Quantum Systems in Bernoulli Potentials

by

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A Dissertation Submitted to the Faculty of the

Department of Mathematics

In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

In the Graduate College

The University of Arizona

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Michael Bishop entitled
Quantum Systems in Bernoulli Potentials
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

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Michael Bishop
April 8, 2013

Acknowledgments

For my father Ira, my mother Sue, and my sister Kathryn.

Dedication

There are many people to thank for their years of support.

I would like to thank my father Ira Bishop, my mother Sue Markham, and my sister Kathryn Bishop, for all their love and encouragement. I would not be here today if it were not for them.

I would like to thank my advisor Professor Jan Wehr for sharing his experience and expertise. His insight and intuition were invaluable in my development as a mathematician. Whenever I am frustrated with research, I keep in mind Jan's optimistic outlook that "something more interesting is happening."

I would like to thank the University of Arizona for hosting me as a graduate student. I would like to thank the Department of Mathematics for their support and training in mathematics, teaching, and research. I would especially like to thank my committee of Professor Leonid Friedlander, Professor Robert Sims, and Professor Ken McLaughlin. I would like to thank the graduate head Professor Thomas Kennedy, the Department Head Professor William McCallum as well as thanks to Professor William Faris, Professor Sunder Sethuraman, and Professor Joseph Watkins. I would like to thank the office staff of Sandra Sutton, Brooke Valmont, Sylvia Anderson, Jennifer Garcia, and David Gonzalez for their help navigating through the bureaucracy. And of course, I would like to thank the department for their financial support in the form of teaching assistantships, research assistantships in the NSF VIGRE Program, and the NSF GK-12 Fellowship.

I would like to thank my fellow graduate students for being pleasant and helpful colleagues school: Enrique Acosta, Michael Gilbert, Yaron Haddad, Martin Leslie, Shane Passon; my office mates: John Gemmer, Derek Seiple, Priya Prasad, Sarah Mann, Steven Rosenthal, Jon Edwards, Colin Clark; as well as Dr. Tom LaGatta, Dr. Thomas Occhipinti, Dr. Victor Piercey. There are many other wonderful graduate

students to recognize.

I would like to thank my collaborators in Barcelona: Dr. Maciej Lewenstein, Dr. Anna Sanpéra, Dr. Julia Stasińska, and Dr. Pietro Massignan. I benefited greatly from their insight and expertise.

I have make a great many friends in Tucson that I will not forget. I would like to especially thank Jason and Jordan Kennedy, the brothers who have been like my brothers. Thank you for the many nights spent relaxing, praying, chatting, and just being. Most importantly, thank you for keeping me humble by giving as little credit possible for my mathematical ability. I would like to thank my roommates over the years, especially Matthew Pennybacker.

I would like to thank Central City Assembly (CCA) for being my spiritual home in Tucson. You have been a place for those with are unsettled to find rest on their journey through life. I would like to thank Pastor Dave Ferrari for his consistency in bringing Christ to those who need Him. I would like to thank my worship leaders Kai Eilert and Annette Peru-Eilert; Mr. Ed, Papa John, Wynn and Judi Thomson, Brett Martin, and Robb Erickson, and everyone else in the congregation.

I would like to thank Graduate Christian Fellowship (GCF) for the constant moral and spiritual support they provided. I would like to especially thank Dan Alfred, Amy Newhouse, Samuel Jayankathan, Bradley Christoffersen, Jeff and Tova Brown, Jess Vriesema and Christine Calderon-Vriesema, Jennifer Hendryx-Parsons, Bo Miller, and Phil Alderink for their personal investment into my life. My prayer is that GCF continues to be a refuge for graduate students and academics in need of someone who knows both the peace of Christ and the pressures of academia.

I would be remiss if I did not thank God in this dissertation for His support over the last six years. This dissertation would not have come to be if not for the hope I have in Jesus Christ and the love He has for me. While wanting to quit seemed a daily consideration, there were a few times over the last six years when I was so distraught and hopeless that I was planning my exit. In those times, God spoke His love me,

reminded me of His Son, and encouraged me to press on. He comforted me when I was depressed, calmed me when I was frustrated, corrected me when I was wrong, forgave me when I was contrite, and was merciful at all times. This dissertation is but one of many fruit in my life His love for me has produced.

I would also like to thank anyone who takes the time to read this lengthy work summarizing my graduate career. I may not meet you, but I appreciate you for appreciating this work.

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Abstract

Quantum mechanics is a theory developed to explain both particle and wave-like properties of small matter such as light and electrons. The consequences of the theory can be counter-intuitive but lead to mathematical and physical theory rich in fascinating phenomena and challenging questions. This dissertation investigates the nature of quantum systems in Bernoulli distributed random potentials for systems on the one dimensional lattice $\{0, 1, \ldots, L, L+1\} \subset \mathbb{Z}$ in the large system limit $L \to \infty$. For single particle systems, the behavior of the low energy states is shown to be approximated by systems where positive potential is replaced by infinite potential. The approximate shape of these states is described, the asymptotics of their eigenvalues are calculated in the large system limit $L \to \infty$, and a Lifschitz tail estimate on the sparsity of low energy states is proven. For interacting multi-particle systems, a Lieb-Liniger model with Bernoulli distributed potential is studied in the Gross-Pitaevskii approximation. First, to investigate localization in these settings, a general inequality is proven to bound from below the support of the mean-field. The bound depends on the per particle energy, number of particles, and interaction strength. Then, the ground state for the one-dimensional lattice with Bernoulli potential is studied in the large system limit. Specifically, the case where the product of interaction strength and particle density is near zero is considered to investigate whether localization can be recovered.

Chapter 1
Introduction

Quantum mechanics is a theory developed to explain both particle and wave-like properties of small matter such as light and electrons. The consequences of the theory can be counter-intuitive but lead to mathematical and physical theory rich in fascinating phenomena and challenging questions. For example, if an electron travels through an imperfect copper wire with random impurities, how do the random impurities affect the electron's travel? It turns out that random environments have a startlingly powerful effect on the behavior of a single quantum particle. As opposed to the behavior in clean environments (i.e. pure copper wires), individual quantum particles in random environments localize to small regions of space and do not easily transport through system. This effect is called Anderson localization. Chapter 2 and Chapter 3 discuss the effects of a random Bernoulli potentials on the low energy states of these single particle systems.

A copper wire may have many electrons in it; physical models describing this system must be able to model the many electrons in the system. If individual particles are often indistinguishable, models should have symmetries to describe this. Bosons have the simplest such symmetry: particles can be interchanged without changing the state of the system. This symmetry leads to the possibility that a given system of bosons may have macroscopic numbers of particles in the same individual state. In fact, the physical form of this state, called a Bose-Einstein condensate, was first created in 1995 by Cornell and Wieman at the University of Colorado. They were able to do this in a laboratory by cooling atoms to within a couple hundred nanoKelvin of absolute zero. These cold atom experiments are robust enough to explore and test theories on a wide range of quantum phenomena, including systems with inter-particle

interactions which are difficult to study.

My research studies the interplay in quantum many-body systems between the delocalization due to inter-particle interaction and the localization due to a random external potential. In these systems, the ground state, which is the energy minimizer, describes typical behavior in a given system. Chapter 5 will discuss this interplay in a wide range of quantum many-body systems with Bernoulli potential as well as giving specific results for the ground state for states on a one-dimensional lattice.

This introduction will give some background in quantum mechanics, random Schrödinger operators, random Schrödinger operators with Bernoulli potential, quantum many-body systems, and quantum many-body systems in disordered potentials. It is intended as a brief introduction to concepts and ideas, see the references for more thorough works on the topics below.

1.1 A Brief Introduction to Quantum Mechanics

Quantum mechanics is the theory which describes systems with both particle-like and wave-like qualities of matter, such as the behavior of electrons orbiting an atom or the photons of light illuminating a room. It is daunting as a physical theory because it is incredibly difficult to picture in one's mind and often counter-intuitive to the way the world is viewed in everyday life. Yet it has proven to be incredibly accurate predictor of physical behavior at the small scales. It is a theory rich in both mathematical and physical questions; its counter-intuitive nature leaves a large collection of strange and wondrous phenomena to be discovered. The goal of this section is provide physical motivation for Schrödinger's formulation of quantum mechanics. This discussion is intended for someone with some undergraduate background in physics, calculus, linear algebra, and differential equations.

The theory of quantum mechanics seeks to describe systems where classical mechanics fails. Classical mechanics describes the motion of objects such as a cannonball

flying through the air to the orbit of a satellite around the Earth to the vibration of a guitar string. In classical mechanics, a system is described by the positions and momenta of particles, indexed by i, each with position q_i and momentum p_i which may depend on time t. For example, the motion of a cannonball is described by a trajectory $\boldsymbol{x}(t)$, where $\boldsymbol{x}(t)$ is position of the center of mass of the cannonball and $\boldsymbol{p}(t)$ is its momentum, typically the product of its mass and velocity vector. The motion would be described in terms of a differential equation, such as Newton's equation $\boldsymbol{F} = m\boldsymbol{a}$. An alternate formulation would be based on the fact that a frictionless system conserves energy, $\frac{dE}{dt} = 0$, i.e. the energy E does not change. The total energy $E(t)$ is the sum of kinetic energy $\frac{1}{2m}\boldsymbol{p} \cdot \boldsymbol{p}$ and potential energy $V(\boldsymbol{x}(t))$. Any motion $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$ must satisfy $\frac{dE}{dt} = 0$.

Classical mechanics also describe the motion of a wave, such as the displacement of a guitar string. In the case of the string, the wave function $\phi(x, t)$ describes the vertical (perpendicular) displacement of the string at position x along the string at time t. The wave motion on the string is derived from Hooke's law, where the force of a spring satisfies the equation $\boldsymbol{F} = -c(\boldsymbol{y} - \boldsymbol{y_0})$ for spring constant c and displacement $\boldsymbol{y}$ from equilibrium point $\boldsymbol{y_0}$. The wave motion is given by the wave equation

$$\frac{d^2\phi}{dt^2} = c^2 \frac{d^2\phi}{dx^2} \tag{1.1.1}$$

where c is the wave speed and $\frac{d^2}{dx^2}$ is the one-dimensional Laplacian operator Δ. In the case of the guitar string with length ℓ and with Dirichlet boundary conditions $\phi(0) = \phi(\ell) = 0$ which correspond to the ends of the string being fixed in place, solutions to the wave equation are sine waves. The wave equation is a linear equation, which means that for any set of solutions ϕ_j to the wave equation, the linear combination $\sum c_j \phi_j$ is also a solution, where c_j are real or complex numbers depending on the setting. In physics, this property is called the Superposition Principle, where different waves are simply added together where they overlap. For comparison, when two particles overlap at the same point and time, a collision occurs and certain interaction

conditions must be satisfied.

Waves exhibit a property called diffraction. Consider a water wave traveling into a harbor with a sea wall. If the water wave had a particle-like trajectory, the water wave would not exist in the shadow of the sea wall. But the waves travel around the sea wall and into its 'shadow.' This bending of the wave around the barrier is called diffraction. If waves travel through a pair of small openings in the barrier, the diffraction caused by the slits leave a sinusoidal pattern of the wave on the shoreline, caused by the superposition of the diffraction effects from each of the openings. In the similar experiments with light, the light showed this sinusoidal pattern of dark and bright spots on a screen beyond the slits, implying a wave-like behavior. In comparison, if particles travel through a slit, they continue on a linear trajectory and form a single spot (or tight grouping of spots) on the screen.

Two problems with classical mechanics led to the development of quantum mechanics, both dealing with the structure of the atom. The first problem was the idea that an electron orbited a nucleus of protons and neutrons in a manner similar to the Earth orbiting the Sun. The idea of a large nucleus with a cloud of electrons was confirmed by experiment by Thompson. However, an accelerating charged particle emits electromagnetic waves causing it to lose energy. An electron in orbit should lose energy as it emits an electromagnetic wave, causing it to eventually crash into the nucleus. Thus this model of the atom is fundamentally unstable.

The second problem was due to the photoelectric effect. To induce electrons to emit from a metal plate, a light is projected onto the plate. The light transmits energy to the electrons, causing them to 'escape' their nuclei and emit. In theory, if the light has sufficient energy, the electrons should be hit by that energy and caused to emit. In classical mechanics, the energy of a wave is dependent on its amplitude, so a brighter light should eventually cause electrons to be emitted. In experiment, if a low frequency high intensity light (such as infrared light) is shone onto the plate, the electrons are not emitted. However, if the frequency of the light is increased, the plate

will emit electrons, even if the intensity is relatively low. This confounded physicists until Einstein offered up the idea of 'quanta:' light is not merely a wave, but a series of packets of waves. Their energy of a quanta is directly proportional to its frequency. The intensity of the light corresponds to number of quanta emitted, not the energy of individual quanta of light, called photons. In the photoelectric effect, higher intensity light merely sent more photons into the plate, so if the the frequency of the light was not sufficiently large, the photons could not cause the plate to emit electrons. The energy of a photon is given by $E = \hbar\omega$, where $\hbar$ is Planck's constant divided by 2π and ω is the frequency. Photons also have momentum, similar to a particle, given by $\boldsymbol{p} = \hbar\boldsymbol{k}$, where $\boldsymbol{k}$ is the direction of the photon divided by the wavelength of the photon.

Both these problems inspired physicists such as Heisenberg and Schrödinger to formulate quantum theories. Later, Dirac later showed the equivalence of these formulations. In Schrödinger's formulation, he first considered a wave function of the form $\psi(x,t) = e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ and substituted the equation for energy and momentum of a photon, $\psi(x,t) = e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)/\hbar}$. Taking the gradient ∇,

$$\nabla\psi = \frac{i\boldsymbol{p}}{\hbar}\psi \tag{1.1.2}$$

he redefined momentum as an operator on the wave function with the form

$$\hat{p} = -i\hbar\nabla \tag{1.1.3}$$

where the $\hat{}$ denotes that the term is an operator on the wave function rather than a scalar. The kinetic energy in the classical formulation $-\frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}$, the quantum analog has the form

$$\frac{\hat{p}\cdot\hat{p}}{2m} = \frac{-\hbar^2}{2m}\nabla\cdot\nabla = \frac{-\hbar^2}{2m}\Delta \tag{1.1.4}$$

where Δ is the Laplacian operator. Taking the derivative with respect to time,

$$\frac{d\psi}{dt} = \frac{-iE}{\hbar}\psi \tag{1.1.5}$$

or $E = i\hbar\frac{d\psi}{dt}$. If the photon is free, i.e. $V(x,t) \equiv 0$,

$$i\hbar\frac{d\psi}{dt} = \frac{-\hbar^2}{2m}\Delta\psi \tag{1.1.6}$$

which is Schrödinger's time-dependent equation . If there is an external potential $V(x,t)$ due to gravity or some other environmental effect, Schrödinger's equation becomes

$$i\hbar\frac{d\psi}{dt} = \frac{-\hbar^2}{2m}\Delta\psi + \hat{V}\psi \tag{1.1.7}$$

where $\hat{V}$ is the multiplication operator which multiplies ψ by $V(x,t)$. A more general formulation is

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi \tag{1.1.8}$$

where $\hat{H}$ is the Hamiltonian, an operator describing the energy of a system being studied. In the case above, the operator $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V$ is called the Schrödinger operator.

If $\hat{H}$ is time independent, i.e. V is independent of time in the Schrödinger operator, then the eigenvectors of $\hat{H}$ provide a straightforward solution to the time-dependent Schrödinger equation. For any vector $\psi(x)$ that solves the eigenvalue equation, known as the time-independent Schrödinger equation, for some real number E,

$$\hat{H}\psi = E\psi \tag{1.1.9}$$

there exists a solution $e^{-iEt/\hbar}\psi(x)$ to the time-dependent Schrödinger equation. This solution to the time-dependent equation implies that the spectrum of $\hat{H}$ is important for understanding quantum systems. To simplify calculations, the units for the system are chosen such that $\frac{\hbar^2}{2m}$ is equal to 1.

The mathematical formulation of quantum mechanics is built upon the theory of self-adjoint operators on Hilbert spaces. A Hilbert space is a vector space, which is complete (all Cauchy sequences converge in the space) with respect to the norm $\|\cdot\|$ associated with the inner product $\langle\cdot,\cdot\rangle$. A state of the system is a function in

the Hilbert space normalized to one, which will also be referred to as a vector. An operator $\hat{A}$ is linear map from the Hilbert space to itself, or from a subset $D(\hat{A})$ of the Hilbert space into the Hilbert space. The adjoint $\hat{A}^*$ of an operator $\hat{A}$ is the operator which satisfies

$$\langle \hat{A}u, v\rangle = \langle u, \hat{A}^* v\rangle \tag{1.1.10}$$

for all $u \in D(\hat{A})$, where the set of all v which satisfy this equation is the domain of $\hat{A}^*$, denoted $D(\hat{A}^*)$. An operator is self-adjoint if $D(\hat{A}) = D(\hat{A}^*)$ and $\hat{A} = \hat{A}^*$.

Operators acting on the Hilbert space correspond to observables of the system, experimental quantities that can be measured such as position, momentum, or energy. In the spaces above, the position operator $\hat{x}$ is multiplication by the position vector $\boldsymbol{x}$, the momentum operator$\hat{p}$ is the differential operator $-i\hbar\nabla$, and the energy operator is the Schrödinger operator. When measuring a state ϕ for an observable $\hat{A}$, the 'expected' outcome is $\langle \phi, \hat{A}\phi\rangle$. If a state is an eigenvector of an operator corresponding to a specific observable, then the experimental quantity is exactly the eigenvalue A, i.e. if $\hat{A}\phi = A\phi$, then $\langle \phi, \hat{A}\phi\rangle = A\|\phi\|^2$. If a state is not an eigenvector but a linear combination of eigenstates ϕ_j with eigenvalues A_j, $\sum_j c_j\phi_j$, then the experimental quantity is $\langle \phi, \hat{A}\phi\rangle = \sum_j A_j|c_j|^2\|\phi_j\|^2$. In this interpretation, the probability of measuring a system in state ϕ and receiving the outcome A_k is

$$\frac{|c_k|^2\|\phi_j\|^2}{\sum_j |c_j|^2\|\phi_j\|^2} \tag{1.1.11}$$

States are normalized both to simplify the above expression and to clarify the probabilistic picture. After normalization, the magnitude $|\phi(\boldsymbol{x})|^2$ an be interpreted as a probability density for the position of the particle.

For this dissertation, the Hilbert space of interest will be $\ell^2(\mathbb{Z})$, the set of all functions defined from the discrete space $\mathbb{Z}$ to $\mathbb{C}$ which are square-integrable with inner product $\langle \phi, \psi\rangle = \sum_{\boldsymbol{x}\in\mathbb{Z}} \overline{\phi(\boldsymbol{x})}\psi(\boldsymbol{x})d\boldsymbol{x}$. The other Hilbert space that will be considered at times is $L^2(\mathbb{R}^d)$, the set of all functions from the continuous space $\mathbb{R}^d$ to $\mathbb{C}$ which are square-integrable with inner product $\langle \phi, \psi\rangle = \int_{\mathbb{R}^d} \overline{\phi(\boldsymbol{x})}\psi(\boldsymbol{x})d\boldsymbol{x}$. The

discrete Schrödinger operator is the operator on $\ell^2(\mathbb{Z})$ with the form

$$(H\phi)(j) = \sum_{|\boldsymbol{i}-\boldsymbol{j}|=1} (\phi(j) - \phi(i)) + V(j)\phi(j) \tag{1.1.12}$$

where $\sum_{|\boldsymbol{i}-\boldsymbol{j}|=1}(\phi(j) - \phi(i))$ is the discrete Laplacian $-\Delta$. This operator is related to the energy of the system; an eigenvector and the eigenvalue corresponds to a state in a specific energy levels.

1.2 Quantum Mechanics for Single Particles in Disordered Systems

The quantum systems studied in experiments rarely exist in perfectly clean and ordered environments: wires conducting electricity have impurities and the electric and magnetic fields in experimental settings have some outside interference. Conventional wisdom suggests that if these problems are small, their effect on an experiment is small. In quantum mechanics, one could model these impurities as a random potential in the Schrödinger equation and study how 'small' the randomness needs to be to have no effect. The hope is that conventional wisdom is correct and some noise can be admitted without effect.

In his seminal 1958 paper [11], Anderson discovered that randomness had an incredibly strong insulating effect on the transport of a single electron traveling in a one-dimensional space. In [48], Mott explained how randomness affected the conductivity of materials, providing methods to classify whether a material was a metal which conducted electrons or an insulator which did not conduct electrons. In 1974, Kosterlitz and Thouless demonstrated the basic scaling principles that determined conductance in these systems [40].

At the heart of Anderson's argument was the idea that disorder, however minor, has a cumulative insulating effect. A quantum particle with a certain energy E has an associated frequency $\omega = E/\hbar$. For the particle to occupy a certain location, the

location must be conducive to a state of that frequency: this requires the potential to have a rather specific shape. Since the potential at a given site is random, the probability that this shape would repeat over and over in the space is exponentially small.

The transport of a single particle through the potential environment can be thought of as a person dancing to music. A particle dances with a specific rhythm, determined by the frequency associated to the particle's energy. If the rhythm of the dance matches the music being played, then the particle will dance to that music. If the rhythm of the dance does not match the music, then the particle will not dance to that music, or at the very least, struggle to do so. Music is analogous to the shape of the potential V. If the music is random, then in most places, it is not likely to match the rhythm of the particle, which implies the particle will not exist there with significant probability. Since particles do exist in these systems, there must be regions in the random potential where particles of a specific frequency can dance.

A random potential is defined on a probability space Ω equipped with a probability measure P defined on a specific σ-algebra on Ω. For each $\omega \in \Omega$, V_ω is a function from $\mathbb{Z}$ to $\mathbb{R}$. In general, these random potentials are defined by independent identically distributed (i.i.d.) random variables at each site $x \in \mathbb{Z}$. This makes the discrete Schrödinger operator, $H = -\Delta + V$, a random Schrödinger operator $H_\omega = -\Delta + V_\omega$. The ω notation is used here to clarify between the family of potentials and a specific potential and will be dropped because this dissertation generally focuses on the behavior of a system with a specific random potential function V_ω. Once determined for specific potentials, the probabilistic behavior of a given quantity is determined by the probability distribution on V and related random variables. It will be assumed that the support of the distribution of V, the set of values that $V(\cdot)$ can take in $\mathbb{R}$, will be bounded. In this case, H is a bounded and self-adjoint operator.

Localization can be considered both as a dynamical property and spectral property [39]. For a state to be dynamically localized, the probability of finding a particle

in a compact set is strictly positive for all time in the large system limit. For a state initially in the form ϕ, dynamical localization is mathematically expressed as

$$P\left[\lim_{L\to\infty}\sup_{t>0}\|e^{itH}\phi\chi_{B_L(0)}\| = 1\right] = 1 \tag{1.2.1}$$

where e^{itH} is the time evolution operator, $B_L(0)$ is a ball of radius L around the point 0, and $\chi_{B_L(0)}$ is the indicator function of $B_L(0)$. Dynamical localization implies that if a particle starts in a small part of the space, it will not transport very far. The cumulative effect of randomness prevents it from traveling away from its starting region. State of this form are commonly referred to as a bound states [39]. Dynamical delocalization describes the property that a particle will not be found in a compact box for all time in the large system limit. This can be written as

$$P\left[\lim_{L\to\infty}\sup_{t>0}\|e^{itH}\phi\chi_{B_L^c(0)}\| = 1\right] = 1 \tag{1.2.2}$$

In this case, the particle will escape any bounded box. States of this form are referred to as extended states.

Spectral localization describes a similar behavior through the analysis of the time-independent operator in the Schrödinger equation. This analysis is done on the spectrum of an operator, to be defined in the following discussion and following from the much more thorough presentation in [41]. The spectrum is a generalization of the concept of eigenvalues of a linear operator on a finite-dimensional vector space. This discussion will restrict to bounded operators and symmetric operators, i.e. self-adjoint operators with $D(A)$ equal to the entire Hilbert space. The spectrum of a bounded and symmetric operator H, denoted $\sigma(H)$, is defined as the complement of the resolvent set. The resolvent set of an operator H acting on the Hilbert space $\mathbb{H}$ is the subset $\{z\}$ in the complex plane $\mathbb{C}$ such that $(z - H)$ is invertible. If H is symmetric and bounded, the spectrum is a compact subset of the real line.

The operator H can be represented by a projection-valued measure defined on its spectrum. For any continuous function f from $\sigma(H)$ to $\mathbb{R}$, there is a sequence

of polynomials q_n which converge uniformly to f by the Weierstrass theorem. If $q(z) = \sum a_j z^j$ with $a_j \in \mathbb{R}$, then there is a well-defined operator $q(H) = \sum a_j H^j$, this operator is bounded and symmetric. Using the uniform convergence of $q_n \to f$, there is a well-defined operator $f(H) = \lim_{n\to\infty} q_n(H)$. Given two points x and y in the Hilbert space $\mathbb{H}$, a linear functional L can be defined on the continuous functions f by

$$L_{x,y}(f) = \langle f(H)x, y\rangle \tag{1.2.3}$$

By the Riesz representation theorem, there exists a unique measure $m_{x,y}$ such that,

$$\langle f(H)x, y\rangle = \int f(z) dm_{x,y} \tag{1.2.4}$$

For each Borel subset of $\sigma(H)$ S, $m_{x,y}(S)$ is a bounded, skew-symmetric, sesquilinear functional in x and y. These functionals define a unique bounded symmetric operator $Proj(S)$ defined on $\mathbb{H}$ such that

$$m_{x,y}(S) = \langle Proj(S)x, y\rangle \tag{1.2.5}$$

Each $Prof(S)$ is projection onto a subspace of $\mathbb{H}$. The projection of the empty set is the zero operator,

$$Proj(\emptyset) = 0$$

the projection of the spectrum of H is the identity,

$$Proj(\sigma(H)) = \mathbb{I}$$

the family of operators commute

$$Proj(S)Proj(T) = Proj(T)Proj(S)$$

and factor over intersections

$$Proj(S \cap T) = Proj(S)Proj(T)$$

and are orthogonal for disjoint $S,\ T$

$$Proj(S)Proj(T) = 0$$

This family of operators $Proj(S)$ defined on the Borel σ-algebra and acting on $\mathbb{H}$ is called an orthogonal projection-valued measure. The operators $f(H)$ can be represented using the $Proj(S)$:

$$f(H) = \int_{\sigma(H)} f(z) dProj \tag{1.2.6}$$

where $\int_S dProj = Proj(S)$ recovers the original projection; H is explicitly represented by $f(z) = z$,

$$H = \int_{\sigma(H)} z dProj \tag{1.2.7}$$

Using these projections and measures, the Hilbert space $\mathbb{H}$ can also be decomposed into orthogonal subspaces. For a given x and y in $\mathbb{H}$, the Lebesgue decomposition theorem decomposes $m_{x,y}$ into a sum of measures

$$m_{x,y} = m_{x,y}^{(pp)} + m_{x,y}^{(sc)} + m_{x,y}^{(ac)} \tag{1.2.8}$$

The measure $m_{x,y}^{(pp)}$ is a point measure supported on a countable subset of $\sigma(H)$, $m_{x,y}^{(sc)}$ is a singular measure supported on a Lebesgue measure zero subset of $\sigma(H)$, $m_{x,y}^{(ac)}$ is an absolute continuous measure with respect to Lebesgue measure. The supports of these measures partition $\sigma(H)$. Each of these measures is bounded, skew-symmetric, sesquilinear functional in x and y, so just as with $m_{x,y}$, there exist orthogonal projection-valued measures such that

$$\begin{aligned} m_{x,y}^{(pp)}(S) &= \langle Proj(S)^{(pp)} x, y\rangle \\ m_{x,y}^{(sc)}(S) &= \langle Proj(S)^{(sc)} x, y\rangle \\ m_{x,y}^{(ac)}(S) &= \langle Proj(S)^{(ac)} x, y\rangle \end{aligned}$$

These projections have the same properties as the projection measure above. Each of these projections, when evaluated on the set that supports the associated measure,

are orthogonal to each other and decompose the Hilbert space $\mathbb{H}$ into orthogonal subspaces

$$\begin{aligned} \mathbb{H}^{(pp)}(S) &= Proj^{(pp)}\mathbb{H} \\ \mathbb{H}^{(sc)}(S) &= Proj^{(sc)}\mathbb{H} \\ \mathbb{H}^{(ac)}(S) &= Proj^{(ac)}\mathbb{H} \end{aligned}$$

where each subspace is defined by the spectrum associated with it.

In the works by Ruelle [55], Amrein and Georgescu [9], Enss [25], referred altogether as the RAGE Theorem [19], each type of the spectrum corresponds to different qualitative behavior. The pure point spectrum and the subspace $\mathbb{H}^{(pp)}$ correspond to localized bound states of the system. The absolutely continuous spectrum corresponds to extended states or generalized eigenstates, solutions to $H\phi = E\phi$, where $\phi \notin \mathbb{H}$. These are delocalized states: the probability that a particle with state in the subspace $\mathbb{H}^{(ac)}$ will not be in a compact set is zero in the infinite system limit. In essence, states associated with this part of the spectrum will escape any compact set. Singular continuous spectrum is also associated with extended states which also escape a given compact set, but not necessarily at the same rate as states associated with the absolutely continuous spectrum. Localization can be proved by the absence of both absolute and singular continuous spectrum for the operator H.

Dynamical localization is a stronger condition than spectral localization; in fact, it implies spectral localization. It is also the physically more relevant quantity. In a system that has dynamical localization, a particle starting in any state will stay in a slightly larger compact set high probability. However, spectral localization is commonly studied because it reduces analysis to the operator H and its spectrum, avoiding the difficulties related to the complex time evolution $i\hbar\frac{d}{dt}$.

Localization has been proved in many settings, both on $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$, with the focus on results for lattice models. For one-dimensional lattices, Goldsheid, Molchanov, and Pastur proved localization of the spectrum can be proven through the use a transfer matrices and Lyapunov exponents [34]. For higher dimensional lattices, mathematical proofs of localization rely on various bounds on the Green's

function, to be defined later. Localization can be proved for sufficiently high disorder or on the edges of the spectrum using multi-scale analysis pioneered by Fröhlich and Spencer [31] or fractional moments method pioneered by Aizenman and Molchanov [5]. In both the case of high disorder and the case of energies near the edges of the spectrum, the potential barriers strongly block transport of the particle.

Many physicists believe that just as in one dimension, the spectrum in two dimensions must be completely pure point and all states localize, albeit in a weaker form than in one dimension [1]. In three dimensions, there should exist absolutely continuous spectrum away from the edges of the spectrum in the case of low disorder. The spectral values determining whether an energy corresponds to a bound state or extended state is separated by a 'mobility edge.' The intuition follows from the idea that given enough dimensions, a quantum particle should be able to find a path out of any given region of space if it has enough energy and with small enough random potential barriers. In the analogy of the music with the dancer, there should be at least on path of music in the random environment to allow the particle to comfortably dance out. The localization in two dimensions is due to the space not having enough paths for the particle to escape, thus being too restrictive to allow transport.

For the discrete random Schrödinger operator in one dimension, localization can be proven through the use a transfer matrices representation and Lyapunov exponents. For a given energy E and i.i.d. random potential function $V(\cdot)$, the state $u(j) \in \ell^2(0, \ldots)$ satisfies the discrete Schrödinger equation

$$Eu(j) = 2u(j) - u(j-1) - u(j+1) + V(j)u(j) \tag{1.2.9}$$

Solving for $u(j+1)$,

$$u(j+1) = (2 + V(j) - E)u(j) - u(j-1) \tag{1.2.10}$$

which is a sequence determined by $u(0)$ and $u(1)$ and the realization of the potential

$V(\cdot)$ through the transfer matrix equation

$$\begin{bmatrix} u(j+1) \\ u(j) \end{bmatrix} = T_j^E T_{j-1}^E \dots T_1^E \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{1.2.11}$$

for transfer matrices

$$T_j^E = \begin{bmatrix} 2 + V(j) - E & -1 \\ 1 & 0 \end{bmatrix} \tag{1.2.12}$$

These form an ergodic family of random operators. The subadditive ergodic theorem [23, 38] states that the limit

$$\lim_{L \to \infty} \frac{1}{L} \ln \left(\| T_L^E \dots T_0^E \| \right) = \gamma(E) \tag{1.2.13}$$

has a deterministic limit $\gamma(E)$, which is called the Lyapunov exponent. If $\gamma(E) > 0$, then $|u(L)| \leq e^{-\gamma L}$ and the state exponentially localizes. Most proofs in one dimension work to show the positivity of $\gamma(E)$ by taking expectations and through various bounds on the probability density of the distribution of V.

There are two general methods of proof of localization in higher dimensional spaces. Both rely on showing an exponential bound on the expectation of the Green's function. The Green's function describes the spectral behavior of an operator and is defined as

$$G(x, y, E) = \langle e_x, (E - H)^{-1} e_y \rangle \tag{1.2.14}$$

where e_x is the vector equal to 1 at x and zero elsewhere and $(E - H)^{-1}$ is the resolvent of H. The Green's function estimates how the magnitudes of a state with energy E between the positions x and y. Both methods show the exponential decay of the Green's function to show the absence of continuous spectrum and exponential localization of states. If the Green's function decays exponentially as $|G(x, y, E)| \leq e^{-\gamma |x-y|}$, then states must also decay exponentially, precluding extended states.

The first method was introduced by Fröhlich and Spencer in [31] and referred to as multi-scale analysis. First, an initial length scale determined by the density of states of the random Schrödinger operator. The density of states is a function $k(\epsilon)$ defined

as the large system limit of the number of eigenstates restricted to a box with energy less than ϵ per unit volume. It is used to measure the number of low energy states a given box can support for the operator acting on functions restricted to the box. For a box of this initial length scale, the probability that a state with energy near E being supported on this box bounded. From this initial length scale, the lattice is tiled by boxes of this initial length scale. Using the probabilistic estimate for the smaller boxes, a similar probabilistic estimate is derived for larger boxes composed of smaller boxes. This process is iterated to show localization in the infinite volume limit, see [62, 39] for more introductory work.

The second method is the fractional moments method, introduced by Aizenmann and Molchanov in 1993 [5]. The method takes advantage of the integrability of the fractional moment of the Green's function, $\mathbb{E}\left[|G(x,y,E)|^s\right]$ for $s \in (0,1)$, to derive exponential bounds. The method uses two main ingredients [63]. The first step is an a-priori bound on the fractional moment of the Green's functions for two neighboring sites. This bound has the form $C\lambda^{-s}$, where C is a constant, λ is a measure of the strength of disorder, and s is the fractional moment. The second step is to derive a decoupling lemma from the bounded and compactly supported probability density of the potential. These two bounds combine to bound the fractional moment of the Green's function for sites x and y by the product of the bounds on fractional moments on the shortest path of neighboring sites from x to y. This product is bounded by some constant times a factor of the form $(C'\lambda^{-s})^{|x-y|}$, which will decay exponentially in distance if the disorder is large enough. This exponential bound on the Green's function implies an exponential localization of states as well as the existence of only pure point spectrum. The methods have been generalized in [3] and for a basic introduction, see [63].

1.3 Quantum Mechanics in Bernoulli Disorder

The localization methods described in the previous section are generally proved for random potentials with 'nice' probability distributions: for any site j on the lattice and set B in the Borel σ-algebra of $\mathbb{R}$, the probability that $V(j)$ is in B has the following form,

$$P[V(x) \in B] = \int_B p(x)dx \tag{1.3.1}$$

where $p(x)$ is a bounded and compactly supported function on $\mathbb{R}$, referred to as the probability density. For any function f of the random potential at site j, $V(j)$, its expectation is the integral

$$\mathbb{E}[f(V(j))] = \int f(x)p(x)dx \tag{1.3.2}$$

and for functions F of the potential V at sites in the subset of the lattice Λ, the expectation is

$$\mathbb{E}[f(V(\cdot)\chi_\Lambda)] = \int f(\boldsymbol{x}) \prod_{j\in\Lambda} p(x_j)dx_j \tag{1.3.3}$$

where $\boldsymbol{x}$ is the vector of x_j, $j \in \Lambda$ and the integral is over the Cartesian product of $|\Lambda|$ copies of the support of $p(x)$. This product of the individual $p(x_j)$ is due to the independence of the $V(j)$. Because the density is bounded and compactly supported, integrals of bounded functions with respect to the probability distribution are also bounded.

This fact is used in the methods above to bound the Green's functions. The Green's function can be expressed as an integral, either over sites, functions, or even as a contour integral. The expectation of the Green's function can then be expressed as a double integral, with one of the integrals evaluated with respect to the probability distribution $\prod p(x_j)dx_j$. Using the Fubini-Tonelli Theorem, the order of integration can be switched in cases where the overall integral is finite. The integrability of the fractional moment of the Green's function makes the fractional moments method more straight-forward than the multi-scale analysis method. By switching the order

of integration, the Green's function is integrated with the bounded and compactly-supported density function first which allows for all the localization estimates.

The Bernoulli random variable does not have a 'nice' density function. In this dissertation, a Bernoulli random variable takes values 0 and b with probability p and $q = 1 - p$, respectively. Its 'density function' has the form $p(x) = p\delta_0(x) + q\delta_b(x)$, where $delta_y$ is a Dirac delta at the point y. This density is really a distribution and not a function. The fact that it is not bounded and supported on a set of Lebesgue measure zero makes the above method much more difficult to use and makes many other estimates necessary to prove localization harder to demonstrate.

These difficulties contrast with the relatively simple nature of the Bernoulli random variable. When introduced to probability, students first see the Bernoulli random variable. A quantum system with random Bernoulli distributed potential is easiest form of disorder to express. The mathematics community has derived necessary estimates on Bernoulli distributions for localization in certain settings. In 1987, Carmona, Klein, and Martinelli were able to show Lifschitz tails and localization on the lattice [17]. In 2002, Damanik, Sims, and Stolz proved localization for continuum Bernoulli models in one-dimension [20]. In 2005, Bourgain and Kenig proved localization in continuous settings for higher dimensions [16]. In 2005, Germinet, Hislop, Klein proved localization for Poisson distributed potentials in continuous settings [32].

Recently, [4] argued the importance of localization in the Bernoulli distributed random potentials. Their argument noted that if every random variable has a Bernoulli component, localization for the Bernoulli distributed random variables should extend to every random variable. In this case, the result in [16] was a not a solution to pathological case of Anderson localization, but as a starting point for more general proofs for localization.

This dissertation focuses on discrete Random Schrödinger operators with Bernoulli distributed potential. Whereas this type of potential makes the methods described above more difficult, the Bernoulli potential leads to more straightforward geometric

approach to the description of states and eigenvalues. In several works including [37], the general intuition is that low energy states should be supported largely on intervals of low potential, sometimes referred to as 'islands.' This follows from the intuition that the low energy states must have low potential energy, i.e. states are largely supported on sites of low potential. These states must also have low kinetic energy. The eigenvalues of the Laplacian $-\Delta$ decrease as the size of the support of the states increases. So lower energy states will be supported on larger intervals of low potential. With continuous probability densities, distinguishing between low potential values and high potential values is not obvious because the potential takes each value between the bottom and top of its support. With Bernoulli probability distributions, low potential sites is clearly defined because the potential only takes two values. Chapters 2 and 3 describe this behavior by studying the eigenvalues at the bottom of the spectrum.

For the small eigenvalues, these results imply that the system with Bernoulli distributed potential and finite barrier height b is closely approximated by the system with infinite barrier height $b = \infty$. This is due to the fact that low energy states cannot significantly occupy the sites with positive potential. For an state with energy much less than the value of barrier height b, each site with positive potential is an effective barrier to transport.

1.4 Quantum Many-body Systems

The discussion above focuses on single particle behavior; many quantum systems discussed involve large systems of particles. In [39], Kirsch notes that these one particle descriptions are likely insufficient to describe many physically relevant systems. Theoretical studies of quantum many-body systems in the physics communities go back to the works of Bose and Einstein [15], Fermi [65], and [22]. The works of Gross [35] and Pitaevskii [50] established mean-field methods for studying multi-particle systems.

Penrose and Onsager defined a straightforward particle number formulation of condensation [45]. The Lieb-Liniger model [43] was introduced to model these systems. Lieb, Seiringer, and Yngvason have rigorously shown convergence of the ground state of a bosonic system with nonrandom harmonic potential to the mean-field approximation [24, 46, 47, 44]. The dynamics of such mean-field approximations and their stability has been studied in detail [54, 27, 26, 28].

In 1995, Cornell and Wieman were able to conduct cold atom experiments realizing Bose-Einstein condensates[10]. These condensates are a special quantum state of matter where a large number of particles overlap and have the same single particle wave function. The cold atom experiments cool hydrogen-like atoms, i.e. atoms listed in the first column of the periodic table such as rubidium, to within a few hundred nanoKelvin of absolute zero, where the effects of temperature-related motion stops. This allows particles to be in the same low energy state; if the temperature of the system is too high, then particles will be in higher energies states with significant probability. Their discovery has led to new research in both experimental and theoretical work on quantum mechanical systems.

In a system of many quantum particles, the multi-particle wave function generally has two symmetries due to the inability to distinguish particles [12]. The first symmetry, first theorized by Bose and Einstein, is that states should be symmetric under permutation of indices. This is, if $\boldsymbol{x_i}$ is the position of the i-th particle, then the wave function Ψ of N particles should be invariant under any permutation π of the indices,

$$\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N}) = \Psi(\boldsymbol{x_{\pi(1)}}, \boldsymbol{x_{\pi(2)}}, \ldots, \boldsymbol{x_{\pi(N)}}) \tag{1.4.1}$$

with the vector normalized to one in the symmetric tensor product space

$$\otimes_{1\ symm}^{N} \ell^2(\{0, \ldots, L+1\}^d)$$

Systems with this symmetry are said to obey Bose-Einstein statistics; particles obeying this symmetry are called bosons in honor of Bose.

The other general symmetry is that particles should be antisymmetric under permutation of indices. For a permutation π,

$$\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N}) = (-1)^{|\pi|}\Psi(\boldsymbol{x_{\pi(1)}}, \boldsymbol{x_{\pi(2)}}, \ldots, \boldsymbol{x_{\pi(N)}}) \tag{1.4.2}$$

where $|\pi|$ is the number of flips required to generate π in the symmetric group. The vector is normalized to one in the antisymmetric tensor product space

$$\otimes_1^N{}_{asymm}\ell^2(\{0, \ldots, L+1\}^d)$$

Systems with this symmetry are said to obey Fermi-Dirac statistics; particles obeying this symmetry are called fermions in honor of Fermi. The antisymmetry precludes overlap between two states, if the position of the i-th and j-th particles are the same, i.e. $\boldsymbol{x_i} = \boldsymbol{x_j}$,

$$\Psi(\boldsymbol{x_i}, \boldsymbol{x_j}) = -\Psi(\boldsymbol{x_j}, \boldsymbol{x_i}) \tag{1.4.3}$$

which must necessarily equal zero, meaning the probability two particles will be at the same position is zero. In contrast, the positions of bosons can overlap.

There are other more complicated symmetries for particles in $\mathbb{R}^2$ due to the nontrivial fundamental groups of complement of a set of points. Particles obeying symmetries corresponding to these loop groups are called anyons, see [49].

A useful basis for the tensor product space $\otimes_1^N\ell^2(\{0, \ldots, L+1\})$is

$$\phi_1(\boldsymbol{x_1}) \otimes \phi_2(\boldsymbol{x_2}) \otimes \cdots \otimes \phi_N(\boldsymbol{x_N}) \tag{1.4.4}$$

which can be interpreted as the j-th particle has wave function ϕ_j. The basis vectors for the symmetric tensor product space is symmetrization of the vector by taking the sum of the vector under permutation of the indices and then normalizing the vector. For the anti-symmetric space, a similar procedure is done where each term in the sum is multiplied by the sign of the permutation of the indices. While these basis vectors provide some intuition for the nature of the multi-particle system, the individual wave vector ϕ_j cannot be easily determined from system wave vector $\Psi(\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_N})$.

The operators can be defined both as operators on the general function space or through a tensor product of operators on basis vectors. For example, the position of the j-th particle is given by the multiplicative operator $\hat{x_j}\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N}) = \boldsymbol{x_j}\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N})$. It can be alternatively expressed by the operator

$$\mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \hat{x_j} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \tag{1.4.5}$$

where $\mathbb{I}$ is the identity and $\hat{x_j}$ is in the j-th spot in the tensor product. It can be written as the 'position operator acting on the j-th particle. The first expression is simpler to write, the second expression is simpler to interpret. The momentum operator for the j-th particle can be defined similarly. Of particular interest is defining the multi-particle (MP) Schrödinger operator in such situations. It can be expressed as

$$H^{MP} = \sum_{j=1}^{N} H_j = \sum_{j=1}^{N} -\nabla^2_{\boldsymbol{x_j}} + \hat{V}(\boldsymbol{x_j}) \tag{1.4.6}$$

where H_j denotes the single particle Schrödinger operator defined acting on the j-th particle. The gradient $\nabla_{\boldsymbol{x_j}}$ is the gradient with respect to the j-th particle; in the discrete setting, this is just the vector of discrete derivatives in each direction. In the tensor product notation, H^{MP} can be written as

$$H = \sum_{j=1}^{N} \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes H_j \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \tag{1.4.7}$$

If the ϕ_j are eigenvectors of H_j, i.e. $H_j\phi_j = E_j\phi_j$, then

$$H\phi_1(\boldsymbol{r_1}) \otimes \phi_2(\boldsymbol{r_2}) \otimes \cdots \otimes \phi_N(\boldsymbol{r_N}) = \left(\sum_{j=1}^{N} E_j \right) \phi_1(\boldsymbol{r_1}) \otimes \phi_2(\boldsymbol{r_2}) \otimes \cdots \otimes \phi_N(\boldsymbol{r_N}) \tag{1.4.8}$$

which expresses the total energy of the system as the sum of the energies of each particle in the system.

The Schrödinger operator H^{MP} defined above describes a system of noninteracting particles, but many physical systems include interactions. This dissertation will consider pairwise interactions, operator terms which depend on exactly two indices.

If the interaction is the function of the positions of particles, then it is expressed by an operator $\hat{U}(\boldsymbol{x_j}, \boldsymbol{x_k})$. The operator for the interacting systems takes the form

$$H^{int} = \sum_{j=1}^{N} H_j + \sum_{j \neq k} \hat{U}(\boldsymbol{x_j}, \boldsymbol{x_k}) \tag{1.4.9}$$

Commonly considered interactions include Coulomb interactions of the form

$$\hat{U}(\boldsymbol{x_j}, \boldsymbol{x_k}) = \frac{1}{|\boldsymbol{x_j} - \boldsymbol{x_k}|} \tag{1.4.10}$$

which models the repulsive interaction due to charge between two electrons.

The interaction considered in this dissertation will be 'soft core' repulsive interactions. When the positions of two particles intersect, they repel each other. The interaction between the j-th and k-th particles takes the form

$$H^{int} = \hat{U}(\boldsymbol{x_j}, \boldsymbol{x_k}) + g\delta(\boldsymbol{x_j}, \boldsymbol{x_k}) \tag{1.4.11}$$

where g is the interaction strength and $\delta(\cdot, \cdot)$ is the Kronecker delta, a function equal to one when the inputs are equal and zero otherwise. This interaction causes a single particle with state ϕ to act as a external potential of the form $g|\phi|^2$ on all other wave functions. If $g > 0$, the interaction is repulsive and particles push against each other and separate. If $g < 0$, then the interaction is attractive and particles overlap. The case where the interaction constant g is finite is called 'soft core' interaction because it allows for wave functions of individual particles to overlap. The case $g = \infty$ is called 'hard core' interaction because the individual particle wave functions cannot overlap. Multi-particle systems with this type of interaction is referred to as the Lieb-Liniger model [43].

These operators are incredibly difficult to analyze. First, the tensor product space is enormous. Second, single particle states cannot be easily determined or expressed from a general multi-particle state. For interacting systems of bosons, the Gross-Pitaevskii mean-field approximation [35, 50] makes the questions related to these systems more tractable. In this approximation, the domain of admissible functions is

restricted and the operator is rewritten for this new domain. This method analyzes a related nonlinear operator on a smaller function space instead of the full multi-particle linear operator on the larger tensor product of functions spaces.

The set of admissible states is restricted to mean-field states.

Definition 1.4.12. A mean-field state is a multi-particle state

$$\Psi \in \otimes_{1\ symm}^{N} \ell^2(\{0, \ldots, L+1\})$$

with the form

$$\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N}) = \prod_{j=1}^{N} \psi(\boldsymbol{x_j}) \tag{1.4.13}$$

where ψ is the single particle state in $\ell^2(\{0, \ldots, L+1\})$ normalized to one.

The mean-field is a natural family of states to consider in systems with Bose-Einstein statistics. For systems with Fermi-Dirac statistics, all mean-field states must be equal to zero.

When restricted to mean-field states, the interacting multi-particle operator H^{int} takes the form

$$H\Psi = N\left(-\Delta\psi + V\psi + \frac{g(N-1)}{2}|\psi|^2\psi\right) \tag{1.4.14}$$

where ψ is the associated single particle state. This is a form of the nonlinear Schrödinger equation. It is natural to consider the associated energy functional. In this approximation, the associated energy per particle for a mean-field state with single particle state ψ is

$$\langle \Psi, H\Psi \rangle / N = \sum_{\boldsymbol{r}} \sum_{|\boldsymbol{s}-\boldsymbol{r}|=1} |\psi(\boldsymbol{r}) - \psi(\boldsymbol{s})|^2 + V(\boldsymbol{r})|\psi(\boldsymbol{r})|^2 + \frac{g(N-1)}{2}|\psi(\boldsymbol{r})|^4 \tag{1.4.15}$$

$$\langle \Psi, H\Psi \rangle / N = \int \left(|\nabla\psi(\boldsymbol{r})|^2 + V(\boldsymbol{r})|\psi(\boldsymbol{r})|^2 + \frac{g(N-1)}{2}|\psi(\boldsymbol{r})|^4\right) d\boldsymbol{r} \tag{1.4.16}$$

The Gross-Pitaevskii limit is the limit where gN is fixed as $N \to \infty$ [44].

Rigorous proofs for the existence of Bose-Einstein condensation demonstrate that the Gross-Pitaevskii approximation is accurate. A large body work on these proofs

by Lieb, Seiringer, Solovej, Yngvason, and others [24, 46, 47, 44, 2], and see [45] for general reference.

Condensation in the quantum many body setting is a property where most particles are in the single particle mean-field state. For simplicity, the temperature will be set equal to zero; condensation is a low temperature phenomena in these models. The following is an overview of the proof from [44] of condensation of the multi-particle ground state to a mean-field ground state for systems in $\mathbb{R}^3$, the three-dimensional continuum setting. The ground state is the state associated with the smallest eigenvalue of the continuous analogue H^{MP} in equation (1.4.11) with a trapping potential that is locally bounded and divergent as $|x| \to \infty$ and an interaction potential which is not a Dirac δ, but a locally supported function which converges in distribution to the Dirac δ in the appropriate rescaling limit. These experimental realizations study the low energy states making the ground state natural choice to study. In optimizing the associated inner product, $\langle H^{MP}\Psi, \Psi\rangle$, Ψ must have the same complex phase. If Ψ has different complex phase at a given site from the phase at the neighboring sites, apply a complex rotation to Ψ at that site to decrease the kinetic energy while leaving the potential and interaction energy unchanged. Without loss of generality, Ψ is real and non-negative.

To define condensation, the density matrix must be defined for the ground state Ψ of the operator H^{MP} in equation(1.4.11) [44].

Definition 1.4.17. For a multi-particle symmetric wave function Ψ, the one particle density matrix is defined γ as

$$\gamma(x, x') = N \int \Psi(r, X)\Psi(x', X)dX \tag{1.4.18}$$

where $X = (x_2, \ldots, x_n)$ and $dX = \prod_{j=2}^{N} d^3x_j$.

This particle density matrix is a representation of projections of the multi-particle state onto possible single particle states. Condensation is the property that the ground

state Ψ is closely approximated by the mean-field ground state, defined as follows:

Definition 1.4.19. Let Ψ be the ground state of H^{MP}. In the limit where $L = 1$, gN is constant,

$$\lim_{N\to\infty} \frac{\gamma(x,x')}{N} = f(x)f(x') \tag{1.4.20}$$

where f is the condensate wave function.

With this definition, the authors of [44] prove the ground state condensation of the form

$$\lim_{N\to\infty} \frac{\gamma(x,x')}{N} = \Psi^{GP}(x)\Psi^{GP}(x') \tag{1.4.21}$$

where $Proj^{GP}$ is the projection operator of Ψ^{GP}, the ground state of the operator H^{GP}(1.4.14). The limit above is meant both in the trace sense

$$\lim_{N\to\infty} Tr|\frac{\gamma}{N} - Proj^{GP}| = 0 \tag{1.4.22}$$

and in the L^2 sense,

$$\lim_{N\to\infty} \int \left(\frac{\gamma(x,x')}{N} - \Psi^{GP}(x)\Psi^{GP}(x')|^2 \right) d^3x d^3x' \tag{1.4.23}$$

This result states that the true multi-particle ground state converges to the Gross-Pitaevskii mean-field ground state in the Gross-Pitaevskii limit. Most theory including [54, 27, 26, 28] use techniques derived from the above method. In [58], this technique is used to show condensation for similar interacting systems in random potentials.

1.5 Quantum Many-body Systems in Disordered Potential

The work of Mott [48] has inspired many questions and much research on the effects of random potentials on interacting quantum multi-particle systems. These questions are natural extensions of the study of random Schrödinger operators described in section 1.2 and by the study of interacting quantum systems described in section

1.4. Both studies developed a wealth of intuition, experience, and methods that inform research on these interacting quantum systems in disordered potential. The behavior of these systems can depend on a complicated interplay of the nature of the disorder, the form of interaction, the effects of temperature, and the number of particles, see [8, 13, 29, 33, 56, 57, 30] for a few of the many related works on the topic.

Much of the research into these systems has focused on the ground state, the lowest possible energy state of the system. Physically, the ground state describes the typical behavior in a given due to the tendency of physical systems to prefer energetically efficient states. Theoretically, the ground study is more straightforward to study due to its variational definition. In these disordered systems, the ground state may take different forms and different phases which are wildly different depending on size and structure of disorder and strength of the interaction. These phases may be described not just by the shape of the state but also the conductance, the effective resistance to an applied current. Mathematically and numerically, the conductance is defined by the change in energy as a function of the phase shift in imposed complex boundary conditions on opposite sides of lattice.

In a system with high disorder, i.e. large barriers, and no interactions, the system is non-conducting and the mean-field should be a localized state. With high disorder and very weak interaction, there should be a 'Lifschitz Glass' or 'Bose Glass,' where the mean-field is localized to a few isolated parts of the space and the state is also non-conducting. With small disorder and some interaction strength, there should be a 'super-fluid' state where the localization due to the disorder is overcome by the repulsive interaction. A super-fluid state is a quantum state where the particles flow freely, similar a fluid without viscosity. A super-fluid has no effective resistance to an applied current, whereas Bose Glasses should have strong resistance. In a system with high interaction strength, there is no longer a super-fluid state because mean-field approximation is not accurate because it is too energetically expensive for states

to overlap. In this case, each particle partitions the region to support its state. (This can extend to finite and disjoint sets of particles sharing a common space together, i.e. each site has three particles supported on that site and nowhere else). This state is referred to as a 'Mott Insulator' and has similarly strong resistance to applied current as the Bose glass. It is not clear how the shape of the ground state changes as both disorder strength and interaction strength are both increased to large values.

Mathematicians have recently renewed investigating these systems. There are generally two approaches for studying these interacting disordered systems. The first approach is motivated by disordered systems techniques. In this approach, start with a single exponentially localized state in a random potential and iterate by adding new particles to the system in this same state. With each iteration, the process is to study how the exponential bound changes in an attempt to study the exponential localization of the condensate. The bounds tend to break down in the infinite particle limit; this implies that localization as it is understood in single particle systems is not likely to hold in interacting condensates. The second approach is to start with a condensate in an interacting system in models discussed in 1.4 without a random potential and study the stability of the condensate when a random potential is added. Seiringer, Yngvason, and Zagrebnov considered Bose-Einstein condensation in systems with randomly placed point scatterers in the continuous setting on the unit lattice in one dimension [58]. They demonstrated the existence of a condensate in the Gross-Pitaevskii limit as well as a general description of the mean-field state in different presence of a phase transition.

In Chapter 4, the mean-field ground state is studied on the lattice with Bernoulli potential. In the first part, an inequality is derived from the Cauchy-Schwarz inequality for both the discrete and continuous spaces to show that a mean-field state with a large number of particles cannot both have low energy and localize to a small region of space. In the second part, the ground state and the ground state energy for the mean-field in Bernoulli random potential are described in the large system limit for

low particle density and low interaction strength.

Chapter 2
Single Particle Ground State Energy

The energy of a single particle is determined by the random Schrödinger operator $H = -\Delta + V$ operating on the particle's wave function ϕ in $\ell^2\{0, ..., L+1\}$ with Dirichlet boundary conditions $\phi(0) = \phi(L+1) = 0$. The operator $-\Delta$ is the (positive definite) discrete Laplacian of the form $(-\Delta\phi)(j) = 2\phi(j) - \phi(j-1) - \phi(j+1)$. The multiplication operator V is a random function defined at each point $j \in \{0, \ldots, L+1\}$ by an independent Bernoulli random variable with the distribution

$$V(j) = \begin{cases} 0 \text{ with probability } p \\ b \text{ with probability } q = 1 - p \end{cases} \tag{2.0.1}$$

for $b > 0$. The ground state energy is the smallest eigenvalue of this self-adjoint operator. The equivalent variational definition is used in this chapter.

Definition 2.0.2. The finite system size ground state energy E_0^L is defined as

$$E_0^L = \min_{\{\|\phi\|=1\}} \langle \phi, H\phi \rangle \tag{2.0.3}$$

The ground state energy is strictly positive for each L because $-\Delta$ is positive definite and V is non-negative. The corresponding eigenvector is referred to as the ground state.

This chapter describes the dependence of the ground state energy on the longest interval of consecutive sites with potential equal to zero, i.e. the longest set of consecutive points $j \in \{1, \ldots, L\}$ such that $V(j) = 0$. The length of this interval is denoted ℓ_L. It is natural to expect that $\ell_L \to \infty$ as $L \to \infty$ with probability one and is addressed in the appendix. The longest interval is not necessarily unique. The probability that the (finite volume) potential has more than one interval with maximal length ℓ_L is positive. However, the non-uniqueness is inconsequential for the result in

this chapter. The intervals as sets are denoted I_i with length denoted L_i. For each system size L, the number of intervals is a random variable denoted n. This system can be probabilistically approximated by a system where n is deterministic, the intervals have geometric distribution, and L is the random variable. See the appendix for details.

For every system size L and for every realization of the random potential, there exists a ground state vector Ψ_L. The ground state vector Ψ_L can be assumed to be real and nonnegative without loss of generality. The argument for this is as follows. The energy of the state can be written as $\langle \Psi_L, H\Psi_L \rangle = \sum_j V(j)|\Psi_L(j)|^2 + |\Psi_L(j+1) - \Psi_L(j)|^2$. Assume $\Psi_L(j+1)$ and $\Psi_L(j)$ are off by a phase factor. The energy is strictly smaller when one of terms is multiplied by a complex phase factor correction. This contradicts the energy-minimizing property of Ψ_L.

Theorem 2.0.4. *Let E_0^L be the ground state energy of the random Schrödinger operator with independent identically Bernoulli distributed potentials on $\ell^2\{0, 1, ..., L, L+1\}$ with Dirichlet boundary conditions. With probability one,*

$$\lim_{L\to\infty} \frac{E_0^L}{(\frac{\pi^2}{(\ell_L+1)^2})} = 1. \tag{2.0.5}$$

Remark 2.0.6. In the proof of this result, the random potential is used only to show that $\ell_L \to \infty$ with probability one. The argument for both the upper and the lower bounds on the ground state energy apply for any two-value (0 and b) potential realizations which satisfy the condition $\lim_{L\to\infty} \ell_L = \infty$.

Remark 2.0.7. The results for the ground state naturally extend to the largest eigenvalue. The largest eigenvalue $\tilde{E}^L$ behaves as $b - \frac{\pi^2}{(\ell'_L+1)^2}$, where ℓ'_L is the longest interval of b potential. To prove this, conjugate the operator H with the unitary operator $U : \phi(j) \to (-1)^{|j|}\phi(j)$, which takes $-\Delta$ to $4 + \Delta$ and leaves the potential operator V unchanged, so the bottom of the spectrum of $b - V - U^{-1}(-\Delta)U$ is the top

of the spectrum of $-\Delta + V$. The claim follows from the proof of the above theorem. This argument is due to [59].

2.1 Upper Bound on Ground State Energy

Lemma 2.1.1. *For any realization of the potential V and for any system size L,*

$$E_0^L \leq \frac{\pi^2}{(\ell_L + 1)^2} \tag{2.1.2}$$

.

Proof. The variational definition the ground state energy

$$E_0^L = \min_{\{\|\phi\|=1\}} < H\phi, \phi >$$

implies that the ground state energy is bounded above by the energy of any normalized test function. For this lemma, the test function is a sine wave supported on the longest interval of zero potential. The energy of this test function equals $4\sin^2(\frac{\pi}{2(\ell_L+1)})$, which is bounded by the right-hand side of the claimed inequality. This test function is motivated by the intuition that finite potential barriers are similar to infinite potential barriers in Bernoulli distributed potentials. This proof restricts the choice of functions to those that are zero on sites with $V(j) = b$ and then minimizes the kinetic energy associated with the discrete Laplacian $-\Delta$. This simplifies the task to finding the test function with the smallest eigenvalue on a given interval which satisfies Dirichlet boundary conditions on neighboring sites of positive potential. The remainder of the proof justifies this assertion that the energy is minimized by the test function $\sqrt{\frac{2}{L+1}}\sin(\frac{\pi j}{L+1})$, and that the intervals which minimize this quantity are the longest ones.

The minimization presented here follows from [61]. Consider a wave function on $\{0, \ldots, L\}$. This set is treated as $\mathbb{Z}/(L+1)\mathbb{Z}$, the cyclic group of order $L+1$. The Dirichlet boundary conditions are $\phi(0) = \phi(L+1) = 0$, which is really a single

condition because the point $L+1$ is identified with 0 in the cyclic group. For a given wave function ϕ, its Finite Fourier Transform is

$$\widetilde{\phi}(p) = \frac{1}{\sqrt{L+1}}\Sigma_{j=0}^{L}\phi(j)e^{\frac{2\pi ipj}{L+1}} \tag{2.1.3}$$

The Finite Fourier Transformation diagonalizes the discrete Laplacian; the Finite Fourier Transform of the discrete Laplacian acting on a vector ϕ is

$$\widetilde{(-\Delta\phi)}(p) = \frac{1}{\sqrt{L+1}}\Sigma_{j=0}^{L}\left[2\phi(j) - \phi(j-1) - \phi(j+1)]e^{\frac{2\pi ipj}{L+1}}\right]$$

$$= \frac{1}{\sqrt{L+1}}\left[2\Sigma_{j=0}^{L}\phi(j)e^{\frac{2\pi ipj}{L+1}} - \Sigma_{j=-1}^{L-1}\phi(j)e^{\frac{2\pi ip(j+1)}{L+1}} - \Sigma_{j=1}^{L+1}\phi(j)e^{\frac{2\pi ip(j-1)}{L+1}}\right]$$

Using the fact that $\phi(0) = \phi(L+1) = 0$ and $\phi(-1) = \phi(L)$:

$$= \frac{1}{\sqrt{L+1}}\left[2\Sigma_{j=0}^{L}\phi(j)e^{\frac{2\pi ipj}{L+1}} - \Sigma_{j=0}^{L}\phi(j)e^{\frac{2\pi ip(L+1)}{L+1}} - \Sigma_{j=0}^{L}\phi(j)e^{\frac{2\pi ip(j-1)}{L+1}}\right]$$

$$= \frac{1}{\sqrt{L+1}}\left[2 - e^{\frac{2\pi ip}{L+1}} - e^{-\frac{2\pi ip}{L+1}}\right]\Sigma_{j=0}^{N-1}\phi(j)e^{\frac{2\pi ipj}{L+1}}$$

$$= 4\sin^2(\frac{p\pi}{L+1})\widetilde{\phi}(p) \tag{2.1.4}$$

This implies that the eigenvalues of $-\Delta$ are $4\sin^2(\frac{p\pi}{L+1})$ for $p = 0, \ldots, L$. The smallest eigenvalue of the discrete Laplacian without considering boundary conditions correspond to $p = 0$. The corresponding eigenfunction is a constant which is equal to zero on the interval due to the Dirichlet boundary conditions.

Therefore, the minimal energy is given by a linear combination of the eigenvectors corresponding to $p = 1$ and $p = L$, $e^{\frac{i\pi}{L+1}}$ and $e^{\frac{-i\pi}{L+1}}$. The vector should be $\phi(j) = \sin(\frac{\pi j}{L+1})$ which satisfies the assumptions of non-negativity and Dirichlet boundary conditions. The norm of the vector is

$$\|\sin(\frac{\pi j}{L+1})\|^2 = \sum_{j=0}^{L}|\sin^2(\frac{\pi j}{L+1})|^2$$

$$\sum_{j=0}^{L}|\sin^2(\frac{\pi j}{L+1})|^2 = \sum_{j=0}^{L}\frac{1}{2}\left(1 - \cos(\frac{2\pi j}{L+1})\right) = \frac{L+1}{2} \tag{2.1.5}$$

The normalized eigenvector minimizing the discrete Laplacian supported on an interval length L with Dirichlet boundary conditions has the form $\sqrt{\frac{2}{L+1}}\sin(\frac{\pi j}{L+1})$. Note that the above calculation assumed the vectors were defined on the cyclic group $\mathbb{Z}/(L+1)\mathbb{Z}$ rather than on $\{0,\ldots,L+1\}$. However, since the discrete sine wave is zero at the boundary points, the inner product will not include the boundary terms of the Laplacian acting on the vector where the sets differ in definition. Thus, this vector also the minimizes the eigenvalue of $-\Delta$ for functions defined on $\{0,1,\ldots,L,L+1\}$ with Dirichlet boundary conditions.

The associated eigenvalue of the test functions, $4\sin^2\left(\frac{\pi}{2(L+1)}\right)$, is decreasing in L for $L>1$. The test function restricted to the longest interval of zero potential, with the form $\sqrt{\frac{2}{\ell_L+1}}\sin(\frac{\pi j}{\ell_L+1})$ on a longest interval and equal to zero everywhere else. The energy of this test function is bounded by

$$\langle H\psi,\psi\rangle = 4\sin^2\left(\frac{\pi}{2(\ell_L+1)}\right) \leq \frac{\pi^2}{(\ell_L+1)^2} \tag{2.1.6}$$

The ground state energy must be less than the energy of the test function and the bound is proven.

□

Remark 2.1.7. This upper bound holds for all realizations of the Bernoulli potential for any system size.

2.2 Lower Bound on Ground State Energy

For each realization of potential and for every L, there exists a specific nonnegative ground state, denoted Ψ_L. The lower bound derived in this section will have the same leading order term in ℓ_L as the upper bound in the previous section in the limit $\ell_L\to\infty$. This proves the limit

$$\lim_{\ell_L\to\infty}\frac{E_0^L}{\left(\frac{\pi^2}{(\ell_L+1)^2}\right)} = 1 \tag{2.2.1}$$

The lower bound calculated here is a simpler version of the form that appears in [14], which will remain a monument to the suffering caused by a suboptimal choice of variables.

For each interval of zero potential I_i, denote the values of Ψ_L on the sites neighboring I_i with b potential as $m_i\delta_L^i$ and $m_i\delta_R^i$,the left and the right boundary values, and denote the norm of Ψ_L restricted to I_i as m_i, i.e. $m_i^2 = \|\Psi_L|_{I_i}\|^2 + (m_i\delta_L^i)^2 + (m_i\delta_R^i)^2$. The fact that Ψ_L is the energy minimizer will be used repeatedly to estimate its energy from below.

The following lemma quantifies the intuition that the norm of ground state restricted to b potential must be small. It is a corollary that follows from the upper bound derived in the previous section.

Corollary 2.2.2. *Let Ψ_L be the ground state vector for a given realization of potential. Let $B = \{j : V(j) = b\}$. Then $\|\Psi_L|_B\|^2 \leq \frac{\pi^2}{b(\ell_L+1)^2}$, where $\Psi_N|_B$ is the restriction of Ψ_N to the set B.*

Proof. The potential energy of Ψ_L is bounded above by the quantity in 2.1.1. The potential energy is expressed as

$$\begin{aligned} & b\|\Psi_L|_B\|^2 \\ &= \langle \Psi_L|_B, V\Psi_L|_B\rangle \\ &\leq \langle \Psi_L, H\Psi_L\rangle \\ &\leq \frac{\pi^2}{(\ell_L+1)^2} \end{aligned}$$

Therefore, $\|\Psi_L|_B\|^2 \leq \frac{\pi^2}{b(\ell_L+1)^2}$. □

It follows from the corollary that the norm of all the δ boundary points is bounded by:

$$\sum_{i=1}^{n}(|m_i\delta_L^i|^2 + |m_i\delta_R^i|^2) \leq \frac{2\pi^2}{b(\ell_L+1)^2} \tag{2.2.3}$$

where the extra factor of 2 is introduced to cover the case in which the boundary points may be shared by intervals. Thus, the norm of Ψ_L is bounded below on sites of zero potential by $\|\Psi_L|_{B^c}\|^2 \geq 1 - \frac{\pi^2}{b(\ell_L+1)^2}$.

Next, fix a $\gamma \in (0,1)$. Its specific value does not play a crucial role, but it will enter the convergence rate in our main result.

Definition 2.2.4. A heavy interval of Ψ_L is an interval I_i such that

$$\max(\delta_L^i, \delta_R^i)^2 \leq \frac{1}{(L_i+1)(\ell_L+1)^{1-\gamma}} \tag{2.2.5}$$

and an interval is light if

$$\max(\delta_L^i, \delta_R^i)^2 > \frac{1}{(L_i+1)(\ell_L+1)^{1-\gamma}} \tag{2.2.6}$$

Heavy intervals are intervals where the δ_i boundary conditions are small relative to the function on the interval. This will imply that the ground state restricted to a heavy interval is approximated well by a sine wave, and more importantly, its kinetic energy can be bounded below. Light intervals have the opposite properties: they have relatively large boundary conditions. The ground state restricted to a light interval is approximated by a linear function and its kinetic energy is difficult to bound below. For a light interval,

$$m_i^2 < m_i^2 \delta_i^2 (L_i+1)(\ell_L+1)^{1-\gamma}$$

where the δ_i^2 is the maximum of the two boundary values (the ground state is non-negative, so there are no issues with complex values). Applying the upper bound on the norm of the δ boundary values, the norm of the ground state restricted to a light interval is bounded above by

$$m_i^2 \leq \frac{2\pi^2(L_i+1)}{b(\ell_L+1)^{1+\gamma}} \leq \frac{2\pi^2}{b(\ell_L+1)^{\gamma}} \tag{2.2.7}$$

meaning the ground state restricted to a light interval does not contribute significantly to the norm of the ground state. Here, the parameter $\gamma \in (0,1)$ determines the

dependence of the bound on ℓ_L. The following lemma shows that this holds for the sum over all light intervals and the norm of Ψ_L restricted to heavy intervals converges to one in the limit as $\ell_L \to \infty$.

Lemma 2.2.8. *Let M be the index set of all heavy intervals I_i. Then*

$$\Sigma_{i\in M}\|\Psi_L|_{I_i}\|^2 \geq 1 - \frac{3\pi^2}{b\ell_L^\gamma}$$

Proof. First, Ψ_L is the orthogonal sum of itself restricted to disjoint sets B and B^c, where $B = \{j : V(j) = b\}$.

$$1 = \|\Psi_L\|^2 = \|\Psi_L|_B\|^2 + \|\Psi_L|_{B^c}\|^2$$

Next, $\Psi_L|_{B^c}$ is the sum of two orthogonal vectors, Ψ_L restricted to heavy intervals and light intervals.

$$\|\Psi_l|_{B^c}\|^2 = \Sigma_{i\in M}\|\Psi_L|_{I_i}\|^2 + \Sigma_{i\in M^c}\|\Psi_L|_{I_i}\|^2$$

By the definition of light intervals, the norm of Ψ_L restricted to light intervals is bounded above as follows.

$$\begin{aligned}\Sigma_{i\in M^c}\|\Psi_L|_{I_i}\|^2 &< \Sigma_{i\in M^c}\max(m_i\delta_L^i, m_i\delta_R^i)^2(L_i+1)(\ell_L+1)^{1-\gamma}\\ &\leq \Sigma_{i\in M^c}\max(m_i\delta_L^i, m_i\delta_R^i)^2(\ell_L+1)^{2-\gamma}\end{aligned}$$

where the fact that $L_i + 1 \leq \ell_L + 1$ is used. Since $\Sigma_{i\in M^c}\max(m_i\delta_L^i, m_i\delta_R^i)^2 \leq 2\|\Psi_L|_B\|^2 \leq \frac{2\pi^2}{b(\ell_L+1)^2}$ by definition (2.2.3), this quantity is bounded above by

$$\frac{2\pi^2}{b(\ell_L+1)^\gamma}$$

Thus,

$$\begin{aligned}1 &= \|\Psi_L|_B\|^2 + \Sigma_{i\in M}\|\Psi_L|_{I_i}\|^2 + \Sigma_{i\in M^c}\|\Psi_L|_{I_i}\|^2\\ &\leq \frac{\pi^2}{b(\ell_L+1)^2} + \frac{2\pi^2}{(\ell_L+1)^\gamma} + \Sigma_{i\in M}\|\Psi_L|_{I_i}\|^2\\ &\leq \frac{3\pi^2}{b(\ell_L+1)^\gamma} + \Sigma_{i\in M}\|\Psi_L|_{I_i}\|^2\end{aligned}$$

Subtracting appropriate terms, the bound is shown. □

The lemma guarantees there exists at least one heavy interval when $\frac{3\pi^2}{b(\ell_L+1)^\gamma} < 1$. The following theorem is the main technical result that bounds the energy below by estimating the contribution of energy of Ψ_L restricted to the heavy intervals.

Theorem 2.2.9. *If $\ell_L > \frac{3\pi^2}{b}^{\frac{1}{\gamma}} - 1$, then the lower bound on the energy contributed by heavy intervals is*

$$\left(1 - \frac{3\pi^2}{b\ell_L^\gamma}\right)\left(1 - \frac{\sqrt{2}}{\ell_L^{(1-\gamma)/2}}\right)^2\left(\frac{\pi^2}{(\ell_L+1)^2} + O(\frac{\pi^4}{(\ell_L+1)^4})\right) \tag{2.2.10}$$

and thus E_0^L is also bounded below by this quantity.

Proof. The first step is to find an explicit form of the state Ψ_L restricted to a heavy interval by minimizing the energy on that heavy interval. The heavy interval exists by the supposition of the theorem. By using the Lagrange multiplier method with fixed nonnegative values of $m_i\delta_L^i$ and $m_i\delta_R^i$ as well as m_i, the norm of the restriction of the state to the i-th interval, the form of the ground state is found.

To find the energy-minimizing vector under these conditions, consider the ground state restricted to an interval with length L_i, with energy functional $E = \langle f, Hf\rangle = \Sigma_{j=1}^{L_i}(-\Delta f(j))f(j)$ subject to the constraints $\Sigma_{j=0}^{L_i+1} f(j)^2 = m_i^2$, $f(0) = m_i\delta_L^i$, and $f(L_i+1) = m_i\delta_R^i$. Because the ground state was assumed to non-negative without loss of generality, the state restricted to the interval is nonnegative as well.

Introducing a Lagrange multiplier constant λ, the minimizing vector satisfies

$$\nabla\langle f, Hf\rangle = \lambda\nabla\Sigma_{j=1}^{L_i} f(j)^2 \tag{2.2.11}$$

where ∇ is the directional derivative taken with respect to $f(j)$ for each j. This gives the following set of equations.

$$\frac{\partial E}{\partial f(1)} = 4f(1) - 2m_i\delta_L^i - 2f(2) = 2\lambda f(1)$$

$$\frac{\partial E}{\partial f(j)} = 4f(j) - 2f(j-1) - 2f(j+1) = 2\lambda f(j)$$

$$\frac{\partial E}{\partial f(L_i)} = 4f(L_i) - 2m_i\delta_R^i - 2f(L_i - 1) = 2\lambda f(L_i)$$

With some rearranging, these equations give the eigenvalue equation

$$-\Delta f(j) = \lambda f(j)$$

The critical points are eigenvectors of $-\Delta$ with boundary conditions $f(0) = m_i\delta_L^i$ and $f(L+1) = m_i\delta_R^i$. The set of eigenvectors contains $f_0(j) = \frac{m_i\delta_R^i - m_i\delta_L^i}{L_i+1}j + m_i\delta_L^i$ and $f_k(j) = \frac{c_k m_i}{\sqrt{L_i}}\sin(s_k\frac{k\pi}{L_i+1}j + t_k)$, where $k = 1, ..., L$ is the frequency index, c_k is a normalizing constant, and s_k and t_k are stretch and shift constants to make the sine wave satisfy the $f(0) = m_i\delta_L^i$ and $f(L_i+1) = m_i\delta_R^i$ boundary conditions. Since I_i is a heavy interval, $\|\Psi_L|_{I_i}\|^2 \geq m_i^2 \max(\delta_L^i, \delta_R^i)^2 L_i(\ell_L+1)^{1-\gamma}$. The square of the norm of the linear function f_0 satisfies $\|f_0\|^2 < m_i^2 \max(\delta_L^i, \delta_R^i)^2 L_i$ and is strictly smaller than $m_i^2 \max(\delta_L^i, \delta_R^i)^2 L_i(\ell_L+1)^{1-\gamma}$, so the minimizing vector must be one of the sine waves. Since Ψ_N is strictly positive, $s_k k$ must be less than 1. In fact, $s_k k$ converges to one in the limit $\ell_L \to \infty$. This means the frequency is approximately one making it notationally convenient to set $k = 1$. The k will be dropped from the notation. By evaluating the kinetic energy and ignoring the potential energy contributions of the end points of this minimizer, the energy of any state satisfying the boundary conditions and heavy interval conditions must satisfy

$$\langle f, Hf\rangle \geq \langle c\sin(s\frac{\pi}{L_i+1}j + t), -\Delta c\sin(\frac{s\pi}{L+1}j + t)\rangle$$

$$= 4m_i^2\sin^2\left(\frac{s\pi}{2(L_i+1)}\right) = m_i^2\frac{s^2\pi^2}{(L_i+1)^2} + O\left(\frac{s^4\pi^4}{(L_i+1)^4}\right) \tag{2.2.12}$$

which gives a lower bound for the energy contribution of $\Psi_N|_{I_i}$.

The energy of the ground state is estimated through analyzing the dependence of the coefficients s, t and c on the boundary conditions δ_L^i and δ_R^i, the norm restricted to the interval m_i, and the length of the interval L_i. In particular, the stretch constant s will be bounded below. The boundary conditions for the sine wave satisfy

$$\frac{cm_i}{\sqrt{L_i+1}}\sin(t) = m_i\delta_L^i$$
$$\frac{cm_i}{\sqrt{L_i+1}}\sin(s\pi+t) = m_i\delta_R^i \tag{2.2.13}$$

To solve for s, the left boundary condition is solved using $\arcsin(t)$ for t, where the positivity of the state restricts t to $[0, \pi/2]$.

$$t = \arcsin\left(\frac{\delta_L^i\sqrt{L_i+1}}{c}\right) \tag{2.2.14}$$

The right boundary condition satisfies a similar equation. Since I_i is a heavy interval, f cannot be monotone because that property would contradict the definition of a heavy interval. The state is not monotone on the interval and the sine wave achieves its maximum. Solving the right boundary condition for $s\pi + t$ requires taking the inverse of $y = \sin(x)$ on $[\pi/2, \pi]$, which is $x = -\arcsin(y) + \pi$. Therefore,

$$s\pi + t = -\arcsin\left(\frac{\delta_R^i\sqrt{L_i+1}}{c}\right) + \pi \tag{2.2.15}$$

Subtracting the two equations and solving for s gives

$$s = 1 - \frac{1}{\pi}\left(\arcsin\left(\frac{\delta_L^i\sqrt{L_i+1}}{c}\right) + \arcsin\left(\frac{\delta_R^i\sqrt{L_i+1}}{c}\right)\right) \tag{2.2.16}$$

For heavy intervals, the inputs $\frac{\delta^i\sqrt{L_i+1}}{c}$ converge to zero in the limit $\ell_L \to \infty$. The definition of a heavy interval gives the bound

$$\frac{\max(\delta_L^i, \delta_R^i)\sqrt{L_i+1}}{c} \leq \frac{1}{c(\ell_L+1)^{(1-\gamma)/2}} \tag{2.2.17}$$

The normalization constant c satisfies the equation

$$\frac{c^2}{L_i+1}\sum_{j=0}^{L_i+1}\sin^2\left(\frac{\pi sj}{L_i+1} + t\right) = 1 \tag{2.2.18}$$

The sum of $\sin^2(\cdot)$ takes values between 0 and $L_i + 2$ since $0 \leq \sin^2(\cdot) \leq 1$. Then $c^2 \geq \frac{1}{2}$ since $L_i \geq 0$.

$$\frac{\max(\delta_L^i, \delta_R^i)\sqrt{L_i+1}}{c} \leq \frac{\sqrt{2}}{(\ell_L+1)^{(1-\gamma)/2}} \tag{2.2.19}$$

which converges to zero in the limit.

The function $\arcsin(\cdot)$ is bounded by $\frac{\pi x}{2}$ for $x \in [0,1]$, therefore

$$s \geq 1 - \frac{\sqrt{L_i+1}}{2c}\left(\delta_L^i + \delta_R^i\right) \tag{2.2.20}$$

Since $\delta_L^i + \delta_R^i \leq 2\max(\delta_L^i, \delta_R^i) \leq \frac{2}{\sqrt{(L_i+1)}(\ell_L+1)^{(1-\gamma)/2}}$ for heavy intervals, s satisfies

$$s \geq 1 - \frac{\sqrt{2}}{(\ell_L+1)^{(1-\gamma)/2}} \tag{2.2.21}$$

Because the energy contribution of any state restricted to each of the heavy intervals from (2.2.12) is bounded below by

$$m_i^2\left(\frac{s_i^2\pi^2}{(L_i+1)^2} + O\left(\frac{\pi^4}{(L_i+1)^4}\right)\right)$$

the bound of s_i gives

$$m_i^2\left(\frac{\left(1-\frac{\sqrt{2}}{\ell_L^{(1-\gamma)/2}}\right)^2\pi^2}{(\ell_L+1)^2} + O\left(\frac{\pi^4}{(\ell_L+1)^4}\right)\right)$$

where $L_i \leq \ell_L$ is used and s_i is dropped from the fourth order term since $s \leq 1$. The norm of the ground state restricted to heavy intervals is bounded below by $\left(1-\frac{3\pi^2}{b(\ell_L+1)^\gamma}\right)$. The energy contribution of a state restricted heavy intervals is bounded below by

$$\left(1-\frac{3\pi^2}{b\ell_L^\gamma}\right)\left(1-\frac{\sqrt{2}}{\ell_L^{(1-\gamma)/2}}\right)^2\left(\frac{\pi^2}{(\ell_L+1)^2} + O(\frac{\pi^4}{(\ell_L+1)^4})\right)$$

Since the ground state energy is bounded below by the energy contribution of heavy intervals, we have the desired lower bound.

□

2.3 Summary of Results

The ground state energy is bounded below by

$$\left(1-\frac{3\pi^2}{b\ell_L^{\gamma}}\right)\left(1-\frac{\sqrt{2}}{\ell_L^{(1-\gamma)/2}}\right)^2\left(\frac{\pi^2}{(\ell_L+1)^2}+O(\frac{\pi^4}{(\ell_L+1)^4})\right) \tag{2.3.1}$$

and above by

$$\frac{\pi^2}{(\ell_L+1)^2} \tag{2.3.2}$$

implying that $E_0^L \approx \frac{\pi^2}{(\ell_L+1)^2}$ as $\ell_L \to \infty$. More formally,

$$\lim_{\ell_L\to\infty}\frac{E_0^L}{\left(\frac{\pi^2}{(\ell_L+1)^2}\right)}=1 \tag{2.3.3}$$

From the appendix, given

$$P\left[\lim_{L\to\infty}\ell_L=\infty\right]=1 \tag{2.3.4}$$

it follows that

$$P\left[\lim_{L\to\infty}\frac{E_0^N}{\left(\frac{\pi^2}{(\ell_L+1)^2}\right)}=1\right]=1 \tag{2.3.5}$$

As pointed out in the proof of the upper bound, the argument used there essentially compares the given system to a system with infinite potential barriers rather than barriers of height b. The existence of an asymptotically equivalent lower bound means physically that the two systems are not very different as far as the ground state energy is concerned. Consequently, leading order behavior does not depend on b. In the next chapter, an analogous result is proven for the other small eigenvalues of the system.

Chapter 3
Sparsity of States

In Chapter 2, the ground state energy was shown to be approximately the energy of a sine wave supported on the largest interval of zero potential. In this chapter, this result will be extended to the excited state energies, the other eigenvalues of the random Schrödinger operator less than some small positive parameter ϵ. These energies correspond to sine waves of various frequencies on long intervals. The goal of the chapter is to describe the sparsity of states at the end of the spectrum, that is, the rarity of eigenvalues near the bottom of the spectrum.

The sparsity of states at the ends of the spectrum for random Schrödinger operators was first noted by Lifschitz [59]. His intuition was simple: to have a state with low energy, the kinetic energy requires that the state be supported on a large region with shape similar to a box or a ball. For the state to have low potential energy, the random potential must take small values on most of this large set. For a random potential which is independent at each site, the probability of this event is exponentially small. This means that regions that support a state with low energy occur rarely.

The random Schrödinger operator $H = -\Delta + V$ defined on $\ell^2(\{0, \ldots, L+1\}^\nu)$ is self-adjoint and has ordered eigenvalues $E_0^{(L)} \leq E_1^{(L)} \leq \cdots \leq E_{L^\nu - 1}^{(L)}$ when V is real-valued and bounded. The density of states is the proportion of states with energy less than some parameter ϵ.

Definition 3.0.1. The number of eigenvalues less than ϵ for H restricted to a box with side length L is denoted $N_L(\epsilon)$. The density of states is the proportion of eigenvalues less than ϵ in the large system limit:

$$k(\epsilon) = \lim_{L \to \infty} L^{-\nu} N_L(\epsilon) \tag{3.0.2}$$

where $N_L(\epsilon)$ is the number of eigenvalues less that ϵ for an operator restricted to function on the box $\{0, \ldots, L+1\}^\nu$.

For random potentials V which are i.i.d. at each site in the box, the function $k(\epsilon)$ is well-defined and independent of the realization with probability one [59]. The proof from [59] follows from Lifschitz's intuition and also requires the density function of the potential probability distribution to be continuous at the ends of its support. The kinetic energy is determined by the discrete Laplacian $-\Delta$ which can be estimated using a Dirichlet-Neumann bracketing technique, described as follows. Let $-\Delta_\Omega$ be the Laplacian operating on functions supported on a domain $\Omega \subset \mathbb{Z}^\nu$. If the sub-domains Ω_k partition Ω, define operators $\oplus_k(-\Delta^D_{\Omega_k})$ and $\oplus_k(-\Delta^N_{\Omega_k})$ which act on functions defined on Ω with Dirichlet and Neumann boundary conditions on the boundaries of Ω_k, respectively. The Dirichlet boundary conditions restrict the class of functions, fixing them on the boundaries whereas the Neumann boundary conditions essentially remove the restrictions by making the functions free on the boundaries. This translates into an operator bound of the form $\oplus_k - \Delta^D_{\Omega_k} \geq -\Delta_\Omega \geq \oplus_k - \Delta^N_{\Omega_k}$ which also bounds corresponding eigenvalues $E^D_j \geq E_j \geq E^N_j$.

The domain of $\mathbb{Z}^\nu$ is typically partitioned into boxes of side length L because the eigenvectors of $-\Delta$ restricted to a box are products of sine waves. Using the Dirichlet upper bound on energy, a box supports a state with kinetic energy less than $\epsilon/2$ if it has side length greater than $\frac{\pi\sqrt{2\nu}}{\sqrt{\epsilon}}$. The probability a box has $V(j) \leq \epsilon/2$ for most j in the box is exponentially small, so the probability of such occurrences in the large system limit is correspondingly small. This probability bounds the expected number of states from below. Using the Neumann lower bound on energy, a length is chosen such that only the lowest eigenvalue can have energy less than ϵ and second lowest eigenvalue cannot. For a box of this size, if the potential function is greater than a given ϵ for too many sites, the box cannot support a state with energy less than ϵ. These two conditions restrict the number of states with energy less than ϵ to the

number of boxes. Using a large deviation estimate, the probability that a box has an eigenvalue less than ϵ is bounded above by the probability at least a sufficient number of sites have probability less than $\frac{\epsilon}{2}$ which is exponentially small.

The density of states satisfies the following limit with probability one,

$$\lim_{\epsilon \downarrow 0} \frac{\ln(-\ln(k(\epsilon)))}{\ln(\epsilon)} = -\nu/2 \tag{3.0.3}$$

where ν is the dimension of the space. This behavior of the density of states is called a Lifschitz tail. Informally, the density of states is approximated asymptotically as $\epsilon \to 0$ by $k(\epsilon) \sim \exp\left[-c\epsilon^{-\nu/2}\right]$. It implies that low energy states occur with exponentially small probability. The proof requires the use of several logarithms to account for large class of random potentials considered.

In this chapter, the same Hamiltonian is used as in Chapter 2: $H = -\Delta + V$ acting on $\ell^2\{0, \ldots, L+1\}$ with Dirichlet boundary conditions, where $-\Delta$ is the discrete Laplacian and V is the Bernoulli random potential. This result is analogous to the result in [59] where boxes are not a general partition of the space but specifically long intervals of zero potential which takes advantage of the clear geometric nature of the Bernoulli distribution. Likewise, the Dirichlet-Neumann bracketing is essentially replaced by weighted boundary conditions accounting for the potential energy costs.

Just as in Chapter 2, the system with fixed length L is approximated by a system fixed interval number n, see the appendix for details.

Theorem 3.0.4. *Let $k(\epsilon)$ be the density of states and let $\epsilon < b$. Then $k(\epsilon)$ satisfies*

$$k(\epsilon) \leq \limsup_{L \to \infty} \frac{N(\epsilon)}{L} \leq \frac{q p^{\frac{\pi}{\sqrt{\epsilon}} - \frac{2}{b}}}{p(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} \tag{3.0.5}$$

$$k(\epsilon) \geq \liminf_{L \to \infty} \frac{N(\epsilon)}{L} \geq \frac{q^2 p^{\frac{\pi}{\sqrt{\epsilon}}}}{p(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} \tag{3.0.6}$$

This is a stronger and more specific result for the Lifschitz tail. The proof proceeds as follows. First, the eigenvalues of H are bounded above by the eigenvalues of H

with domain restricted to functions which are zero on sites with positive potential, essentially the case with $b = \infty$. The E_k are bounded above by the energies of sine waves on long intervals. Using the joint probability distribution of the intervals, the lower bound on $k(\epsilon)$ is calculated in the large system limit. Second, a lower bound is constructed by studying the contribution of an eigenstate on a given interval. For $\epsilon < b$, each eigenstate with energy less than ϵ must be at least partly supported on sites of zero potential. These eigenstates must be sine waves on the intervals of zero potential. Using a similar optimization to Chapter 2, the energy of a sine wave with approximate frequency ω on an interval length ℓ is bounded below. This restricts the minimum length of intervals that can support a state with frequency ω and energy less than ϵ and also provides a lower bound on the length of intervals significantly supporting any state with energy less than the upper bounds above. This bounds the dimension of the vector space of eigenstates with energy less than ϵ above. The distribution of interval lengths is used again to bound $k(\epsilon)$ above.

3.1 Upper Bounds for Excited State Energies

The excited state energies are bounded above by the energies sine waves on intervals of zero potential using a variation on Dirichlet-Neumann bracketing, see [51, 52, 53]. The random Schrödinger operator is defined as before, $H = -\Delta + V$, where $-\Delta$ is the discrete Laplacian and V is the Bernoulli random potential; it acts on functions in $\mathcal{H} = \ell^2(\{0, \ldots, L+1\})$ with Dirichlet boundary conditions. This operator is self-adjoint for every realization of the potential and therefore has (ordered) eigenvalues $E_0^{(L)} \leq E_1^{(L)} \leq \cdots \leq E_{L-1}^{(L)}$, referred to as excited state energies.

For a specific realization of the potential V, define set $B(V)$ by $B(V) = \{x : V(x) = b\}$. Define $D(V)$ as the subspace of $\ell^2(\{0, \ldots, L+1\})$ where $\phi(x) = 0$ for every $x \in B(V)$. The operator H restricted to this subspace can be thought as the operator with sites with infinite potential, i.e. $b = \infty$. This operator has

(ordered) eigenvalues $\tilde{E}_0^{(L)} \leq \tilde{E}_1^{(L)} \leq \cdots \leq \tilde{E}_{L'-1}^{(L)}$, where L' is the number of sites where $V(x) = 0$ in the specific potential.

The variational formulation for the eigenvalues is used to bound $E_k^{(L)}$; it is a generalization of the variational formulation of the ground state used in chapter 2.

Theorem 3.1.1. *The excited state energies E_k, ordered from smallest to largest by $E_k \leq E_{k+1}$, have an equivalent variational formulation*

$$E_k = \max_{S_{k-1}} \min_{\|\phi\|=1, \phi \perp S_{k-1}} \langle \phi, H\phi \rangle \tag{3.1.2}$$

where S_k is a k-dimensional subspace of $\ell^2\{0, \ldots, L+1\}$ with Dirichlet boundary conditions. The vector ψ_k is defined as the eigenvector corresponding to the eigenvalue E_k and called an excited state.

These eigenvalues are bounded above by the eigenvalues of the operator restricted to $D(V)$, a subspace of the domain.

Theorem 3.1.3. *Let V be a specific realization of Bernoulli potential and $D(V)$ the subspace of $\mathcal{H}$ where $\phi(x) = 0$ if $V(x) = 0$. If $k \leq L'$, then*

$$E_k \leq \max_{S_{k-1} \subset D(V)} \min_{\phi \in D(V),\ \phi \perp S_{k-1}, \|\phi\|=1} \langle \phi, H\phi \rangle \tag{3.1.4}$$

Proof. The domain can be decomposed into a direct sum $\mathcal{H} = D(V) \oplus D(V)^\perp$. The condition $k \leq L'$ is required for there to exist k dimensional subspaces in $D(V)$ making the variational formulation of eigenvalues on the restricted domain is well-defined. Since each test function in $D(V) \cap S_{k-1}^\perp$ can be mapped into $\mathcal{H}$ in span$(S_{k-1}^\perp$, the following inequality holds:

$$\max_{S_{k-1} \subset D(V)} \min_{\phi \in D(V),\ \phi \perp S_{k-1}} \langle \phi, H\phi \rangle \tag{3.1.5}$$

$$= \max_{S_{k-1} \subset D(V)} \min_{\phi \in D(V),\ \phi \perp \mathrm{span}_{\mathcal{H}}(S_{k-1}, D(V)^\perp)} \langle \phi, H\phi \rangle \tag{3.1.6}$$

where the normalization condition $\|\phi\| = 1$ is dropped. For a given $k-1$ dimensional subspace T_{k-1} of $\mathcal{H}$, there exists a subspace S_{k-1} such that $T_{k-1} \subset \mathrm{span}(S_{k-1}, D(V)^\perp)$.

Since the minimum is taken over a larger class of functions

$$\max_{S_{k-1}\subset D(V)} \min_{\phi\in D(V),\ \phi\perp \mathrm{span}_{\mathcal{H}}(S_{k-1},D(V)^{\perp})} \langle\phi, H\phi\rangle \tag{3.1.7}$$

$$\geq \max_{T_{k-1}\subset\mathcal{H}} \min_{\phi\in\mathcal{H},\ \phi\perp T_{k-1}} \langle\phi, H\phi\rangle \tag{3.1.8}$$

The second expression is E_k, so the E_k is bounded above. □

Be using the above inequality, the eigenvalues of H are bounded above by a similar operator with Dirichlet boundary conditions on each site x where $V(x) = b$. This method is a refined version of the test function method used to find the upper bound in Chapter 2; the upper bound is given by the sine waves on zero potential. These sine waves have energies with the form

$$\tilde{E}_0 = \frac{\omega_0^2\pi^2}{(\ell_{0,1}+1)^2}$$

$$\tilde{E}_1 = \frac{\omega_1^2\pi^2}{(\ell_{0,2}+1)^2}$$

$$\vdots$$

$$\tilde{E}_k = \frac{\omega_k^2\pi^2}{(\ell_{\alpha,\beta}+1)^2}$$

where $\ell_{\alpha,\beta}$ is the β-th interval of length ℓ_α. Notationally different from Chapter 2, ℓ_0 is the length of the longest intervals rather than ℓ_L. The ℓ_α are ordered by $\ell_\alpha > \ell_{\alpha'}$ if $\alpha < \alpha'$. These sine waves also include frequencies ω_k which are positive integers. The ω_k are not a frequency in the traditional physical sense, but rather a count for the relative extrema on the sine wave. The number of eigenstates with finite eigenstates is equal to the number of sites of zero potential in a specific realization of potential, approximately pL.

3.2 Lower Bound on Lifschitz Tail

Using the upper bounds on the excited state energies of the previous section, the density of states is bounded below. An eigenstate is guaranteed to have energy less

than ϵ if its upper bound obeys the inequality $\frac{\omega^2\pi^2}{(\ell_{\alpha,\beta}+1)^2} \leq \epsilon$, implying that intervals of zero potential with length $\ell+1 \geq \frac{\pi}{\sqrt{\epsilon}}$ can support an eigenstate with energy less than ϵ. For such an eigenstate to have frequency ω, the interval much have length $\ell+1 \geq \frac{\pi\omega}{\sqrt{\epsilon}}$. For a specific interval, the largest frequency ω determines the number of states a given interval can support with energy less than ϵ. There could be more eigenstates with energy less than ϵ; the upper bounds only guarantee a minimum number of energies less than epsilon. The following sum counts the number of intervals that can support a state with that frequency.

$$N(\epsilon) \geq \sum_{\omega=1}^{L} \#\{L_i : \frac{\omega^2\pi^2}{(|L_i|+1)^2} \leq \epsilon\} \tag{3.2.1}$$

The number of intervals obeying this bound is a random variable, see the appendix for details. With probability one, in the limit as $L \to \infty$, the previous quantity equals

$$= \sum_{\omega=1}^{L} P\left[L_i + 1 \geq \frac{\omega\pi}{\sqrt{\epsilon}}\right] * (pqL + o(pqL)) \tag{3.2.2}$$

$$\geq \sum_{\omega=1}^{L} p^{\frac{\omega\pi}{\sqrt{\epsilon}}} * (pqL + o(pqL)) \tag{3.2.3}$$

$$= \frac{(1 - p^{\frac{\pi L}{\sqrt{\epsilon}}})p^{\frac{\pi}{\sqrt{\epsilon}}}}{p^2(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} * (pqL + o(pqL)) \tag{3.2.4}$$

where $p^{\lfloor x \rfloor} \geq p^x$. In the limit,

$$\liminf_{L\to\infty} \frac{N(\epsilon)}{L} \geq \frac{qp^{\frac{\pi}{\sqrt{\epsilon}}}}{p^2(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} \tag{3.2.5}$$

3.3 Upper Bound for the Lifschitz Tail

Rather than bounding individual energies from below to derive an upper bound of the Lifschitz tail, this proof bounds the number of intervals that can support an eigenstate with approximate frequency equal to ω and energy less than ϵ. Bounds for individual energies can be calculated, but the method that follows in simpler. For an

interval to significantly support a sine wave with frequency ω with energy less than ϵ, it must have a length greater than $\frac{\pi\omega}{\sqrt{\epsilon}} - \frac{2}{b}$.

First, the norm of eigenstates with energy less than ϵ restricted to sites with zero potential V must be positive.

Lemma 3.3.1. *If $\epsilon < b$ and the energy of a normalized state ϕ is less than ϵ, then $\|\phi|_{\{x:V(x)=0\}}\|^2 > 1 - \frac{\epsilon}{b}$.*

Proof. The potential energy of ϕ must be less than ϵ,

$$b\|\phi|_{\{x:V(x)=b\}}\|^2 < \epsilon \tag{3.3.2}$$

Because $\|\phi|_{\{x:V(x)=0\}}\|^2 + \|\phi|_{\{x:V(x)=b\}}\|^2 = 1$, the inequality follows. □

Next, the energy contribution of the eigenstate restricted to an interval of zero potential is bounded below. On intervals of zero potential, $H = -\Delta$, so the eigenstate restricted to this interval must be a sine wave. Given this restriction, the goal is to optimize the energy contribution of a sine wave with approximate frequency ω on the interval length ℓ (the interval has ℓ consecutive sites where $V = 0$ with neighboring sites on each side with $V = b$) with the potential energy contributions of the boundary values included. This lower bound does not depend on the behavior of a state outside the interval and its boundary. If the interval cannot support a state with energy less than ϵ without the restrictions imposed by its global behavior, then the interval cannot support a state with those restrictions imposed. On an interval length ℓ, the state has the form

$$f(x) = \frac{cm}{\sqrt{\ell+1}} \sin\left(\frac{\alpha\pi x}{\ell+1} + t\right) \tag{3.3.3}$$

The constant m is the ℓ^2-norm of the sine wave restricted to the interval and c is a normalization constant. The constant α determines the frequency of the state. Define s and ω by the relations $\omega = \lfloor\alpha\rfloor + 1$ and $\alpha = s\omega$. The integer ω counts the approximate number of extrema of the sine wave, s is a stretching constant. The

constant $s = \frac{\alpha}{\lfloor\alpha\rfloor+1}$ and takes values between $\frac{\lfloor\alpha\rfloor}{\lfloor\alpha\rfloor+1}$ and 1. The shifting constant t takes values between $-\pi/2$ and $\pi/2$ and determines the left boundary value. The constants c, s, and t are determined by the boundary values of the sine wave on the sites of positive potential, denoted $m\delta^L$ and $m\delta^R$, which may be denoted as δ in expressions which are true for both boundary values. The boundary values are rescaled with norm m to simplify the expression for s. The energy contribution of this sine wave is

$$4m^2 \sin^2\left(\frac{s\pi\omega}{2(\ell+1)}\right) + bm^2\left((\delta^L)^2 + (\delta^R)^2\right) \tag{3.3.4}$$

where the $4\sin^2(\cdot/2)$ is the eigenvalue of a discrete sine wave. The process is to first bound s below, then use this to bound the energy contribution from below, then optimize the lower bound over boundary values. The boundary conditions for these sine waves satisfy

$$\frac{cm}{\sqrt{\ell+1}}\sin(t) = m\delta^L$$
$$\frac{cm}{\sqrt{\ell+1}}\sin(\pi\omega s + t) = m\delta^R$$

Because $t \in [-\pi/2, \pi/2]$, the left boundary condition becomes

$$t = \arcsin\left(\frac{\delta^L\sqrt{\ell+1}}{c}\right)$$

The right boundary condition requires the inverse of $y = \sin(x)$ on $[\omega\pi - \pi/2, \omega\pi + \pi/2]$, which is $x = (-1)^\omega \arcsin(y) + \omega\pi$. The boundary condition becomes

$$s\pi\omega + t = (-1)^\omega \arcsin\left(\frac{\delta^R\sqrt{\ell+1}}{c}\right) + \omega\pi$$

Solving for s,

$$s = 1 - \frac{1}{\omega\pi}\left(\arcsin\left(\frac{\delta^L\sqrt{\ell+1}}{c}\right) + (-1)^{\omega-1}\arcsin\left(\frac{\delta^R\sqrt{\ell+1}}{c}\right)\right) \tag{3.3.5}$$

The choice of rescaling the boundary values by m makes the value of s independent of the norm. It isolates the curvature of the sine wave from the norm of the sine wave.

With this expression of s, the next step is to minimize s by finding optimal $\delta^L,\ \delta^R$. To substitute x for $\arcsin(x)$, the value $\frac{\delta\sqrt{\ell+1}}{c}$ must be small. The magnitude of the boundary points should be less than the average magnitude of the state due to energy-minimization considerations, i.e. $|\delta| \leq \frac{1}{\sqrt{\ell+1}}$. Likewise, as a contribution to the energy, the potential energy from the boundary values $\delta^L,\ \delta^R$ must be less than ϵ. The contribution to the eigenvalue is $b|\delta|^2$, which makes $\delta \leq \sqrt{\frac{\epsilon}{b}}$. For intervals long enough, this condition is insufficient to make $|\delta|\sqrt{\ell+1}$ small. The upper bounds from the previous section could be used, but the goal here is to find the shortest interval length that supports states with energy less than ϵ, not bound every eigenvalue below. Given that $|\delta| \leq (\ell+1)^{-1/2}$, the correction to the energy by stretching the sine wave is at best a constant factor s, meaning the order of shortest interval lengths is at best $O(\epsilon^{-1/2})$. Therefore, $|\delta|\sqrt{\ell+1} \leq \epsilon^{1/4}$ which means the substitution $x \approx \arcsin(x)$ is asymptotically accurate.

After this substitution, two other optimizations are straightforward. The optimal s will occur when δ^R is the same sign as δ^L for ω odd and the opposite sign for ω even. The other optimization is that s is optimized when $|\delta^L| = |\delta^R|$ under the constraint that the potential energy contribution is fixed, i.e. when the sine wave is balanced in its left and right boundary values.

In the case where $s\omega \leq \frac{1}{2}$, the sine wave may not achieve an extreme value on the interval which will make the following calculation incorrect. In this case, the values of the sine wave are bounded above and below by the boundary values, which are small, thus making the norm of the sine wave restricted to the interval small. Any sine wave with these properties cannot have norm close to one, meaning it cannot significantly contribute to an eigenstate, making this case irrelevant in finding a minimal interval length.

The bound on s can be rewritten as

$$s \geq 1 - \frac{2\delta\sqrt{\ell+1}}{\omega\pi} + O(\delta^2(\ell+1))$$

where $O(\delta^2(\ell+1)) \approx \frac{\sqrt{\epsilon}}{b}$. This gives the following lower bound on the energy,

$$E \geq \left(1 - \frac{2\delta\sqrt{\ell+1}}{\omega\pi}\right)^2 \frac{\pi^2\omega^2}{(\ell+1)^2} + 2b\delta^2$$

where the second term is the potential energy contribution of the left and right boundary points. Differentiating this with respect to δ and solving for zero,

$$\delta = \frac{2\pi\omega}{(\ell+1)^{3/2}(2b + \frac{4\pi\omega}{(\ell+1)})}$$

The energy contribution is bounded below by the kinetic energy term of the minimum:

$$\left(1 - \frac{2}{b(\ell+1)}\right)^2 \frac{\pi^2\omega^2}{(\ell+1)^2}$$

If this lower bound is greater than ϵ, then an interval cannot support a state with energy less than ϵ and $s\omega \geq \frac{1}{2}$. Then $s\omega \leq \frac{1}{2}$, the sine wave does not achieve a maximum and is therefore monotone with small norm.

From this lower bound on the energy contribution, the minimum interval length that supports a sine wave with frequency ω and energy less than ϵ must satisfy

$$\left(1 - \frac{2}{b(\ell+1)}\right)^2 \frac{\pi^2\omega^2}{(\ell+1)^2} \leq \epsilon$$

An equivalent condition is

$$\sqrt{\epsilon}(\ell+1)^2 - \omega\pi(\ell+1) + \frac{2\omega\pi}{b} \geq 0$$

which requires $\ell+1$ to be greater than the greater root,

$$\ell + 1 \geq \frac{\omega\pi + \sqrt{\omega^2\pi^2 - \frac{8\sqrt{\epsilon}\omega\pi}{b}}}{2\sqrt{\epsilon}}$$

Using the Taylor series $\sqrt{x_0 - x} = \sqrt{x_0} + \frac{x}{2\sqrt{x_0}} + O(x^2)$, this condition requires

$$\ell + 1 \geq \frac{\omega\pi}{\sqrt{\epsilon}} - \frac{2}{b} + O(\epsilon/b)$$

This inequality measures the effect of δ boundary conditions on reducing the kinetic energy contribution of a sine wave on a given interval. The dependence on potential

height b is straightforward: as b decreases, the potential barriers are effectively weaker allowing for greater δ boundary values and lower kinetic energies.

The lower bound on the interval energy contribution provides the upper bound on the Lifschitz tail. An interval must be longer than $\frac{\omega\pi}{\sqrt{\epsilon}} - \frac{2}{b}$ to support a state with approximate frequency ω and energy less than ϵ. All energy levels less than ϵ must be supported on intervals longer than $\frac{\pi}{\sqrt{\epsilon}} - \frac{2}{b}$, effectively restricting the total eigenstates possible. The calculation below follows the same pattern as the calculation for the lower bound on the Lifschitz tail.

$$N(\epsilon) \leq \sum_{\omega=1}^{L} \#\{L_i + 1 \geq \frac{\omega\pi}{\sqrt{\epsilon}} - \frac{2}{b}\} \tag{3.3.6}$$

$$= \sum_{\omega=1}^{L} P\left[L_i + 1 \geq \frac{\omega\pi}{\sqrt{\epsilon}} - \frac{2}{b}\right] * (n + O(\sqrt{n})) \tag{3.3.7}$$

$$\leq \sum_{\omega=1}^{L} p^{\frac{\omega\pi}{\sqrt{\epsilon}} - 2 - \frac{2}{b}} * (pqL + O(\sqrt{pqL})) \tag{3.3.8}$$

$$= \frac{(1 - p^{\frac{\pi L}{\sqrt{\epsilon}}}) p^{\frac{\pi}{\sqrt{\epsilon}} - \frac{2}{b}}}{p^2(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} * (pqL + O(\sqrt{pqL})) \tag{3.3.9}$$

This bound the density of states above

$$k(\epsilon) \leq \limsup_{L\to\infty} \frac{N(\epsilon))}{L} \leq \frac{qp^{\frac{\pi}{\sqrt{\epsilon}} - \frac{2}{b}}}{p(1 - p^{\frac{\pi}{\sqrt{\epsilon}}})} \tag{3.3.10}$$

Remark 3.3.11. This approach makes no use of localization results such as [17] or [18], where states localize exponentially as $e^{|x-y|/\ell}$ with ℓ denoting the 'localization length.' Applying this result, if the intervals were far enough away, the states would be exponentially small. The inner product of states supported on these distant intervals would be exponentially small and the orthogonality condition could be relaxed to develop a lower bound. However, in the limit $L \to \infty$, $\frac{\pi}{\sqrt{\epsilon}}$ is constant, meaning there are order L intervals of this length. The number of shorter intervals is proportional to the system size, meaning that they are separated by an average distance independent

of the system size. The exponential decay of a state would not be significant from one interval to the next; the inner product of two states would not be necessarily small.

Chapter 4

Ground State Energy for the Interacting System in Bernoulli Distributed Potential

4.1 Introduction and Intuition

For interacting systems in random potentials, Anderson localization effects compete with delocalization effects caused by repulsive interaction. It is not at all obvious which of the two mechanisms dominates the system's behavior, even in the ground state. Two methods of approach descend from the two research areas studying these phenomena. From random Schrödinger operator theory, mathematicians tend to approach this system by starting from a noninteracting system and treat interaction as a perturbation. A system of bosons with no interaction places each particle in the single particle ground state. The method attempts to control the interactions when it is relatively weak. From condensation theory in interacting quantum systems, mathematicians start with a system of interacting bosons and add a random potential, controlling the effect of the random potential [58].

The goal of this chapter is to study effects of disorder in a system of bosons which interact with a weak repulsive 'soft core' force as the number of particles and system size are taken to infinity, proving two results. Theorem 4.1.13 is a general statement about delocalization effects of such interaction, applying to both discrete and continuous versions of the model and any bounded potential (without loss of generality,the infimum of the support of the distribution is assumed to equal zero). It is then applied, together with detailed analysis of the energy functional, to the one-dimensional system with a Bernoulli distributed potential—Theorems 4.1.17 and 4.1.19. For those theorems, the asymptotic behavior of the ground state energy (per particle) is derived for the limit as the product of the interaction strength and the particle density

goes to zero. The geometry of each realization of the random potential determines the ground state wave function. The methods are inspired by an adaptation of the technique used in [14].

In the discrete setting, the state of $N+1$ bosons in the lattice cube $\Lambda = \{0, \ldots, L+1\}^d$ with length L is described by a normalized wave function $|\Phi(x_1, \ldots, x_{N+1})\rangle$, a function in the symmetric subspace of $\otimes^{N+1} L^2(\Lambda, \mu)$ with Dirichlet boundary conditions, where the measure μ is the counting measure and x_i are positions of the particles in Λ. The Hamiltonian of the system is given by

$$H = \sum_{i=1}^{N+1} H_i + \sum_{i \neq j} U(x_i, x_j) \tag{4.1.1}$$

where H_i is the single particle Hamiltonian acting on the i-th particle and U is the potential of the interaction between particles. The single particle Hamiltonian is the (random) Schrödinger operator

$$H = -\Delta + V \tag{4.1.2}$$

where Δ is the discrete Laplacian and V is a (random) multiplication operator, which in this paper will always be bounded below (without loss of generality the bottom of the support is 0). The interaction U is a 'soft core' interaction of the form

$$g\delta_{x_i, x_j} \tag{4.1.3}$$

where the δ is the Kronecker delta and the coupling constant g is positive, making the interaction repulsive. This repulsion makes it energetically unfavorable for bosons to occupy the same space, but does not exclude the possibility. In the the case of 'hard core' interactions where $g = \infty$, bosons cannot overlap spatially. Because of the difficulty of working in a large tensor space, the multi-particle bosonic states are approximated by the Gross-Pitaevskii mean-field wave function [12]. In this approximation, each boson is assumed to be in the same state ϕ which defines the state of

the whole system: $|\Phi\rangle = |\phi(x_1)\rangle \dots |\phi(x_{N+1})\rangle$. The Hamiltonian becomes

$$(N+1)H_i + \frac{g(N+1)(N)}{2}|\phi|^2, \tag{4.1.4}$$

a discrete nonlinear random Schrödinger operator. The nonlinearity is due to the second term, where the state acts as its own external potential. The approximation exchanges the difficulties arising from Bose statistics for the nonlinearity of the new problem. The associated per particle energy functional is

$$E[\Phi] = \sum_{x\in\Lambda}\sum_{|y|=1} |\phi(x+y)-\phi(x)|^2 + V(x)|\phi(x)|^2 + \frac{gN}{2}|\phi(x)|^4 \tag{4.1.5}$$

Remark 4.1.6. In the infinite volume limit, if the particle number increases proportionally to the system size, the total energy diverges. The natural quantity to study in these systems is energy per particle, which can be shown to be finite. In this chapter, any mention of energy refers to the per particle energy.

In the continuous setting, the state of $N+1$ bosons in the box $\Omega = [0, L]^d$ with linear size L is described by the wave function $|\Phi(x_1, \dots, x_{N+1})\rangle$, a function in the symmetric subspace of $\otimes^{N+1} L^2(\Omega, \mu)$ with Dirichlet boundary conditions, where the measure μ is the Lebesgue measure, and x_i are the positions of the particles in Ω. The Hamiltonian of the system is given by

$$H = \sum_{i=1}^{N+1} H_i + \sum_{i\neq j} U(x_i, x_j) \tag{4.1.7}$$

where H_i is the single particle Hamiltonian acting on the i-th particle and U is the interaction potential between particles. The single particle Hamiltonian is given by the (random) Schrödinger operator

$$H = -\Delta + V_i \tag{4.1.8}$$

where Δ is the Laplacian and V is a (random) multiplication operator. The potential is bounded below (without loss of generality by 0). The interaction U is a 'soft core'

interaction of the form

$$g\delta(x_i - x_j) \tag{4.1.9}$$

where the δ is the Dirac delta and $g > 0$ corresponds to a repulsive interaction. In the mean-field approximation, each boson is assumed to be in the same state ϕ which defines a state of the whole system: $|\Phi\rangle = |\phi(x_1)\rangle \dots |\phi(x_{N+1})\rangle$. The Hamiltonian in this approximation is

$$(N+1)H_i + \frac{g(N+1)(N)}{2}|\phi|^2, \tag{4.1.10}$$

a nonlinear Schrödinger operator. The associated per particle energy functional is

$$E[\Phi] = \int_\Omega dx \left[|\phi'(x)|^2 + V(x)|\phi(x)|^2 + \frac{gN}{2}|\phi(x)|^4 \right] \tag{4.1.11}$$

In section 4.2, the localization and delocalization of states is described for both the discrete and continuous systems. For any state ϕ and $\epsilon \in (0,1)$, define the set

$$X_{>\epsilon}(\phi) = \{x : |\phi(x)| > \frac{\epsilon}{L^{d/2}}\} \tag{4.1.12}$$

as the set of points where the absolute value of $\phi(x)$ is greater than its average magnitude per site, multiplied by a constant ϵ. This is a natural set in the study of localization of low energy states.

Theorem 4.1.13. *Assume that $V \geq 0$, for both the discrete and continuous settings, for a state ϕ with energy $E' = E[\phi]$, the measure of the set $X_{>\epsilon}(\phi)$ obeys the lower bound*

$$\mu(X_{>\epsilon}(\phi)) \geq \frac{gN(1-\epsilon^2)^2}{2E'} \tag{4.1.14}$$

This theorem implies that Anderson-type localization for low energy states, $E' \approx 0$, requires gN to be small. For a large number of particles or strong interaction, the wave function must fill a significant amount of space, meaning the effects of repulsion dominate the Anderson localization effects caused by the random potential V. The repulsive interaction, though local, forces overlap to become energetically

expensive if too many bosons occupy the same place, causing the ground state to spread. If a low energy state is localized to some length ℓ, gN must be smaller than ℓ^d. To recover an Anderson type localization, a necessary condition is $gN \leq O(E'\ell^d)$. In physical experiments, the interaction constant g is a controlled parameter rather than a variable dependent on the particle number and the particle density is approximately constant ($N \approx \rho L^d$) to ensure the existence of thermodynamic limits. In these systems, a low-energy state must occupy a nonzero fraction of the volume rather than being localized with some lower order localization length ℓ.

In section 4.3, the above theorem motivates the description of the ground state defined on the one-dimensional lattice with the Bernoulli distributed potential V, small interaction constant g, and approximately constant particle density, i.e. $N+1 \approx \rho L$. The ground state minimizes the energy (per particle) given by

$$E(\phi) = \sum_{x=1}^{L} \left(|\phi'(x)|^2 + V(x)|\phi(x)|^2 + \frac{g\rho L}{2}|\phi(x)|^4 \right) \tag{4.1.15}$$

where $\phi \in \ell^2\{0, \ldots, L+1\}$ with Dirichlet boundary conditions and normalized to one. As in the previous chapters, the random potential V is a multiplication operator represented by a function $V(x)$ defined for each $x \in \{1, \ldots, L\}$, where $V(x)$ are i.i.d. Bernoulli random variables with $P[V(x) = 0] = p$ and $P[V(x) = b] = 1 - p = q$ for fixed constant $b > 0$.

Definition 4.1.16. The finite volume ground state is defined as the normalized state $\phi_0^{(L)}$ which minimizes the energy functional $E[\phi]$ in (4.1.15) and the corresponding energy, $E_0^{(L)}$, is called the ground state energy.

These systems are ergodic and have a nonrandom limit for a given set of parameters.

Theorem 4.1.17. *For any g, ρ, p, and b,*

$$\lim_{L\to\infty} E_0^{(L)} = E_0 \tag{4.1.18}$$

where E_0 is a nonrandom function of the above parameters.

In an attempt to recover the noninteracting system behavior from the interacting system, the behavior of the infinite volume ground state is studied in the limit of weak interaction.

Theorem 4.1.19. *For the one-dimensional lattice Gross-Pitaevskii model with a Bernoulli distributed potential, the ground state energy E_0 satisfies the following conditions with probability one.*

$$0 < \liminf_{g\rho\to 0} E_0 \log_p^2(g\rho) \leq \limsup_{g\rho\to 0} E_0 \log_p^2(g\rho) < \infty \tag{4.1.20}$$

Theorem 4.1.19 is an illustration of Theorem 4.1.13. In Theorem 4.1.19, the ground state is approximated by sine waves supported on intervals of zero potential longer than some minimum interval length; the ground state is approximated by zero everywhere else. The ground state energy is bounded above by $\frac{C_+}{(\log_p(g\rho))^2}$ for some constant C_+. Using Theorem 4.1.13, $\mu(X_{>\epsilon}(\phi))$ is bounded below by

$$\frac{g\rho L(1-\epsilon^2)^2(\log_p(g\rho))^2}{2C_+} \tag{4.1.21}$$

which is proportional to the system size. The proof of Theorem 4.1.19 confirms that for the ground state this lower bound is asymptotically accurate for $L \to \infty$ then $g\rho \to 0$.

Remark 4.1.22. In Theorem 4.1.19, the constants in upper and lower limits are not proven to be equal. One should be able to close this gap between the lower and upper bounds using a more accurate variational function than the sine wave, the solution of the discrete nonlinear Schrödinger equation on an interval with Dirichlet boundary conditions. This can be thought of as a discrete version of the Jacobi elliptic sine function, which is the solution to the (continuous) nonlinear Schrödinger equation on an interval with Dirichlet boundary conditions.

4.2 Proof of Theorem 4.1.13

Proof. The ℓ^2-norm (or L^2-norm) of the state ϕ restricted to $X_{\leq\epsilon}$ is bounded by

$$\|\phi(x)|_{X_{\leq\epsilon}}\|^2 \leq \mu(X_{\leq\epsilon})\frac{\epsilon^2}{L^d} \leq \epsilon^2$$

where the system restricted to the finite box determines the bound $\mu(X_{\leq\epsilon}) \leq L^d$. Because the state ϕ has norm equal to one,

$$\|\phi(x)|_{X_{>\epsilon}}\| \geq 1 - \epsilon^2$$

The energy of ϕ is bounded below by its interaction energy when restricted to the set $X_{>\epsilon}(\phi)$. The interaction energy is bounded below using Schwarz's Inequality:

$$\left(\int_{X_{>\epsilon}} |f|^2 d\mu\right)^2 \leq \left(\int_{X_{>\epsilon}} d\mu\right)\left(\int_{X_{>\epsilon}} |f|^4 d\mu\right)$$

which bounds the ℓ^4-norm (or L^4-norm) below by

$$\int_{X_{>\epsilon}(\phi)} |f|^4 d\mu \geq \frac{(1-\epsilon^2)^2}{\mu(X_{>\epsilon}(\phi))}$$

The interaction energy is thus bounded below by

$$\frac{gN(1-\epsilon^2)^2}{2\mu(X_{>\epsilon}(\phi))}$$

and bounded above by E'. It follows that

$$\frac{gN(1-\epsilon^2)^2}{2\mu(X_{>\epsilon})} \leq E'$$

which gives the desired lower bound:

$$\mu(X_{\geq\epsilon}) \geq \frac{gN(1-\epsilon^2)^2}{2E'} \tag{4.2.1}$$

and completes the proof. □

Remark 4.2.2. By shifting the energy, one can easily generalize the above theorem to arbitrary potentials bounded below. If V_{min} is the infimum of the support of the potential V,

$$\mu(X_{>\epsilon}(\phi)) \geq \frac{gN(1-\epsilon^2)^2}{2(E' - V_{min})}$$

4.3 Interacting Ground State Estimates for Bernoulli Potentials

Theorems 4.1.17 and 4.1.19 are specifically for the system on the one-dimensional lattice with Bernoulli potential. The proof of Theorem 4.1.17 follows from Kingman's subadditive ergodic theorem. It suffices to use the version from the standard probability reference [23]; Kingman's original formulation [38] would also be sufficient.

Theorem 4.3.1 (Kingman's Subadditive Ergodic Theorem)**.** *Suppose $X_{m,n}$, $0 \le m < n$ satisfy:*

(i) $X_{0,m} + X_{m,n} \ge X_{0,n}$.

(ii) $\{X_{nk,(n+1)k},\ n \ge 1\}$ *is a stationary sequence for each* k.

(iii) The distribution of $\{X_{m,m+k},\ k \ge 1\}$ *does not depend on* m.

(iv) $\mathbb{E}[X_{0,1}^+] < \infty$ *and for each* n, $\mathbb{E}[X_{0,n}] \ge \gamma_0 n$, *where* $\gamma_0 > -\infty$.

Then

(a) $\lim_{n\to\infty} \mathbb{E}[X_{0,n}]/n = \inf_m \mathbb{E}[X_{0,m}]/m \equiv \gamma$.

(b) $X = \lim_{n\to\infty} X_0/n$ *exists almost surely and in* L^1, *so* $\mathbb{E}[X] = \gamma$.

(c)If all the stationary sequences in (ii) are ergodic then $X = \gamma$ *almost surely.*

Proof of Theorem 4.1.17. The ground state energy is expressed in a slightly different manner as follows.

$$E_0^{(L)} = \min_{\|\psi\|=1} \sum_{j=0}^{L-1} |\psi(j+1) - \psi(j)|^2 + V(j)|\psi(j)|^2 + \frac{g\rho L|\psi(j)|^4}{2} \tag{4.3.2}$$

where $\psi \in \ell^2\{0, \dots L\}$ with Dirichlet boundary conditions. With the substitution $\phi = \sqrt{L}\psi$, $E_0^{(L)}$ can be rewritten as

$$E_0^{(L)} = \frac{1}{L} \min_{\|\phi\|=\sqrt{L}} \sum_{j=0}^{L-1} |\phi(j+1) - \phi(j)|^2 + V(j)|\phi(j)|^2 + \frac{g\rho|\phi(j)|^4}{2} \tag{4.3.3}$$

Denote the quantity on the left side of the equation as $\frac{1}{L}X_{0,L}$, where

$$X_{0,L} = \min_{\|\phi\|=\sqrt{L}} \sum_{j=0}^{L-1} |\phi(j+1) - \phi(j)|^2 + V(j)|\phi(j)|^2 + \frac{g\rho|\phi(j)|^4}{2} \tag{4.3.4}$$

The process $X_{0,L}$ satisfies the assumptions of Kingman's theorem. For (i), $X_{0,L}$ is subadditive since the only difference between $X_{0,M} + X_{M,L}$ and $X_{0,L}$ is the restriction $\phi(M) = 0$ in the definition of the former. Properties (ii) and (iii) hold because the $V(j)$ are independent. For (iv), $X_{0,1}^{+} < 2 + b + \frac{g\rho}{2}$ and $X_{0,n} \geq 0$. Therefore, with probability one,

$$\lim_{L\to\infty} E_0^{(L)} = \lim_{L\to\infty} \frac{X_{0,L}}{L} = E_0 \tag{4.3.5}$$

□

Since the infinite volume ground state is a nonrandom function of the parameters, Theorem 4.1.19 says essentially that $E_0^{(L)}$ in the limit $L \to \infty$ has leading order of the form $\frac{C}{(\log_p(g\rho))^2}$ for some constant C and for $g\rho$ sufficiently small. The proof does not provide the precise value of C, but this value is likely determined by the discrete nonlinear Schrödinger ground state. In any case, the proof does provide a good approximation of the true ground state.

The asymptotic behavior described in Theorem 4.1.19 was discovered as follows. An upper bound on the ground state energy can be generated by evaluating the energy functional on any test function. The desired test function should have asymptotic behavior similar to a demonstrable lower bound. The process is iterative; first, a test function is evaluated to find an upper bound on the ground state energy. Second, this upper bound is used to bound the norm of the ground state restricted to various sets, such as the set of sites of positive potential. The goal is to isolate the subset where the ℓ^2-norm of the ground state is concentrated. Third, the energy of the ground state on these sets is minimized to find a lower bound. This lower bound should give intuition for a better choice of test function for the upper bound. The process repeats until

the upper and lower bounds on the ground state energy are asymptotically similar. The proof below is simply the final iteration of this process.

Proof of Theorem 4.1.19 Upper Bound. The energy of any test function will bound the ground state energy from above. Consider the test function ψ defined as follows: for an interval of zero potential with length L_i, the function ψ restricted to the interval is a sine wave $m_i\sqrt{\frac{2}{L_i+1}}\sin(\frac{\pi x}{L_i+1})$, where m_i is the L^2-norm of the function restricted to the interval. On intervals of high potential, ψ is zero. For intervals of zero potential with length $L_i > \log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)$, we let

$$m_i^2 = \frac{L_i}{\sum_{L_i > \log_p(g\rho)+\log_p\left(\log_p(g\rho)\right)} L_i} \tag{4.3.6}$$

and for shorter intervals, $m_i = 0$. This makes the kinetic energy term and interaction energy term have the same asymptotic order as $g\rho \to 0$. This test function also satisfies the normalization criterion $\sum_i m_i^2 = 1$. The kinetic energy of the discrete sine wave on a specific interval of zero potential is bounded above by $\frac{m_i^2\pi^2}{(L_i+1)^2}$, see Chapter 2. The interaction energy of the sine wave on this interval is

$$\frac{g\rho L}{2}\sum_x \|\phi(x)\|^4 = \frac{3g\rho L m_i^4}{4L_i} \tag{4.3.7}$$

Summing the upper bound on kinetic energy and the interaction energy of test function ψ over the space, the total energy of the test function ψ is bounded above by

$$\frac{3g\rho L}{4\sum_{L_i > \log_p(g\rho)+\log_p\left(\log_p(g\rho)\right)} L_i} + \frac{\pi^2}{\left(\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right) + 1\right)^2} \tag{4.3.8}$$

where the second term is an overestimate of the kinetic energy, treating each interval as the shortest interval admitted. Both L and $\sum_{L_i > \log_p(g\rho)+\log_p\left(\log_p(g\rho)\right)} L_i$ depend on the realization of the potential, (see the appendix) with probability one in the limit $n \to \infty$,

$$L = \frac{n}{pq} + o(n) \tag{4.3.9}$$

and using $\lfloor x \rfloor \geq x - 1$,

$$\sum_{L_i > \log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)} L_i$$

$$\geq \frac{n}{pq} g\rho \log_p(g\rho) \left(p + q \log_p(g\rho) + q \log_p\left(\log_p(g\rho)\right)\right) + o(n) \tag{4.3.10}$$

by the appendix. The interaction energy is bounded in the limit as $n \to \infty$ with probability one by

$$\frac{3}{4q \log_p(g\rho) \left[\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right) + \frac{p}{q}\right]}$$

In the limit as $g\rho \to 0$, the leading order term of the upper bound on the ground state energy is

$$\frac{C'}{\log_p^2(g\rho)}$$

with $C' = \frac{3}{4q} + \pi^2$. Thus, $\limsup_{g\rho \to \infty} E_0 \log_p^2(g\rho) < \infty$. □

Proof of Theorem 4.1.19 Lower Bound. For each n and $g\rho$, the ground state ϕ_0 exists, is well-defined, but is not explicitly known. The lattice will be partitioned into four sets: the set of sites of high potential, denoted M_b; the set of sites on intervals of zero potential longer than $\log_p(g\rho)$, denoted M_{long}; the set of sites on intervals of zero potential shorter than $\log_p(g\rho)$ where the kinetic energy cannot be easily bounded below, denoted M_{light}; and the set of sites on intervals of zero potential shorter than $\log_p(g\rho)$ where the kinetic energy can be easily bounded below, denoted M_{heavy}. Both M_{light} and M_{heavy} are defined similar to their definition in Chapter 2.

The lower bound is shown as follows. The ground state energy is bounded below by a lower bound of the ground state energy restricted to M_{heavy}. The kinetic energy for a given interval in M_{heavy} is bounded below by Lemma 4.3.16. The interaction energy is bounded below by the bound derived in the proof of Theorem 4.1.13. Lemma 4.3.16 provides a lower bound on the norm of the ground state ϕ_0 restricted to M_{heavy}, which converges to one in the limit $g\rho \to 0$. Using Lagrange multipliers, the lower bound on

kinetic and interaction energy on each interval is minimized over the m_i, the norms of restrictions of the state to each interval. This minimization determines a minimal interval length for any interval in M_{heavy}. The number of sites in M_{heavy} is estimated above by the number of sites on intervals longer that this minimal interval length. Using this and the lower bound of the norm of ϕ_0 restricted to M_{heavy}, we obtain the desired lower bound of the ground state energy.

Without loss of generality, the ground state wave function can be assumed to be non-negative. By a standard argument: if the ground state does not have the same complex phase at each site, the kinetic energy can by reduced by making the phases equal. Since the potential and interaction energy only depend on the magnitude of the state at each site and not on the complex phase, the energy of the resulting state is strictly smaller. Therefore the ground state must have the same complex phase and can be assumed to be positive.

To define M_{light} and M_{heavy}, the ground state energy on an interval of zero potential is approximated by that of the the minimizer of the kinetic energy on the interval—the discrete sine wave. For a given interval with length L_i, the ground state determines boundary values $m_i\delta_i^L$ and $m_i\delta_i^R$ on the sites of high potential to the left and right of the interval, respectively, where m_i is the norm of the ground state on the interval. The boundary values are positive by the above argument. The kinetic-energy-minimizing state is of the form

$$\frac{c_i m_i}{\sqrt{L_i+1}} \sin\left(\frac{s_i \pi x}{L_i+1} + t_i\right) \tag{4.3.11}$$

where the sine wave is normalized to m_i by $c_i \in [1, \sqrt{2}]$, stretched by $s_i \in (0,1)$, and shifted by t_i; all three are determined by the δ_i's and m_i. Heavy intervals have relatively small δ_i's which determine a lower bound on kinetic energy. Light intervals have large δ_i's which do not admit a good lower bound on kinetic energy, but do allow an upper bound on the norm of the ground state on this intervals.

Definition 4.3.12. For intervals of zero potential with length less than $\log_p(g\rho)$, an

interval is in M_{heavy} if for the ground state ϕ_0,

$$\max(\delta_i^L, \delta_i^R) \leq \frac{1}{2\sqrt{L_i}} \tag{4.3.13}$$

An interval is in M_{light},

$$\max(\delta_i^L, \delta_i^R) > \frac{1}{2\sqrt{L_i}} \tag{4.3.14}$$

These definitions can be restated using m_i and are thus directly determined by the values of the ground state wave function on the sites adjacent to the zero potential intervals. The definition is stated above without m_i in order to separate the shape and curvature of the sine wave from its norm and to facilitate estimates on energy contributions. The norm of ϕ_0 restricted to a heavy interval is bounded below by

$$m_i \geq m_i 2\sqrt{L_i} \max(\delta_i^L, \delta_i^R) \geq \sqrt{L_i} \max(m_i\delta_i^L, m_i\delta_i^R) \tag{4.3.15}$$

where the last term is the norm of the constant function $m_i \max(\delta_i^L, \delta_i^R)$ on the interval. This means that a sine wave with this norm achieves its maximum rather than being nearly flat.

Lemma 4.3.16. *The kinetic energy of the ground state restricted to an interval in M_{heavy} is bounded below by*

$$m_i^2 \left(1 - \frac{1}{\sqrt{2}}\right)^2 \frac{\pi^2}{(L_i+1)^2} \tag{4.3.17}$$

Proof of Lemma 4.3.16. The kinetic energy of a heavy interval is bounded below by the kinetic energy of the sine wave with norm m_i and boundary conditions $m_i\delta_i^L$ and $m_i\delta_i^R$. The energy of the minimizer $\frac{c_i m_i}{\sqrt{L_i+1}} \sin(\frac{s_i \pi x}{L_i+1} + t_i)$ is $4m_i^2 \sin^2(\frac{s_i\pi}{2(L_i+1)})$. To solve for s_i, note that the function must satisfy the boundary conditions

$$\frac{c_i m_i}{\sqrt{L_i+1}} \sin(t_i) = m_i\delta_i^L \tag{4.3.18}$$

$$\frac{c_i m_i}{\sqrt{L_i+1}} \sin(s_i\pi + t_i) = m_i \delta_i^R \tag{4.3.19}$$

The left boundary condition is solved using the inverse of sine on $[0, \frac{\pi}{2}]$, $\arcsin(x)$. The right boundary condition is solved using the inverse of sine on $[\frac{\pi}{2}, \frac{3\pi}{2}]$, $-\arcsin(x)+\pi$. The boundary conditions are rewritten as

$$t_i = \arcsin\left(\frac{\delta_i^L\sqrt{L_i+1}}{c_i}\right) \tag{4.3.20}$$

$$s_i\pi + t_i = \pi - \arcsin\left(\frac{\delta_i^R\sqrt{L_i+1}}{c_i}\right) \tag{4.3.21}$$

Solving for s_i,

$$s_i = 1 - \frac{1}{\pi}\left(\arcsin\left(\frac{\delta_i^L\sqrt{L_i+1}}{c_i}\right) + \arcsin\left(\frac{\delta_i^R\sqrt{L_i+1}}{c_i}\right)\right) \tag{4.3.22}$$

Since $\arcsin(\theta) \leq \frac{\pi\theta}{2}$ for $\theta \leq 1$,

$$\begin{aligned} s_i &\geq 1 - \frac{\max(\delta_i^L, \delta_i^R)\sqrt{L_i+1}}{c_i} \\ &\geq 1 - \frac{1}{\sqrt{2}} \end{aligned} \tag{4.3.23}$$

where $c_i \geq 1$, $L_i \geq 1$, and the definition of a heavy interval is used. □

Lemma 4.3.24. *In the limit as $n \to \infty$, with probability one,*

$$\liminf_{n\to\infty} \|\phi_0|_{M_{heavy}}\|^2 \geq 1 - O\left(\frac{1}{\sqrt{\log_p(g\rho)}}\right) - O\left(\frac{1}{4b\log_p(g\rho)}\right) \tag{4.3.25}$$

where $O(\cdot)$ is taken with respect to the limit $g\rho \to 0$.

Proof of Lemma 4.3.24. Because the potential energy of the ground state must be less than the upper bound on the ground state energy, the norm of the ground state restricted to high potential is bounded as follows:

$$\|\phi_0|_{M_b}\|^2 \leq \frac{C_+}{b\left(\log_p(g\rho)\right)^2} \tag{4.3.26}$$

An upper bound on the norm of the ground state on intervals longer than $\log_p(g\rho)$ follows from the fact that the interaction energy must be bounded above by the upper bound on the ground state energy. The minimum of the interaction energy depends on the number of sites occupied and the norm restricted to the set of these sites. It was shown in Theorem 4.1.13 that for $\|\phi\| = m'$, the minimum of $\sum_{i=1}^{L} |\phi(i)|^4$ is $\frac{m'^4}{L}$. If $\|\phi_0|_{M_{long}}\|$ is the norm of $\phi_0|_{M_{long}}$, then the interaction energy for the sites in M_{long} is bounded below using the lower bound derived in Theorem 4.1.13.

$$\frac{g\rho L\|\phi_0|_{M_{long}}\|^4}{\sum_{L_i>\log_p(g\rho)} L_i} \geq \frac{g\rho\|\phi_0|_{M_{long}}\|^4(\frac{n}{pq} - o(n))}{\frac{ng\rho}{pq}(p + pq\log_p(g\rho)) + o(n)} \tag{4.3.27}$$

Using the upper bound on the ground state energy, the left hand side of the above inequality is bounded above by $\frac{C_+}{\left(\log_p(g\rho)\right)^2}$. In the limit as $n \to \infty$,

$$\|\phi_0|_{M_{long}}\|^2 \leq \sqrt{\frac{pqC_+}{\log_p(g\rho)} + \frac{pC_+}{(\log_p(g\rho))^2}} \tag{4.3.28}$$

For light intervals,

$$\frac{m_i^2}{4L_i} < m_i^2 \max(\delta_i^L, \delta_i^R)^2 \tag{4.3.29}$$

The upper bound on the norm of the ground state restricted to sites of high potential bounds the norm on the boundary points

$$\sum_i m_i^2 \max(\delta_i^L, \delta_i^R)^2 \leq \frac{2C_+}{b\left(\log_p(g\rho)\right)^2} \tag{4.3.30}$$

where the extra factor of 2 is included to cover the cases where two intervals of zero potential are separated by a single site of positive potential. The bound on the norm of the ground state restricted to light intervals, which by definition are shorter than $\log_p(g\rho)$, is

$$\begin{aligned}\frac{1}{\log_p(g\rho)} \sum_{light\ intervals} m_i^2 &\leq \sum_{light\ intervals} \frac{m_i^2}{L_i} \\ &\leq \frac{1}{4} \sum_{light\ intervals} m_i^2 \max(\delta_i^L, \delta_i^R)^2\end{aligned}$$

$$\leq \frac{C_+}{4b\left(\log_p(g\rho)\right)^2} \tag{4.3.31}$$

This means that the norm on light intervals is bounded above by

$$\|\phi_0|_{M_{light}}\|^2 \leq \frac{C_+}{4b\log_p(g\rho)} \tag{4.3.32}$$

The normalization condition requires

$$\|\phi_0|_{M_b}\|^2 + \|\phi_0|_{M_{long}}\|^2 + \|\phi_0|_{M_{light}}\|^2 + \|\phi_0|_{M_{heavy}}\|^2 = 1 \tag{4.3.33}$$

Using the upper bounds on the three other terms, the norm of the ground state restricted to M_{heavy} gives the desired lower bound

$$\|\phi_0|_{M_{heavy}}\|^2 \geq 1 - O\left(\frac{1}{\sqrt{\log_p(g\rho)}}\right) - O\left(\frac{1}{b\log_p(g\rho)}\right)$$

□

The energy for a heavy interval is bounded below by

$$\frac{g\rho L m_i^4}{2L_i} + m_i^2\left(1-\frac{1}{\sqrt{2}}\right)^2\frac{\pi^2}{(L_i+1)^2} \tag{4.3.34}$$

where the first term is the minimum of the interaction energy and the second term is the minimum of the kinetic energy. Using Lagrange multiplier method, the choice of m_i, where i labels the heavy intervals, which minimize this lower bound

$$\sum_i \frac{g\rho L m_i^4}{2L_i} + m_i^2\left(1-\frac{1}{\sqrt{2}}\right)^2\frac{\pi^2}{L_i^2} \tag{4.3.35}$$

under normalization constraint

$$\sum_i m_i^2 = 1 - O\left(\frac{1}{\sqrt{\log_p(g\rho)}}\right) - O\left(\frac{1}{b\log_p(g\rho)}\right) \tag{4.3.36}$$

must satisfy

$$\frac{\partial}{\partial m_i}\left[\sum_i \frac{g\rho L m_i^4}{2L_i} + m_i^2\left(1-\frac{1}{\sqrt{2}}\right)^2\frac{\pi^2}{L_i^2}\right]$$

$$= \lambda \frac{\partial}{\partial m_i} \sum_i m_i^2 \tag{4.3.37}$$

This equation implies that

$$m_i = 0 \tag{4.3.38}$$

or

$$m_i^2 = \frac{L_i}{g\rho L} \left(\lambda - \left(1 - \frac{1}{\sqrt{2}} \right)^2 \frac{\pi^2}{L_i^2} \right) \tag{4.3.39}$$

For an interval to contribute to the minimization of the lower bound, the kinetic energy of a heavy interval must be less than λ. For the kinetic energy to meet this bound, the heavy interval must have length

$$L_i > \left(1 - \frac{1}{\sqrt{2}} \right) \frac{\pi}{\sqrt{\lambda}} \tag{4.3.40}$$

It follows from Lemma 4.3.24 that the normalization condition requires λ to satisfy

$$\sum_{L_i > \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\pi}{\sqrt{\lambda}}} \frac{\lambda L_i}{g\rho L} = 1 - O\left(\frac{1}{\sqrt{\log_p(g\rho)}} \right) - O\left(\frac{1}{b \log_p(g\rho)} \right) \tag{4.3.41}$$

From the appendix and approximating $\lfloor x \rfloor$,

$$\sum_{L_i > x} L_i = \left(\frac{n}{pq} + o(n) \right) \left(\lfloor x \rfloor q p^{\lfloor x \rfloor + 1} + p^{\lfloor x \rfloor + 1} \right) \tag{4.3.42}$$

which requires

$$\frac{\lambda}{g\rho \left(\frac{n}{pq} + o(n) \right)} \left(\frac{n}{pq} + o(n) \right) \left(\left(1 - \frac{1}{\sqrt{2}} \right) \frac{\pi}{\sqrt{\lambda}} q p^{\left(1 - \frac{1}{\sqrt{2}}\right) \frac{\pi}{\sqrt{\lambda}}} + p^{\left(1 - \frac{1}{\sqrt{2}}\right) \frac{\pi}{\sqrt{\lambda}}} \right) \approx 1 \tag{4.3.43}$$

The asymptotic behavior of the parameter λ is determined by $g\rho$. If λ is constant or taken to infinity as $g\rho \to 0$, the normalization will not hold because the left side of (4.3.43) goes to infinity. If λ converges to zero, the dominant term is the exponential. The correct asymptotic solution for λ as $g\rho \to 0$ is

$$\left(1 - \frac{1}{\sqrt{2}} \right) \frac{\pi}{\sqrt{\lambda}} = \log_p(g\rho) + \log_p \left(\log_p(g\rho) \right) + O(1) \tag{4.3.44}$$

This substitution will make left hand side of (4.3.43) converge to a constant. Therefore, the heavy intervals must be longer than $\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right) + O(1)$. The size of M_{Heavy} is bounded above by the number of sites on intervals longer than this lower bound. This number is bounded above by

$$ng\rho \log_p(g\rho)\left(\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)\right) + o(n) \tag{4.3.45}$$

The interaction energy of any state supported on this number of sites is bounded below by

$$\frac{g\rho\left(\frac{n}{pq} + o(n)\right)}{2ng\rho \log_p(g\rho)[\log_p(g\rho) + \log_p\left(\log_p(g\rho)\right)] + o(n)} \tag{4.3.46}$$

The lower bound follows. □

Chapter 5
Future Research Directions

There are many research directions which follow from the work in this dissertation. These are simply a few possible avenues for those who are curious about quantum systems with disordered potentials, with interactions, or with both.

The results for the ground state energy and excited state energies in Bernoulli potentials may extend to higher dimensions. In one dimension, the states essentially corresponded to specific long intervals of zero potential. In two dimensions, preliminary numerical work done in collaboration with Dr. Julia Stasińska suggests shape analogous to the intervals of zero potential is approximately the balls of zero potential. Here is the general idea: Suppose that potential barriers are effectively infinite in two dimensions as the intuition from one dimension suggests. Then the norm of low energy states should be concentrated on sites of zero potential. The low energy states should then be supported on sets of zero potential that minimize the small eigenvalues of the Laplacian. Due to isoperimetric considerations, this shape should be approximately an ℓ^2 ball. It would follow that the low energy states should correspond to states supported the large balls of zero potential. This intuition should be correct for two dimensions, but misses issues that occur in higher dimensions.

The effective barrier caused by sites of positive potential in the Bernoulli distributed potential depends on the dimension of the lattice that states are supported on. Assume the ground state or an excited state is supported on a large ball with volume approximately equal to ℓ^d, where d is the dimension of the lattice and ℓ is the approximate diameter of the ball. The state will have an approximate average magnitude of the order $\ell^{-d/2}$ on the ball. The small eigenvalues of the Laplacian on that ball should have order $d\ell^{-2}$ as $\ell \to \infty$. If there is a single site of positive potential

somewhere in the interior of the ball, the potential contribution for the state on that site is approximately $b|\phi(j)|^2 \approx b\ell^{-d}$. Using the smallest eigenvalue of the Laplacian as an upper bound on energy, the following inequality should hold as $\ell \to \infty$:

$$b\ell^{-d} \leq d\ell^{-2} \tag{5.0.1}$$

For $d = 1$, this inequality will not hold as $\ell \to \infty$, implying that there cannot be a site of positive potential in the support of the low energy state. For $d = 2$, these terms have the same asymptotic order and the inequality may or may not hold, implying that there may be a few sites of positive potential in the support of the low energy state. For $d = 3$, the inequality should hold in the limit even if there are a significant number of sites of positive potential in the support.

The work by Sznitman [64] on the diffusion process in a field of random barriers yields information about the ground state of the random Schrödinger operator in any dimension. The diffusion process is released in a box with random potential of barriers scattered throughout, each barrier a compactly supported function centered on the points distributed by Poisson-point process. The infinitesimal generator of this diffusion process is the random Schrödinger operator with random potential defined above. The ground state of this operator determines the invariant distribution of the process. In Sznitman's work, the ground state is localized to a large ball of zero potential with a few points of high potential. His approximation replaces the true potential with a new potential where isolated points of high potential are replaced by zero potential and large subsets of positive potential are replaced by infinite potential barriers. Sznitman's work accounts for this dimensionality issue in the approximation of the potential in the diffusion process. From his work, a similar method could be derived for the excited states which could allow for results similar to the results in Chapters 3 and 4.

The results of Chapters 3 and 4 may also extend to different random potential distributions, both in the sense of individual site distributions and in the sense of

interval distributions. This dissertation considered the distribution of the potential at a given site to be Bernoulli distributed. As stated in the introduction, most works on the topic consider distributions with compact support and bounded density functions. The sharp nature of the Bernoulli distribution makes it difficult to work with analytically, yet here it provides a clean distinction for low and high potential. Contrast this with a continuously-distributed random variable, such as a uniform random variable on the interval $[0, \lambda]$: there is no clear cut-off between low and high potential values for the low energy states. It may be possible to use the intuition derived from Bernoulli potentials to calculate cut-offs between low, high, and medium potential heights. These cut-offs would partition the space into intervals that could correspond to excited states similar to the Bernoulli distributed potential. From there, Lifschitz tail behavior could possibly be derived and proved.

The results of this dissertation also depended heavily on the distribution of interval lengths rather than the potential distribution at an individual site. The distribution was derived from the independence of the potential values at each site, essentially becoming a geometric (exponential) distribution. In the results of Chapters 3 and 4, a new interval distribution function could be substituted for the cases where potential distributions are dependent on the values of the potential on other sites. If various sums converge, then different behavior may appear. For example, consider a distribution for interval lengths that depends on a power of the length, i.e. $P[L_i = x] = CL^{-\gamma}$, rather than an exponential function of the length. In this case, the density of states may not be exponential in form, but take the form of a related power law. Likewise, the interacting system studied in Chapter 4 would have different asymptotic behavior as $g\rho \to 0$.

The results of Chapter 4 only describe a small part of the phase diagram for the mean-field ground state in an interacting system with Bernoulli potential. As described in introduction section 1.5, there should be a phase diagram depending on the interaction strength g, particle density ρ, barrier height b, and approximate

proportion of zero potential sites p. The collaborative work [60] was an investigation into this phase diagram. For $g\rho$ small, the mean-field is in a 'Bose glass' state, a spatially fragmented state approximated by a sine wave on many of the longer intervals of zero potential. In the case where both $g\rho$ and pb were fairly large, the kinetic energy term $-\Delta$ should be relatively small and subsequently dropped. In that approximation, if $g\rho >> pb$, the mean-field is approximately constant on the whole space because the repulsive interaction dominates the behavior and spreads the mean-field evenly. For the reverse inequality, there should be a Bose Glass like state.

The mean-field approximation is unlikely to completely describe the true ground state in the full tensor product space but should still approximate the true ground state. For example, consider two particles in a one-dimensional lattice with length L and zero potential at each site. The mean-field ground state for large g would be approximately constant with energy per particle approximately $\frac{g}{2L}$. A test function with lower per particle energy could be attained by each particle having the form of a sine wave supported on a different half of the interval; this state has energy per particle $\frac{\pi^2}{(L/2)^2}$. It is clear the second test function is energetically favorable for large enough g. The mean-field in this case would predict that the states are evenly spread in the space. There should be a result similar to Theorem 4.1.13 of the form:

Let $X_{>\epsilon}(\Phi) = \{x : \sum_{\boldsymbol{x}\in\{0,\dots,L+1\}^{N+1}} |\Phi(\boldsymbol{x})|^2 \delta_{x,x_i} > \frac{\epsilon}{L^{1/2}}\}$ *be the set of points where the average particle density is greater than an average spatial density. For fixed interaction strength* g*, particle number* N*, lattice length* L*, and energy per particle* E'*,*

$$\#(X_{>\epsilon}(\Phi)) \geq \frac{gNL(1-\epsilon^2)^2}{E'} \tag{5.0.2}$$

A result like this would be quite ambitious in the full tensor product space. The result would require bounds on the interaction energy. The interaction energy is determined by the spatial overlap of the states of different particles. It may also require

bounds on the kinetic energy which depend on the support of an individual particle state. This spacial overlap is quite difficult to calculate from a general symmetric multi-particle wave function.

In a given realization of the potential, the system is partitioned into intervals of zero potential and positive b potential. For each interval of zero or positive potential, the mean-field approximation would likely still estimate the particle density of the true ground state in a given subset. The hypothetical result could then describe the nature of the ground state on that interval; whether individual particle states overlap in a mean-field or separate into non-overlapping states. If such results can be proven, a true ground state phase diagram may be feasible.

Appendix A

Throughout this dissertation, the random potential V is represented by a sequence of L independent identically distributed (i.i.d.) Bernoulli random variables. The structure of the ground state and low energy states depend on the lengths of the intervals. These intervals of zero and positive potential are each geometrically distributed variables. If interval lengths were independent, the i-th interval of zero potential with length denoted L_i would have geometric distribution $P[L_i = x] = qp^{x-1}$ for $x = 1,\ 2,\ \ldots$ and the i-th interval of b potential with length $\tilde{L}_i$ would have geometric distribution $P[L_i = x] = pq^{x-1}$. Because the sum of intervals of zero potential and intervals of positive potential must equal L, these intervals are dependent random variables. If L is determined, the number of intervals n is a random variable, possibly with an extra correction interval or two on either end of the system, denoted by the random variable L', and should be thought of as an extra geometric random variable with a maximum value. It is probabilistically much simpler to consider a sequence of $(2)n$ independent geometrically-distributed intervals of zero and positive potential, where there are n of each type of interval. This makes the system size L the random variable. With probability one in the large system limit, these two different probabilistic pictures are the same. The following discussion makes use of standard techniques in probability theory, see [23] for a general reference.

For determined L, the random variable n converges to pqL with probability one. The probability that n deviates from its expectation by δL for $\delta > 0$ is determined by the probabilities

$$P\left[\{\omega : n < \lfloor (pq - \delta)L \rfloor\}\right] \tag{A.0.1}$$

and

$$P\left[\{\omega : n > \lfloor (pq + \delta)L \rfloor\}\right] \tag{A.0.2}$$

which are bounded above by

$$P\left[\{\omega : \sum_{i=1}^{\lfloor (pq-\delta)L\rfloor+1} L_j + \tilde{L_j} + L' > N\}\right] \tag{A.0.3}$$

and

$$P\left[\{\omega : \sum_{i=1}^{\lfloor (pq+\delta)l\rfloor} L_j + \tilde{L_j} < N\}\right] \tag{A.0.4}$$

respectively, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. These bounds are demonstrated by showing the events above are subsets of the events below. Consider a specific realization of the potential with n bounded above and below as in the events above. Adding an extra interval should make the sum of the variables greater than L in the first case; in the second case, removing an interval should be strictly less than L.

Using the exponential Chebyshev inequality and the moment generating functions, the probabilities converge to 0 in the large system limit at a rate on the order of $e^{-\delta L}$. With this convergence rate and the Borel-Cantelli Lemmas, the n converges to pqL with probability one in the limit L goes to infinity. Likewise, consider the case where n is a determined variable L is the random variable equal to a sum of $(2)n$ independent variables, $L = \sum L_j + \tilde{L_j}$. By the Law of Large numbers $L \to n\mathbb{E}[L_j + \tilde{L_j}] + O(\sqrt{n})$ with probability one. Thus, these two formulations are equivalent in the large system limit. That is, these two systems do not deviate significantly in the large system limit. This allows the fixed system size to be approximated by the system with fixed interval number. In the rest of the appendix, this estimate is used to calculate asymptotics for random variables used throughout this dissertation.

For Chapter 2, the limiting behavior of the longest interval of zero potential must go to ∞. Denoted ℓ_L, it is approximately $\log_p(pqL)$ as $L \to \infty$. The following theorem describes the asymptotic behavior of the maximum of a set of i.i.d. random variables [42].

Theorem A.0.5. *The maximum M_n of a set of n i.i.d. random variables X_j is determined by the following relations as $n \to \infty$:*

$$n(1 - P[X_1 \leq u_n(\tau)]) \to \tau \Rightarrow P[M_n \leq u_n(\tau)] \to e^{-\tau} \tag{A.0.6}$$

where $u_n(\tau)$ approximates the limiting behavior of M_n.

To determine the distribution of ℓ_L, the system with fixed length L will be approximated by the system of independent interval lengths, the probability these two systems differ converges to zero. Using this approximation, the maximum is calculated in the independent interval setting. There is a difficult technical issue due to the fact that these random variables must take discrete values which will make using the theorem above more difficult. There is a term in the limit that is an irrational rotation that adds a correction factor which makes a limiting distribution impossible. Instead, there are a family of subsequences that converge to limiting distribution determined by the limit of the irrational rotation of that subsequence. The individual distributions are geometric which implies the maximum should asymptotically behave as $-\log_p(n)$. For each τ, let

$$u_{n_k}(\tau) = -\log_p(n) + \log_p(\tau) \tag{A.0.7}$$

In the precondition above,

$$\begin{aligned} n(1 - P[X_1 \leq u_n(\tau)]) &= p^{\lfloor u_n(\tau) \rfloor - u_n(\tau) + u_n(\tau)} \\ &= \tau p^{\lfloor u_n(\tau) \rfloor - u_n(\tau)} \end{aligned} \tag{A.0.8}$$

The fractional (non-integer) part of $u_n(\tau)$, denoted $\{u_n(\tau)\}$ is an irrational rotation precluding a limit. Since it is on a compact set, there are convergent subsequences $u_{n_k}(\tau)$ if the subsequence $-\{u_{n_k}(\tau)\}$ converges monotonically from above or from below to θ on the unit circle, $\theta \in [0, 1]$. When $\{-\log_p n_k\} \downarrow \theta \in [0, 1)$ the subsequence has the following limiting distribution:

$$P[M_{n_k} \leq u_{n_k}(\tau)] \to \exp[-p^{\tau - \{\tau + \theta\}}] + O(1/n_k) \tag{A.0.9}$$

Therefore, there is no limiting distribution, but the possible sub-sequential limits are parameterized by θ. It follows that $\ell_L \to \infty$ with probability one.

For Chapters 3 and 4, the distribution of the intervals is needed for various estimates. The distribution of these intervals is determined by the Glivanko-Cantelli Theorem which states that the empirical distribution of random variables converges uniformly to the distribution of the random variables. Let us thus fix the number of independent intervals of zero and positive potential to be $2n$. This makes $L = \sum_{i=1}^{n}(L_i + \tilde{L}_i)$ a random variable, the sum of n geometrically distributed intervals of zero potential and n geometrically distributed intervals of positive potential. There intervals of zero potential have lengths L_i with the distribution

$$P[L_i = x] = qp^{x-1} \tag{A.0.10}$$

for integer values of x greater than or equal to one (an interval must have at least one site of zero potential). Likewise, lengths of there intervals of positive potential are distributed according to

$$P[\tilde{L}_i = x] = pq^{x-1} \tag{A.0.11}$$

The variable i indexes the intervals and takes values $1, \ldots, n$. The total system size L is the sum of the random interval lengths. The system size L has expected value

$$\mathbb{E}\left[\sum_{i}^{n}(L_i + \tilde{L}_i)\right] = \frac{n}{pq} \tag{A.0.12}$$

By the Law of Large Numbers [23], with probability one in the limit $n \to \infty$, the difference of both L and $\sum_{L_i > x} L_i$ and their expectations has order less than n. These controls occur with probability one in the limit $n \to \infty$, further referred to as:

$$|L - \frac{n}{pq}| = o(n) \tag{A.0.13}$$

$$|\sum_{L_i \geq x} L_i - \mathbb{E}[\sum_{L_i \geq x} L_i]| = o(n) \tag{A.0.14}$$

where

$$\begin{aligned}\mathbb{E}\left[\sum_{L_i>x} L_i\right] &= n \sum_{y\geq \lfloor x\rfloor+1} yP\left[L_i = y\right] \\ &= \frac{n}{pq}\left(\lfloor x\rfloor q p^{\lfloor x\rfloor+1} + p^{\lfloor x\rfloor+1}\right) \end{aligned} \tag{A.0.15}$$

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