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NONLINEAR WAVE MIXING BETWEEN ATOMIC AND OPTICAL FIELDS

by

Michael Glen Moore

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1999
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Michael G. Moore entitled Nonlinear Wave Mixing Between Atomic and Optical Fields and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director  Pierre Meystre  

12/2/99
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SIGNED: Michael L. Moore
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ABSTRACT

The interaction between Bose-Einstein condensates (BECs) and coherent light fields is treated within the framework of nonlinear wave mixing and studied using techniques developed in the fields of nonlinear and quantum optics. We focus in particular on two situations involving a BEC driven by a strong off-resonant 'pump' laser. First, we consider the case where the laser light is scattered into a single mode of an optical ring cavity. We then consider the case where the light is scattered into the continuum of vacuum modes of the electric field. The first problem is an extension of recent theoretical and experimental work on the so-called collective atomic recoil laser (CARL), whereas the second corresponds to a recent condensate-superradiance experiment performed at MIT.

In the cavity situation, we develop a CARL model in which both the atomic and optical fields are treated fully quantum mechanically. We first show that the previous CARL model, which treats the atomic motion classically, breaks down at the recoil temperature due to the effects of matter-wave diffraction. We then show that when combined with a BEC the CARL can be viewed as a device which parametrically amplifies atomic and optical fields. The existence of entanglement and non-classical intensity correlations between the amplified atomic and optical fields is demonstrated, as well as the ability to manipulate the quantum statistical properties of the matter and light waves by injecting a weak laser field into the
optical cavity to trigger the device.

By replacing the cavity mode with a continuum of modes, we are able to formulate a quantum theory of condensate superradiance in which the scattered light field is eliminated in the Markov approximation. This model shows that condensate depletion leads to mode competition which prevents light scattering in all but the preferred direction(s). The outcome of the mode-competition is highly sensitive to the quantum fluctuations which trigger the phenomenon, resulting in large run-to-run variations in the angular pattern of the superradiant light pulse, an effect which is observed experimentally.
CHAPTER 1

INTRODUCTION

The discovery and development of the optical laser has made a tremendous impact not only on the world of physics, but also, through its many technological applications, on the everyday world as well. Many of the milestones in the history of laser science involve the development of new types of lasers, such as those which operate on novel principles, or which operate in a previously inaccessible region of the electromagnetic spectrum. Even more intriguing, however, is the possibility to achieve 'lasing' using particles other than photons. Quantum physicists have long known that the underlying principle of the laser, bosonic stimulation, is not an exclusive property of photons, but applies to the general class of bosonic particles characterized by an integer spin quantum number. Drawing on this generalization, researchers in several fields have worked on devising laser-like sources of various bosonic particles, two examples being phonons (Asher and Scully 1973; Rivlin 1993) and excitons (Snoke et al. 1990; Link and Baym 1992; Hasuo et al. 1993) in solid state materials. Perhaps the most striking achievement along such lines, however, is the recent demonstration of Bose-Einstein condensates (BEC's) in dilute gases of alkali-metal atoms (Anderson et al. 1995; Davis et al. 1995;
Bradley et al. 1995), which, with the addition of various output coupling mechanisms, are routinely referred to as 'atom lasers'. The development of the atom laser can be expected to revolutionize the field of atom optics, leading to many new developments in areas such as nonlinear atom optics and quantum atom optics. In addition to studying nonlinear and quantum phenomena involving the interaction of matter-waves only, the coexistence of lasers and atom lasers suggests another possibility: to observe direct nonlinear wave mixing and nonclassical phenomenon \textit{between} optical and atomic lasers. The purpose of this dissertation is to explore this new possibility by developing a theoretical framework and analyzing two important applications: the collective atomic recoil laser and condensate superradiance. Particular emphasis is given to the manipulation of quantum statistical properties and the creation of quantum entanglement between atomic and optical fields.

The principal similarity between a BEC and an optical laser is the presence of a large number of identical bosons occupying a single quantum state. In a laser, this high phase-space density leads to the generation of high intensity, monochromatic light. It was the ability to generate such light fields that led to the first experimental demonstration of nonlinear second harmonic generation by Peter Franken and coworkers in 1961 (Franken et al. 1961), the experiment which pioneered the field of nonlinear optics (Shen 1984). In analogy to these developments in optics, the invention of the atom laser has led to the realization of \textit{nonlinear atom optics}, with the first experimental observation of atomic four-wave mixing having
been performed by the group of W. D. Phillips at NIST (Deng et al. 1999).

In the standard approach to nonlinear optics, the nonlinear (atomic) medium is formally eliminated, resulting in effective nonlinear equations which describe the evolution of the electromagnetic field. Under a different set of conditions, and for the special case of an atomic vapor, it has been shown (Lenz et al. 1993; Lenz et al. 1994; Zhang et al. 1994; Zhang and Walls 1994; Castin and Mølmer 1995) that one can reverse the roles of light and matter, and formally eliminate the dynamics of the electromagnetic field, resulting in a nonlinear set of equations which govern the evolution of the atomic field. This then defines the regime of nonlinear atom optics, an area of significant theoretical and more recently experimental interest. Between the regimes of nonlinear optics and nonlinear atom optics lies an intermediate regime where neither the atomic nor optical fields are readily eliminated and must instead be treated on equal footing. This is similar to solving the set of Maxwell-Bloch equations (Meystre and Sargent 1999), only with atomic center-of-mass motion and the effects of photon recoil included. In recent years, we have been among the first to explore the interaction of quantum degenerate atomic and optical fields in this intermediate dynamical regime.

The remainder of this chapter will serve as an introduction and background to research conducted during the preparation of this dissertation. First, an overview of the field of atom optics is presented in Sec. 1.1. This is followed by a discussion of Bose-Einstein condensation and atom lasers in Sec. 1.2. In Sec. 1.3 the topics of nonlinear and quantum atom optics are introduced, with emphasis on the example of atomic four-wave mixing. We then focus on the specific topic of the dissertation
in Sec. 1.4, were the basic formalism for treating the interaction between atomic and optical fields is introduced. Lastly, sections 1.4.1 and 1.4.2 give background information regarding the collective atomic recoil laser and condensate superradiance, the two specific systems which we have investigated in detail.

1.1 Matter waves and atom optics

The concepts of nonlinear atom optics and atom lasers were first proposed by theorists working in the field of atom optics. Just as optics is the study of the generation and manipulation of electromagnetic waves, atom optics is the study of the generation and manipulation of atomic matter waves. Atom optics is typically, but not necessarily, concerned with the case of cold atomic beams where the wave nature of atomic motion plays an important role. In analogy with electromagnetic waves, atom optics too has a ray-optics approximation in which the propagation of the field is determined by following deterministic trajectories in phase space. For massive particles such as atoms, the trajectories of the ray-atom-optics approximation obey Newton's laws of motion. Thus 'semiclassical' theories which treat atomic motion classically are simply the ray-optics approximation to the wave description of atomic motion. Just as in ordinary optics, the ray approximation of matter wave optics breaks down when the characteristic wavelength of the atomic field, which for a thermal atomic sample is the de Broglie wavelength, becomes comparable to the length scale of any atom-optical element in the system.
The notion that massive particles exhibit wave-like behavior was first pos-
tulated by Louis de Broglie in 1923, and served as one of the founding principles
of quantum mechanics. De Broglie's hypothesis was subsequently refined by Er-
win Schrödinger in 1926, who successfully formulated the basic equation governing
matter-wave propagation

$$i\hbar \frac{d}{dt} \psi(\mathbf{r}, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right) \psi(\mathbf{r}, t).$$

(1.1)

When extended to include atomic internal degrees of freedom, Schrödinger's equa-
tion (1.1) serves as the central wave equation in atom optics, analogous to Maxwell's
equations in ordinary optics.

In a complete reversal of the roles played by light and matter in ordinary
optics, it is electromagnetic fields which form the atom-optical elements which guide
and manipulate atomic matter waves. In addition to a reversal of roles, there
are some interesting fundamental differences between atom optics and ordinary
optics. One important difference is that whereas photons have only two 'internal'
polarization states with different angular momentum, atoms have an infinite number
of internal states with different energies and angular momenta. The majority of
these states are unstable due to the interaction with the vacuum and tend decay into
a ground state. Most atoms, however, have degenerate ground states with different
spin angular momentum, all of which are stable and hence behave analogously
to photon polarization states. A second important difference between atoms and
photons is their free-space dispersion relations. The linear dispersion relation of the
photon means that in a vacuum, the group velocity for wavepackets of light is always
c, the speed of light. The quadratic dispersion relation for atoms, on the other
hand, means that in free space atomic wavepackets may propagate with different group velocities, hence, in contrast to laser pulses, atom laser wavepackets can be stationary. This can lead to very long interaction times for nonlinear processes. We remark that similar situations can now be achieved in optics, where coherent manipulation of the optical medium results in very low group velocities for optical pulses (Hau et al. 1999).

The general technique to manipulate the center-of-mass motion of atoms is through the interaction of laser light or magnetic fields with the atomic internal degrees of freedom. Many atom-optical elements, for example, rely on the AC Stark shift or the Zeeman effect (Cohen-Tannoudji et al. 1977), which lead to internal energy shifts when an atom moves into a region of higher (or lower) light or magnetic field intensity. Energy conservation then requires that the internal energy shifts be accompanied by corresponding changes in the atomic velocity, i.e. the electromagnetic fields produce mechanical effects on the atomic motion (Minogin and Letokhov 1987; Kazantsev et al. 1990).

As an example, consider the familiar case of a two-level atom driven far from resonance by a quasi-monochromatic electric field. In this case the wavefunction has two components, labeled $g$ and $e$ for the ground and excited states respectively. In the rotating-wave approximation (Meystre and Sargent 1999) Schrödinger's equation (1.1) then gives

$$i\hbar \frac{d}{dt} \psi_g(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi_g(r, t) - d^* \cdot \mathbf{E}^*(r, t) e^{i\omega t} \psi_e(r, t),$$  

(1.2)
and
\[ i\hbar \frac{d}{dt} \psi_e(r,t) = -\frac{\hbar^2}{2m} \nabla^2 \psi_e(r,t) + \hbar \omega_a \psi_e(r,t) - d \cdot \mathbf{E}(r,t)e^{-i\omega_0 t} \psi_g(r,t). \tag{1.3} \]

where \( d \) is the atomic dipole moment, and the electric field has been decomposed as \( \mathbf{E}(r,t) = \mathbf{E}(r,t)e^{-i\omega_0 t} + \mathbf{E}^*(r,t)e^{i\omega_0 t} \). \( \mathbf{E}(r,t) \) is thus a slowly varying function of time relative to the central frequency \( \omega_0 \).

In order to solve Eq. (1.3), we first express the time-dependent terms as Fourier integrals, giving
\[
\int_{-\infty}^{+\infty} d\omega \left[ \hbar \left( \omega + \Delta + \frac{\hbar}{2m} \nabla^2 \right) \bar{\psi}_e(r,\omega + \omega_0) - \bar{f}(r,\omega) \right] e^{-i(\omega + \omega_0)t} = 0, \tag{1.4} \]

where \( \Delta = \omega_0 - \omega_a \) is the detuning, and \( \bar{f}(r,\omega) \) is the Fourier transform of \( f(r,t) = -d \cdot \mathbf{E}(r,t)\psi_g(r,t) \). For ultracold atoms tuned far away from resonance, the kinetic energy term in Eq. (1.4) is small compared to the detuning \( \Delta \) and can be neglected, in which case Eq. (1.4) has the solution
\[
\bar{\psi}_e(r,\omega + \omega_0) = C(r)\delta(\omega + \Delta) + \frac{\bar{f}(r,\omega)}{\hbar(\omega + \Delta)}. \tag{1.5} \]

The delta function gives to the free evolution of the excited state, the function \( C(r) \) being a free parameter determined by the initial conditions. We then make the assumption that \( f(r,t) \) varies on a time scale which is much longer than \( 1/\Delta \), i.e. we assume that \( \bar{f}(r,\omega) \approx 0 \), unless \( \omega \ll \Delta \). With this approximation, it follows that
\[
\psi_e(r,t)e^{i\omega_0 t} = \psi_e(r,0)e^{i\Delta t} - \frac{f(r,0)}{\hbar \Delta} e^{i\Delta t} + \frac{f(r,t)}{\hbar \Delta}. \tag{1.6} \]

The first term in Eq. (1.6) vanishes provided that the atom is initially in the ground state, while the second term can be neglected as it is rapidly oscillating and thus its
time-averaged effect will be negligible relative to the third term. We keep, therefore, only the third term in Eq. (1.6), which when substituted back into Eq. (1.2) gives

\[ i\hbar \frac{d}{dt}\psi_g(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi_g(r, t) + \frac{|d \cdot \mathcal{E}(r, t)|^2}{\hbar \Delta} \psi_g(r, t). \]  

(1.7)

This shows that the effect of the light fields is to introduce a so-called ‘optical potential’, which guides the propagation of the ground state wavefunction.

These methods of manipulating atomic trajectories are not new. They can be traced back to the earliest days of atomic physics, a well-known example being the magnetic beam splitter demonstrated by Stern and Gerlach in 1922 (Gerlach and Stern 1922). In more recent years there has been considerable progress and refinement of these ideas. Spectacular progress in laser cooling (Cohen-Tannoudji 1992) has had a tremendous impact, and many atom-optical elements have been demonstrated experimentally. Lenses and mirrors for atomic beams, for example, have been devised using the intensity profile of a focused laser beam (Bjorkholm et al. 1978). Atom mirrors have also been demonstrated using evanescent light fields (Kasevich et al. 1990; Aminoff et al. 1993) and magnetic fields (Roach et al. 1995). Wave guides for neutral atoms have likewise received considerable attention. One technique is to guide atoms using the evanescent light field inside a hollow optical fiber (Renn et al. 1995; Renn et al. 1996). Another interesting technique involves guiding cold atoms in the magnetic fields generated by current carrying wires microfabricated on a chip (Dekker et al. 1999). This latter technique opens the door to integrated atom optics circuits. Lastly, we mention the possibility to form matter-wave resonators, e.g. two atomic mirrors can be combined to form a Fabry Pérot resonator for atoms (Balykin and Letokhov 1989a; Wilkens et al.
16

1993). More generally, any type of trap for neutral atoms can be considered as a matter-wave resonator and since a BEC is the atom-optics analog to the laser, it is interesting to note the important and analogous role that resonator cavities play in both devices.

In this research, the principal atom optical element employed is an atomic matter-wave diffraction grating (Moskowitz et al. 1983; Martin et al. 1988). Diffraction gratings for atomic beams can be formed, for example, from the standing wave light fields generated by counterpropagating laser beams. The optical potential forms a periodic structure which then affects the atomic center-of-mass motion much the same way as a periodic grating alters the propagation of light waves. The diffraction of ground state atoms by a far off-resonant standing wave results in the creation of momentum sidemodes separated in momentum space from the initial atomic momentum distribution by integer multiples of twice the photon recoil. In the near field, the quantum interference of these momentum sidemodes results in a periodic modulation of the atomic density, whereas in the far field, the various diffraction orders become spatially separated. There are several excellent review articles and books on atom optics and/or atom interferometry. Good examples are (Balykin and Letokhov 1989b; Adams et al. 1995), as well as (Berman 1996).
1.2 BEC and atom lasers

Physicists studying atom optics realized early on the importance of a laser-like source of atomic matter waves. While theorists worked in pursuit of this goal, several groups of experimentalists were attempting to create Bose-Einstein condensates (Bose 1924; Einstein 1924; Einstein 1925) out of alkali-metal atomic vapors. Eventually both groups came together, and the BEC is currently accepted as the atomic analog to the laser. A BEC is a unique macroscopic quantum state of matter in which a large number of identical bosonic particles occupies a single quantum state. In the alkali-metal atomic vapors the BEC is formed by evaporatively cooling atoms in a magnetic trap (although efforts are underway to achieve BEC in purely optical traps). As the temperature falls below a critical temperature $T_c$ a phase transition occurs and the BEC begins to develop. At first, the fraction of atoms in the BEC is very small, however, well below $T_c$ there is no measurable non-condensed fraction, and for most purposes, the sample can be regarded as a pure condensate. While Bose-Einstein condensate is also accepted as the mechanism for superfluidity in He$^4$, the achievement of BEC in atomic vapors is a milestone because it allows for the first time to observe the phenomenon in a weakly interacting system.

Around the same time as the initial discovery of BEC in alkali-metal atomic vapors (Anderson et al. 1995; Davis et al. 1995; Bradley et al. 1995), we and other theorists proposed various schemes for a coherent atomic beam generator, or atom laser (Spreeuw et al. 1995; Wiseman and Collett 1995; Guzman et al. 1996; Holland et al. 1996). At that time there was considerable debate over whether or not a BEC
qualified as an atom laser. While both clearly involved a macroscopic occupation of atoms in a single quantum state, this was not necessarily a sufficient criterion. It was thought that if a laser is defined more by its coherence properties than by simply being an intense, monochromatic source of photons, that an atom laser should generate a matter wave in a coherent state. The main point of the debate, therefore, involved the number statistics of the BEC. One argument was that, as a thermal equilibrium state, the BEC should have number fluctuations which scale roughly as the mean atom number, even in the limit of zero temperature. This is borne out by statistical calculation using the grand canonical ensemble, an approach which clearly applies to an ensemble of condensates with identical temperatures and mean atom numbers. It was then pointed out that as a closed system a condensate, together with the noncondensed fraction of atoms, has a fixed number of atoms, so that the canonical ensemble was the proper choice (Wilkens and Weiss 1997; Grossmann and Holthaus 1997). This gives the result that the number fluctuations become small compared to the mean atom number as the temperature falls well below the critical temperature. This approach would seem to apply to repeated measurements on a single condensate, for which the total number of condensed and noncondensed atoms remains fixed (ignoring whatever loss mechanisms operate in a nonideal magnetic trap). Both results, however, seem quite different from the coherent state of an ordinary laser.

As these issues were being debated at an atom laser workshop in Tucson, Hans Meisner, a post-doc in the group of Wolfgang Ketterle at MIT, made the
announcement that the MIT group had outcoupled blobs of atoms from a magnetically trapped BEC using rf pulses. This was advertised to the world as the first pulsed atom laser (Mewes et al. 1997). At this point, Ketterle's group had also successfully demonstrated interference between two BEC's (Andrews et al. 1997) in a separate experiment, thus proving at least first-order coherence. With these results in mind, most in the field followed Ketterle's lead, and accepted that a macroscopically occupied, single-mode atomic field, exhibiting any type of coherence would be considered as an atom laser. The idea of generating a coherent atomic beam by outcoupling atoms from a BEC was subsequently taken up and refined by several groups (Hagley et al. 1999), with quasi-CW (Bloch et al. 1999) and mode-locked (Anderson and Kasevich 1998) atom lasers recently demonstrated in a series of beautiful experiments.

While no longer part of the atom laser debate, the issue of the coherence properties of a BEC clearly remains an important area of research. Theorists have taken up the challenge of characterizing the coherence of atomic matter-waves (Goldstein and Meystre 1998a; Goldstein et al. 1998; Naraschewski and Glauber 1999). The question of the utility of these theoretical developments, however, still awaits the arrival of experiments aimed at determining the higher order coherence properties of BEC's. There has been some experimental progress in this direction, with somewhat indirect evidence of higher-order coherence seen in 3-body collision experiments performed at JILA (Burt et al. 1997). One of the aims of the present work is to develop a means to control the coherence properties of atomic matter waves. As we will see, this can be achieved through the interaction of a BEC with
coherent optical fields.

There exists a considerable body of work dealing with the phenomenon of Bose-Einstein condensation in alkali-metal atomic vapors. BECs have been demonstrated using sodium (Davis et al. 1995; Hau et al. 1997; Kozuma et al. 1999), rubidium (Anderson et al. 1995; Anderson and Kasevich 1998; Ernst et al. 1998; Esslinger et al. 1998; Han et al. 1998; Söding et al. 1999), lithium (Bradley et al. 1995), and most recently atomic hydrogen (Fried et al. 1998). Lithium condensates are particularly interesting because they have a negative scattering length and are therefore unstable above a critical density. Topics of considerable current interest to both experimentalists and theorists include: elementary excitations (Jin et al. 1996; M.-O. et al. 1996; Edwards et al. 1996; Singh and Rokhsar 1996; Stringari 1996; Stamper-Kurn et al. 1999), multi-component condensates (Goldstein and Meystre 1997; Hall et al. 1998; Matthews et al. 1998; Meisner et al. 1999; Stenger et al. 1998), vortices (in addition to being an atom laser, a BEC is also a superfluid) (Dalfovo and Stringari 1996; Dodd et al. 1997; Marzlin et al. 1997; Rokhsar 1997; Dum et al. 1998; Goldstein and Meystre 1998b; Feder et al. 1999; Matthews et al. 1999; Pu et al. 1999), and nonlinear atom optics (Goldstein et al. 1995; Goldstein et al. 1996; Trippenbach et al. 1998; Goldstein and Meystre 1999a; Goldstein and Meystre 1999b; Deng et al. 1999). For an excellent review of experimental aspects of BEC see (Ketterle et al. 1999) and the many references therein. For a review of the theory of Bose-Einstein condensation in dilute trapped gases see (Pethick and Smith 1997) and (Dalfovo et al. 1999) and references therein.
1.3 Nonlinear and quantum atom optics

Many of the ideas in atom optics involve reversing the roles light and atoms play in traditional optics. It was thinking along these lines which led to the original formulation of nonlinear atom optics (Lenz et al. 1993; Lenz et al. 1994; Zhang et al. 1994; Zhang and Walls 1994; Castin and Mølmer 1995). In standard nonlinear optics (Shen 1984), the formal elimination of the atomic dynamics leads to effective multi-photon interactions, and hence the optical field propagation becomes nonlinear. The pioneers of nonlinear atom optics realized that under a different set of conditions the dynamics of the electromagnetic field could be formally eliminated, in which case the atomic field would obey a nonlinear equation describing effective atom-atom interactions.

At first, nonlinear atom optics focussed on the situation of an atomic beam propagating in the presence of a laser light field, which was then shown to induce a nonlinearity due to the long-range dipole-dipole interaction. In this scheme care must be taken to avoid spontaneous emission which can lead to heating and decoherence of the atomic center-of-mass motion. Fortunately, whereas photon-photon interactions generally occur only in strongly polarizable media, atom-atom interactions occur even in free space. This means that a laser field is not needed to realize nonlinear atom optics. Collisions between ground state atoms in the absence of light are sufficient. As all atomic collisions are in fact mediated by the electromagnetic field, it is possible to view the electromagnetic vacuum itself as a nonlinear medium for the atomic field.
The dominant collision mechanism between ground-state atoms is the two-body Van der Waals interaction. It is a very short range interaction, decaying as $1/r_{12}^6$, where $r_{12}$ is the interatomic separation. For ultracold atoms, it is a well-known theoretical result that only s-wave scattering plays a significant role (Landau and Lifshitz 1958), in which case the interaction potential can be approximated by a zero-range pseudo-potential (Huang 1987)

\[ \langle r_1 r_2 | V | \psi \rangle = \frac{4\pi \hbar^2}{m} \delta^3(r_1 - r_2) \frac{\partial}{\partial r_{12}} \langle r_{12} | r_1 r_2 | \psi \rangle, \quad (1.8) \]

where $r_{12} = |r_1 - r_2|$, and $a$ is the scattering length of the exact potential. If $\langle r_1 r_2 | \psi \rangle$ is regular close to $r_1 = r_2$ then the interaction potential is simply a delta function. If $\langle r_1 r_2 | \psi \rangle$ diverges as $1/r_{12}$, as in the case of an outgoing spherical wave, e.g., then the pseudo-potential is regularized to eliminate the divergent part of $|\psi\rangle$. The use of the pseudo-potential approximation (1.8) is valid provided that the atomic de Broglie wavelength is large compared to the scattering length. In the recent experimental demonstration of nonlinear atom optics performed at NIST (Deng et al. 1999), sodium atoms ($a = 2.5 \times 10^{-9}$ m) traveling at the photon recoil velocity ($\lambda \approx 10^{-7}$ m) were used, in which case Eq. (1.8) can be safely employed.

Strictly speaking, both nonlinear optics and nonlinear atom optics are only truly nonlinear when considered as classical fields. When treated fully quantum-mechanically, the linear superposition principle always applies to the quantum state vector. Traditionally, the word ‘classical’ when applied to massive particles refers to Newtonian mechanics, which we now understand as the ray-optics approximation to the mean-field theory. At this point, we introduce a slightly non-standard terminology, and argue that the mean-field theory is also ‘classical’, in the sense
that it is the atom-optics analog to the classical theory of electromagnetic waves. It is quite common to see the classical equations for electromagnetic waves 'derived' from the equations for quantized fields by taking $\hat{E}(\mathbf{r}, t) \rightarrow E(\mathbf{r}, t)$, in which case $E(\mathbf{r}, t)$ is often interpreted as the expectation value of the quantized field. This is also the quick and dirty method to 'derive' a nonlinear Schrödinger equation (NLSE) from the quantum field theory of interacting atoms, i.e. replacing $\hat{\Psi}(\mathbf{r}, t)$ with the c-number $\varphi_0(\mathbf{r}, t)$. Despite the fact that the NLSE equation, which we derive shortly, contains $\hbar$, it is clearly analogous to the classical field equations of nonlinear optics. In most cases the classical-field approximation requires that the fields be strongly degenerate, e.g. in optics we generally ascribe classicality to a mode, or Hilbert-space state, whose Fock-space state is a Glauber coherent state (Glauber 1963) with a large mean occupation number. Such a mode would then be described as a c-number, even while other parts of the field are treated quantum mechanically.

We now treat as an example the case of a scalar atomic field, i.e. one involving only a single internal atomic energy level. We start with a full many-body theory in the formalism of second quantization, and then briefly discuss the approximations which lead to a 'classical', or mean-field theory. We will also discuss under which conditions the atomic field ceases to be well-described 'classically', thus introducing the regime of quantum atom optics.

The second-quantized Hamiltonian for atoms subject to the two-body
pseudo-potential \((1.8)\) is given by
\[
\hat{\mathcal{H}} = \int d^3r \hat{\Psi}^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + \frac{2\pi \hbar^2 a}{m} \hat{\Psi}^\dagger(r)\hat{\Psi}(r) \right] \hat{\Psi}(r),
\]
where we have assumed that the regularization of the delta-function potential is not required. The potential \(V(r)\) includes any atom-optical elements, e.g. a trapping potential and/or diffraction grating. Assuming bosonic atoms, the atomic field operators obey the commutation relations
\[
\begin{align*}
\left[ \hat{\Psi}(r), \hat{\Psi}^\dagger(r') \right] & = \left[ \hat{\Psi}^\dagger(r), \hat{\Psi}^\dagger(r') \right] = 0, \\
\left[ \hat{\Psi}(r), \hat{\Psi}(r') \right] & = \delta^3(r - r'),
\end{align*}
\]
from which, together with Eq. \((1.9)\), the Heisenberg equation of motion for the atomic Schrödinger field operator
\[
\frac{d}{dt} \hat{\Psi}(r) = \frac{i}{\hbar} \left[ \hat{\mathcal{H}}, \hat{\Psi}(r) \right]
\]
\[
= i \left[ \frac{\hbar}{2m} \nabla^2 - \frac{V(r)}{\hbar} - \frac{4\pi \hbar a}{m} \hat{\Psi}^\dagger(r)\hat{\Psi}(r) \right] \hat{\Psi}(r)
\]
can be derived. As mentioned, the quickest way to derive a mean field theory is to take the expectation value of Eq. \((1.11)\), and then make the factorization ansatz
\[
\langle \hat{\Psi}^\dagger(r)\hat{\Psi}(r)\hat{\Psi}(r) \rangle \approx \langle \hat{\Psi}^\dagger(r) \rangle \langle \hat{\Psi}(r) \rangle \langle \hat{\Psi}(r) \rangle.
\]
With the additional assumption that all atoms are in the state \(\varphi_0(r)\) we then take
\[
\langle \hat{\Psi}(r) \rangle = \sqrt{N_0} \varphi_0(r),
\]
where \(N_0\) is the mean number of atoms. One then arrives at the time-dependent Gross-Pitaevskii equation (GPE) (Gross 1961; Gross 1963; Pitaevskii 1961)
\[
\frac{d}{dt} \varphi_0(r) = i \left[ \frac{\hbar}{2m} \nabla^2 - \frac{V(r)}{\hbar} - \frac{4\pi \hbar a}{m} N_0 |\varphi_0(r)|^2 \right] \varphi_0(r).
\]
This type of equation has been used extensively to model BEC experiments, and is generally a very good approximation, provided that the mean occupation number \( N_0 \) is large and that fraction of non-condensed atoms is negligible. We note that the ansatz (1.13) is exact for the case where the field mode \( \varphi_0(r) \) is in a coherent state. Such a state will not, however, remain a coherent state when it evolves under the GPE (1.14), leading to effects such as collapses and revivals (Wright et al. 1996) on sufficiently long time scales.

Typically, corrections for states other than coherent states scale like \( 1/N_0 \), so that for fields with high mean occupation numbers, Eq. (1.14) is generally valid for the mean field, regardless of the precise nature of the quantum state. In some cases it is known \( a \) priori that the mean field is precisely zero. In these cases, the mean field theory may still be employed provided that \( \langle \hat{\psi}(r) \rangle \) is only used as an intermediate quantity, appearing in the factorization of non-zero expectation values such as \( \langle \hat{\psi}^\dagger(r)\hat{\psi}(r) \rangle \). This is the spontaneous symmetry breaking ansatz, familiar from laser theory as well as many other branches of physics involving quantum field theory.

The GPE (1.14) is analogous to the optical paraxial wave equation in a third-order nonlinear optical medium. In the optical version, however, \( t \) is replaced by \( z \), the axis of propagation, and the three-dimensional Laplacian is replaced by the two-dimensional transverse Laplacian. From this analogy it can be inferred that the GPE supports bright soliton solutions (Scott et al. 1973; Lamb 1980; Agrawal 1995) in the self-focusing regime \( (\alpha < 0) \). There have been many papers on atomic solitons, including many of the earliest papers on nonlinear atom optics (Lenz et al.
1994; Zhang et al. 1994). A particularly interesting soliton proposal was recently put forward by Zobay and coworkers (Zobay et al. 1999) in which a periodic optical potential is used to induce a negative effective atomic mass near the band edge. This allows the formation of bright gap solitons in the more common case of a positive scattering length \( a > 0 \).

Our goal is to consider the interaction between atomic and optical fields as a type of nonlinear four-wave mixing, hence we shall briefly review the example of atomic four-wave mixing so as to establish some basic concepts. Considerable insight into the nonlinear phenomenon of four-wave mixing can be obtained by expanding the mean field \( \varphi_0(r) \) onto a set of basis functions. As an example, consider an expansion onto plane waves,

\[
\varphi_0(r, t) = V^{-1/2} \sum_k e^{i\mathbf{k} \cdot \mathbf{r}} c_k(t),
\]

where \( V \) is the quantization volume. Substituting this expression into Eq. (1.14) and setting \( V(r) = 0 \) yields the following equation for the coefficient \( c_k(t) \)

\[
\frac{d}{dt} c_k = -i \frac{\hbar k^2}{2m} c_k - i \frac{4 \pi \hbar a N_0}{V} \sum_{lmn} \delta_{k+l,m+n} c_l^* c_m c_n
\]

From this equation we see that nonlinearity combines the three modes \( l, m, \) and \( n \) to influence the dynamics of the mode \( k \) provided that the phase-matching condition \( k + l = m + n \), i.e. momentum conservation, is satisfied. In particular, if only three fields are present initially, than a fourth field may be generated spontaneously. This is the process referred to as four-wave mixing. A straightforward application of energy and momentum conservation laws will show that all four-wave mixing governed by Eq. (1.16) is degenerate, i.e. in some frame of reference all four
wavevectors have the same magnitude and come in counterpropagating pairs.

In many situations the classical field theory no longer suffices. At best, it simply says nothing about phenomena which cannot be explained in terms of the mean field alone. At worst, it no longer gives even the correct mean field dynamics due to a breakdown of the factorization ansatz. In order to go beyond the mean-field approximation one must enter the regime of quantum atom optics, in which the full quantum state of the atomic field is taken into account. In quantum optics, the definition of classical states is generalized to include not only coherent states, but also states which can be described as a statistical mixture of coherent states. According to this definition a non-classical state is any state which cannot be so described. Using a fully quantum mechanical approach, a main result of this dissertation is to show that such states are generated by the interaction of coherent atomic and optical fields.

Just as with the optical field, the atomic field may consist of separate components, some of which are treated classically and some of which are treated fully quantum mechanically, In this case the field amplitudes to be quantized are reverted back to operator form, where the annihilation operator for a specific quantum state \( \phi_m(\mathbf{r}) \) is defined by

\[
\hat{c}_m = \int d^3\mathbf{r} \phi_m^*(\mathbf{r}) \hat{\Psi}(\mathbf{r}).
\]  

(1.17)

By making use of (1.10), it is seen that the atomic field operators obey the commutation relations

\[
[\hat{c}_m, \hat{c}_n^\dagger] = [\hat{c}_m^\dagger, \hat{c}_n^\dagger] = 0,
\]
This shows how bosonic commutation relations generalize when applied to nonorthogonal states.

One type of non-classical state which has received a great deal of attention in recent years is the entangled state. An entangled state of a two-mode field is one whose density matrix cannot be factorized into a product of density operators for the individual modes. The principal device used for the generation of entangled states between optical fields is the optical parametric amplifier (Walls and Milburn 1994), i.e. a down-conversion crystal. It is used, e.g., in fundamental tests of quantum mechanics (Einstein et al. 1935; Clauser et al. 1969; Aspect et al. 1982), quantum cryptography (Ekert 1991; Rarity et al. 1994), and quantum teleportation (Bouwmeester et al. 1997; Braunstein and Kimble 1998). One of the primary results of this research is to devise a similar device to create entanglement between atomic and optical fields. The full quantum model, in addition to giving complete information about the state of the system, also shows how the system is triggered from quantum noise, and allows us to find the correct initial conditions for the mean-field equations.

1.4 Four-wave mixing with atoms and light

Now that we have reviewed the important aspects of atom optics, atom lasers, and four-wave mixing, we are ready to address our central topic, the interaction between quantum-degenerate atomic and optical fields. We begin by casting the light-atom
interaction in the form of a cubic nonlinearity which describes four-wave mixing between atomic and optical fields. In Sec. 1.4.1 we then consider the interaction of an ultracold atomic sample with two counterpropagating running wave light fields. One of these light fields, the pump, is an externally injected strong laser field, and the other, the probe, is an excited mode of an optical ring-cavity. In Sec. 1.4.2 we alter the model by replacing the probe mode with the continuum of vacuum modes of the electromagnetic field. These two situations, which correspond to the limiting cases where there is either an ideal cavity with no losses, or no cavity at all, are examined in detail in this dissertation. The intermediate regime, a cavity with significant losses, shares features with both extremes and has yet to be explored extensively. Combining the approaches used in the limiting cases, however, will be sufficient to model this intermediate regime.

In all cases, we assume that the pump laser is detuned far enough away from the atomic resonance so that the incoherent effects of spontaneous emission may be safely neglected. The starting point for either of the two cases is then Eq. (1.7), which describes the interaction between ground state atoms and a far-off resonant electric field. The first step is to decompose the electric field into pump and probe fields according to

$$d \cdot \mathbf{E}(\mathbf{r}, t) = \sum_{\ell} \hbar g_{\ell} a_{\ell}(t) e^{i \mathbf{k}_\ell \cdot \mathbf{r}},$$

(1.19)

where $g_{\ell}$ is the atom-pump coupling coefficient for the $\ell$th mode, and $a_{\ell}(t)$ and $k_\ell$ are the corresponding dimensionless field amplitude and wavenumber, respectively. For the case of a single probe cavity mode, the summation in (1.19) includes two modes, the pump and the probe, while in the free space limit it includes the pump
laser as well as the continuum of vacuum modes, in which case the sum should technically be replaced by an integral.

Substituting expression (1.19) in Eq. (1.7) and replacing \( \psi_g(\mathbf{r}) \) by the mean-field \( \varphi_0(\mathbf{r}) \) then gives the classical equation of motion for the atomic field

\[
\frac{d}{dt} \varphi_0(\mathbf{r}, t) = i \frac{\hbar}{2m} \nabla^2 \varphi_0(\mathbf{r}, t) - i \sum_{\ell m} \frac{g^2 g_m}{\Delta} e^{-i(k_r - k_m) \cdot \mathbf{r}} a_\ell^*(t) a_m(t) \varphi_0(\mathbf{r}, t).
\] (1.20)

Treating the atomic and optical fields on equal footing, we next expand the atomic field onto a set of orthonormal basis states according to

\[
\varphi_0(\mathbf{r}, t) = \sum_k \phi_k(\mathbf{r}) c_k(t).
\] (1.21)

The equations of motion for the amplitudes \( \{c_k(t)\} \) then take the form

\[
\frac{d}{dt} c_k(t) = -i \sum_\ell T_{k\ell} c_\ell(t) - i \sum_{\ell m n} V_{k\ell mn} a_\ell^*(t) a_m(t) c_n(t),
\] (1.22)

where

\[
T_{k\ell} = -\frac{\hbar}{2m} \int d^3 \mathbf{r} \phi_k^*(\mathbf{r}) \nabla^2 \phi_\ell(\mathbf{r}),
\] (1.23)

and

\[
V_{k\ell mn} = \frac{g^2 g_m}{\Delta} \int d^3 \mathbf{r} \phi_k^*(\mathbf{r}) e^{-i(k_r - k_m) \cdot \mathbf{r}} \phi_n(\mathbf{r}).
\] (1.24)

For the optical field amplitudes, the Maxwell wave equation (Cohen-Tannoudji 1989) takes the form

\[
\frac{d}{dt} a_\ell(t) = -i \delta_\ell a_\ell(t) - i \sum_{k m n} V_{k\ell mn} c_k^*(t) a_m(t) c_n(t),
\] (1.25)

where \( \delta_\ell = c|k_\ell| - \omega_0 \) is the detuning between the frequency of the \( \ell \)th mode and that of the pump laser. By comparing Eqs. (1.22) and (1.25) to Eq. (1.16), we see
clearly that the interaction between the optical and atomic fields takes the form of a third-order nonlinearity, and under appropriate circumstances can therefore lead to four-wave mixing between atomic and optical fields.

Equations (1.22) and (1.25) are the 'classical' mean-field equations. They can readily be quantized, however, by replacing the atomic and optical field expansion coefficients with bosonic annihilation and creation operators. It is not necessary, however, to treat all of the field modes quantum mechanically. It is sufficient to treat the pump laser classically, for example, due to the fact that is in a coherent state of high intensity. Because it is an external CW light field and we are working far-off resonance, its amplitude can furthermore be approximated as constant in space and time. In the case where a BEC is present, the atomic field mode corresponding to the BEC ground-state wavefunction may also be treated classically.

The effect of the light fields is to transfer atoms out of the condensate ground state and into other states via photon recoil. The condensate, unlike the pump laser, is not being continuously replenished from the outside, hence the condensate field amplitude will generally decrease with time. For times short enough that condensate depletion remains negligible, however, we can still make an 'undepleted-pump' approximation and neglect the time dependence of the condensate ground state. With both the pump laser and condensate ground-state amplitudes taken as constant c-numbers, the equations for the other atomic and optical field modes then contain linear terms which give the dominant contribution. Thus the regime of negligible condensate depletion is in fact the linear regime, where
the dynamics of the remaining field modes must be either sinusoidal or exponential.

It is the case of exponential solutions which is most interesting, as it is this regime where four-wave mixing can amplify quantum fluctuations to create new fields which then grow to macroscopic size. If, as in the NIST atomic four-wave mixing experiment, three fields are initially present, then the cubic nonlinearity can generate a fourth field. If only two fields are present initially, four-wave mixing can still occur. Both the third and fourth waves, however, must be generated simultaneously. In the opto-atomic four-wave mixing considered here, matter and light fields are generated in pairs through the amplification of quantum fluctuations in both the electric field and the atomic density. Each of these new fields then interacts with the original two, resulting in mutual amplification via four-wave mixing. The situation can become slightly more complex due to the fact that more than two new fields can be spontaneously generated. This leads to mode competition, as is seen in the free-space model described in Sec. 2.3.

1.4.1 The BEC Collective Atomic Recoil Laser

The first situation we consider is that of an atomic field interacting only with the pump laser and a single optical cavity mode. This configuration was first proposed by Bonifacio and coworkers in Milan (Bonifacio and De Salvo 1994; Bonifacio et al. 1994), who called it the Collective Atomic Recoil Laser (CARL). The CARL has been studied extensively in the ray-atom-optics regime by Bonifacio and collaborators (Bonifacio and De Salvo 1995; De Salvo et al. 1995; Bonifacio and Verkerk
As they considered the atoms as classical particles, the generation of light in the probe mode was not interpreted as being due to four-wave mixing. Instead, an alternate but complementary picture was devised. In this interpretation the atoms move in the periodic potential of the counterpropagating pump and probe fields, resulting in a bunching of the atoms as they move towards the potential minima. The atoms thus form a density grating, off which pump photons scatter into the probe mode. As the probe amplitude then increases, the optical potential becomes deeper. This leads to increased bunching and even stronger photon scattering. This positive feedback mechanism leads to exponential growth of both the probe field and the atomic bunching. The mechanism of the CARL is therefore analogous to that of the Free Electron Laser (FEL) (Brau 1990), with the difference that the FEL uses a periodic magnetic field in place of the pump laser. In the rest frame of the relativistic electron beam, however, the magnetic 'wiggler' field appears as a classical running wave electric field, just as the pump laser is seen by the atoms in the CARL.

Various experiments involving the CARL in the ray-atom-optics regime have been conducted recently. Using a sample of hot (room temperature) sodium atoms, Lippi et al (Lippi et al. 1996; Verkerk et al. 1997) observed amplification of an injected probe laser, which they interpreted in terms of scattering off an atomic density grating resulting from atomic center-of-mass motion. Also in a sodium atomic cell, Hemmer et al (Hemmer et al. 1996) reported probe gain in the absence of an injected signal, and interpreted it as resulting from the CARL mechanism. There has, however, been some controversy regarding the interpretation of these
experiments (Brown et al. 1997), mostly concerning the presence of a large Doppler
broadening, and the possibility of gain mechanisms not necessarily related to atomic
recoil.

These difficulties can be surmounted by using ultracold atoms for which
Doppler broadening effects are negligible. This, however, requires going to subre­
coil temperatures, at which point the ray-atom-optics approximation breaks down.
This breakdown can be understood by noting that the atom-optical element in­
volved is a matter-wave diffraction grating whose period is given roughly by the
optical wavelength. As the de Broglie wavelength of a subrecoil cooled atom is
longer than the optical wavelength, the ray-atom-optics approximation is clearly no
longer applicable. Given the spectacular recent progress witnessed by atomic cool­
ing techniques, it is highly likely that CARL experiments using ultracold atomic
samples can and will be performed in the future. For this reason we have devel­
oped a wave-atom-optics theory of the CARL (Moore and Meystre 1998), where
we confirmed that the ray-optics version did indeed break down for temperatures
of the order of the recoil temperature or below. In fact, at \( T = 0 \), a second thresh­
old for the existence of the exponential instability was discovered, occurring when
the bunching process is overcome by matter-wave diffraction. For \( T > T_R \), the
recoil temperature, it was shown that ray-atom-optics theories make indistinguish­
able predictions from the quantum theory. We remark that while in this work the
atomic center of mass motion was treated quantum mechanically, the light fields
were still treated classically, hence predictions concerning the quantum statistics of
either the atomic or optical fields could not be made.
With the ultimate intent of extending the CARL theory into the BEC regime so that the unique coherence properties of condensates might be further understood and exploited by the interaction with dynamical light fields, a fully quantum mechanical model was then formulated (Moore and Meystre 1999a). In that paper, the subjects of manipulating quantum statistics and and atom-photon entanglement were first addressed. This work was later extended and refined in a more detailed manuscript (Moore et al. 1999). As we shall see, when the atomic motion is treated fully quantum mechanically the CARL can be considered as a tool for generating entanglement between coherent atomic and optical fields. We note that an alternate attempt at a fully quantum $T = 0$ model (Bonifacio 1998) failed because the authors relied too heavily on the ray-atom-optics approximation. They assumed that atoms followed classical trajectories, but with 'quantum' fluctuations about them included. That technique, however, can not be applied at $T = 0$, as that approach then requires that the atoms be localized in both position and momentum to an extent which violates the Heisenberg uncertainty principle.

Within the framework of the wave-atom-optics model, a new interpretation of the CARL gain mechanism was formulated. Assuming that the atomic field consists initially of a trapped BEC, the spontaneous scattering of a photon from the pump field into the probe mode transfers an atom from the condensate ground state to a new state that is shifted in momentum space by the two-photon recoil. This new state constitutes a second condensate component, which can be considered as a momentum sidemode to the original condensate. As this sidemode is populated it begins to interfere with the original condensate. The interference fringes are then
seen by the pump and probe fields as a spatial density grating, which enhances the photon scattering process. This interplay between interference fringes and scattering can act as a positive feedback mechanism, which leads to instability and exponential growth. In nonlinear optics, such an instability is commonly referred to as a 'modulational instability'. Any small signal, including quantum noise, will be sufficient to trigger the instability, resulting in the generation of exponentially growing sidemode and probe fields. Of course this exponential growth is eventually reversed by high intensity effects, so that the long-time dynamics is characterized by large-amplitude nonlinear oscillations.

The fully quantum model is similar in many ways to a system first studied by Zeng and coworkers (Zeng et al. 1995), and later extended by Kuang (Kuang 1998), in which the principle of manipulating the quantum statistics of a condensate by its interaction with a quantized light field was first proposed. These papers, however, do not recognize the existence of unstable (exponential) solutions nor the fact that the system can be triggered from quantum noise, both crucial components of this present work. Lastly, we mention the connection to recent work on matter-wave amplification by Law and Bigelow (Law and Bigelow 1998), which also explores the interaction between condensates and quantized light fields. In that work, however, the light field is assumed to be heavily damped, thus allowing for its dynamical elimination. As a result, only the properties of the atomic field are studied in detail. We adopt a similar approach in the next section, but rather than consider the scattering of light into a single, heavily-damped cavity mode, we consider the free-space situation having a continuum of modes. In the work of Law and Bigelow
the optical ring cavity selects the photon-scattering direction, thus precluding a description of the condensate superradiance effect which we now turn to.

1.4.2 Condensate superradiance

"It was late in the night," begins Wolfgang Ketterle in describing his recent condensate superradiance experiment (Inouye et al. 1999) to a writer for Physics Today. "We saw beams of atoms shooting out of the condensate. In five to ten minutes, we concluded that we had a new phenomenon. It was really a surprise."

What Ketterle's group had observed was the directional Rayleigh scattering of far-detuned laser light by a highly asymmetrical cigar-shaped BEC. As the experiment involved the transfer of photons from a pump mode into different modes of the electromagnetic field, the phenomenon is an example of four-wave mixing between atomic and optical fields. What differentiates this system from the CARL is the absence of an optical cavity to provide feedback for the scattered light fields.

In order to describe these experimental results, we have modified the CARL theory to include a continuum of 'probe' fields (Moore and Meystre 1999b). These are then formally eliminated à la Wigner-Weiskoff to account for the fact that the scattered photons leave the condensate 'instantaneously'. Thus, in contrast to the CARL, coherence is no longer stored in the light field. The high degree of spatial and temporal coherence of the condensate, however, allows for 'memory' of the scattered photons to be stored in the form of interference fringes. This is similar to Dicke superradiance (Dicke 1954; Dicke 1964), only now the memory is stored
in the coherence between center-of-mass states rather than between internal atomic states. The surprising feature of the experiment is that the asymmetrical shape of the condensate introduces a directional dependence of the gain, which when combined with mode competition results in light being emitted along the long axis of the BEC only.

The time a scattered photon takes to exit the condensate is of the order of $10^{-13}$ s. This time is 'infinitely' fast when compared to the time scales on which the atomic field evolves. This is known as the Markov property and it allows the scattered light field to be formally eliminated from the equations of motion for the atomic field. Situations such as this are known as system-reservoir interactions and are encountered often in quantum optics. In the Heisenberg picture, the formal elimination of the reservoir to second order in perturbation theory leads to a set of quantum Langevin equations, where the effect of the reservoir is to introduce irreversibility and noise into the system dynamics. In the condensate superradiance problem, recoiling atoms and scattered photons are created in pairs, a process described by a Hamiltonian similar to

$$\hat{H} = \hbar \omega \hat{c}^\dagger \hat{c} + \int d\omega \hbar \omega \hat{a}^\dagger (\omega) \hat{a}(\omega) + \int d\omega \hbar \left[ g(\omega) e^{-i\omega t} \hat{c}^\dagger \hat{a}(\omega) + g^*(\omega) e^{i\omega t} \hat{c} \hat{a}(\omega) \right],$$

(1.26)

where $\hat{c}$ is the system operator, and the set of operators $\hat{a}(\omega)$ describe the reservoir. The Hamiltonian (1.26) is therefore a simplified version of the superradiance Hamiltonian and will thus serve to illustrate our approach. We note that the natural frequency of the system $\Omega$ and the driving frequency $\omega_0$ are assumed to satisfy $\Omega \ll \omega_0$, as is the case in the condensate superradiance problem, where $\Omega$ is roughly
the atomic recoil frequency and $\omega_0$ is an optical frequency.

Introducing the rotating variables $\hat{C} = \hat{c}e^{i\Omega t}$ and $\hat{A}(\omega) = \hat{a}(\omega)e^{i\omega t}$, we use (1.26) to obtain the Heisenberg equations of motion for the operators $\hat{C}$ and $\hat{A}(\omega)$, yielding

$$\frac{d}{dt}\hat{C} = -i\int d\omega g(\omega)\hat{A}^\dagger(\omega)e^{i(\Omega+\omega-\omega_0)t}. \quad (1.27)$$

and

$$\frac{d}{dt}\hat{A}(\omega) = -ig(\omega)\hat{C}^\dagger e^{i(\Omega+\omega-\omega_0)t}. \quad (1.28)$$

Formally integrating the equations of motion for the optical field modes (1.28) gives

$$\hat{A}(\omega, t) = \hat{A}(\omega, 0) - ig(\omega)\int_0^t d\tau \hat{C}^\dagger(\tau)e^{i(\Omega+\omega-\omega_0)\tau}. \quad (1.29)$$

where the first and second terms correspond to the free and radiated fields respectively. Inserting Eq. (1.29) into Eq. (1.27), we find a closed equation for the system operators

$$\frac{d}{dt}\hat{C}(t) = -i\int d\omega g(\omega)\hat{A}^\dagger(\omega, 0)e^{-i(\Omega+\omega-\omega_0)\tau} + \int d\omega |g(\omega)|^2 \int_0^t d\tau \hat{C}^\dagger(\tau)e^{i(\Omega+\omega-\omega_0)(t-\tau)}, \quad (1.30)$$

where the first term describes quantum fluctuations while the second gives the dynamical response of the system.

In order to evaluate the second term on the r.h.s. of Eq. (1.30), which we call $I(t)$, we first expand $\hat{C}(\tau)$ onto its Fourier components, giving

$$I(t) \equiv \int d\omega \int d\omega' |g(\omega)|^2 \hat{C}(\omega')e^{i(\Omega+\omega-\omega_0)t} \int_0^t d\tau e^{-i(\Omega+\omega-\omega_0+\omega')\tau}. \quad (1.31)$$
When $t$ is large compared to the inverse bandwidth of the reservoir we can approximate the integral of the exponential as

$$\int_0^t d\tau e^{-i(\Omega + \omega - \omega_0 + \omega')\tau} \approx \pi \delta(\Omega + \omega - \omega_0 + \omega) - i\mathcal{P} \frac{1}{(\Omega + \omega - \omega_0 + \omega)},$$

(1.32)

where $\mathcal{P}$ indicates the principal value. Inserting this expression into Eq. (1.31) and integrating with respect to $\omega$ then gives

$$I(t) = \pi(1 - i) \int d\omega' |g(\omega_0 - \Omega - \omega')|^2 \hat{C}(\omega')e^{-i\omega't}.$$  

(1.33)

Our next assumption is that the system evolves on a time scale which is much smaller than the driving frequency $\omega_0$. With the second assumption that $g(\omega_0 - \Omega - \omega') \approx g(\omega_0)$ is over the range of frequencies for which $\hat{C}(\omega')$ is non-negligible, we can rewrite Eq. (1.30) as

$$\frac{d}{dt}\hat{C}(t) = (1 - i)\chi \hat{C}(t) + \hat{f}(t),$$

(1.34)

where $\chi = \pi|g(\omega_0)|^2$ and

$$\hat{f}(t) = i \int d\omega g^*(\omega) \hat{A}(\omega, 0)e^{i(\Omega + \omega - \omega_0)\tau}$$

(1.35)

is a quantum noise operator which, among other things, guarantees that the commutation relation $[\hat{C}, \hat{C}^\dagger] = 1$ remains valid for all times. The real part of the term involving $\chi$ will lead to exponential growth, whereas the imaginary part gives the Lamb shift due to the reservoir interaction. To evaluate expectation values containing the $\hat{f}(t)$ we assume that at $t = 0$ the reservoir modes are all in the ground state. As Eq. (1.34) corresponds to a driven linear system, the solution is readily found to be

$$\hat{C}(t) = \hat{C}(0)e^{(1-i)\chi t} + \int_0^t d\tau \hat{f}(t - \tau)e^{(1-i)\chi \tau}.$$  

(1.36)
In the case where the system begins in the ground state, we find that the mean field $\langle \hat{C}(t) \rangle$ is zero at all times. The mean occupation number $\langle \hat{C}^\dagger(t)\hat{C}(t) \rangle$, on the other hand, is found to be

$$\langle \hat{C}^\dagger(t)\hat{C}(t) \rangle = \langle \hat{C}\hat{C}^\dagger \rangle - 1 = e^{2xt} - 1.$$  \hspace{1cm} (1.37)

Thus the mean occupation number grows exponentially, even when the system begins in the ground state. We recall that it was the noise operators which guaranteed that the commutation relation used in (1.37) remains valid at all times. It can readily be shown that if the noise operator is not included in Eq. (1.34) the mean occupation number vanishes, which shows that it is the quantum fluctuations of the reservoir which trigger the exponential growth of the system population.

1.5 Dissertation format

The remainder of this dissertation consists of three published research papers relating to the topic of nonlinear wave mixing between quantum-degenerate atomic and optical fields. Each paper appears unedited as an appendix, while Chapter 2 summarizes and gives conclusions for each paper. The research and writing was done mainly by myself under the supervision of Pierre Meystre, with limited collaboration from Oliver Zobay on the topic of squeezing between atomic and optical fields.

The papers are presented in chronological order, the first being "Effects of atomic diffraction on the collective atomic recoil laser" (Moore and Meystre 1998),
In which the wave-atom-optics theory of the CARL is first introduced. Here the focus is on temperature dependence and the transition between the ray and wave regimes of atom optics.

The second paper, "Quantum optics of Bose-Einstein condensate coupled to quantized light field" (Moore et al. 1999), specializes to the case of a BEC, presenting a full quantum field theory of the linear response of the BEC/CARL system. Collisions are included in the CARL theory using the s-wave scattering approximation (1.8), and issues of quantum entanglement, nonclassical correlations, and the manipulation of coherence properties are addressed. Particular emphasis is placed on the analogy with the optical parametric amplifier.

In the last paper, "Theory of superradiant scattering of laser light from Bose-Einstein condensates" (Moore and Meystre 1999b), the model is extended by replacing the single-mode probe field with the continuum of electromagnetic vacuum modes. This paper develops a quantum field theory description of a recent experimental demonstration of four-wave mixing between atomic and optical fields performed at MIT. This paper differs from the previous two also in that it is extended into the nonlinear regime.
CHAPTER 2

PRESENT STUDY

The theoretical approach, results, and conclusions of my dissertation research are presented in the three papers appended to this thesis. The following is a summary of the most important aspects of these papers.

2.1 Effects of atomic diffraction on the collective atomic recoil laser: summary and conclusions

Appendix A contains the manuscript "Effects of atomic diffraction on the collective atomic recoil laser". It is co-authored by myself and Pierre Meystre and appeared in the October 1998 issue of Physical Review A. The main object of this paper is to derive a wave-atom-optics (WAO) theory of the collective atomic recoil laser (CARL), and to compare and contrast this model with the previously published ray-atom-optics (RAO) model (Bonifacio and De Salvo 1994; Bonifacio et al. 1994). As outlined in Chapter 1, ray atom optics corresponds to treating the atoms as classical point particles subject to Newton's laws of motion, whereas wave atom optics treats their center-of-mass motion quantum mechanically.

The paper begins by reviewing the RAO model in order to establish the
notation and set the stage for the comparison with the WAO theory. In the RAO model only the internal atomic degree of freedom is quantized. Hence, Pauli pseudospin operators are introduced which describe the internal atomic states. In the WAO theory both the internal and center-of-mass motions are treated quantum mechanically, the internal and external atomic degrees of freedom now being described by 'single particle' density operators. An example of a single particle density operator is $\hat{\rho}_{ee}(k,k) = \hat{c}_e^\dagger(k)\hat{c}_e(k)$, whose expectation value gives the mean number of excited atoms with momentum $\hbar k$. Here $\hat{c}_e^\dagger(k)$ and $\hat{c}_e(k)$ are the corresponding atom creation and annihilation operators, respectively.

In both models, the excited internal atomic state is adiabatically eliminated under the assumption of far-detuned light fields and the pump laser is treated in the classical and undepleted approximations. The atomic variables are then reexpressed as the sum of their initial values plus a small perturbation. With the probe also considered as a small perturbation, the equations of motion are linearized and their quantum mechanical expectation values are taken. Assuming that entanglement exists only between the probe field and the perturbation part of the atomic field, results in a closed set of c-number equations for the various expectation values.

The linearized equations yield either stable (sinusoidal) or unstable (exponential) solutions. It is the unstable solutions which allow the CARL to serve its primary purpose as an amplifier for atomic and optical waves. One of the main goals of the paper, therefore, is to determine the range of parameters for which unstable solutions exist. For both the RAO and WAO models, the exponential growth rate, when it exists, depends on the pump-probe detuning, the effective atom-probe
coupling strength, and the temperature of the system. We note that the effective atom-probe coupling parameter depends on the pump intensity and detuning, as well as on the total number of atoms in the system. By using a Laplace transform technique, transcendental equations are derived from which the exponential growth rates for both models are determined numerically.

Comparing the results for the RAO and WAO models clearly demonstrates that the ray-atom-optics picture indeed breaks down at the atomic recoil temperature $T_R$, as was initially predicted (recall that for the geometry at hand, $T_R$ is the temperature where the atomic de Broglie wavelength equals the spacing of the diffraction grating formed by the counterpropagating pump and probe fields). Particular attention is paid to the $T \ll T_R$ limit, as this is the limit of negligible Doppler broadening, and hence of strongest exponential growth. In the earlier work by the Milan group, a so-called “CARL cubic equation” was derived for the case $T = 0$. From this equation the CARL operating regime and gain were determined. By using the WAO model, the correct $T = 0$ cubic equation is presented in this paper. This new equation takes properly into account atomic diffraction, which tends to counteract the formation of a grating in the atomic density, thus inhibiting or reducing the CARL instability. In particular, the new zero-temperature theory predicts a reduction in the maximum growth rate by roughly one half, as well as the appearance of a new threshold for instability below which matter-wave diffraction stabilizes the system. As a result there is a significant reduction in the operating regime of the CARL, which is illustrated in Fig. 3 of the manuscript.
2.2 Quantum optics of a Bose-Einstein condensate coupled to a quantized light field: summary and conclusions

Appendix B contains the manuscript “Quantum optics of a Bose-Einstein condensate coupled to a quantized light field”. Co-authored by myself, Oliver Zobay, and Pierre Meystre, this paper appeared in the August 1999 issue of Physical Review A. This paper was prepared as a detailed elaboration and extension of a shorter article (Moore and Meystre 1999a) entitled “Optical control and entanglement of atomic Schrödinger fields, which was published as a Rapid Communication in the March 1999 issue of Physical Review A. The main purpose of these two papers is to extend the theory of the CARL system to the specific case of a Bose-Einstein condensate. The goal is to study the interaction of coherent matter waves with coherent light, in order to better understand and if possible to manipulate the coherence properties of the BEC.

Well below the critical temperature, a BEC is for all intents and purposes a single-mode atomic field. The central working idea for this paper is to therefore construct a few-mode model of the BEC/CARL, which can then be treated and interpreted using standard quantum-optical techniques. Whether or not a few-modes model is adequate in this case is a question of time scales. To understand this, consider what happens to a condensate atom as it scatters a photon from the pump field into the probe. By scattering the photon, the atom acquires a two-photon recoil kick. This changes the wavefunction of the atom from that of the condensate ground state into a new state, which retains the spatial envelope of the condensate
ground state, but which is shifted in momentum space by the two-photon recoil momentum. We call this state a momentum sidemode to the initial BEC. The state is actually a non-stationary wavepacket, and will begin to move with respect to the condensate at the two-photon recoil velocity. It will also begin to spread out, but due to the extremely small momentum spread of the condensate, this will occur on a much longer time scale than the translational motion. As the atom moves off to the side, its overlap with the momentum sidemode decreases. This continues until a time given by the length of the condensate divided by the recoil velocity, after which it no longer has any probability of being found in the original sidemode state. This time scale is quite long, from tens of microseconds to milliseconds. Provided, therefore, that we consider only times small compared to this decay time, we can neglect the translational motion, and consider the momentum sidemodes as stationary states. For longer times, the loss of atoms could be modeled by adding a loss term to the sidemode, so that its population decays at the appropriate rate.

Treating the BEC in this manner, we make the additional approximations of adiabatically eliminating the excited internal atomic state, and treating both the pump laser and the condensate in the classical and undepleted approximations. This treatment of the BEC is valid for times short enough that the fraction of the condensate transferred into the momentum sidemodes remains negligible. Keeping only terms linear in the (weakly populated) sidemode and probe field operators, we then derive a linearized set of Heisenberg equations of motion. Rather than solve simply for their mean values, as we did in Appendix A, we now determine the exact solution, from which all expectation values of the sidemode and probe
field operators can be obtained in a straightforward manner. Atomic collisions are included in the CARL model using the s-wave scattering approximation. We find that for positive scattering lengths, the collisions tend to oppose the bunching and reduce the operating regime of the CARL. This effect is readily understandable as bunching increases the local density, and hence the mean field energy.

From this analytical solution, we then compute the probe electric-field and atomic-density mean values and variances, as well as the probe and sidemode mean occupation numbers and number variances. We assume that the probe field begins in a very weak coherent state \( \alpha \), corresponding to the injection of a small laser field into the ring cavity at the probe frequency. By considering the case \( \alpha = 0 \), triggering from vacuum fluctuations alone is considered as well. We find that for the vacuum case the fields develop with no mean values (phase symmetry) and large fluctuations, characteristic of a thermal field, while for a large injected field, there is a well defined phase, and the fluctuations are reduced to those consistent with a coherent state. Furthermore, we find that by varying \( \alpha \), it is possible to generate fields whose quantum statistics vary continuously between the thermal and coherent limits.

In addition to investigating the coherence properties of the probe field and the condensate momentum sidemodes, we also explore the question of atom-photon entanglement. This is accomplished by computing two measures: the two-mode intensity correlation function and the quadrature component of the superposed atomic and optical fields, the latter being a common diagnostic to identify the presence of
two-mode squeezing. Squeezing occurs when the variance of one of a pair of conjugate observables is less than the square root of the Heisenberg uncertainty limit, thus giving the 'appearance' of violating the uncertainty principle. We find that in the case where the system is triggered by quantum fluctuations alone, there is very nearly the maximum violation of classical two-mode intensity correlations allowed by the laws of quantum mechanics. This is the same result found the nondegenerate optical parametric amplifier (OPA), the tool of choice for the generation of entangled quantum optical states. In addition, we find strong squeezing in the quadrature components which is similar to the OPA, but exhibits novel features such as oscillations and a finite squeezing time. These effects are understood as resulting from of coupling to nonresonant momentum sidemodes upsetting a delicate balance necessary for squeezing to occur.

The emphasis on squeezing and entanglement indicates the future direction we plan to take with this system. In the OPA, it is not only the intensity correlations between the two light fields which are utilized for exploring fundamental aspects of quantum mechanics, but the creation of maximal entanglement between the internal states of the pairs of generated photons, i.e. quantum correlations between the polarization states of the emitted photons. Our conjecture is that by extending our model to more realistic atoms with multiple ground states, maximally entangled states might be generated between the ground hyperfine states of the atomic momentum sidemode and the polarization state of the probe field. Work in this direction is currently underway.
2.3 Theory of superradiant scattering of laser light from Bose-Einstein condensates: summary and conclusions

Our most recent paper on nonlinear wave mixing between atomic and optical fields, entitled "Theory of superradiant scattering of laser light from Bose-Einstein condensates", is found in Appendix C. It was co-authored by myself and Pierre Meystre, and has been accepted for publication in Physical Review Letters. The purpose of this paper is to develop a theoretical framework to further understand the physics behind a recent experiment performed by the group of Wolfgang Ketterle at MIT (Inouye et al. 1999). In the experiment, highly directional Rayleigh scattering was only observed when a far-detuned pump laser was aimed onto a cigar-shaped Bose-Einstein condensate. The effect was observed when the polarization and direction of propagation of the laser were perpendicular to the long axis of the BEC. When the polarization was turned parallel to the long axis a standard dipole radiation pattern was instead observed. In addition, no effect was seen above the critical temperature for Bose condensation, thus demonstrating that the high degree of spatial and temporal coherence of the BEC plays an important role in the phenomenon.

Ketterle and coworkers correctly attributed the effect to a new form of Dicke superradiance where the memory of previous scattering events is stored in the center-of-mass state of the atoms in the form of interference fringes between the states of recoiling atoms and the original condensate. The theory presented in the experimental paper, however, is semi-classical and leaves room for a more technical and fully quantum mechanical model to be devised.
The main part of this paper is dedicated to constructing a manybody quantum theory using the operator formalism of quantum field theory. The model is constructed by suitably adapting the Wigner-Weiskopf theory of spontaneous emission to include the center-of-mass states of a system of $N$ identical bosons. With the assumption that all of the atoms begin in the condensate state, and with the adiabatic elimination of the field of the scattered photons, a multimode theory of condensate superradiance is derived.

In order to understand the directionality of the scattered light, consider the effect of photon recoil on an atom which scatters a photon from the pump (with wavevector $k_0\hat{y}$) into an arbitrary direction $\hat{k}$. The atom is transferred from the initial condensate ground state $\varphi_0(r)$ into a 'momentum sidemode', whose state is given by $\varphi_0(r)\exp[ik_0(\hat{y} - \hat{k}) \cdot r]$. If the scattering direction is changed slightly, than the overlap between the corresponding momentum sidemodes may remain close to unity due to the finite spatial extent of $\varphi_0(r)$. If this is the case than the momentum sidemodes are not distinct and for practical purposes correspond to the same state. As the difference in scattering directions is increased, however, a point is reached where the overlap drops rapidly towards zero. This boundary defines a solid angle corresponding to photon scattering directions for which the atom recoils into a single, relatively distinct quantum state, or 'quasi-mode'. As this solid angle is finite and the total available solid angle is $4\pi$, the number of quasi-modes is therefore finite. This means that while we consider a continuum of final states for the scattered photons, we only need to consider a discrete and finite set of quasi-modes to describe the recoiling atoms. The solid angle which
we have defined for each quasi-mode gives a measure of the 'number' of photon scattering directions corresponding to that state. Taking into account the dipole-radiation pattern, which weights each scattering direction, the quasi-modes with the largest solid angle should thus experience the largest gain. From the geometry of the BEC it is readily seen that this leads to maximum gain along the long axis of the condensate. Mode competition then squelches scattering in all but a subset of modes, termed 'endfire' modes by Dicke, which share the maximum gain.

A key feature of the theory is its ability to describe the triggering of the superradiant scattering by quantum fluctuations. The quantum fluctuations are amplified by subsequent scattering events because the presence of a bosonic atom in a particular recoil state enhances the probability for a subsequent scattering in the same direction. In classical nonlinear optics, this type of effect would be interpreted as scattering of photons off the atomic density grating set up by the interference between the recoiling atoms and the remaining condensate. In the experiment, large run-to-run fluctuations were observed in the amplitude, number, and direction of the scattered light pulses. Despite the high degree of directionality, the precise direction varies within the geometrical angle, defined as the ratio of the width of the BEC to its length. Within this angle, all quasi-modes have roughly the same gain and mode competition is random, but 'fair'. The light pulses are triggered by quantum noise, hence the observed fluctuations are amplified quantum fluctuations. A linearized quantum theory predicts that for times short enough that condensate depletion may be safely neglected, each recoil mode evolves into a chaotic state, with a Poissonian number distribution. The mean value of the distribution is largest for
recoil modes corresponding to photons scattered into the endfire modes.

This model differs from the earlier work on the CARL in that the theory is now extended into the nonlinear regime where condensate depletion begins to play a significant role. In the nonlinear regime, a 'classical' model is developed, in which the expectation values of products of intensity operators are assumed to factorize. This assumption is reasonable once the field populations are large compared to unity, as higher order correlation functions then effectively factorize. The initial conditions for these 'classical' fields are taken from the quantum probability distribution, thus taking into account the fact that the process is triggered by quantum noise. In the nonlinear regime it is seen that mode competition occurs as the various momentum sidemodes compete for condensate atoms. This competition strongly favors those modes with the highest scattering probability, i.e. the endfire modes. We determine, however, that there are many hundreds of endfire modes, which compete fairly, and thus a random pattern of radiation scattered with the geometric angle is predicted by the theory and observed in the experiment.

As in the work on the CARL, we hope to extend this model to include multiple ground hyperfine states to explore the possibility to generate correlated atom-photon pairs whose internal states are entangled.
REFERENCES


Goldstein, E. V. and P. Meystre (1999b). Quantum theory of atomic four-wave


Stenger, J., S. Inouye, D. Stamper-Kurn, H.-J. Miesner, A. Chikkatur, and 
Nature 369, 345–348.

Rev. Lett. 77, 2360–2363.


generation of a longitudinal atomic density grating in sodium vapor - Com­

Springer-Verlag.


Lett. 77, 2158–2161.


Zhang, W. P. and D. F. Walls (1994). Quantum-field theory of interaction of 
ultracold atoms with a light wave: Bragg scattering in nonlinear atom optics. 


Effects of atomic diffraction on the collective atomic recoil laser

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We formulate a wave-atom-optics theory of the collective atomic recoil laser (CARL) where the atomic center-of-mass motion is treated quantum mechanically. By comparing the predictions of this theory with those of the ray-atom-optics theory, which treats the center-of-mass atomic motion classically, we show that for the case of a far off-resonant pump laser the ray-optics model fails to predict the linear response of the CARL when the temperature is of the order of the recoil temperature or less. This is due to the fact that in this temperature regime one cannot longer ignore the effects of wave–wave diffraction on the atomic center-of-mass motion. [S1050-2947(98)00510-1]

PACS numbers: 42.55.-t, 42.50.Vc, 03.75.-b

I. INTRODUCTION

The collective atomic recoil laser, or CARL, is the atomic equivalent of the free-electron laser [1]. Developed theoretically by Bouyer and co-workers [2–6], the CARL device has three main components: (1) the active medium, which consists of a gas of two-level atoms. (2) a strong pump laser which drives the two-level atomic transitions, and (3) a ring cavity which supports an electromagnetic mode (the probe) counterpropagating with respect to the pump. Under suitable conditions, the operation of the CARL results in the generation of a coherent probe field due to the following mechanism. First, a weak probe field is initiated by some, either optical in the form of spontaneously emitted light, or atomic in the form of density fluctuations in the atomic gas which backscatter the pump. Once initiated, the probe counterpropagates with the pump field to form a weak standing wave which acts as a periodic optical potential (light shift). This causes the motion of the atoms on this potential to remain in a bunching (modulation) of their density, very much like the combined effects of the wiggler and the light field leads to electron bunching in the free-electron laser. This bunching process is then seen as a probe laser field of the appearance of a polarization grating in the active medium, which results in stimulated backscattering into the pump field. The resulting increase in the probe strength further increases the magnitude of the standing wave field, resulting in more bunching followed by an increase in stimulated backscattering, etc. This positive feedback mechanism results in an exponential growth of both the probe intensity and the atomic bunching. This leads to the perhaps surprising result that the presence of the ring cavity turns the ordinarily unstable systems of an atomic gas driven by a strong pump laser into an unstable system.

Various experiments related to the CARL have been conducted recently. Using hot sodium atoms, Lefèvre et al. [7] observed amplification of an injected probe laser, which they interpreted in terms of scattering off an atomic density grating resulting from atomic recoil. Also using hot sodium atoms, Hammer et al. [8] reported spontaneous probe oscillations in the absence of an injected signal. These were also interpreted as resulting from the CARL mechanism. There has, however, been some controversy regarding the interpretation of these experiments [9], mostly concerning the presence of a large Doppler broadening, and the possibility of gain mechanisms not necessarily related to atomic recoil. An unambiguous demonstration of the recoil related gain mechanism was performed by Courtois et al. [10], who observed small signal probe gain in a gas of cold cesium atoms. The absence of a ring cavity for probe feedback in this experiment, however, means that the observed gain was mainly a single-atom recoil effect, not the collective gain of the CARL system.

The CARL theory developed by Bouyer et al. considers the atoms either as classical point particles moving in the optical potential generated by the light fields, or, in a "hybrid" version [11], as particles whose center of mass is labeled by their classical positions, but with quantum fluctuations about those positions included. From an atom-optics point of view, such theories can be described as "ray-atom-optics" treatments of the atomic field, in analogy with the ordinary ray-optics treatment of electromagnetic fields.

Like ordinary ray optics, the ray-atom-optics description of CARL is expected to be valid provided that the characteristic wavelength of the matter-wave field remains much smaller than the characteristic length scale of any atomic optical elements in the system. The characteristic wavelength of the atomic field is its de Broglie wavelength, determined by the atomic mass and the temperature $T$ of the atomic gas. The central atom-optical element of the CARL is the periodic optical potential, which acts as a diffraction grating for the atoms, and has the characteristic length scale of half the optical wavelength. Hence the classical "ray-atom-optics" description is intuitively expected to be valid provided that the temperature is high enough that the thermal de Broglie wavelength is much smaller than the optical wavelength. This gives the condition $T \gg T_B$, the recoil temperature of the atoms, as the domain of ray-optics. In particular, it is certainly expected to hold under the temperature conditions of the experiments performed so far.

However, the recent progress made by atomic cooling techniques makes it likely that CARL experiments using ultracold atomic samples can and will be performed in the future. In particular, subkelvin temperatures can now be achieved almost routinely. The purpose of this paper is to extend the CARL theory to this "wave-atom-
APPENDIX A

EFFECTS OF ATOMIC DIFFRACTION ON THE COLLECTIVE
ATOMIC RECOIL LASER

M. G. Moore and P. Meystre

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ABSTRACT

We formulate a wave atom optics theory of the Collective Atomic Recoil Laser, where the atomic center-of-mass motion is treated quantum mechanically. By comparing the predictions of this theory with those of the ray atom optics theory, which treats the center-of-mass atomic motion classically, we show that for the case of a far off-resonant pump laser the ray optics model fails to predict the linear response of the CARL when the temperature is of the order of the recoil temperature or less. This is due to the fact that in this temperature regime one can no longer ignore the effects of matter-wave diffraction on the atomic center-of-mass motion.

I. INTRODUCTION

The Collective Atomic Recoil Laser, or CARL, is the atomic equivalent of the Free Electron Laser [1]. Developed theoretically by Bonifacio et al [2-6], the CARL device has three main components: (1) the active medium, which consists of a gas of
two-level atoms, (2) a strong pump laser which drives the two-level atomic transition, and (3) a ring cavity which supports an electromagnetic mode (the probe) counter-propagating with respect to the pump. Under suitable conditions, the operation of the CARL results in the generation of a coherent probe field due to the following mechanism. First, a weak probe field is initiated by noise, either optical in the form of spontaneously emitted light, or atomic in the form of density fluctuations in the atomic gas which backscatters the pump. Once initiated, the probe combines with the pump field to form a weak standing wave which acts as a periodic optical potential (light shift). The center-of-mass motion of the atoms on this potential results in a bunching (modulation) of their density, very much like the combined effects of the wiggler and the light field leads to electron bunching in the free-electron laser. This bunching process is then seen by the pump laser as the appearance of a polarization grating in the active medium, which results in stimulated backscattering into the probe field. The resulting increase in the probe strength further increases the magnitude of the standing wave field, resulting in more bunching followed by an increase in stimulated backscattering, etc. This positive feedback mechanism results in an exponential growth of both the probe intensity and the atomic bunching. This leads to the perhaps surprising result that the presence of the ring cavity turns the ordinarily stable system of an atomic gas driven by a strong pump laser into an unstable system.

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regarding the interpretation of these experiments [9], mostly concerning presence of a large Doppler broadening, and the possibility of gain mechanisms not necessarily related to atomic recoil. An unambiguous demonstration of the recoil related gain mechanism was performed by Courtois et al [10], who observed small signal probe gain in a gas of cold Cesium atoms. The absence of a ring-cavity for probe feedback in this experiment, however, means that the observed gain was mainly a single-atom recoil effect, not the collective gain of the CARL system.

The CARL theory developed by Bonifacio et al considers the atoms either as classical point particles moving in the optical potential generated by the light fields, or, in a "hybrid" version [11], as particles whose center-of-mass is labeled by their classical position, but with quantum fluctuations about that position included. From an atom optics point of view, such theories can be described as "ray atom optics" treatments of the atomic field, in analogy with the ordinary ray optics treatment of electromagnetic fields.

Like ordinary ray optics, the ray atom optics description of CARL is expected to be valid provided that the characteristic wavelength of the matter-wave field remains much smaller than the characteristic length scale of any atom-optical element in the system. The characteristic wavelength of the atomic field is its de Broglie wavelength, determined by the atomic mass and the temperature $T$ of the atomic gas. The central atom-optical element of the CARL is the periodic optical potential, which acts as a diffraction grating for the atoms, and has the characteristic length scale of half the optical wavelength. Hence the classical "ray atom optics" description is intuitively expected to be valid provided that the temperature is high enough that the thermal de Broglie wavelength is much smaller than the optical wavelength. This gives the condition $T \gg T_R$, the recoil temperature of the atoms, as the domain of ray atom
optics. In particular, it is certainly expected to hold under the temperature conditions of the experiments performed so far.

However, the spectacular recent progress witnessed by atomic cooling techniques makes it likely that CARL experiments using ultracold atomic samples can and will be performed in the future. In particular, subrecoil temperatures can now be achieved almost routinely. The purpose of this paper is to extend the CARL theory to this "wave atom optics" regime [12]. In this regime it becomes necessary to treat the atomic center-of-mass motion fully quantum mechanically, in order to preserve the wave nature of the atomic motion. Thus the interaction between the atoms and the standing wave light field should no longer be thought of in terms of particles moving in a periodic potential, but instead as diffraction of matter-waves by a grating. As the grating amplitude is initially assumed to be zero (or at least infinitesimal), the system starts in the Bragg regime, where the free space evolution (kinetic energy term) plays an important dynamical role. Similar in origin to the free space diffraction which limits the spot size of a focused beam, this type of diffraction effect can counteract the bunching process by which the atomic density grating is established. It is our goal to determine the precise limitations which this aspect of matter-wave diffraction imposes on CARL operation.

The wave optics theory of the CARL is similar to the analysis of atomic diffraction by standing waves [13], except that the electromagnetic field is now treated as a dynamical variable. It is also similar to the theory of recoil induced resonances [14], which describes the stimulated scattering of light off a standing wave induced polarization grating, but the absence of a feedback mechanism for the probe feedback in that case means that it models only single-atom gain effects, and not the collective gain of the CARL.
In this paper we focus on the case of a far off-resonant pump laser, thus permitting us to neglect the excited state population and therefore to ignore the effects of spontaneous emission (except as a hypothetical source of noise for probe initiation). We further concentrate on the linear regime, where both the probe field and the atomic bunching are considered as infinitesimal quantities, since it is this regime that determines whether or not the exponential instability occurs. Finally, we restrict our analysis to atomic densities low enough that collisions between atoms may be ignored, and neglect the transverse motion of the atoms, which in the absence of collisions is decoupled from the longitudinal degree of freedom along which bunching occurs.

We note at the outset that our theory is semiclassical in that it treats the electromagnetic field classically. While this approximation can not fully describe the statistical properties of the CARL output, it is sufficient to describe the small-signal gain of the system, provided that one makes the implicit assumption that small fluctuations will trigger it, an approach familiar from conventional laser theory and nonlinear optics. We also emphasize that it is not inconsistent to treat the matter waves quantum mechanically while treating the light classically, since the limits under which a quantum description is required are independent. For light, this limit is usually associated with weak intensities, while for matter waves it is normally a low temperature limit.

The rest of this paper is organized as follows: Section II briefly reviews the ray atom optics model of the CARL, establishing the notation and setting the stage for a comparison of its predictions with those of the wave atom optics theory, which is introduced in section III. Section IV discusses the collective instability leading to CARL operation, compares the ray atom optics and the wave atom optics predictions, and determines the domain of validity of the former theory. Section V is a discussion
and section VI offers a conclusion and outlook.

II. RAY ATOM OPTICS MODEL

The Ray Atom Optics (RAO) model of the CARL has been developed and extensively studied by Bonifacio et al. [2-6] It begins with the classical $N$-particle Hamiltonian

$$H_N = \sum_{j=1}^{N} H_1(z_j, p_j),$$

where $z_j$ and $p_j$ are the classical position and momentum of the $j$th atom, obeying the canonical equations of motion $dz_j/dt = \partial H_N/\partial p_j$ and $dp_j/dt = -\partial H_N/\partial z_j$. The single-particle Hamiltonian $H_1$ is given explicitly by

$$H_1(z_j, p_j) = \frac{p_j^2}{2m} + \frac{\hbar \omega_0}{2} \sigma_{z_j} + i\hbar \left[ g_1 a_i^* e^{-ik_1 z_j} \sigma_{-j} + g_2 a_2^* e^{-ik_2 z_j} \sigma_{-j} - \text{c.c.} \right],$$

where $m$ is the atomic mass, $\omega_0$ is the natural frequency of the atomic transition being driven by the pump and probe lasers, and $g_1$ is the atom-probe electric dipole coupling constant. It is given by $g_1 = \mu_1 [\varepsilon (k_1^2/2\hbar \omega_0 V)]^{1/2}$, where $\mu_1$ is the projection of the atomic dipole moment along the probe polarization, $k_1$ is the probe wavenumber, and $V$ is the quantization volume. The atom-pump coupling constant $g_2$ is defined analogously to $g_1$, but depending on $\mu_2$, the projection of the atomic dipole moment along the pump polarization, and $k_2$ the pump wavenumber. The normal variables $a_1$ and $a_2$ describe the probe and pump laser fields, respectively. They obey Maxwell's equation

$$\frac{d}{dt} a_i = -ic|k_i| a_i + g_i \sum_{j=1}^{N} e^{-ik_j z_j} \sigma_{-j},$$
where \( c|k_i| \) is the natural frequency of the probe \((i = 1)\) or the pump \((i = 2)\) field. Note that these equations are also valid for quantized electromagnetic fields, provided that \( a_i \) are interpreted as annihilation operators, but we describe the light fields classically in this paper.

The variables \( \sigma_{-j} \) and \( \sigma_{zj} \) are the expectation values of the quantum mechanical Pauli pseudo-spin operators which describe the internal state of the \( j \)th atom. They obey the familiar optical Bloch equations, appropriately modified to include the center-of-mass motion of the atoms and with spontaneous emission neglected *.

\[
\frac{d}{dt} \sigma_{-j} = -i\omega_0 \sigma_{-j} + [g_1^* a_1 e^{ik_1 z_j} + g_2^* a_2 e^{ik_2 z_j}] \sigma_{zj},
\]

and

\[
\frac{d}{dt} \sigma_{zj} = -2 [g_1^* a_1 e^{-ik_1 z_j} + g_2^* a_2 e^{-ik_2 z_j}] \sigma_{-j} + c.c.
\]

It is convenient to introduce slowly varying variables via the transformations \( a_1 = a_1'e^{-i\omega_2 t}, a_2 = a_2'e^{-i\omega_2 t}, \) and \( \sigma_{-j} = \sigma_{-j}' e^{-i(\omega_2 t - k_2 z_j)}, \) where \( \omega_2 \) is the pump frequency. The relation between \( \omega_2 \) and \( k_2 \) will be derived shortly in a self-consistent manner, so as to include the dispersive effects of the polarized atoms on the pump propagation.

These new variables obey the equations of motion

\[
\frac{d}{dt} z_j = \frac{p_j}{m},
\]

\[
\frac{d}{dt} p_j = -\hbar [g_1 k_1 a_1'^* e^{-i(k_1 - k_2)z_j} + g_2 k_2 a_2'^* ] \sigma_{-j}' + c.c.,
\]

*Spontaneous emission is neglected in anticipation of the future approximation that the pump lasers are far-off resonant, and therefore the excited state population may be safely neglected.
\[ \frac{d}{dt} a'_1 = i(\omega_2 - c|k_1|)a'_1 + g_1 \sum_{j=1}^{N} e^{-i(k_1 - k_2)z_j} \sigma'_{-j}, \]  
(8)

\[ \frac{d}{dt} a'_2 = i(\omega_2 - c|k_2|)a'_2 + g_2 \sum_{j=1}^{N} \sigma'_{-j}, \]  
(9)

\[ \frac{d}{dt} \sigma_{zj} = -2 \left[ g_1 a'_1^* e^{-i(k_1 - k_2)z_j} + g_2 a'_2^* \right] \sigma'_{-j} \]  
\[ + \text{c.c.,} \]  
(10)

and

\[ \frac{d}{dt} \sigma'_{-j} = i(\omega_2 - \omega_0 - \frac{k_2}{m} p_j)\sigma'_{-j} \]  
\[ + \left[ g'_1 a'_1 e^{i(k_1 - k_2)z_j} + g'_2 a'_2 \right] \sigma_{zj}, \]  
(11)

In the case where the lasers are tuned far off resonance, and the atoms are initially in the ground state, the excited state population remains small and can be neglected. This is equivalent to describing the atoms as classical Lorentz atoms, and is accomplished by setting \( \sigma_{zj} = -1 \) in Eq. (11). Assuming further that the detuning \( \omega_2 - \omega_0 \) is much larger than any other frequency in Eq. (11), allows one to adiabatically eliminate \( \sigma'_{-j} \) with

\[ \sigma'_{-j} \approx -\frac{i}{(\omega_2 - \omega_0)} \left[ g'_1 a'_1 e^{i(k_1 - k_2)z_j} + g'_2 a'_2 \right], \]  
(12)

where we have in addition neglected the Doppler shift \( k_2 p_j/m \) compared to \( \omega_2 - \omega_0 \). This leads to the reduced set of equations

\[ \frac{d}{dt} z_j = \frac{p_j}{m}, \]  
(13)

\[ \frac{d}{dt} p_j = -i\frac{2\hbar k_0}{(\omega_2 - \omega_0)} \left[ g'_1 g'_2 a'_2^* a'_1 e^{i2k_0 z_j} - \text{c.c.} \right], \]  
(14)
\[
\frac{d}{dt}a'_1 = i \left[ \omega_2 - \frac{N|g_1|^2}{(\omega_2 - \omega_0)} - c|k_1| \right] a'_1 \\
- i \frac{g_2^*g_1}{(\omega_2 - \omega_0)} a'_2 \sum_{j=1}^{N} e^{-i2k_0z_j},
\] (15)

and

\[
\frac{d}{dt}a'_2 = i \left[ \omega_2 - \frac{N|g_2|^2}{(\omega_2 - \omega_0)} - c|k_2| \right] a'_2 \\
- i \frac{g_1^*g_2}{(\omega_2 - \omega_0)} a'_1 \sum_{j=1}^{N} e^{i2k_0z_j},
\] (16)

where we have introduced \( k_0 = (k_1 - k_2)/2 \).

We now introduce the undepleted pump approximation, valid in the linear regime where \( a'_1 \) remains small. This is achieved by dropping the term proportional to \( a'_1 \) in Eq. (16). This yields

\[
\frac{d}{dt}a'_2 = i \left[ \omega_2 - \frac{N|g_2|^2}{(\omega_2 - \omega_0)} - c|k_2| \right] a'_2,
\] (17)

which has the steady state solution \( a'_2(t) = a_2(0) \) provided that the dispersion relation

\[
c|k_2| = \omega_2 - \frac{N|g_2|^2}{(\omega_2 - \omega_0)}.\] (18)

is satisfied. Equation (18) thus determines the magnitude of the pump wavenumber as a function of the pump frequency and other known experimental parameters.

To proceed analytically past this point, it is convenient to introduce the dimensionless variables \( \theta_j \equiv 2k_0z_j, \ P_j = p_j/\hbar k_0, \ A = g_1^*g_2a_2^*(0)a'_1/[2\omega_r(\omega_2 - \omega_0)] \) and \( \tau = 4\omega_r t \), where the recoil frequency \( \omega_r \) is given by

\[
\omega_r = \hbar k_0^2/2m.
\] (19)

These variables obey the equations of motion

\[
\frac{d}{d\tau} \theta_j = P_j,
\] (20)
\[
\frac{d}{d\tau} P_j = -iA e^{i\theta_j} + \text{c.c.,} \quad (21)
\]
and
\[
\frac{d}{d\tau} A = i\Delta A - i\alpha \frac{1}{N} \sum_{j=1}^{N} e^{-i\theta_j}, \quad (22)
\]
where we have introduced the dimensionless control parameters
\[
\Delta = (\omega_2 - \frac{N|g_1|^2}{\omega_2 - \omega_0} - c|k_1|)/4\omega_r \quad (23)
\]
and
\[
\alpha = N|g_1|^2|g_2|^2|a_2(0)|^2/8\omega_r^2(\omega_2 - \omega_0)^2. \quad (24)
\]
We note that both $\Delta$ and $\alpha$ are real numbers, and furthermore that $\alpha \geq 0$. The term $c|k_1| + N|g_1|^2/(\omega_2 - \omega_0)$ in Eq. (23) is simply the natural frequency of the probe plus the shift to the atomic dispersion, i.e. it is the frequency of the probe field $\omega_1$. This means that $\Delta = (\omega_2 - \omega_1)/4\omega_r$, is simply the pump-probe detuning in units of $4\omega_r$.

We seek solutions of these equations which are perturbations about the case $A = 0$. Thus we make the substitutions
\[
\theta_j = \theta_j(0) + P_j(0)\tau + \delta \theta_j, \quad (25)
\]
and
\[
P_j = P_j(0) + \delta P_j, \quad (26)
\]
where $\theta_j(0)$ is randomly taken from a uniform distribution, and $P_j(0)$ is randomly taken from the initial momentum distribution. The new variables $\delta \theta_j$ and $\delta P_j$ give the perturbations on the atomic center-of-mass motion due to a nonzero $A(0)$. We introduce finally the linearized velocity group bunching parameter and its "conjugate" momentum according to
\begin{align}
B(k) &= \frac{1}{N} \sum_{j=1}^{N} \delta_{P_j(0), 2k} (1 - i\delta \theta_j) e^{-i(\theta_j(0) + P_j(0)r)}, \\
\text{and} \\
\Pi(k) &= \frac{1}{N} \sum_{j=1}^{N} \delta_{P_j(0), 2k} \delta P_j e^{-i(\theta_j(0) + P_j(0)r)} \\
&\quad + 2kB(k).
\end{align}

We note that

\begin{equation}
\sum_k B(k) = \langle e^{-i2k_0z} \rangle,
\end{equation}

and the amplitude of (29) is a measure of the degree of bunching of the atomic gas. A magnitude of zero indicates no bunching, while a magnitude of one indicates maximum bunching. This leads to the equations

\begin{align}
\frac{d}{d\tau} B(k) &= -i\Pi(k), \\
\frac{d}{d\tau} \Pi(k) &= i \left[ 4k^2 B(k) - 4k \Pi(k) - \frac{N(k)}{N} A \right],
\end{align}

and

\begin{equation}
\frac{d}{d\tau} A = i \left[ \Delta A - \alpha \sum_k B(k) \right],
\end{equation}

where \(N(k)\) is the number of atoms in the velocity group with momentum \(2\hbar k_0 k\) and we have assumed that

\begin{equation}
\sum_{j=1}^{N} \delta_{P_j(0), 2k} e^{-i2\theta_j(0)} = 0,
\end{equation}

an assumption that requires that \(N(k) \gg 1\). Note that this formulation implies a discretization of the initial momentum distribution, and furthermore assumes that the atomic positions in each velocity group are initially randomly distributed along the CARL cavity. Fluctuations in the initial distributions can of course readily be included into the initial conditions of the perturbation variables.
III. WAVE ATOM OPTICS MODEL

In order to quantize the center-of-mass motion of a gas of Bosonic atoms, one may either utilize first quantization, and replace the variables $z_j$ and $p_j$ in the $N$-particle Hamiltonian (1) with operators satisfying the canonical commutation relations $[\hat{z}_j, \hat{p}_{j'}] = i\hbar \delta_{jj'}$, or equivalently we can second-quantize the single particle Hamiltonian (2), introducing creation and annihilation operators for excited and ground state atoms of a given center-of-mass momentum. It is this second method which we will adopt in deriving the Wave Atom Optics (WAO) model. In the absence of collisions, the second-quantized Hamiltonian is simply

$$\hat{H} = \sum_k \hat{H}(k),$$

where $\hat{H}(k)$ is given by

$$\hat{H}(k) = \frac{\hbar^2 k^2}{2m} \hat{c}_g^\dagger(k)\hat{c}_g(k) + \left( \frac{\hbar^2 k^2}{2m} + \hbar \omega_0 \right) \hat{c}_e^\dagger(k)\hat{c}_e(k)$$

$$+ i\hbar \left[ g_1 a_1^\dagger \hat{c}_g^\dagger(k - k_1) \hat{c}_e(k) + g_2 a_2^\dagger \hat{c}_g^\dagger(k - k_2) \hat{c}_e(k) 
- H.c. \right].$$

The field operator $\hat{c}_g(k)$ annihilates a ground state atom of momentum $\hbar k$, and $\hat{c}_e(k)$ annihilates an excited atom of momentum $\hbar k$. We assume that the atoms in the sample are bosonic, so that these operators obey the commutation relations

$$[\hat{c}_g(k), \hat{c}_g^\dagger(k')] = [\hat{c}_e(k), \hat{c}_e^\dagger(k')] = \delta_{kk'},$$

all other commutators being equal to zero.

With the atomic polarization now expressed in terms of field operators, Maxwell's equations (3) for the classical laser fields become

$$\frac{d}{dt} a_i = -ic|k_i|a_i + g_1 \sum_k (\hat{c}_g^\dagger(k - k_i)\hat{c}_e(k)).$$
Hence, all that is required to determine the field evolution are the expectation value of bilinear combinations of atomic creation and annihilation operators. The evolution of these expectation values is easily obtained by introducing the "single-particle" atomic density operators
\[
\hat{\rho}_{gg}(k, k') = \hat{c}^\dagger_{k'}(k')\hat{c}_k(k),
\]
\[
\hat{\rho}_{eg}(k, k') = [\hat{\rho}_{ge}(k', k)]^\dagger = \hat{c}^\dagger_{k'}(k')\hat{c}_e(k),
\]
and
\[
\hat{\rho}_{ee}(k, k') = \hat{c}^\dagger_{k'}(k')\hat{c}_e(k).
\]
Note that e.g. the expectation value of the diagonal operator $\langle \hat{\rho}_{gg}(k, k) \rangle$ gives the mean number of ground state atoms with momentum $\hbar k$. The expectation values of these operators obey the equations of motion
\[
\frac{d}{dt}\rho_{jj'}(k, k') = \frac{i}{\hbar} \langle [\hat{H}, \hat{\rho}_{jj'}(k, k')] \rangle
\]
where $\rho_{jj'}(k, k') = \langle \hat{\rho}_{jj'}(k, k') \rangle$. The full form of these equations is given in the Appendix. The important point is that they depend only on $\rho_{jj'}(k, k')$, hence they form a closed set of equations which describe the response of the atomic field to the driving laser fields. We note that had we included collisions in our model, this would no longer be the case.

Introducing in analogy to the ray optics description the rotating variables $a_1 = a_1' e^{-i\omega_2 t}$, $a_2 = a_2' e^{-i\omega_2 t}$, and $\rho_{eg}(k, k') = \rho_{eg}(k - k_2, k') e^{-i\omega_2 t}$, neglecting the excited state population, and solving adiabatically for $\rho_{eg}(k, k')$ yields

\[\text{These are single-particle operators in the sense of many-body theory, since they only involve the annihilation of an atom in a given state and its creation in some other state.}\]
\[ \rho_{eg}(k, k') \approx -\frac{i}{(\omega - \omega_0)} [g_1^* a_1^* \rho_{eg}(k - 2k_0, k') + g_2^* a_2 \rho_{eg}(k, k')] . \] (42)

Substituting Eq. (42) into Maxwell's equation (37) for the pump and making once more the undepleted pump approximation leads to the solution \( a'_p(t) = a_2(0) \) provided that \(|k_2|\) satisfies the dispersion relation (18). We then substitute Eq. (42) into the equation of motion for \( \rho_{eg}(k, k') \), and introduce the dimensionless wavenumber \( \kappa = \frac{k}{2k_0} \) and the mean density \( \rho(\kappa, \kappa') = \rho_{eg}(k, k')/N \), in addition to the dimensionless variables already defined in the ray atom optics model. We arrive at the wave optics equations of motion

\[
\frac{d}{d\tau} \rho(\kappa, \kappa') = -i(\kappa^2 - \kappa'^2) \rho(\kappa, \kappa') \\
+ \frac{i}{2} A^*[\rho(\kappa, \kappa' - 1) - \rho(\kappa + 1, \kappa')] \\
- \frac{i}{2} A[\rho(\kappa - 1, \kappa') - \rho(\kappa, \kappa' + 1)], \quad (43)
\]

and

\[
\frac{d}{d\tau} A = i\Delta A - i\alpha \sum_\kappa \rho(\kappa, \kappa + 1), \quad (44)
\]

where the parameters \( \Delta \) and \( \alpha \) are given by Eqs. (23) and (24), respectively.

As in Sec. II, we seek a solution which is a perturbation about the case \( \Delta = 0 \). From Eq. (43), the unperturbed solution is readily found to be

\[
\rho(\kappa, \kappa', \tau) = \rho(\kappa, \kappa', 0)e^{-i(\kappa^2 - \kappa'^2)\tau}. \quad (45)
\]

We consider specifically an atomic sample initially in thermal equilibrium, so that Eq. (45) becomes

\[
\rho(\kappa, \kappa', \tau) = \frac{N(\kappa)}{N} \delta_{\kappa, \kappa'}. \quad (46)
\]
where \( N(\kappa) \), the number of atoms with initial wavenumber \( 2k_0\kappa \), is given by a thermal distribution function. We introduce the perturbation variables \( \delta \rho(\kappa, \kappa') \) according to

\[
\rho(\kappa, \kappa') = \frac{N(\kappa)}{N} \delta_{\kappa, \kappa'} + \delta \rho(\kappa, \kappa').
\]

(47)

and observe that Maxwell's equation (44), which becomes

\[
\frac{d}{d\tau} A = i \Delta A - i \alpha \sum_{\kappa} \delta \rho(\kappa, \kappa - 1),
\]

(48)

together with the linearized equation

\[
\frac{d}{d\tau} \delta \rho(\kappa, \kappa - 1) = -i(2\kappa - 1) \delta \rho(\kappa, \kappa - 1)
\]

\[
+ i \frac{[N(\kappa) - N(\kappa - 1)]}{2N} A.
\]

(49)

form a closed set of equations which underlies the dynamics of the CARL in the linear regime of wave atom optics. We note that the sum of the spatial coherence terms has the physical interpretation

\[
\sum_{\kappa} \delta \rho(\kappa, \kappa - 1) = \langle e^{-2i k_0 z} \rangle,
\]

(50)

which in analogy to Eq. (29) measures of the degree of bunching of the atomic gas.

IV. COLLECTIVE INSTABILITY

The most important feature of the CARL is the appearance of a collective instability, which gives rise to exponential gain under appropriate parameter settings. This instability is characterized by an imaginary frequency component in the spectrum of the probe field \( A(\tau) \). As has been demonstrated in Ref. [4], one needs not solve the complete set of equations derived in the previous sections in order to determine the necessary conditions for the collective instability. Instead, by taking the Laplace
transform of these equations one can derive a "characteristic equation" which allows
one to determine whether exponential gain occurs, and if so what the exponential
growth rate is.

For the Ray Atom Optics model, the Laplace transform of Eq. (32) yields

\[ \tilde{A}_{R}(s) = \frac{A(0)}{R(s)}, \]

where \( R(s) \) is given by

\[ R(s) = \left[ s - i\Delta - i\alpha \int \frac{f(k)dk}{(s + i2k)^2} \right]. \]

In obtaining this result we have taken the continuum limit and assumed that \( B(k) \) and
\( \Pi(k) \) vanish at \( \tau = 0 \), which corresponds to an initially homogeneous (unbunched)
distribution of atoms. Here \( f(k) \) is simply the normalized thermal distribution func­
tion for the dimensionless wavenumber \( k \). We note that in original units an atom
with dimensionless wavenumber \( k \) has momentum \( 2\hbar k_0 k \). The roots of \( R(s) \) give
the characteristic exponents of the CARL. Stability requires that all roots be purely
imaginary. When the collective instability occurs, however, there will be one root
with a positive real part. This real part is the RAO exponential growth rate \( \Gamma_R \).
This result is identical to that obtained by Bonifacio et al [4].

The Wave Atom Optics model, which includes the effects of atomic diffraction,
yields the Laplace transform

\[ \tilde{A}_{W}(s) = \frac{A(0)}{W(s)}. \]

\( W(s) \) is given by

\[ W(s) = \left[ s - i\Delta - i\alpha \int \frac{f(k)dk}{(s + i(2k - 1))(s + i(2k + 1))} \right], \]

where we have again taken the continuum limit and assumed that \( \delta \rho(k, k+1) \) vanishes
at \( \tau = 0 \), which corresponds to an initially unbunched atomic sample. If a root of
$W(s)$ with a positive real part exists, that real part is the WAO exponential growth rate $\Gamma_w$.

We see by comparing Eqs. (51) and (53) that the effect of atomic diffraction is to lift the degeneracy of the singularity under the integral. This expression also leads us immediately to the conclusion that if the width of the momentum distribution $f(k)$ is large compared to $2k$, then the singularity will appear as essentially degenerate, and the effects of matter waves diffraction will be negligible. Thus the RAO and WAO models should agree for large enough temperatures.

**A. Finite temperatures**

In the absence of quantum degeneracies, the thermal momentum distribution is given by the Maxwell-Boltzmann distribution

$$f(k) = \frac{2\beta}{\sqrt{\pi}} e^{-4k^2\beta^2},$$ (55)

where $\beta^2 = T_r/T$ and $T_r = \hbar/\kappa B$ is the recoil temperature, $K_B$ being the Boltzmann constant. By substituting Eq. (55) into Eq. (52) and using the Fourier convolution theorem we find that the RAO exponential growth rate $\Gamma_R$ is determined by the equation

$$s - i\Delta - i\alpha \int_0^\infty pe^{-p^2/4\beta^2 - ps} dp = 0,$$ (56)

which can be integrated to give the transcendental equation

$$s - i\Delta - i2\alpha\beta^2 + i2\sqrt{\pi}\alpha\beta^3 e^{\beta^2 s^2} \text{erfc}(\beta s) = 0.$$ (57)

In contrast, substituting Eq. (55) into Eq. (54) and again using the convolution theorem we find that the WAO exponential growth rate $\Gamma_w$ is determined by equation
By examining Eq. (58) we see that in the case \( \beta \ll 1 \) we are justified in expanding \( \sin(p) \) to lowest order in \( p \). This exactly reproduces Eq. (56), thus showing that the WAO and RAO descriptions make indistinguishable predictions about the exponential growth rate in the limit \( T \gg T_R \). However, for temperatures comparable to or less than the recoil temperature, we will see that the RAO theory fails to correctly predict the behavior of the CARL in the linear regime. Physically, this is due to the fact that it does not account for the effects of atomic diffraction, which tends to counteract the bunching process. Finally, we note that upon integration, Eq. (58) becomes the transcendental equation

\[
s - i\Delta - i\alpha \int_0^{\infty} e^{-p^2/4\beta^2 - \rho^2} \sin(p) dp = 0, \quad (58)
\]

In the next subsection we will examine in more detail the precise manner in which diffraction interferes with the bunching process for the special case of a zero temperature atomic gas. But before turning to this extreme situation, we present numerical results comparing RAO and WAO models at non-zero temperature, as determined by solving Eqs. (57) and (59).

Figures 1(b-d) compare \( \Gamma_R \) with \( \Gamma_W \) at \( \alpha = 10 \) for the three different temperature regimes, \( T = T_R \), \( T = 10T_R \), and \( T = 100T_R \) respectively. Figures 2(b-d) shows the same comparison for \( \alpha = 10^{-1} \). While we see that the behavior of \( \Gamma_R \) and \( \Gamma_W \) depends strongly on \( \alpha \) (recall that \( \alpha \) is proportional to both the pump intensity and the atomic density), the discrepancies between the two models as a function of temperature are very similar. At \( T = T_R \) there are significant differences between the predictions of
FIG. 1. Comparison of the exponential growth rate as a function of pump-probe detuning $\Delta$ between the RAO (solid line) and the WAO (dashed line) models, for the case $\alpha = 10$. (a) shows the results for $T = 0$ (see Sec. III.b), (b) shows the case $T = T_R$, (c) shows $T = 10T_R$, and (d) shows $T = 100T_R$. We see that the ray atom optics model gives the correct result only in the limit $T \gg T_R$. 
FIG. 2. Figure 2 is identical to Figure 1, except we have now taken $\alpha = 10^{-1}$. Since $\alpha$ gives the strength of the bunching process, when it is small the effects of atomic diffraction play a larger role, leading to stronger discrepancies between the predictions of wave atom optics and ray atom optics. However, we see that the RAO limit, given by $T \gg T_R$, is independent of $\alpha$. 
the RAO and WAO models, but these differences become minimal at $T = 10T_R$, and insignificant at $T = 100T_R$. We also observe that the differences are more pronounced for lower values of $\alpha$, meaning that at lower densities and/or pump intensities, the quantum mechanical behavior becomes more apparent. The reason for this is that at high intensities the bunching process, driven by the probe field, dominates, while at low intensities the anti-bunching effects of atomic diffraction play a larger role.

B. The $T = 0$ limit

For a typical atom, the recoil temperature is of the order of microkelvins, e.g. for sodium we have $T_R = 2.4 \mu K$. However, recent advances in cooling techniques have led to measured temperatures as low as the picokelvin regime. At these extreme temperatures the condition $T \ll T_R$ is satisfied, i.e. we are effectively in the $T \rightarrow 0$ limit. In this section we study the $T = 0$ case in detail in order to gain further insight into the exact role of matter wave diffraction in the CARL system.

For the RAO model, we have a single velocity group at $k = 0$. Thus by differentiating Eq. (30) with respect to $\tau$ and using Eq. (31), we see that the bunching parameter $B \equiv B(0) = \exp(-i2k_0z)$ obeys the equation of motion

$$\frac{d^2}{d\tau^2} B = -A, \quad (60)$$

where we have taken $P(0) = 0$ and $N(0) = N$ to indicate that all atoms are initially at rest.

In the WAO description, setting $N(\kappa)/N = \delta_{\kappa,0}$ in Eq. (49) shows that two variables are coupled to the probe field, $\delta \rho(-1,0)$, and $\delta \rho(0,1)$. They describe the recoil of atoms initially at rest as a result of their interaction with the light fields. We proceed then by introducing the new variable $B \equiv \delta \rho(-1,0) + \delta \rho(0,1)$, which has
the same physical meaning as in the RAO model, namely \( B = \langle \exp(-i2k_0z) \rangle \). But in contrast to that case, the time evolution of \( B \) is now governed by the equation of motion

\[
\frac{d^2}{d\tau^2}B = -B - A. \tag{61}
\]

This result shows that in contrast to the predictions of classical mechanics, where the bunching parameter \( B \) has dynamics similar to a free particle driven by the probe field \( A \), quantum mechanically \( B \) behaves as a simple harmonic oscillator of frequency \( 4\omega_r \) (in original time units), and subject to that same driving force. In the linear regime, \( B \) is assumed to be a small perturbation about its initial value of zero, and the forces resulting from a non-zero probe field \( A \) tend to cause \( B \) to increase. But this mechanism is opposed by the "restoring force" due to atomic diffraction.

In addition to opposing any increase in the magnitude of \( B \), the diffraction term also modifies its phase, which may upset any phase relation between \( A \) and \( B \) which might be required for the collective instability to occur.

The RAO model only makes accurate predictions at \( T = 0 \) in the limit \( \omega_r \rightarrow 0 \). Therefore, if we were to increase the mass of the atoms, thus decreasing \( \omega_r \), the behavior at \( T = 0 \) would become more and more classical. This is because heavier atoms suffer less diffraction than lighter atoms. We also note that the correspondence principle states that quantum mechanics should agree with classical mechanics in the limit \( \hbar \rightarrow 0 \), which would also cause \( \omega_r \) to tend to zero. These considerations can also be derived from the statement that the RAO model is valid when \( T \gg T_R \), if we note that as \( \omega_r \rightarrow 0 \) the recoil temperature also goes to zero.

In both the RAO and WAO models, the probe field \( A \) obeys the equation

\[
\frac{d}{d\tau}A = i(\Delta A - \alpha B). \tag{62}
\]
For the RAO model we combine Eq. (62) with (60), and find that the solutions are exponentials with exponents given by the roots of the cubic equation

\[ s^3 - i\alpha s - i\alpha = 0. \]  

(63)

This is exactly the "cold-beam" cubic equation of Bonifacio et al [4]. However with the inclusion of atomic diffraction effects, we now see that the correct "cold-beam" cubic equation, derived from Eqs. (62) and (61), is given by the WAO model to be

\[ s^3 - i\alpha s^2 + s - i(\alpha + \Delta) = 0. \]  

(64)

These equations can also be derived from the Laplace transform method of Sec. III, with the substitution \( f(k) = \delta(k) \), indicating a zero temperature momentum distribution.

From these cubic equations it is possible to determine the point of transition between the stable and the unstable regimes of the CARL. For the RAO model the collective instability occurs provided that the threshold condition

\[ \alpha > \frac{4\Delta^2}{27} \]  

(65)

is satisfied, and above threshold the exponential growth rate is given by

\[ \Gamma_R = \frac{\sqrt{3}}{2} \left( \frac{\alpha}{4} \right)^{1/3} \left| (1 + \sqrt{C})^{2/3} - (1 - \sqrt{C})^{2/3} \right|, \]  

(66)

where \( C = 1 - 4\Delta^3/27\alpha \). For the WAO theory the threshold condition is

\[ \alpha > \frac{2}{27} \left[ (3 + \Delta^2)^{3/2} - 9\Delta + \Delta^3 \right], \]  

(67)

and above threshold the exponential growth rate is given by

\[
\Gamma_W = \frac{\sqrt{3}}{2} \left( \frac{\alpha}{4} \right)^{1/3} \left[ \left( (1 + \sqrt{D})^2 + \frac{4}{27\alpha^2} (1 - \Delta^2)^2 \right)^{1/3} \right. \\
- \left. \left( (1 - \sqrt{D})^2 + \frac{4}{27\alpha^2} (1 - \Delta^2)^2 \right)^{1/3} \right],
\]  

(68)
where

\[ D = 1 - \frac{4\Delta}{3\alpha} \left( 1 - \frac{\Delta^2}{9} \right) - \frac{4}{27\alpha^2} \left( 1 - \Delta^2 \right)^2. \]  \hspace{1cm} (69)

In Figure 3(a) we examine the CARL operating regime, defined as the region in parameter space where the exponential instability occurs, at \( T = 0 \) as it would be if Ray Atom Optics were valid. We contrast this with Figure 3(b) which shows the actual CARL operating regime at \( T = 0 \), as calculated using Wave Atom Optics. From this figure we see that the operating regime of the CARL is drastically reduced at low pump intensities and/or atomic densities when the effects of atomic diffraction are included.

Figure 1(a) compares \( \Gamma_R \) with \( \Gamma_W \) for the case \( \alpha = 10 \) at \( T = 0 \), and Fig. 2(a) shows the same comparison for \( \alpha = 10^{-1} \). We see that atomic diffraction leads to the appearance of a second threshold below which the collective instability does not occur. From Fig. 2(a) we see that this second threshold may even be above \( \Delta = 0 \) for low intensities and/or densities. In fact, the threshold crosses \( \Delta = 0 \) at precisely \( \alpha = 2/3\sqrt{3} \).

Figure 2(a) shows that in the limit of weak pump intensities and/or atomic densities the peak gain for the WAO model tends to \( \Delta = 1 \), while that of the RAO model is at \( \Delta = 0 \). This result can actually be understood quite simply: The atomic center-of-mass dispersion curve tells us that the absorption of a pump photon and the emission of a probe photon by an atom initially at rest creates an energy defect of \( 4\omega_r \) due to atomic recoil. This defect can be compensated by a detuning between the pump and probe, which in dimensionless units occurs at \( \Delta = 1 \). Therefore, the fact that \( \Gamma_W \) is a sharply peaked function around \( \Delta = 1 \) is simply an expression of energy-momentum
FIG. 3. The CARL operating regime (shaded region) as predicted by the RAO model (a), and the actual operating regime (b), as given by the WAO model.
conservation. If we are to take the Ray Atom Optics model seriously at $T = 0$, then we must concede that we are in the limit where $\omega_r \to 0$, therefore, energy-momentum conservation would predict the maximum of $\Gamma_R$ to occur at $\Delta = 0$. In other words, in that limit the center-of-mass dispersion curve is flat over the range of a few photon momenta.

V. DISCUSSION

We have argued that the classical description of atomic center-of-mass motion actually corresponds to a ray-optics description of the atomic Schrödinger field. At high temperatures, it adequately describes the CARL dynamics, however, at sub-recoil temperatures the wave nature of the atomic field becomes apparent, and the ray-optics approximation no longer suffices. Even at high temperatures, where the ray and wave pictures make indistinguishable predictions, their physical interpretations are different: the first considers the atoms as localized distinguishable particles which follow trajectories in phase-space, and the other considers the collection of atoms as a quantum Schrödinger field in which the various normal modes of oscillation are coupled via the atom-photon interactions.

Quantum thermodynamics tells us that in thermal equilibrium, the uncertainty in position of each atom completely fills the volume of its container, independent of the temperature. Therefore, from the quantum field point of view, even at high temperatures it makes no sense in principle to consider the atoms as localized or even as distinguishable particles, even though in practice such a picture works quite well. The differences between the two points of view lead to different physical interpretations of CARL behavior, even though quantitatively they agree completely in the proper limit.
For example, in the classical model the dynamics of a single atom differs vastly from the dynamics of a large number of atoms, hence leading to a distinction between effects which rely on the presence of many atoms and effects which would occur for even a single atom. We note that in our derivation of the RAO model we have made averaging approximations which assume that the atom number is very large, thus it can not be used as a single atom theory just by setting $N = 1$. In contrast, in the quantum picture the atom number appears only as the amplitude of the Schrödinger field, and due to the fact that the atoms are delocalized, no averaging is necessary and the WAO model can be used as a single atom theory by simply setting $N = 1$. Thus we see that, excluding high intensity effects and collisions not included in our model, a single atom CARL will exhibit all possible CARL behavior, provided that the pump intensity is increased to compensate for the small atom number.

Using the quantum picture, it is relatively easy to understand the effects of Doppler broadening on the CARL. These effects have been studied in detail within the framework of the RAO model by Bonifacio et al [4,9,6], and we present arguments here only to illustrate the utility of the quantum picture as well as to discuss what happens to the Doppler broadening effect as one enters the sub-recoil regime.

The fundamental interaction which gives rise to probe amplification in the CARL involves the absorption of a pump photon and the emission of a probe photon, together with a transfer of an atom from the initial mode with dimensionless wavenumber $k$ to a final mode $k - 1$. As can be seen from Eq. (49) this transition rate is proportional to the population difference between an initial and final atomic field modes. This population difference is maximized when the initial atomic field mode coincides with the maximum population gradient, given by $k = -\sigma_k + 1/2$, where $\sigma_k = \sqrt{T/4T_R}$ is the half-width of the Maxwell-Boltzman distribution function. However, this interaction
carries an energy defect given in units of $4\hbar \omega_r$ by $\Delta E = 2\sigma_k + \Delta$, where $\Delta$ is the pump-probe detuning defined by Eq. (23). Therefore, the maximum gain will occur when the pump-probe detuning is chosen to conserve energy for transitions between atomic field modes centered around the maximum population gradient. This leads to the condition $\Delta = 2\sigma_k = \sqrt{T/T_R}$ as maximizing the exponential growth rate for the CARL.

The previous argument applies only to atomic fields with a velocity spread much larger than the recoil velocity. For fields where the spread in velocity is small compared to the recoil velocity a slightly different mechanism occurs. Here, a transition involving atomic field modes separated by two recoil momenta can no longer be centered around the maximum population gradient, as this would result in both levels involved having virtually zero population. Instead, the maximum population difference occurs between the mode with the largest population ($k = 0$) and a virtually empty mode ($k = -1$). This transition carries an energy defect given by $\Delta E = 1 - \Delta$, so the maximum exponential growth rate occurs for pump-probe detunings $\Delta = 1$, as discussed in Sec. IV.B.

Thus we see that the physics of the CARL at high temperatures is different from that at low temperatures. For $T \gg T_R$ maximum gain comes from transitions centered on the maximum population gradient, characterized by the condition $\Delta = \sqrt{T/T_R}$, while for $T \ll T_R$ maximum gain comes from transitions starting from the mode with the largest population, characterized by the condition $\Delta = 1$.

Because the wave-optics picture involves transitions between center-of-mass modes with different atomic velocities, it raises the question of whether or not a $T \approx 0$ samples of atoms is actually destroyed due to the interaction with the pump and probe lasers. Consider that before the interaction, the atoms are practically at rest,
but after absorbing a pump photon and emitting a probe photon, they are moving at
twice the recoil velocity due to atomic recoil, and therefore will eventually leave the
initial sample. This results in atom losses, causing the atomic bunching to decay and
reducing the CARL gain. We argue, however, that these losses will remain negligible
for a realistic experiment involving a BEC with a size on the order of 100 µm. Here
the decay rate should be approximately equal to the atomic velocity divided by the
condensate size. For sodium, which has a recoil velocity of 3 cm/s, the loss rate works
out to $10^2$-$10^3$ s$^{-1}$. The CARL growth rate, on the other hand can be of the order
$4\omega_r$ or larger, which for sodium is of the order $10^5$ s$^{-1}$ or larger. Therefore, on the
time scale of the exponential growth in the CARL, the loss rate due to atoms leaving
the sample is negligible. Additionally, we note that in the linear regime we have
shown that the population of the various atomic field modes remains unchanged, the
effect of the light-matter interaction being solely to generate a spatial coherence, i.e.
bunching, in the form of nonzero off-diagonal density matrix elements $\rho(k, k - 1)$. This
means that while the increase in the atomic bunching is a linear effect, the shift
in atomic population is a second-order effect. The fractional population shift is in
fact given by the square of the bunching parameter. So in the linear regime, we can
consider the sample to experience bunching (spatial coherence) effectively without
population transfer. Of course in the non-linear regime this would no longer be the
case. But here we also need to consider probe reabsorption, which plays a crucial
role in the saturation behavior of CARL, and can reverse the spread in the atomic
sample.
VI. CONCLUSION AND OUTLOOK

The CARL system represents an experimentally realizable non-trivial example of dynamically coupled Maxwell and Schrödinger fields. Thus the theory of the CARL is actually a hybrid which combines ordinary non-linear optics with non-linear atom optics. In ordinary non-linear optics, one benefits greatly when using laser light (characterized by coherent, "single-mode" optical fields) as opposed to incoherent light. By analogy, one should greatly benefit in atom optics when using coherent atomic fields. This has been the primary motivation behind the recent interest in developing the "atom laser" as a source of coherent atomic fields. While the search for a CW atom laser continues, the current state of the art in coherent atomic field generation involves the creation of Bose-Einstein Condensates. A trapped BEC can be thought of as a stationary "atom laser" pulse, and as such is ideal for studying systems of coupled atomic and electromagnetic fields, such as the CARL. However, the temperature of a Bose-Einstein Condensate typically falls well below the atomic recoil temperature, and thus outside of the regime of the current CARL theory.

The main result of this paper has been to develop the Wave Atom Optics model of the CARL, valid in the sub-recoil regime, and to compare this theory to the previous Ray Atom Optics model. We have shown that, as expected, in the sub-recoil regime the behavior of the CARL is strongly influenced by matter-wave diffraction, which tends to counteract the atomic bunching process and reduce the operating regime of the system. However, for temperatures large compared to the recoil temperature we have shown that the two models make indistinguishable predictions.

The present theory quantizes the matter wave, but not the electromagnetic field. It will be of considerable interest to extend it to regimes where both fields need to be quantized. An analysis of the density regime where quantum degeneracy becomes im-
important will also be a fascinating extension, in particular when two-body collisions are included. This study will allow one to investigate to which extent a Bose-Einstein condensate can be manipulated and modified in a far off-resonant CARL configuration. An intriguing possibility would be to generate in this fashion a coupled laser-"atom laser" system. The study of the coherence properties of this system will be the object of future investigations. Finally, a comparison between bosonic and fermionic CARL systems in the quantum degenerate regime should also be considered.

ACKNOWLEDGMENTS

We have benefited from discussions with R. Bonifacio and L. De Salvo, who brought the CARL system to our attention. This work is supported in part by the U.S. Office of Naval Research Contract No. 14-91-J1205, by the National Science Foundation Grant PHY95-07639, by the U.S. Army Research Office and by the Joint Services Optics Program.

APPENDIX: HEISENBERG EQUATIONS OF MOTION FOR DENSITY OPERATORS

The full equations of motion for the expectation values of the density operators are

\[
\frac{d}{dt}\rho_{gg}(k, k') = -\frac{i\hbar}{2m}(k^2 - k'^2)\rho_{gg}(k, k') + g_1a_1^*\rho_{eg}(k - k_1, k') + g_2a_2^*\rho_{eg}(k - k_2, k') + g_1^*a_1\rho_{ge}(k, k' - k_1) + g_2^*a_2\rho_{ge}(k, k' - k_2),
\]

\[
\frac{d}{dt}\rho_{eg}(k, k') = -i \left[ \frac{\hbar}{2m}(k^2 - k'^2) + \omega_0 \right] \rho_{eg}(k, k')
\]
\[ + g_1^* a_1 [\rho_{ee}(k, k' - k_1) - \rho_{gg}(k + k_1, k')] \]
\[ + g_2^* a_2 [\rho_{ee}(k, k' - k_2) - \rho_{gg}(k + k_2, k')] , \]  
(A2)

and

\[ \frac{d}{dt} \rho_{ee}(k, k') = \frac{-i\hbar}{2m} (k^2 - k'^2) \rho_{ee}(k, k') \]
\[ - g_1 a_1^* \rho_{eg}(k, k' + k_1) - g_2 a_2^* \rho_{eg}(k, k' + k_2) \]
\[ - g_1^* a_1 \rho_{ge}(k + k_1, k') - g_2^* a_2 \rho_{ge}(k + k_2, k') . \]  
(A3)


Quantum optics of a Bose-Einstein condensate coupled to a quantized light field

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We consider the interaction between a Bose-Einstein condensate and a single-mode quantized light field in the presence of a strong far-off-resonant pump laser. The dynamics is characterized by an exponential instability, hence the system acts as an atom-photon parametric amplifier. Triggered by a small injection probe field, or simply by quantum noise, entangled atom-photon pairs are created which exhibit nonclassical correlations similar to those seen between photons in the optical parametric amplifier. In addition, the quantum statistics of the master and light field depend strongly on the initial state which triggers the amplifier. Thus, by preparing different initial states of the light field, one can generate matter waves in a variety of quantum states, demonstrating optical control over the quantum statistics of matter waves. [S1050-2947(99)06708-6]

I. INTRODUCTION

In many ways, recently developed Bose-Einstein condensates (BEC's) of trapped alkali-metal atomic vapors [1-3] are the atomic analog of the optical laser. In fact, with the addition of an output coupler, they are frequently referred to as "atom lasers" [3]. Despite many astounding and important differences, the chief similarity behind the analogy is that both optical lasers and atomic BEC's involve large numbers of identical bosons occupying a single quantum state. As a result, the physics of lasers and BEC's involves stimulated processes, which due to Bose enhancement often completely dominate the spontaneous processes which play central roles in the nondegenerate regime.

Just as the discovery of the laser led to the development of nonlinear optics, so too has the advent of BEC's led to remarkable experimental successes in the once theoretical field of nonlinear atom optics [4-6]. Nonlinear optics typically involves the study of multimode mixing, operational by phenomena such as parametric down-conversion and phase conjugation. Due to the presence of collisions, the evolution of the atomic field is also nonlinear, and multimode mixing has been predicted [7-9] and observed in multicomponent condensates [10] as well as in scalar condensates [11].

At the root of most optical phenomena is the dynamical interaction between optical and atomic fields. Under certain circumstances, one can formally eliminate the dynamics of the atomic field, resulting in effective interactions between light waves. Under a different set of conditions, one can eliminate the electromagnetic field dynamics, resulting in effective atom-atom interactions. These are the regimes of nonlinear optics and nonlinear atom optics, respectively. These regimes, therefore, represent limiting cases, where either the atomic or optical field is not dynamically independent, and instead follows the other field in some adiabatic manner which allows for effective elimination.

Outside of these two regimes the atomic and optical fields are dynamically independent, and neither field is readily eliminated. In this paper we investigate the dynamics of coupled quantum degenerate atomic and optical fields in this intermediate regime. In particular, we investigate a system which is analogous to the nondegenerate optical parametric amplifier (OPA) [17,18]. However, whereas the OPA involves the creation of correlated photon pairs, this system involves the generation of correlated atom-photon pairs. The purpose of this paper is to develop a detailed theory for the interaction of quantized atomic and optical fields, with an emphasis on the manipulation and control of their quantum statistics and the generation of quantum correlations and entanglement between matter and light waves.

The specific system we consider consists of a Bose-Einstein condensate driven by a strong far-off-resonant pump laser which interacts with a single mode of an optical ring cavity counterpropagating with respect to the pump. The strong pump laser is treated in the usual manner as a classical, undeviated light field, and furthermore it is assumed to be detuned far enough away from resonance that spontaneous emission may be safely neglected. The cavity field, henceforth referred to as the "probe," is assumed to be weak relative to the pump, and is treated fully quantum mechan-ically as a dynamical variable. It is the dynamical interplay between this probe field and the atomic field which is the subject of interest. The pump serves as a sort of catalyst, inducing a strong atomic dipole moment, thus significantly enhancing the atom-probe interaction.

Assuming that the probe field begins in or near the vacuum state, and the atomic field consists initially of a trapped BEC, the initial dynamics is dominated by a single process: the absorption of a pump photon by a condensate atom followed by the emission of a probe photon. We remark that in the far-off-resonant configuration, this is a two-photon virtual transition in which the excited atomic state population remains negligible. Due to atomic recoil, the absorption-emission process transfers the atom from the condensate ground state to a new state that is shifted in momentum space by the two-photon recoil. This new state constitutes a second condensate component, which can be considered as a momentum side mode to the original

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APPENDIX B

QUANTUM OPTICS OF A BOSE-EINSTEIN CONDENSATE COUPLED TO A QUANTIZED LIGHT FIELD

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ABSTRACT

We consider the interaction between a Bose-Einstein condensate and a single-mode quantized light field in the presence of a strong far off-resonant pump laser. The dynamics is characterized by an exponential instability, hence the system acts as an atom-photon parametric amplifier. Triggered by a small injected probe field, or simply by quantum noise, entangled atom-photon pairs are created which exhibit non-classical correlations similar to those seen between photons in the optical parametric amplifier. In addition, the quantum statistics of the matter and light fields depend strongly on the initial state which triggers the amplifier. Thus by preparing different initial states of the light field, one can generate matter waves in a variety of quantum states, demonstrating optical control over the quantum statistics of matter waves.
I. INTRODUCTION

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At the root of most optical phenomena is the dynamical interaction between optical and atomic fields. Under certain circumstances, one can formally eliminate the dynamics of the atomic field, resulting in effective interactions between light waves. Under a different set of conditions, one can eliminate the electromagnetic field dynamics, resulting in effective atom-atom interactions. These are the regimes of nonlinear optics and nonlinear atom optics, respectively. These regimes, therefore, represent limiting cases, where either the atomic or optical field is not dynamically indepen-
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process: the absorption of a pump photon by a condensate atom followed by the emission of a probe photon. We remark that in the far off-resonant configuration, this is a two-photon virtual transition in which the excited atomic state population remains negligible. Due to atomic recoil, the absorption/emission process transfers the atom from the condensate ground state to a new state that is shifted in momentum space by the two-photon recoil. This new state constitutes a second condensate component, which can be considered as a momentum side mode to the original condensate.

* As this side mode is populated, it begins to interfere with the original condensate, resulting in fringes [21]. These fringes are seen by the pump and probe fields as a spatial density grating, which then enhances the photon scattering process.

This interplay between interference fringes and scattering can act as a positive feedback mechanism, in which case the system is unstable, and is characterized by exponential growth. Any small signal, including quantum noise, will be sufficient to trigger the instability, resulting in the generation of exponentially growing side mode and probe fields. Of course this exponential growth is eventually reversed by high intensity effects, so that the long-time dynamics is characterized by large-amplitude

* This new state may or may not be in the same internal ground state as the original condensate, depending on the polarizations of the pump and probe photons. If the magnetic sublevels are different, then the transition would be termed a Raman transition, if the sublevels are the same, than it may be thought of as Rayleigh scattering or two-photon Bragg scattering [19,20]. As the states are already distinguished by their center-of-mass momentum states, to further distinguish them by an additional quantum number would add nothing. Our model deals specifically with the Bragg scheme, however, with only minimal modifications it could be applied to the Raman scheme as well.
nonlinear oscillations.

At the present time, we focus on the small-signal regime, characterized by exponentially growing fields. In this regime, we demonstrate that the quantum state of the probe and side mode fields depends strongly on the initial conditions, so that, e.g., by injecting a small coherent light field into the probe, one can create an entirely different quantum state than that generated from the amplification of quantum vacuum fluctuations. The differences are manifested in both the quantum statistics of the individual field modes, as well as in non-classical correlations and entanglement between them.

This rest of this paper is organized as follows. Section II gives the background and relates the current theory to previous works in a variety of fields. Section III outlines the basic model for a quantized many-body atomic field interacting with a strong classical pump laser and a quantized optical cavity mode. In Sec. IV, coupled-mode equations are developed for the condensate and its momentum side modes. These equations are then linearized in Sec. V, resulting in a three-mode model which is exactly solvable. Section VI then discusses the exponential instability, with emphasis on the effects of collisions. In Sec. VII the quantum statistics of the atomic and electric fields are investigated, and the extent to which they can be manipulated is determined. In Sec. VIII atom-photon entanglement is discussed, including an examination of two-mode squeezing between atomic and optical fields. Lastly, section IX is a discussion and conclusion, which includes estimates of the important physical parameters, as well as potential experimental obstacles.
II. BACKGROUND

The system we describe is in fact an extension into the ultracold regime of the theoretical work of Bonifacio and coworkers on the Collective Atomic Recoil Laser (CARL) [22,23]. The original CARL theory treated the atomic center-of-mass motion classically, an approximation certainly valid for hot atoms, but not sufficient to describe ultracold samples such as BEC’s. Within this framework, the feedback mechanism which gives rise to the exponential instability in the CARL was outlined using a slightly different, but complementary physical picture, where the classical atomic center-of-mass motion in the optical potential of the counterpropagating pump and probe fields is responsible for the grating formation. The theory was extended to the limit of zero temperature by assuming that all of the (classical) atoms begin from rest, leading to the discovery of the so-called ‘CARL cubic equation’, which gives the exponential growth rate of the instability in terms of the relevant system parameters. Out of a desire to better understand the quantum statistics of the probe field, an attempt at a quantum $T = 0$ theory was made [24]. However, this attempt explicitly assumed that the wavefunctions of the individual atoms could be localized in both momentum and position space to an extent which violates the Heisenberg uncertainty principle. Thus rather than being a true quantum theory, it still treated the atoms as following ‘classical’ trajectories, but now with small ‘quantum’ fluctuations included.

Both the original classical CARL model, as well as the later ‘quantum’ model, fall within the ray-optics approximation for the atomic field. Clearly then, one would expect such models to break down as soon as the atomic deBroglie wavelength becomes comparable to the period of the optical potential formed by the pump and probe fields. As the wavelength of the optical potential is twice the optical wavelength, this breakdown should occur near the atomic recoil temperature, which for typical alkali
atoms is on the order of microKelvins. As subrecoil temperature atomic vapors are achieved routinely through a variety of cooling techniques, a theory which properly treats the quantum motion of the atoms is required if one desires to investigate the behavior of the CARL in this regime.

With the ultimate intent of extending the CARL theory into the BEC regime, so that the unique coherence properties of condensates might be further understood and exploited by the interaction with dynamical light fields, a quantum model of the atomic motion was formulated [25], where it was confirmed that the ray-optics versions did indeed break down for temperatures of the order of the recoil temperature or below. In fact, at $T = 0$, a second threshold for the existence of the exponential instability was discovered, occurring when the bunching process is overcome by matter-wave diffraction. For $T > T_R$, however, it was shown that the previous theories make indistinguishable predictions from the quantum theory. We remark that while in this work the atomic center of mass motion was treated quantum mechanically, the light fields were still treated classically, hence predictions concerning the quantum statistics of either the atomic or optical fields could not be made.

A full quantum model of both the atomic and optical fields was recently outlined in [26], where the subjects of manipulating quantum statistics and atom-photon entanglement were first addressed. The present paper is a detailed elaboration and extension of that work, including significant new physics. For example, utilizing the familiar s-wave scattering approach of BEC theory, the effects of atom-atom collisions are incorporated into the CARL theory for the first time. Also, in an extension of the OPA analogy, the existence of two-mode squeezing is shown to occur between a condensate side mode and the probe optical field. The current approach also differs from earlier work in that the familiar spontaneous symmetry breaking technique is
no longer applied to the condensate. Instead it is assumed that a condensate well below the critical temperature is better described by a number state than a coherent state, as recent work appears to demonstrate [27,28].

The fully quantum model is similar in many ways to a system first studied by Zeng and coworkers [29], and later extended by Kuang [30], in which the principle of manipulating the quantum statistics of a condensate by its interaction with a quantized light field was first proposed. These papers, however, do not recognize the existence of unstable (exponential) solutions nor the fact that the system can be triggered from quantum noise. We note that the unstable (exponential) solutions, and the possibility to initiate them from quantum vacuum fluctuations are both crucial components of this present work. Lastly, we mention the connection to recent work on matter-wave amplification by Law and Bigelow [31], which also explores the interaction between condensates and quantized light fields. In that work, however, the light field is assumed to be heavily damped, thus allowing for its dynamical elimination. As a result, only the properties of the atomic field are studied in detail.

CARL theory, including the present version, is also closely related to the theory of Recoil Induced Resonances (RIR) [32], in which the effects of atomic recoil on the pump-probe spectroscopy of an atomic vapor is investigated. This theory treats the atomic center-of-mass motion quantum mechanically. The probe field, however, is not typically treated as a dynamical variable. Hence, it does not include the effects of probe feedback, which are necessary for exponential behavior. A detailed comparison of the RIR and CARL theories is given in [33].
III. THE BASIC MODEL

In this section we derive a fully quantized model of a gas of bosonic two-level atoms which interact with a strong, classical, undepleted pump laser and a weak, quantized optical ring cavity mode, both of which are assumed to be tuned far away from atomic resonances. As a result, single-photon transitions between atomic internal ground and excited states are highly non-resonant and the excited state population remains negligible. In this case, one can safely neglect the effects of spontaneous emission as well as the two-body dipole-dipole interaction.

We must still, however, allow for two-photon virtual transitions in which the atomic internal state remains unchanged, but due to recoil may result in a change in the atom’s center-of-mass motion. For example, an atom which absorbs a pump photon and emits a probe photon experiences a recoil kick equal to the difference of the momenta of the two photons (which for nearly counterpropagating pump and probe beams is of the order of two optical momenta). These transitions, therefore, couple different states of the atomic center-of-mass motion. Due to the quadratic dispersion relation of the atoms, these transitions will in general be non-resonant. For very cold atoms, the resultant detunings are typically on the order of the recoil frequency, i.e. much smaller than the natural linewidth of the atomic transition, $\gamma_a$, whereas the one-photon transitions which we are neglecting have a detuning many orders of magnitude larger than $\gamma_a$.

Our theory begins with the second-quantized Hamiltonian

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{probe}} + \hat{H}_{\text{atom-probe}} + \hat{H}_{\text{atom-pump}} + \hat{H}_{\text{atom-atom}},$$

(1)

where $\hat{H}_{\text{atom}}$ and $\hat{H}_{\text{probe}}$ give the free evolution of the atomic field and the probe
mode respectively, $\hat{H}_{\text{atom-probe}}$ and $\hat{H}_{\text{atom-pump}}$ describe the dipole coupling between the atomic field and the probe mode and pump laser, respectively, and $\hat{H}_{\text{atom-atom}}$ contains the two-body s-wave scattering collisions between ground state atoms.

The free atomic Hamiltonian is given by

$$\hat{H}_{\text{atom}} = \int d^3r \left[ \hat{\Psi}_g^\dagger(r) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_g(r) \right) \hat{\Psi}_g(r) + \hat{\Psi}_e^\dagger(r) \left( -\frac{\hbar^2}{2m} \nabla^2 + \hbar \omega_a + V_e(r) \right) \hat{\Psi}_e(r) \right].$$

where $m$ is the atomic mass, $\omega_a$ is the atomic resonance frequency, $\hat{\Psi}_e(r)$ and $\hat{\Psi}_g(r)$ are the atomic field operators for excited and ground state atoms respectively, and $V_g(r)$ and $V_e(r)$ are their respective trap potentials. The atomic field operators obey the usual bosonic equal time commutation relations $[\hat{\Psi}_j(r), \hat{\Psi}_j^\dagger(r')] = \delta_{j,j'} \delta^3(r - r')$, and $[\hat{\Psi}_j(r), \hat{\Psi}_{j'}(r')] = [\hat{\Psi}_j^\dagger(r), \hat{\Psi}_{j'}^\dagger(r')] = 0$, where $j, j' = \{e, g\}$.

The free evolution of the probe mode is governed by the Hamiltonian

$$\hat{H}_{\text{probe}} = \hbar ck \hat{A}^\dagger \hat{A},$$

where $c$ is the speed of light, $k$ is the magnitude of the probe wave number $k$, and $\hat{A}$ and $\hat{A}^\dagger$ are the probe photon annihilation and creation operators, satisfying the boson commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$. The probe wavenumber $k$ must satisfy the periodic boundary condition of the ring cavity, $k = 2\pi \ell / L$, where the integer $\ell$ is the longitudinal mode index, and $L$ is the length of the cavity.

The atomic and probe fields interact in the dipole approximation via the Hamiltonian

$$\hat{H}_{\text{atom-probe}} = -i\hbar g \hat{A} \int d^3r \hat{\Psi}_e^\dagger(r) e^{ikr} \hat{\Psi}_g(r)$$

$$+ H.c.,$$

(4)
where \( g = d[ck/(2\hbar e_0 LS)]^{1/2} \) is the atom-probe coupling constant. Here \( d \) is the magnitude of the atomic dipole moment, and \( S \) is the cross-sectional area of the probe mode in the vicinity of the atomic sample (where it is assumed to be approximately constant across the length of the atomic sample).

In addition, the atoms are driven by a strong pump laser, which is treated classically and assumed to remain undepleted. The atom-pump interaction Hamiltonian is given in the dipole approximation by

\[
\hat{H}_{\text{atom-pump}} = \frac{\hbar \Omega_0}{2} e^{-i\omega_0 t} \int d^3r \hat{\Psi}_e^\dagger (r) e^{i\mathbf{k}_0 \cdot \mathbf{r}} \hat{\Psi}_g (r)
\]

\[
+ H.c.,
\] (5)

where \( \Omega_0 \) is the Rabi frequency of the pump laser, related to the pump intensity \( I_0 \) by \( |\Omega_0|^2 = 2d^2 I_0/\hbar^2 e_0 c \), \( \omega_0 \) is the pump frequency, and \( k_0 \approx \omega_0/c \) is the pump wavenumber. The approximation indicates that we are neglecting the index of refraction inside the atomic gas, as we assume a very large detuning \( \Delta = \omega_0 - \omega_a \) between the pump frequency and the atomic resonance frequency.

Finally, the collision Hamiltonian is taken to be

\[
\hat{H}_{\text{atom-atom}} = \frac{2\pi \hbar^2 \sigma}{m} \int d^3r \hat{\Psi}_g^\dagger (r) \hat{\Psi}_g (r) \hat{\Psi}_g (r),
\] (6)

where \( \sigma \) is the atomic s-wave scattering length. This corresponds to the usual s-wave scattering approximation, and leads in the Hartree approximation to the standard Gross-Pitaevskii equation for the ground state wavefunction (in the absence of the driving optical fields).

We limit ourselves to the case where the pump laser is detuned far enough away from the atomic resonance that the excited state population remains negligible, a condition which requires that \( \Delta \gg \gamma_a \). In this regime the atomic polarization adiabatically follows the ground state population, allowing the formal elimination of
the excited state atomic field operator. We proceed by introducing the operators
\( \hat{\psi}_e^f(r) = \hat{\psi}_e(r)e^{i\omega_0 t} \) and \( \hat{\omega} = \hat{\omega}e^{i\omega_0 t} \), which are slowly varying relative to the optical driving frequency. The new excited state atomic field operator obeys then the Heisenberg equation of motion

\[
\frac{d}{dt}\hat{\psi}_e^f(r) = i\Delta \hat{\psi}_e^f(r) - \left[ \frac{i\Omega_0}{2} e^{ik_0 r} + g\hat{\omega} e^{ikr} \right] \hat{\psi}_g(r),
\]

where we have dropped the kinetic energy and trap potential terms under the assumption that the lifetime of the excited atom, which is of the order \( 1/\Delta \), is so small that atomic center-of-mass motion may be safely neglected during this period. For the same reason, we are justified in neglecting collisions between excited atoms, or between excited and ground state atoms in the collision Hamiltonian (6).

We now adiabatically solve for \( \hat{\psi}_e^f(r) \) by formally integrating Eq. (7) under the assumption that \( \hat{\psi}_g(r) \) varies on a time scale which is much longer than \( 1/\Delta \). This yields

\[
\hat{\psi}_e^f(r, t) \approx \frac{1}{\Delta} \left[ \frac{\Omega_0}{2} e^{ik_0 r} - ig\hat{\omega}(t)e^{ikr} \right] \hat{\psi}_g(r, t) - \frac{1}{\Delta} \left[ \frac{\Omega_0}{2} e^{ik_0 r} - ig\hat{\omega}(0)e^{ikr} \right] \hat{\psi}_g(r, 0)e^{i\Delta t} + \hat{\psi}_e^f(r, 0)e^{i\Delta t}.
\]

The third term on the r.h.s. of Eq. (8) can be neglected for most considerations if we assume that there are no excited atoms at \( t = 0 \), so that this term acting on the initial state gives zero. The second term may also be neglected, as it is rapidly oscillating at frequency \( \Delta \), and thus its effect on the ground state field operator is negligible when compared to that of the first term, which is non-rotating. 

\[\text{We note that in much of the literature the second and third terms are simply ignored. We}\]
Dropping the unimportant terms, and then substituting Eq. (8) into the equation of motion for \( \hat{\Psi}_g(\mathbf{r}) \), we arrive at the effective Heisenberg equation of motion for the ground state field operator

\[
\frac{d}{dt} \hat{\Psi}_g(\mathbf{r}) = i \left[ \frac{\hbar}{2m} \nabla^2 - \frac{V_g(\mathbf{r})}{\hbar} - \frac{4\pi\hbar\sigma}{m} \hat{\Psi}_g(\mathbf{r}) \right. \\
\left. - \frac{g|\Omega_0|}{2|\Delta|} \left( \hat{a} e^{iK \cdot \mathbf{r}} + \hat{a}^\dagger e^{-iK \cdot \mathbf{r}} \right) \\
- \left( \frac{|\Omega_0|^2}{4\Delta} + \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \right) \right] \hat{\Psi}_g(\mathbf{r}),
\]

where \( K = k - k_0 \) is the recoil momentum kick the atom acquires from the two-photon transition, and we have introduced the new slowly-varying probe field operator \( \hat{a} = -i(\Omega_0 \Delta/(|\Omega_0| |\Delta|)) \hat{\Psi}_g(\mathbf{r}) \), which still obeys the boson commutation relation \([\hat{a}, \hat{a}^\dagger]\) = 1. Here, the second to last term is simply the optical potential formed from the counterpropagating pump and probe light fields, and the last term gives the spatially independent light shift potential, which can be thought of as cross-phase modulation between the atomic and optical fields.

To complete our model, in addition to Eq. (9), we also require the equation of motion for the slowly varying probe field operator. By again substituting Eq. (8), we find that it obeys

\[
\frac{d}{dt} \hat{a} = i\delta' \hat{a} - i\frac{g|\Omega_0|}{2|\Delta|} \int d^3 \mathbf{r} \hat{\Psi}_g^\dagger(\mathbf{r}) e^{-iK \cdot \mathbf{r}} \hat{\Psi}_g(\mathbf{r}),
\]

where \( \delta' = \omega_0 - \omega \), is the detuning between the pump and probe fields. The probe frequency is given by \( \omega \approx ck \), again assuming that the index of refraction inside the condensate is negligible.

choose to keep them temporarily to demonstrate that the commutation relation for \( \hat{\Psi}_e(\mathbf{r}) \) is preserved (to order \( 1/\Delta \)) by the procedure of adiabatic elimination.
IV. COUPLED-MODE EQUATIONS

We assume that the atomic field is initially in a Bose-Einstein condensate with mean number of condensed atoms $N$. Furthermore, we assume that $N$ is very large, and that the condensate temperature is small compared to the critical temperature. These assumptions allow us to neglect the noncondensed fraction of the atomic field. Thus our model does not include any effects of condensate number fluctuations.

We now introduce the atomic field operator which annihilates an atom in the condensate ground state

$$\hat{c}_0 = \int d^3 r \varphi_0^*(r) \hat{\psi}_g(r),$$

where $\varphi_0(r) = \langle r | \varphi_0 \rangle$ satisfies the time-independent Gross-Pitaevskii equation

$$\left( \frac{\hbar}{2m} \nabla^2 - \frac{V_0(r)}{\hbar} - \frac{4\pi \hbar \sigma}{m} N|\varphi_0(r)|^2 + \frac{\mu}{\hbar} \right) \varphi_0(r) = 0,$$

$\mu$ being the chemical potential. By differentiating Eq. (11) with respect to time, and inserting Eqs. (9) and (12) we find that the equation of motion for $\hat{c}_0$ is

$$\frac{d}{dt} \hat{c}_0 = -i \left( \frac{\mu}{\hbar} + \frac{\Omega_0^2}{4\Delta} + \frac{g^2}{\Delta} \hat{a} \hat{a} \right) \hat{c}_0$$

$$+ i \frac{4\pi \hbar \sigma}{m} \int d^3 r \varphi_0^*(r) \left( N|\varphi_0(r)|^2 - \hat{\psi}_g^*(r) \hat{\psi}_g(r) \right)$$

$$- i \frac{g|\Omega_0|}{2|\Delta|} \int d^3 r \varphi_0^*(r) \left( \hat{a} e^{iK \cdot r} + \hat{a}^\dagger e^{-iK \cdot r} \right) \hat{\psi}_g(r).$$

From this equation we see that the effect of the optical fields is to couple the condensate mode to two side modes, whose wavefunctions are given by

$$\langle r | \varphi_{\pm} \rangle = \varphi_0(r) e^{\pm iK \cdot r}.$$
condensate \( T = 0 \) we assume that collisions alone do not populate any new atomic states.

Defining the field operators for the (first-order) condensate side modes as

\[
C_\pm = \int d^3r \langle \varphi_\pm | \mathbf{r} \rangle \hat{\psi}_\pm (\mathbf{r}),
\]

(15)

allows us to reexpress the equation of motion for the condensate mode field operator as

\[
\frac{d}{dt} \hat{c}_0 = -i \left( \frac{\mu}{\hbar} + \frac{|\Omega_0|^2}{4 \Delta} + \frac{g^2}{\Delta} \hat{a}^\dagger \hat{a} \right) \hat{c}_0
\]

\[+ i \frac{4 \pi \hbar \sigma}{m} \int d^3r \varphi_0^* (\mathbf{r}) \left[ N |\varphi_0(\mathbf{r})|^2 - \hat{\psi}_\pm^\dagger (\mathbf{r}) \hat{\psi}_\pm (\mathbf{r}) \right] \hat{\psi}_\pm (\mathbf{r}) \]

\[+ i \Omega_0 \left( \hat{a}_- \hat{c}_+ + \hat{a}^\dagger \hat{c}_- \right),
\]

(16)

where the operators \( \hat{c}_j \) obey the bosonic commutation relations

\[
[\hat{c}_j, \hat{c}^\dagger_{j'}] = \langle \varphi_j | \varphi_{j'} \rangle; \quad j, j' = \{-, 0, +\},
\]

(17)

all other commutators being equal to zero.

We note that the three states, \( |\varphi_0\rangle \), and \( |\varphi_\pm\rangle \) are not mutually orthogonal, as their overlap integrals are given by

\[
\langle \varphi_\pm | \varphi_\pm \rangle = \int d^3r |\varphi_0(\mathbf{r})|^2 e^{\pm i \mathbf{K} \cdot \mathbf{r}}
\]

(18)

For most condensate sizes and trap configurations, however, these integrals are many orders of magnitude smaller than unity. As a result, for 'typical' condensates, the orthogonality approximation

\[
\langle \varphi_j | \varphi_{j'} \rangle = \delta_{jj'},
\]

(19)
yields accurate results. The range of validity of this approximation is discussed in Appendix A, where we briefly examine how the theory should be modified to properly take this non-orthogonality into account. In the following, however, we assume the validity of Eq. (19), so that the states $|\varphi_0\rangle$, and $|\varphi_{\pm}\rangle$ can be considered as well defined and distinct modes of the atomic field.

We now derive the Heisenberg equations for the momentum side mode field operators, found by differentiating Eq. (15) with respect to time and again inserting Eq. (9), yielding

\[
\frac{d}{dt} \hat{c}_{-,+} = -i \left( \frac{\mu}{\hbar} + \frac{\hbar K^2}{2m} + \frac{|\Omega_0|^2}{4\Delta} + \frac{g_2^2}{\Delta} \right) \hat{c}_{-,+} + \frac{4\pi \hbar \sigma}{m} \int d^3r \langle \varphi_{-,+}(r)| (N|\varphi_{0}(r)|^2 - \hat{\psi}_g^\dagger(r) \hat{\psi}_g(r)) \hat{\psi}_g(r) \\
- \frac{i g_2|\Omega_0|}{2|\Delta|} (\hat{a}^\dagger \hat{c}_{0,+2} + \hat{a} \hat{c}_{-2,0}) + \frac{\hbar K k \hat{b}_{-,+}}{m},
\]

where we have introduced four new field operators $\hat{c}_{\pm 2}$ and $\hat{b}_{\pm}$.

The operators $\hat{c}_{\pm 2}$, which have the definitions

\[
\hat{c}_{\pm 2} = \int d^3r \langle \varphi_{\pm 2}(r)| \hat{\psi}_g(r),
\]

are the annihilation operators for the second-order side modes

\[
\langle r|\varphi_{\pm 2}\rangle = \varphi_{0}(r)e^{\pm i2K \cdot r},
\]

These modes will be optically coupled to third-order side modes, and so on so that a full theory of the nonlinear response of the system should include the entire manifold of side modes. In this paper, however, we focus on the linear regime, where only the first-order side modes contribute significantly.

The operators $\hat{b}_{\pm}$ have the definitions

\[
\hat{b}_{\pm} = \int d^3r \langle \varphi_{\pm}(r)| \hat{\psi}_g(r),
\]
where

\[ (r|\phi_{\pm}) = (Kk_c)^{-1} e^{\pm iK \cdot r} (\pm iK \cdot \nabla \varphi_0(r)). \]  

Here \( k_c \) is the momentum width of the condensate state along \( K \), and is roughly given by \( k_c \sim 1/W_c \), where \( W_c \) is the size of the condensate along \( K \). The factor \( (Kk_c)^{-1} \) is simply a normalization coefficient.

To understand the physical meaning of the \( \hat{b}_{\pm} \) term in Eq. (20), consider what happens to a single atom after it is transferred into the state \( \psi(r) = \varphi_0(r) \exp(-iK \cdot r) \) at time \( t = 0 \). Under free evolution the wavepacket of the atom, which initially has the shape of the condensate ground state, will move with group velocity \( \hbar K/m \) and spread at the velocity \( \hbar k_c/m \). This evolution is described by the propagation equation

\[ \psi(r, t) = \exp[i(t\hbar/2m)\nabla^2] \psi(r, 0), \]

which for short enough times becomes

\[ \psi(r, t) \approx \left(1 - it\frac{\hbar K^2}{2m}\right)(r|\varphi_-) + t\frac{\hbar}{m}(K \cdot \nabla \varphi_0(r))e^{-iK \cdot r} + it\frac{\hbar}{2m}(\nabla^2 \varphi_0(r))e^{-iK \cdot r}. \]  

The first term on the r.h.s. of Eq. (26) gives a phase shift due to the kinetic energy of the atom, the second term contributes a translational shift, and the third and final term describes spreading. If we include the effects of the trap potential and collisions, this last term vanishes as all spreading effects are balanced by the trap potential for the ground state \( \varphi_0(r) \). From Eqs. (26) and (24) we see that the state of the atom at time \( t \) can then be viewed as a coherent superposition of the state \( |\varphi_-\rangle \) and the state \( |\phi_-\rangle \). Thus the coupling to \( \hat{b}_- \) in Eq. (20) corresponds physically to translational motion of the side mode wavepacket at the recoil velocity \( v_r = \hbar K/m \). Since the probability at time \( t \) that the atom is still in the ground state is simply the overlap
between $\psi(r,t)$ and $\psi(r,0)$, it is clear that for times $t \ll W_c/v_r$ this probability will be essentially unity, and the coupling to $\hat{b}_-$ can be ignored.

V. LINEARIZED THREE-MODE MODEL

From Eq. (20), we see that the first-order side modes are optically coupled to both the condensate mode and to second-order side modes. For times short enough that the condensate is not significantly depleted, the coupling back into the condensate is subject to Bose enhancement due to the presence of $\sim N$ identical bosons in this mode. The coupling to the second-order side mode, in contrast, is not enhanced. Hence for these time scales, the higher-order side modes are not expected to play a significant role. In addition, we consider only times $t \ll W_c/v_r$, so that the translational coupling can be neglected.

These arguments suggest developing an approach where the three atomic field operators $\hat{c}_0$, $\hat{c}_-$, and $\hat{c}_+$ play a predominant role. Therefore, we expand the atomic field operator as

$$\hat{\psi}_p(r) = (r|\varphi_0)\hat{c}_0 + (r|\varphi_-)\hat{c}_- + (r|\varphi_+)\hat{c}_+ + \hat{\psi}(r),$$

where the field operator $\hat{\psi}(r)$ acts only on the orthogonal complement to the subspace spanned by the state vectors $|\varphi_0\rangle$, $|\varphi_-\rangle$, and $|\varphi_+\rangle$. As a result, $\hat{\psi}(r)$ commutes with the creation operators for the three central modes.

In the next step, we use Eq. (27) to expand the atomic polarization and collision terms in Eqs. (10), (16), and (20), with the eventual goal of deriving a closed set of operator equations which fully describes the system dynamics. At present, we are considering four dominant modes, the condensate and first-order side modes, as well as the optical probe mode. In the linear regime, however, we will see that the
condensate mode can be dynamically eliminated, resulting in an effective three-mode model.

In expanding the polarization and collision terms by means of Eq. (27), there are two principal considerations in determining which are the dominant terms. The first is Bose enhancement, which, in the regime of negligible condensate depletion, strongly selects transitions involving the condensate mode. In order to estimate this effect, we assign a weight of $\sqrt{N}$ for each occurrence of the operators $c_0$ and $c_0^\dagger$ in a given term. The second consideration is momentum conservation, which comes from the spatial integration in the polarization and collision terms. Integrals over slowly varying functions such as $|\varphi_0(r)|^2$, or $|\varphi_0(r)|^4$ are 'momentum selected' and dominate over integrals of rapidly oscillating functions such as $|\varphi_0(r)|^2 \exp(-iK \cdot r)$.

With this approach we find that the equation of motion for the probe field operator (10) becomes

$$\frac{d}{dt} \hat{a} = i \hat{\sigma} \hat{a} - i \frac{g|\Omega_0|}{2|\Delta|} \left[ \hat{c}_0^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_+ \right]. \tag{28}$$

Thus we see that the probe annihilation operator is coupled to the bilinear atomic field operators $\hat{c}_0^\dagger \hat{c}_0$ and $\hat{c}_0^\dagger \hat{c}_+$. These operators correspond physically to interference fringes, i.e. a periodic modulation of the atomic density, which appear because the atoms are in a coherent superposition of the side mode and condensate states. Gain in the probe can thus be interpreted as Bragg scattering of the pump due to the presence of interference fringes.

By inserting (27) into Eq. (16) we further find that the equation of motion for $\hat{c}_0^\dagger \hat{c}_0$ is given to leading order in the collision and optical terms by

$$\frac{d}{dt} \hat{c}_0^\dagger \hat{c}_0 = \frac{i}{m} \frac{8\pi \hbar F_0}{m} \hat{c}_- \hat{c}_0 \hat{c}_0^\dagger \hat{c}_0 + \frac{i}{2|\Delta|} g|\Omega_0| \hat{a}^\dagger (\hat{c}_- \hat{c}_0 - \hat{c}_0^\dagger \hat{c}_+) + H.c., \tag{29}$$
where

\[ F_0 = \int d^3r |\varphi_0(r)|^4. \]  

(30)

Similarly we find that the operators \( \hat{c}_0^\dagger \hat{c}_0 \) and \( \hat{c}_0^\dagger \hat{c}_+ \) obey the equations

\[
\frac{d}{dt} \hat{c}_0^\dagger \hat{c}_0 = \frac{i\hbar K^2}{2m} \hat{c}_0^\dagger \hat{c}_0 + \frac{4\pi \hbar \sigma F_0}{m} \left( \hat{c}_0^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_+ \right) \hat{c}_0^\dagger \hat{c}_0 \\
+ i \frac{g|\Omega_0|^2}{2|\Delta|} \hat{a} \hat{c}_0^\dagger \hat{c}_0, 
\]  

(31)

and

\[
\frac{d}{dt} \hat{c}_0^\dagger \hat{c}_+ = -\frac{i\hbar K^2}{2m} \hat{c}_0^\dagger \hat{c}_+ - \frac{4\pi \hbar \sigma F_0}{m} \left( \hat{c}_0^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_+ \right) \hat{c}_0^\dagger \hat{c}_+ \\
- i \frac{g|\Omega_0|^2}{2|\Delta|} \hat{a} \hat{c}_0^\dagger \hat{c}_0. 
\]  

(32)

We assume that all \( N \) atoms are initially in the condensate mode, so that

\[
|\psi\rangle_{t=0} = \frac{1}{\sqrt{N!}} \left( \hat{c}_0^\dagger \right)^N |0\rangle, 
\]  

(33)

|0\rangle being the vacuum state. We proceed by linearizing the atomic field operators around their initial expectation values, which can be determined from Eq. (33), together with the approximate commutation relations given by Eq. (19). This yields

\[
\hat{c}_0^\dagger \hat{c}_0 = N(1 + \hat{\delta}_0), 
\]  

(34)

\[
\hat{c}_0^\dagger \hat{c}_- = N\hat{\delta}_-, 
\]  

(35)

and

\[
\hat{c}_0^\dagger \hat{c}_+ = N\hat{\delta}_+, 
\]  

(36)

where \( \hat{\delta}_0, \hat{\delta}_- \) and \( \hat{\delta}_+ \) are therefore infinitesimal operators. In addition, we introduce a rescaled probe field operator
\[ \dot{\hat{a}} = \frac{\hat{a}}{\sqrt{N}}, \]

which, provided that the mean number of photons in the probe mode is small compared to \( N \), is also infinitesimal. This constraint is consistent with the assumption of negligible condensate depletion.

Inserting these definitions into the equation of motion (29) for \( \hat{c}_0^\dagger \hat{c}_0 \), and keeping only terms linear in the infinitesimal operators, gives

\[ \frac{d}{dt} \dot{\hat{c}}_0 = 0, \]

which has the trivial solution \( \dot{\hat{c}}_0 = 0 \). As a result, this operator can be dropped from the linearized equations for \( \dot{\hat{c}}_- \) and \( \dot{\hat{c}}_+ \). This leads to a set of three coupled infinitesimal operators whose linearized equations of motion can be expressed as

\[ \frac{d}{d\tau} \ddot{\delta} = i\mathbf{M}\ddot{\delta}, \]

where \( \ddot{\delta} = (\dot{\hat{c}}_a, \dot{\hat{c}}_-, \dot{\hat{c}}_+) \), the matrix \( \mathbf{M} \) is given by

\[ \mathbf{M} = \begin{pmatrix} \delta & \chi & -\chi \\ \chi & (1 + \beta) & \beta \\ -\chi & -\beta & -(1 + \beta) \end{pmatrix}, \]

and we have introduced the dimensionless time \( \tau = \omega_r t \), \( \omega_r = \hbar K^2/2m \) being the atomic recoil frequency, as well as the dimensionless control parameters

\[ \chi = \frac{g|\Omega_0|\sqrt{N}}{2|\Delta|\omega_r}, \]

\[ \delta = \delta'/\omega_r, \]

and

\[ \beta = \frac{4\pi \hbar \sigma NF_0}{m\omega_r}. \]
Here $\chi$ is a dimensionless atom-probe coupling constant, $\delta$ is the pump-probe detuning in units of the atomic recoil frequency, and $\beta$ gives the strength of collisions between the side modes. In Appendix B we give the effective Hamiltonian from which Eq. (39) can be derived.

The solution to Eq. (39) is then given by

$$\tilde{\delta}(\tau) = e^{i M \tau} \tilde{\delta}(0).$$  \hspace{1cm} (44)

From this we see that the time dependence of the infinitesimal operators is determined by the eigenvalue spectrum of the matrix $M$. For certain values of the control parameters $\chi$, $\delta$, and $\beta$, one of the eigenvalues contains a negative imaginary part. When this occurs, the infinitesimal operators undergo exponential growth. This exponential instability is the focus of the next section, where we discuss its properties in detail.

VI. EXPONENTIAL INSTABILITY

The eigenvalues of $M$ are determined by the characteristic equation

$$\omega^3 - \delta \omega^2 - (1 + 2\beta) \omega + (1 + 2\beta) \delta + 2\chi^2 = 0,$$ \hspace{1cm} (45)

which has either three real solutions, or one real and a pair of complex conjugate solutions. In the first case, the system is stable and exhibits only small oscillations around its initial state. In the second case, the system is unstable and grows exponentially, even from noise.

From Eq. (45) one finds that exponential instability occurs when

$$\chi^2 > [(3 + 6\beta + \delta^2)^{3/2} + \delta^3 - 9\delta(1 + 2\beta)]/27.$$ \hspace{1cm} (46)
FIG. 1. Exponential instability region in the $\delta$-$\chi^2$ plane. The shaded area gives the unstable domain in the absence of collisions ($\beta = 0$). The dashed curves show how the boundaries change as collisions are included. They correspond to the cases $\beta = .3$ (dashed) and $\beta = 1.0$ (dotted).
In Fig. 1 we plot the region of instability as a function of $\delta$ and $\chi^2$. The shaded region of Fig.1 corresponds to the instability region in the absence of collisions ($\beta = 0$). As collisions are added, the boundaries shift, illustrated by the dashed and dotted curves, which show the boundaries for the cases $\beta = .3$ and $\beta = 1.0$.

From Fig. 1, we see that for positive $\delta$ the boundary asymptotically reduces to

$$\chi^2 > 2\delta^3/27 - (1 + 2\beta)\delta/6$$  \hspace{1cm} (47)

i.e., the threshold value of $\chi^2$ increases with the third power of the detuning and is only weakly influenced by the presence of collisions. On the other hand, for negative $\delta$ the asymptotic behavior is given by

$$\chi^2 > -(1 + 2\beta)\delta/2,$$  \hspace{1cm} (48)

which only grows linearly with $\delta$ and is strongly affected by interatomic collisions, which have the effect of reducing the unstable region. In earlier work [25], it was shown that this lower threshold occurs in the absence of collisions when atomic diffraction overcomes the bunching process. For positive scattering lengths, we note that formation of a density grating increases the mean field energy. Collisions, therefore, should join diffraction in opposing the bunching process, resulting in a higher threshold for the instability. For negative scattering lengths, on the other hand, bunching reduces the mean field energy; hence collisions should enhance the bunching process and oppose diffraction, thus lowering the threshold. Lastly, we note that the instability region for $\chi^2 \ll 1$ is centered around $\delta = \sqrt{1 + 2\beta} \approx 1 + \beta$. This is consistent with energy conservation in the scattering of a pump photon into the probe by an atom initially at rest.

Once we have found the eigenvectors and eigenvalues of $M$, we can reexpress the solution (44) in the form
\[ \delta'(\tau) = U e^{i\Omega \tau} U^{-1} \delta(0), \]  
(49)

where \( U \) is the matrix of eigenvectors of \( M \), such that \( U_{ij} \) is the \( i \)th component of the \( j \)th eigenvector, and \( \Omega \) is the diagonal matrix of eigenvalues, such that the \( i \)th diagonal element of \( \Omega \) is the \( i \)th eigenvalue of \( M \). In the unstable regime, we have the eigenvalues \( \omega_1, \omega_2 = \Omega + i\Gamma \), and \( \omega_3 = \Omega - i\Gamma \), where \( \omega_1 \) and \( \omega_2 \) are real, and \( \Gamma \) is real and positive. Thus \( \omega_1 \) corresponds to an oscillating solution, \( \omega_2 \) an exponentially decaying solution, and \( \omega_3 \) corresponds to an exponentially growing solution. Eventually, this exponentially growing solution will dominate, at which time we can neglect the other two terms, yielding the approximate solution

\[ \tilde{\delta}_j(\tau) = \sum_k \zeta_{jk} \tilde{\delta}_k(0) e^{(\Gamma+i\Omega)\tau}, \]
(50)

where \( \zeta_{jk} = U_{j3} U_{3k}^{-1} \). The range of validity for this approximation is roughly \( 1 < \Gamma \tau \ll \ln(\sqrt{N}) \), where the lower limit is set by the requirement that the exponentially growing terms dominate, and the upper limit comes from the requirement that the side mode populations remain a small fraction of the total atom number. This condition therefore formally defines the exponential growth regime. We note that for very short times \( \tau \ll \Gamma^{-1} \), the behavior is of course not exponential. In this transient regime the instantaneous growth rate is not well approximated by \( \Gamma \).

The rate of exponential growth \( \Gamma \) has the explicit form

\[ \Gamma = \frac{\sqrt{3}}{2} \left[ r + \sqrt{q^3 + r^2} \right]^{1/3} - \left[ r - \sqrt{q^3 + r^2} \right]^{1/3}, \]
(51)

where

\[ r = -\frac{1}{3} \delta(1 + 2\beta) - \chi^2 + \frac{\delta^3}{27}, \]
(52)

and
FIG. 2. Exponential growth rate $\Gamma$ as a function of the scaled pump-probe detuning $\delta$ and coupling parameter $\chi$, for various values of the collision parameter $\beta$. Figure 2a shows the case $\beta = 0$, while Figs. 2b and 2c show the cases $\beta = 0.3$ and $\beta = 1.0$, respectively.
FIG. 3. The dashed lines show the growth rate $\Gamma$ as a function of $\chi^2$ for the case $\delta = 1+\beta$, with the values of $\beta$ specified in the figure. The solid line is the approximate expression given by Eq. (54).
\[ q = -\frac{1}{9}(3 + 6\beta + \delta^2). \] (53)

As this equation is complicated and does not provide much insight. In Fig. 2 we have plotted \( \Gamma \) as a function of \( \delta \) and \( \chi^2 \) for three different values of \( \beta \). Figure 2a shows the limit of negligible collisions \( \beta = 0 \), and Figs. 2b and 2c show the cases \( \beta = .3 \) and \( \beta = 1 \) respectively. From these figures, we observe that the most significant effect of collisions is to shift the lower threshold. The values of \( \Gamma \) in the vicinity of the maximum (for fixed \( \chi \)), on the other hand, show less pronounced variations.

In cases where \( \chi^2 \gg |\delta^3|, |\beta| \) we have \( r \approx -\chi^2 \) and \( \sqrt{\chi^3 + r^2} \approx \chi^2 \). In this case Eq. (51) reduces simply to

\[ \Gamma \approx \sqrt{3}(\chi^2)^{2/3}. \] (54)

Among other things, this shows that the gain scales as the number of atoms in the condensate to the 1/3 power. In Fig. 3, the growth rate \( \Gamma \) is plotted versus \( \chi^2 \) with \( \delta = 1 + \beta \), roughly maximizes \( \Gamma \) for fixed \( \chi \). The three dashed curves correspond to different values of the collision parameter \( \beta \), while the solid line gives the approximate result (54). This shows that the approximation is a relatively accurate estimate of the maximum gain for all values of \( \chi^2 \).

VII. QUANTUM STATISTICS

In this section we use the solution (49) to compute some of the quantum statistical properties of the system. This, however, first requires a more detailed discussion of the physical meaning of the infinitesimal operators. The first, \( \hat{\delta}_\alpha = \hat{a}/\sqrt{N} \), is clearly just a rescaling of the photon annihilation operator. From it one can compute all properties of the electric field and/or the photon statistics of the probe mode. The atomic side mode operators \( \hat{\delta}_- = \hat{c}_- \hat{c}_0 / N \) and \( \hat{\delta}_+ = \hat{c}_0^{\dagger} \hat{c}_+ / N \), however, are not simply
rescalings of atom annihilation operators. Rather, they are directly related to the atomic density, \( \hat{\rho}(r) = \hat{\Psi}_g^\dagger(r)\hat{\Psi}_g(r) \).

To illustrate this point, we expand \( \hat{\rho}(r) \) according to Eq. (27) and linearize, yielding
\[
\hat{\rho}(r) = N|\varphi_0(r)|^2 \left[ \frac{1}{2} + e^{i\mathbf{K} \cdot r} (\hat{\delta}_- + \hat{\delta}_+) + H.c. \right].
\] (55)

From this expression we see that the side mode operators \( \hat{\delta}_\pm \) indeed describe the appearance of a density modulation with wavelength \( 2\pi/K \).

In addition to the atomic density, one can also express the number operators for the side modes in terms of \( \hat{\delta}_\pm \). For example, we have after linearization
\[
N\hat{\delta}_-\hat{\delta}_+^\dagger = \frac{\hat{c}_-^\dagger \hat{c}_0 \hat{c}_+ \hat{c}_-}{N} \rightarrow \hat{c}_-^\dagger \hat{c}_- \frac{(N + 1)}{N}.
\] (56)

Hence with \( N + 1 \approx N \) the number operator for the ' -' side mode can be expressed as
\[
\hat{c}_-^\dagger \hat{c}_- \approx N\hat{\delta}_-\hat{\delta}_+^\dagger.
\] (57)

Similarly the number operator for the ' + ' side mode is given by
\[
\hat{c}_+^\dagger \hat{c}_+ \approx N\hat{\delta}_+^\dagger\hat{\delta}_+.
\] (58)

From these number operators, one can therefore compute the number statistics of the side modes in the linear regime.

From the analytical solution (49) it is straightforward to compute the properties of the atomic and optical fields for an arbitrary initial condition. We focus on two conditions which appear readily accessible experimentally. In the first one, the probe field and the atomic side modes all begin in the vacuum state. In this case the exponential growth is triggered by vacuum fluctuations in both the probe field and the
atomic density. A second possible triggering mechanism involves injecting of a weak laser field into the probe mode. Both initial situations are investigated by assuming that the probe mode is initially in the coherent state $\alpha$, such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, the vacuum case corresponding to $\alpha = 0$. In addition, we assume throughout that the condensate side modes begin in the vacuum state. Hence, the initial state of the three-mode system can be expressed as $|\alpha, 0, 0\rangle$, where the first index refers to the probe mode, and the second and third indices give the states of the momentum side modes.

A. Electric field and atomic density

The expectation value of the operator $\hat{\delta}_{\alpha}$ is sufficient to compute the mean electric field, and likewise, the mean values of $\hat{\delta}_{-}$ and $\hat{\delta}_{+}$ are sufficient to compute the mean atomic density. We now give analytic solutions for these physical quantities and their quantum mechanical uncertainties in the exponential growth regime, where all but the leading exponential terms can be safely neglected.

The electric field operator for the probe field is given by

$$\hat{E}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathcal{E}(\mathbf{r}, \tau)\sqrt{N}\delta_{\alpha}(\tau) + H.c.,$$  \hspace{1cm} (59)

where $\varepsilon(\mathbf{r})$ is the polarization unit vector, and

$$\mathcal{E}(\mathbf{r}, \tau) = -\sqrt{\frac{\hbar \omega}{2\varepsilon_0|\Omega_0|\Delta}}\varphi_E(\mathbf{r})^* e^{-i(\omega_0/\omega_0)r}$$  \hspace{1cm} (60)

contains all constants of proportionality, the normalized spatial wavefunction of the probe mode $\varphi_E(\mathbf{r})$, and the oscillation at the pump frequency $\omega_0$. The mean electric field is obtained by inserting Eq. (50), and taking the quantum mechanical expectation value with respect to the initial state $|\alpha, 0, 0\rangle$, which yields
\( \mathcal{E}(r) \cdot \langle \hat{E}(r) \rangle = \mathcal{E}(r, \tau) \zeta_{aa} \alpha e^{(\Gamma + i\Omega)\tau} + c.c. \quad (61) \)

This corresponds to an oscillating mean field with amplitude

\[ E_0(r) = 2|\mathcal{E}(r, \tau)||\zeta_{aa}||\alpha|e^{\Gamma\tau}. \quad (62) \]

From this expression, we see that there is a nonzero mean field provided only that \( \alpha \neq 0 \). We also see from Eq. (61) that the mean field amplitude grows exponentially at the rate \( \Gamma \), and that its frequency is shifted by \( -\Omega \) away from the pump frequency. Its phase, on the other hand, has a somewhat complicated dependence on the system parameters. An analytic expression for this phase can be computed directly from Eq. (61), but as we draw no specific conclusions from it, we do not give the explicit expression here.

The variance in the electric field can also be computed in a straightforward manner from Eq. (50), yielding

\[ \Delta E(r) = \sqrt{2}|\mathcal{E}(r, \tau)||\zeta_{a-}|e^{\Gamma\tau}. \quad (63) \]

This shows that the fluctuations also grow exponentially in time, irrespective of whether the mean field vanishes or not, and are in fact independent of \( \alpha \). Hence these fluctuations can be attributed solely to the amplification of quantum noise, i.e. vacuum fluctuations in the probe electric field as well as atomic density fluctuations. While the mean field and the fluctuations both grow exponentially in time, the relative uncertainty, on the other hand, is constant in time, given by

\[ \frac{\Delta E(r)}{E_0(r)} = \frac{f(\delta, \chi, \beta)}{\sqrt{2}|\alpha|}. \quad (64) \]

Here, we have introduced the fluctuation function

\[ f(\delta, \chi, \beta) = \frac{|U^{\dagger}_{3-1}|}{|U^{\dagger}_{3a}|}. \quad (65) \]
FIG. 4. Comparison of the fluctuation function $f(\delta, \chi, \beta)$ as a function of $\delta$ and $\chi^2$ for various values of the collision parameter $\beta$. Figure 4a gives the limit of negligible collisions ($\beta = 0$), where Figs. 4b and 4c show the cases $\beta = 0.3$ and $\beta = 1.0$, respectively.
which can be computed directly from the eigenvectors of the matrix \( M \), and therefore depends only on the parameters \( \chi, \delta, \) and \( \beta \). Figure 4 plots \( f(\delta, \chi, \beta) \) above the \( \delta-\chi^2 \) plane for various values of the collision parameter \( \beta \). Figure 4a shows the limit of negligible collisions (\( \beta = 0 \)), where we see that \( f(\delta, \chi, 0) \) is nearly flat in the vicinity of maximum gain (\( \delta \approx 1 \)), where it has a value somewhere between 1 and 2. It steadily increases from this value as the pump-probe detuning \( \delta \) moves in the negative direction. Figures 4b and 4c show the cases \( \beta = 0.3 \) and \( \beta = 1.0 \) respectively. From these we see that the effect of increasing the collision parameter is to flatten \( f(\delta, \chi, \beta) \) as a function of \( \chi \) and \( \delta \).

In a similar manner, we next calculate the mean value and variance of the atomic density. By inserting Eq. (50) into Eq. (55) and taking the expectation value with respect to the initial state, we find the expectation value of the atomic density to be

\[
\langle \hat{\rho}(r) \rangle = N|\varphi_0(r)|^2 + \rho_0(r, \tau) \cos[\mathbf{K} \cdot \mathbf{r} + \Omega \tau + \phi]
\]

(66)

where the amplitude \( \rho_0(r) \) of the density modulation is given by

\[
\rho_0(r, \tau) = \sqrt{N}|\varphi_0(r)|^2|\zeta_{-\alpha} + \zeta_{+\alpha}|\alpha|e^{i\tau r}.
\]

(67)

Thus we see that the mean atomic density is the sum of two contributions, the initial density of the condensate plus a density modulation which grows exponentially in time, provided of course that \( \alpha \neq 0 \). Together with Eq. (61), this shows that the phase symmetry of the system is broken by the phase of the injected field. Only in the case \( \alpha = 0 \) does the symmetry remain unbroken. We note that for the atomic side modes, this is not symmetry breaking in the commonly used sense of non-zero mean fields. Rather, it is the mean atomic density modulation which acquires a nonzero phase. Note also that the mean density modulation is not stationary, as its phase is given by \( \Omega \tau + \phi \), where \( \Omega \), depends only on the system parameters \( \delta, \chi, \) and \( \beta, \) and
\( \phi \) depends on both the system parameters and the phase of the injected probe field arg(\( \alpha \)).

The variance in the atomic density can also be readily computed in the exponential growth approximation, yielding

\[
\Delta \rho(r) = 2\sqrt{N}|\varphi_0(r)|^2|\zeta_- + \zeta_+|e^{\Gamma t},
\]

which shows that the density fluctuations grow exponentially in time, even in the case \( \alpha = 0 \). The ratio between the density variance and the modulation amplitude is constant in time, and is given by the same expression as that for the probe, i.e.,

\[
\frac{\Delta \rho(r)}{\varrho_0(r)} = \frac{f(\delta, \chi, \beta)}{\sqrt{2}|\alpha|}.
\]

From Eqs. (64) and (69), we see that in the case \( f(\delta, \chi, \beta) \ll |\alpha| \), both the mean electric field and the mean atomic density modulation are quantum mechanically well-defined, meaning that the quantum noise is small compared to their mean values. In this regime, both quantities could be adequately treated as classical (c-number) fields. Outside of this regime, however, the quantum fluctuations play a significant role, and a classical description no longer suffices.

The main implication of these results is that by varying the system control parameters, and in particular the injected field intensity and phase, one can vary the mean electric field and atomic density modulation continuously between two limits. For \( |\alpha| \ll f(\delta, \chi, \beta) \) the fields are dominated by quantum fluctuations, and we can expect to find important non-classical effects. In the limit of a 'strong' injected field, however, the fluctuations are not significant, and the atomic and optical fields behave classically.
B. Intensities

We now turn to the number statistics of the three field modes, concentrating on the mean atom/photon numbers and their variances. It is convenient to reexpress the mode number operators given by Eqs. (56)-(58), as

\[ \hat{N}_j = N \hat{\delta}_j^\dagger \hat{\delta}_j - \delta_{j-}, \]  

(70)

where the index \( j \) is again the mode label \( a, -, \) or \( + \), and the \( \delta \)-function accounts for the fact that we have normally ordered the infinitesimal field operators. It is then straightforward to derive the full time-dependent solution for the mean occupation numbers \( \bar{N}_j \equiv \langle \hat{N}_j \rangle \) as

\[ N_j = |a_j(\tau)|^2 |\alpha|^2 + |a_{j-}(\tau)|^2 - \delta_{j-}, \]  

(71)

where

\[ a_{ij}(\tau) = \sum_{k=1}^3 U_{ik} U_{kj}^{-1} e^{i\omega_k \tau}. \]  

(72)

The first term in Eq. (71) can be interpreted as the stimulated contribution to the intensity,

\[ [N_j]_{st} = |a_j(\tau)|^2 |\alpha|^2, \]  

(73)

whereas the second term gives the spontaneous contribution,

\[ [N_j]_{sp} = |a_{j-}(\tau)|^2 - \delta_{j-}, \]  

(74)

present even in the case \( \alpha = 0 \). In the exponential growth regime Eq. (71) reduces to

\[ N_j = \left( |\zeta_j|^2 |\alpha|^2 + |\zeta_{j-}|^2 \right) e^{2F \tau}, \]  

(75)
FIG. 5. Logarithmic plot of the probe intensity $N_a$ as a function of time. The thick curves show the exact solution, given by Eq. (71), and the corresponding thin lines give the approximate solution of Eq. (75). The parameters chosen are $\delta = 1$, $\chi^2 = 1$, and $\beta = 0$. Each pair of curves corresponds to a different value of $|\alpha|^2$, as specified in the figure.
which shows that the mode occupation grows exponentially, even in the spontaneous case, where it was seen that the mean electric field and mean density modulation both vanish. The validity of the exponential approximation (75) is demonstrated in Fig. 5, where we have plotted the logarithm of the probe intensity as a function of time. The parameters chosen for the plot are $\delta = 1$, $\chi^2 = 1$, and $\beta = 0$. The thick lines give the full solution (71) for three different values of the initial probe intensity $|\alpha|^2 = 0, 1, 10$. The corresponding thin lines are the approximate solutions given by Eq. (75), which show good agreement for $\tau > 1$.

Turning now to quantum fluctuations in the occupation numbers, we find that the relative uncertainties are given by

$$\frac{\Delta N_j}{N_j} = \sqrt{1 - \frac{|a_j(\tau)|^4}{N_j^2}} + \frac{1}{N_j}. \quad (76)$$

From Eq. (71) it is clear that the second term under the radicand in the above expression is $\leq 1$, which means that as a function of time, the relative uncertainty is always between $\sqrt{1/N}$, characteristic of the fluctuations found in a coherent state, and $\sqrt{1 + 1/N}$, which is the signature of thermal number fluctuations.

While for very short times the relative uncertainty may fluctuate between the thermal and coherent limits, once the exponentially growing terms dominate, the relative uncertainty eventually reaches a steady-state value given by

$$\frac{\Delta N_j}{N_j} = \sqrt{1 - \frac{|\alpha|^4}{[|\alpha|^2 + f^2(\chi, \delta, \beta)]^2}}. \quad (77)$$

Thus we see that when $|\alpha|^2 \gg f^2(\chi, \delta, \beta)$, the relative uncertainty tends towards zero, while in the opposite case, it tends toward one. These limits can be labeled as the stimulated and spontaneous limits respectively. In the intermediate regime, the fluctuations can be varied continuously between the thermal and coherent limits,
FIG. 6. The side mode number variance $\Delta N_-/N_-$ is plotted versus time (thick solid line). Also shown are the approximate solution given by Eq. (77) (thick dashed line) as well as the variances for a thermal state (thin solid line) and a coherent state (thin dashed line) with the same mean value $N_-$. The parameters chosen are $\delta = 1$, $\chi^2 = 1$, $\beta = 0$, and $\alpha = 1$. 
FIG. 7. The long time limit of $\Delta N_j/N_j$ is plotted against the initial probe intensity $|\alpha|^2$ (thick solid line). It varies continuously between thermal (upper dashed line) and coherent (lower dashed line) limits. The parameters chosen are $\Delta = 1$, $\chi^2 = 1$, and $\beta = 0$. 
e.g. by varying the injected laser intensity, thus achieving optical control over the quantum statistics of matter waves.

The behavior of the particle number variances is illustrated by Figs. 6 and 7. In Fig. 6 the full time dependence of $\Delta N_\gamma / N_\gamma$ is shown (thick solid line). Also shown are the approximate solution given by Eq. (77) (thick dashed line) as well as the variances for a thermal state (thin solid line) and a coherent state (thin dashed line) with the same mean value $\langle N_\gamma \rangle$. The parameters chosen are $\delta = 1$, $\chi^2 = 1$, $\beta = 0$, and $\alpha = 1$. Thus we see that the variance always falls between those of thermal and coherent fields. We also see that the long time behavior is well approximated by Eq. (77). In Fig. 7 the steady state value for large $\tau$ is plotted as a function of the initial probe intensity $|\alpha|^2$. Thus we see that it is possible to vary the output continuously over the whole range between thermal and coherent limits, simply by varying the initial probe intensity. The parameters chosen for the figure are $\delta = 1$, $\chi^2 = 1$, and $\beta = 0$.

VIII. ATOM-PHOTON ENTANGLEMENT

We have previously discussed the analogy between the present system and the non-degenerate optical parametric amplifier (OPA). One of the most interesting applications of the OPA is the generation of entangled quantum optical states. We show that similar entanglements occur in the present system, but they are now between atomic and optical field modes. We first examine the two-mode intensity correlation functions, which give a measure of entanglement, and can be used to determine whether or not non-classical correlations exist between the three field modes. We then discuss the issue of two-mode squeezing, and show how this phenomenon manifests itself in the present system.
A. Two-mode intensity correlations

The equal-time intensity correlation functions are defined in the usual manner as

\[ g_{ij}^{(2)} = \frac{\langle \hat{N}_i \hat{N}_j \rangle - \delta_{ij} \langle \hat{N}_j \rangle}{\langle \hat{N}_i \rangle} \]  

The two-mode correlation functions \( i \neq j \) arise, e.g., if we consider a measurement of the intensity difference between two modes, described by the operator

\[ \hat{N}_{ij} = \hat{N}_i - \hat{N}_j, \]  

whose variance is given by

\[ (\Delta N_{ij})^2 = (\Delta N_i)^2 - 2N_i N_j \left( g_{ij}^{(2)} - 1 \right) + (\Delta N_j)^2. \]  

For uncorrelated fields, \( g_{ij}^{(2)} = 1 \), and we have the usual rule for the addition of uncorrelated noise sources

\[ (\Delta N_{ij})^2 = (\Delta N_i)^2 + (\Delta N_j)^2. \]  

If, however, there are correlations between the fluctuations in the intensities of the two modes, then we have \( g_{ij}^{(2)} > 1 \), and the variance \( \Delta N_{ij} \) will be less than that given by Eq. (81).

For classical fields (positive \( P \)-function), the two-mode \( (i \neq j) \) correlations are constrained by the Cauchy-Schwartz inequality

\[ g_{ij}^{(2)} \leq \left[ g_{ii}^{(2)} \right]^{1/2} \left[ g_{jj}^{(2)} \right]^{1/2}. \]  

Quantum mechanical fields, however, can violate this inequality and are instead constrained by

\[ g_{ij}^{(2)} \leq \left[ g_{ii}^{(2)} + \frac{1}{\langle \hat{N}_i \rangle} \right]^{1/2} \left[ g_{jj}^{(2)} + \frac{1}{\langle \hat{N}_j \rangle} \right]^{1/2}, \]  

\[ (\Delta N_{ij})^2 = (\Delta N_i)^2 - 2N_i N_j \left( g_{ij}^{(2)} - 1 \right) + (\Delta N_j)^2. \]
which reduces to the classical result in the limit of large intensities.

We focus our attention on the spontaneous case \( \alpha = 0 \). Here the single-mode intensity correlation functions are those of thermal fields, \( g_i^{(2)}(\tau) = 2 \). In this case, the equal-time intensity cross-correlation functions are found to be

\[
g_{a-}^{(2)} = g_{-+}^{(2)} = \left[ 2 + \frac{1}{N_a + N_+} \right]^{1/2} \left[ 2 + \frac{1}{N_-} \right]^{1/2},
\]

\( g_{a+}^{(2)} = 2 \).

From Eq. (84) we see that both \( g_{a-}^{(2)}(\tau) \) and \( g_{-+}^{(2)}(\tau) \) violate the Cauchy-Schwartz inequality, while \( g_{a+}^{(2)}(\tau) \) is consistent with classical cross-correlations. Furthermore, the explicit evaluation of the \( \zeta_{ij} \)'s shows that \( I_+(\tau) \ll I_i(\tau) \), which implies that \( g_{a-}^{(2)}(\tau) \) is very close to the maximum violation of the classical inequality consistent with quantum mechanics, whereas for \( g_{-+}^{(2)}(\tau) \) the violation is not close to the allowed maximum.

In the two-mode parametric amplifier, the two-mode correlation function shows the maximum violation of the Cauchy-Schwartz inequality consistent with quantum mechanics. In the three-mode system, however, the two-mode cross-correlation functions involve a trace over the third mode, hence it is not surprising that they are not maximized. This is illustrated in Fig. 8a, where we have plotted the correlation function \( g_{a-}^{(2)} \) as a function of time (solid line). Also shown for comparison are the quantum mechanical upper limit given by Eq. (83) (upper dashed line), as well as the classical upper limit given by Eq. (82) (lower dashed line). The parameters are \( \delta = 1 \), \( \chi^2 = 1 \), \( \beta = .1 \), and \( \alpha = 0 \). We see that when the system is triggered by vacuum fluctuations alone, \( g_{a-}^{(2)} \) is virtually indistinguishable from the maximum allowed by quantum mechanics.

If we now allow for an injected coherent probe field (\( \alpha \neq 0 \)), we first note that the intensities are increased by approximately \( |\alpha|^2 \), which means that the time scale on which the classical and quantum upper limits (82) and (83) converge is reduced.
FIG. 8. The cross-correlation function $g^{(2)}_{-\alpha}$ is plotted as a function of time (solid line). The upper dashed line gives the maximum allowed by quantum mechanics, while the lower dashed line gives the upper limit for classical correlations only. The parameters chosen are $\delta = 1.1$, $\chi = 1$, and $\beta = .1$. Figure 8a corresponds to triggering from noise ($|\alpha|^2 = 0$), while Figs. 8b and 8c show the cases $|\alpha|^2 = 1$ and $|\alpha|^2 = 10$ respectively.
by $1/|\alpha|^2$, making an experimental confirmation of quantum correlations more difficult. In addition, whereas for the spontaneous case $\alpha = 0$, numerics show the cross-correlation $g_{ij}^{(2)}$ follows almost exactly the quantum upper limit (83) for all $t > 0$, for $\alpha \neq 0$, it lies somewhere in between the quantum (83) and classical (82) limits. As $\alpha$ is increased, it falls ever closer to the classical upper limit, and furthermore, we see that this upper limit approaches unity. Thus in the limit of very large $\alpha$, the fields can be treated as classical and uncorrelated. The dependence of $g_{ij}^{(2)}$ on $\alpha$ is illustrated by Figs. 8b and 8c, which are identical to 8a, except that now we have taken $|\alpha|^2 = 1$ and $|\alpha|^2 = 10$ respectively. We remark that even when the probe field initially contains only one photon on average (Fig. 8b), it is enough to significantly reduce the quantum correlations from the quantum mechanical maximum.

**B. Two-mode squeezing**

To complete the study of atom-photon entanglement we complement the investigation of intensity cross-correlations with a discussion of phase-sensitive two-mode correlations. Drawing again on the similarity between the current model and the two-mode parametric amplifier we expect the correlations between the cavity mode and the atomic side-mode "-" to be of particular interest as a "squeezing-like" behavior may occur. In analogy to the parametric amplifier we thus introduce the quadrature components of the superposed atomic and optical fields

$$X_\theta = \sqrt{\frac{N}{2}} (\delta_a e^{i\theta} + \delta_- e^{-i\theta} + h.c.).$$

The variance of $X_\theta$ is given by

$$V(X_\theta) = \langle X_\theta^2 \rangle - \langle X_\theta \rangle^2$$

$$= |a_- (\tau) e^{i\theta} + a_- (\tau) e^{-i\theta}|^2.$$
Note that this variance (86) is independent of the injected signal strength $\alpha$, just as with the electric field, atomic density, and mode intensities. It follows from Eq. (86) that the angle $\theta_{\text{min}}(\tau)$ that minimizes the quadrature variance is determined by

$$\arg[a_-(\tau)a^*_-(\tau)] + 2\theta_{\text{min}}(\tau) = \pi$$

(87)

so that the corresponding minimum of $V$ for fixed $\tau$ is given by

$$V_{\text{min}}(\tau) = |a_-(\tau)|^2 - |a_-(\tau)|^2$$

(88)

The Heisenberg uncertainty principle gives $V(X_{\theta})V(X_{\theta+\pi/2}) \geq 1$, hence, a quadrature component is squeezed provided $V(X_{\theta}) < 1$.

In Fig. 9, we display $V_{\text{min}}(\tau)$ for various values of the system parameters. We find that for $\delta \approx 1$, i.e., maximum exponential growth rate, $V_{\text{min}}$ is a concave function of $\tau$ displaying a single (global) minimum which is typically of the order of $10^{-1}$ (cf. the full and dashed curves). The maximum squeezing time $\tau_m$, i.e., the largest $\tau$ for which $V_{\text{min}}(\tau) = 1$, is given by $\tau_m = 3.5$ for $\chi = 1$, $\delta = 1$, and $\beta = 0$ whereas for $\chi = 10$ $\tau_m$ decreases to a value of 0.20. The reduction is due to the increase in the exponential growth rate $\Gamma$ with larger $\chi$. From Fig. 9, we see that squeezing does indeed occur over a broad range of parameters, however it only persists over intermediate time scales. For long times, $V_{\text{min}}(\tau)$ is dominated by the exponential behavior, which eventually leads to the violation of the squeezing condition. This is in contrast to the two-mode OPA, where the quadrature component remains squeezed for all time.

To understand how the presence of a third mode quenches the squeezing, we reexpress the squeezing condition with the help of Eq. (74), yielding

$$([N_a]_{sp} - [N_-]_{sp})^2 < 2 [N_a]_{sp}$$

(89)
FIG. 9. Minimum variance $V_{\text{min}}$ of the quadrature components $X_\theta$ as a function of $\tau$ for parameter values $\chi = 1.0, \delta = 1.0, \beta = 0.0$ (full curve); $\chi = 3.0, \delta = 1.0, \beta = 0.0$ (dashed); $\chi = 3.0, \delta = -17.0, \beta = 0.0$ (dotted).
As our analysis of squeezing so far has not made explicit use of the particular form of the matrix $M$, Eq. (89) can also be applied to the standard two-mode optical parametric amplifier. One simply chooses a suitable $M$ where the third mode is decoupled. In this case, the solution is well known [18], and the two modes are symmetrical, with equal populations. This means that the l.h.s. of Eq. (89) is always zero, and the squeezing condition is satisfied for all times. However, once the third mode is included, it introduces a small imbalance between the populations of the two original modes, since momentum conservation requires that $N_{a} + N_{+} = N_{-}$. On long time scales, the l.h.s. of Eq. (89) grows like $\exp(i4\Gamma \tau)$, whereas the r.h.s. only grows as $\exp(i2\Gamma \tau)$. Therefore, the squeezing condition (89) must eventually cease to be satisfied.

The introduction of moderate collisional interactions, i.e., $g < 1$, only leads to quantitative modifications of $V_{\min}$ without altering the characteristic features. An interesting change in behavior occurs, however, if $\delta$ is tuned closer to the borders of the amplification range (dotted curve in Fig. 9). Under these circumstances squeezing still occurs but $V_{\min}$ displays an oscillating behavior as a function of $\tau$. These oscillations are caused in part by the reduction of the exponential growth rate and a simultaneous increase of the imaginary part of the eigenvalues of the matrix $M$.

The extremal angle $\theta_{\min}$ varies as time evolves, eventually attaining a constant value when the behavior of $V_{\min}(\tau)$ is dominated by exponential growth. However, in many cases $V(X_{\theta})$ is well approximated by $V_{\min}(\tau)$ if $\theta$ is chosen in the vicinity of $\theta_{\min}(\tau)$. 
IX. DISCUSSION AND CONCLUSION

The fundamental time scale in the system is the inverse growth rate $\Gamma^{-1}$. It is estimated by Eq. (54) in units of the inverse recoil frequency $\omega_r^{-1}$ which, for sodium is given by $\omega_r \approx 1.7 \, \mu s$. The estimated growth rate is then of the order of

$$\chi^{2/3} = (\sqrt{2n_eN} g/\omega_r)^{2/3},$$

where $n_e = (I_0/8I_{sat})(\gamma/\Delta)^2$. Here $I_0$ is the pump intensity, $I_{sat}$ the atomic saturation intensity, and $\gamma$ the atomic spontaneous decay rate. For sodium atoms we have $I_{sat} = 6.33 \text{mW/cm}^2$. The parameter $n_e$ equals the fraction of excited atoms, hence, under the far-off resonance conditions we are considering, we have $n_e \ll 1$. If we chose the ring cavity length $L = 0.1 \text{m}$ and effective cross section $S = 10^{-9} \text{m}^2$, the atom-cavity coupling constant $g$ is of the order $10^6 \text{ s}^{-1}$ so that $g/\omega_r \approx 1$. As one has a great latitude in choosing the values of $n_e$ and the total number $N$ of atoms in the BEC it should be possible to vary the exponential growth rate over a wide range.

Our choice of $N$ and $n_e$ is constrained, however, by the requirement that the spontaneous heating rate $\mathcal{L}$ be much smaller than the exponential growth rate $\Gamma$, so that spontaneous emission can in fact be neglected. The heating rate (in units of $\omega_r$) is given by $\mathcal{L} = n_e \gamma/\omega_r$. As $\gamma/\omega_r \approx 10$ the condition $\Gamma \gg \mathcal{L}$ translates into $N \gg 10^3 n_e^2$. As $n_e \ll 1$, this condition is practically always fulfilled. Also related to spontaneous emission is the two-body dipole-dipole interaction, which acts in addition to ground-state collisions. For very cold atoms whose de Broglie wave length is large in comparison to the pump laser wavelength the dipole-dipole interaction can be approximated as a contact interaction [34]. Comparing the strength of this interaction to that ground-state collisions one finds that the former is negligible under the condition $n_e \ll \frac{8\hbar k_0^3}{m\gamma}$, which translates to $n_e \ll 10^{-3}$ for sodium atoms. If this condition is not met the dipole-dipole interaction can still be accounted for
to a good degree of approximation by modifying the scattering length according to
\[ \sigma \rightarrow \sigma - I_0 \gamma^3 / 32 I_{sat} \Delta^2 \omega_r. \]

Another important parameter is the collision parameter \( \beta \). Estimating the quantity \( NF_0 \) to be of the order of the atomic density we find the collision parameter \( \beta \) to lie in the range 0.1–1 for 'typical' densities around \( 10^{15} \text{ cm}^{-3} \). If \( \beta \) could be measured by, e.g. observing the boundaries of the instability regime, than this could be used as a novel means to determine the atomic s-wave scattering length.

In a realistic optical cavity, the lifetime of a photon is of the order of 1-10ns, corresponding to a decay rate of \( 10^3 \) to \( 10^4 \) in units of the recoil frequency. This tells us how large the growth rate \( \Gamma \) would have to be to result in a buildup of photons in the cavity. Since \( \Gamma \) goes like \( N^{1/3} \) we would likely need a very large condensate, with \( 10^{12} \) or more atoms, to achieve this. In the future, larger condensates and better cavities, will be available, at which point the theory could presumably be tested. Future research will study the quantum statistics of the system under the influence of probe field damping, which will relate the model more closely to current experiments. For example, recent experiments by W. Ketterle's group involving a condensate driven by a far-off resonant pump laser have demonstrated the appearance of distinct momentum side modes as a consequence of spontaneous emission. While these experiments involve at most one spontaneous photon at a time in the condensate, they clearly demonstrate many aspects of the theory we have described, as well as the importance of continuing to develop a nonlinear/quantum optics approach to the theory of optically driven Bose-Einstein condensates.
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APPENDIX A: NON-ORTHOGONALITY

By properly taking into account the non-orthogonality of the atomic field modes, it can be shown that the only surviving effect in the linearized theory is the modification of the atomic polarization term in the equation of motion for the probe field (10) to include a second scattering mechanism in which a condensate scatters a photon without changing its center of mass state. As a consequence of momentum conservation, this process is suppressed by a factor \(|\varphi_0|\varphi_-|\) relative to the process which transfers the atom to the side mode state. Bose enhancement, on the other hand, is stronger for this transition by a factor \(\sqrt{N}\), because we now have \(N\) identical bosons in both the initial and final states. Thus it is the product \(\sqrt{N}|\varphi_0|\varphi_-|\) which must be negligible if we are to make the orthogonality approximation.

More precisely, we find that the quantum statistics must be modified by taking \(\alpha \to \alpha - s\), where

\[
s = \frac{\chi(\varphi_0|\varphi_-)|\sqrt{N}}{\left(\Omega - i\Gamma\right)},
\]

in order to account for this additional scattering mechanism. Thus when \(\alpha\) is comparable to \(f(\delta, \chi, \beta)\), the condition \(|s| \ll |\alpha|\) allows us to neglect \(s\). For the case \(\alpha \ll f(\delta, \chi, \beta)\), one the other hand, requires \(|s| \ll f(\delta, \chi, \beta)\). Since \(f(\delta, \chi, \beta)\) is typically of order 1, then the condition \(|s| \ll 1\) is typically sufficient to satisfy both conditions.
For a dilute condensate, the ground state wave function \( \varphi_0(r) \) is given approximately by the single-particle ground state. For a harmonic trap with \( \mathbf{K} \) taken along the \( z \)-axis, one has thus

\[
\langle \varphi_0 | \varphi_- \rangle = e^{-\frac{1}{2}(K W_c)^2},
\]

where \( K \) approximately twice the optical wave number, and \( W_c \) is the condensate width along \( \mathbf{K} \). For a dilute condensate which is an order of magnitude or larger than the optical wavelength, this integral is clearly vanishingly small, and \( s \approx 0 \) for all reasonable values \( \lambda \) and the system parameters \( \delta, \chi, \) and \( \beta \).

In the case of a dense condensate the density distribution can be described with the help of the Thomas-Fermi approximation, i.e.,

\[
|\phi_0(r)|^2 = [\mu - V_0(r)]m/(4\pi \hbar^2 \sigma N),
\]

where the chemical potential \( \mu \) is determined from the normalization requirement. Starting from Eq. (92) and choosing the recoil wave vector \( \mathbf{K} \) to lie parallel the \( z \)-axis, the overlap integral is found to be

\[
\langle \varphi_0 | \varphi_- \rangle = 15 j_2(K W_c)/(K W_c),
\]

where \( j_2 \) is the modified Bessel function, and the \( W \) is the condensate width along \( \mathbf{K} \). For condensates which are large compared to the optical wavelength, Eq. (93) behaves like \(-15 \sin(K W_c)/(K W_c)^3\). So assuming that \( K W_c = 100 \), which corresponds to a 10\( \mu \)m condensate, we would need \( \chi \sqrt{N}/(\Omega - i \Gamma) \) to be of the order \( 10^6 \) in order achieve an appreciable value of \( s \), and this is much larger than is currently feasible. However, if, for example, \( \mathbf{K} \) is oriented parallel to a trap axis with tight confinement \( K W_c \) need not be too large and the effects of non-orthogonality might be observed.

A more efficient way to increase \( s \) consists in confining the condensate in three-dimensional rectangular trap potential. In this case the Thomas-Fermi approximation
gives $|\varphi_0(r)|^2 = 1/V$, $V$ being the volume of the trap. With $K$ chosen along the $z$-axis of the trap, and $W_c$ being the width of the box in this direction, one obtains for the overlap integral

$$\langle \varphi_0 | \varphi_- \rangle = e^{-iKW_c/2} \frac{\sin(KW_c/2)}{KW_c/2}. \quad (94)$$

For large $KW_c I$ now decays only as $(KW_c)^{-1}$.

**APPENDIX B: EFFECTIVE HAMILTONIAN**

If we define $\hat{C}_- = \sqrt{N}d^\dagger_-$, and $\hat{C}_+ = \sqrt{N}d^\dagger_+$, then these operators obey approximately bosonic commutation relations, and the system of equations (39) can be derived from the effective Hamiltonian

$$\mathcal{H}_{eff} = (1 + \beta)\hat{C}^\dagger_- \hat{C}_- + (1 + \beta)\hat{C}^\dagger_+ \hat{C}_+ - \delta \hat{a}^\dagger \hat{a}$$

$$+ \beta(\hat{C}^\dagger_- \hat{C}^\dagger_+ + \hat{C}^\dagger_+ \hat{C}^\dagger_-)$$

$$+ \chi(\hat{a}^\dagger \hat{C}^\dagger_- + \hat{a} \hat{C}^\dagger_+ + \hat{a}^\dagger \hat{C}^\dagger_+ + \hat{a} \hat{C}^\dagger_-). \quad (95)$$


THEORY OF SUPERRADIANT SCATTERING OF LASER LIGHT FROM BOSE-EINSTEIN CONDENSATES

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ABSTRACT

In a recent MIT experiment, a new form of superradiant Rayleigh scattering was observed in Bose-Einstein condensates. We present a detailed theory of this phenomenon in which condensate depletion leads to mode competition, which, together with the directional dependence of the scattering rate, is ultimately responsible for the observed effects. The nonlinear response of the system is shown to be highly sensitive to initial quantum fluctuations which cause large run to run variations in the observed superradiant pulses.

With the recent advent of Bose-Einstein condensation (BEC) in low-density alkali vapors [1,2], a laser-like source of coherent monochromatic atomic matter-waves is now readily available. As the electromagnetic vacuum itself provides a nonlinear medium for atomic fields, an atomic BEC is thus an ideal system to study nonlinear
wave-mixing and related phenomena. Indeed, nonlinear atom optics [3–5] is now an experimental reality with the recent observation of atomic four-wave mixing in condensate systems [6,7]. In addition to wave-mixing between atomic matter waves, the ability to generate laser-like atomic fields also raises the possibility to observe direct wave-mixing between atomic and optical fields.

In the case where the atoms interact only with far off-resonant optical fields, the dominant atom-photon interaction is two-photon Rayleigh scattering. When the atoms are described as matter waves, Rayleigh scattering is formally equivalent to a cubic nonlinearity, and therefore leads to four-wave mixing between atomic and optical fields. In recent work [8–10] along these lines, the scattering of light by a condensate from a strong pump laser into a weak quantized optical cavity mode was considered. This work was an extension of the collective atomic recoil laser (CARL) [11,12] into the regime of BEC, and focused on exploiting an instability in the light-matter interaction to parametrically amplify atomic and optical waves as well as to optically manipulate matter-wave coherence properties and generate entanglement between atomic and optical fields.

Recent experiments by Ketterle and coworkers at MIT [13], however, have demonstrated that this instability can play an important role also in the case in which laser light is scattered into the vacuum modes of the electromagnetic field. In these experiments, a variation of Dicke superradiance [14] was observed in which the role of electronic coherence, which stores the memory of previous scattering events, is replaced by coherence between center-of-mass momentum states, i.e. interference fringes in the atomic density. In this paper we present a multi-mode theory of condensate superradiance. Beginning with the elimination of the radiated light field as in the Wigner-Weiskopf theory of spontaneous emission, we then derive a linearized
model which describes amplification of quantum fluctuations. This is then coupled to a 'classical' nonlinear model in which mode competition quenches scattering in all but the direction(s) of maximum gain. The initial quantum fluctuations are shown to strongly influence superradiant pulse formation, and lead to large fluctuations between runs with identical experimental parameters. In the MIT experiments this is clearly demonstrated by the presence of random spots in the angular distribution of the scattered photons.

Our model consists of a Schrödinger field of two-level atoms coupled via the electric-dipole interaction to a far-off resonant pump laser field, as well as to the vacuum modes of the electromagnetic field. The pump laser has frequency $\omega_0$, wavevector $k_0 = (\omega_0/c)\hat{y}$, and its polarization is taken along the $\hat{x}$ axis. Due to the large detuning between the pump frequency and the atomic transition frequency $\omega_a$, we can eliminate the excited state field, and describe the atoms as a scalar field of ground state atoms. This atomic field is self-interacting, due to ground-ground collisions, however, we note that collisions which transfer populations are generally nonresonant and should make only a small contribution to the dynamics. The remaining collisions then simply give a mean field shift to the resonance frequency for quasiparticle excitations. As the effect of these shifts on our model are negligible, at present we include collisions only implicitly in the determination of the condensate wavefunction.

The effective Hamiltonian which describes the coupling of the atomic and electromagnetic fields is given by

$$\hat{H} = \int d^3r \hat{\Psi}^\dagger(r) H_0(r) \hat{\Psi}(r) + \int d^3k \hbar \omega(k) \hat{b}^\dagger(k) \hat{b}(k)$$

$$+ \int d^3k d^3r \left[ \hbar g(k) \hat{\Psi}^\dagger(r) \hat{b}^\dagger(k) e^{i(k_0 - k) \cdot r} \hat{\Psi}(r) \right]$$

$$+ H.c.]$$

(1)

where $\hat{\Psi}(r)$ is the atomic field operator, and $\hat{b}(k)$ is the annihilation operator for
a photon in mode \( k \) in the frame rotating at the pump frequency \( \omega_0 \). The photon energy in this frame is given by \( \omega(k) = c|k| - \omega_0 \). The single-atom Hamiltonian is given by

\[
H_Q(r) = -(\hbar^2/2m)\nabla^2 + V(r) + \hbar|\Omega_0|^2/2\Delta, \quad V(r) \text{ being the trap potential.} \]

\( \Omega_0 \) the pump Rabi frequency, and \( \Delta = \omega_0 - \omega_a \) the pump detuning. The Hamiltonian (1) includes only scattering of pump photons, i.e. multiple scatterings between vacuum modes are neglected. The coupling coefficient for Rayleigh scattering between the pump and vacuum modes is

\[
g(k) = \frac{|\Omega_0|}{2|\Delta|} \frac{c|k|d^2}{2\hbar \epsilon_0 (2\pi)^3} |k \times \hat{x}|, \quad (2)
\]

where \( d \) is the magnitude of the atomic dipole moment for the transition involved.

The atomic field is initially taken to be a number state in which \( N \) atoms occupy the trap ground state \( \varphi_0(r) \), which satisfies \([H_Q(r) - \hbar \mu]|\varphi_0(r) = 0, \mu \text{ being the energy of the trap ground state.} \) The effect of atomic recoil during Rayleigh scattering between the pump and the vacuum mode \( k \) is therefore to transfer atoms into the state \( \varphi_0(r) \exp[i(k_0 - k) \cdot r] \). This suggest to expand the atomic field operator onto quasi-modes according to

\[
\hat{\Psi}(r, t) = \sum_q \langle r|q \rangle e^{-i(\omega_q + \mu)t} \hat{c}_q(t), \quad (3)
\]

where \( \langle r|q \rangle = \varphi_0(r) \exp(iq \cdot r) \), and \( \omega_q = \hbar|q|^2/2m \). This is similar to the slowly varying envelope approximation from optical physics, the envelope being given here by \( \varphi_0(r) \).

A discrete quantization of the \( q \) values follows from the requirement that the operators \( \{\hat{c}_q\} \) obey boson commutation relations \([\hat{c}_q, \hat{c}_q^\dagger] = \langle q|q' \rangle \approx \delta_{q,q'} \). Due to the finite size of the ground state wavefunction \( \varphi_0(r) \), this means that \( q \text{ and } q' \text{ must be separated in } k \text{-space.} \) Hence, the summation in Eq. (3) is taken to include the condensate mode \( q = 0 \) as well as a grid of \( q \)-values as closely spaced as is consistent
with orthogonality. Clearly, this expansion is not rigorously orthogonal and complete, however, it is sufficient to account for the quantum statistical effects which occur above the critical phase-space density.

An important aspect of BEC superradiance is the generation of families of higher-order sidemodes due to the scattering of pump photons by the first-order sidemodes. For the scope of this paper, however, we consider a simplified model containing only the primary Rayleigh scattering process whereby a condensate atom is transferred to a first-order sidemode by scattering a pump photon. With this simplification, we insert the expansion (3) into Eq. (1) and arrive at the effective Hamiltonian

\[ \hat{H} = \int d^3k \hat{h}\omega(k) \hat{\beta}^\dagger(k) \hat{\beta}(k) \]

+ \sum_{q \neq 0} \int d^3k \left[ \hat{h}_g(k) \rho_q(k) e^{i\omega q t} \hat{c}_q^\dagger \hat{c}_q(0) + H.c. \right], \tag{4} \]

where \( \rho_q(k) = \int d^3r |\varphi_0(r)|^2 \exp[-i(k - k_0 + q) \cdot r] \) is the Fourier transform of the ground state density distribution centered at \( k = k_0 - q \).

From the Hamiltonian (4) it is straightforward to derive the equation of motion for \( \hat{\beta}(k) \), which upon formal integration yields

\[ \hat{\beta}(k, t) = \hat{\beta}(k, 0) e^{-i\omega(k)t} - i \sum_{q \neq 0} g(k) \rho_q(k) e^{i\omega q t} \]

\[ \times \int_0^t d\tau e^{-i(\omega(k) + \omega q)\tau} \hat{c}_q^\dagger(t - \tau) \hat{c}_0(t - \tau), \tag{5} \]

where the first term gives the free electromagnetic field, i.e. vacuum fluctuations, and the second term is the radiation field due to Rayleigh scattering. A nonzero expectation value of the coherence operator \( \hat{c}_q^\dagger \hat{c}_0 \) indicates the presence of interference fringes, hence the radiated field increases as fringes build up. This term therefore leads to an instability where the memory of previous scattering events, stored in the matter-wave interference fringes, enhances the present rate of Rayleigh scattering.
Equation (5) is then substituted into the equation of motion for \( \hat{c}_q \). In the Markoff approximation, familiar from the Wigner-Weisskopf theory of spontaneous emission, this yields

\[
\frac{d}{dt} \hat{c}_q = -i \int d^3k g(k) \rho_q(k) \hat{b}^\dagger(k, 0) e^{i(\omega(k) + \omega_q) t} \hat{c}_0 + \frac{G_q}{2} \hat{c}_0 \hat{c}_q, \tag{6}
\]

where

\[
G_q = 2\pi \int d^3k |g(k)|^2 |\rho_q(k)|^2 \delta(\omega(k) + \omega_q) \tag{7}
\]

is the single-atom gain. In deriving Eq. (6) we have used the orthogonality of the states \( \{|q\}\) to make the approximation \( \rho_q^*(k) \rho_{q'}(k) \approx |\rho_q(k)|^2 \delta_{qq'} \), and neglected the principal part which accompanies the \( \delta \)-function. If included, the principle part would contribute additional ground-state collisions due to the dipole-dipole interaction.

For a closed atomic system, the total number of atoms is conserved, hence \( \hat{c}_0^\dagger \hat{c}_0 = N - \sum_{q \neq 0} \hat{c}_q^\dagger \hat{c}_q \). For very short times we can therefore take \( \hat{c}_0^\dagger \hat{c}_0 \approx N \). In this case Eq. (6) reduces to

\[
\frac{d}{dt} \hat{c}_q = \frac{G_q}{2} N \hat{c}_q + \hat{f}^\dagger_q(t). \tag{8}
\]

where \( \hat{f}_q(t) \) is a noise operator whose correlation functions are given in the Markoff approximation by

\[
\langle \hat{f}^\dagger_q(t) \hat{f}_q(t') \rangle = 0,
\]

\[
\langle \hat{f}_q(t) \hat{f}^\dagger_q(t') \rangle = G_q N \delta(t - t'). \tag{9}
\]

These noise operators allow the system to be triggered by quantum fluctuations, and hence describe "spontaneous" scattering which occurs in the absence of any sidemode population.
Equation (8) can be solved exactly, giving

\[ \hat{c}_q(t) = e^{(G_q/2)Nt} \hat{c}_q(0) + \int_0^t d\tau e^{(G_q/2)N\tau} \hat{f}_q(t - \tau) . \]  

(10)

From Eq. (10) it is possible to compute the probability \( P_q(n, t) \) of having \( n \) atoms in mode \( q \) at time \( t \), assuming zero population at \( t = 0 \). To accomplish this we first compute the anti-normally ordered characteristic function, \( \chi_q(\eta) = \langle \exp[-\eta^* \hat{c}_q] \exp[\eta \hat{c}_q^\dagger] \rangle \), yielding

\[ \chi_q(\eta) = e^{-|\eta|^2(\bar{n}_q(t) + 1)} , \]

(11)

where \( \bar{n}_q(t) = \exp(G_qNt) - 1 \) is the mean population of mode \( q \) at time \( t \). We can identify expression (11) as corresponding to a chaotic field [15]. The number distribution for a chaotic field is given by

\[ P_q(n, t) = \frac{1}{\bar{n}_q(t)} \left( 1 + \frac{1}{\bar{n}_q(t)} \right)^{-(n+1)} , \]

(12)

which for \( \bar{n}_q(t) \gg 1 \) is well approximated by \( \exp[-n/\bar{n}_q(t)]/\bar{n}_q(t) \).

When the mean population of a field mode is sufficiently large, correlation functions effectively factorize to all orders, and it becomes possible to formulate a 'classical' description of the field dynamics. In the classical theory, we can consider the sidemode populations as \( c \)-numbers and neglect the influence of the quantum noise operators as spontaneous scattering of atoms is now negligible in comparison to the stimulated contribution. With the inclusion of condensate depletion, the sidemode populations then obey the equations

\[ \frac{d}{dt} n_q = G_q(N - \sum_{q' \neq 0} n_{q'}) n_q . \]

(13)

Momentum conservation tells us that for each atom scattered into the sidemode \( q \), there is a photon scattered roughly in the direction \( k = k_0 - q \). Hence, \( N \times dI_q/dt \).
is the ideal photon count rate generated by the $|0\rangle \rightarrow |q\rangle$ atomic center-of-mass transition.

The 'classical' nonlinear model is applicable when $\bar{n}_q(t) \gg 1$, whereas the linearized quantum theory requires $\sum_{q \neq 0} \bar{n}_q(t) \ll N$. Provided that the number of active modes is not comparable to $N$, there is a significant overlap in the validity regimes of these two models. We then join them by choosing initial conditions for Eqs. (13) from $P_q(n, t_{cl})$, where $t_{cl}$ satisfies $1 \ll \bar{n}_q(t_{cl}) \ll N$. Because the response is still linear at time $t_{cl}$ the resulting nonlinear evolution does not depend on the particular choice of $t_{cl}$.

We now analyze the geometrical dependence of the single-atom gain given by Eq. (7), and show that it is largest for radiation along the long-axis of the condensate. We first note that $G_q$ depends on $g(k)$, which contains the dipole radiation pattern, as well as on $\rho_q(k)$, which depends on the geometry of the initial condensate. For a cigar-shaped condensate, aligned along the $\hat{z}$-axis, $\rho_q(k)$ is a disc which lies parallel to the $\hat{x}$-$\hat{y}$-plane in $k$-space. The dimensions of the disc in $k$-space are roughly the inverse of the condensate dimensions in $r$-space. Thus for a condensate whose dimensions are large compared to an optical wavelength, the dimensions of $\rho_q(k)$ are small compared to $k_0$.

Since $g(k)$ is slowly varying compared to $\rho_q(k)$, it can be removed from the integral in Eq. (7), and evaluated at the center of $\rho_q(k)$. In addition we neglect the recoil shift $\omega_q$ in the $\delta$-function as it has negligible effect on the value of $G_q$. The remaining integral then defines the solid angle $\Omega_q$ for the scattered radiation associated with the $q$th mode according to

$$\Omega_q = \frac{1}{k_0^2} \int d^2k |\rho_q(k)|^2 \delta(|k| - k_0),$$

which shows that only $q$ values for which the center of $\rho_q(k)$ lies at a distance $k_0$
from the origin experience gain, a consequence of energy conservation. Thus for every active quasi-mode $q$ there is a corresponding radiation direction $\hat{k}$, such that $q = k_0(y - \hat{k})$.

We can obtain a good estimate for $\Omega_q$ by taking $|\rho_q(k)|^2$ to be an ellipsoid solid with the inverse dimensions of the condensate. This gives

$$\Omega_q = \frac{4\pi}{k_0^4 W^2} \left[ \cos^2 \theta_{k,\hat{z}} + \left( \frac{L}{W} \right)^2 \sin^2 \theta_{k,\hat{z}} \right]^{-1/2},$$

where $L$ is the length of the condensate along the $\hat{z}$ axis, $W$ is the radial diameter, and $\theta_{k,\hat{z}}$ is the angle between the radiation direction and the long-axis of the condensate. Thus $\Omega_q$ is maximized for $\theta_{k,\hat{z}} = 0, \pi$, corresponding to radiation along $\hat{z}$ and $-\hat{z}$, where it is given by $\Omega_q = 4\pi/k_0^4 W^2$. As $\theta_{k,\hat{z}}$ moves away from the $\hat{z}$ axis, $\Omega_q$ is relatively flat until it reaches the geometric angle $W/L$, after which it drops off rapidly.

It is important to note that for the isotropic case $L = W$ there is no preferred direction, and a ring of radiation is instead observed.

Taking into account all of these considerations, the expression for the single-atom gain becomes

$$G_q = \mathcal{G} \frac{\sin^2 \theta_{k,\hat{z}}}{\sqrt{\cos^2 \theta_{k,\hat{z}} + \left( \frac{L}{W} \right)^2 \sin^2 \theta_{k,\hat{z}}}},$$

where $\mathcal{G} = 3|\Omega_q|^2 \Gamma / 8|\Delta|^2 k_0^2 W^2$ is the maximum single-atom gain, $\Gamma = k_0^2 d^2 / 3\pi \hbar \varepsilon_0$ being the single-atom spontaneous decay rate, and $\theta_{k,\hat{z}}$ is the angle between the radiation and polarization directions. For the parameters of the MIT experiment we find $\mathcal{G} \sim 4 \times 10^{-4} \cdot I$, where $I$ is the laser intensity in mW/cm$^2$, and $\mathcal{G}$ is given in Hz. A rough estimate of the duration of a superradiant pulse for the case $N = 10^6$ and $I = 100$ mW/cm$^2$ is $t = \ln(N)/\mathcal{G} N \sim 150$ $\mu$s, in excellent agreement with experimentally observed time scales.
The interplay between the dependence of $G_q$ on the radiation direction and the nonlinearity in Eq. (13) leads to mode competition, the outcome of which depends sensitively on the initial quantum fluctuations. When modes with different values of $G_q$ compete, the competition is 'unfair' and the mode with the largest $G_q$ generally
FIG. 1. A typical numerical simulation of condensate superradiance. The scattered photon intensity is plotted as seen by a detector array located at a distance $Z$ along the symmetry axis of the cigar-shaped BEC. The black regions correspond to maximum, and the white regions to negligible intensity, respectively. The width of the figure is $2ZW/L$, corresponding to radiation within the geometric angle of the condensate.
depletes all of the condensate atoms before the populations of the other modes have
a chance to grow. Modes with the same $G_q$, such as the quasi-modes corresponding
to radiation along the $\hat{z}$ and $-\hat{z}$ directions, instead compete 'fairly', and while the
interplay is highly sensitive to the random initial conditions, neither mode necessarily
wins, i.e. there is no real 'winner-takes-all' effect, and often a 'tie' will occur.

The angular dependence of the scattered light from a typical simulation is shown
in Fig. 1, where we have plotted the photon count density as recorded by an ideal
detector array located at a distance $Z$ from the center of the condensate along the $\hat{z}$
axis. The dark regions correspond to maximum light intensity, while the white regions
indicate negligible intensity. The center of the figure lies along the symmetry axis of
the BEC, and half-width of the box, given by $ZW/L$, corresponds to the geometric
radiation angle. The simulation was performed for a condensate with $N = 10^6$ atoms
in a BEC with a width of 10 $\mu$m and a length of 100 $\mu$m, in rough agreement with the
MIT experiment. Only matter-wave quasimodes corresponding to radiation within
twice the geometric angle of $+\hat{z}$ and $-\hat{z}$ were included in the simulation. This set of
576 modes, is more than sufficient, however, as a buildup of significant population in
modes corresponding to radiation outside of the geometric angle was never observed.
This can be attributed to the fact that these modes have a much smaller gain, and thus
compete 'unfairly' with the 'endfire' modes. Within $\theta_{k,z} < W/L$ we see the results of
the interplay between quantum fluctuations and 'fair' mode competition. The pattern
of dark spots indicates multimode superradiance, and exhibits variation on the scale
of $Z/k_0W$, corresponding to the solid angle of radiation for an endfire mode. The
pattern, which arises from the amplification of quantum fluctuations, varies randomly
from run to run, an effect which has been directly observed experimentally.

In conclusion, we note that the quasi-mode populations will experience losses as
the recoiling atoms eventually propagate out of the condensate volume. The lifetime of the quasi-mode, however, is on the order of \( T_q \equiv m L_q / \hbar |q| \), where \( L_q \) is the length of the condensate along \( q \). These losses tend to destroy the coherence between condensate and quasi-mode, which accounts for the observation of a threshold for superradiance in the MIT experiment: for insufficient laser power, the growth of matter-wave coherence cannot overcome the losses. As this threshold is very small, we have considered only the situation far above threshold, in which case the losses may be neglected.

In general, the rate of matter-wave decoherence in superradiance or CARL type experiments is given by the ratio between the recoil velocity and the matter-wave coherence length. As a BEC is maximally coherent, its coherence length is given, as above, by its spatial extent. For a noncondensed atomic cloud, however, it is instead given by the thermal DeBroglie wavelength, which is in general small compared to that of the BEC. As a result, the threshold for superradiance is significantly larger, for which reason superradiance was not observed above \( T_c \) in the MIT experiment. We remark that in the case of the CARL, the presence of an optical cavity provides additional feedback, which can compensate for the lack of atomic coherence and allow for instability and gain.

Lastly, we remark that the Hamiltonian (1) describes the creation of correlated atom-photon pairs, and is therefore analogous to the optical parametric amplifier (OPA), which generates entangled two-photon states for a variety of applications, e.g. tests of Bell's inequality, quantum cryptography, and quantum teleportation. It should therefore be possible to perform analogous experiments using entangled atom-photon states generated in a BEC superradiance experiment.

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