \( y \) and \( z \) are found by substituting Eqn. 2-9b into Eqn. 2-2 and making the appropriate substitutions for \( a \) and \( b \) to get

\[
y(\theta) = \frac{FD \tan(\theta)}{\sqrt{4F^2 \tan(\theta)^2 + 1}} \tag{2-11}
\]

\[
z(\theta) = \frac{FD}{\sqrt{4F^2 \tan(\theta)^2 + 1}} \tag{2-12}
\]

Next, we compute the derivatives and group terms. The tangential radius of curvature as a function of the paraxial radius of curvature, \( r_0 \), and fineness ratio is
The curvature in the sagittal plane follows a different behavior. The centers of curvature, instead of moving away from the $z_0$ axis, degenerate to a line. This is because a single dome surface is nothing more than an ellipse rotated about its major axis. Any cross-section of the surface taken in the $x$-$y$ plane must be circular with the $z_0$ axis at its center.

The length of $r_s$ is calculated by computing the distance between a point $P$ on the ellipse and its major axis, measured along the surface normal. The angle between the normal and the major axis is known. From Figure 2-1 and Eqn. 2-10, we have

$$\tan(\alpha) = \tan(\theta) = \frac{y(\theta)}{\Delta z}$$

where $y(\theta)$ is the radial position on the surface and $\Delta z$ is the axial separation between a line dropped perpendicular to the $z$ axis from $P$ and the $z$-axis intercept of the surface normal. $Y(\theta)$ is known from Eqn. 2-11. The points $P$, $z_p$, and $z_n$ form a right triangle with

$$r_s(\theta) = \sqrt{y(\theta)^2 + \Delta z^2}$$

Using $\Delta z = y(\theta)/4F \tan(\theta)$ and $r_0 = D/4F$ we are left with

$$r_s(\theta) = r_0 \left( \frac{16F^4 \tan(\theta)^2 + 1}{4F^2 \tan(\theta)^2 + 1} \right)^{\frac{3}{2}}$$

Both the tangential and sagittal radii of curvatures have a minimum at $\theta = 0$ (or $\pi$) and maximum at $\theta = \pm \pi/2$. Substituting $\theta = 0$ in Eqns. 2-13 and 2-15, gives $r_t(0) = r_s(0) = r_0$. To calculate the maximum tangential radius of curvature, multiply both the
numerator and denominator of Eqn. 2-13 through by $1/\cos(\theta)$ to eliminate the discontinuity at $\theta = \pm \pi/2$. The result is that $r_T(\pm \pi/2) = 8F^3 r_0$. The same technique applied to Eqn. 2-16 yields $r_S(\pm \pi/2) = 2Fr_0$. Figure 2-4 and Figure 2-5 contain plots of Eqn. 2-13 and Eqn. 2-16 for several values of $F$. 
Figure 2-3 Tangential centers of curvature. The upper curve shows a portion of an ellipse ($F = 1$). The lower curve shows the location of the caustic. The centers of curvature lie on this curve. The lengths of the lines connecting the two are calculated from Eqn. 2-13. The figure is drawn to scale.
Figure 2-4 Tangential radius of curvature as a function of look angle for several values of $F_a$. The outer surface of the dome has a diameter of 50 mm. 

Top Calculated from Eqn. 2-13. Bottom Eqn. 2-13 divided by $r_0$. 
Figure 2-5 Sagittal radius of curvature as a function of look angle for several values of $F_a$. The outer surface of the dome has a diameter of 50 mm.

Top Calculated from Eqn. 2-16. Bottom Eqn. 2-16 divided by $r_0$. 
2.4 First-Order Characteristics

The first-order, or paraxial, parameters of the dome will be discussed in detail only for the on-axis case because they lack a clear interpretation for nonzero look angles. Rotating the aperture stop violates a number of key assumptions used to make paraxial calculations. Specifically, a paraxial ray trace assumes an infinitesimally small aperture stop, so the paraxial ray bundle is nearly colinear with the optical axis. In this limit, optical surfaces are represented as planes with aspheric contributions being ignored. The small angle approximation, \( \sin(\theta) = \theta \) for refraction and transfer is valid in this regime. Rotational symmetry is assumed, with the result being that tilts and decenters are ignored.

On boresight the parameters of interest are the effective focal length (EFL) of the dome, the location of its cardinal points and planes, its entrance and exit pupil locations, and its symmetric shape factor. For nonzero look angles, line of sight deviation (LSD) and the astigmatic focal properties of the dome are its most significant first-order characteristics and will be discussed in detail.

A number of references describing the first-order characteristics of asymmetric optical systems already exist. These include developments using Hamiltonian optics [Stone 1991, 1992], a general theory for optical systems having small tilts and decenters [Buchroeder 1976], and computational methods [Hopkins 1985].
2.4.1 Line of Sight Deviation

Line of sight deviation manifests itself as a pointing error resulting from the bending of the optical axis ray as it passes through a dome. By convention, the optical axis ray after refraction by the dome is colinear with the pointing of the gimbal. In other words, the OAR is defined as the ray that is both perpendicular to the aperture stop and passes through its center. This convention has been adopted because the orientation of the gimbal platform with respect to the missile body axis is always known. It provides a more readily determined reference than the alternative, which would be to associate the look angle of the seeker with the angle of the OAR with respect to the missile body axis in object space, before refraction and deviation by the dome.

With these conditions, it is necessary to trace the OAR backward from the aperture stop and through the dome to calculate the line of sight deviation. The OAR strikes the inner surface of the dome at some angle, $\alpha$. It is then refracted by both the inner and outer surfaces and exits the dome at some other angle $\beta$. Thus, we have (see Figure 2-6),

$$\text{LSD} = (\alpha - \beta) - \Delta n$$

(2-17)

where $\alpha$ and $\beta$ are the angles of incidence and refraction from the inner and outer surfaces, respectively, and $\Delta n$ is the angular difference between the normals of the two surfaces.
Figure 2-6 Line of sight deviation (LSD). The refracted ray is bent away from the look angle: LSD $> \theta$. Sign conventions are shown.
2.4.2 Angle of Incidence

The angle of incidence on the inner surface of the dome is \( \alpha = \theta - i_n \). Its sign convention is defined such that \( \alpha < 0 \) for \( \theta, i_n > 0 \). That is, the OAR must be rotated counterclockwise to reach the surface normal. In a similar fashion, \( \alpha > 0 \) for \( \theta, i_n < 0 \) because the OAR must be rotated clockwise to reach the normal.

Application of Eqn. 2-10 gives \( i_n \) for the inner surface of the dome. When the OAR is traced outwards from the aperture stop and is incident first on the inner surface, the angle \( \theta \) is the look angle. Likewise, the fineness ratio used in computing \( i_n \) is that of the inner surface. Figure 2-7 shows a plot of \( \alpha \) against look angle for a number of fineness ratios. It is interesting to note that the maximum angle of incidence for any fineness ratio lies along the line \( \alpha_{\text{MAX}} = 2\theta - \pi 2 \). The look angle, \( \theta_{\text{MAX}} \) at which this occurs can be calculated by differentiating \( \alpha \) with respect to \( \theta \), setting the result equal to zero, and solving for \( \theta_{\text{MAX}} \). This has the result

\[
\theta_{\text{MAX}} = \arctan(1/2F)
\]  \hspace{1cm} (2-18)

which is plotted in Figure 2-8.
Figure 2-7 Angle of incidence as a function of look angle for several $F_a$ with the stop at the geometric origin of the ellipse. The look angle at which the maximum AOI occurs is along the line passing through the maximum value of $|a|$. 
Figure 2-8 Maximum angles. Maximum value of $|\alpha|$ occurs at $|\theta_{\text{MAX}}|$
2.4.3 Computing $\Delta n$

The difference between the surface normals is calculated once again by application of Eqn. 2-11. In this case however, the fineness ratio is that of the outer surface and $\tan(\theta) = y_{outer}/z_{outer}$. Computation of the ray intercept points $y_{outer}$ and $z_{outer}$ is non-trivial because it requires a cumbersome calculation of the difference between the refracted ray angle and the normal to the inner surface. Rather than developing an explicit formulation for this transfer and determining the angle of the outer surface normal, the difference between the normals of the inner and outer surfaces can be assessed qualitatively.

The magnitude of $i_n$ relative to $\theta$ is determined as follows. Recall that for a hemispherical dome, $F_a = 0.5$, so the quantity $4F_a^2$ in Eqn 2-11 is 1. For any ellipse, $F_a > 0.5$ so $4F_a^2$ must be greater than 1. Therefore, $|i_n|$ must always be equal to or greater than $\sin\theta$. As a result, the OAR refracts towards the $z$-axis of the ellipse following its intersection with the inner surface.

As a result, for nonzero dome thickness, the OAR always intercepts the outer surface at $\tan(\theta') < \tan(\theta)$. Additionally, the outer surface of the dome has a smaller fineness ratio because the diameter of the inner surface is decreasing faster than its length: $F_{so} = L/D < F_{si} = (L-t)/(D-2t)$. As a result $|i_{n1}| < |i_{n1}|$. Thus, in the region near the OAR, the dome appears as a wedge with its apex pointing towards the $z$-axis of the dome. Consequently, the angle of incidence on the outer surface is smaller than that on the inner
by the amount \( \Delta n \) and the refracted angle \( \beta \) is also smaller. The net effect is that the OAR refracts away from the z-axis of a dome at some angle greater than the look angle.

2.4.4 Asymmetric Pupil

This deviation has an interesting effect on the mapping of the entrance pupil. To illustrate this effect, consider an object at infinity and a look angle of \(-\theta\). The incoming ray bundle will consist of a series of rays parallel to the OAR in object space. The upper and lowermost rays of the bundle are the marginal rays. The upper marginal ray strikes the dome at \( y_{m^+} > y_{OAR} \) and passes through the upper edge of the aperture stop. Its deviation may be larger or smaller than that of the OAR. The lower marginal ray intercepts the dome at \( y_{m^-} < y_{OAR} \), passing through the lower edge of the aperture stop. The amount of its deviation may also be either larger or smaller than that of the OAR and different than that of the upper marginal ray. As a result, the transmitted ray bundle is neither parallel to nor symmetric about the OAR in image space. This difference in refraction between the three rays implies an asymmetric entrance pupil. This effect is illustrated in Figure 2-9.
Figure 2-9 Asymmetry of the entrance pupil resulting from LSD differences between the upper and lower marginal rays and the OAR. In this figure the pupil must be expanded so that the marginal rays intercept the edges of the stop.
2.4.5 Dome Optical Power

In the paraxial limit, for $\theta = 0$, the dome is represented as a convex meniscus lens. The radius of curvature of both the inner and outer surfaces will depend on the diameter, fineness ratio, and thickness of the dome. These are calculated using Eqns. 2-7 and 2-8. Further, the optical power of a pair of surfaces is derived by tracing a paraxial marginal ray through the dome. The result is the following [Smith 1992]

$$\Phi_{ab} = \phi_a + \phi_b - \tau \phi_a \phi_b$$  \hspace{1cm} (2-19)

where

$$\phi_j = (n_{j+1} - n_j)c_j = \frac{(n_{j+1} - n_j)}{r_j}$$  \hspace{1cm} (2-20)

is the optical power of surface $j$, and

$$\tau = \frac{t}{n_{j+1}}$$  \hspace{1cm} (2-21)

is the reduced thickness of the element. For a lens in air with an index of refraction $n$, $\phi_a= (n-1)/r_a$ and $\phi_b = (1-n)/r_b$. The effective focal length (EFL) of the lens is $1/\Phi_{ab}$.

For small values of $F$, $n$, and $t$, a dome can be likened to a weakly powered negative meniscus lens. However, for sufficiently large values of any one of the three parameters, the dome transitions from weakly negative to afocal to strongly positive (see Figures 2-11 to 2-14). For most applications, the desired dome is afocal or has low optical power because this minimizes the change in focal length between the boresight and maximum look angle positions, thereby easing the requirements for image correction.
placed on the imaging system. This requirement has the effect of bounding the parameter space for more easily achieved optical correction.

2.4.6 Cardinal Points and Planes

The EFL of the dome is the distance from the second principal plane at P' to the infinity focus at F', the location of which may be derived from a paraxial marginal ray from an object at infinity. The sign convention used here is that the ray is traced from left to right and the EFL > 0 if F' is located to the right of P'. Similarly, the front focal point F of the dome is obtained by tracing a paraxial marginal ray from infinity backward through the dome. For a lens in air, the EFL = P'F' = PF. The nodal points N and N' of a lens in air are coincident with P, P' respectively. The back focal length (BFL) is the distance from the vertex V' of the inner surface to F'. The separation between the points V'P' = δ = EFL - BFL = \( \phi_a \sigma \Phi_{ab} \) [Grievenkamp 1992]. Likewise, the front focal length (FFL) of the dome can be obtained and related to the vertex of the outer surface in a similar fashion, the result being VP = \( \delta = -(EFL) - BFL = \phi_b \sigma \Phi_{ab} \). The separation of the principal planes is PP' = t - δ + δ. These relationships and appropriate sign conventions are illustrated in Figure 2-10.

The principal planes of a convex meniscus lens are located on one side of the lens. If the focal length is negative, then the points P and P' are to the right of the lens. Conversely, P and P' are located to the left of a positive meniscus.

The symmetric shape factor of a lens describes its bending. That is, the curvatures of the front and rear surfaces may change while maintaining the same overall focal
length. The shape factor is primarily of interest from the perspective of aberration correction, as the aberration content of the transmitted wavefront can be altered and minimized by the proper distribution of power between the two lens surfaces. Lower values generally imply less aberration. It is plotted in Figure 2-15.
Figure 2-10 Cardinal points and planes for the dome at $\theta = 0$. Positive distances are measured from left to right.
Figure 2-11 Optical power of the dome as a function of dome thickness and fineness ratio for \( n = 1.5 \).
Figure 2-12 Optical power of the dome as a function of dome thickness and fineness ratio for $n = 2.0$. 
Figure 2-13 Optical power of the dome as a function of dome thickness and fineness ratio for $n = 3.0$. 
Figure 2-14 Optical power of the dome as a function of dome thickness and fineness ratio for $n = 4.0$. 

Dome Optical Power, $n = 4.0$

Fineness Ratio

- $0.5$
- $1$
- $1.5$
- $2$
- $2.5$
- $3$

$t$

$1/EFL$
Figure 2-15 Symmetric shape factor, $X$, of the dome as a function of thickness and fineness ratio. $X = (c_1 + c_2)/(c_1 - c_2)$. 
2.5 Concluding Remarks

This chapter investigated many of the important geometric aspects of an ellipsoidal dome. First, expressions for the conic constant and the base radius of curvature of an ellipse were derived in terms of the fineness ratio, thickness, and outer surface diameter. Further, the angle between the normal to a surface at the intersection of the surface and the optical axis ray and the major axis of an ellipse was derived. It was shown to be proportional to the square of the fineness ratio, $F$. Using this information, the line of sight deviation of a dome was shown to always be greater than or equal to the look angle. The variation of the angle of incidence of the OAR on a dome surface was derived as well. The last element of the geometric analysis was to calculate the radius of curvature of an ellipsoidal surface in both the sagittal and tangential planes in terms of the fineness ratio and look angle.

Next, the first-order characteristics of a dome, including the locations of the principal points and planes, were derived. Only the boresight position was investigated, since paraxial calculations assume a spherical surface and ignore the effects of tilts and decenters. At boresight, a dome is analogous to a concave meniscus lens. It was shown that for modest values of $F$, $t$, and $n$, a dome is a weak negative element. As $F$ and $n$ become large (i.e. $>2$), the effective focal length increases, making the transition from weakly negative to strongly positive.

Using this foundation we will now proceed to the analysis of the aberration function of a dome. In the following chapter the effect that varying the look angle has on
the quality of the transmitted wavefront will be discussed. For reference, the ideal image is assumed to lie at the paraxial sagittal focus. This convention has been chosen because a dome exhibits bilateral symmetry in the sagittal plane containing the optical axis ray. In this plane, the first-order imaging characteristics of a dome are analogous to those of a conventional convex meniscus lens (conic surfaces are assumed spherical in the paraxial region). As a result, the upper and lower paraxial marginal rays in the sagittal plane share a common focus where they intercept the optical axis ray. This is true even though the sagittal curvature varies as a function of look angle because it does not vary across the paraxial aperture. Thus, although the paraxial image location shifts with look angle, it is uniquely determined by first-order calculations that take into account the changing curvature.

A slice through a dome in the tangential plane does not exhibit the same symmetry about the optical axis ray, however. The curvatures seen by the upper and lower marginal rays are different so the rays do not always intercept the axis ray at a common point. Because of this, the paraxial image appears aberrated and the resulting first-order calculations are ambiguous.
Chapter 3: Third-Order Aberrations

3.1 Introduction

In Chapter 2 some important first-order imaging characteristics of a dome with its gimbal at the boresight position were discussed. The first-order treatment allows us to locate the position and determine the size of an image but does not describe the image quality. In this chapter we will address the problem of image quality at the boresight position, and perhaps more importantly, as a function of look angle by examining the behavior of three of the third-order aberrations: spherical, coma, and astigmatism. Distortion is related to line of sight deviation, but is not discussed in detail because the emphasis of this chapter is primarily on those aberrations that degrade the quality of an image rather than affect its position.

This chapter develops a theory that describes the behavior of the aberration function of an ellipsoidal dome that is based on both its symmetry and geometry. Its development follows a stepwise approach that illustrates the need for an increasingly more complex function as it is adapted to the geometry of a dome and gimbal system.

This process begins in Section 3.2.1 with a review of elements of the standard wavefront aberration polynomial describing the aberrations of an axially symmetric optical system. The polynomial coefficients of an axially symmetric system may be computed using Seidel sums, which are reviewed in Section 3.2.3. This section is useful in part because it introduces the use of Seidel sums to compute aberration coefficients
and also because it illustrates how large the coefficients become for highly aspheric surfaces.

The aberration function of a dome is not, however, adequately described by the basic theory because it does not address the plane symmetry of a dome when the look angle is rotated away from the boresight position. An extension of the axially symmetric aberration function to plane symmetric optical systems follows. This development does not account for the variation of the aberration coefficients as a function of the look angle.

The geometry of an ellipsoidal dome imposes a number of symmetry conditions on the form of the aberration function and its variation as a function of look angle. These conditions are investigated for the particular case of a dome with the stop located at the common geometric origin of its two surfaces in Section 3.2.5. This section does not yet introduce specific aberration terms. This final step is incorporated into Section 3.3, with the emphasis being on the aberration content of the on-axis field of view of the imaging system. Section 3.4 expands on this by including plots that show the magnitude of various aberration terms over a finite field of view for a number of look angles.

As was alluded to in the previous chapter, the first-order properties of the dome have a different interpretation as the gimbal is rotated away from the boresight position. The problem of locating an ideal image plane (or surface) is still relevant to the discussion of image quality because a reference from which to determine wavefront degradation and optical path difference (OPD) still must be established. For any given look angle, the dome exhibits bilateral symmetry in the sagittal plane. To take advantage of this symmetry, all aberration calculations refer to departures from an ideal image point
located at the paraxial focus in the sagittal plane. Because the sagittal radius of curvature changes as a function of look angle, the paraxial focus position varies as well. This focus error is removed by allowing the image plane location to float. This is important for the discussions to follow.

3.2 Optical Aberrations

3.2.1 Wavefront Aberrations

A perfect imaging system forms a spherical wavefront at the exit pupil that is concentric with the paraxial image point. Ignoring diffraction, the ideal wavefront converges to a point. This is known as stigmatic or point-to-point imaging. In the exit pupil, the wavefront forms a segment of the Gaussian reference sphere [Wolf 1980].

Consider the diagram in Figure 3-1, with a point \((x_o,y_o,z_o)\) in a media of index \(n\), located a distance \(s\) along the optical axis from a refracting surface. A ray emanating from the object point with some angle \(u\) to the optical axis strikes the surface and is refracted by the change in index to \(n'\) towards the point \((x_i,y_i,z_i)\) in the image plane. The refracted ray has an angle \(u'\) with respect to the optical axis, and the separation of the refracting surface and the image surface is \(s'\).

Departures of the actual transmitted wavefront from the reference sphere are the aberrations of the optical system. Aberrations measure the optical path difference (OPD), the change in optical path length between the chief ray and any other ray emitted from the same object point. The chief ray, or as it is sometimes called, the principal ray, is the ray emitted from a point on the object that passes through the center of the aperture stop.
Figure 3-1 Stigmatic imaging. The reference sphere is converging to the point \((x_i, y_i)\). The chief ray is at an angle \(u\) and \(u'\) with respect the axis of symmetry (z-axis) in object and image space, respectively. Both \(u\) and \(u'\) are positive, as drawn.
Since the entrance and exit pupils of the optical system are conjugate images of the aperture stop in object and image space respectively, the chief ray necessarily passes through the center of the pupils as well. The chief ray establishes the optical path length from object to image so the OPD of any chief ray is by definition zero.

The reference sphere for any point \((x_o, y_o, z_o)\) of the object not on the optical axis is itself inclined with respect to the optical axis. It is centered on the chief ray concentric to a point \((x_i, y_i, z_i)\) in the paraxial image plane separated by some transverse distance from the optical axis. The \(z\) positions, measured from the principal planes, are given by the familiar \( \frac{1}{f} = \frac{1}{z_i} - \frac{1}{z_o} \) where \(f\) is the effective focal length of the optical system and \(z_i\) and \(z_o\) are the image and object distances, respectively. For small angles and an object at infinity, \((x_i, y_i) = (f \cdot \tan(u_x), f \cdot \tan(u_y))\), where \(u_x\), \(u_y\) are the \(x\) and \(y\) components of the field angle between the chief ray and the optical axis. For small angles the paraxial approximation \(\tan(u) = u\) may be used. At finite object distances, the image height is found by applying the optical invariant, \(h n u = h' n' u' = k\), a constant.

In the exit pupil, the OPD is represented by a polynomial expansion \(W(H, \rho)\) about the field and pupil coordinates, respectively [Shack 1992]. \(W\) is scalar and reflects the symmetries of the optical system. For example, a system that is rotationally symmetric in both the field and the pupil will have an expansion that is also rotationally symmetric:

\[
W(\vec{H}, \vec{\rho}) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} W_{2k+m+2n+k}(\vec{H} \cdot \vec{H})^k (\vec{\rho} \cdot \vec{\rho})^m (\vec{H} \cdot \vec{\rho})^n
\]  (3-1)
Note that the expansion does not depend specifically on the absolute orientation of the field and pupil vectors but rather on their magnitudes and the dot product $H \cdot \rho$.

Thus, $W(H,\rho) = W(H,\rho,\phi)$, with $H$ and $\rho$ being the scalar field and aperture components and $\phi$ the angle between them. The coefficient $W_{2k+n, 2m+n, n}$ gives the maximum departure from the reference sphere for a particular aberration. It is typically measured in waves: the physical departure of the wavefront from the reference sphere divided by the wavelength of interest. The subscripts $2k+n, 2m+n,$ and $n$ are associated with the powers of $H, \rho,$ and $\cos \phi$, respectively.

$W(H,\rho)$ is typically ordered by even powers of the field and aperture components. In other words, the sums $(2k+n) + (2m+n) = 2, 4, 6,$ etc. make up the second-, fourth-, and sixth-order wavefront deformations.

It is not uncommon to develop the wavefront expansion in a Cartesian reference frame when the optical system exhibits plane symmetry, as the dome does for nonzero look angles. However, because a circular pupil and field are assumed, the notation listed in the previous paragraphs will be retained. The aberration terms added by the plane symmetry will simply be added to the expansion in Eqn. 3-1.

### 3.2.2 Transverse Ray Aberrations

Historically, the OPD is not measured directly but is instead calculated by tracing select rays through the optical system [Smith 1992]. An approximation to the transmitted wavefront of a surface is derived by expanding $\sin(\theta)$ in a Taylor series,
\[
\sin(I) = I - \frac{I^3}{3!} + \frac{I^5}{5!} - \frac{I^7}{7!} + \ldots
\]  

(3-2)

where \( I \) is the angle of incidence of a ray on the surface. First-order elements are those terms such as tilt, focus, and magnification that are linear in \( I \). The third-order aberrations: spherical, coma, astigmatism, field curvature, and distortion are associated with the cubic terms in \( I \). The aberration terms are additive, so that the total aberration may be approximated by summing contributions across all the surfaces in an optical system. The third-order terms are the first to affect image quality rather than location. They tend to have a larger effect on image quality compared to fifth and higher orders.

The image formed by an aberrated wavefront is generated by tracing rays from points across the exit pupil, normal to the wavefront, to the image plane. The ray distribution is called a spot diagram. The distribution of rays in the diagram is calculated by calculating the divergence of \( W \) with respect to \( H \) and \( \rho \), yielding the transverse ray aberration function \( \nabla W \).

In other words, there exists an exact correlation between the ray trace and wavefront error. Where a particular ray pierces the image plane relative to the chief ray is its ray intercept. It is assumed, from paraxial calculations, the intersection of chief ray with the image plane locates the ideal image. Therefore, the difference \((e_x, e_y)\) at the image plane between the chief ray and any other ray emanating from the same object point is the result of some defect in the imaging characteristics of an optical system. The transverse ray aberrations may be computed using Eqns. 3-3a, b.
Transverse ray aberrations are mapped with ray intercept curves and plotted, in
general, as a function of field and pupil in the following fashion. Each field point has two
plots associated with it. The first plots the transverse component in the tangential plane,
with $\rho$ (normalized to the pupil radius) varying from -1 to 1. It is generated by tracing a
fan of rays from a single object point across the full dimension of the pupil in the
tangential plane. The second plot is generated in a similar fashion except the ray fan is
traced in the sagittal plane.

A great deal of information can be extracted from the ray intercept plots. For example, the shape of the curve is an indication of the type of aberration generating it.
Spherical aberration has a curve that is cubic in $\rho$ and constant across field. A difference
in slopes of the sagittal and tangential fans at $\rho=0$ shows astigmatism. Coma has a curve
that goes as $\rho^2$ and increases linearly with field. Spot size is given by the peak-to-valley
of the curve. The slope of the curve is an indication of ray density. A shallow slope is an
indication that many rays are clustered about a single point. Conversely, a steep slope
indicates that a small change in the pupil results in a large transverse ray error and a less
dense spot distribution at the image.
Figure 3-2 $W(H, \rho, \phi)$, the optical path difference (OPD), is the difference between the actual and reference wavefronts. Transverse ray aberrations are the displacement of rays away from the paraxial image point in the paraxial image plane (PIP). The reference sphere is concentric to the paraxial image point and perpendicular to the OAR.
3.2.3 Seidel Sums

The third (fourth) order contributions of the dome at the boresight position can be computed directly, using Seidel sums, as follows [Shack 1992]:

<table>
<thead>
<tr>
<th>Aberration</th>
<th>Coefficient</th>
<th>Seidel Sum, $S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical aberration</td>
<td>$W_{040} = -(1/8)S_1$</td>
<td>$\Sigma(A^2y\Delta(u/n) + \delta_k)$</td>
</tr>
<tr>
<td>Coma</td>
<td>$W_{131} = -(1/2)S_{II}$</td>
<td>$\Sigma(AA_by\Delta(u/n) + \delta_k)$</td>
</tr>
<tr>
<td>Astigmatism</td>
<td>$W_{222} = -(1/2)S_{III}$</td>
<td>$\Sigma(A_b^2y\Delta(u/n) + \delta_k)$</td>
</tr>
<tr>
<td>Petzval curvature</td>
<td>$W_{220} = -(1/4)S_{IV}$</td>
<td>$\Sigma[(A_b^2y\Delta(u/n) + K^2c\Delta(1/n)) + \delta_k]$</td>
</tr>
<tr>
<td>Distortion</td>
<td>$W_{311} = -(1/2)S_V$</td>
<td>$\Sigma([(A/A_b)^2[ K^2c\Delta(1/n)+A_b^2y\Delta(u/n)] + \delta_k)$</td>
</tr>
</tbody>
</table>

Table 3-1 Seidel sums for the third-order aberrations of an axially symmetric system.

where $y$ is the ray height, $i$ is the angle of incidence, $A = ni = n'i'$, $u$ is the angle of the ray with respect to the optical axis, $c$ is the curvature of the surface, and $K$ is the optical invariant. The sum is taken over all surfaces. The subscript $b$ refers to quantities relating to the chief ray. Primed quantities are those after refraction. The difference operator, $\Delta(x) = x' - x$, is the change in $x$ resulting from refraction. The addition of an asphere (conic) introduces additional contributions, $\delta_k$, to each surface. These are listed in Table 3-2.

Most significant is the contribution to spherical aberration that results from the addition of a conic. For small $y$, the contribution is small. However, as $y$ gets large, its contribution, relative to that of the base sphere, i.e. $\delta_kS_{n}/S_{n}$, grows as the cube of the aperture. Coma is slightly less affected with the contribution growing as the square of the
aperture. Astigmatism and distortion show a linear and constant dependence on aperture, respectively.

<table>
<thead>
<tr>
<th>Aberration</th>
<th>Description</th>
<th>Aspheric Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>$\delta_k S_I$</td>
<td>$ky^4 c^3 \Delta(n)$</td>
</tr>
<tr>
<td>Coma</td>
<td>$\delta_k S_{II}$</td>
<td>$(y/y_k)ky^4 c^3 \Delta(n)$</td>
</tr>
<tr>
<td>Astigmatism</td>
<td>$\delta_k S_{III}$</td>
<td>$(y/y_k)^2 ky^4 c^3 \Delta(n)$</td>
</tr>
<tr>
<td>Field curvature</td>
<td>$\delta_k S_{IV}$</td>
<td>0</td>
</tr>
<tr>
<td>Distortion</td>
<td>$\delta_k S_V$</td>
<td>$(y/y_k)^3 ky^4 c^3 \Delta(n)$</td>
</tr>
</tbody>
</table>

Table 3-2 Aspheric surface contributions to the Seidel sums.

3.2.4 Aberrations of a Plane Symmetric Optical System

Although the Seidel sums provide a useful tool for evaluating the dome at $\theta = 0$, it is the variation of look angle that makes the analysis of the aberration function of the ellipsoidal dome a difficult and interesting task. On boresight, the optical system has axial symmetry. For nonzero look angles, however, the axial symmetry is broken. Instead, it is replaced with bilateral symmetry in both the pupil and field. Figure 3-3 illustrates the symmetries imposed by the dome. The symmetry axis is in the plane perpendicular to the rotation axis of the optical system.

A number of forms of the aberration function for a plane symmetric system have been derived [Sands 1972]. Tatian has shown that the complexity of the aberration function grows from 13 to 73 when plane symmetry is invoked [Tatian 1985]. Tatian also suggests a split form for the aberration function that is the sum of Zernike polynomials (axial) and Chebychev polynomials for both the sagittal and tangential components of the
imaging system. Sasian has suggested a particular ordering scheme that follows from the standard axially symmetric aberration function [Sasian 1988]. The primary difference between these prior investigations and the current analysis is the variation of the coefficients themselves. By examining the symmetry conditions, it is possible to rule out certain functional forms.

The geometry of the dome requires bilateral symmetry about the tangential plane. Following the notation of Sasian, the wavefront expansion of a plane symmetric system becomes

$$W(\vec{H}, \vec{\rho}) = \sum_{(k,m,n,p,q)} W_{2k+n+p,2m+n+q,n,p,q} (\vec{H} \cdot \vec{H})^k (\vec{\rho} \cdot \vec{\rho})^m (\vec{H} \cdot \vec{\rho})^n (\vec{i} \cdot \vec{H})^p (\vec{i} \cdot \vec{\rho})$$

(3-4)

Table 3-3 lists the third-order aberration terms generated by the expansion. The first nine terms in the table are characteristic of plane symmetric systems. These will become significant for nonzero look angles. The last five terms in the table are the traditional third-order aberrations for an axially symmetric system. They are obtained by setting the subscripts $p$ and $q$ equal to zero.

On boresight, the expansion reduces to the typical rotationally symmetric aberration function, as it should be. It contains various orders of spherical aberration, coma, field curvature, astigmatism, and distortion. Any expansion about the look angle must converge to this form for look angles of zero and 180 degrees. For nonzero look angles, bilateral symmetry requires that the function be even in both field and pupil coordinates about the symmetry axis. That is, $F(\theta, \alpha, \beta) = F(\theta, -\alpha, -\beta)$, where $\theta$ is once
again the look angle and $\alpha$ and $\beta$ are the rotations of the aperture and field vectors from the symmetry axis. The symmetry in look angle requires the addition of $\pi$ to $\alpha$ and $\beta$ with $F(-\theta, \pi-\alpha, \pi-\beta) = F(\theta, \alpha+\pi, \beta+\pi)$. The coordinate system used to define these symmetry conditions is defined such that the symmetry vector is always pointed up with respect to the pupil of the imaging system.
<table>
<thead>
<tr>
<th>Vector Form</th>
<th>Scalar Form</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{02002}(i-p)^2$</td>
<td>$W_{02002} \rho^2 \cos^2(\beta)$</td>
<td>Constant astigmatism</td>
</tr>
<tr>
<td>$W_{11011}(i-H)(i-p)$</td>
<td>$W_{11011} \rho \cos(\alpha) \cos(\beta)$</td>
<td>Anamorphism</td>
</tr>
<tr>
<td>$W_{20020}(i-H)^2$</td>
<td>$W_{20020} \rho^2 \cos^2(\alpha)$</td>
<td>Quadratic piston</td>
</tr>
<tr>
<td>$W_{03001}(i-p)(p-p)$</td>
<td>$W_{03001} \rho^3 \cos(\beta)$</td>
<td>Constant coma</td>
</tr>
<tr>
<td>$W_{12101}(i-p)(H-p)$</td>
<td>$W_{12101} \rho \cos(\phi) \cos(\beta)$</td>
<td>Linear astigmatism</td>
</tr>
<tr>
<td>$W_{12010}(i-H)(p-p)$</td>
<td>$W_{12010} \rho \cos(\beta)$</td>
<td>Field tilt</td>
</tr>
<tr>
<td>$W_{21001}(i-p)(H-H)$</td>
<td>$W_{21001} \rho \cos(\beta)$</td>
<td>Quadratic distortion I</td>
</tr>
<tr>
<td>$W_{21110}(i-H)(H-p)$</td>
<td>$W_{21110} \rho \cos(\phi) \cos(\alpha)$</td>
<td>Quadratic distortion II</td>
</tr>
<tr>
<td>$W_{30010}(i-H)(H-H)$</td>
<td>$W_{30010} \rho \cos(\alpha)$</td>
<td>Cubic piston</td>
</tr>
<tr>
<td>$W_{04000}(p-p)^2$</td>
<td>$W_{04000} \rho^4$</td>
<td>Spherical aberration</td>
</tr>
<tr>
<td>$W_{13100}(H-p)(p-p)$</td>
<td>$W_{13100} \rho \cos(\phi)$</td>
<td>Linear coma</td>
</tr>
<tr>
<td>$W_{22200}(H-p)^2$</td>
<td>$W_{22200} \rho \cos^3(\phi)$</td>
<td>Quadratic astigmatism</td>
</tr>
<tr>
<td>$W_{22000}(H-H)(p-p)$</td>
<td>$W_{22000} \rho^2$</td>
<td>Field curvature</td>
</tr>
<tr>
<td>$W_{31100}(H-H)(H-p)$</td>
<td>$W_{31100} \rho \cos(\phi)$</td>
<td>Cubic distortion</td>
</tr>
<tr>
<td>$W_{40000}(H-H)^2$</td>
<td>$W_{40000} \rho^4$</td>
<td>Quartic piston</td>
</tr>
</tbody>
</table>

Table 3-3 Aberrations of a plane symmetric system [Sasian 1993].
Figure 3-3 Symmetries of the dome. The exit pupil is projected onto the surface of the dome for plus and minus look angles. Symmetry requires that the aberration function be the same at the four positions shown. The angles of the field and pupil vectors are measured from the symmetry vector, i, which always point upwards.
3.2.5 Aberration Function of an Ellipse

The symmetry conditions with regard to the look angle are especially important. Recall that the look angle is defined relative to the missile body axis. Thus, shape of the transmitted wavefront for a positive look angle is the mirror image of that for a negative look angle. This results from the addition of both the even and the odd terms that comprise the aberration function. Summing all of the component terms describes the total aberration function, $F(\theta)$, of an ellipse. The behavior of the total aberration, or magnitude (such as peak-to-valley deformation or RMS wavefront error), of $F$ as a function of look angle is the topic of this section. Note that in the development which follows, the aperture stop is located at the common geometric origin of the two dome surfaces.

For further development, consider the ellipse to be a closed surface that extends through the full range of look angles from 0 to $2\pi$. Because much of what follows concerns the behavior of both the ellipse and the aberration function in angle space, it is useful to make a transformation to polar coordinates. In this reference frame, the ellipse is periodic, continuous, and even, with

$$R_E(\theta) = R_E(\theta + m\pi) = R_E(-\theta) = R_E(-\theta + m\pi)$$

(3-5)

$m$ an integer and $R_E$ the radial distance of the surface (curve) from the origin. The polar equation for an ellipse is derived by computing $y(\theta)$ and $z(\theta)$ and reducing the result $R_E(\theta) = (y^2 + z^2)^{1/2}$, which yields

$$R_E(\theta) = \frac{FD\sqrt{\tan^2(\theta) + 1}}{\sqrt{4F^2\tan^2(\theta) + 1}}$$

(3-6)
The aberration function should also exhibit similar behavior. It will, in general, have a more complicated form because of the complex relationship between the look angle and the angle of incidence, which depends on the fineness ratio, thickness, and index of refraction of the dome. As was indicated with the Seidel sums, the aberrations of a system are largely dependent on the angles of incidence. For an ellipse, the fact that the angle of incidence is zero at integer multiples of $\pi/2$ requires that the aberration function pass through a local minimum (or maximum) at these positions as well.

Like the ellipse, the aberration function exhibits four-fold symmetry about the major and minor axes. It also has the same periodicity as $R_E(\theta)$, with $F(\theta) = F(\theta + m\pi)$, $m$ an integer. This is confirmed by rotating the ellipse 180 degrees about the gimbal axis. From the perspective of the aperture stop the dome is unchanged.

Figure 3-4 and Figure 3-5 illustrate the periodic nature of both the ellipse and the magnitude of the aberration function. Figure 3-4 is a polar plot showing an ellipse with a fineness ratio of 1, thickness of 2 mm, and an index of refraction of 1.5. Superimposed over the ellipse is the peak-to-valley aberration function, $F$, in waves at a wavelength of 1.0 micron. Note that the maximum magnitude of $F$ is located in the region of the maximum angle of incidence, which, from Eqn. 2-18, occurs at 26.6 degrees. Figure 3-5 illustrates the same information on a linear plot of $F$ and $R_E$ as a function of look angle. The aberration function has the same period as $R_E$ and a more complicated shape.
Figure 3-4 Polar plot of an ellipse and the magnitude of the aberration function (either P-V or RMS error) it generates (in waves at 1.0 microns). Read the plot by tracing a line radially outward from the origin. The intersection of the line with the ellipse gives its radial dimension. The intersection of the line with the aberration curve gives the magnitude of the aberration for that particular look angle. This figure assumes the stop is located at the geometric origin of the dome surfaces.
Figure 3-5 Linear plot of Y, Z, R for the outer surface of a dome with F= 1, t = 2 mm, and n = 1.5 having an aberration function, F. The aberration function has the same period as R. The aperture stop is at the geometric origin of the dome surfaces.
In summary, the periodicity of the ellipse with look angle suggests an aberration function whose coefficients that depend on the sine and cosine of the look angle. In fact, any periodic function may be exactly represented as an expansion of the sums of sine and cosine terms. This is the fundamental theorem of Fourier analysis. To continue, we can write the function

\[ F(\theta) = A + \sum_{m} B_m \cos(m\theta) + \sum_{n} C_n \sin(n\theta) \]  

(3-7)

where \( F \) is the aberration function, \( A, B, \) and \( C \) are coefficient multipliers for the respective terms, and the summations are made from 0 to infinity. Because \( F \) has a period length of \( \pi \), the odd multipliers of both the sine and cosine terms vanish, and the even terms are retained. For the cosine terms, this is shown by substituting \( \theta + \pi \) in place of \( \theta \) and using the identity

\[ \cos(u + v) = \cos u \cos v - \sin u \sin v \]  

(3-8)

with

\[ \cos(m\theta + m\pi) = \cos(m\theta)\cos(m\pi) - \sin(m\theta)\sin(m\pi) = \cos(m\theta), \text{ m even.} \]

Likewise, the sine terms are reduced in a similar fashion with the identity

\[ \sin(u + v) = \sin u \cos v - \cos u \sin v \]  

(3-9)

and making the appropriate substitutions for \( u = m\theta \) and \( v = m\pi \).

Extending the summation to infinity requires an impractical number of coefficients to compute. This number may be reduced significantly by taking advantage of Fourier analysis. The Fourier transform, \( \mathcal{F}(F) \), yields its frequency components and their relative magnitudes. Figure 3-6a shows the real and imaginary parts of the Fourier
transform of the aberration function plotted in Figures 3-4 and 3-5. Of note are a number of important features. The first is that the imaginary parts are all identically zero. This should be expected because $F(\theta)$ is real and even. Second, the odd frequency components, $(2n+1)/\pi$, $n = 0, 1, 2, \ldots$ are all zero as well, confirming the assertions of the previous paragraphs. The last point of note is illustrated in Figure 3-6b and 3-6c. These show the function $F(\theta)$ on the left, and on the right, the computed value of $F(\theta)$ using coefficients obtained from the real part of $\Im \{F\}$ and truncating the series after seven terms. The implication of this is that the aberration function generated by the ellipse truncates after only a small number of orders. Therefore, the aberrations introduced by the ellipse may be completely corrected by aspheric elements of similar order and sufficient number of degrees of freedom.

It is evident from the previous analysis that the aberration function can be decomposed into a simple sum of a constant and a finite number of cosine terms. That there are no sine terms is a consequence of its definition. Recall that $F$ is the total aberration function, a single number describing the peak-to-valley wavefront error or RMS wavefront error. It is necessarily real and even. Its constituents, that is, the aberrations that describe the error in the transmitted wavefront, may be odd or even functions of the look angle. In other words, $F(\theta)$ can be more generally split into even and odd functions, with

\[
F(\theta) = F(\theta)_{\text{even}} + F(\theta)_{\text{odd}} \tag{3-10a}
\]

\[
F(\theta)_{\text{even}} = A + \Sigma(B_{2m}\cos(2m\theta)), \ m = 1, 2, 3 \ldots \tag{3-10b}
\]

\[
F(\theta)_{\text{odd}} = \Sigma(C_{2m}\sin(2m\theta)), \ m = 1, 2, 3\ldots \tag{3-10c}
\]
In fact, it can be argued that for reasons of symmetry some terms must be odd. The transmitted wavefront is a mirror image of itself for positive and negative look angles. The magnitude of the error is unchanged for $F(\theta)$ and $F(-\theta)$. Rather, the wavefront is effectively rotated 180 degrees about the optical axis ray in both the field and the pupil. Such a rotation would be impossible to achieve without the inclusion of odd aberrations.

The terms which comprise $F(\theta)_{\text{even}}$ are those which contain even powers of the field vector $H$ (ref. Table 3-3) including field curvature, astigmatism, and spherical aberration. $F(\theta)_{\text{odd}}$ is likewise comprised of odd powers of $H$, including coma, tilt (line of sight deviation), and distortion.

Regardless of whether or not the coefficients (i.e. $W_{klmpq}$) are odd or even, their magnitudes all must obey the same symmetry requirements and periodicity imposed on the total aberration function. That is,

$$|W_{klmpq}(\theta+m\pi)| = |W_{klmpq}(m\pi-\theta)|,$$

$m$ an integer \hspace{1cm} (3-11)

As a result, the analysis results from this section may be applied to any coefficient with the understanding that some must necessarily be odd functions.
Figure 3-6 a) The real and imaginary parts of the Fourier transform of the aberration function $F$. The frequency components are integer multiples of $1/\pi$. Figures b) and c) are the function $F$, and its reconstruction of $F$ using a cosine series expansion truncated at the seventh real term, respectively.
3.3 Variation of Third-Order Astigmatism, Coma, and Spherical Aberration

Although Section 3.2 developed a number of important concepts regarding the form of the aberration function, it did little to identify which aberration terms are present or to develop expressions for the coefficients themselves in terms of the parameters that describe the dome: fineness ratio, index of refraction, and thickness. The difficulty in this development lies in the variability of the aberration coefficients as a function of look angle and in the complexity dome surface seen by the pupil.

Identifying which aberration terms are present (or dominant since the total aberration function cannot be completely described with a small number of terms) is accomplished by examining the transverse ray fan plots for a particular dome. These are shown in Figure 3-7 for a number of look angles. The dominant aberration terms, in order of significance, are third-order astigmatism, coma, and spherical.

Relating the aberration to the geometry of the dome surfaces is one method of approximating the variation of the coefficients for these aberration terms as a function of look angle. Following the work of [Nelson 1982] on the third-order aberrations of conic mirrors with segments having decentered apertures, the surfaces of the dome can be described in a coordinate frame that is oriented such that the new z-axis is aligned with the surface normals. Aberrations introduced by the surface are assumed to be proportional to the aspheric shape of the surface section [Mahajan 1991]. Expressions for astigmatism and coma are developed in Sections 3.3.2 and 3.3.3 respectively. No explicit
calculation is made for spherical aberration, but the effect it has on the design of an imaging system is discussed in Section 3.3.4.

The coordinate transformations applied by the authors are rather involved, so only the results are presented. The expressions for the coefficients are not exact. This is because of a difference in application (i.e. refracting, instead of reflecting surfaces) and a difference in coordinate reference frames. The optical axis ray, rather than the surface normal defines the reference frame for this analysis. As has been shown, there can be a large angular difference between the two. This change affects the numerical accuracy of the model, but is not enough to hide the gross variations in the derived coefficients as a function of look angle. It is the latter which describes the true utility of the coefficient expressions that follow.
Figure 3-7 Transverse ray aberration plots for $\theta = 0$, 24, and 90 degrees of a dome with $F = 1$, $n = 1.5$, $t = 2$ mm. For $\theta = 0$ degrees, the dome introduces primarily spherical aberration. At $\theta = 24$ degrees, the tangential ray fan shows astigmatism and coma and the sagittal ray fan shows spherical. At $\theta = 90$ degrees, the tangential fan shows only astigmatism (coma is zero) and the sagittal fan shows spherical. Note that the plots reflect a repositioning of the image plane to the paraxial sagittal focus prior to tracing the ray fans.
3.3.1 Review of Relevant Terms

A list of terms used in the equations which follow is:

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>conic constant</td>
</tr>
<tr>
<td>( r_a )</td>
<td>radius of the aperture stop</td>
</tr>
<tr>
<td>R</td>
<td>paraxial radius of curvature</td>
</tr>
<tr>
<td>y</td>
<td>( y ) intercept of the OAR on the surface of the dome</td>
</tr>
<tr>
<td>n</td>
<td>index of refraction of the dome</td>
</tr>
<tr>
<td>F</td>
<td>fineness ratio of the outer surface of the dome</td>
</tr>
<tr>
<td>D</td>
<td>diameter of the outer surface of the dome</td>
</tr>
<tr>
<td>L</td>
<td>semi-major axis length of the outer surface of the dome</td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td>angle between the normal and the major axis of the dome</td>
</tr>
<tr>
<td>t</td>
<td>thickness</td>
</tr>
</tbody>
</table>

Table 3-4 Summary of terms used in aberration coefficient calculations.

The subscripts \( i \) and \( o \) will refer to the inner and outer surfaces, respectively. The following terms are common to the different aberration calculations. They have been derived earlier in the text, but are included here for reference.

The \( y \) intercept on the inner surface of the dome is

\[
y_i(\theta) = \frac{F_i D_i \tan(\theta)}{\sqrt{4 F_i^2 \tan(\theta)^2 + 1}} \tag{3-12}
\]

with

\[
F_i = \frac{(F_o D_o - t)}{(D_o - 2t)} \tag{3-13}
\]

\[
D_i = D_o - 2t \tag{3-14}
\]
The y-intercept of the OAR on the outer surface is approximated as

$$y_o(\theta) = y_i(\theta) - \frac{t \sin(\theta)}{\cos(in_i(\theta) - \theta)}$$

(3-16)

The cosine factor accounts for the difference in length between the physical thickness of the dome and the optical path followed by the OAR, and the minus sign accounts for the established sign convention. The OAR is traced outward from the stop to more accurately determine its intercepts on the surface without the added difficulty of correcting for the line of sight deviation. The normal to the inner surface has an angle $\text{in}_i$ with respect to the major axis of the ellipse, given by

$$\text{in}_i = \tan^{-1}(4F_i^2 \tan(\theta))$$

(3-17)

As with the inner surface, we have the following expressions for fineness ratio, conic constant, and base radius of curvature:

$$F_o = \frac{L}{D_o}$$

(3-18)

$$k_o = \frac{1}{4F_o^2} - 1$$

(3-19)

$$R_o = \frac{D_o}{4F_o}$$

(3-20)
3.3.2 Astigmatism

The amount of quadratic astigmatism, $W_{222}$, generated by the dome is proportional to the sum of that generated by the individual surfaces,

$$W_{222} = (n-1)a_{220} + (1-n)a_{221} = (n-1)(a_{220} - a_{221}) \quad (3-21)$$

where

$$a_{22} = \frac{r_a^2}{R} \frac{k(y/R)^2}{4(1-k(y/R)^2)^{3/2}} \quad (3-22)$$

With the appropriate substitutions, the analytic form of $W_{222}$ is calculated and plotted in Figure 3-9a. Beneath it is the variation of $W_{222}$ as determined by an optical ray trace program. Both plots are similar in both shape and magnitude to the total aberration function, indicating that astigmatism is the largest contributor to the total of the three third-order aberrations considered here. This is to be expected because the dome is increasingly cylindrical as the look angle is increased. Further, the sagittal paraxial image is always located further from the dome than the tangential focus. The tangential wavefront is converging more rapidly than the sagittal, so the OPD is positive, which is consistent with Figure 3-9b.

That the two curves differ slightly is likely the result of the difference between a paraxial calculation and a real ray trace. The oblique angle between the OAR and the surface normal at any given field angle also contributes to the error, since it was assumed in the analytic model that the coordinate reference frame was aligned with the local surface normal.
3.3.3 Coma

Like astigmatism the coefficient for coma is determined by adding the surface contributions,

\[ W_{311} = (n-1)(a_{3110} - a_{311}) \]  

(3-23)

where

\[ a_{31} = \frac{r_a^3}{R^2} \left( \frac{k}{R} \right)^3 \frac{4 - k \left( \frac{y}{R} \right)^2}{\sqrt{1 - (k + 1) \left( \frac{y}{R} \right)^2}} \left( 1 - k \left( \frac{y}{R} \right)^2 \right) \]  

(3-24)

and once again the appropriate substitutions are made. The result is plotted in Figure 3-10a, with the coma coefficient generated from the ray trace plotted below it in Figure 3-10b. As with the astigmatism calculation, the two curves show the same general form. Since coma is linear in the field, it is an odd function of the look angle and necessarily changes sign as it passes through the origin. However, the analytic calculation from Eqn. 3-24 is clearly in error, as it has a much higher maximum coefficient value.

Coma may be expanded into multiple terms for nonzero look angles. From Table 3-3 we have \( W_{\text{coma}} = \text{constant coma} + \text{linear coma} \), or,

\[ W_{\text{coma}} = W_{03001} \rho^3 \cos(\beta) + W_{13100} H \rho^3 \cos(\phi) \]  

(3-25)

The full field displays show that both terms exist. Figure 3-12 shows the boresight condition, where \( W_{03001} \) is zero. \( W_{13100} \) shows its linear dependence on field in this position, with the comatic flares pointed outward, away from the center of the field of view. However, as the look angle is increased, the sagittal component becomes
vanishingly small compared to the tangential component, which is nearly constant over the field of view.

3.3.4 Spherical Aberration

Although it is in general not a large contributor to the aberration function, spherical aberration is significant because of the way it affects correction. The curvature of a dome is largest on boresight. As a result, \( W_{040} \) is largest at or near the boresight position. Further, coma and astigmatism are zero for the on-axis field at the boresight position. Away from boresight, the dome flattens out so the magnitude of the coefficient is reduced and remains nearly constant over look angle. The difficulty this poses in the course of designing a real imaging system inside a dome is that the correctors and image-forming optics will necessarily introduce spherical aberration of the opposite sign to correct the system on boresight. Recall that the image-forming optics cannot correct the variable aberration component of the transmitted wavefront. They can correct spherical aberration. The consequence of correcting the spherical aberration on axis and on boresight is that the correctors must add to the spherical aberration of a dome at large look angles to compensate so that the wavefront entering the image-forming optics is independent of look angle.

The variation of \( W_{040} \) is shown Figure 3-11. There is an indication that the angle of incidence of the marginal ray is changing in a manner similar to that discussed in Chapter 2 with regard to the OAR. Recall from Chapter 2 that the angle of incidence of the OAR rapidly increases in magnitude as \( \theta \) departs from zero, passes through some maximum, and gradually declines to zero as \( \theta \) approaches 90 degrees.
Figure 3-8a) Ray trace of dome used for aberration calculations shown in the figures which follow. The dome has a fineness ratio $F_a = 1.0$, $t = 2.0$ mm, $n = 1.5$. The exit pupil is 25 mm, one half the dome diameter of 50 mm. A perfect lens (EFL = 50 mm) focuses the transmitted wavefront for analysis. b) $\theta = 50$ degrees, tangential (y-z) plane view. c) $\theta = 50$ degrees, sagittal (y-z) plane view. The image plane is located at the paraxial focus of the sagittal ray fan. The reference sphere is concentric about this point.
Astigmatism ($W_{222}$).
$F=1$, $z=2.0$ mm, $n=1$.

**Figure 3-9a** Analytic calculation of quadratic astigmatism as a function of look angle. **b**) Determination from an optical ray trace. $W_{222} = 0$ for $\theta = 0$. Three fields of view are plotted: 0, +10 mrad in x, +10 mrad in y. The coefficient changes due to the field of view are small compared to those induced by the look angle. It is the largest contributor to the aberration function.
Figure 3-10a) Analytic determination of coma. b) Optical ray trace calculation of linear coma. $W_{131} = 0$ for $\theta = 0$. Three fields of view are plotted: 0, +10 mrad in x, +10 mrad in y. The coefficient changes due to the field of view are small compared to those induced by the look angle. It is an odd function of the look angle, as required by the symmetry conditions.
Figure 3-11 Spherical aberration as a function of look angle. Three fields of view are plotted: 0, +10 mrad in x, +10 mrad in y. The coefficient changes due to the field of view are small compared to those induced by the look angle.
3.4 Full Field Displays

Figures 3-12 to 3-17 are included to illustrate two properties exhibited by the third-order aberrations. The first is to show the bilateral symmetry of the dome as the look angle is changed. The second effect is to show that the field aberrations are dominated by the rotation of the stop through the field of regard. The contributions from a finite field of view tend to be much smaller, so that the resulting aberrations appear to have a large, almost constant bias across the field of view of the optical system. The theory of aberration fields was proposed in an analysis of tilted and decentered optical systems [Buchroeder 1976]. It was formalized later by Thompson in his Ph.D. dissertation [Thompson 1980].

The figures show the magnitude of the third-order aberrations across a finite field of view. The field plotted is square with field angles of ±10 mrad in both the x and y directions. The size of a symbol is proportional to the magnitude of the aberration. For field-dependent aberrations, the symbols are oriented so that they are pointing towards the center of the aberration field. A scale bar to the left of each row is included and is the same for every plot in the row. The coefficients used are from the Fringe Zernike polynomial set. They are: $Z_5$, $Z_6$ (astigmatism), $Z_7$, $Z_8$ (coma), $Z_9$ (spherical + defocus + piston). For coma and astigmatism, the individual terms are plotted first and the sum plotted last in the row. Spherical aberration has no field dependence, so it is plotted only once. The reference sphere is concentric about the paraxial sagittal focus and changes as a function of look angle. The image plane is perpendicular to the OAR.
Figure 3-12 Full field display plots for a 0 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.
Figure 3-13 Full field display plots for a 5 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.
Figure 3-14 Full field display plots for a 10 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.
Figure 3-15 Full field display plots for a 15 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.
Figure 3-16 Full field display plots for a 30 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.
<table>
<thead>
<tr>
<th>Aberration Type</th>
<th>Tangential</th>
<th>Sagittal</th>
<th>Combined</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astigmatism</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
<td><img src="image3" alt="Graph" /></td>
</tr>
<tr>
<td>140 waves</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coma</td>
<td><img src="image4" alt="Graph" /></td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
<tr>
<td>4 waves</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spherical</td>
<td><img src="image7" alt="Graph" /></td>
<td><img src="image8" alt="Graph" /></td>
<td><img src="image9" alt="Graph" /></td>
</tr>
<tr>
<td>(Z9)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5 waves</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 3-17** Full field display plots for a 45 degree look angle. The size of the symbol in each plot is correlated with the magnitude of the aberration coefficient.

\[ \theta = -45 \text{ degrees} \]
3.5 Concluding Remarks

This chapter developed a number of tools for assessing the transmitted wavefront error of an ellipsoidal dome. It began with a review of general aberration theory for axially symmetric optical elements. A dome satisfies this condition only at the boresight position. The large departure from sphericity of the dome surfaces results in significant aspheric contributions that degrade the wavefront.

To describe the wavefront error introduced by a dome at nonzero look angles, the axially symmetric aberration function was expanded to include the aberrations of plane symmetric systems. While the expanded form is sufficient to determine the aberrations at a single, fixed position, it does not adequately address the dynamic variation of the coefficients as a function of look angle.

To better understand how the coefficients vary, the symmetry properties of the dome were imposed on both the aberration function and its constituent terms. It was shown that because of the periodic nature of the ellipse, the aberrations vary periodically as a function of look angle as well. The functional form for even aberrations is a sum of several cosine terms and a constant. Odd aberrations are similarly represented by a sum of sine terms, but without the constant. Only even multiples of the look angle satisfy the symmetry conditions.

With this knowledge, the aberration coefficients for third-order astigmatism and coma were derived. The basis of the derivation is a Taylor series expansion of a conic surface around a new coordinate system, defined with respect to the normal to the surface
at the point of interest [Nelson 1980]. The Taylor series also takes the form of a summation of sine and cosine terms that are analogous to aberration coefficients. The derived coefficients were compared to actual ray trace data and found to illustrate the form of the variation over look angle well, but did not accurately reproduce the magnitude of the real aberration coefficient.

The chapter concluded with illustrations of the aberration fields for astigmatism, coma, and spherical aberration at number of look angles. It was shown that the tangential components of astigmatism and coma are the dominant aberrations. The sagittal contribution from the field of view is, in general, small in comparison. Spherical aberration is not constant over look angle, but its contribution is smaller than that of either coma or astigmatism.

This chapter emphasized the variability of the aberrations over look angle. It did not explicitly address the effect that changing the fineness ratio, thickness, and index of refraction will have on the aberration terms. These are elements that must be included in the analysis as well because they are often the result of system-level trades that are external to the optical problem. For example, operational environments may dictate a thick dome for durability. In the following chapter, a parametric trade study is developed to examine the impact that fineness ratio, dome thickness, and index of refraction, have on wavefront quality over the full range of look angles.
Chapter 4: Calculating the Transmitted Wavefront

4.1 Introduction

In Chapters 2 and 3 a discussion of the first order characteristics and the behavior of the primary aberrations for a particular dome was presented. In the design of the dome are three parameters that significantly affect both the first order properties and the aberration content of the dome and transmitted wavefront. As has been stated previously, these are fineness ratio, thickness, and index of refraction. The intent of this chapter is to present a parametric study that, from a predetermined starting configuration, varies each of the three parameters independently to determine the impact of each on the aberration function.

To facilitate the analysis, the use of Zernike polynomials will be introduced in section 4.2. A small subset of these terms will be used to evaluate the transmitted wavefront as a function of both the look angle and the variable parameter. First, third, and fifth order contributions will be evaluated in this fashion. A brief discussion of the relationships between the Zernike coefficients, $Z_{lm}$, and the aberration coefficients, $W_{0lm}$, be presented as well [Wyant 1992].

Section 4.3 will contain a large number of plots and figures illustrating the effect the varying the fineness ratio, thickness, and index of refraction has on the aberration content of the transmitted wavefront.
4.2 Zernike Polynomials

The Zernike polynomials are an orthogonal basis set of polynomials described over the unit circle. They are a set of polynomials fit, in a least squares sense, to an aberrated wavefront. They were first derived by Frits Zernike in the 1930s and extended to the computation of non-rotationally symmetric aberrations by B. R. A. Nijboer in the late 1940s. Since then, they have been employed, with great success, to the analysis of optical systems.

The Zernike polynomials are defined over a circular pupil and may be decomposed into radial and azimuthal products [Kim 1987],

$$Z_n^m(\rho, \theta) = R_n(\rho) \ B_m(\theta) \quad (4-1)$$

$B_m(\theta)$ is a radial function with a period of $2\pi$ and whose form is independent of rotation of the coordinate system. The function $B_m(\theta) = e^{im\theta}$, $m$ any integer or zero, satisfies this requirement. In actual use, $B_m(\theta)$ is decomposed into its constituent sine and cosine functions. $R_n(\rho)$ is a radial polynomial of degree $n$, which contains no power of $\rho$ less than $m$. It has the generating function

$$R_n^m(\rho) = \sum_{s=0}^{s=m} \frac{(n-s)!}{(n+m+s + \frac{n-m+2s}{2}) \cdot s!} \quad (4-2)$$

The order $n$ of $R$ is correlated with the aberration order. Thus, $R_n(\rho)$ is related to a fourth order deformation of the wavefront, or, equivalently, third-order spherical aberration. It contains, in addition, the lower order terms $\rho^2$ and $\rho^0$. The magnitude and sign of these terms tell exactly what would be required to balance highest order
aberration term with lower orders to achieve the minimum wavefront variance. This property is a consequence of the fit and is a very useful tool for determining the optimum aberration balance to achieve the best image quality in the presence of higher order aberrations.

The use of Zernike polynomials requires that the pupil be circular and unobstructed. Failure to satisfy this requirement leads to terms that may no longer be orthogonal. Cross terms are introduced so that the introduction of higher order terms influences the fit of lower orders. As was indicated in Chapter 2, this may not necessarily be the case for the dome at nonzero look angles. Recall that the optical axis and the upper and lower marginal rays in the tangential plane are all refracted at different angles by the dome. Since the marginal rays, by definition, must pass through the maximum aperture of the stop, the entrance pupil must be compressed or expanded asymmetrically to satisfy this requirement. As a result, the entrance pupil is no longer circular. To address this problem in the analysis, the Zernike terms are fit to the exit pupil. Since there are no imaging elements between it and the focal plane, it is both coincident with the aperture stop at the gimbal center, and also by definition, circular.

A second point to remember is that the Zernike polynomials are fit to the aberrated wavefront of a single field, or object point. As a result, explicit information regarding the field dependence is lost. In this sense, the Zernike polynomials provide only a snapshot of the aberration function. Any field dependence must be interpolated from the computation of the Zernike coefficients for a number of different object (field)
positions. With these constraints, the Zernike polynomials may be used to fit any wavefront generated by the optical system if a sufficient number of orders are used. The Zernike polynomial terms used in the parametric analysis are a subset of the so-called Fringe Zernikes. These are a subset consisting of 37 standard Zernike terms, ordered in such a way as to better represent the first-, third-, fifth-, and higher order aberrations as contiguous sets. Of these, the first 16 terms are used. These are listed in the Table 4-1. Figure 4-1 illustrates the orientation of the exit pupil assumed by the fitting routine. Although this topic will be discussed in greater detail in later sections, it was found that the first 16 Fringe Zernike terms are sufficient to accurately fit the transmitted wavefront for most dome configurations. Further, the fifth-order terms, i.e., $Z_{10}$ and higher are, in general, small compared to the third-order aberrations. Exceptions to this rule occur for domes having some combination of a large fineness ratio ($F_s \geq 2$), excessive thickness relative to the aperture of the dome and optical system, or a high index of refraction ($n \geq 2.5$). In these instances, sixth and higher orders of spherical aberration exist for $\theta = 0$, and higher orders of astigmatism tend to dominate at larger look angles.
<table>
<thead>
<tr>
<th>Fringe</th>
<th>Standard</th>
<th>Polynomial</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z1</td>
<td>Z1</td>
<td>1</td>
<td>Piston</td>
</tr>
<tr>
<td>Z2</td>
<td>Z2</td>
<td>$\rho \cos(\alpha)$</td>
<td>Tilt about Y</td>
</tr>
<tr>
<td>Z3</td>
<td>Z3</td>
<td>$\rho \sin(\alpha)$</td>
<td>Tilt about X</td>
</tr>
<tr>
<td>Z4</td>
<td>Z5</td>
<td>$2\rho^2 - 1$</td>
<td>Defocus</td>
</tr>
<tr>
<td>Z5</td>
<td>Z4</td>
<td>$\rho^2 \cos(2\alpha)$</td>
<td>Quadratic astigmatism, X</td>
</tr>
<tr>
<td>Z6</td>
<td>Z6</td>
<td>$\rho^2 \sin(2\alpha)$</td>
<td>Quadratic astigmatism, Y</td>
</tr>
<tr>
<td>Z7</td>
<td>Z8</td>
<td>$(3\rho^3 - 2\rho)\cos(\alpha)$</td>
<td>Linear coma, X</td>
</tr>
<tr>
<td>Z8</td>
<td>Z9</td>
<td>$(3\rho^3 - 2\rho)\sin(\alpha)$</td>
<td>Linear coma, Y</td>
</tr>
<tr>
<td>Z9</td>
<td>Z13</td>
<td>$6\rho^4 - 6\rho^2 + 1$</td>
<td>3rd Spherical aberration</td>
</tr>
<tr>
<td>Z10</td>
<td>Z7</td>
<td>$\rho^3 \cos(3\alpha)$</td>
<td>Trefoil, X</td>
</tr>
<tr>
<td>Z11</td>
<td>Z10</td>
<td>$\rho^3 \sin(3\alpha)$</td>
<td>Trefoil, Y</td>
</tr>
<tr>
<td>Z12</td>
<td>Z12</td>
<td>$(4\rho^4 - 3\rho^2)\cos(2\alpha)$</td>
<td>Oblique spherical, X</td>
</tr>
<tr>
<td>Z13</td>
<td>Z14</td>
<td>$(4\rho^4 - 3\rho^2)\sin(2\alpha)$</td>
<td>Oblique spherical, Y</td>
</tr>
<tr>
<td>Z14</td>
<td>Z18</td>
<td>$(10\rho^5 - 12\rho^3 + 3\rho)\cos(\alpha)$</td>
<td>Elliptical coma, X</td>
</tr>
<tr>
<td>Z15</td>
<td>Z19</td>
<td>$(10\rho^5 - 12\rho^3 + 3\rho)\sin(\alpha)$</td>
<td>Elliptical coma, Y</td>
</tr>
<tr>
<td>Z16</td>
<td>Z25</td>
<td>$20\rho^6 - 30\rho^4 + 12\rho^2 - 1$</td>
<td>5th Spherical aberration</td>
</tr>
</tbody>
</table>

Table 4-1 Fringe Zernike terms used in the parametric analysis. The description refers to the highest order wavefront aberration term that the Zernike polynomial will fit. The X terms are associated with $\cos(m\alpha)$ and have a maximum at $\alpha = 0$, along the x-axis. The Y terms are associated with $\sin(m\alpha)$ and have a minimum at $\alpha = 0$. The polynomial expressions are multiplied by the associated coefficient $Z_n$ in the fitting routine. Note that coefficients $Z_n$ are not normalized; the polynomial set used in the analysis is orthogonal, but not orthonormal.
Figure 4-1 Coordinate frame for Fringe Zernike polynomial fit. The function is normalized to unit radius. Angles are measured from the x-axis.
4.3 Relating Zernike and Wavefront Polynomials

Because the fitting process used to obtain Zernike polynomials balances lower and higher order terms to obtain the minimum wavefront variance, there is no one-for-one correlation between the Zernike coefficients $Z_n$ and the wavefront aberration polynomials $W_{ab}$. Formulations relating Zernike coefficients with the wavefront polynomial coefficients have been developed [Wyant 1992].

The derivation will not be repeated but highlighted to illustrate the process, with the results summarized in Table 4-2. The first nine Zernike polynomials are those associated with the first- and third-order aberrations. These are grouped by powers of the radial component, $\rho$.

\[
W'(\rho,\phi) = Z_1 + Z_2 + Z_9
+ (Z_2 - 2Z_7)\rho\cos(\phi) + (Z_3 - 2Z_8)\rho\sin(\phi)
+ (2Z_4 - 6Z_9 + Z_5\cos(2\phi) + Z_6\sin(2\phi))\rho^2
+ 3(Z_7\cos(\phi) + Z_8\sin(\phi))\rho^3
+ 6Z_9\rho^4
\]

(4-3)

$W'$ is set equal to the field independent terms of $W(\rho,\theta)$

\[
W(\rho,\phi) = W_{11}\rho\cos(\phi) + W_{20}\rho^2 + W_{40}\rho^4 + W_{31}\rho^3\cos(\phi) + W_{22}\rho^2\cos(\phi)^2
\]

(4-4)

With the help of the identity

\[
a\cdot\cos(\alpha) + b\cdot\sin(\alpha) = \sqrt{a^2 + b^2}\cdot\cos\left(\alpha - \arctan\left(\frac{b}{a}\right)\right)
\]

(4-5)

the following table can be constructed.
Table 4-2 Converting Fringe Zernike coefficients into corresponding Seidel terms. The angle gives the rotation of the aberration term in the exit pupil [Wyant 1992].

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
<th>Magnitude</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{11}$</td>
<td>tilt</td>
<td>$\sqrt{(Z_2 - 2Z_7)^2 + (Z_3 - 2Z_8)^2}$</td>
<td>$\tan^{-1}\left(\frac{Z_3 - 2Z_8}{Z_2 - 2Z_7}\right)$</td>
</tr>
<tr>
<td>$W_{20}$</td>
<td>focus</td>
<td>$2Z_4 - 6Z_9 \pm [Z_5^2 + Z_6^2]^{1/2}$</td>
<td></td>
</tr>
<tr>
<td>$W_{22}$</td>
<td>astigmatism</td>
<td>$\mp [Z_5^2 + Z_6^2]^{1/2}$</td>
<td>$\frac{1}{2} \tan^{-1}(Z_6/Z_5)$</td>
</tr>
<tr>
<td>$W_{31}$</td>
<td>coma</td>
<td>$[Z_7^2 + Z_8^2]^{1/2}$</td>
<td></td>
</tr>
<tr>
<td>$W_{40}$</td>
<td>spherical</td>
<td>$6Z_9$</td>
<td></td>
</tr>
</tbody>
</table>

4.4 Parametric Analysis

For a particular look angle, $\theta$, the aberration coefficient depends on the three parameters, the fineness ratio (F), thickness (t), and index of refraction (n) in some complicated fashion. The intent of the parametric analysis was to quantify these dependencies by freezing two parameters and varying the third, and repeating this for each variable. As it turns out, however, the investigation showed that even though the parameters themselves may be considered independent variables, the value of an aberration coefficient always depends on a particular combination of the three. In other words, the behavior of the aberration can be predicted only when two variables are frozen and the variation of the coefficient with the third is known. For example, consider the variation of the coefficient $W_{lm}(F, t, n, \theta)$. Let $t$ be variable and the remainder of the terms be fixed and known. The coefficient value for different values of $t$ is obtained. This data
is then plotted as a function of the thickness. A polynomial curve is fit to the data and the subsequent expression may be used to generate values of the coefficient in between data points to very high accuracy. If, however, a second parameter is changed, such as the index, the original data can no longer be used to predict the behavior of the coefficient unless the process (vary one, freeze two, calculate the coefficient) is repeated. This is true even when the dome has been characterized for each of the three parameters.

The reasons the coefficients exhibit this sort of complex relationship are many. These are best explained by way of example. Recall, from Chapter 3, Eqns. 3-21 and 3-22

\[ W_{222} = (n-1)a_{220} + (1-n)a_{221} = (n-1)(a_{220} - a_{221}) \]  

\[ a_{22} = \frac{r_{a}^{2}}{R} \cdot \frac{k(y/R)^{2}}{4(1-k(y/R)^{2})^{3/2}} \]

which describe the astigmatism component of the transmitted wavefront. From Eqns. 3-18, 3-19, and 3-20 we have \( k \) and \( R \), which depend on the geometry of the dome surface

\[ F_{o} = \frac{L}{D_{o}} \]

\[ k_{o} = \frac{1}{4F_{o}^{2}} - 1 \]

\[ R_{o} = \frac{D_{o}}{4F_{o}} \]

Further the \( y \) height on the dome surface depends on both the geometry and the look angle.
\[ y(\theta) = \frac{FD \tan(\theta)}{\sqrt{4F^2 \tan(\theta)^2 + 1}} \]

Substituting these values into Eqn. 3-22 yields (for a single surface)

\[ a_{22} = \frac{Fr^2 (n-1)}{D} \left( \frac{1}{4F^2 - 1} \right) \left( \frac{4F^2 \tan(\theta)}{\sqrt{4F^2 \tan(\theta)^2 + 1}} \right)^2 \left( 1 - \left( \frac{4F^2 \tan(\theta)}{\sqrt{4F^2 \tan(\theta)^2 + 1}} \right)^2 \right)^{\frac{1}{2}} \]  

(Eq-6)

Eqn. 4-6 is intended to illustrate the manner in which the astigmatism coefficient depends on the fineness ratio and index of refraction. The dome thickness, in conjunction with the fineness ratio and outer diameter, is used to calculate \( F, R, \) and \( k \) for the second (usually the inner) surface, so it, too, is included in the calculation of the transmitted wavefront error. As a result of the interrelationships exhibited by Eqn. 4-6, the determination of the astigmatism coefficient will always depend on the combination of the three variables. The effect each has cannot be determined independent of the others. Differentiating Eqn. 4-6 with respect to the fineness ratio to determine its sensitivity to change yields a polynomial having terms up to sixth-order, \( F^6 \).

Despite the difficulties presented by Eqn. 4-6, it is possible to make some general statements about each of the three parameters. For zero thickness, any coefficient \( W_{\text{in}}(F,0,n,\theta) = 0 \). The total aberration is equal to zero as well. This is one consequence of requiring the dome to have a constant thickness because for \( t = 0 \), the inner and outer surfaces are coincident and the aberrations generated by each exactly cancel one another.
Likewise, for $n = 1$, $W_{lm}(F, t, l, \theta) = 0$ as well. In this case, the dome has no refractive power whatsoever; it is made of air.

The dependence on fineness ratio is somewhat more complex than that of the latter two parameters. In this application, $F \geq 0.5$. For $F < 0.5$, the dome surfaces are no longer conic sections but are instead oblate ellipsoids with the conic constant $k > 0$. These surfaces no longer exhibit rotational symmetry about the gimbal axis. So while it is possible to define a surface with a fineness ratio of less than 0.5 in a mathematical sense, the surface defined no longer exhibits the same symmetries as the ellipse and therefore cannot be treated in the manner developed in this dissertation. However, for $F = 0.5$, the dome is hemispherical and the resulting aberration function is constant for all look angles and fixed values of $n$ and $t$.

Figures 4-2 to 4-19 illustrate the effect that the parameters $F$, $t$, and $n$ have on the aberration content of the transmitted wavefront. The nominal dome has the following characteristics: $F = 1$, $t = 2$ mm, $n = 1.5$. In the analysis, two parameters are frozen and the third is varied. The dome has a diameter of 50 mm and the exit pupil diameter is held constant at 25 mm. The entrance pupil dimensions are variable, sized to maintain a circular exit pupil with the proper diameter. Within the field of view, only the on-axis field is plotted. The $y$ component of the field of view is essentially a change in the look angle, $\theta$, shifting the curve from side to side. Except for $\theta = 0$, the $x$ component of the field of view introduces aberrations that are small compared to those resulting from the change in look angle, so it is ignored.
4.4.1 Vary Fineness Ratio

The variation of the coefficient $Z_n$ as a function of the fineness ratio for a fixed look angle can be fit with a sixth order polynomial curve. The coefficients of the polynomial curve are themselves a function of the look angle and must therefore be recalculated at each position in the field of regard. For $F = 0.5$, the coefficients $Z_n$ are constant. Those coefficients associated with field aberrations are zero, but those that are independent of field will have a constant value that is not zero. As $F$ is increased, the magnitude of the aberrations increases as well.

The sensitivity to changes in the fineness ratio can be explained by considering the changes in the angles of incidence. The fineness ratio determines the shape of the dome, which significantly alters the angles of incidence on the dome surfaces, particularly at small look angles. As was shown in Chapter 2, the maximum angle of incidence of the OAR increases with fineness ratio. Aberrations scale nonlinearly with angle of incidence. For large angles, and therefore, large $F$, the effect is magnified. Further, the look angle at which the maximum angle of incidence is attained gets smaller as the fineness ratio increases. As a result, the aberrations will peak for small look angles and will, for any $F$, converge towards a single value as the look angle is increased.

Put another way, with increasing fineness ratio the dome is increasingly more tapered. The region of the dome undergoing the most significant change is the tip. Beyond this region the dome is increasingly more cylinder-like. The tangential radius of curvature of a point in the latter region is long compared to the base radius of curvature. In addition, the surface normal is nearly perpendicular to the missile body axis. As a
result, changes in the angle of incidence vary proportionally with changes in the look angle. They are approximately linear and small. Also, in this region the angle of the surface normal for different values of $F$ is nearly the same, so that the difference between the amount of aberration generated is small.

Figures 4-2 to 4-6 are plots of selected Fringe Zernike terms up to $Z_{16}$ as a function of look angle and for a number of values of the fineness ratio. Those terms not included are negligibly small or zero. On the on-axis (center of the field of view) field is plotted. In addition, Figure 4-7 shows the peak RMS wavefront plotted against fineness ratio. Up to $F = 2.5$, the increase in RMS error is proportional to $\sqrt{F}$. Beyond this it increases dramatically. For $F > 2.5$, spherical aberration has a very large contribution at the boresight position, particularly when the aperture stop diameter is a significant fraction of the dome diameter.

### 4.4.2 Vary Center Thickness

The variation of the coefficients $Z_n$ due to changes in the center thickness of the dome is plotted in the next series of figures. Unlike the fineness ratio and, as will be seen, the index of refraction, the change in the aberration function depends nearly linearly on the dome thickness. This behavior is duplicated for any value of $F$ and $n$, so the changes in the thickness have a predictable effect even if the dome configuration has changed. This is not necessarily true in an absolute sense, but rather in the sense that the aberration is directly proportional to the thickness. If, for example, the coefficient $Z_n(F,n,t)$ is known, then the coefficient is scaled by a factor of two for $Z_n(F,n,2t)$. This linear
dependence is valid for small values of the center thickness. As the dome thickness increases, the change in the coefficient takes on a weak quadratic form. That is, its behavior is almost linear, but the addition of a \( t^2 \) term is necessary to exactly fit the data.

Changes in thickness lead to changes to the \( y \)-intercept of a ray as it travels between the two surfaces (ref. Eqns. 3-21 and 3-22) and to changes of the fineness ratio of the inner surface. The change in \( y \) height is given by the first order transfer equation:

\[
y' = y - tu',
\]

where \( u' \) is the angle of the refracted ray with respect to the optical axis. The change in fineness ratio is given by

\[
F_{inner} = \frac{(L-t)}{(D-2t)}.
\]

Since \( t \) tends to be small compared to \( D \) and \( L \), \( F_{inner} \), although larger, is not significantly different from \( F \). Therefore, the change of the aberration coefficient depends more on the change in \( y \) height and will share its nearly linear dependence.

Like the previous sequence of figures, Figures 4-8 to 4-12 contain plots of selected Fringe Zernike terms up to \( Z_{16} \) as a function of look angle and for a number of values of the center thickness of the dome. Those terms not included are either negligibly small or zero. Only the on-axis (center of the field of view) field is plotted. Figure 4-13 illustrates the maximum RMS wavefront error plotted against dome thickness. It, too, illustrates nearly linear dependence on the thickness.

### 4.4.3 Vary Index of Refraction

The last parameter investigated was the index of refraction, \( n \). The variation of the coefficient \( Z_n \) as a function of \( n \) has the same complexity as the variation due to changes of the fineness ratio. It, too, requires a sixth-order polynomial to fit the data for a
particular look angle. The effect is analogous to that of the fineness ratio in the sense that
n has an effect on the angles of incidence. However, instead of modifying it at the outer
surface, as the fineness ratio does, n alters the angle of incidence on the inner surface of
the dome. The arguments with regard to the effects of the angle of incidence have on the
coefficient $Z_n$ (developed in Section 4.3.1) apply here as well.

There are, however, a few notable exceptions. The first is that, for defocus and
astigmatism, rather than converging to a single value for large look angles for any value
of n, the line plots for different values of n are distinct. On the contrary, the convergent
point for the different values of F appears to scale with the index of refraction. As n
increases, so too does the magnitude of the aberration coefficient. Defocus and
astigmatism are unique because they are both manifestations of a focus error. The second
effect is that the contribution to the total aberration from the inner surface is of a larger
magnitude, so that coefficient $Z_n$ changes sign.

Like the previous sequence of figures, Figures 4-14 to 4-18 contain plots of
selected Fringe Zernike terms up to $Z_{16}$ as a function of look angle and for a number of
values of the index of refraction of the dome. Those terms not included are either
negligibly small or zero. Only the on axis (center of the field of view) field is plotted.
Figure 4-19 shows that the maximum RMS wavefront error is proportional to $\sqrt{n-1}$.
Figure 4-2 Variation due to changes in the fineness ratio (t = 2 mm, n = 1.5, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z3, tilt about X. Shows line of sight deviation through the dome. (Bottom) Fringe Zernike Z4, defocus from the sagittal image plane.
Figure 4-3 Variation due to changes in the fineness ratio ($t = 2 \text{ mm}, \, n = 1.5, \, D = 50 \text{ mm}, \, \text{ExPD} = 25 \text{ mm}$). (Top) Fringe Zernike Z5, tangential astigmatism. Astigmatism peaks near the maximum angle of incidence on the dome. It is the largest aberration term. (Bottom) Fringe Zernike Z8, coma. A large contributor that peaks at small look angles and decreases to zero at $\theta = \pi/2$. 
Figure 4-4 Variation due to changes in the fineness ratio (t = 2 mm, n = 1.5, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z9, third-order spherical aberration. (Bottom) Fringe Zernike Z16, fifth-order spherical aberration. The spherical aberration coefficients $W_{040}$ and $W_{060}$ are generally constant as a function of look angle. The variation in these plots is due to the fact that the curvature of the dome decreases with look angle.
Figure 4-5 Variation due to changes in the fineness ratio\(t = 2\ \text{mm}, \ n = 1.5, \ D = 50\ \text{mm},\ \text{ExPD} = 25\ \text{mm}\). (Top) Fringe Zernike Z11, trefoil. (Bottom) Fringe Zernike Z12, trefoil. Trefoil is a variant of astigmatism, with three lobes around the pupil. Its contribution is, in general, much smaller than the contribution from the quadratic and constant astigmatism terms.
Figure 4.6 Variation due to changes in the fineness ratio (t = 2 mm, n = 1.5, D = 50 mm, ExPD = 25 mm). Fringe Zernike Z15, elliptical coma. Like trefoil, its contribution is, in general, much smaller than the contribution from third-order tangential coma.
Figure 4-7 (Top) Maximum RMS wavefront error plotted against fineness ratio. (Bottom) The solid line depicts the angle at which the maximum error occurs. The dashed line is a plot of the maximum angle of incidence as a function of F. (t = 2 mm, n = 1.5, D = 50 mm, ExPD = 25 mm)
Figure 4-8 Variation due to changes in the center thickness \( (F = 1, n = 1.5, D = 50 \text{ mm}, \text{ExPD} = 25 \text{ mm}) \). (Top) Fringe Zernike Z3, tilt about X. Shows line of sight deviation through the dome. (Bottom) Fringe Zernike Z4, defocus from the sagittal image plane.
Figure 4-9 Variation due to changes in the center thickness (F = 1, n = 1.5, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z5, tangential astigmatism. Astigmatism peaks near the maximum angle of incidence on the dome. It is the largest aberration term. (Bottom) Fringe Zernike Z8, coma. A large contributor that peaks at small look angles and decreases to zero at $\theta = \pi/2$. 
Figure 4-10 Variation due to changes in the center thickness (F = 1, n = 1.5, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z9, third-order spherical aberration. (Bottom) Fringe Zernike Z16, fifth-order spherical aberration. The spherical aberration coefficients $W_{040}$ and $W_{060}$ are generally constant as a function of look angle. The variation in these plots is due to the fact that the curvature of the dome decreases with look angle.
Figure 4-11 Variation due to changes in the center thickness (F = 1, n = 1.5, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z11, trefoil. (Bottom) Fringe Zernike Z12, trefoil. Trefoil is a variant of astigmatism, with three lobes around the pupil. Its contribution is, in general, much smaller than the contribution from the quadratic and constant astigmatism terms.
Figure 4-12 Variation due to changes in the center thickness \((F = 1, n = 1.5, D = 50 \text{ mm}, \text{ExPD} = 25 \text{ mm})\). Fringe Zernike Z15, elliptical coma. Like trefoil, its contribution is, in general, much smaller than the contribution from third-order tangential coma.
Figure 4-13 Maximum RMS wavefront error as a function of dome thickness.
($F = 1, n = 1.5, D = 50 \text{ mm}, \text{ExPD} = 25 \text{ mm}$)
Figure 4-14 Variation due to changes in the index of refraction (F = 1, t = 2 mm, D = 50 mm, ExPD = 25 mm). (Top) Fringe Zernike Z3, tilt about X. Shows line of sight deviation through the dome. (Bottom) Fringe Zernike Z4, defocus from the sagittal image plane.
Figure 4-15 Variation due to changes in the index of refraction ($F = 1, t = 2\, \text{mm}, D = 50\, \text{mm}, \text{ExPD} = 25\, \text{mm}$). (Top) Fringe Zernike Z5, tangential astigmatism. Astigmatism peaks near the maximum angle of incidence on the dome. It is the largest aberration term. (Bottom) Fringe Zernike Z8, coma. A large contributor that peaks at small look angles and decreases to zero at $\theta = \pi/2$. 

<table>
<thead>
<tr>
<th>Fringe Zernike Z5: Astigmatism</th>
<th>$n = 1.50$</th>
<th>$n = 2$</th>
<th>$n = 2.50$</th>
<th>$n = 3$</th>
<th>$n = 3.50$</th>
<th>$n = 4$</th>
</tr>
</thead>
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<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

![Graph of Fringe Zernike Z5: Astigmatism](image1.png)

<table>
<thead>
<tr>
<th>Fringe Zernike Z8: Coma, Y</th>
<th>$n = 1.50$</th>
<th>$n = 2$</th>
<th>$n = 2.50$</th>
<th>$n = 3$</th>
<th>$n = 3.50$</th>
<th>$n = 4$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

![Graph of Fringe Zernike Z8: Coma, Y](image2.png)
Figure 4-16 Variation due to changes in the index of refraction ($F = 1$, $t = 2$ mm, $D = 50$ mm, ExPD = 25 mm). (Top) Fringe Zernike Z9, third-order spherical aberration. (Bottom) Fringe Zernike Z16, fifth-order spherical aberration. The spherical aberration coefficients $W_{040}$ and $W_{060}$ are generally constant as a function of look angle. The variation in these plots is due to the fact that the curvature of the dome decreases with look angle.
Figure 4-17 Variation due to changes in the index of refraction ($F = 1$, $t = 2$ mm, $D = 50$ mm, ExPD = 25 mm). (Top) Fringe Zernike Z11, trefoil. (Bottom) Fringe Zernike Z12, trefoil. Trefoil is a variant of astigmatism, with three lobes around the pupil. Its contribution is, in general, much smaller than the contribution from the quadratic and constant astigmatism terms.
Figure 4-18 Variation due to changes in the index of refraction (F = 1, t = 2 mm, D = 50 mm, ExPD = 25 mm). Fringe Zernike Z15, elliptical coma. Like trefoil, its contribution is, in general, much smaller than the contribution from third-order tangential coma.
Figure 4-19 Maximum RMS wavefront error as a function of index of refraction. (F = 1, t = 2 mm, D = 50 mm, ExPD = 25 mm)
4.5 Concluding Remarks

This chapter has demonstrated the complex behavior of the transmitted wavefront aberration resulting from changes in the fineness ratio, index of refraction, and the center thickness of the dome. The analysis focused on the use of the Fringe Zernike polynomials to fit the transmitted wavefront. The first 16 terms were used. The resulting fit errors were on the order of a few percent or less.

The dominant aberration is tangential astigmatism, owing to the large differences between the tangential and sagittal curvatures. The contributions of the third-order aberrations ($Z_5$ to $Z_9$) are considerably larger than those introduced by $Z_{10}$ and higher. In many instances, the first nine terms alone are adequate to fit the transmitted wavefront. The exception to this rule is that steeply curved, high index, thick domes have a large spherical aberration component on boresight. This requires the inclusion of $Z_{16}$ or higher to accurately fit the wavefront. As a result, the analysis is limited to domes with a fineness ratio $F = 3$ or less.
Chapter 5: Conclusions

5.1 Summary

This dissertation has been written in an attempt to explain the optical behavior of an ellipsoidal dome. It began with a first order development that was extended to include some aberration theory, and finally concluded with a parametric analysis to investigate how changing certain physical properties of the dome would affect the transmitted wavefront. Much of what has been achieved would not have been possible were it not for the development in recent years of new optical design, manufacturing, and testing technology that makes it possible not only to conceive of an aerodynamic dome, but to incorporate it into an optical system as well. As of this writing, the world's first conformal dome and imaging system has been fabricated, aligned, and tested. The dome itself has a modest fineness ratio, is thin, and has a low index of refraction. Its optical performance is near diffraction limited and has met or exceeded predictions.

This milestone has been achieved with a great deal of perseverance, patience, and in some instances, luck. It is a testament to the power of state of the art optical design software. However, reliance on design codes to find the optimum solution does little to increase understanding of the problem. The question is not only one of how, but one of why. If the optical properties of the dome are well understood, that is, the why, then the how becomes a much simpler task. It is the question of why that prompted the early work that has since evolved into this dissertation.

An ellipsoidal dome was selected because it satisfies the aerodynamic requirement of having a low cross-sectional area in the direction of flight and because it
is easily modeled in optical design software. Ellipsoidal surfaces are easier to test than more generalized aspheric surfaces. Its symmetry and periodic behavior provide insight into the imaging characteristics of a dome. In addition, the fineness ratio of an ellipsoidal dome is easily defined based on the geometric constructs of its outer surface. Further, Chapter 2 showed how the fineness ratio is used to calculate the base radius of curvature and conic constant of the dome surfaces that allows specification of the dome in optical design software. Also in Chapter 2, expressions for a number of first order properties were developed.

It was shown that the line of sight deviation (pointing error) of a dome is related to the angle of incidence of the optical axis ray on the dome surfaces. The angle of incidence was shown to have a maximum value that increases with fineness ratio but occurs at look angles that decrease with increasing fineness ratio. The paraxial optical power of a dome was also investigated. This calculation is limited to the boresight position because first-order characteristics do not adequately describe the image size and position for nonzero look angles. It was shown that a dome is a weakly negative element for low index materials. Increasing the index of refraction eventually moves the dome to a strong positive element.

Chapter 3 developed an aberration theory of a dome. It began with a review of basic aberration theory as it relates to axially symmetric optical systems. It was developed further to include the aberrations of plane symmetric systems.

An ellipsoidal dome, however, has a number of unique features that may be applied to the development of its own aberration function. Chief among these are the
symmetries with respect to look angle. An ellipse exhibits periodic behavior. It was shown that the total aberration function, either RMS or peak-to-valley wavefront error, has the same periodicity. As a result, the functional form of the aberration function, as well as individual coefficients, can be decomposed into sums of sine and cosine terms. Odd aberrations in look angle, such as coma and tilt, are given by an expansion of sine terms. These are necessarily zero when the gimbal is aligned with either the major or minor axis of the dome surfaces. Even aberrations, such as defocus, spherical, and astigmatism, are given by an expansion of cosine terms.

The technique used to model the behavior of third-order astigmatism and coma was an elaboration of work in which a conic surface is defined in a coordinate frame relative to the intercept of the OAR with the surface. It assumes the aberrations generated by a surface are proportional to its shape. Expressions showing the variation of astigmatism and coma were developed. The variation of spherical aberration was not derived, but its impact on the design of imaging systems inside a conformal dome was discussed. All three were plotted as a function of look angle and compared to real data where applicable. There are significant differences between the predicted values for astigmatism and coma and those obtained from real ray trace data, particularly with respect to the magnitude of the coefficients. However, both data sets exhibited the same functional form with regard to look angle, an important result.

Chapter 4 expanded further on the aberration function of the dome by examining the parametric behavior of a number of Fringe Zernike coefficients. The fineness ratio, thickness, and index of refraction are parameters that, in combination, uniquely define a
dome. In the analysis, each one was varied independently of the remaining two to determine its effect on the aberration coefficients.

It was shown that both the fineness ratio and index of refraction exhibit complex, nonlinear, behavior. For a particular look angle, the changes introduced by either of the two parameters could be fit only with a sixth-order polynomial. It was hypothesized that these results are better understood by considering the effect that each has on the maximum angle of incidence on a dome. A more useful figure of merit may be the change in the maximum RMS wavefront error, independent of look angle as a function of parameter change. For the index of refraction, \( RMS_{\text{max}} \propto \sqrt{n} \) up to \( n = 4 \). The same relation holds for \( F < 3 \). As the fineness ratio increases beyond 3, the resulting error increases rapidly, proportional to \( F^6 \), perhaps suggesting a practical limit for an optically useful dome. In contrast to \( n \) and \( F \), thickness variations were shown to have a nearly linear effect on both the aberration coefficient and RMS, so that doubling the thickness doubles its value.

The parametric analysis also showed that, aside from thickness, it is difficult to separate out the effect each parameter has on a particular coefficient independent of the others. The aberration content of the transmitted wavefront could not be predicted if more than one parameter is changed from its original value.

### 5.2 Suggestions for Future Work

Developing techniques for correcting dome-induced aberrations is a logical next step. Key to the imaging problem is reducing the large amount of variable focus error. It
has two components: astigmatism, resulting from the splitting of sagittal and tangential
optical power over look angle; and a focus shift, resulting simply from the change in
curvature (sagittal or tangential) as a function look angle as well. Both depend
fundamentally on the geometry (or shape) of a dome. Astigmatism is the largest single
aberration term. Focus shift, although not strictly an image defect, is also significant and
requires compensation as well. Correction schemes may be devised that work on both
simultaneously.

Coma, which is proportional to the angle of incidence $\alpha$ of the OAR on the dome
surfaces, is the next most significant defect. Reducing $\alpha$ reduces the coma contribution.
Likewise, minimizing the variation of $\alpha$ will have a similar effect on the variation of
coma as well.

Spherical aberration is significant primarily at the boresight position because the
curvatures of the dome surfaces are largest in this region. It becomes the limiting
aberration for large values of $F$. For $F > 2$ or 3, the fifth- and seventh-order spherical
aberration terms become significant. This is an exception to the rule that the transmitted
wavefront error is mainly attributable to third order contributions. The difficulty with
correcting spherical aberration is that its contribution to the aberration shrinks rapidly
with increasing look angle, forcing the correcting optics to have the same characteristic
behavior.

This dissertation has described the optics of ellipsoidal domes. It has provided a
glimpse into why, until recently, the concept of a conformal dome was an interesting
abstraction, but of little practical value because of the difficulties it posed for optical
correction and imaging. The use of highly aspheric surfaces is further complicated by the rotation of a large, relative to the dome, gimbaled collecting aperture, pointing the line of sight through different portions of a dome. The problem of correcting the aberrations introduced by a dome is material for future work. However, with the basic understanding of the optical properties of an ellipsoidal dome put forth in this dissertation, the form the correction optics need to take is more evident.
Appendix A: Summary of Equations

A.1 Geometry of an Ellipse

Eqn. A-1.1 is a Cartesian representation of an ellipse; a, b are the semi-major and semi-minor axis lengths, respectively

\[
\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{A-1.1}
\]

The eccentricity \( e \) of an ellipse is a measure of its elongation.

\[
e^2 = 1 - \frac{b^2}{a^2} \quad \text{A-1.2}
\]

The sag \( Z \) is measured from the vertex of the ellipse. \( r_0 \) is the base radius of curvature, \( k = -e^w \) is the conic constant (-1 < \( k < 0 \)) for an ellipse.

\[
Z_s = \frac{cv \cdot R^2}{1 + \sqrt{1 - (k + 1) \cdot cv^2 \cdot R^2}} \quad \text{A-1.3}
\]

\[
R^2 = x^2 + y^2 \quad \text{A-1.4}
\]

\[
(cv = 1/r_0) \quad \text{A-1.5}
\]

The fineness ratio \( F \) is a measure of aerodynamic efficiency. Higher values of \( F \) reduce the coefficient of drag on the surface. It is the ratio of the length to the diameter of the ellipse.

\[
F = \frac{L}{D} = \frac{a}{2b} \quad \text{A-1.6}
\]

The conic constant, \( k \), may be defined in terms of the fineness ratio.

\[
k = \frac{1}{4F^2} - 1 \quad \text{A-1.7}
\]
The base radius of curvature acts as a scaling factor for a dome. It does not change its relative shape, but does change its size. It is related to the fineness ratio and diameter of a dome in the following manner.

\[ r_0 = \frac{D}{4F} \quad \text{(A-1.8)} \]

Other representations include,

\[ r_0 = FD(1+k) \quad \text{(A-1.9)} \]
\[ r_0 = a \left(1 - \sqrt{1 - \frac{b^2}{a^2}}\right) \quad \text{(A-1.10)} \]

For a constant thickness dome with a thickness \( t \), the inner surface parameters are related to the outer surface.

\[ k_i = \frac{(D-2t)^2}{4(L-t)} - 1 \quad \text{(A-1.11)} \]
\[ r_i = \frac{(D-2t)^2}{L-t} \quad \text{(A-1.12)} \]

where the following substitutions have been made.

\[ D_i = D_o - 2t \quad \text{(A-1.13)} \]
\[ L_i = L_o - t \quad \text{(A-1.14)} \]
\[ F_i = \frac{L_i}{D_i} = \frac{L-t}{D-2t} \quad \text{(A-1.15)} \]

The sag of the surface can be rewritten in terms of the fineness ratio and diameter.

\( r_0 \) is positive if the center of curvature is to the right of the vertex.
A.2 Normal to the Surface of an Ellipse

The normal to a surface in a Cartesian coordinate system is given by

\[ \vec{n} = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

where

\[ f(x, y, z) = 1 - \left( \frac{x^2 + y^2}{b^2} + \frac{z^2}{a^2} \right) = 0 \]

with the result

\[ \vec{n}(x_0, y_0, z_0) = \left( \frac{-2x_0}{b^2}, \frac{-2y_0}{b^2}, \frac{-2z_0}{a^2} \right) \]

The angle between the normal and the z-axis (major axis) of the ellipse is

\[ \tan(i_n) = \frac{\partial f/\partial y}{\partial f/\partial z} = \frac{a^2}{b^2} \frac{y_0}{z_0} \]

which is equivalent to

\[ \tan(i_n) = 4F^2 \tan(\theta) \]

where \( \theta \) is the look angle. The difference between the look angle and the normal is the angle of incidence of the optical axis ray on the surface of an ellipse.

\[ \alpha = \theta - i_n \]

Differentiating Eqn A-2.6 with respect to \( \theta \) and setting the result equal to zero yields an expression for the look angle at which the maximum angle of incidence is observed.
\[ \theta_{\text{MAX}} = \tan^{-1}\left(\frac{1}{2F}\right) \]  

A-2.7

The maximum angle of incidence occurs on the line

\[ \alpha_{\text{MAX}} = \frac{\pi}{2} + 2\theta_{\text{MAX}} \]  

A-2.8

A.3 Curvature of the Surface of an Ellipse

The curvature of an ellipse differs in the sagittal and tangential planes. It may be determined by rewriting the equation in a polar, parametric form in term of the look angle, \( \theta \),

\[ f(\theta) = (y(\theta), z(\theta)) \]  

A-3.1

The expressions for \( y \) and \( z \) are found by substituting

\[ z(\theta) = \frac{y(\theta)}{\tan(\theta)} \]  

A-3.2

into Eqn. A-1 to get

\[ \frac{4y(\theta)^2}{D^2} + \frac{y(\theta)^2}{F^2D^2\tan(\theta)^2} = 1 \]  

A-3.3

Solving now for \( y \) and substituting that result into Eqn. A-3.2 to find \( z \) leaves us with the following results

\[ y(\theta) = \frac{FD\tan(\theta)}{\sqrt{4F^2\tan(\theta)^2 + 1}} \]  

A-3.4

\[ z(\theta) = \frac{FD}{\sqrt{4F^2\tan(\theta)^2 + 1}} \]  

A-3.5

The radius of curvature of a parametric function is
Computing the derivatives with respect to $\theta$ and simplifying leaves the following result for the tangential radius of curvature of an ellipse.

$$r_T = \frac{(z^2 + y^2)^{\frac{3}{2}}}{|z'y'' - y'z'|} \quad \text{(A-3.6)}$$

The sagittal radius of curvature is computed in a different fashion. Because an ellipsoid is a surface of revolution about the major axis, the centers of curvatures in the sagittal plane must be located on the major axis. To compute the sagittal curvature at a point $P$ on the surface, drop a line from the point that is perpendicular to the major axis. Then form a triangle by extending the normal until it, too, intersects the major axis. The sagittal radius of curvature is the length of the hypotenuse. Using Eqns. A-2.5 and A-2.8, solve for $r_s = y / \sin(i_n)$ to get

$$r_s(\theta) = r_0 \left( \frac{16F^4 \tan(\theta)^2 + 1}{4F^2 \tan(\theta)^2 + 1} \right)^{\frac{3}{2}} \quad \text{(A-3.8)}$$
Appendix B: Code V™ Sequence Files

This appendix contains a list of some of the Code V sequence files used to generate and plot data.

B.1 Sample dome lens sequence file: fa100th200n150.seq

```
RD M;LEN  "VERSION: 8.30 A  LENS VERSION: 48  Creation Date:  5-Sep-1999"

TITLE 'fa100th200n150' ! F = 1.0, t = 2.0 mm, n = 1.5
EPD  23.2740300424  ! Entrance pupil diameter (XPD = 25.0 mm)
DIM M  ! units are mm
WL  1000.0  ! wavelength in nm
REF  1
WTW  1
INI 'KSE'
XAN 0.0 0.0 0.573  ! field angles in degrees
YAN 0.0 0.573 0.0

! vignetting factors for zoom position 1
VUX 0.834857072363e-9 -0.33935023974e-4 0.00292635001184
VLX 0.834857072363e-9 -0.33935023974e-4 -0.00313877073728
VUY 0.834857072363e-9 0.00292635001191 -0.339350237801e-4
VLY 0.834857072363e-9 -0.00313877073728 -0.339350237801e-4

! private catalog to reset dome index of refraction
PRV
  PWL 1000.0
    'dome' 1.5
END

! Start surface definition
SO  0.0 0.1e14  ! object at infinity, SO
S  0.0 75.0 AIR  ! dummy surface for lens views, S1
S  0.0 0.0 0.0 AIR  ! dummy surface sets origin, S2
S  0.0 -75.0 AIR  ! tilt surface, S3
S  0.0 25.0

! Start NSS surface range
N 12.5 AIR REFR 'dome'  ! outer surface of dome, S5
  CON
  K -0.75  ! conic constant
  DAR
  CIR 24.9999999975  ! aperture radius
  CIR EDG 24.9999999975  ! edge aperture radius
```

153
N 11.0208333333 'dome' REFR AIR  ! inner surface of dome, S6
    CON
    K -0.770399305556
    DAR
    XDE 0.0; YDE 0.0; ZDE 2.0
    CIR 22.9999999975
    CIR EDG 22.9999999975

N -11.0208333333 AIR REFR 'dome'  ! 2nd inner dome surface, S7
    CON
    K -0.770399305556
    DAR
    XDE 0.0; YDE 0.0; ZDE 98.0
    CIR 22.9999999975
    CIR EDG 22.9999999975

N -12.5 'dome' REFR AIR  ! 2nd outer dome surface, S8
    CON
    K -0.75
    DAR
    XDE 0.0; YDE 0.0; ZDE 100.0
    CIR 24.9999999975
    CIR EDG 24.9999999975

N 0.0 AIR REFR ! Exit port of NSS range, S9
    DAR
    XDE 0.0; YDE 0.0; ZDE 50.0
    CIR 13.125
    CIR EDG 13.125

S 0.0 0.0  ! Dummy surface @ stop, S10

S 0.0 0.0 AIR  ! Dummy surface @ stop, S11
    RET S2  ! Return to original coordinate system

S 0.0 50.0 AIR  ! Aperture stop, S12
    STO
    CIR 12.5

SI 0.0 7.83673717991 ! Image plane, S13
    THC 0

AFI 50.0  ! Perfect lens (for afocal lenses only)

NSS S5..9  ! Define NSS range
NSP S9 GL2  ! Define exit conditions for NSS range

ZOO 16  ! 16 zoom positions (tilt through look angle)

! Zoomed EPD dimensions

ZOO EPD 23.2740300424 24.1254949138 25.0563893814 25.6450935322
     25.8427988725
! Zoom vignetting factors

ZOO  VUY F1
ZOO  VLY F1
ZOO  VUY F2
ZOO  VLY F2
ZOO  VUY F3
ZOO  VLY F3
ZOO  VUX F1
ZOO  VLX F1
ZOO  VUX F2
ZOO  VLX F2
ZOO  VUX F3
ZOO  VLX F3

! Tilt stop through look angles
! ADE rotates about X-axis

ZOO  ADE S3 0.0 -6.0 -12.0 -18.0 -24.0 -30.0 -36.0 -42.0 -48.0 -54.0 & -60.0 -66.0 -72.0 -78.0 -84.0 -89.0
ZOO  ADC S3 100 100 100 100 100 100 100 100 100 100 100 100 100 100 & 100 100

! Undo tilts

ZOO  ADE S9 0.0 6.0 12.0 18.0 24.0 30.0 36.0 42.0 48.0 54.0 60.0 66.0 & 72.0 78.0 84.0 89.0
ZOO  ADC S9 100 100 100 100 100 100 100 100 100 100 100 100 100 100 & 100 100

! zoom image distance (from focus of AFI lens to paraxial sagittal focus)


! Make image distance variable for each zoom position

ZOO  THC S13 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

! Pickups to define 2nd inner and outer surface parameters

PIK  K S7 Z1 K S6 Z1
PIK  RDY S8 Z1 RDY S5 Z1 -1.000000
PIK  K S8 Z1 K S5 Z1

GO  ! Execute lens entry
B.2 Macro to Modify Dome Lens File: fa.seq

!fa.seq

! This macro changes the shape or thickness of an ellipsoidal dome.
! The starting lens is in nonsequential (NSS) format. It is a closed
! ellipse. It is assumed that the lens file already exists and is the
! active file.
! See Appendix B.1 for a sample lens file
!
!Input parameters:
!
! #1 == ^fa ==> new fineness ratio
! #2 == "th ==> new center thickness
! #3 == "dia ==> outer diameter of dome
! #4 ==> exit pupil radius
! #5 == ^sav ==> save new lens or not?
!
! Remaining variables
! ^s1 ==> first dome surface (outer)
! ^s2 ==> second dome surface (inner)
! ^r ==> base radius of curvature
! ^cc ==> conic constant

rfd 1 2 50 12.5 'y'

! Create and initialize variables

lcl num ^s1 ^s2 ^a ^b ^c
lcl num ^r ^cc ^th ^fa ^dia
lcl str ^file ^tit
^s1 == 5
^s2 == 6
^fa == #1
^th == #2
^dia == #3
^sav == #5

! Write parameters (fineness ratio, thickness, index) into title
! for future reference.

^tit == concat('fa',substr(num_to_str(#1*100),1,3))
^tit == concat('tit','th')
^tit == concat('tit',substr(num_to_str(#2*100),1,3))
^tit == concat('tit','n')
^tit == concat('tit',substr(num_to_str((ind w1 s6)*100),1,3))
tit ^tit

! Begin surface modifications

thi s1 1.5*"fa*"dia  ! s1 is a dummy surface
thi s3 -(thi s1)      ! move from stop back to s1
thi s4 (thi s1)-(fa^dia) ! move to first dome surface

^cc == (1/(fa^2)**2)-1 ! calculate k of outer surface
^r == dia*fa*(1+cc) ! calculate r of inner surface
rdy s^s1 ^r
k s^s1 ^cc

! inner surface

^k == (((dia-2*th)**2 / (4*(fa^dia-th)**2))-1 ! k of inner surface
^r == (((dia-2*th)**2 / (4*(fa^dia-th)))) ! r of inner surface
rdy s^s2 ^r
k s^s2 ^k
zde s^s2 th ! set z position of inner surface

! resize apertures

del ape sa

cir s^s1 0.9999999999*(dia/2) ! outer surfaces
cir edg s^s1 0.9999999999*(dia/2)
cir s^s1+3 0.9999999999*(dia/2)
cir edg s^s1+3 0.9999999999*(dia/2)
cir s^s2 0.9999999999*(dia/2)-th ! inner surfaces
cir edg s^s2 0.9999999999*(dia/2)-th
cir s^s2+1 0.9999999999*(dia/2)-th
cir edg s^s2+1 0.9999999999*(dia/2)-th

zde s8 2*(fa^dia) ! s8 is the 2nd outer surface, flipped about x
zde s7 (zde s8)-th! s7 is the inner surface, flipped about x
zde s9 (fa^dia) ! s9 is the exit port, coincident with the ! aperture stop

out n ! suppress screen output
sto s12 ! aperture stop is on surface 12
cir ss #4 ! sets aperture stop radius
cir s9 #4*1.05
cir s9 edg (cir s9)
yan 0 ! delete all fields
xan 0

for ^f 1 (num f) ! delete all vignetting factors
    zoo vly f^f 0
    zoo vlx f^f 0
    zoo vuy f^f 0
    zoo vux f^f 0
end for

! Optimize image distance to locate paraxial sagittal focus
frz sa;zoo thc si 0;epd 1;aut;sag;wtf 1;go;

! Reset fields
yan 0 0.573 0  ! angles in degrees
xan 0 0.573 0  ! angles in degrees
set epd  ! set entrance pupil diameter
set vig  ! set vignetting factors

! Save lens if requested (this is the default operation)
if "sav = 'Y' or "sav = 'y'
  ^file == concat('d:\scott\lensfiles\',^tit)
  ^file == concat(^file, '.len')
  sav ^file
end if
out t  ! turn screen output back on

B.3 Macros to Calculate, Print, and Plot Data

B.3.1 Calculate Fringe Zernike Coefficients: getzem.seq

! getzem.seq
! Gets zernike coefficients for dome analysis.
! Run global.seq first to define global variables
!
! Coefficients 5,6,10,11,12,13 are negative because Code V calculates
! coefficients with respect to the tangential paraxial image plane.
! For this analysis, the reference image plane is at the
! paraxial sagittal focus.
!
! Input parameters
! #1 == nrd == number of rays across pupil diameter for polynomial fit
!
! Local variables
! ^f ==> field counter
! ^z ==> zoom position counter
!
! Global variables
! ^zn(^z,^f) == array of coefficient values for coefficient n
!
! Begin macro

rfd 51

lcl num ^z ^f ^nrd
^nrd == #1
out t
wri "Getting zernike coefficients..."
out n  ! screen output off
for ^f 1 (num f)  ! Loop through fields
  for ^z 1 (num z)  ! Loop through zoom positions
    ^z1(^z,^f) == zfrcoef(1,^f,^z,1,^nrd,37,'EXP')  ! get coeff.data...
    ^z2(^z,^f) == zfrcoef(1,^f,^z,2,^nrd,37,'EXP')
    ^z3(^z,^f) == zfrcoef(1,^f,^z,3,^nrd,37,'EXP')
    ^z4(^z,^f) == zfrcoef(1,^f,^z,4,^nrd,37,'EXP')
B.3.2 Convert Fringe Zernike Coefficients to Wavefront Polynomials: getwave.seq

\[ \text{out t} \text{ End loop through zooms} \]
\[ \text{out n} \text{ End loop through fields} \]
\[ \text{screen output on} \]

getwave.seq

! Requires prior execution of global.seq and getzem.seq

! This macro converts select wavefront aberration coefficients

! from fringe zernike coefficients.

! Reference: [Wyant 1992] or Chapter 4.2

! Local variables

! \^z == zoom position counter

! \^f == field counter

! lcl num \^z \^f

out t
wri "Converting zemikes into wave expansion coefficients"
out n

for \^z 1 (num z) ! Begin loop through zoom positions
for \^f 1 (num f) ! Begin loop through fields

! first order (piston and tilt)

\[ ^w00(\^z,\^f) = ^z1(\^z,\^f)-^z4(\^z,\^f)+^z9(\^z,\^f) \]
\[ ^w11(\^z,\^f) = \sqrt{^z2(\^z,\^f)-2*^z7(\^z,\^f)} + ^z9(\^z,\^f) \]

! third order and defocus

! Calculate square root terms

\[ ^\text{rootp} = 2*^z4(\^z,\^f)-6*^z9(\^z,\^f)+\sqrt{^z(\^z,\^f)^2+^z6(\^z,\^f)^2} \]
\[ ^\text{rootm} = 2*^z4(\^z,\^f)-6*^z9(\^z,\^f)-\sqrt{^z(\^z,\^f)^2+^z6(\^z,\^f)^2} \]

! Sign of roots determine sign of focus and astigmatism (they are
! opposite one another)

if absf(^\text{rootm}) >= absf(^\text{rootp})
\[ ^w20(z, f) = \sqrt{2} \quad \text{defocus} \]
\[ ^w22(z, f) = -2 \cdot \sqrt{z5(z, f)^2 + z6(z, f)^2} \quad \text{quadratic astig} \]
\[ \text{else} \]
\[ ^w20(z, f) = \sqrt{2} \quad \text{defocus} \]
\[ ^w22(z, f) = 2 \cdot \sqrt{z5(z, f)^2 + z6(z, f)^2} \quad \text{quadratic astig} \]
\[ \text{end if} \]
\[ ^w40(z, f) = z9(z, f) \cdot 6 \quad \text{3rd spherical} \]
\[ ^w31(z, f) = 3 \cdot \sqrt{z7(z, f)^2 + z8(z, f)^2} \quad \text{tang. coma} \]
\[ \text{if } z8(z, f) > 0 \quad \text{check sign} \]
\[ ^w31(z, f) = -^w31(z, f) \quad \text{end if} \]
\[ \text{end for} \quad \text{end loop through fields} \]
\[ \text{end for} \quad \text{end loop through zooms} \]
\[ \text{out t} \quad \text{screen output on} \]

B.3.3 Print Zernike Coefficients to Formatted Table and File: printzrn.seq

! printzrn.seq
!
! Macro to print coefficient data to a formatted table.
! Requires prior execution of global.seq, getzern.seq
! Prints table to the screen and stores it to a separate data file
! coded to surface parameters F, t, and n
!
! Define variables
lcl str ^data ^title !string variables for file name
lcl num ^f ^z !counters
!
! Name output file based on lens title
^file == concat('d:\scott\lensfiles\',(tit))
^file == concat('^file,\'.lis')
out t ^file ! output to screen and data file
wri (tit) ! write lens title
wri 'EPD z1:' (EPD z1) ! write EPD
for ^f 1 (num f) ! loop through fields (1 table for each field)
  wri
  ^data == 'F'd' ('d.dd','d.dd') mrad' ! write field angle
  wri Q^data ^f (xan f^f)*17.453292 (yan f^f)*17.453292
!
! write table header
wri 'LookAngle Z1 Z2 Z3 Z4 Z5 Z6 Z7 Z8* &
  Z9 Z10 Z11 Z12 Z13 Z14 Z15 Z16 *
for ^z 1 (num z) ! loop through zooms
!
! set output format
^data == **'d.dd' 'ddd.ddd' 'ddd.ddd' 'ddd.ddd' 'ddd.ddd'
B.3.4 Plot Zernike coefficients vs. fineness ratio: plotfa.seq

! plotfa.seq

! Macro to plot variation of Fringe Zernike coefficients as a function of look angle and fineness ratio. Similar macros exist for the variation of n and t as well.

! Requires prior execution of global.seq, getzem.seq, printzm.seq

lcl str ^tit ^file ! define string variables
lcl num ^a ^fa ! ^a is a counter, ^fa = fineness ratio

for ^a 0 6 ! ^a is a loop counter
  out n ! screen output off
  ^fa == .5*(^a+1) ! set value for ^fa

! Build search string to locate correct file

  ^tit == concat('fa',substr(num_to_str(^fa*100),1,3))
  ^tit == concat('tit', 'th')
  ^tit == concat('tit', substr(num_to_str(2*100),1,3))
  ^tit == concat('tit', 'n')
  ^tit == concat('tit', substr(num_to_str(150),1,3))
  ^file == concat('d:\scott\lensfiles\', ^tit)
  ^file == concat( ^file, '.lis')
  out t;wri "Getting data from " ^file; out n

! Import file into data buffer

  buf imp bl lis ^file
  buf mov bl il jl ! move to first cell
  buf fnd bl "LookAngle" ! find cell called "LookAngle"
  if (buf.fnd) ! if found move to next row, 1st column
    buf mov bl ic+1 jl
  ! Data is stored in a table
! Column 1 contains look angle data
! Read down the table to get variation of coefficient zn vs. look angle
! Read across table to get different coefficients
for ^c 1 16
  ! loop through look angles
  ^z1(\(^c,^a+1\)) == (buf.num bl ic j2) ! zernike coefficient z1..16
  ^z2(\(^c,^a+1\)) == (buf.num bl ic j3)
  ^z3(\(^c,^a+1\)) == (buf.num bl ic j4)
  ^z4(\(^c,^a+1\)) == (buf.num bl ic j5)
  ^z5(\(^c,^a+1\)) == (buf.num bl ic j6)
  ^z6(\(^c,^a+1\)) == (buf.num bl ic j7)
  ^z7(\(^c,^a+1\)) == (buf.num bl ic j8)
  ^z8(\(^c,^a+1\)) == (buf.num bl ic j9)
  ^z9(\(^c,^a+1\)) == (buf.num bl ic j10)
  ^z10(\(^c,^a+1\)) == (buf.num bl ic j11)
  ^z11(\(^c,^a+1\)) == (buf.num bl ic j12)
  ^z12(\(^c,^a+1\)) == (buf.num bl ic j13)
  ^z13(\(^c,^a+1\)) == (buf.num bl ic j14)
  ^z14(\(^c,^a+1\)) == (buf.num bl ic j15)
  ^z15(\(^c,^a+1\)) == (buf.num bl ic j16)
  ^z16(\(^c,^a+1\)) == (buf.num bl ic j17)
  buf mov bl ic+1 j1 ! move to next row, repeat
end for ! end loop through look angle
end if
buf del bl
end for ! end loop through fineness ratio
out t
gra plotfa ! plot data to screen and plot file
out n

! Open user defined graphics subroutine
UGR
  ^tit == "Fringe Zernike Z3: Tilt About X"
  TIT ^tit ! plot title
  XLA "Look Angle [deg]" ! x axis label
  ^flabel == "Waves @ 1.0 micron"
  YLA ^flabel ! y axis label
  xde $6 ! x increment (6 deg.)
  XMA 90 ! maximum x value (90 deg.)
for ^a 0 6 ! loop through fineness ratio
  ^fa == .5*(\(^a+1\)) ! calculate F
  ^tit == concat('F = ',substr(num_to_str(\(^fa\)),1,4))
  DPO ^tit ! legend label
  SPL PNT ^a+1 bla ! spline fit, symbols, black solid line
  FOR ^z 1 (NUM Z) ! plot coefficient vs. look angle
    (ade s9 z^z) ^z3(\(^z,^a+1\))
  END FOR
end
end for ! end loop through fineness ratio
go ! plot the curves

! Remaining coefficient plots (Z2 to Z16) omitted for brevity
REFERENCES


REFERENCES—Continued


REFERENCES-Continued


