Integrable Nonlinear Relativistic Equations

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A Dissertation Submitted to the Faculty of the

Department of Mathematics

In Partial Fulfillment of the Requirements
For the Degree of

Doctor of Philosophy

In the Graduate College

The University of Arizona

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

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Integrable Nonlinear Relativistic Equations

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Statement by Author

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Dedication

For my grandmother Miriam,
who foretold this moment three decades ago.

Acknowledgments

The last six years were a fascinating journey for me with several life-changing events and perplexing crossroads. It is therefore impossible for me to express gratitude to all the people who contributed to this project. Nevertheless, some personal thanks.

I would like to thank my wonderful family who supported me endlessly throughout this process. Their unfailing love is what allowed me to pursue my dream of studying the world around us.

I wish to thank my advisor Vladimir Zakharov from whom I learned a great deal. He is the one who revealed to me the infinitude of science, an infinitude that is sometimes hidden underneath the surface of the smallest problem. His teaching goes well beyond mathematics, physics or even science in general. As the biggest lesson I learned from him is that even the greatest scientist should be first and foremost a modest person. Without his enlightening guidance on this path, I'd probably still be buried underneath the math books.

There was one door I had to pass through dozens of times on my path. This was the entrance to the office of Lennie Friedlander who, with his endless patience and incorruptible rigor, helped me with every question or when in any doubt. Lennie's analysis classes inspired me to pursue research in this vast beautiful field. Thank you.

It is a pleasure thanking Misha Stepanov for his direction in my RTG project. He was the first who opened my eyes to the inverse transform of Einstein's equation, a step, without which this work would never have come into existence. The numerical work described here is the result of our joint work at the beginning of my graduate studies.

My gratitude to Nicholas Ercolani for his assistance during the last two years. Several sections of this work would not have been written without his help. His profound knowledge of so many distinct fields is a pool of inspiration and great support.

I thank Andrei Lebed for reviewing this work and taking the important role of my physics minor advisor.

Finally, I would like to thank Tom Kennedy, Klaus Lux, Sandra Sutton and Alan Newell for their guidance and help. Alan assisted me when I was standing at the biggest crossroad, and introduced me to my mentor Vladimir Zakharov.

Thank you all.

Table of Contents

TABLE OF CONTENTS—*Continued*

List of Figures

List of Tables

Abstract

This work focuses on three nonlinear relativistic equations: the symmetric Chiral field equation, Einstein's field equation for metrics with two commuting Killing vectors and Einstein's field equation for diagonal metrics that depend on three variables.

The symmetric Chiral field equation is studied using the Zakharov-Mikhailov transform, with which its infinitely many local conservation laws are derived and its solitons on diagonal backgrounds are studied. It is also proven that it is equivalent to a novel equation that poses a fascinating similarity to the Sinh-Gordon equation.

For the $1+1$ Einstein equation the Belinski-Zakharov transformation is explored. It is used to derive explicit formula for N gravitational solitons on arbitrary diagonal background. In particular, the method is used to derive gravitational solitons on the Einstein-Rosen background. The similarities and differences between the attributes of the solitons of the symmetric Chiral field equation and those of the $1+1$ Einstein equation are emphasized, and their origin is pointed out.

For the $1+2$ Einstein equation, new equations describing diagonal metrics are derived and their compatibility is proven. Different gravitational waves are studied that naturally extend the class of Bondi-Pirani-Robinson waves. It is further shown that the Bondi-Pirani-Robinson waves are stable with respect to perturbations of the spacetime. Their stability is closely related to the stability of the Schwarzschild black hole and the relation between the two allows to conjecture about the stability of a wide range of gravitational phenomena. Lastly, a new set of equations that describe weak gravitational waves is derived. This new system of equations is closely and fundamentally connected with the nonlinear Schrödinger equation and can be properly called *the nonlinear Schrödinger-Einstein equations*. A few preliminary solutions are constructed.

Chapter 1
Introduction

1.1 The special theory of relativity

Up until the end of the 19th century, the luminiferous aether was widely accepted as the medium in which light travels. Failed attempts to detect the aether, most notably the Michelson-Morley experiment [1], hinted the physics community that there is something wrong with the notion of aether. Following the work of Hendrik Lorentz, Henri Poincaré and others [2], there were several incomplete attempts to develop new theories of aether that explain the new experimental results.

The situation changed drastically in 1905, when Albert Einstein published the paper [3] on what is now called 'the Special Theory of Relativity'. The paper postulates two fundamental principles (nowadays firmly supported by experiments):

1. The laws of nature are identical in all inertial systems of references (the principle of relativity).

2. In empty space, light always propagates with a definite velocity c which is independent of the state of motion of the emitting body (the principle of the constancy of the speed of light).

These two principles, together with an assumption on the isotropy and homogeneity of space, proved that the notion of the aether was superfluous, and provided theoretical results that are fully compatible with experiments.

In special relativity, space and time ($\equiv$ spacetime) are considered to be a continuum composed of events, where each event occurs at a certain location (x, y, z) in space at an instant of time t. Consider a ray of light emitted from the point (x_1, y_1, z_1)

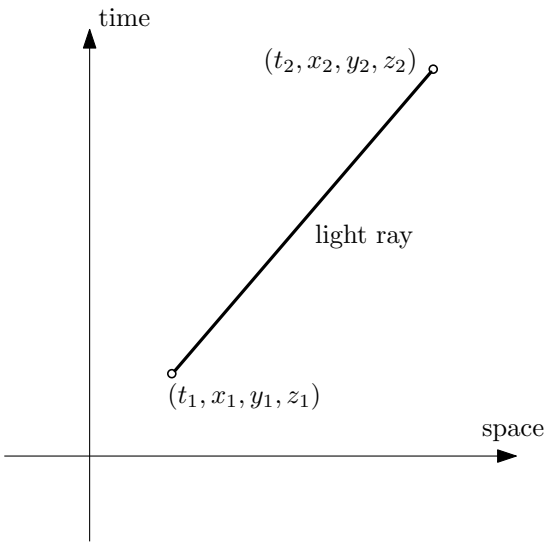

FIGURE 1.1. The path traced by a light ray emitted from the point (x_1, y_1, z_1) at time t_1 and arriving to (x_2, y_2, z_2) at time t_2 (in flat spacetime)

at time t_1 and arriving to (x_2, y_2, z_2) at time t_2 (figure (1.1)). Euclidean geometry implies that

$$c(dt) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{1.1.1}$$

where $dt = t_2 - t_1$, $dx = x_2 - x_1$, $dy = y_2 - y_1$ and $dz = z_2 - z_1$. Introducing the notation

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z \tag{1.1.2}$$

and the matrix

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.1.3}$$

allows one to write Eq. (1.1.1) in the abbreviated form

$$ds^2 = 0 \tag{1.1.4}$$

where ds^2 is the *spacetime interval*, defined by

$$\begin{aligned} ds^2 &:= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \end{aligned} \tag{1.1.5}$$

Throughout this work with the only exception of chapter 4, Einstein's summation convention will follow on Greek indices $\mu, \nu, \ldots$, assuming to take values $0, 1, 2, 3$.

The principle of relativity and the principle of the constancy of the speed of light reveal that Eq. (1.1.4) must hold for all inertial observers. Thus in the theory of relativity, the spacetime interval ds^2 is of crucial importance. It can be shown that the sign of the spacetime interval divides spacetime into three parts which determine whether two events can be causally related (see figure (1.2)):

- Time-like: This includes all events for which $ds^2 < 0$. Time-like events are causally related and lie inside the light-cone.

- Space-like: This includes all events for which $ds^2 > 0$. Space-like events are not causally related and lie outside of the light-cone.

- Light-like: This includes all events for which $ds^2 = 0$. The spatial distance between two light-like events is exactly enough for a light-ray to connect them. This means that light-like events are only causally related by light-rays (or other massless particles in general).

The implications of the special theory of relativity go much further than this. Indeed, the notions of simultaneity, absolute space, absolute time, mass and energy, together with the Galilean transformations for velocity were all altered as a result of the two principles in the foundations of the special theory of relativity. This is beyond the scope of the current work and the reader should refer to [4] for a discussion of these topics.

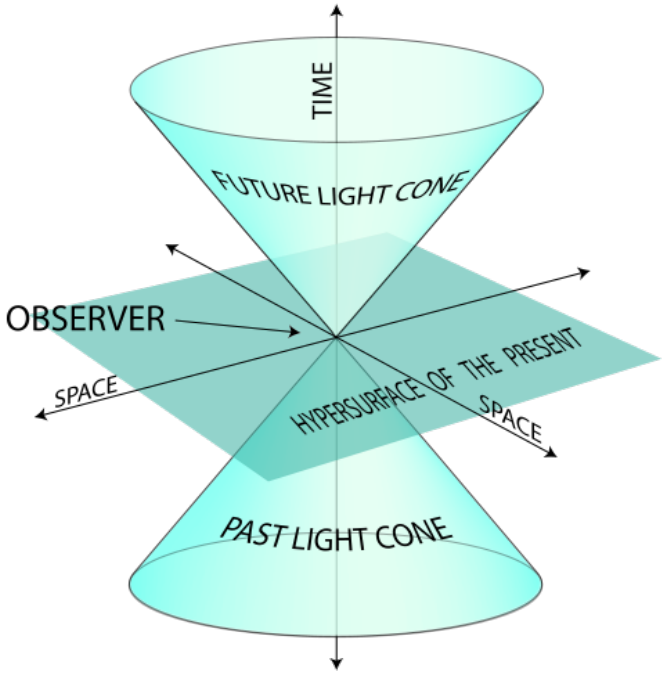

FIGURE 1.2. A light-cone centered at an observer separates spacetime into three causally disjoint subsets. The light-cone itself is the set of all light-like events with $ds^2 = 0$. The interior of the light-cone $ds^2 < 0$ (time-like events) corresponds to events which can be causally related by a material body to the observer. The exterior of the light-cone $ds^2 > 0$ are space-like events, that don't have any causal relation to the observer.

1.2 The general theory of relativity

The general theory of relativity was developed by Albert Einstein between 1907 and 1916, after he completed his special theory of relativity. Einstein realized that Newton's theory of gravity is not consistent with special relativity since it invokes instantaneous influence of one body on another. This motivated him to seek a new theory of gravity [4]. However, instead of modifying Newton's theory of gravitation to make it compatible with special relativity, he decided to follow a completely different path.

The key idea was that all bodies are influenced by gravity. In fact, all bodies are influenced by gravity in precisely the same way, independently of the their com-

position. This notion, known as *the equivalence principle* is manifested in Newton's law of universal gravitation by the statement that the gravitational force on a body is proportional to its inertial mass, and thus as long as only gravity is involved, the mass of a body does not affect its dynamics. Therefore, instead of describing gravity as a force 'acting at a distance', a notion that even Newton himself found to be unsatisfactory, the equivalence principle suggests that gravity might have to do with the structure of spacetime itself and for this reason influences all bodies in an equivalent way. This led Einstein to dismiss the notion of a 'gravitational force'. Instead, his theory of general relativity postulates that:

1. Spacetime is a four-dimensional Lorentzian manifold $(M, g_{\mu\nu})$. This is a manifold M equipped with a pseudo-Riemannian metric $g_{\mu\nu}$ that is not positive-definite, but instead has signature $(-+++)$. The coordinates of spacetime are denoted by $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Here and for the rest of this work, the speed of light is taken to be unity $c = 1$. The spacetime interval is then given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.2.1}$$

2. Spacetime curves in the presence of energy and/or matter according to *Einstein's field equation*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (8\pi G)\, T_{\mu\nu} \tag{1.2.2}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric of spacetime, $R = R^\mu_\mu$ is the scalar curvature, G is the gravitational constant[1] and $T_{\mu\nu}$ is the stress-energy tensor that represents the distribution of matter and energy in spacetime.

This work only investigates Einstein's equation in empty space ($T_{\mu\nu} = 0$), and in this case Eq. (1.2.2) reduces to the *Einstein vacuum equation*

$$R_{\mu\nu} = 0 \tag{1.2.3}$$

[1] $G = 6.67384(80) \times 10^{-11}\, m^3\, kg^{-1}\, s^{-2}$

3. In the absence of any forces (remember, gravity is not considered to be a force anymore!) matter travels along geodesics. Geodesics are the curved-space analogs of straight lines and satisfy the *geodesic equation*

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.2.4}$$

where $x^\mu = x^\mu(\tau)$ is the four-position of the particle in spacetime, τ is its proper time ($\equiv$ time as measured by an observer traveling with the particle) and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols for the Levi-Civita connection of $g_{\mu\nu}$. The four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ satisfies

$$u^\mu u_\mu = -1 \tag{1.2.5}$$

and therefore the world-line of a material particle ($\equiv$ the path $x^\mu(\tau)$ traces in spacetime) is always timelike.

Rather similarly, light rays travel along null-geodesics. If $x^\mu = x^\mu(s)$ is a path of a light ray, it satisfies the same geodesic Eq. (1.2.4) (upon substitution $\tau \leftrightarrow s$, as it is common to use a different notation for the parameterization of null-geodesics). Null-geodesics are fundamentally different than geodesics in their norm, as the world-line of a light ray is always *null* and therefore satisfies

$$u^\mu u_\mu = 0 \tag{1.2.6}$$

instead of Eq. (1.2.5) for a material particle.

In many practical cases, the stress-energy tensor $T_{\mu\nu}$ is known and one solves Einstein's equation for the metric tensor $g_{\mu\nu}$. This is a symmetric tensor, which in four-dimensional spacetime has ten unknown entries, and generally depends on the four spacetime coordinates $x^\mu = (t, x, y, z)$. The simple form of Einstein's Eq. (1.2.2) might give the impression that it is a simple equation for $g_{\mu\nu}$, but this is far from being the case. Both the Ricci curvature tensor $R_{\mu\nu}$ and the scalar curvature R

depend nonlinearly on the Christoffel symbols that in turn depend nonlinearly on the metric $g_{\mu\nu}$. The Ricci curvature tensor is defined by

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} \tag{1.2.7}$$

The Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ depend on the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \tag{1.2.8}$$

This reveals that actually, the Einstein's field Eq. (1.2.2) is a highly nonlinear set of ten second order partial differential equations for the entries of the metric tensor $g_{\mu\nu}$. It can be shown [5] that four of the ten degrees of freedom can be dismissed by the use of gauge symmetry, making the Einstein equation a nonlinear system of ten partial differential equations for six unknowns depending on four variables. This is an overdetermined system, but its compatibility is guaranteed by the Bianchi identities [5]. We will come back to this important issue in the last chapter.

The vacuum Einstein field Eq. (1.2.3) can also be derived using the principle of least action. This work was done by David Hilbert himself [6] and was published before Einstein's work on general relativity. Define the Einstein-Hilbert action

$$S = \frac{1}{2} \int_{M} R\sqrt{-g} d^4x \tag{1.2.9}$$

where R is the scalar curvature tensor, $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and the integration is taken over the whole spacetime M. The variational derivative of the action (1.2.9) with respect to the spacetime metric

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{1.2.10}$$

is equivalent to the Einstein vacuum Eq. (1.2.3). This fact will be used later in this work to find a Hamiltonian structure for Einstein's equation.

1.3 Integrability

In mathematics and physics various distinct notions of integrability are used. These distinct notions are the result of a number of characteristics that certain differential equations have in common, such as Lax pairs, Bäcklund transformations, infinite number of conservation laws, existence of soliton solutions and more. However, it is still not clear what are the precise conditions under which these characteristics are equivalent [7]. The goal of this section is to define and differentiate the various notions of integrability that will be used, and in this way to also clarify the title of this work. The definitions used here are in no way sufficient to describe all the cases of integrability discussed in the literature. They are only here to ensure that this work will be self-contained.

Consider a nonlinear partial differential equation

$$F[u] = 0 \tag{1.3.1}$$

for the function $u = u(t, x)$.

Definition 1.3.2. The nonlinear PDE (1.3.1) is called *Lax-integrable*, if there exists a pair of linear PDE's,

$$L_i(\psi; u; t, x, \lambda) = 0 \qquad\qquad (i = 1, 2) \tag{1.3.3}$$

such that the system (1.3.3) is compatible for ψ if and only if Eq. (1.3.1) is satisfied. In such a case the system (1.3.3) is called a *Lax pair* for Eq. (1.3.1) with *Lax operators* L_i. The Lax operators L_i are assumed to depend on u and on an additional (spectral) parameter λ.

Given a PDE (1.3.1) it is sometimes the case that it has certain conservation laws or integrals of motion. A *conservation law* for Eq. (1.3.1) is an equation of the form

$$\frac{\partial}{\partial t}T[u] + \frac{\partial}{\partial x}X[u] = 0 \tag{1.3.4}$$

that is satisfied whenever u is a solution of Eq. (1.3.1). An *integral of motion (constant of motion)* is a quantity $I[u]$ such that

$$\frac{d}{dt}I[u] = 0 \tag{1.3.5}$$

whenever u is a solution of Eq. (1.3.1). Namely, $I[u]$ is independent of time. Every conservation law gives rise to an integral of motion once certain boundary conditions are met. To see this, integrate Eq. (3.4.6) with respect to x over $\mathbb{R}$. If $X[u] \to 0$ as $x \to \pm\infty$ then one obtains the integral of motion

$$I[u] = \int_{-\infty}^{\infty} T[u]dx \tag{1.3.6}$$

satisfying Eq. (2.2.6).

Definition 1.3.7. The nonlinear PDE (1.3.1) is called *Liouville-integrable* if it has infinitely many conservation laws. We will further say that it is *completely integrable* if its integrals of motion completely determine the evolution of the system at any time t [1]. Otherwise, the system is only *partially integrable*.

Definition 1.3.8. Any PDE (1.3.1) that is either Lax-integrable or Liouville-integrable (completely or partially) will be referred to as an *integrable system*.

So for example, the famous Korteweg-de-Vries equation was proven to be Lax-integrable in the work [8]. Later, it was proven to be completely integrable in [9]. In both cases, it deserves the title 'an integrable system.'

One of the most important aspects of integrability of nonlinear partial differential equations is the existence of a certain class of exact solutions called solitons. We will not attempt to give a precise definition of solitons here, but instead, give a loose definition that distinguishes two types of solitons:

[1] More precisely, if there exists a maximal set of Poisson commuting integrals of motion. The reader may consult [7] for a broader discussion of complete integrability.

- A *(mathematical) soliton* is any solution of a nonlinear PDE (1.3.1) using the dressing method that corresponds to a pole of the dressing matrix (see section 1.6 for an example and a discussion of the dressing method).
- A *(physical) soliton* is an object that has the following properties [10]:
  - It is of a permanent form
  - It is localized within a region of space
  - It can interact with other solitons, but emerges from the collision unchanged (except for a phase shift)

In chapter 2 examples of mathematical and physical solitons are provided. Chapter 3 discusses mathematical solitons that are *not* physical solitons, and thus proves that the two notions are not equivalent.

1.4 Chiral fields

In this section we take a short detour from the theory of relativity and integrability to discuss other closely akin relativistic systems: chiral fields and in particular the principal chiral field.

Chiral fields on Lie groups represent equivalence classes of integrable relativistic two-dimensional systems. Zakharov and Mikhailov [11, 12] showed that classical spinor fields (i.e. the classical analogs of fermion fields prior to quantization) are connected with each such system. Two examples of such classical spinor fields are the Nambu-Jona-Lasinio model [13] and the Vaks-Larkin model [14], both appearing naturally in the theory of solid state physics.

Two-dimensional systems may be expressed in spacetime coordinates t and x. However, it is convenient to use light-cone variables, defined by the transformation

$$t = \zeta - \eta \qquad x = \zeta + \eta \tag{1.4.1}$$

Consider relativistic systems in two dimensions that can be expressed as the compatibility condition of the following overdetermined linear system

$$\begin{aligned} \partial_\zeta \psi &= U\psi \\ \partial_\eta \psi &= V\psi \end{aligned} \tag{1.4.2}$$

where U, V are rational functions of the parameter λ,

$$\begin{aligned} U &= U_0 + \sum_{s=1}^{N} \frac{U_s}{\lambda - \lambda_s} \\ V &= V_0 + \sum_{s=1}^{N} \frac{V_s}{\lambda + \lambda_s} \end{aligned} \tag{1.4.3}$$

the poles λ_s are arbitrary complex constants, $\lambda_s + \lambda_{\tilde{s}} \neq 0$ and $\psi = \psi(\lambda; \zeta, \eta)$. In general U, V and ψ are complex square matrices of any dimension, but in this work they are always assumed to be 2×2 matrices.

The compatibility conditions for Eqs. (1.4.2) are of the form

$$\begin{aligned} \partial_\eta U_0 - \partial_\zeta V_0 + [U_0, V_0] &= 0 \\ \partial_\eta U_s + \left[U_s, \Phi_s^U\right] &= 0 \\ \partial_\zeta V_s + \left[V_s, \Phi_s^V\right] &= 0 \end{aligned} \tag{1.4.4}$$

where

$$\begin{aligned} \Phi_s^U &= V_0 + \sum_{\tilde{s}=1}^{N} \frac{V_m}{\lambda_{\tilde{s}} + \lambda_s} \\ \Phi_s^V &= U_0 - \sum_{\tilde{s}=1}^{N} \frac{U_m}{\lambda_{\tilde{s}} + \lambda_s} \end{aligned} \tag{1.4.5}$$

Since Eqs. (1.4.4) only involve linear operations and commutators, one may assume that the matrices U and V belong to a Lie algebra $\mathfrak{g}$. Moreover, if the initial conditions are given in some subalgebra $\mathfrak{g}' \subseteq \mathfrak{g}$, then the process of evolution will keep the fields in the same subalgebra $\mathfrak{g}'$. The problem of restricting an integrable system to a given

invariant manifold is called *the reduction problem*, and indeed the restriction of Eqs. (1.4.4) to a certain subalgebra is an example of such a reduction problem.

Let $\Omega(\lambda;\zeta,\eta)$ be any non-degenerate matrix function. The change of variables

$$\psi \to \Omega\psi \tag{1.4.6}$$

together with

$$\begin{aligned} U &\to (\partial_\zeta\Omega)\Omega^{-1} + \Omega U\Omega^{-1} \\ V &\to (\partial_\eta\Omega)\Omega^{-1} + \Omega V\Omega^{-1} \end{aligned} \tag{1.4.7}$$

keeps the form of Eqs. (1.4.2) intact. This justifies naming such a transformation a *gauge transformation*. A choice of gauge matrix Ω for which $U_0 = V_0 = 0$ is called a *canonical gauge*.

The simplest case of a relativistic integrable system that raises from Eqs. (1.4.2) is when U and V have a single simple pole. In this case, one may assume that the pole is the complex unity $\lambda = 1$ for simplicity, and then Eqs. (1.4.2) in canonical gauge are

$$\begin{aligned} \partial_\zeta\psi &= \frac{A}{\lambda-1}\psi \\ \partial_\eta\psi &= \frac{B}{\lambda+1}\psi \end{aligned} \tag{1.4.8}$$

where $A = U_1$ and $B = V_1$. This is *the Zakharov-Mikhailov system* and its compatibility conditions (1.4.4) are

$$\begin{aligned} 0 &= \partial_\eta A + \partial_\zeta B + [A,B] \\ 0 &= \partial_\eta A - \partial_\zeta B \end{aligned} \tag{1.4.9}$$

By setting $\lambda = 0$ in Eqs. (1.4.8) and using the compatibility conditions (1.4.9) it follows that the matrix $g(\zeta,\eta) = \psi\,|_{\lambda=0}$ satisfies the matrix equation

$$(g_{,\zeta}g^{-1})_{,\eta} + (g_{,\eta}g^{-1})_{,\zeta} = 0 \tag{1.4.10}$$

where the commas represent partial differentiation $g_{,\zeta} \equiv \partial_\zeta g$. Eq. (1.4.10) is called *the principal chiral field equation* (on the group under consideration). Various reductions of the principal chiral field equation were studied in the literature. In particular, the reduction on the special unitary group $SU(N)$ was studied in [11] and is connected to the Nambu-Jona-Lasinio model. The reductions on the real symplectic group $Sp(2N, \mathbb{R})$ and on the special orthogonal group $SO(N)$ were studied in [12] and are connected to the Vaks-Larkin model. A very extensive study of the algebraic structure of chiral models was given in [15].

Note that Eqs. (1.4.8) are a Lax pair for the principal chiral field Eq. (1.4.10). One may thus use the powerful dressing method on the Zakharov-Mikhailov Eqs. (1.4.8) to find soliton solutions of the principal chiral field Eq. (1.4.10). However, it is shown in the next section that this is merely a special case of the Belinski-Zakharov transform which will be studied thoroughly there. In particular, we shall see that if one further assume that the matrix g is real-valued and symmetric, Eq. (1.4.10) is a very special case of the Einstein vacuum Eq. (1.2.3).

1.5 The Belinski-Zakharov transform

The Belinski-Zakharov transform is a transformation of the nonlinear vacuum Einstein equation to an equivalent linear set of equations.

Due to the strong nonlinearity of the Einstein field Eq. (1.2.2), only few physically relevant exact solutions are known [16]. In 1978, Belinski and Zakharov worked on this problem [17], [18] and developed what is now commonly called *the Belinski-Zakharov transform*. The Belinski-Zakharov transform gives a practical way to generate new solutions of Einstein's vacuum Eq. (1.2.3) from known solutions. This technique can be used for fields depending on time and one spatial coordinate [17], or alternatively for stationary axially symmetric fields [18]. Although these two cases are fundamentally different on physical grounds, one may use a generic tensorial notation that

treats both cases simultaneously [19]. Such a notation will be used next, although the careful reader should note that the notation used here will be slightly different than in [19].

Consider a spacetime interval of a block diagonal form

$$ds^2 = g_{ij}dx^i dx^j + g_{ab}dx^a dx^b \tag{1.5.1}$$

where we use the summation convention on $i, j, k, \cdots = 0, 3$; $a, b, \cdots = 1, 2$; and assume g_{ij} and g_{ab} depend only on the coordinates x^k. By using a proper change of coordinates, one can always cast the matrix g_{ij} to the conformally flat form $g_{ij} = f\eta_{ij}$ where $f > 0$ and

$$\eta_{ij} = \begin{bmatrix} -e & 0 \\ 0 & 1 \end{bmatrix} \tag{1.5.2}$$

is a constant matrix with $e = \pm 1$. The notation used here for η_{ij} is consistent with the one used in section 1.1. $\eta_{\mu\nu}$ is the metric of a flat Minkowski 4-dimensional spacetime, while η_{ij} is the metric of its 2-dimensional subspace. η^{ij} denotes the inverse of η_{ij}.

It will be convenient to denote by g the 2×2 matrix with components g_{ab},

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{1.5.3}$$

and for its determinant, the notation

$$\det g_{ab} = e\alpha^2 \tag{1.5.4}$$

Thus in matrix form, the metric is

$$g_{\mu\nu} = \begin{bmatrix} -ef & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{bmatrix} \tag{1.5.5}$$

To keep the discussion more definite, the reader should keep two concrete physical examples in mind:

- *Non-stationary spacetimes*: These are metrics that depend on time and one spatial coordinates [17], and are typically expressed using Cartesian coordinates in which $x^i = \{t, z\}$ and $x^a = \{x, y\}$. Here $e = 1$, and the spacetime interval is

$$ds^2 = f(t, z)(-dt^2 + dz^2) + g_{ab}(t, z)dx^a dx^b \tag{1.5.6}$$

  Such spacetimes describe cosmological solutions of general relativity, gravitational waves and their interactions.

- *Stationary and axially-symmetric spacetimes*: These are metrics that depend on two spatial coordinates [18] and can be conveniently expressed in cylindrical coordinates $x^i = \{\rho, z\}$, $x^a = \{t, \varphi\}$ with $\varepsilon = -1$. Then the spacetime interval is

$$ds^2 = f(\rho, z)(d\rho^2 + dz^2) + g_{ab}(\rho, z)dx^a dx^b \tag{1.5.7}$$

  Such spacetimes describe fields of stationary compact objects (e.g. black holes).

The forms (1.5.6) and (1.5.7) are of crucial importance in the general theory of relativity, as they include many important solutions of Einstein's equation as special cases. In particular, they include the Bondi-Pirani-Robinson plane gravitational waves [20], cylindrical-wave solutions as the Einstein-Rosen waves [21], homogeneous cosmological models of Bianchi types I through VII [16], the Weyl's axially symmetric solution [16], the Kasner cosmological solution [22] and most importantly the Schwarzschild and Kerr solutions together with their NUT-generalizations [23, 24, 25, 26]. The reader is encouraged to consult [16] for a thorough discussion of all solutions that belong to these classes.

In order to proceed and reduce Einstein's vacuum Eq. (1.2.3), one needs to compute the Ricci curvature tensor (1.2.7) in terms of f and g_{ab}. The reader may consult appendix B for the values of the various curvature tensors of the metric (1.5.1), where one can see that the complete system of Einstein's vacuum Eq. (1.2.3) decomposes

into two sets of equations. The equation $R_{ab} = 0$ can be written as a single tensorial (matrix) equation

$$\eta^{ij} \left(\alpha g_{ac,i} g^{cb} \right)_{,j} = 0 \tag{1.5.8}$$

where the commas denote the (usual) partial differentiation with respect to the variable in the subscript. Eq. (1.5.8) will be used extensively in this work, and we shall refer to it as the *reduced Einstein equation*. The trace of the reduced Einstein equation is the *trace equation*:

$$\eta^{ij} \alpha_{,ij} = 0 \tag{1.5.9}$$

To fully comprehend the meaning of Eqs. (1.5.8) and (1.5.9), it is useful to consider again the two separate physical scenarios. When considering these equations for the first example depicted above, of a non-stationary spacetime $\varepsilon = 1$, Eq. (1.5.8) is

$$(\alpha g_{,t} g^{-1})_{,t} - (\alpha g_{,z} g^{-1})_{,z} = 0 \tag{1.5.10}$$

and α satisfies the wave equation

$$\alpha_{,tt} - \alpha_{,zz} = 0 \tag{1.5.11}$$

Quite differently, for the stationary axially symmetric case $\varepsilon = -1$, Eq. (1.5.8) is

$$(\alpha g_{,\rho} g^{-1})_{,\rho} + (\alpha g_{,z} g^{-1})_{,z} = 0 \tag{1.5.12}$$

and α satisfies Laplace's equation

$$\alpha_{,\rho\rho} + \alpha_{,zz} = 0 \tag{1.5.13}$$

Despite the fact that the stationary metric can always be transformed to a non-stationary metric (and vice-versa) by a simple complex transformation, the last two pairs of equations reveal a fundamental mathematical difference between the two cases. A non-stationary metric satisfies a hyperbolic equation with a trace that satisfies the canonical example of such equation: *the wave equation*. On the other hand,

a stationary metric satisfies an elliptic equation with a trace that satisfies *Laplace's equation.*

It will prove very convenient to express Eq. (1.5.8) as a system of two first-order matrix equations. If one defines the two matrices A_i by

$$A_i = e\alpha g_{,i} g^{-1} \tag{1.5.14}$$

then it is evident that the reduced Einstein Eq. (1.5.8) is

$$\eta^{ij} A_{i,j} = 0 \tag{1.5.15}$$

In appendix C, we prove that the integrability condition for Eqs. (1.5.14) is

$$\varepsilon^{ij} \left(A_{i,j} + \frac{1}{\alpha} A_i A_j - \frac{1}{\alpha} \alpha_{,j} A_i \right) = 0 \tag{1.5.16}$$

The second set of equations are

$$\eta^{ij} \left[(\ln f)_{,ij} + (\ln \alpha)_{,ij} + \frac{1}{4} g^{ab} g^{cd} g_{bc,i} g_{da,j} \right] = 0 \tag{1.5.17}$$

and

$$\eta^{jk} \left[\frac{1}{2} (\ln f)_{,i} (\ln \alpha)_{,k} + \frac{1}{2} (\ln f)_{,k} (\ln \alpha)_{,i} \right] - \frac{1}{2} \delta_i^j \eta^{kl} (\ln f)_{,k} (\ln \alpha)_{,l} \tag{1.5.18}$$
$$-\eta^{jk} (\ln \alpha)_{,ik} + \frac{1}{2} \delta_i^j \eta^{kl} (\ln \alpha)_{,kl} - \frac{1}{4\alpha^2} \eta^{jk} Tr(A_i A_k) + \frac{1}{8\alpha^2} \delta_i^j \eta^{kl} Tr(A_k A_l) = 0$$

Eq. (1.5.17) is just a second-order scalar equation for the function f, while Eq. (1.5.18) is a first-order matrix equation for f. It is easy to verify that the trace of Eq. (1.5.18) vanishes identically, and therefore Eq. (1.5.18) only consists of two independent relations for f. Eq. (1.5.17) is not a new independent equation, as can be proven using the Bianchi identities. In fact, one can show that Eqs. (1.5.18) are compatible precisely when Eq. (1.5.17) is satisfied.

Since Eq. (1.5.8) is decoupled from Eqs. (1.5.17) and (1.5.18), we can focus our attention on Eq. (1.5.8). In fact, we will later see that once the matrix g is known,

the function f can be easily computed by quadratures from Eqs. (1.5.18). This means that the real challenge in solving Einstein's equation lies in solving the highly nonlinear Eq. (1.5.8) for the matrix g. This is the focus of the rest of this section.

Before proceeding, note that there is one case that deserves a special attention when treating gravitational phenomena. If the determinant of g is fixed, α may be canceled out from Eq. (1.5.12), reducing it to the known principal chiral field equation considered in section 1.4. It follows then from the two relations in Eq. (1.5.18) that in this case g is a constant matrix. In such a case the function f can be determined from Eq. (1.5.17) which reduces to the wave equation with d'Alembert solution $f = \exp[f_1(x^0 + x^3) + f_2(x^0 - x^3)]$. However, a simple coordinate transformation reduces f to a constant as well. In other words, *any solution of the reduced Einstein equation with $\alpha =$ constant is diffeomorphic to the trivial Minkowski (flat) spacetime.* This shows that the principal chiral field Eq. (1.4.10) is a special case of the reduced Einstein Eq. (1.5.8) corresponding to $\alpha = 1$ and has no physical implications in the theory of gravity.

We proceed to solve the reduced Einstein Eq. (1.5.8) using the *inverse scattering (spectral) transform.* This is done by representing Eqs. (1.5.15) and (1.5.16) as the compatibility condition of an overdetermined linear system of matrix equations that depends on an additional spectral parameter λ. Intuitively, one can think of this process as replacing a *nonlinear* equation in two variables x^k with *two linear* equations in three variables λ and x^k. The introduction of the new variable λ (and consequently increased dimensionality) is the price one has to pay to eliminate the nonlinearity. With this in mind, we proceed by introducing some notation.

Let ε be the 2×2 matrix

$$\varepsilon_{ab} = \varepsilon^{ab} = \varepsilon_{ij} = \varepsilon^{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.5.19}$$

and let β be a 'conjugate' function to α, in the sense that it satisfies the following

relation

$$\partial_i \beta = -e\eta_{ij}\varepsilon^{jk}\partial_k \alpha \tag{1.5.20}$$

It can easily be verified that Eq. (1.5.20) implies that β is a second independent solution of the trace Eq. (1.5.9). We also introduce the following linear operators

$$D_i = \partial_i + P_i \partial_\lambda \tag{1.5.21}$$

where

$$P_i = (2e\lambda)\Lambda_i{}^j \partial_j \alpha \tag{1.5.22}$$

and the matrix $\Lambda_i{}^j$ is defined by

$$\Lambda_i{}^j = \frac{\lambda \varepsilon_i{}^j - \alpha \delta_i^j}{\lambda^2 - e\alpha^2} \tag{1.5.23}$$

Here $\varepsilon_i{}^j = \eta_{ik}\varepsilon^{kj}$ and δ_i^j is the Kronecker delta. This means that when written explicitly, the operators under consideration are

$$D_i = \partial_i - \frac{2\lambda(\lambda\beta_{,i} + e\alpha\alpha_{,i})}{\lambda^2 - e\alpha^2}\partial_\lambda \tag{1.5.24}$$

and it is shown in appendix D that these operators D_i commute,

$$[D_i, D_j] = 0 \tag{1.5.25}$$

Consider the system of the linear *Belinski-Zakharov* equations

$$D_i \psi = \Lambda_i{}^j A_j \psi \tag{1.5.26}$$

for the complex matrix function $\psi(\lambda; x^k)$, where the matrices A_i are real and do not depend of the parameter λ. In it easy to verify that Eqs. (1.5.26) are compatible if and only if Eqs. (1.5.15) and (1.5.16) are both satisfied. This means that the Belinski-Zakharov Eqs. (1.5.26) are solvable when the matrix g satisfies the reduced Einstein Eq. (1.5.8) and therefore the reduced Einstein Eq. (1.5.8) is Lax-integrable. In fact, a solution $\psi(\lambda; x^k)$ of Eqs. (1.5.26) automatically yields a solution $g(x^k)$ of

Eq. (1.5.8). To see this, we notice that by Eq. (1.5.21) and Eq. (4.2.8) the limit $\lambda \to 0$ gives $D_i \to \partial_i$ and $\Lambda_i^{\ j} \to \frac{e}{\alpha}\delta_i^j$. Therefore, taking the limit $\lambda \to 0$ in Eqs. (1.5.26) gives

$$\psi_{,i}\psi^{-1} \mid_{\lambda=0} = g_{,i}g^{-1} \tag{1.5.27}$$

and the matrix g has the same logarithm derivative as the matrix function $\psi \mid_{\lambda=0}$. Therefore,

$$g(x^k) = \psi(0; x^k)U \tag{1.5.28}$$

where U is a constant matrix with $\det U = 1$. The appearance of U is a remnant of the symmetry $g \to gU$ of the reduced Einstein Eq. (1.5.8) and for the rest of this work U will be taken to be the identity matrix for simplicity. One should always keep in mind however, that we can obtain more solutions by multiplying a given solution g from the right by such a special matrix U.

We may therefore complete the integration of the reduced Einstein Eq. (1.5.8) by solving Eqs. (1.5.26). Since Eqs. (1.5.26) depends on g itself through A_i, one should not hope to solve Eqs. (1.5.26) in full generality. Nevertheless, one can obtain a solution of Eqs. (1.5.26) by assuming the knowledge of one particular solution $g^{(0)}$ of Eq. (1.5.8) using the *dressing method*.

Let $g^{(0)}(x^k)$ be a particular solution of the *reduced Einstein* Eq. (1.5.8). In the following $g^{(0)}$ will be called the *seed (background) metric*. Using definition (1.5.14), one can compute the matrices $A_i^{(0)}$ and integrate Eqs. (1.5.26) to obtain the generating function $\psi^{(0)}(\lambda; x^k)$. We substitute

$$\psi = \chi\psi^{(0)} \tag{1.5.29}$$

into Eqs. (1.5.26), where χ is a 2×2 matrix called the *dressing matrix*. Since $\psi^{(0)}$ satisfies Eqs. (1.5.26), the dressing matrix χ satisfies the *dressing equations*,

$$D_i\chi = \Lambda_i^{\ j}\left(A_j\chi - \chi A_j^{(0)}\right) \tag{1.5.30}$$

Recall however, that for physical reasons the matrix g must be real and symmetric. This can be guaranteed by imposing two additional constraints on the generating function ψ and on the dressing matrix χ. To ensure g is real, we require both χ and ψ to be real on the λ-real axis,

$$\overline{\chi}(\overline{\lambda}) = \chi(\lambda), \qquad \overline{\psi}(\overline{\lambda}) = \psi(\lambda) \tag{1.5.31}$$

The second condition to assure that g is symmetric is less obvious. If χ satisfies Eqs. (1.5.30) then a newly defined matrix

$$\chi' = g\left(\left(\chi(e\alpha^2/\lambda)\right)^{-1}\right)^T (g^{(0)})^{-1} \tag{1.5.32}$$

also satisfies Eqs. (1.5.30) when g is symmetric. We can therefore guarantee that g is symmetric if we require $\chi' = \chi$, or more explicitly the *symmetry constraint* is

$$g = \chi(\lambda) g^{(0)} \chi^T(e\alpha^2/\lambda) \tag{1.5.33}$$

If we further require that the dressing matrix tend to the unit matrix at infinity of the λ-plane,

$$\chi \xrightarrow{\lambda\to\infty} I \tag{1.5.34}$$

we obtain from Eq. (1.5.28) that the new solution is

$$g_{\text{non-ph}} = \chi(0) g^{(0)} \tag{1.5.35}$$

A different normalization of χ at infinity is equivalent to a modification of the constant matrix U in Eq. (1.5.28).

There is a caveat in the above scheme that the reader should be aware of, which is the reason for the subscript 'non-physical' in Eq. (1.5.35). The solution g must of course satisfy $\det g = e\alpha^2$. We assumed that the function α is the same for both the determinant of g and the determinant of the particular solution $g^{(0)}$. From Eq. (1.5.35), we see that this means that one must impose yet another condition on the dressing matrix,

$$\det \chi(0) = 1 \tag{1.5.36}$$

However, for practical reasons it is easier not to worry about this condition. Instead, we normalize the solution to obtain the *real (physical) solution* g from the non-physical solution $g_{\text{non-ph}}$ according to

$$g = \frac{e\alpha}{(e \det g_{\text{non-ph}})^{1/2}} g_{\text{non-ph}} \tag{1.5.37}$$

at the end of the procedure. The matrix g again satisfies the reduced Einstein Eq. (1.5.8), but this time has the right determinant $\det g = e\alpha^2$.

To summarize, one may generate a solution of the reduced Einstein Eq. (1.5.8) from a given known solution $g^{(0)}$ by solving the dressing Eqs. (1.5.30) for the dressing matrix χ, where χ is constrained by Eqs. (1.5.31) and (1.5.33).

1.6 Construction of N gravitational solitons

Soliton solutions of the reduced Einstein's Eq. (1.5.8) correspond to the case in which the determinant of the dressing matrix χ has poles (see section 1.3). Therefore a purely solitonic solution corresponds to a dressing matrix χ that has the form of a rational function in the parameter λ with a finite number of poles N, which we take to be simple poles. A similar analysis can be done in the more general case of poles of arbitrary order but will not be given here.

There are three more conditions that one needs to consider to get the explicit form of χ. The first is the symmetry constraint in Eq. (1.5.33). Taking its determinant and recalling that the matrices g and $g^{(0)}$ are independent of λ, one sees that there is a certain symmetry with respect to the circle $|\lambda|^2 = \alpha^2$. To guarantee the analyticity of the right-hand side of Eq. (1.5.33) (as the left-hand side is analytic), each pole μ_s of χ must have a corresponding zero of χ^{-1} at α^2/μ_s. Therefore this shows that if χ has N poles at $\lambda = \mu_s(x^k)$ for $k = 1, 2, \ldots, N$, then χ^{-1} has N poles at $\lambda = \alpha^2/\mu_s(x^k)$. Moreover, from the reality constraint (1.5.31) one sees that all poles must either lie on the real axis, or be paired to their complex conjugate (namely, the set of poles

$\{\mu_s\}$ satisfies $\{\mu_s\} \subseteq \overline{\{\mu_s\}}$. Finally, the normalization condition (1.5.34) shows that the dressing matrix χ must tend to unity at infinity $\lambda \to \infty$.

We therefore established the fact that for gravitational solitons, the dressing matrix has the form

$$\chi = I + \sum_{s=1}^{N} \frac{R_s}{\lambda - \mu_s} \tag{1.6.1}$$

where the *residue matrices* R_s and the *poles* μ_s depend on x^k only (and are independent of λ). Due to Eq. (1.5.35) one can now obtain the N-solitonic solution of the reduced Einstein's Eq. (1.5.8) simply by evaluating χ at the origin $\lambda = 0$ and multiplying it by the background matrix $g^{(0)}$. This means that all there is left to do is to find the explicit form of the residue matrices R_s and the poles μ_s. This is the focus of the rest of this section.

Consider the dressing Eqs. (1.5.30). As long as $\mu_s \neq \alpha$, their left-hand side has only first order poles $\lambda = \mu_s$, while the right-hand side have second order poles $\lambda = \mu_s$ due to the differentiation by D_i. This means that equality can hold only if the coefficient of each power $(\lambda - \mu_s)^{-2}$ vanishes, and therefore the poles must satisfy the *pole trajectory equations*:

$$\partial_i \mu_s = P_i \mid_{\lambda=\mu_s} = -\frac{2\mu_s(\mu_s \partial_i \beta + e\alpha \partial_i \alpha)}{\mu_s^2 - e\alpha^2} \tag{1.6.2}$$

In appendix E we show that the solutions of Eqs. (1.6.2) are precisely the roots of the quadratic equation,

$$\mu_s^2 + 2(\beta - \omega_s)\mu_s + e\alpha^2 = 0 \tag{1.6.3}$$

where ω_s are arbitrary complex constants of integration. For a given constant ω_s, the quadratic formula yields two solutions, μ_s and α^2/μ_s. The second root α^2/μ_s should not be of a surprise, as it represents the symmetry with respect to the circle $|\lambda|^2 = \alpha^2$ mentioned before. In fact, the symmetry constraint (1.5.33) shows that the second root α^2/μ_s is a pole of the inverse dressing matrix χ^{-1}.

The solutions of Eq. (1.6.3) give the trajectory of the poles μ_s as a function of x^k,

$$\mu_s = \omega_s - \beta \pm \sqrt{(\omega_s - \beta)^2 - e\alpha^2} \tag{1.6.4}$$

which are (by definition) the poles of the dressing matrix χ and its inverse χ^{-1} respectively. For readers experienced with the inverse scattering transform, we note that here we have an atypical situation in the study of integrable systems and their dressing matrices. The poles of the dressing matrix χ depend on x^k, and are therefore floating in the λ-complex plane. Since the poles of the dressing matrix determine the properties of the solitons, such as their amplitude, velocity and phase, the fact that they are moving means that the solitons' features change with x^k. In particular, gravitational solitons do not preserve their amplitude. This reveals one fundamental difference between gravitational solitons and their classical counterparts that we will come back to later in this work: *gravitational solitons are not physical solitons* (see section 1.3).

To get a better intuition for the motion of the floating poles μ_s, it helps visualizing them in the λ complex plane. Since the circle $|\lambda|^2 = \alpha^2$ plays a central role in their geometric description, and α varies with x^k, it is convenient to normalize the poles so that the circle turns into the unit circle. Thus instead of the poles μ_s, consider the *normalized poles* $\lambda_s = \mu_s/\alpha$ defined by Eq. (1.6.4) as

$$\lambda_s = \frac{\omega_s - \beta}{\alpha} \pm \sqrt{(\frac{\omega_s - \beta}{\alpha})^2 - e} \tag{1.6.5}$$

Thus a normalized floating pole depend only on the parameter $z_s = \frac{\omega_s - \beta}{\alpha}$ and can be studied as a function of this parameter alone. To understand its behavior consider its inverse,

$$z_s = \frac{1}{2}\left(\lambda_s + \frac{e}{\lambda_s}\right) \tag{1.6.6}$$

Remarkably, for the case of non-stationary metrics $e = 1$ this is just the Joukowsky transform (up to a factor of $\frac{1}{2}$). It is a famous function in the theory of aerodynamics

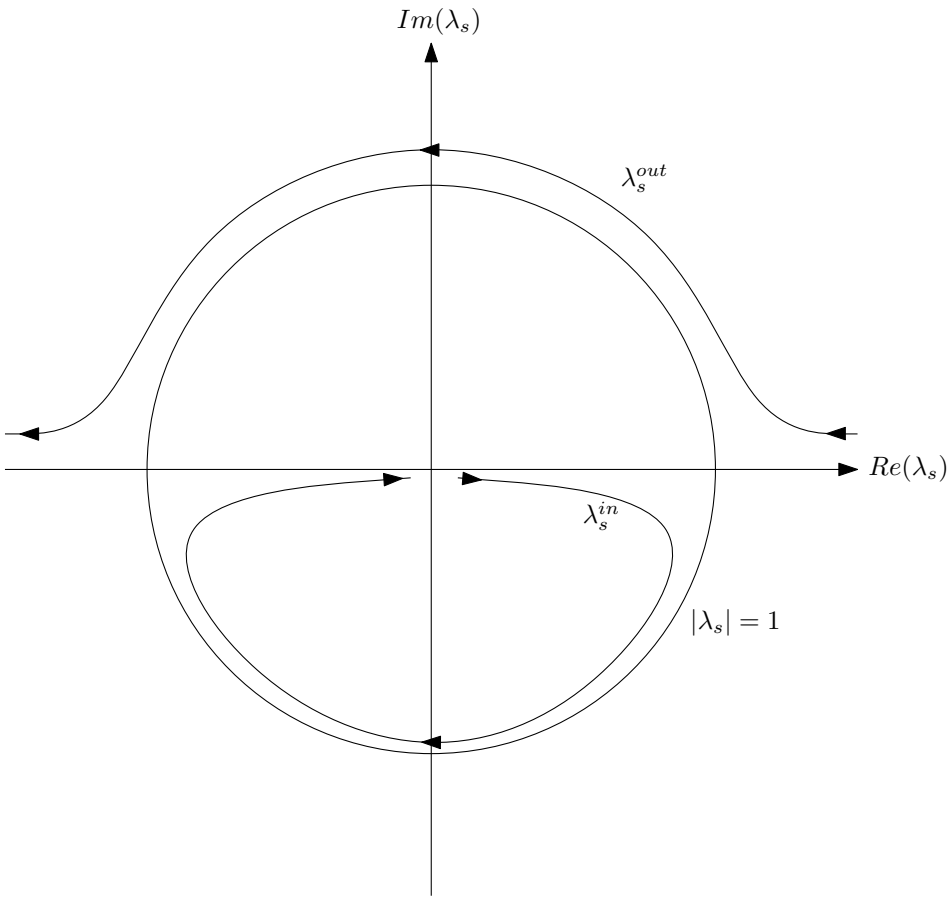

FIGURE 1.3. Each normalized pole λ_s of the dressing matrix χ is given by Eq. (1.6.5). When a non-stationary metric with $e = 1$ is considered, it is either always inside or always outside the unit circle $|\lambda_s| = 1$. For an outside pole λ_s^{out}, its trajectory nearly traces a horizontal line as $z_s \to \infty$ while bending near the unit circle. On the other hand, the inside pole λ_s^{in} can be obtained by the relation $\lambda_s^{in} = (\lambda_s^{out})^{-1}$. In this diagram the arrows point in the direction of increased β (or decrease of the real part of z_s). It is important to realize that the behavior of the non-normalized poles μ_s is more complex, as they depend on α as well. For fixed α, the trajectory of a non-normalized pole μ_s is the same as λ_s qualitatively, but as α varies the size of their defining circle varies as well.

[27], as it is a conformal map that transforms a (non-unit) circle in the λ_s-plane to an airfoil shape in the z_s-plane. The function maps the outside of the unit circle to the entire complex plane, and the unit circle itself to its real part, the set of real numbers

between -1 and 1. Thus, one can interpret it as squashing the unit circle vertically. In particular, this gives the behavior of the normalized pole λ_s as its inverse, mapping horizontal lines in the z_s-plane to paths that are asymptotically horizontal, but curve near the unit sphere in the λ_s-plane. As mentioned earlier, the sign choice in Eq. (1.6.5) gives one pole λ_s^{out} floating outside the unit circle $|\lambda_s| = 1$, while the other sign choice corresponds to a pole $\lambda_s^{in} = (\lambda_s^{out})^{-1}$ floating inside the unit circle. This is depicted in figure (1.3).

If $e = -1$ and we are dealing with stationary metrics, the inverse of the normalized pole is not the Joukowsky transform anymore. Nevertheless, it can be transformed to the Joukowsky transform by two simple rotations: a counter clockwise rotation by 90 degrees of the λ_s-plane, followed by a clockwise rotation of the z_s-plane by 90 degrees. This can be seen from the transformation,

$$z_s = \frac{1}{2}(\lambda_s - \frac{1}{\lambda_s}) \xrightarrow{\lambda_s \to i\lambda_s} \frac{i}{2}(\lambda_s + \frac{1}{\lambda_s}) \tag{1.6.7}$$

This means that if the Joukowsky transform squashes the unit circle vertically to a line segment, the inverse of the normalized poles for stationary metrics squashes the unit circle horizontally. One can use a similar argument to the one given before to understand the behavior of the normalized pole λ_s in this case, as is shown in figure (1.4). In this case the sign choice in Eq. (1.6.5) doesn't necessarily correspond to a pole being completely outside / inside the unit circle, as the poles of stationary metrics might pass through the unit circle (see figure (1.4)).

We are now ready to consider the residue matrices R_s. Rewriting Eqs. (1.5.30) as

$$(D_i\chi)\chi^{-1} + \Lambda_i{}^j \chi A_j^{(0)} \chi^{-1} = \Lambda_i{}^j A_j \tag{1.6.8}$$

makes it clear that the right-hand side is analytic at the poles $\lambda = \mu_s$. This means that the residues at each of the poles μ_s of the left-hand side must vanish as well, yielding

$$(\partial_i R_s)\chi^{-1}(\mu_s) + \Lambda_i{}^j(\mu_s) R_s A_j^{(0)} \chi^{-1}(\mu_s) = 0 \tag{1.6.9}$$

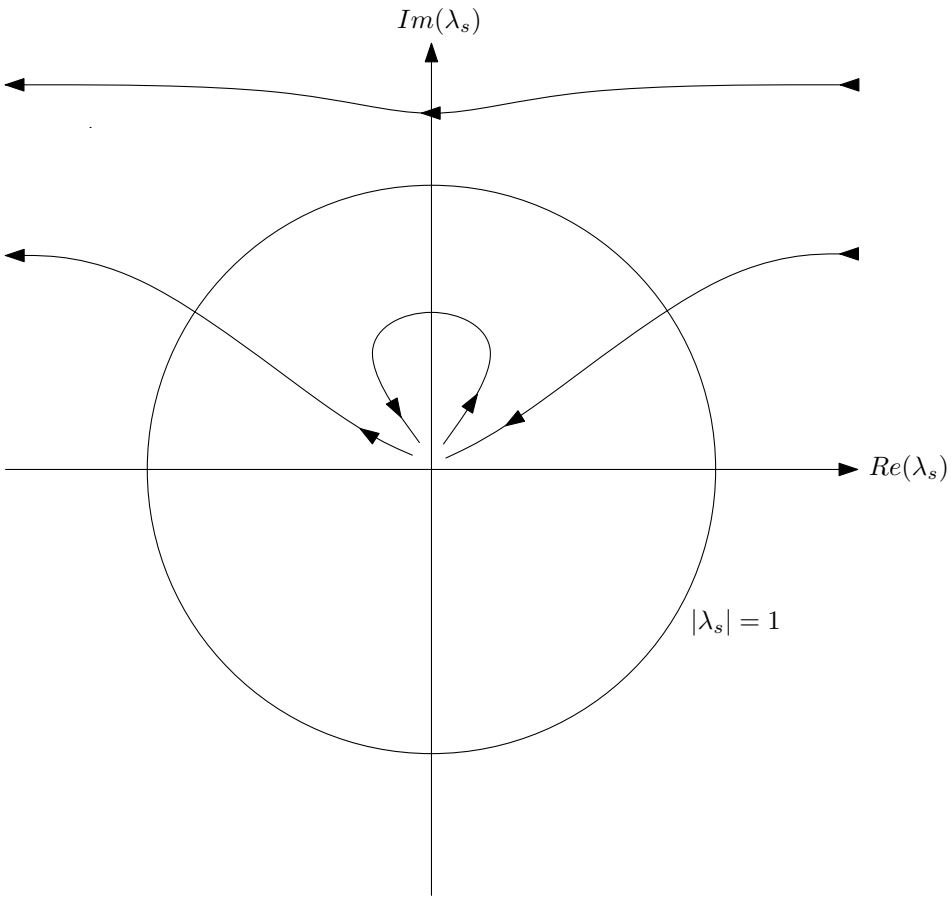

FIGURE 1.4. Each normalized pole λ_s of the dressing matrix χ is given by Eq. (1.6.5). In this case of a stationary metric $e = -1$, a pole traveling from infinity (horizontally) has its trajectory being distorted when approaching the unit circle. If it passes through the unit circle, it finishes its motion at the origin. Alternatively, a pole may start at the origin and travel to the left outside of the unit circle to become an asymptotically straight line, or travel completely inside the unit circle to traverse an oval-like shape. Here the arrows point in the direction of increased β (or decrease of the real part of z_s).

where we used the fact that

$$R_s \chi^{-1}(\mu_s) = 0 \tag{1.6.10}$$

following from the identity $\chi\chi^{-1} = I$ at the poles $\lambda = \mu_s$. However, taking the determinant of Eq. (1.6.10) shows that both R_s and $\chi^{-1}(\mu_s)$ are degenerate matrices and therefore have linearly dependent columns/rows. As two-dimensional matrices,

we can thus write them in the form of a tensor product,

$$R_s = \vec{n}_s \otimes \vec{m}_s \qquad \chi^{-1}(\mu_s) = \vec{q}_s \otimes \vec{p}_s \tag{1.6.11}$$

where $\vec{n}_s$, $\vec{m}_s$, $\vec{q}_s$ and $\vec{p}_s$ are two-dimensional column vectors. Substituting Eq. (1.6.11) into Eqs. (1.6.9) gives the evolution of the vectors m_s:

$$\left[\partial_i \vec{m}_s^T + \Lambda_i^{\ j}(\mu_s)\vec{m}_s^T A_j^{(0)}\right]\vec{q}_s = 0 \tag{1.6.12}$$

Fortunately, the solution of these equations can be given in terms of the particular solution $\psi^{(0)}$ of Eqs. (1.5.26). Since D_i is a linear first-order operator, it operates on the inverse matrix $M = (\psi^{(0)})^{-1}$ just as any other derivative operator,

$$D_i M = -M(D_i\psi^{(0)})M \tag{1.6.13}$$

Therefore the Belinski-Zakharov Eqs. (1.5.26) are equivalent to the *inverse Belinski-Zakharov equations*

$$D_i M + \Lambda_i^{\ j} M A_j^{(0)} = 0 \tag{1.6.14}$$

Fixing $\lambda = \mu_s$ in the inverse Belinski-Zakharov Eqs. (1.5.26) shows that the matrix $M_s = M\,|_{\lambda=\mu_s}$ satisfies

$$\partial_i M_s + \Lambda_i^{\ j}(\mu_s) M_s A_j^{(0)} = 0 \tag{1.6.15}$$

Comparing this result with Eq. (1.6.12) proves that the matrices M_s provide us with the general solution of Eq. (1.6.12)

$$\vec{m}_s = M_s^T \vec{c}_s \qquad \left(\vec{m}_s^T = \vec{c}_s^T M_s\right) \tag{1.6.16}$$

where each $\vec{c}_s$ is an arbitrary constant two-dimensional column vectors. They must be chosen to be real when the corresponding pole μ_s is real, or pairwise-complex-conjugate otherwise.

In order to fully determine the matrices R_s, we are therefore left to determine the vectors $\vec{n}_s$ defined in Eq. (1.6.11). For this purpose, we recall that the dressing

matrix χ satisfies the symmetry constraint Eq. (1.5.33). Substituting the dressing matrix χ from Eqs. (1.6.1) & (1.6.11) into Eq. (1.5.33) and evaluating this expression at the poles μ_s gives a linear algebraic equation for the vectors $\vec{n}_s$. These equations can be written in the form

$$\sum_{\tilde{s}=1}^{N} \frac{1}{\mu_{\tilde{s}}} \Gamma_{s\tilde{s}} \vec{n}_{\tilde{s}}^{T} = \vec{m}_s^{T} g^{(0)} \tag{1.6.17}$$

where Γ is an $N \times N$ symmetric matrix with elements

$$\Gamma_{s\tilde{s}} = \frac{\mu_s \mu_{\tilde{s}}}{\mu_s \mu_{\tilde{s}} - e\alpha^2} \vec{m}_s^{T} g^{(0)} \vec{m}_{\tilde{s}} \tag{1.6.18}$$

and $s, \tilde{s} = 1, \ldots, N$ is the number of solitons. The reader should notice that the vector sign used for the vector $\vec{n}_s$ means that for each given soliton index s, $\vec{n}_s$ is a two-vector. The same holds for $\vec{m}_s$, meaning that Eq. (1.6.17) in fact gives two sets of N linear algebraic equations for the components of $\vec{n}_s$ (as two N-vectors).

If Γ^{-1} is the inverse of the matrix Γ, the solution of Eq. (1.6.17) can be written explicitly as

$$\vec{n}_s = \mu_s \sum_{\tilde{s}=1}^{N} (\Gamma^{-1})_{s\tilde{s}} \vec{L}_{\tilde{s}} \tag{1.6.19}$$

where

$$\vec{L}_s = g^{(0)} \vec{m}_s \tag{1.6.20}$$

This completes the determination of the matrices R_s and consequently of the dressing matrix χ, from which the solution of the vacuum Einstein's equation can be obtained by the procedure described in the previous section. Using the expression for R_s from Eqs. (1.6.11), (1.6.16) and (1.6.19) in Eq. (1.5.35) yields the matrix $g^{(N)}_{\text{non-ph}}$:

$$g^{(N)}_{\text{non-ph}} = g^{(0)} - \sum_{s,\tilde{s}=1}^{N} (\Gamma^{-1})_{s\tilde{s}} \vec{L}_s \otimes \vec{L}_{\tilde{s}} \tag{1.6.21}$$

However, as mentioned earlier the matrix $g_{\text{non-ph}}$ does not satisfy the determinant condition and must therefore be normalized. One can easily show that after normal-

ization by Eq. (1.5.37) the physical solution with determinant $\det g = e\alpha^2$ is

$$g^{(N)} = \Pi \left[g^{(0)} - \sum_{s,\tilde{s}=1}^{N} (\Gamma^{-1})_{s\tilde{s}} \vec{L}_s \otimes \vec{L}_{\tilde{s}} \right] \tag{1.6.22}$$

where we defined the normalization factor

$$\Pi = \prod_{s=1}^{N} \left| \frac{\mu_s}{\alpha} \right| \tag{1.6.23}$$

Eq. (1.6.22) gives the N-gravitational soliton solution $g^{(N)}$ on the background metric $g^{(0)}$. Its form is manifestly symmetric as required by the symmetry constraint (1.5.33).

From a computational point of view, the evaluation of $g^{(N)}$ requires calculating the inverse of the $N \times N$ matrix Γ which can be a tedious task for large number of solitons. Next, we follow the work of Alekseev [19] to show how the computation of the inverse can be replaced with a much simpler calculation of determinants of four $N \times N$ matrices.

One can always write the 2×2 components of Eq. (1.6.22) in the form

$$g^{(N)}_{ab} = \Pi \left[g^{(0)}_{ab} - k_a k_b H_{ab} \right] \quad \text{(no summation over } a, b) \tag{1.6.24}$$

where

$$H_{ab} = \frac{1}{k_a} \frac{1}{k_b} \sum_{s,\tilde{s}=1}^{N} (\Gamma^{-1})_{s\tilde{s}} (\vec{L}_s)_a (\vec{L}_{\tilde{s}})_b \tag{1.6.25}$$

and k_a is an arbitrary two-dimensional complex vector with non-zero components. In fact, the components k_a can depend on x^k. Clearly, the metric elements $g^{(N)}_{ab}$ do not depend at all of the vector $\vec{k}$. But nevertheless, a wise choice of $k_a \neq 0$ can greatly simplify the computations involved.

Consider the last equation for fixed indices a and b. Each H_{ab} is the sum over all elements of an inverse matrix

$$H_{ab} = \sum_{s,\tilde{s}=1}^{N} (G^{-1}_{ab})_{s\tilde{s}} \tag{1.6.26}$$

where each $N \times N$ matrix G_{ab} has elements

$$(G_{ab})_{s\tilde{s}} = k_a k_b \frac{1}{(\vec{L}_s)_a (\vec{L}_{\tilde{s}})_b} \Gamma_{s\tilde{s}} \tag{1.6.27}$$

By the matrix determinant lemma, given a non-singular $N \times N$ matrix G,

$$\sum_{s,\tilde{s}=1}^{N} (G^{-1})_{s\tilde{s}} = -1 + \frac{\det(1+G)}{\det(G)} \tag{1.6.28}$$

where in the numerator of the right-hand side, unity is added to each element $G_{s\tilde{s}}$. In particular, one can use this lemma on each of the matrices G_{ab} to obtain H_{ab} from Eq. (1.6.26). Consequently, Eq. (1.6.24) gives the *determinant form* of the N-soliton solution:

$$g_{ab}^{(N)} = \Pi \left[g_{ab}^{(0)} + k_a k_b - k_a k_b \Delta_{ab}/\Delta \right] \quad \text{(no summation over } a, b) \tag{1.6.29}$$

where we define Δ_{ab} and Δ to be the determinants of the $N \times N$ matrices

$$\begin{aligned} \Delta &= \det \Gamma_{s\tilde{s}} \\ \Delta_{ab} &= \det \left(\Gamma_{s\tilde{s}} + k_a^{-1} k_b^{-1} (\vec{L}_s)_a (\vec{L}_{\tilde{s}})_b \right) \end{aligned} \tag{1.6.30}$$

(to make the notation unambiguous, the subscripts $s, \tilde{s}$ are also written explicitly, and one should understand these formulas as computing the determinants with respect to these indices while keeping the indices a and b fixed.)

Finally, recall that the spacetime interval in Eq. (1.5.1) also depends on the function f. The function f can be obtained explicitly by integration of Eq. (1.5.18),

$$f^{(N)} = C_0 \frac{\alpha^{-N^2/2} \left(\prod_{s=1}^{N} \mu_s \right)^{N-1}}{\prod_{s>\tilde{s}=1}^{N} (\mu_s - \mu_{\tilde{s}})^2} \Delta f^{(0)} \tag{1.6.31}$$

where C_0 is an arbitrary real constant and $f^{(0)}$ is the corresponding metric element for the background metric.

1.7 An algorithm for construction of N gravitational solitons

As we showed in section 1.6, the construction of the N-soliton solution of the reduced Einstein Eq. (1.5.8) is carried out in several simple steps. Beyond the lengthy derivation, it is convenient to state the steps required to obtain the N gravitational solitons in an algorithmic manner, which is the purpose of this section. The construction of N gravitational solitons is given by the following steps:

1. Take a particular solution of the Einstein vacuum Eq. (1.2.3) of the form (1.5.1),

$$ds^2 = f^{(0)}\eta_{ij}dx^i dx^j + g^{(0)}_{ab}dx^a dx^b \tag{1.7.1}$$

   where $f^{(0)} > 0$ and

$$\eta_{ij} = \begin{bmatrix} -e & 0 \\ 0 & 1 \end{bmatrix} \tag{1.7.2}$$

   is a constant matrix with $e = \pm 1$. This solution $\left\{f^{(0)}, g^{(0)} = g^{(0)}_{ab}\right\}$ will be referred to as the *seed (background) metric.*

2. Denote by α the determinant $\det g_{ab} = e\alpha^2$. Define the function β to be any solution of Eq. (1.5.20),

$$\beta_{,i} = -e\eta_{ij}\varepsilon^{jk}\alpha_{,k} \tag{1.7.3}$$

   where ε is the 2×2 matrix

$$\varepsilon_{ab} = \varepsilon^{ab} = \varepsilon_{ij} = \varepsilon^{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.7.4}$$

3. Introduce a set of N real or pairwise-complex conjugate function μ_s (here and in the following $s, \tilde{s}, \cdots = 1, 2, \ldots, N$) according to Eq. (1.6.4):

$$\mu_s = \omega_s - \beta \pm \sqrt{(\omega_s - \beta)^2 - e\alpha^2} \tag{1.7.5}$$

   Here $\{\omega_s\}$ are any set of complex constants that are chosen to be either real or appear together with their complex-conjugate (i.e. $\{\omega_s\} = \overline{\{\omega_s\}}$).

4. Determine the 2×2 matrix-valued function M (of x^μ and a complex parameter λ) by solving the *inverse Belinski-Zakharov* Eqs. (1.6.14)

$$D_i M + \Lambda_i{}^j M A_j^{(0)} = 0 \tag{1.7.6}$$

where the commuting operators D_i are

$$D_i = \partial_i - \frac{2\lambda(\lambda\beta_{,i} + e\alpha\alpha_{,i})}{\lambda^2 - e\alpha^2}\partial_\lambda \tag{1.7.7}$$

and the matrices $\Lambda_i{}^j$ and $A_i^{(0)}$ are

$$\begin{aligned} \Lambda_i{}^j &= \frac{\lambda\varepsilon_i{}^j - \alpha\delta_i^j}{\lambda^2 - e\alpha^2} \\ A_j^{(0)} &= e\alpha g_{,j}^{(0)} \left(g^{(0)}\right)^{-1} \end{aligned} \tag{1.7.8}$$

Here $\varepsilon_i{}^j = \eta_{ik}\varepsilon^{kj}$ (the order of matrices does matter) and δ_i^j is the Kronecker delta.

5. Define a set of N two-dimensional complex column vectors $\vec{m}_s$ by Eq. (1.6.16),

$$\vec{m}_s = M_s^T \vec{c}_s \tag{1.7.9}$$

where each $\vec{c}_s$ is an arbitrarily chosen constant two-dimensional column vector, and $M_s = M\,|_{\lambda=\mu_s}$. The vectors $\vec{c}_s$ must be chosen to be real when the corresponding pole μ_s is real, or pairwise-complex-conjugate otherwise.

6. Compute the function Π, the vectors $\vec{L}_s$ and the matrix function Γ_{ab} according to

$$\begin{aligned} \Pi &= \prod_{s=1}^{N} \left|\frac{\mu_s}{\alpha}\right| \\ \vec{L}_s &= g^{(0)}\vec{m}_s \\ \Gamma_{s\tilde{s}} &= \frac{\mu_s\mu_{\tilde{s}}}{\mu_s\mu_{\tilde{s}} - e\alpha^2}\vec{m}_s^T g^{(0)}\vec{m}_{\tilde{s}} \end{aligned} \tag{1.7.10}$$

7. Compute the determinants of the $N \times N$ matrices

$$\begin{aligned} \Delta &= \det \Gamma_{s\tilde{s}} \\ \Delta_{ab} &= \det \left(\Gamma_{s\tilde{s}} + k_a^{-1} k_b^{-1} (\vec{L}_s)_a (\vec{L}_{\tilde{s}})_b \right) \end{aligned} \tag{1.7.11}$$

where k_a is an arbitrary two-dimensional vector with nonzero components. Note that although k_a appears explicitly in the solution below, the solution is in fact independent of k_a. However, by well-choosing the value of $k_a \neq 0$ one can greatly simplify the calculations involved (this will be demonstrated in the next two chapters).

8. Finally, the N-soliton solution of Einstein's vacuum Eq. (1.2.3) is given by the metric,

$$ds^2 = f^{(N)} \eta_{ij} dx^i dx^j + g^{(N)}_{ab} dx^a dx^b \tag{1.7.12}$$

with

$$\begin{aligned} g^{(N)}_{ab} &= \Pi \left[g^{(0)}_{ab} + k_a k_b (1 - \Delta_{ab}/\Delta) \right] \\ f^{(N)} &= C_0 \frac{\alpha^{-N^2/2} \left(\prod_{s=1}^{N} \mu_s \right)^{N-1}}{\prod_{s>\tilde{s}=1}^{N} (\mu_s - \mu_{\tilde{s}})^2} \Delta f^{(0)} \end{aligned} \tag{1.7.13}$$

where C_0 is an arbitrary constant. Here, no summation is implied on the indices a and b.

Clearly, the most crucial step in the procedure described above is step 4, in which one must integrate a system of first order *linear* partial differential equations (1.6.14) instead of the highly *nonlinear* equation (1.5.8) for the matrix g. In fact, once a seed solution is prescribed, all steps but step 4 involve simple algebraic computations. Later in this work we will see that in the case of a diagonal metric, step 4 can be skipped completely to produce a closed-form expression of the N-soliton solution on a general diagonal background.

Chapter 2

The symmetric principal chiral field equation

As we saw in the previous chapter, the principal chiral field equation

$$(g_{,\zeta}g^{-1})_{,\eta} + (g_{,\eta}g^{-1})_{,\zeta} = 0 \tag{2.0.1}$$

is a special case of the reduced Einstein Eq. (1.5.8) when the 2×2 matrix g has a constant determinant $\alpha = 1$. It is Lax-integrable through the Zakharov-Mikhailov Eqs. (1.4.8),

$$\partial_\zeta \psi = \frac{A}{\lambda - 1}\psi \qquad \partial_\eta \psi = \frac{B}{\lambda + 1}\psi \tag{2.0.2}$$

where

$$A = -g_{,\zeta}g^{-1} \qquad B = g_{,\eta}g^{-1} \tag{2.0.3}$$

Before proceeding, we will state that the Zakharov-Mikhailov Eqs. (2.0.2) are a special case of the Belinski-Zakharov Eqs. (1.5.26) when $\alpha = 1$. It can be seen by changing variables in Eqs. (1.5.26) to the light-cone coordinates

$$x^0 = \zeta - \eta \qquad x^3 = \zeta + \eta \tag{2.0.4}$$

Due to the axioms of general relativity (see section 1.2) the matrix g must be real and symmetric. Therefore we study a particular case of the reduction problem 1.4 on 'symmetric spaces' [11, 12]. The symmetric space considered here is the invariant manifold of symmetric matrices sitting in the Lie group $SL(2, \mathbb{R})$. This invariant space is *not* a Lie group, but we will later show that it can be identified with a Hyperboloid in Minkowsi spacetime.

The study of Einstein's equation in this case of a constant determinant has long been neglected for physical reasons: this case corresponds to flat Minkowski spacetime. To see this, notice that if α is fixed, Eq. (1.5.18) implies that $A_i^2 = 0$ are

traceless matrices. This can only happen when the matrix g is constant. Solving for f then shows that in general, $f = \exp[f_1(x^0 + x^3) + f_2(x^0 - x^3)]$ where f_1 and f_2 are two arbitrary functions. However, a simple change of coordinates reduces the coefficient f to a constant as well and the metric to a flat Minkowski spacetime.

Nevertheless, the structure of the soliton solutions of the symmetric principal chiral field Eq. (2.0.1) is interesting in its own right outside of gravitational phenomena. Furthermore, even when gravity is concerned, the properties of the solitons obtained for the symmetric principal chiral field equation assist in clarifying the meaning of the gravitational solitons in the less trivial case $\alpha \neq$ constant. This more laborious case will be the topic of the next chapter.

This chapter is divided to three more sections. In the first section, we show that the symmetric principal chiral field may be casted to a form of a single scalar equation. This equation has similar structure to the famous Sine-Gordon equation [31], and is Lax-integrable merely from being equivalent to the symmetric principal chiral field. In the second section, we focus on one of the most important features (or perhaps, meanings...) of integrability, the existence of infinitely many conservation laws and their associated integrals of motion. Finally, the last section focuses on solitons of the symmetric principal chiral field equation. In particular, *all* of its soliton solutions on diagonal backgrounds are derived explicitly, and their properties, behavior and interactions are studied.

2.1 Reduction to a single scalar equation

By the spectral theorem, the real and symmetric matrix g can be diagonalized,

$$g = RDR^T \tag{2.1.1}$$

where D is a diagonal matrix

$$D = \begin{bmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{bmatrix} \tag{2.1.2}$$

and R is an orthogonal matrix

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{2.1.3}$$

Since g is a real and symmetric matrix with $\det g = 1$, it has two degrees of freedom. In this alternative representation the two degrees of freedom are the fields Λ and ϕ. These new fields can be interpreted as determining the eigenvalues of g and measuring the deviation of g from being a diagonal matrix (respectively). Written explicitly, g is

$$g = \begin{bmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi \sinh\Lambda & \cosh\Lambda - \cos 2\phi \sinh\Lambda \end{bmatrix} \tag{2.1.4}$$

and the principal chiral field Eq. (2.0.1) is equivalent to

$$\begin{aligned} 0 &= \Lambda_{,\zeta\eta} - 2\phi_{,\zeta}\phi_{,\eta}\sinh 2\Lambda \\ 0 &= (\phi_{,\zeta}\sinh^2\Lambda)_{,\eta} + (\phi_{,\eta}\sinh^2\Lambda)_{,\zeta} \end{aligned} \tag{2.1.5}$$

(this last fact is proven in section 3.2 by considering the more general case of arbitrary α.)

In terms of the fields Λ and ϕ the 2×2 matrices A and B defined in (2.0.3) have determinants

$$\begin{aligned} \det A &= -\Lambda_{,\zeta}^2 - 4\phi_{,\zeta}^2 \sinh^2\Lambda \\ \det B &= -\Lambda_{,\eta}^2 - 4\phi_{,\eta}^2 \sinh^2\Lambda \end{aligned} \tag{2.1.6}$$

Differentiating these determinants with respect to η and ζ respectively and using the field Eqs. (2.1.5) gives

$$\begin{aligned} \frac{\partial}{\partial\eta}(\det A) &= 0 \\ \frac{\partial}{\partial\zeta}(\det B) &= 0 \end{aligned} \tag{2.1.7}$$

Hence the determinant of A is a function of ζ only, and similarly $\det B$ is a function of η only. Physically, this means that each one of the determinants is constant along one boundary of the light-cone.

We can therefore rescale the coordinates ζ, η by defining new coordinates $\bar{\zeta}, \bar{\eta}$ according to

$$\bar{\zeta} = \int d\zeta \sqrt{-\det A} \qquad \bar{\eta} = \int d\eta \sqrt{-\det B} \tag{2.1.8}$$

where the square roots are always real-valued (this can be seen from Eqs. (2.1.6).) In these new coordinates, Eqs. (2.1.6) give

$$1 = \Lambda_{,\bar{\zeta}}^2 + 4\phi_{,\bar{\zeta}}^2 \sinh^2 \Lambda \qquad 1 = \Lambda_{,\bar{\eta}}^2 + 4\phi_{,\bar{\eta}}^2 \sinh^2 \Lambda \tag{2.1.9}$$

and therefore the first partial derivatives of ϕ can always be expressed in terms of Λ and its first partials,

$$\phi_{,\bar{\zeta}} = \pm \frac{1}{2 \sinh \Lambda} \sqrt{1 - \Lambda_{,\bar{\zeta}}^2} \qquad \phi_{,\bar{\eta}} = \pm \frac{1}{2 \sinh \Lambda} \sqrt{1 - \Lambda_{,\bar{\eta}}^2} \tag{2.1.10}$$

Finally, consider the first equation in (2.1.5). The rescaling of ζ, η doesn't affect the form of the equation, which in terms of the coordinates $\bar{\zeta}, \bar{\eta}$ is

$$\Lambda_{,\bar{\zeta}\bar{\eta}} - 2\phi_{,\bar{\zeta}} \phi_{,\bar{\eta}} \sinh 2\Lambda = 0 \tag{2.1.11}$$

Using Eqs. (2.1.10) one may eliminate ϕ from this equation, resulting in a single scalar equation for the field Λ

$$\Lambda_{,\bar{\zeta}\bar{\eta}} \mp \sqrt{(1 - \Lambda_{,\bar{\zeta}}^2)(1 - \Lambda_{,\bar{\eta}}^2)} \coth \Lambda = 0 \tag{2.1.12}$$

Doing the same substituting on the second equation in (2.1.5) results in exactly the same equation.

This means that the entire content of the symmetric principal chiral field Eq. (2.0.1) is contained in a single scalar Eq. (2.1.12). One question remains as to the meaning of the choice of sign in this equation. This sign in fact has no physical significance, as an inversion $\bar{\zeta} \to -\bar{\zeta}$ (or alternatively $\bar{\eta} \to -\bar{\eta}$) gives the same equation with the opposite choice of sign. We may therefore fix the choice of sign,

$$\Lambda_{,\bar{\zeta}\bar{\eta}} - \sqrt{(1 - \Lambda_{,\bar{\zeta}}^2)(1 - \Lambda_{,\bar{\eta}}^2)} \coth \Lambda = 0 \tag{2.1.13}$$

This new equation may be written as a conservation law, as the second equation in (2.1.5) shows

$$(\sqrt{1-\Lambda^2_{,\bar{\zeta}}}\coth\Lambda)_{,\bar{\eta}}+(\sqrt{1-\Lambda^2_{,\bar{\eta}}}\coth\Lambda)_{,\bar{\zeta}}=0 \tag{2.1.14}$$

Since Eq. (2.1.14) is the result of the *integrable* Chiral field equation, it is also integrable. As an equation that is expressed as a conservation law, its first integral of motion is obtained immediately,

$$I_0=\int\sqrt{1-\Lambda^2_{,\bar{\zeta}}}\coth\Lambda d\bar{\zeta} \tag{2.1.15}$$

with

$$\frac{dI_0}{d\bar{\eta}}=0 \tag{2.1.16}$$

This integral of motion is finite as long as the partial derivatives of the field Λ converge to unity at infinity,

$$\frac{\partial\Lambda}{\partial\bar{\eta}}\xrightarrow{\bar{\zeta}\to\pm\infty}1^- \tag{2.1.17}$$

The reader might think that this condition is far too restrictive, as the partial derivatives of Λ may become greater than unity, however, this is not possible due to the rescaling process (see Eqs. (2.1.9)).

It is an interesting observation that Eq. (2.1.13) contains hyperbolic functions and is relativistically invariant. It is clear that it has a very similar form to the Sinh-Gordon equation [7]. Despite being an integrable equation, its Lax pair is still unknown and the author hopes to study this equation in more depth in the future.

2.2 Conservation laws

As mentioned earlier, the symmetric principal Chiral field Eq. (2.0.1) is Lax-integrable through the Zakharov-Mikhailov Eqs. (2.0.2). A key feature of integrability is the existence of infinitely many conservation laws and integrals of motion. The goal of the present section is to derive these conservation laws and thus prove that the symmetric principal Chiral field Eq. (2.0.1) is also Liouville-integrable.

We shall assume without loss of generality that

$$\det A = -1 \tag{2.2.1}$$

This assumption amounts to using the new coordinates $\bar{\zeta}, \bar{\eta}$ instead of the original ζ, η. However, to keep the notation concise we shall omit the bars and denote the new coordinates with ζ, η just like the original coordinates. This notation will be used throughout this section and in this section only.

Consider the Zakharov-Mikhailov Eqs. (2.0.2) operating on a vector $\psi = (\psi_1, \psi_2)^T$, so that ψ_1 and ψ_2 satisfy

$$\begin{aligned} \psi_{1,\zeta} &= \frac{1}{\lambda - 1}\left(A_{11}\psi_1 + A_{12}\psi_2\right) \\ \psi_{2,\zeta} &= \frac{1}{\lambda - 1}\left(A_{21}\psi_1 + A_{22}\psi_2\right) \end{aligned} \tag{2.2.2}$$

as well as

$$\begin{aligned} \psi_{1,\eta} &= \frac{1}{\lambda + 1}\left(B_{11}\psi_1 + B_{12}\psi_2\right) \\ \psi_{2,\eta} &= \frac{1}{\lambda + 1}\left(B_{21}\psi_1 + B_{22}\psi_2\right) \end{aligned} \tag{2.2.3}$$

where A_{ij} and B_{ij} are the coefficients of the matrices A and B. Define the functions

$$P = \frac{\psi_{1,\zeta}}{\psi_1} \qquad Q = \frac{\psi_{1,\eta}}{\psi_1} \tag{2.2.4}$$

which clearly satisfy the conservation law

$$\frac{\partial P}{\partial \eta} = \frac{\partial Q}{\partial \zeta} \tag{2.2.5}$$

This conservation law can be integrated with respect to ζ to give the integral of motion

$$I = \int P d\zeta \tag{2.2.6}$$

such that

$$\frac{\partial}{\partial \eta} I = 0 \tag{2.2.7}$$

Our goal is to express P in a form that does not contain any ζ derivatives, and thus obtain a non-trivial local conservation law of the symmetric principal chiral field Eq. (2.0.1). Since the function P, as well as the quantities I and Q, depend on the spectral parameter λ, this conservation law is in fact a generating function for infinitely many local conservation laws when expanded properly in the λ-plane.

Note that Q can be completely eliminated from Eq. (2.2.5) and expressed in terms of P only. From the first equations in (2.2.2) and (2.2.3)

$$\begin{aligned} P &= \frac{1}{\lambda-1}\left(A_{11}+A_{12}\frac{\psi_2}{\psi_1}\right) \\ Q &= \frac{1}{\lambda+1}\left(B_{11}+B_{12}\frac{\psi_2}{\psi_1}\right) \end{aligned} \tag{2.2.8}$$

By inverting the first relation to express ψ_2/ψ_1 in terms of P and replacing it into the second equation, one gets

$$Q=\frac{1}{\lambda+1}\left(B_{11}-\frac{A_{11}B_{12}}{A_{12}}\right)+\frac{\lambda-1}{\lambda+1}\frac{B_{12}}{A_{12}}P \tag{2.2.9}$$

This equality is valid whenever A is a non-diagonal matrix. The case $A_{12}=0$ is a degenerated case which will be treated separately later on.

It turns out that similarly to other integrable equations [31], the function P satisfies a Riccati-type equation. To see this, we differentiate P with respect to ζ in Eqs. (2.2.8) and use Eqs. (2.2.2). After some simple algebra, one gets

$$P_{,\zeta}=\frac{1}{\lambda-1}\left[A_{12}\left(\frac{A_{11}}{A_{12}}\right)_{,\zeta}-\frac{\det A}{\lambda-1}\right]+\left(\frac{A_{12,\zeta}}{A_{12}}+\frac{\mathrm{tr}A}{\lambda-1}\right)P-P^2 \tag{2.2.10}$$

Note however that A is a traceless matrix in general ($\mathrm{tr}A=0$). Furthermore and as commented in the introduction to this section, one can always assume $\det A=-1$. This gives

$$P_{,\zeta}=\frac{1}{\lambda-1}\left\{A_{12}\left(\frac{A_{11}}{A_{12}}\right)_{,\zeta}+\frac{1}{\lambda-1}\right\}+\frac{A_{12,\zeta}}{A_{12}}P-P^2 \tag{2.2.11}$$

This is a *Riccati equation* for the function P. Since the coefficients of the equation contain the spectral parameter λ, when both sides of the equation are expanded

properly in the λ-plane it results in infinitely many conservation laws for each term in the series.

We therefore proceed by expanding P in a Laurent series about its simple pole at $\lambda = 1$,

$$P = \sum_{n=-1}^{\infty} P_n(\lambda - 1)^n \tag{2.2.12}$$

and use this expansion in Eq. (2.2.11). The second and first order poles in Eq. (2.2.11) give the first two coefficients

$$P_{-1} = 1 \tag{2.2.13}$$

and

$$P_0 = \frac{1}{2}\left[\frac{A_{12,\zeta}}{A_{12}} + A_{12}\left(\frac{A_{11}}{A_{12}}\right)_{,\zeta}\right] \tag{2.2.14}$$

By comparing the coefficients of the non-singular terms one obtains a simple recursive relation for P_{n+1} $(n = 0, 1, 2, \dots)$

$$P_{n+1} = \frac{1}{2}\left[\frac{A_{12,\zeta}}{A_{12}}P_n - P_{n,\zeta} - \sum_{k=0}^{n} P_k P_{n-k}\right] \tag{2.2.15}$$

Expanding Eq. (2.2.9) in powers of $(\lambda - 1)$ as well, shows that Q is analytic about $\lambda = 1$. If one writes

$$Q = \sum_{n=0}^{\infty} Q_n(\lambda - 1)^n \tag{2.2.16}$$

then the coefficients Q_n are given by

$$Q_n = \frac{1}{2}\left[\left(-\frac{1}{2}\right)^n\left(B_{11} - \frac{A_{11} - 1}{A_{12}}B_{12}\right) + \frac{B_{12}}{A_{12}}\sum_{k=0}^{n-1}\left(-\frac{1}{2}\right)^k P_{n-1-k}\right] \tag{2.2.17}$$

The reader should interpret the summation in the last term as being equal to zero whenever the upper bound of the summation is smaller than the lower bound $n-1 < 0$. We will use a similar convention in the rest of this section.

When expanded about $\lambda = 1$, the conservation law in Eq. (2.2.5) decomposes into infinitely many conservation laws for each power of $(\lambda - 1)$. These conservation laws

have the form

$$\frac{\partial P_n}{\partial \eta} = \frac{\partial Q_n}{\partial \zeta} \tag{2.2.18}$$

with corresponding integrals of motion (see Eq. (2.2.6)),

$$I_n = \int P_n d\zeta \tag{2.2.19}$$

satisfying

$$\frac{\partial}{\partial \eta} I_n = 0 \qquad n = -1, 0, 1, \ldots \tag{2.2.20}$$

Applying Eq. (2.2.18) to the first coefficient P_{-1} defined in Eq. (2.2.13) makes it seem as if it is a trivial conservation law. Nevertheless, this is not necessarily the case. Recall that we assume that $\det A = -1$, namely, that we are using the rescaled version $(\bar{\zeta}, \bar{\eta})$ of the light-cone coordinates (ζ, η). In the barred coordinates this conservation law is indeed trivial, but when restoring the original coordinates this is not longer the case, and Eq. (2.2.18) gives the known conservation law from the previous section 2.1.7.

When using Eq. (2.2.15), one can obtain more integrals of motion. They are non-trivial in both coordinate systems (ζ, η) and $(\bar{\zeta}, \bar{\eta})$. The next integral of motion is,

$$I_0 = \frac{1}{2} \int A_{12} \left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} d\zeta \tag{2.2.21}$$

while the other integrals of motion can be computed recursively using Eq. (2.2.15), and satisfy

$$I_{n+1} = \frac{1}{2} \int \left[-A_{12} \left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} P_n - \sum_{k=1}^{n-1} P_k P_{n-k} \right] d\zeta \tag{2.2.22}$$

for $n = 0, 1, 2, \ldots$. Notice the similarity between this recursive formula, and the famous recursive formula for the integrals of motion for the Korteweg-de Vries equation [32]. Written explicitly, the next two non-trivial integrals of motion are

$$I_1 = \frac{1}{8} \int \left[\left(\frac{A_{12,\zeta}}{A_{12}} \right)^2 - A_{12}^2 \left(\left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} \right)^2 \right] d\zeta \tag{2.2.23}$$

and

$$I_2 = \frac{1}{8}\int \left[-\frac{3}{2}\frac{A_{12,\zeta}^2}{A_{12}} + \frac{1}{2}A_{12}^3 \left(\left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} \right)^2 + A_{12,\zeta\zeta} + A_{12} \left(A_{12} \left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} \right)_{,\zeta} \right] \left(\frac{A_{11}}{A_{12}} \right)_{,\zeta} d\zeta \tag{2.2.24}$$

Let us remind the user that as noted earlier, Eqs. (2.2.21), (2.2.22), (2.2.23) and (2.2.24) all assume $\det A = -1$. Namely, that we are using the rescaled coordinates defined in Eqs. (2.1.8). One can easily use Eqs. (2.1.8) to write the conservation laws and integrals of motion generally. This amounts to inserting factors of $\sqrt{-\det A}$ and $\sqrt{-\det B}$ whenever the differentials of ζ and η appear.

We are finally ready to treat the special case mentioned before. Since the coefficients P_n and Q_n contain A_{12} in their denominators, $A_{12} = 0$ is a degenerated case for the conservation laws derived here. However, this is not a real issue as this case only occurs when the matrix g is diagonal and Eqs. (2.1.5) are in fact linear (since $\phi = 0$).

2.3 Solitons on diagonal backgrounds

In this section we apply the Belinski-Zakharov technique from section 1.6 to find the N-soliton solution of the symmetric principal chiral field Eq. (2.0.1) on an arbitrary diagonal background. To do so, we will follow the steps summarized in section 1.7.

Let $g^{(0)}$ be a known diagonal solution of the chiral field equation. One can always write such a solution in the form

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix} \tag{2.3.1}$$

since the determinant of $g^{(0)}$ is unity. In this diagonal case the symmetric principal chiral field equation is linear. In fact, it can be seen from Eq. (2.1.5) that the function $\Lambda^{(0)}$ satisfies the wave equation

$$\partial_\zeta \partial_\eta \Lambda^{(0)} = 0 \tag{2.3.2}$$

Since the determinant of the chiral matrix $g^{(0)}$ is a constant, steps 2 and 3 of the algorithm 1.7 are trivial as the poles of the dressing matrix are fixed $\mu_s = \text{constant}$. Therefore one can proceed to solve the inverse Belinski-Zakharov Eqs. (1.6.14), which simplify greatly to

$$\partial_i M + \Lambda_i{}^j M A_j^{(0)} = 0 \tag{2.3.3}$$

where the matrices $A_j^{(0)}$ are

$$A_j^{(0)} = e\Lambda_{,j}^{(0)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.3.4}$$

Because the matrices $A_j^{(0)}$ are diagonal, so is the inverse dressing matrix M. From Eqs. (2.3.3) one sees that the components of M satisfy a decoupled set of first order linear equations. They can be easily integrated to yield

$$M = \begin{bmatrix} \exp\left[\frac{1}{\lambda^2 - e}(e\Lambda^{(0)} + \lambda\tilde{\Lambda}^{(0)})\right] & 0 \\ 0 & \exp\left[\frac{-1}{\lambda^2 - e}(e\Lambda^{(0)} + \lambda\tilde{\Lambda}^{(0)})\right] \end{bmatrix} \tag{2.3.5}$$

where $\tilde{\Lambda}^{(0)}$ is any function satisfying

$$\partial_i \tilde{\Lambda}^{(0)} = -e\eta_{ij}\varepsilon^{jk}\partial_k \Lambda^{(0)} \tag{2.3.6}$$

This last equation implies that $\tilde{\Lambda}^{(0)}$ is a 'conjugate' solution of the wave equation for $\Lambda^{(0)}$ and plays an identical role as does the function β in relation to α (see Eq. (1.5.20)).

Proceeding according to the algorithm 1.7, we choose $k_a = i(e^{\Lambda^{(0)}/2}, e^{-\Lambda^{(0)}/2})$ to simplify the solution. This yields the N-soliton solution

$$g^{(N)} = \Pi \begin{bmatrix} e^{\Lambda^{(0)}} \frac{\Delta_{00}}{\Delta} & \frac{\Delta_{03}}{\Delta} - 1 \\ \frac{\Delta_{30}}{\Delta} - 1 & e^{-\Lambda^{(0)}} \frac{\Delta_{33}}{\Delta} \end{bmatrix} \tag{2.3.7}$$

where

$$\begin{aligned}
\Delta &= \Pi^2 \det\left(\frac{C_s C_{\tilde{s}} e^{D_{s\tilde{s}}} + e^{-D_{s\tilde{s}}}}{\mu_s \mu_{\tilde{s}} - e}\right) \\
\Delta_{00} &= \det\left(\frac{e C_s C_{\tilde{s}} e^{D_{s\tilde{s}}} + \mu_s \mu_{\tilde{s}} e^{-D_{s\tilde{s}}}}{\mu_s \mu_{\tilde{s}} - e}\right) \\
\Delta_{03} &= \det\left(\frac{\mu_s \mu_{\tilde{s}} (C_s C_{\tilde{s}} e^{D_{s\tilde{s}}} + e^{-D_{s\tilde{s}}})}{\mu_s \mu_{\tilde{s}} - e} - C_s e^{B_s - B_{\tilde{s}}}\right) \\
\Delta_{33} &= \det\left(\frac{\mu_s \mu_{\tilde{s}} C_s C_{\tilde{s}} e^{D_{s\tilde{s}}} + e \cdot e^{-D_{s\tilde{s}}}}{\mu_s \mu_{\tilde{s}} - e}\right)
\end{aligned} \tag{2.3.8}$$

Here the functions B_s are

$$B_s = \frac{1}{\mu_s^2 - e}\left(e\Lambda^{(0)} + \mu_s \tilde{\Lambda}^{(0)}\right) \tag{2.3.9}$$

the function Π is

$$\Pi = \prod_{s=1}^{N} |\mu_s| \tag{2.3.10}$$

and D is the $N \times N$ matrix with coefficients

$$D_{s\tilde{s}} = \Lambda^{(0)} + B_s + B_{\tilde{s}} \tag{2.3.11}$$

Here both $\{C_s\}$ and $\{\mu_s\}$ are sets of complex constants that are chosen to be either real or appear together with their complex-conjugate ($\{C_s\} = \overline{\{C_s\}}$ and $\{\mu_s\} = \overline{\{\mu_s\}}$). Since this case $\alpha = 1$ plays no role in gravitational phenomena, we omit the function f altogether. The reader can easily recognize the form of the background solution $g^{(0)}$ of Eq. (2.3.1) on the diagonal of (2.3.7). This is an explicit formula for the general N-soliton solution on any diagonal background. To understand its physical meaning we will consider next the two simplest cases of one and two solitons. We will only consider these cases for non-stationary metrics $e = 1$.

In order to write the one and two solitons solution explicitly, it is convenient to introduce the notation

$$\begin{aligned}
\tilde{\gamma}_s &= -\ln|\mu_s| \\
\gamma_s &= K_s + \Lambda^{(0)} + 2B_s
\end{aligned} \tag{2.3.12}$$

where $s = 1, 2, \ldots, N$, and $K_s = \ln |C_s| \in \mathbb{R}$ in the case of real constants. Fortunately, the one and two solitons solutions can be written in terms of hyperbolic functions, rather similar to the one and two solitonic solutions of the famous Korteweg-de Vries equation [31].

The one-soliton $N = 1$ is

$$g^{(1)} = \frac{1}{\cosh \gamma_1} \begin{bmatrix} e^{\Lambda^{(0)}} \cosh(\gamma_1 + \tilde{\gamma}_1) & \frac{1-\mu_1^2}{2\mu_1} \\ \frac{1-\mu_1^2}{2\mu_1} & e^{-\Lambda^{(0)}} \cosh(\gamma_1 - \tilde{\gamma}_1) \end{bmatrix} \tag{2.3.13}$$

It is easy to verify that $\det g^{(1)} = 1$ as required. One may also write γ_1 in the form of a traveling wave

$$\gamma_1 = kz + \omega t + K_1 \tag{2.3.14}$$

but the values of the coefficients k and ω depend on the background $\Lambda^{(0)}$ chosen.

To understand the physical significance of this solution, let us consider two cases of important significance. The first, is when the function $\Lambda^{(0)}$ is time-like ($\eta_{ij}\Lambda^{(0)}_{,i}\Lambda^{(0)}_{,j} < 0$). For example, if $\Lambda^{(0)} = t$ (and then $\tilde{\Lambda}^{(0)} = x$ is space-like) then γ_1 is written in the form (2.3.14) with

$$k = \frac{2\mu_1}{\mu_1^2 - 1} \qquad \omega = \frac{\mu_1^2 + 1}{\mu_1^2 - 1} \tag{2.3.15}$$

and therefore the velocity of the soliton is

$$v = \frac{\omega}{k} = \frac{\mu_1^2 + 1}{2\mu_1} > 1 \tag{2.3.16}$$

Namely, we have a traveling superluminal soliton which travels to the left if $\mu_1 > 0$ and to the right if $\mu_1 < 0$. As mentioned in the previous chapter, since $\alpha = 1$ this solution is diffeomorphic to Minkowski spacetime, and therefore there is no violation of Lorentz invariance. The most interesting coefficient in the matrix $g^{(1)}$ is the non-diagonal coefficient $g^{(1)}_{12}$, from which we see that the center-of-mass of the soliton is located at $z_{C.O.M} = vt + \frac{K_1}{k}$ for all moments of time and the soliton's amplitude is $A = \frac{1-\mu_1^2}{2\mu_1}$. Note that the amplitude might be negative, in which case the soliton lies below the z-axis. The diagonal terms travel with the same velocity. Figures (2.1,2.2,2.3) show the elements the matrix $g^{(1)}$.

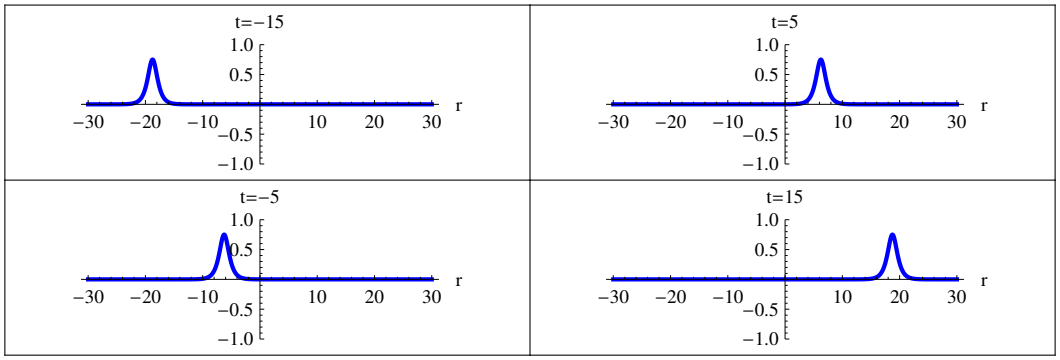

FIGURE 2.1. The non-diagonal coefficient $g_{12}^{(1)}$ for a single soliton of the symmetric chiral field Eq. (2.0.1) or the reduced Einstein Eq. (1.5.8) with $\alpha = 1$. The figure was created for constants $C_1 = 1$ and $\mu_1 = -2$ in Eq. (2.3.13). Here only the non-stationary case $e = 1$ is demonstrated, and as time varies the soliton travels to the right or left depending on the sign of the pole μ_1 of the dressing matrix. If $\Lambda^{(0)}$ is time-like, the soliton travels faster than the speed of light and if $\Lambda^{(0)}$ is space-like it travels slower than the speed of light.

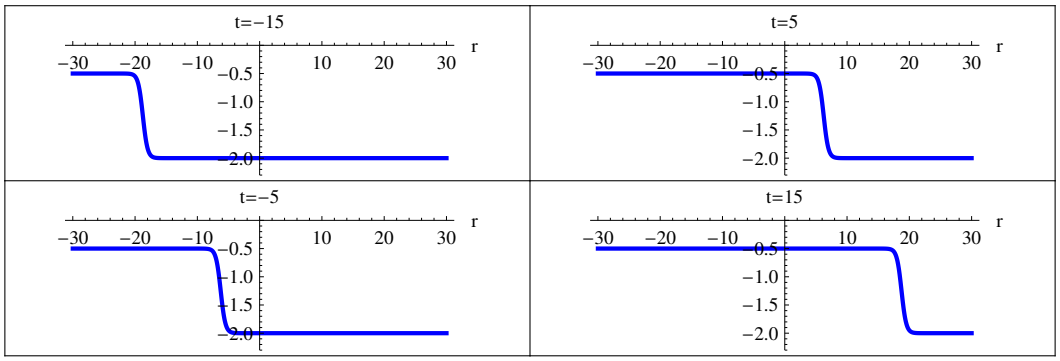

FIGURE 2.2. The diagonal coefficient $g_{11}^{(1)}$ for a single soliton of the symmetric chiral field Eq. (2.0.1) or the reduced Einstein Eq. (1.5.8) with $\alpha = 1$. The figure was created for constants $C_1 = 1$ and $\mu_1 = -2$ in Eq. (2.3.13). Here only the non-stationary case $e = 1$ is demonstrated. One can see from the figure that the spatial derivative of $g_{11}^{(1)}$ is a 'physical soliton' (see section 1.3).

The second interesting case is when the function $\Lambda^{(0)}$ is space-like ($\eta_{ij}\Lambda^{(0)}_{,i}\Lambda^{(0)}_{,j} > 0$). For example, if $\Lambda^{(0)} = x$ (and then $\tilde{\Lambda}^{(0)} = t$ is time-like) then D is written in the form

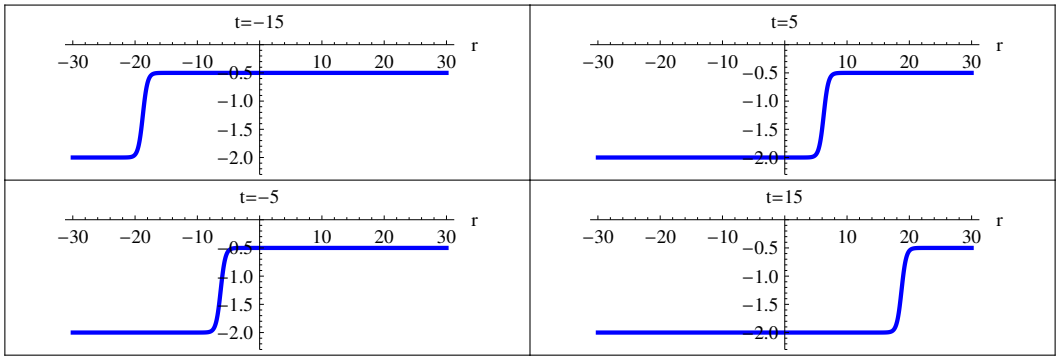

FIGURE 2.3. The diagonal coefficient $g^{(1)}_{22}$ for a single soliton of the symmetric chiral field Eq. (2.0.1) or the reduced Einstein Eq. (1.5.8) with $\alpha = 1$. The figure was created for constants $C_1 = 1$ and $\mu_1 = -2$ in Eq. (2.3.13). Here only the non-stationary case $e = 1$ is demonstrated. One can see from the figure that the spatial derivative of $g^{(1)}_{11}$ deserves the title 'physical soliton' (see section 1.3).

(2.3.14) with

$$k = \frac{\mu_1^2 + 1}{\mu_1^2 - 1} \qquad \omega = \frac{2\mu_1}{\mu_1^2 - 1} \tag{2.3.17}$$

and therefore the velocity of the soliton is

$$v = \frac{\omega}{k} = \frac{2\mu_1}{\mu_1^2 + 1} < 1 \tag{2.3.18}$$

This is a traveling subluminal soliton which may travel again to the left ($\mu_1 > 0$) or to the right ($\mu_1 < 0$). The center-of-mass and amplitude of the soliton is the same as in the other case of $\Lambda^{(0)}$ being time-like and figures (2.1,2.2,2.3) demonstrate most features of its behavior (except for its velocity).

The two soliton ($N = 2$) solution is given by

$$\begin{aligned}
g^{(2)}_{11} &= e^{\Lambda^{(0)}} \frac{(\mu_1-\mu_2)^2 \cosh^2(\frac{\gamma_1+\gamma_2+\tilde{\gamma_1}+\tilde{\gamma_2}}{2}) + (\mu_1\mu_2-1)^2 \sinh^2(\frac{\gamma_1-\gamma_2+\tilde{\gamma_1}-\tilde{\gamma_2}}{2})}{(\mu_1-\mu_2)^2 \cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-1)^2 \sinh^2(\frac{\gamma_1-\gamma_2}{2})} \\
g^{(2)}_{12} &= \frac{(\mu_1-\mu_2)(\mu_1\mu_2-1)}{2\mu_1\mu_2} \cdot \frac{\mu_1(\mu_2^2-1)\cosh\gamma_1 - \mu_2(\mu_1^2-1)\cosh\gamma_2}{(\mu_1-\mu_2)^2 \cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-1)^2 \sinh^2(\frac{\gamma_1-\gamma_2}{2})} \\
g^{(2)}_{22} &= e^{-\Lambda^{(0)}} \frac{(\mu_1-\mu_2)^2 \cosh^2(\frac{\gamma_1+\gamma_2-\tilde{\gamma_1}-\tilde{\gamma_2}}{2}) + (\mu_1\mu_2-1)^2 \sinh^2(\frac{\gamma_1-\gamma_2-\tilde{\gamma_1}+\tilde{\gamma_2}}{2})}{(\mu_1-\mu_2)^2 \cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-1)^2 \sinh^2(\frac{\gamma_1-\gamma_2}{2})}
\end{aligned} \tag{2.3.19}$$

It is also given in terms of hyperbolic functions depending on the functions $\gamma_s, \tilde{\gamma}_s$. In this case, each individual soliton preserves its original features that were studied in the one-soliton case, including its amplitude, velocity and shape. As is usually the case for physical solitons, the non-trivial interaction between the two solitons results in a phase shift. The non-diagonal term $g_{12}^{(2)}$ is shown in figure (2.4) while the diagonal-term $g_{11}^{(1)}$ is depicted in figure (2.5). The other diagonal-term $g_{22}^{(1)}$ is not shown, as it has a rather similar form to $g_{11}^{(1)}$.

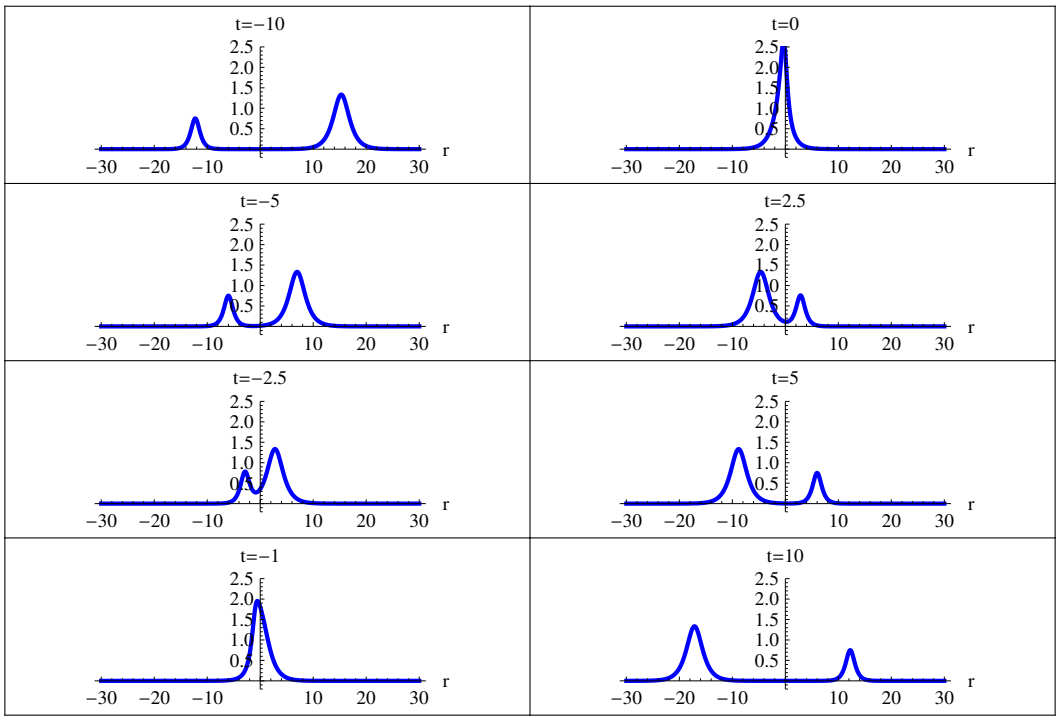

FIGURE 2.4. The non-diagonal coefficient $g_{12}^{(2)}$ for two solitons of the symmetric chiral field Eq. (2.0.1). The first soliton has parameters $\mu_1 = -2$ and $C_1 = 1$ and the second has parameters $\mu_2 = 3, C_2 = 2$. As is usually the case for physical solitons, they interact non-trivially and finish the elastic collision process with a phase shift.

One may proceed in a similar fashion, expressing the N-soliton solution in Eq. (2.3.7) in terms of hyperbolic functions and the functions $\gamma_s, \tilde{\gamma}_s$. This will not be done here, as the general features of the solitons remain the same as described earlier

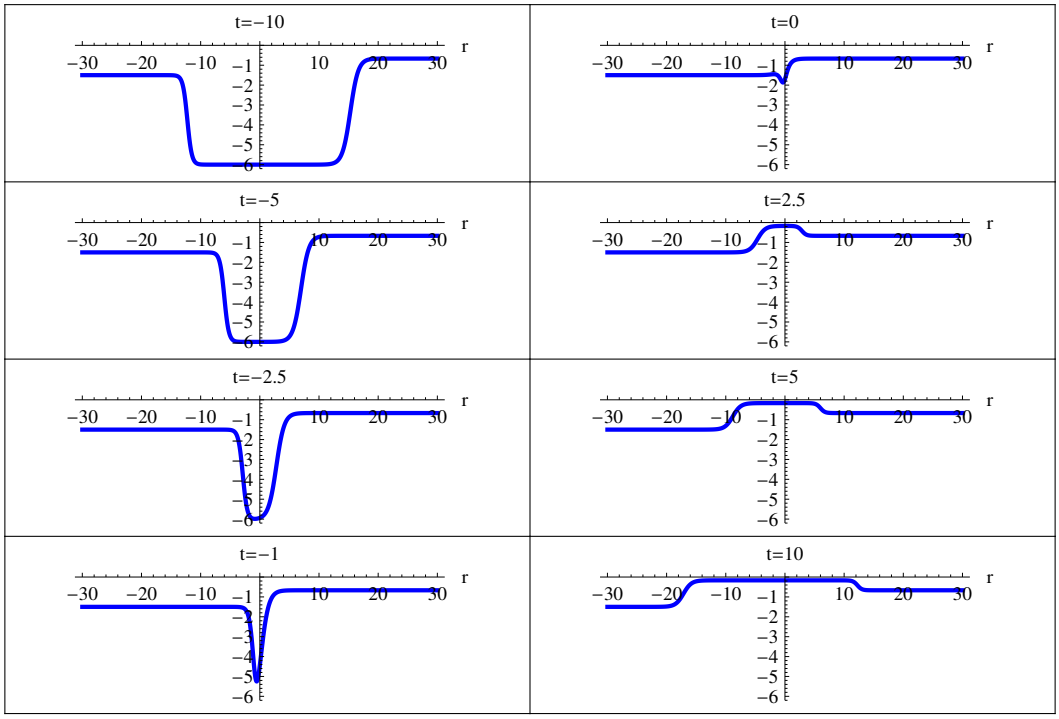

FIGURE 2.5. The diagonal coefficient $g_{11}^{(2)}$ for two solitons of the symmetric chiral field Eq. (2.0.1). The first soliton has parameters $\mu_1 = -2$ and $C_1 = 1$ and the second has parameters $\mu_2 = 3, C_2 = 2$.

and don't deviate in any way from the expected result considering other famous solitonic systems such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation.

It is the goal of the next chapter to study the reduced Einstein Eq. (1.5.8) when $\alpha \neq 1$. We will discover that many of the attributes discussed here in the case $\alpha = 1$ cease to hold. In particular, although we can consider solitons with $\alpha = 1$ as being both mathematical and physical solitons, this will no longer be the case for $\alpha \neq 1$. Indeed, the most striking fact about gravitational solitons studied next is that they are *not physical solitons* at all, but only mathematical solitons. For this reason we differentiated between the two types of solitons (mathematical versus physical) in section 1.3.

Chapter 3
The $1+1$ Einstein equation

In this chapter, Einstein's equation depending on two variables is studied. As explained in section 1.5, in the case of a block diagonal metric this boils down to study the reduced Einstein Eq. (1.5.8):

$$\eta^{ij}\left(\alpha g_{ac,i}g^{cb}\right)_{,j}=0 \tag{3.0.1}$$

where

$$\det g_{ab}=e\alpha^2 \tag{3.0.2}$$

and $e=\pm 1$.

Its solutions can be obtained using the Belinski-Zakharov transform described in section 1.5. In particular, one may use the Belinski-Zakharov transform to obtain new soliton-like solutions of the vacuum Einstein's Equation from known particular solutions of Eq. (3.0.1) (see section 1.7). Such solutions will be referred to as *gravitational solitons*.

To keep the discussion concrete, it will be assumed that the metric is non-stationary. Later on, an additional assumption of cylindrical symmetry will be invoked as well. Both of these assumptions are in no way limiting, as a simple change of variable can transform both of them into the general case of spacetimes depending on two variables as discussed in section 1.5. Thus, one may easily obtain analogous results for stationary metrics in the same way, but they will not be given explicitly here.

The assumption of non-stationary spacetime corresponds to the choice $e=1$ and describes the propagation of gravitational waves. In particular, the spacetime interval of such metrics has the form (see Eq. (1.5.6))

$$ds^2=f(t,z)(-dt^2+dz^2)+g_{ab}(t,z)dx^a dx^b \tag{3.0.3}$$

where the reduced Einstein Eq. (3.0.1) in this case is Eq. (1.5.10):

$$(\alpha g_{,t} g^{-1})_{,t} - (\alpha g_{,z} g^{-1})_{,z} = 0 \tag{3.0.4}$$

and the function f satisfies Eqs. (1.5.18),

$$\begin{aligned} (\ln f)_{,z}(\ln \alpha)_{,z} + (\ln f)_{,t}(\ln \alpha)_{,t} &= (\ln \alpha)_{,zz} + (\ln \alpha)_{,tt} + \frac{1}{4\alpha^2} Tr(A_0^2 + A_3^2) \\ (\ln f)_{,t}(\ln \alpha)_{,z} + (\ln f)_{,z}(\ln \alpha)_{,t} &= 2(\ln \alpha)_{,tz} + \frac{1}{2\alpha^2} Tr(A_0 A_3) \end{aligned} \tag{3.0.5}$$

coupled to Eq. (1.5.17),

$$(\ln f)_{,zz} - (\ln f)_{,tt} = \frac{1}{4\alpha^2} Tr(A_3^2 - A_0^2) - (\ln \alpha)_{,zz} + (\ln \alpha)_{,tt} \tag{3.0.6}$$

Here the matrices A_0 and A_3 were defined in Eq. (1.5.14) to be:

$$A_0 = \alpha g_{,t} g^{-1} \qquad A_3 = \alpha g_{,z} g^{-1} \tag{3.0.7}$$

Alternatively, one may also use light-cone coordinates

$$\zeta = \frac{1}{2}(z + t) \qquad \eta = \frac{1}{2}(z - t) \tag{3.0.8}$$

with which Eq. (3.0.4) is

$$(\alpha g_{,\zeta} g^{-1})_{,\eta} + (\alpha g_{,\eta} g^{-1})_{,\zeta} = 0 \tag{3.0.9}$$

and Eqs. (1.5.18) have the form

$$\begin{aligned} (\ln f)_{,\zeta}(\ln \alpha)_{,\zeta} &= (\ln \alpha)_{,\zeta\zeta} + \frac{1}{4\alpha^2} Tr A^2 \\ (\ln f)_{,\eta}(\ln \alpha)_{,\eta} &= (\ln \alpha)_{,\eta\eta} + \frac{1}{4\alpha^2} Tr B^2 \end{aligned} \tag{3.0.10}$$

together with Eq. (1.5.17)

$$(\ln f)_{,\zeta\eta} = \frac{1}{4\alpha^2} Tr AB - (\ln \alpha)_{,\zeta\eta} \tag{3.0.11}$$

Here the matrices A and B are defined to be

$$A = -\alpha g_{,\zeta} g^{-1} \qquad B = \alpha g_{,\eta} g^{-1} \tag{3.0.12}$$

and generalize their former definition (2.0.3) in the case of the chiral field. The main advantage of the light-cone coordinates is their symmetric form, that treats both coordinates on equal footing and therefore better manifests the principles of the theory of relativity.

In the next section 3.1 the Lagrangian formulation for the reduced Einstein Eq. (3.0.4) is derived. In section 3.2 an alternative formulation of this equation is presented, based on the eigenvalues of the matrix g. Section 3.3 provides a Hamiltonian formulation for the reduced Einstein equation. This gives an alternative definition of the concept of energy than the one typically used in the literature. In section 3.4 a non-trivial local conservation law is obtained. The most important form of spacetime metrics (from a physical point of view) are synchronous metrics. Such metrics and their relation to the Belinski-Zakharov transform is presented in section 3.5. Section 3.6 discusses gravitational solitons, including their physical properties and interactions on diagonal backgrounds. Finally, section 3.7 contains a numerical scheme for solving the reduced Einstein equation and compares its consequences to the results studied in the rest of this chapter.

3.1 Lagrangian formulation

A simple computation based on the formulas in appendix B shows that the Einstein-Hilbert action (1.2.9) of general relativity can be written in this case using a Lagrangian density of a form analogous to the one given in classical Newtonian mechanics. Namely, we can write the action as

$$S = \int \mathcal{L} dt dz \tag{3.1.1}$$

where the Lagrangian density is

$$\mathcal{L} = T - V \tag{3.1.2}$$

The kinetic and the potential energies T and V are defined to be

$$\begin{aligned} T &= -\frac{\alpha_{,t} f_{,t}}{f} - \frac{\Delta^{(t)}}{2\alpha} \\ V &= -\frac{\alpha_{,z} f_{,z}}{f} - \frac{\Delta^{(z)}}{2\alpha} \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} \Delta^{(t)} &= \det\left(\frac{\partial g}{\partial t}\right) = g_{11,t} g_{22,t} - g_{12,t} g_{21,t} \\ \Delta^{(z)} &= \det\left(\frac{\partial g}{\partial z}\right) = g_{11,z} g_{22,z} - g_{12,z} g_{21,z} \end{aligned} \tag{3.1.4}$$

To show that this Lagrangian indeed works, one needs to compute its variation with respect to the fields f and g_{ab}. In appendix F we compute the derivatives of the different terms in the Lagrangian with respect to the matrix g_{ab}. This allows to easily obtain the variations of the Einstein-Hilbert action, giving

$$\frac{\delta S}{\delta f} = \frac{-\alpha_{,tt} + \alpha_{,zz}}{f} \tag{3.1.5}$$

and

$$\begin{aligned} \frac{\delta S}{\delta g_{ab}} &= \frac{\Delta^{(t)}}{4\alpha} g^{ab} + \frac{\alpha}{2} g^{ab} (\ln f)_{,tt} + \alpha_{,tt} g^{ab} + \frac{3}{2} \alpha_{,t} (g^{ab})_{,t} + \frac{\alpha}{2} (g^{ab})_{,tt} \\ &\quad - \frac{\Delta^{(z)}}{4\alpha} g^{ab} - \frac{\alpha}{2} g^{ab} (\ln f)_{,zz} - \alpha_{,zz} g^{ab} - \frac{3}{2} \alpha_{,z} (g^{ab})_{,z} - \frac{\alpha}{2} (g^{ab})_{,zz} \end{aligned} \tag{3.1.6}$$

It is clear that the variation of the action with respect f is zero precisely when α satisfies the wave equation (1.5.11). Moreover, setting Eq. (3.1.6) to zero and contracting it with g_{ab} gives

$$(\ln f)_{,zz} - (\ln f)_{,tt} = \left[\frac{\Delta^{(z)}}{2\alpha^2} - \frac{\alpha_{,zz}}{\alpha}\right] - \left[\frac{\Delta^{(t)}}{2\alpha^2} - \frac{\alpha_{,tt}}{\alpha}\right] \tag{3.1.7}$$

which is equivalent to Eq. (3.0.6). Substituting this result back in Eq. (3.1.6), and performing a few algebraic manipulations gives the matrix Eq. (3.0.4) as one would expect, proving the equivalence of this Lagrangian formulation to Eqs. (3.0.4) and (3.0.6).

Last but not least, we remind the reader that one may also use the light-cone coordinates introduced in Eq. (3.0.8). In these coordinates the Lagrangian density is symmetric with respect to interchanging the two coordinates $\zeta \leftrightarrow \eta$:

$$\mathcal{L} = \frac{\alpha_{,\zeta} f_{,\eta} + \alpha_{,\eta} f_{,\zeta}}{2f} + \frac{1}{4\alpha}\left[g_{11,\zeta} g_{22,\eta} + g_{11,\eta} g_{22,\zeta} - g_{12,\zeta} g_{21,\eta} - g_{12,\eta} g_{21,\zeta}\right] \tag{3.1.8}$$

3.2 The geometric representation

The fact that the 2×2 matrix g is symmetric allows one to diagonalize it for fixed t and z. This means that it may be written in the form

$$g = RDR^T \tag{3.2.1}$$

where R is a rotation matrix,

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{3.2.2}$$

and D is a diagonal matrix

$$D = \alpha \begin{bmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{bmatrix} \tag{3.2.3}$$

This naturally generalizes the diagonalization we performed for the symmetric principal chiral field equation in section 2.1. Written explicitly, Eq. (3.2.1) is

$$g = \alpha \begin{bmatrix} \cosh\Lambda + \sinh\Lambda\cos 2\phi & \sinh\Lambda \sin 2\phi \\ \sinh\Lambda \sin 2\phi & \cosh\Lambda - \sinh\Lambda\cos 2\phi \end{bmatrix} \tag{3.2.4}$$

Eq. (3.2.1) clearly holds for fixed t and z. If t and z vary, then so will ϕ, α and Λ. Therefore one may think of ϕ, α and Λ as three new degrees of freedom replacing the original three degrees of freedom in the symmetric matrix g. This gives an alternative representation of the reduced Einstein Eq. (3.0.4) that has a simpler geometric interpretation. For this reason, we will refer to it as the *geometric representation* [33].

The eigenvalues of the matrix g are easily seen to be αe^{Λ} and $\alpha e^{-\Lambda}$. Therefore when $\phi = 0$, the fields α and Λ control the deformation of spacetime in the x and y

directions. From the form of Eq. (3.2.1), it is clear that the field ϕ is responsible for a rotation of spacetime in the $x-y$ plane.

Inverting Eq. (3.2.4) gives α, Λ and ϕ in terms of the matrix g in a straight-forward manner:

$$\begin{aligned} \alpha &= \sqrt{\det g} && (3.2.5) \\ \cosh\Lambda &= \frac{1}{2\alpha} Tr(g) && (3.2.6) \\ \tan 2\phi &= \frac{2g_{12}}{g_{11}-g_{22}} && (3.2.7) \end{aligned}$$

The last equation shows that indeed ϕ measures the deviation of the matrix g from a diagonal matrix. Small values of ϕ correspond to gravitational waves with a (nearly) single polarization.

Consider the Lagrangian density suggested in the previous section in Eq. (3.1.2). In the geometric representation its kinetic and potential energies (3.1.3) are

$$\begin{aligned} T &= -\frac{\alpha_{,t} f_{,t}}{f} - \frac{\alpha_{,t}^2}{2\alpha} + \frac{1}{2}\alpha\Lambda_{,t}^2 + 2\alpha\phi_{,t}^2 \sinh^2(\Lambda) && (3.2.8) \\ V &= -\frac{\alpha_{,z} f_{,z}}{f} - \frac{\alpha_{,z}^2}{2\alpha} + \frac{1}{2}\alpha\Lambda_{,z}^2 + 2\alpha\phi_{,z}^2 \sinh^2(\Lambda) \end{aligned}$$

Computing the variation of the action with respect to the fields ϕ, Λ, α and f gives:

$$\begin{aligned} \frac{\delta S}{\delta\phi} &= -(4\alpha\phi_{,t}\sinh^2\Lambda)_{,t} + (4\alpha\phi_{,z}\sinh^2\Lambda)_{,z} = 0 && (3.2.9) \\ \frac{\delta S}{\delta\Lambda} &= -(\alpha\Lambda_{,t})_{,t} + (\alpha\Lambda_{,z})_{,z} + 2\alpha\sinh(2\Lambda)(\phi_{,t}^2 - \phi_{,z}^2) = 0 \\ \frac{\delta S}{\delta(\ln f)} &= \alpha_{,tt} - \alpha_{,zz} = 0 \\ \frac{\delta S}{\delta\alpha} &= (\ln f)_{,tt} - (\ln f)_{,zz} + (\ln\alpha)_{,tt} - (\ln\alpha)_{,zz} + \frac{\alpha_{,t}^2 - \alpha_{,z}^2}{2\alpha^2} \\ &\quad + \frac{1}{2}(\Lambda_{,t}^2 - \Lambda_{,z}^2) + 2(\phi_{,t}^2 - \phi_{,z}^2)\sinh^2\Lambda = 0 \end{aligned}$$

This set of four equations is equivalent to the reduced Einstein Eq. (3.0.4) together with the diagonal Eq. (3.0.6) for the function f. In principle, one may solve the non-diagonal Eqs. (3.0.5) for $(\ln f)_{,t}$ and $(\ln f)_{,z}$ to eliminate the function f altogether

from the Lagrangian. This is only possible because the reduced Einstein Eq. (3.0.4) is decoupled from Eqs. (3.0.5) for f in the first place.

The reader might hope that the first equation in Eqs. (3.2.9) is a new conservation law. However, this conservation law is not independent of the matrix elements of the reduced Einstein Eq. (3.0.4) that form four conservation laws for their own sake.

Beyond the convenient interpretation that the geometric representation provides, Eqs. (3.2.9) has another merit, as they make the study of diagonal metrics very simple. In this case, for which $\phi = 0$, the Lagrangian simplifies to

$$\mathcal{L} = -\frac{\alpha_{,t} f_{,t}}{f} - \frac{\alpha_{,t}^2}{2\alpha} + \frac{1}{2}\alpha\Lambda_{,t}^2 + \frac{\alpha_{,z} f_{,z}}{f} + \frac{\alpha_{,z}^2}{2\alpha} - \frac{1}{2}\alpha\Lambda_{,z}^2 \tag{3.2.10}$$

Therefore the functions α and Λ (that completely determine the matrix g in this case) satisfy two linear equations:

$$\begin{aligned} 0 &= \alpha_{,tt} - \alpha_{,zz} \\ 0 &= (\alpha\Lambda_{,t})_{,t} - (\alpha\Lambda_{,z})_{,z} \end{aligned} \tag{3.2.11}$$

3.3 Hamiltonian formulation

In this section we show that the $1+1$ Einstein's equations can also be described as a Hamiltonian system. To do so, we will use the canonical method to obtain the symplectic structure from the Lagrangian given in section 3.1.

Let φ [1] and γ^{ab} be the conjugate momenta to f and g_{ab} respectively. This means that by definition

$$\begin{aligned} \varphi &= \frac{\partial \mathcal{L}}{\partial \dot{f}} \\ \gamma^{ab} &= \frac{\partial \mathcal{L}}{\partial \dot{g}_{ab}} \end{aligned} \tag{3.3.1}$$

[1] Note the different symbols used for the Greek letter phi. In this section we use φ to denote the the conjugate momentum to f while ϕ is the rotation field introduced in the previous section.

where the dot denotes differentiation with respect to time t. To make the notation more concise, we will write the following equations in matrix form. This means that the matrix γ will be identified with γ^{ab} while g is identified with g_{ab} (notice that as tensors, they correspond to different types of tensors). Written explicitly the conjugate momenta are

$$\begin{aligned} \varphi &= -\frac{\dot{\alpha}}{f} \\ \gamma &= \left[-\frac{\dot{f}\alpha}{2f} - \alpha_t + \frac{\alpha}{2} g^{-1}\dot{g}\right] g^{-1} \end{aligned} \tag{3.3.2}$$

Reversing these relations to express $\dot{f}$ and $\dot{g}_{ab}$ in terms of φ and γ^{ab} yields

$$\begin{aligned} \dot{f} &= \frac{f}{\alpha}\left[f\varphi - Tr(\gamma g)\right] \\ \dot{g} &= \frac{1}{\alpha}\left[2g\gamma - Tr(g\gamma) - f\varphi\right] g \end{aligned} \tag{3.3.3}$$

Therefore, the Hamiltonian density

$$\mathcal{H} = \varphi\dot{f} + \gamma^{ab}\dot{g}_{ab} - \mathcal{L} \tag{3.3.4}$$

can be written explicitly as

$$\mathcal{H} = \frac{1}{\alpha}\left[Tr(g\gamma g\gamma) - \frac{1}{2}Tr^2(g\gamma) - f\varphi Tr(g\gamma) + \frac{1}{2}f^2\varphi^2 - \alpha\frac{\alpha_{,z}f_{,z}}{f} - \frac{1}{2}\Delta^{(z)}\right] \tag{3.3.5}$$

where $\Delta^{(z)}$ was defined in Eqs. (3.1.4). It is a simple exercise to verify that with this Hamiltonian density, Einstein's equation are equivalent to the following set of Hamilton's equations

$$\begin{aligned} \dot{f} &= \frac{\delta H}{\delta\varphi} \qquad & \dot{\varphi} &= -\frac{\delta H}{\delta f} \\ \dot{g} &= \frac{\delta H}{\delta\gamma} \qquad & \dot{\gamma} &= -\frac{\delta H}{\delta g} \end{aligned} \tag{3.3.6}$$

where the total Hamiltonian functional is

$$H = \int \mathcal{H} dt dz \tag{3.3.7}$$

In order to properly interpret the Hamiltonian density, it is convenient to use Eqs. (3.3.2) to rewrite the Hamiltonian density only in terms of the metric components f and g. This, of course, doesn't contribute to the analysis of the symplectic structure given, but will reveal that a simple interpretation of the kinetic and potential energy can be given. Using Eqs. (3.3.2) in Eq. (3.3.5) gives

$$\mathcal{H} = -\left[\frac{\alpha_{,t} f_{,t}}{f} + \frac{\Delta^{(t)}}{2\alpha}\right] - \left[\frac{\alpha_{,z} f_{,z}}{f} + \frac{\Delta^{(z)}}{2\alpha}\right] \tag{3.3.8}$$

Comparing the Lagrangian density (3.1.2) to the Hamiltonian density (3.3.8) allows us to interpret the term in the first brackets (together with the minus sign) as the kinetic energy, while the second brackets in the Hamiltonian density (again with the minus sign) as the potential energy. In other words, the Hamiltonian density is exactly

$$\mathcal{H} = T + V \tag{3.3.9}$$

where the kinetic and potential energies were defined in Eqs. (3.1.2). Remarkably, both the Lagrangian and Hamiltonian structures have similar form to their classical analogous (as the difference / sum of the kinetic and potential energy respectively). Moreover, the kinetic energy only contains temporal derivatives as one would hope to be the case from our knowledge of classical mechanics.

One would hope that the Hamiltonian structure provided here will give us a well-defined notion of energy in general relativity. Nevertheless, the following theorem shows that this is not the case:

Theorem 3.3.10. *For every solution g of the reduced Einstein Eq. (3.0.4) coupled to a solution f of Eqs. (3.0.5) and (3.0.6), the Hamiltonian functional is identically zero, namely*

$$H = 0 \tag{3.3.11}$$

Proof. Using the light-cone coordinates (3.0.8) the Hamiltonian density is

$$\mathcal{H} = -\frac{1}{2}\left[\frac{\alpha_{,\zeta} f_{,\zeta}}{f} + \frac{\Delta^{(\zeta)}}{2\alpha} + \frac{\alpha_{,\eta} f_{,\eta}}{f} + \frac{\Delta^{(\eta)}}{2\alpha}\right] \tag{3.3.12}$$

where

$$\Delta^{(\zeta)} = \det\left(\frac{\partial g}{\partial \zeta}\right) = g_{11,\zeta}g_{22,\zeta} - g_{12,\zeta}g_{21,\zeta} \tag{3.3.13}$$
$$\Delta^{(\eta)} = \det\left(\frac{\partial g}{\partial \eta}\right) = g_{11,\eta}g_{22,\eta} - g_{12,\eta}g_{21,\eta}$$

We may eliminate the function f from the Hamiltonian density using Eqs. (3.0.10). This gives,

$$\mathcal{H} = -\frac{1}{2}\left[\alpha_{,\zeta\zeta} - \frac{\alpha_{,\zeta}^2}{\alpha} + \frac{1}{4\alpha}Tr(A^2) + \frac{\Delta^{(\zeta)}}{2\alpha} + \alpha_{,\eta\eta} - \frac{\alpha_{,\eta}^2}{\alpha} + \frac{1}{4\alpha}Tr(B^2) + \frac{\Delta^{(\eta)}}{2\alpha}\right] \tag{3.3.14}$$

Notice however, that the following identities hold

$$Tr(A^2) = 4\alpha_{,\zeta}^2 - 2\Delta^{(\zeta)} \qquad Tr(B^2) = 4\alpha_{,\eta}^2 - 2\Delta^{(\eta)} \tag{3.3.15}$$

and therefore substituting them in Eq. (3.3.14) gives

$$\mathcal{H} = -\frac{1}{2}(\alpha_{,\zeta\zeta} + \alpha_{,\eta\eta}) \tag{3.3.16}$$

Therefore the Hamiltonian density consists of total derivatives only, that vanish upon integration over spacetime. □

The Hamiltonian introduced in this section is different than the famous ADM Hamiltonian of general relativity [34]. The difference lies in the fact that in the ADM formalism no particular reference frame (i.e. gauge) is chosen, and therefore the ADM Hamiltonian requires satisfying the auxiliary Einstein constraint equations [34] to overcome the gauge freedom. Hence the ADM Hamiltonian is a *constrained Hamiltonian*, while the Hamiltonian used here is not constrained. In the context of the ADM formalism, Theorem (3.3.10) is a known result.

3.4 Conservation laws

The existence of infinity many local conservation laws for the symmetric principal chiral field makes one hope that analogous conservation laws can be found for the

reduced Einstein Eq. (3.0.4). Nevertheless, such a result is still unknown and the reduced Einstein Eq. (3.0.4) might not be Liouville-integrable after all.

The most natural approach to find local conservation laws is by considering the Lie symmetries of the Lagrangian in Eq. (3.1.2). However, all of its Lie symmetries, including scaling of f, scaling or rotation of the matrix elements g_{ab} result in trivial conservation laws. Unfortunately, these conservation laws are either one of the matrix elements of the reduced Einstein Eq. (3.0.4) or a consequence of it (e.g. the wave Eq. (1.5.11) for α.) As we saw in the last section, even the Hamiltonian is a trivial integral of motion as it vanishes identically.

Nevertheless, there is one non-trivial local conservation law we can derive immediately. Taking the derivative of the first equation in (3.0.10) with respect to η, and of the second equation in (3.0.10) with respect to ζ and subtracting the results gives:

$$\left[\frac{\alpha_{,\zeta\zeta}}{\alpha_{,\zeta}} - \frac{\alpha_{,\zeta}}{\alpha} + \frac{1}{4\alpha\alpha_{,\zeta}} TrA^2\right]_{,\eta} = \left[\frac{\alpha_{,\eta\eta}}{\alpha_{,\eta}} - \frac{\alpha_{,\eta}}{\alpha} + \frac{1}{4\alpha\alpha_{,\eta}} TrB^2\right]_{,\zeta} \tag{3.4.1}$$

This is of course a conservation law for the matrix g. Since α satisfies the wave Eq. (1.5.11), the first two terms in each of the brackets give a trivial contribution and can therefore be forfeited. Thus we are left with the conservation law,

$$\left[\frac{1}{4\alpha\alpha_{,\zeta}} TrA^2\right]_{,\eta} = \left[\frac{1}{4\alpha\alpha_{,\eta}} TrB^2\right]_{,\zeta} \tag{3.4.2}$$

In the (t, z) variables, this may be written in the canonical form of a conservation law

$$\frac{\partial}{\partial t} T + \frac{\partial}{\partial z} Z = 0 \tag{3.4.3}$$

with

$$\begin{aligned} T &= \frac{\alpha}{\alpha_{,z}^2 - \alpha_{,t}^2} \left[\alpha_{,z} \mathrm{tr}\left(g_{,t} g^{-1} g_{,t} g^{-1} + g_{,z} g^{-1} g_{,z} g^{-1}\right)\right. \\ &\qquad \left. -\alpha_{,t} \mathrm{tr}\left(g_{,t} g^{-1} g_{,z} g^{-1} + g_{,z} g^{-1} g_{,t} g^{-1}\right)\right] \\ Z &= \frac{\alpha}{\alpha_{,z}^2 - \alpha_{,t}^2} \left[\alpha_{,t} \mathrm{tr}\left(g_{,t} g^{-1} g_{,t} g^{-1} + g_{,z} g^{-1} g_{,z} g^{-1}\right)\right. \\ &\qquad \left. -\alpha_{,z} \mathrm{tr}\left(g_{,t} g^{-1} g_{,z} g^{-1} + g_{,z} g^{-1} g_{,t} g^{-1}\right)\right] \end{aligned} \tag{3.4.4}$$

Integrating with respect to z and assuming Z vanishes as $z \to \pm\infty$ gives the integral of motion

$$E = \int T dz \tag{3.4.5}$$

which we will associate with energy.

The definition of energy in general relativity is a complex matter, and this problem has been given a lot of attention in the literature [5]. We shall not dwell on this issue here, but instead only stress that the quantity E defined above is not the famous ADM energy [34]. As mentioned at the end of the last section, the ADM notion of energy is always zero for solutions of the vacuum Einstein equation, but this property is not true for the energy defined in Eq. (3.4.5).

There is another reason why the quantity E is a 'good' notion of energy. In section 3.6 we study metrics with cylindrical symmetry, in which case $\alpha = r$ and the cylindrical coordinates are $x^\mu = \{t, r\}$ and $x^a = \{\phi, z\}$. In such a case the conservation law (3.4.3) is

$$\frac{\partial}{\partial t}\varepsilon + \frac{\partial}{\partial r}F = 0 \tag{3.4.6}$$

where

$$\begin{aligned} \varepsilon(t,r) &= r \cdot \mathrm{tr}\left(g_{,t}g^{-1}g_{,t}g^{-1} + g_{,r}g^{-1}g_{,r}g^{-1}\right) \\ F(t,r) &= -r \cdot \mathrm{tr}\left(g_{,t}g^{-1}g_{,r}g^{-1} + g_{,r}g^{-1}g_{,t}g^{-1}\right) \end{aligned} \tag{3.4.7}$$

The energy is now

$$E = \int_0^\infty \varepsilon(t,r) dr \tag{3.4.8}$$

and if the matrix is diagonal ($\phi = 0$) it is just

$$E = \int_0^\infty (\frac{1}{r} + r(\Lambda_{,t}^2 + \Lambda_{,r}^2)) dr \tag{3.4.9}$$

where Λ was defined in section 3.2. Except for the logarithmic derivative $\frac{1}{r}$ which is independent of Λ, this is nothing other than the canonical notion of energy for the wave equation in polar coordinates (see the second equation in Eqs. (3.2.11).) So the

notion of energy defined above naturally generalizes a familiar concept of energy. We may therefore interpret $\varepsilon(t,r)$ as the energy density of an infinitesimally thin annular cylinder of radius r centered at the z-axis at the moment of time t. Similarly, one may interpret F as the flux of energy through such a cylinder.

We will come back to this conservation law and concept of energy where we study gravitational solitons in section 3.6.

3.5 Synchronous reference frames

The most convenient coordinate system for physical reasons is a synchronous coordinate system. This is a spacetime interval of the form

$$ds^2 = -dt^2 + \sum_{i,j=1}^{3} g_{ij}dx^i dx^j \qquad \text{(No summation convention)} \tag{3.5.1}$$

The importance of such coordinate systems lies in the fact the coordinate t defines the proper time of all comoving observers, and for that reason they are called 'synchronous'. It is an important fact in general relativity that any metric can be transformed to this synchronous form (at least locally), although the synchronous form will not be unique [5].

In this section we study the conditions under which the block diagonal metric in Eq. (3.0.3),

$$ds^2 = f(t,z)(-dt^2 + dz^2) + g_{ab}(t,z)dx^a dx^b \tag{3.5.2}$$

reduces to the synchronous metric

$$ds^2 = -du^2 + A(u,v)dv^2 + g_{ab}(u,v)dx^a dx^b \tag{3.5.3}$$

and suggest a simple method to accomplish this task.

By applying the chain-rule to the transformation $(u,v) \mapsto (t,z)$, one notices immediately that the synchronous metric (3.5.3) originates from the metric (3.5.2) if

and only if the following three equations are satisfied:

$$\begin{aligned} u_{,t}^2 - A v_{,t}^2 &= f \\ u_{,z}^2 - A v_{,z}^2 &= -f \\ u_{,t} u_{,z} - A v_{,t} v_{,z} &= 0 \end{aligned} \tag{3.5.4}$$

If we consider transforming an arbitrary Belinski-Zakharov metric (3.5.2) to the synchronous metric (3.5.3), then Eqs. (3.5.4) form a system of three partial differential equations for three unknown functions $u(t,z)$, $v(t,z)$ and $A(t,z)$. They can be simplified greatly by eliminating the unknown function A.

Let us multiply the first equation with $v_{,z}$ and the third with $v_{,t}$ and subtract the results. Similarly, multiply the second equation by $v_{,t}$ and the third equation by $v_{,z}$ and subtract the results. This gives two linear homogeneous equations for the partial derivatives of the function v:

$$\begin{bmatrix} -u_{,t}u_{,z} & u_{,t}^2 - f \\ u_{,z}^2 + f & -u_{,t}u_{,z} \end{bmatrix} \begin{bmatrix} v_{,t} \\ v_{,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.5.5}$$

As a homogeneous linear system, it has a nontrivial solution if and only if this system's determinant vanishes, namely when

$$u_{,t}^2 - u_{,z}^2 = f \tag{3.5.6}$$

In such a case, the function v has to satisfy the *linear* differential equation

$$u_{,t} v_{,t} - u_{,z} v_{,z} = 0 \tag{3.5.7}$$

This method works for transforming any metric (3.5.2) to a synchronous reference frame. One first needs to solve Eq. (3.5.6) for the function u, and then solve Eq. (3.5.7) for the function v. The solution to these equations is in no way unique.

3.6 Gravitational solitons

We are finally ready to discuss the most important consequence of the Belinski-Zakharov transform 1.5, the existence of gravitational solitons. We start the section

by summarizing the various seeds that were previously studied in the literature. Then we extend the known results by considering general diagonal seeds. In particular, we obtain and study gravitational solitons on the famous Einstein-Rosen metric [21].

3.6.A List of seeds studied in the literature

Table (3.1) contains a list of the seeds studied previously in the literature. In first sight the list might seem too short, but this is the case for several reasons. First, it includes only seeds from which gravitational solitons were derived analytically and in closed form, and omits numerical solutions. Secondly, there are dozens of papers in the literature that discuss possible interpretations of the gravitational solitons listed here, but we omit them as well. Moreover, the table only includes seeds on 4-dimensional spacetime and excludes the more general Alekseev-Belinski-Zakharov transform [30] that includes electromagnetic effects.

(x^0, x^1, x^2, x^3)	α	Seed	Source
(t, x, y, z)	Any	$f^{(0)} = C_0 \alpha_{,\zeta} \alpha_{,\eta} \alpha^{2q^2-1/2}$ $g^{(0)} = \begin{bmatrix} \alpha^{1+2q} & 0 \\ 0 & \alpha^{1-2q} \end{bmatrix}$	1978 [2] [17]
(ρ, t, φ, z)	ρ	$f^{(0)} = 1$ $g^{(0)} = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2 \end{bmatrix}$	1978 [3] [18]
(r, θ, t, z)	r	$f^{(0)} =$ arbitrary $g^{(0)} = \begin{bmatrix} r^2 e^{-\phi} & 0 \\ 0 & e^{\phi} \end{bmatrix}$, $\phi =$arbitrary	1984-5 [4] [35, 36]
(t, x, y, z)	t	$f^{(0)}$ depend on t only $g^{(0)} = l^T(z)\gamma(t)l(z)$	1982 [5] [37]

TABLE 3.1. List of seeds studied in the literature using the Belinski-Zakharov transform. This list only includes seeds in four-dimensional spacetime. In particular, the list only contains analytic work on such seeds and excludes (i) numerical work, and (ii) discussion of possible interpretations.

3.6.B Diagonal backgrounds

In this section we consider diagonal backgrounds. The goal will be to generate the most general solitonic solution on such backgrounds, namely N gravitational solitons independently of the (diagonal) background used. The one and two solitons on diagonal background were studied previously in [35, 36]. We extend this result by (i) deriving explicit formulas for an arbitrary number of gravitational solitons, (ii) study the properties and behavior of such gravitational solitons, (iii) explain the reason for the discrepancy between gravitational solitons and other known solitons, (iv) find exact conditions under which such gravitational solitons are physically meaningful and (v) consider gravitational solitons on Einstein-Rosen background [21]. Unlike [35, 36], it is not assumed that the spacetime metric is stationary.

Consider spacetime intervals of the form,

$$ds^2 = f^{(0)}(-dt^2 + dr^2) + e^{\Lambda^{(0)}}(rd\phi)^2 + e^{-\Lambda^{(0)}}dz^2 \tag{3.6.1}$$

where $f^{(0)} > 0$ and Λ_0 are functions of t, r. This corresponds to the choice of coordinates $x^a = (\phi, z)$ and $x^i = (t, r)$ in the background metric (1.7.12) with $g^{(0)} = \mathrm{diag}(r^2 e^{\Lambda^{(0)}}, e^{-\Lambda^{(0)}})$. The functions α and β from Eq. (1.7.3) are

$$\alpha = r \qquad \beta = t \tag{3.6.2}$$

Taking $\alpha = r$ is more than just a special case, as all other diagonal cases can be obtained from such metrics by a simple change of coordinates [29].

[2]This is the famous Kasner metric [22], which is a Bianchi Type I solution. The one-soliton was obtained for an arbitrary q, while the two-solitons were obtained for $q = 0$ and $q = -1/2$ only.

[3]This is just a simple flat-space time in cylindrical coordinates. They showed that two solitons correspond in this case to the Kerr-NUT solution while $2N$ solitons correspond to N noninteracting black holes remaining fixed along the z-axis

[4]In this work, the author studied *stationary* gravitational solitons on arbitrary diagonal background.

[5]Any Bianchi type I-VI solution of Einstein equation can be casted to such form. In this work they assumed that the solution is a Bianchi II solution, namely that $\frac{dl}{dz} = C^T \varepsilon l$ where C is symmetric 2×2 constant matrix. The matrix ε was defined in Eq. (1.5.19). The work focuses only on a single soliton on such Bianchi II background and was the first example of a gravitational soliton on a non-diagonal background.

Due to the diagonality of the matrix $g^{(0)}$, Eq. (1.5.8) reduces to the cylindrical wave equation for the function $\Lambda^{(0)}$ as can be seen from Eqs. (3.2.9)

$$\Lambda^{(0)}_{,tt} = \frac{1}{r}(r\Lambda^{(0)}_{,r})_{,r} \tag{3.6.3}$$

and the energy and flux densities of the system (see section 3.4) are

$$\begin{aligned} \varepsilon(t,r) &= \frac{2}{r} + 2\Lambda^{(0)}_{,r} + r\left[(\Lambda^{(0)}_{,t})^2 + (\Lambda^{(0)}_{,r})^2\right] \\ F(t,r) &= -2\Lambda^{(0)}_{,t} - 2r\Lambda^{(0)}_{,t}\Lambda^{(0)}_{,r} \end{aligned} \tag{3.6.4}$$

We now prove that in the case of a diagonal background metric (3.6.1), steps 4 and 5 of the above scheme can be skipped, by considering the derivation of Eq. (1.7.9). As is shown in section 1.6, each vector $m^{(k)}$ satisfies Eq. (1.6.12)

$$\left[\partial_i \vec{m}_s^T + \Lambda_i^{\ j}(\mu_s)\vec{m}_s^T A_j^{(0)}\right]\vec{q}_s = 0 \tag{3.6.5}$$

where $q^{(k)}$ is another undetermined vector function. Moreover, from Eq. (1.6.10) one obtains yet another condition

$$m_s^T q_s = 0 \tag{3.6.6}$$

Eqs. (3.6.5, 3.6.6) determine the vectors m_s and q_s, and one can easily verify that the function given in Eq. (1.7.9) indeed satisfies them. However, in the special case of the diagonal metric (3.6.1), Eqs. (3.6.5, 3.6.6) can be solved directly without the knowledge of the matrix M in the spirit of [29] (see section 2.2). This yields

$$m_s = \left(\frac{C_s}{\mu_s}e^{\rho_s}, e^{-\rho_s}\right) \tag{3.6.7}$$

where ρ_s can be find by quadratures of the equations

$$\begin{aligned} \rho_{s,t} &= \frac{r}{\mu_s^2 - r^2}(r\Lambda^{(0)}_{,t} + \mu_s\Lambda^{(0)}_{,r}) \\ \rho_{s,r} &= \frac{r}{\mu_s^2 - r^2}(r\Lambda^{(0)}_{,r} + \mu_s\Lambda^{(0)}_{,t}) \end{aligned} \tag{3.6.8}$$

This allows us to write the general N-soliton solution on the background metric (3.6.1) by proceeding with the procedure described in section 1.7. In this case, the

computation is simpler when setting $k_a = i\left(re^{\Lambda^{(0)}/2}, e^{-\Lambda^{(0)}/2}\right)$. Let D be the $N \times N$ matrix with elements

$$D_{s\tilde{s}} = \Lambda^{(0)} + \rho_s + \rho_{\tilde{s}} \tag{3.6.9}$$

then the N-soliton solution on the background (3.6.1) is given by

$$\begin{aligned}
g^{(N)}_{\phi\phi} &= \frac{r^{2-N}e^{\Lambda^{(0)}}}{\prod_{s=1}^{N}|\mu_s|}\det\left(\frac{\mu_s^2\mu_{\tilde{s}}^2 e^{-D_{s\tilde{s}}} + C_s C_{\tilde{s}} r^4 e^{D_{s\tilde{s}}}}{\mu_s\mu_{\tilde{s}} - r^2}\right)/\Delta \qquad (3.6.10)\\
g^{(N)}_{\phi z} &= r^{1-N}\prod_{s=1}^{N}|\mu_s|\left[\det\left(\frac{\mu_s\mu_{\tilde{s}}e^{-D_{s\tilde{s}}} + C_s C_{\tilde{s}} r^2 e^{D_{s\tilde{s}}}}{\mu_s\mu_{\tilde{s}} - r^2} - \frac{C_s r}{\mu_s}e^{\rho_s - \rho_{\tilde{s}}}\right)/\Delta - 1\right]\\
g^{(N)}_{zz} &= r^N e^{-\Lambda^{(0)}}\prod_{s=1}^{N}|\mu_s|\det\left(\frac{e^{-D_{s\tilde{s}}} + C_s C_{\tilde{s}} e^{D_{s\tilde{s}}}}{\mu_s\mu_{\tilde{s}} - r^2}\right)/\Delta\\
f^{(N)} &= C_0\frac{\prod_{s=1}^{N}|\mu_s|^{N-1}}{r^{N^2/2}\prod_{s>\tilde{s}=1}^{N}(\mu_s - \mu_{\tilde{s}})^2}\Delta f^{(0)}
\end{aligned}$$

where $C_0 \in \mathbb{R}$ and the C_s's are N arbitrary constants in the extended complex plane $\mathbb{C}_\infty$,

$$\Delta = \det\left(\frac{\mu_s\mu_{\tilde{s}}e^{-D_{s\tilde{s}}} + C_s C_{\tilde{s}} r^2 e^{D_{s\tilde{s}}}}{\mu_s\mu_{\tilde{s}} - r^2}\right) \tag{3.6.11}$$

and the functions μ_s are the solutions of Eq. (1.6.4) given by Eq. (1.7.5)

$$\mu_s = \omega_s - t \pm \sqrt{(\omega_s - t)^2 - r^2} \tag{3.6.12}$$

In the following, we shall refer to solutions corresponding to a positive square root in Eq. (3.6.12) as *solitons*, while solutions with a negative square root will be called *anti-solitons* [29].

As stated before, the functions μ_s have to be either real valued or pairwise-complex-conjugate. This, in turn, means that the corresponding constants ω_s and C_s have to be chosen in a similar fashion (real if the corresponding μ_s is real, or pairwise-complex-conjugate if the corresponding μ_s has its complex conjugate).

In order to write the one and two solitons solution explicitly, it is convenient to

introduce the notation

$$\begin{aligned} \tilde{\gamma}_s &= \ln \frac{r}{|\mu_s|} = \mp \cosh^{-1}\left(\frac{\omega_s - t}{r}\right) \\ \gamma_s &= K_s + \Lambda^{(0)} + 2\rho_s + \tilde{\gamma}_s \end{aligned} \tag{3.6.13}$$

where $s = 1, 2, \ldots, N$, $K_s = \ln|C_s| \in \mathbb{R}$ in the case of real constants, and the minus-plus sign in the definition of $\tilde{\gamma}_s$ corresponds to a plus-minus choice of sign in Eq. (3.6.12). The reader should compare this notation with the similar notation used in Eqs. (2.3.12) for the symmetric chiral field equation. Fortunately, also here the one and two solitons solutions can be written in terms of hyperbolic functions, rather similarly to the one and two solitonic solutions of the symmetric principal chiral field equation (as was shown in section 2.3) and the famous Korteweg-de Vries equation [31].

The one soliton ($N = 1$) solution can be written explicitly as the metric (3.6.1) with

$$g^{(1)} = \frac{1}{\cosh\gamma_1}\begin{bmatrix} r^2 e^{\Lambda^{(0)}}\cosh(\gamma_1 + \tilde{\gamma}_1) & \frac{r^2-\mu_1^2}{2\mu_1} \\ \frac{r^2-\mu_1^2}{2\mu_1} & e^{-\Lambda^{(0)}}\cosh(\gamma_1 - \tilde{\gamma}_1) \end{bmatrix} \tag{3.6.14}$$

and

$$f^{(1)} = C_0\sqrt{r}\frac{|\mu_1|}{\mu_1^2 - r^2}\cosh(\gamma_1) f^{(0)} \tag{3.6.15}$$

where C_0 is an arbitrary constant and ρ_1 is given in terms of quadrature of Eqs. (3.6.8).

The reader can easily recognize the familiar form of the one soliton of the symmetric principal chiral field equation written in Eq. (2.3.13). The main apparent difference is the appearance of factors of $\alpha = r$ in Eq. (3.6.14). Nonetheless, in this case the pole μ_1 is not fixed anymore. This is the result of the existence of derivatives with respect to the spectral parameter λ in the Belinski-Zakharov Eqs. (1.5.26). The moving poles have great implications. We already mentioned that a pole μ_s determines the velocity and amplitude of the s-th soliton. However, since now the pole varies in space and time, so will the velocity and amplitude of the soliton

change in space and time. This will be demonstrated in two special cases below. The trajectories of the poles were shown in figure (1.3).

Note that in this case the function μ_1 must be real. This is guaranteed when $\omega_1 \in \mathbb{R}$ and only as long as we are inside a light-cone propagating from the z-axis at time $t = \omega_1$ (this condition is merely $|\omega_1 - t| > r$ as can be seen in Eq. (3.6.12)). As noted in [17] the one soliton solution can always be extended to the entire spacetime using the background metric $g^{(0)}$ outside of the light-cone. However, on the light-cone such an extension will suffer a discontinuity in the first derivative. By transforming $K_1 \to -K_1$ (or $C_1 \to C_1^{-1}$) and $\mu_1 \to r^2/\mu_1$, the one soliton reduces to the same one-soliton solution, only with the opposite sign of the square root in the definition of μ_1 (see Eq. (3.6.12)). This is a result of the symmetry of Eqs. (1.6.2) with respect to the transformation $\mu \to \alpha^2/\mu$.

A very important term in the one-soliton solution is the denominator of $f^{(1)}$ in Eq. (3.6.15) which is the only term that can change sign. One sees that the term $\mu_1^2 - r^2$ might change its sign in different regions of spacetime, depending whether one considers a soliton solution (positive sign in Eq. (3.6.12)) or an anti-soliton solution (negative sign). For physical reasons, one would like to have $f^{(1)} > 0$. We assume $f^{(0)} > 0$. Then for a soliton, one must take $C_0 < 0$ inside the future light-cone $t - \omega_1 > r$, and $C_0 < 0$ in the past light-cone $t - \omega_1 < -r$. For the anti-soliton the situation is the opposite, and one takes $C_0 < 0$ in the past light-cone $t - \omega_1 < -r$ and $C_0 > 0$ in the future light-cone $t - \omega_1 > r$ (see fig. (3.1)).

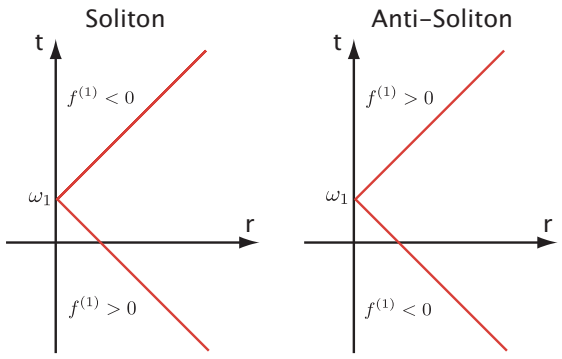

FIGURE 3.1. The spacetime diagram for a single soliton / anti-soliton. To ensure the positivity of $f^{(1)}$, one must take different signs for the constant C_0 on either the future or past light-cone starting at $t = \omega_1$ as noted in the figure. Outside of the light-cone $|t - \omega_1| = r$ the solution can be extended using the background metric $g^{(0)}$.

The two soliton ($N = 2$) solution is given by

$$
\begin{aligned}
g^{(2)}_{\phi\phi} &= r^2 e^{\Lambda^{(0)}} \frac{r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2+\tilde{\gamma_1}+\tilde{\gamma_2}}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2+\tilde{\gamma_1}-\tilde{\gamma_2}}{2})}{r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2}{2})} \\
g^{(2)}_{\phi z} &= r\frac{(\mu_1-\mu_2)(\mu_1\mu_2-r^2)}{2\mu_1\mu_2}\cdot\frac{\mu_1(\mu_2^2-r^2)\cosh\gamma_1 - \mu_2(\mu_1^2-r^2)\cosh\gamma_2}{r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2}{2})} \\
g^{(2)}_{zz} &= e^{-\Lambda^{(0)}} \frac{r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2-\tilde{\gamma_1}-\tilde{\gamma_2}}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2-\tilde{\gamma_1}+\tilde{\gamma_2}}{2})}{r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2}{2})} \\
f^{(2)} &= C_0\frac{\mu_1^2\mu_2^2\left[r^2(\mu_1-\mu_2)^2\cosh^2(\frac{\gamma_1+\gamma_2}{2}) + (\mu_1\mu_2-r^2)^2\sinh^2(\frac{\gamma_1-\gamma_2}{2})\right]}{(\mu_1^2-r^2)(\mu_2^2-r^2)(\mu_1\mu_2-r^2)^2(\mu_1-\mu_2)^2} f^{(0)}
\end{aligned}
\tag{3.6.16}
$$

There are two situations for this solution depending on the nature of the functions μ_1, μ_2 defined in Eq. (3.6.12). Complex valued $\omega_1 = \bar{\omega}_2$ give a solution that is everywhere defined.

The second situation is in which $\omega_1, \omega_2 \in \mathbb{R}$. Without loss of generality, consider the case $\omega_1 = -\omega_2 = \omega$, and one has

$$
\begin{aligned}
\mu_1 &= \omega - t \pm \sqrt{(t-\omega)^2 - r^2} \\
\mu_2 &= -\omega - t \pm \sqrt{(t+\omega)^2 - r^2}
\end{aligned}
\tag{3.6.17}
$$

μ_1, μ_2 are real only inside the intersection of the light-cones $|t-\omega| > r$ and $|t+\omega| > r$. This means that outside of that intersection the solution is non-physical. However, also in this case one can extend the solution outside of this region, using the one soliton solution and the background solution. This is demonstrated in fig. (3.2). However, note that also

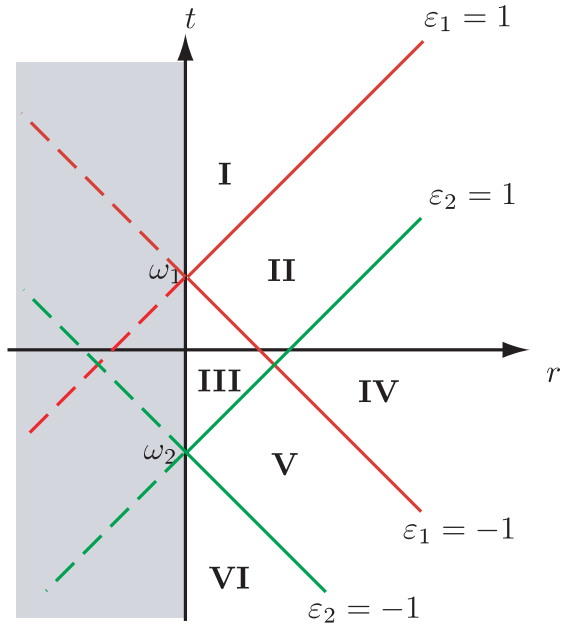

FIGURE 3.2. The spacetime diagram for two gravitational solitons. The solution is properly defined in regions I, III and VI only. It can be extended to either one soliton / anti-soliton in regions II and V that can be extended to the background metric in region IV. The solution is also valid in the grey region, although it corresponds to a non-physical $r < 0$.

in this case the function $f^{(2)}$ may change its sign, depending on the sign of the term $(\mu_1^2 - r^2)(\mu_2^2 - r^2)$ in the denominator, and one has to take the right value for the constant C_0 in each region to ensure the positivity of $f^{(2)}$.

A very important subclass of solutions occur when the matrix g is diagonal. Physically, diagonal metrics represent a single-polarization gravitational wave. This case deserves a special attention as we see from Eq. (3.6.3) that Einstein's equation linearizes. This means that the linear superposition principle holds and any solitons in the solution will *not* interact.

From Eq. (3.6.10), we see that a sufficient condition for the diagonality of the matrix g is that each one of the constants C_s belongs to the set $\{0, \infty\}$. It is enough to study the case where $C_s = 0$ for all s, as a soliton with $C_s = \infty$ can be obtained by changing the sign of the square root in the definition of μ_s (see comment above). Nevertheless, this is not a necessary condition for diagonality. This can be seen for example from Eq. (3.6.16), since in the case $\omega_1 = \omega_2 \in \mathbb{R}$ one has $\mu_1 = \mu_2$ and the diagonal term $g^{(2)}_{r\phi}$ vanishes irrespectively of the values of the constants C_1, C_2. However, a close examination of Eq. (3.6.16) reveals that in fact the case $\mu_1 = \mu_2$ reduces back to the to the background metric $g^{(0)}$.

The *diagonal* N-soliton solution for $C_k = 0$ can be written explicitly from Eq. (3.6.10) as

$$\begin{aligned} g^{(N)} &= \begin{bmatrix} r^2 e^{\Lambda^{(0)}} \prod_{s=1}^{N} \frac{|\mu_s|}{r} & 0 \\ 0 & e^{-\Lambda^{(0)}} \prod_{s=1}^{N} \frac{r}{|\mu_s|} \end{bmatrix} \\ f^{(N)} &= C_0 r^{\frac{N^2}{2} - N} \frac{\prod_{s=1}^{N} |\mu_s|^{N+1}}{\prod_{s,\tilde{s}=1}^{N} (\mu_s \mu_{\tilde{s}} - r^2)} e^{-N\Lambda^{(0)} - 2\sum_{s=1}^{N} \rho_s} f^{(0)} \end{aligned} \tag{3.6.18}$$

A solution of Einstein's equation (1.2.3) can describe a compact physical object when it is asymptotically flat. It was argued in the literature that non-trivial cylindrical metrics are not interesting physically because they are not asymptotically flat [29]. Indeed, a careful examination of Eq. (3.6.18) shows that they cannot be asymptotically flat *for all moments of time*. However, cylindrical metrics may still

be asymptotically flat in the distant past or future ($t \to \pm\infty$) and thus restore their physical importance, as solutions that describe a compact object whose influence at spatial infinity eventually vanishes. A sufficient condition for such flatness of the diagonal metric (3.6.18) is that it will converge to the flat metric $g = \mathrm{diag}(r^2, 1)$ for large times $t \to \pm\infty$. It is easy to see from Eq. (3.6.18) that the diagonal solitonic metric is asymptotically flat when all of the following conditions hold simultaneously:

- $\Lambda^{(0)} \to 0$ as $t \to \pm\infty$
- The number of functions μ_s with a positive square root is equal to the number of functions μ_s with a negative square root.

In fact, the same result holds true for non-diagonal solitonic metrics as well. In the following, we shall refer to solutions corresponding to a positive square root in Eq. (3.6.12) as *solitons*, while solutions with a negative square root will be called *anti-solitons*. Thus the third condition in the above list says that asymptotically flatness can only be achieved when the number of solitons is equal to the number of anti-solitons (one may think of their interactions as canceling one another as time passes). Therefore only solutions with N being even are of physical significant. Moreover, one can see that the only physically significant case for $N = 2$ is when we have two real functions μ_1, μ_2 (corresponding to real $C_1, C_2, \omega_1, \omega_2$). When considering complex functions μ_s, the first physically relevant case is the four-soliton solution $N = 4$. This case will not be studied in this work.

Example: solitons on flat background Consider the case of a flat spacetime represented in cylindrical coordinates by the metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \tag{3.6.19}$$

This is just a special case of the metric (3.6.1) with $\Lambda^{(0)} = 0$ being trivial.

In this case the N-soliton solution has the non-zero components:

$$\begin{aligned} g^{(N)}_{\phi\phi} &= \frac{r^{2-N}}{\prod_{s=1}^{N}|\mu_s|}\det\left(\frac{\mu_s^2\mu_{\tilde{s}}^2+C_sC_{\tilde{s}}r^4}{\mu_s\mu_{\tilde{s}}-r^2}\right)/\Delta \\ g^{(N)}_{\phi z} &= r^{1-N}\prod_{s=1}^{N}|\mu_s|\left[\det\left(\frac{\mu_s\mu_{\tilde{s}}+C_sC_{\tilde{s}}r^2}{\mu_s\mu_{\tilde{s}}-r^2}-\frac{C_sr}{\mu_s}\right)/\Delta-1\right] \\ g^{(N)}_{zz} &= r^{N}\prod_{s=1}^{N}|\mu_s|\det\left(\frac{1+C_sC_{\tilde{s}}}{\mu_s\mu_{\tilde{s}}-r^2}\right)/\Delta \\ f^{(N)} &= C_0\frac{\prod_{s=1}^{N}|\mu_s|^{N-1}}{r^{N^2/2}\prod_{s>\tilde{s}}^{N}(\mu_s-\mu_{\tilde{s}})^2}\Delta \end{aligned} \tag{3.6.20}$$

where $C_0>0$ is a positive constant,

$$\Delta=\det\left(\frac{\mu_s\mu_{\tilde{s}}+C_sC_{\tilde{s}}r^2}{\mu_s\mu_{\tilde{s}}-r^2}\right) \tag{3.6.21}$$

and the functions μ_s were given in Eq. (3.6.12).

For the flat background metric (3.6.19), it is more convenient to write the one and two soliton solutions in a form slightly different than that of Eqs. (3.6.14,3.6.16) that does not invoke hyperbolic functions. The one-soliton solution $N=1$ is

$$g^{(1)}=\frac{sgn(\mu_1)}{\mu_1^2+C_1^2r^2}\begin{bmatrix}\frac{r}{\mu_1}(\mu_1^4+C_1^2r^4) & C_1r(r^2-\mu_1^2)\\ C_1r(r^2-\mu_1^2) & \mu_1r(1+C_1^2)\end{bmatrix} \tag{3.6.22}$$

and

$$f^{(1)}=\frac{C_0}{\sqrt{r}}\cdot\frac{\mu_1^2+C_1^2r^2}{\mu_1^2-r^2} \tag{3.6.23}$$

where in this case μ_1 that was given in Eq. (3.6.12) must be real-valued, and the constants C_1 and ω_1 have to be real as well. The sign of μ_1 appears in the numerator, to guarantee that the matrix g has signature $(++)$ so that the metric has the physical signature $(-+++)$.

The behavior of the soliton is different than that of the anti-soliton, and both are demonstrated in the non-diagonal case $C_1=2$ in figures (3.3), (3.5), (3.4) and (3.4) for the metric elements $g^{(1)}_{\phi\phi}$, $g^{(1)}_{zz}$, $g^{(1)}_{\phi z}$ and $f^{(1)}$ respectively.

Physically, a gravitational soliton travels towards the z-axis at time $t<\omega_1$, completely collapses into the z-axis at time $t=\omega_1$ and recoils a gravitational wave

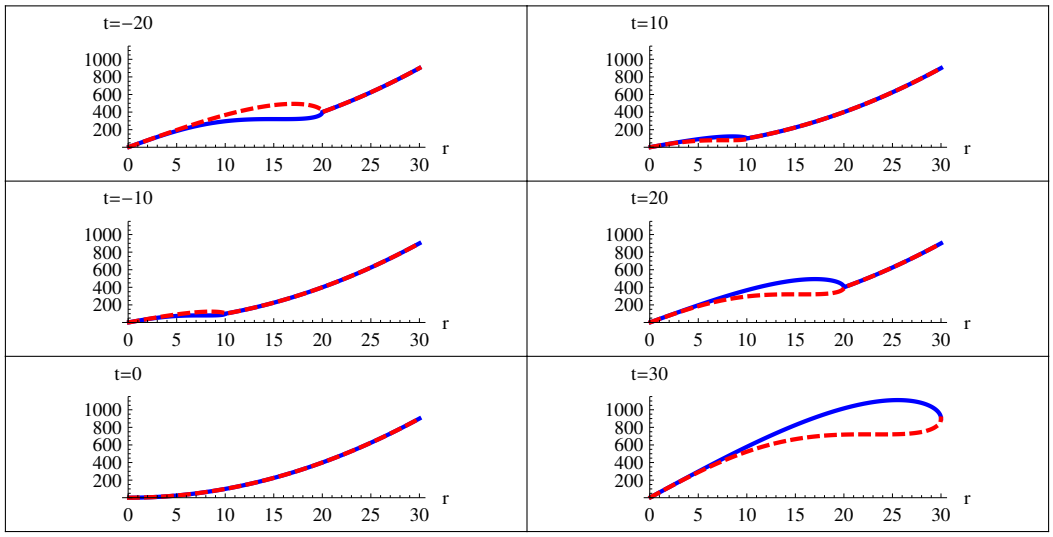

FIGURE 3.3. The diagonal term $g^{(1)}_{\phi\phi}$ for a single soliton or an anti-soliton on the flat background metric (3.6.19). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters in Eq. (3.6.22) where chosen to be $C_1 = 2$, $\omega_1 = 0$

outwards from the z-axis at times $t > \omega_1$ that continues ad infinitum $r \to \infty$. For the diagonal terms, the soliton has a constructive interference effect until time $t = \omega_1$ from which it turns to have a destructive interference effect. For the non-diagonal term $g^{(1)}_{\phi z}$ the soliton is always destructive. For the anti-soliton this effect is always the opposite.

One can learn much by considering the diagonal one-soliton solution $C_1 = 0$. By extending the one-soliton solution (3.6.22) to the entire spacetime, it is:

$$g^{(1)} = \begin{cases} \begin{pmatrix} r|\mu_1| & 0 \\ 0 & \frac{r}{|\mu_1|} \end{pmatrix} & |\omega_1 - t| > |r| \\ \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} & |\omega_1 - t| \leq |r| \end{cases} \tag{3.6.24}$$

where μ_1 is given in Eq. (3.6.12).

By differentiating $g^{(1)}_{\phi\phi}$ with respect to r we see that it has a local minimum/maximum at $r_0 = \sqrt{\frac{3}{4}}|\omega - t|$. This can be considered as the location of our one soliton / anti-

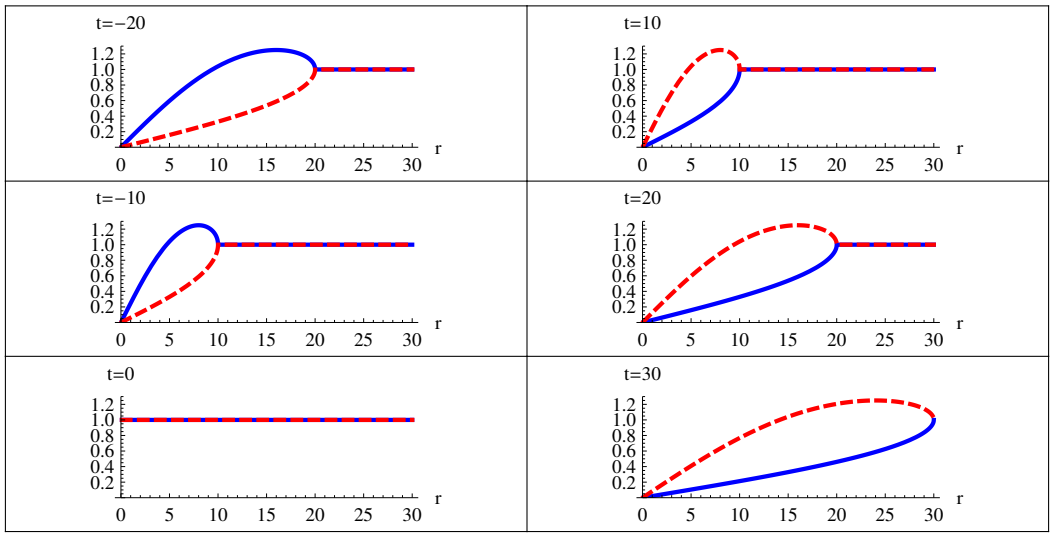

FIGURE 3.4. The diagonal term $g^{(1)}_{zz}$ for a single soliton or an anti-soliton on the flat background metric (3.6.19). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters in Eq. (3.6.22) where chosen to be $C_1 = 2$, $\omega_1 = 0$

soliton. One sees that the soliton travels with subliminal speed $v = \sqrt{\frac{3}{4}}$. At its maximum, the soliton / anti-soliton has amplitude $g^{(1)}_{\phi\phi}\ |_{r=r_0} = \pm\frac{3\sqrt{3}}{4}(\omega - t)^2$. In the non-diagonal case the position, velocity and amplitude of the one-soliton depend on the value of the constant C_1 and have nontrivial forms. Unlike classical solitons, gravitational solitons *do not* preserve their shape and amplitude as they travel. Thus, in the language of section 1.3 gravitational solitons are not physical solitons.

Another interesting fact about the diagonal one soliton can be noticed when putting it in the form $g^{(1)} = \mathrm{diag}(r^2 e^{\Lambda^{(1)}}, e^{-\Lambda^{(1)}})$. In this case, $\Lambda^{(1)}$ is given by

$$\Lambda^{(1)}(t,r) = \begin{cases} \ln\frac{|\mu_1|}{r} & |\omega_1 - t| > |r| \\ 0 & |\omega_1 - t| \leq |r| \end{cases} \tag{3.6.25}$$

Indeed, a simple computation reveals that $\Lambda^{(1)}$ satisfies equation (3.6.3) as expected. This means that in fact, this diagonal case $\Lambda^{(1)}(t,r)$ is nothing other than the Green

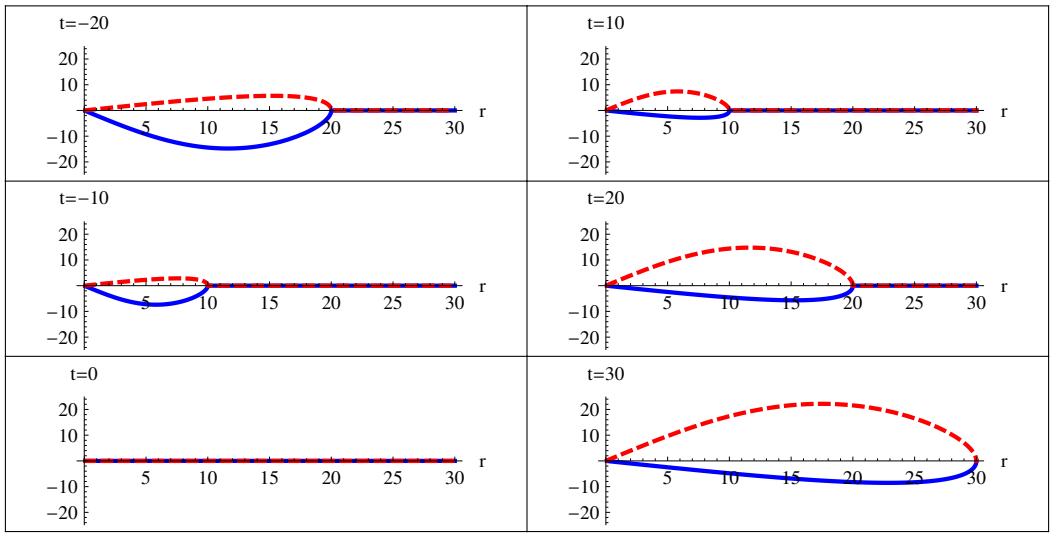

FIGURE 3.5. The non-diagonal term $g^{(1)}_{\phi z}$ for a single soliton or an anti-soliton on the flat background metric (3.6.19). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters in Eq. (3.6.22) where chosen to be $C_1 = 2$, $\omega_1 = 0$

function for the cylindrical wave equation. Namely, it is a solution of the problem:

$$\begin{cases} \frac{\partial^2 \Lambda^{(1)}}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Lambda^{(1)}}{\partial r}\right) \\ \Lambda^{(1)}(\omega_1, r) = 0 \\ \frac{\partial \Lambda^{(1)}}{\partial r}(\omega_1, r) = \mp\delta(r) \end{cases} \tag{3.6.26}$$

where the initial time is understood in the sense that $t \to \omega_1^+$, and the sign of the delta function corresponds to a soliton / anti-soliton solution respectively. Similarly, the diagonal N-soliton solution (3.6.18) for the flat metric (3.6.1) can be written in the form $g^{(N)} = \mathrm{diag}(r^2 e^{\Lambda}, e^{-\Lambda})$ for which Λ corresponds to a superposition of N such Green functions.

Furthermore, in the diagonal limit the energy and flux densities (3.4.7) are

$$\begin{aligned} \varepsilon &= \frac{1}{r} + \frac{2(\omega_1 + t)r}{\mu_1((\omega_1 + t)^2 - r^2)} \\ F &= \frac{2r^2}{\mu_1((\omega_1 + t)^2 - r^2)} \end{aligned} \tag{3.6.27}$$

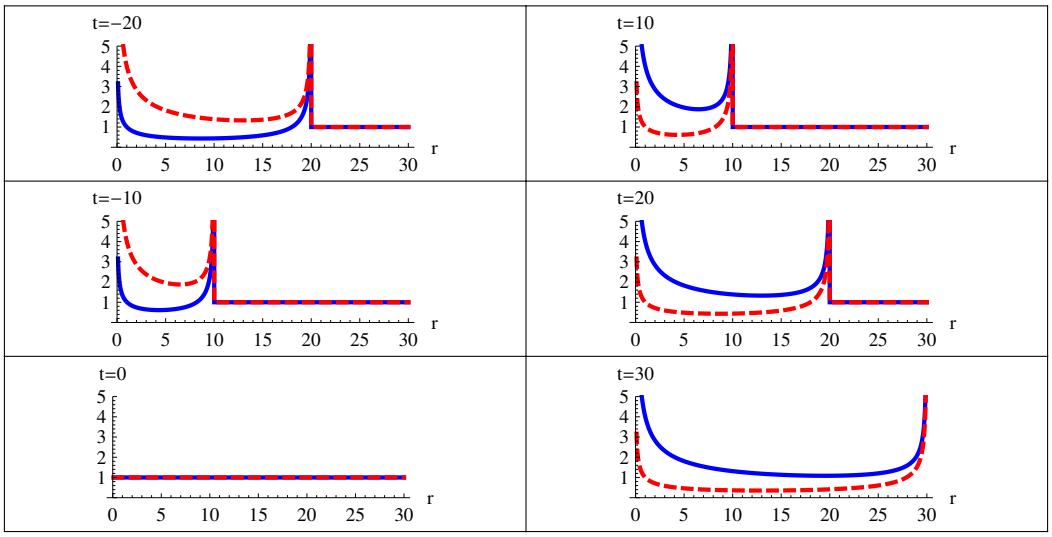

FIGURE 3.6. The term $f^{(1)}$ for a single soliton or an anti-soliton on the flat background metric (3.6.19). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters in Eq. (3.6.22) where chosen to be $C_1 = 2$, $\omega_1 = 0$

and it is easy verify that the conservation law (3.4.6) is satisfied. However, note that in this case the energy (3.4.5) diverges at the edge of the light-cone in the limit $r \to (\omega_1 + t)^-$. Even when E is renormalized with respect to the background metric's energy, it is ill-defined as a global quantity of spacetime.

The components of the two soliton solution on flat background can be written as

$$
\begin{aligned}
g^{(2)}_{\phi\phi} &= \frac{1}{|\mu_1\mu_2|}\frac{(\mu_1-\mu_2)^2(\mu_1^2\mu_2^2+C_1C_2r^4)^2+r^2(\mu_1\mu_2-r^2)^2(\mu_1^2C_2-\mu_2^2C_1)^2}{(\mu_1-\mu_2)^2(\mu_1\mu_2+C_1C_2r^2)^2+(\mu_1\mu_2-r^2)^2(\mu_1C_2-\mu_2C_1)^2} \\
g^{(2)}_{\phi z} &= (\mu_1-\mu_2)(\mu_1\mu_2-r^2)\cdot\frac{C_2(\mu_2^2-r^2)(\mu_1^2+C_1^2r^2)-C_1(\mu_1^2-r^2)(\mu_2^2+C_2^2r^2)}{(\mu_1-\mu_2)^2(\mu_1\mu_2+C_1C_2r^2)^2+(\mu_1\mu_2-r^2)^2(\mu_1C_2-\mu_2C_1)^2} \\
g^{(2)}_{zz} &= |\mu_1\mu_2|\frac{r^2(\mu_1-\mu_2)^2(1+C_1C_2)^2+(\mu_1\mu_2-r^2)^2(C_1-C_2)^2}{(\mu_1-\mu_2)^2(\mu_1\mu_2+C_1C_2r^2)^2+(\mu_1\mu_2-r^2)^2(\mu_1C_2-\mu_2C_1)^2} \\
f^{(2)} &= C_0\frac{|\mu_1\mu_2|}{(\mu_1-\mu_2)^2}\cdot\frac{(\mu_1-\mu_2)^2(\mu_1\mu_2+C_1C_2r^2)^2+(\mu_1\mu_2-r^2)^2(\mu_1C_2-\mu_2C_1)^2}{(\mu_1^2-r^2)(\mu_2^2-r^2)(\mu_1\mu_2-r^2)^2}
\end{aligned}
\tag{3.6.28}
$$

In the case where both functions μ_1, μ_2 are real, a similar situation to the one soliton solution occurs and the solution is only define inside the intersection of the

two light-cones $|\omega_1 + t| > r$ and $|\omega_2 + t| > r$. However, also in this case it can be extended (up to a price of a jump in the first derivative) to the one-soliton solution (with the appropriate function μ_1 or μ_2) outside of each light-cone. This is visualized in figure (3.2), in which we see that region III is the interaction region of the two solitons. The soliton-soliton interaction is demonstrated in figure (3.7).

As commented before, the most interesting case physically is when one has a pair of a soliton and an anti-soliton. In figures (3.8) and (3.9) we see such a soliton-anti soliton interaction.

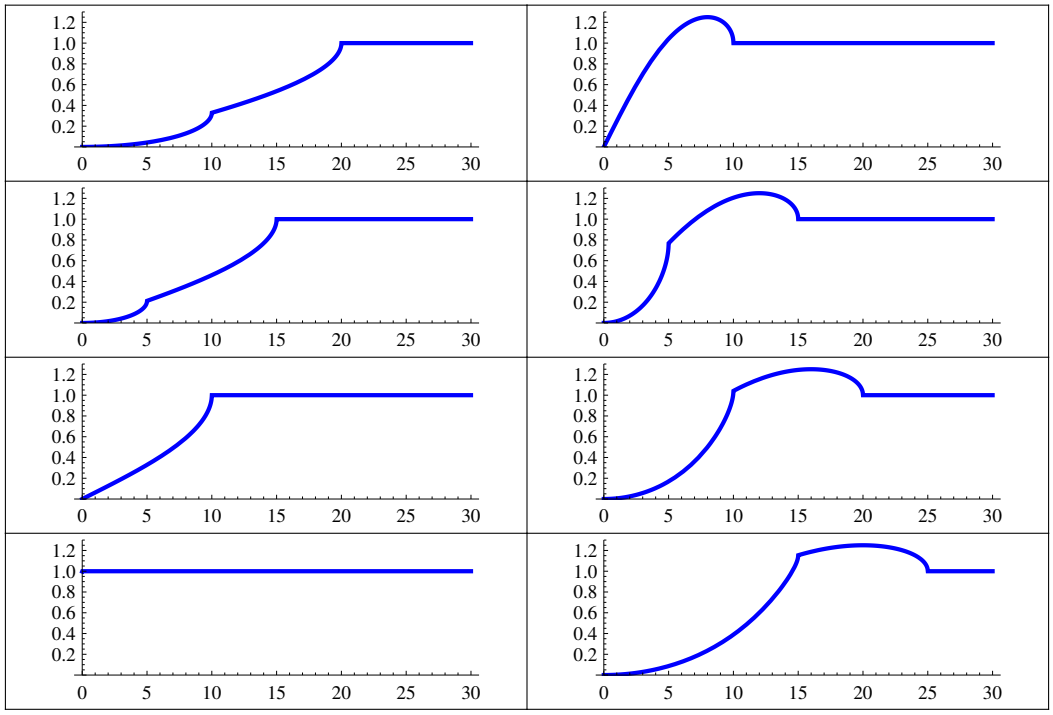

FIGURE 3.7. The diagonal term $g^{(2)}_{\phi\phi}$ for two (real) solitons on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = C_2 = 2$, $\omega_1 = 0$ and $\omega_2 = -10$. The sequence is shown for times $t = -10, -5, 0, 5, 10, 15, 20, 25$ (starting at the top-left corner).

When considering two complex-conjugate functions $\mu_1 = \overline{\mu_2} = Re^{i\theta}$ the situation is rather different. We introduce the notation $C_1 = \overline{C_2} = Ke^{i\theta_0}$ where $R, K, \theta, \theta_0 \in \mathbb{R}$.

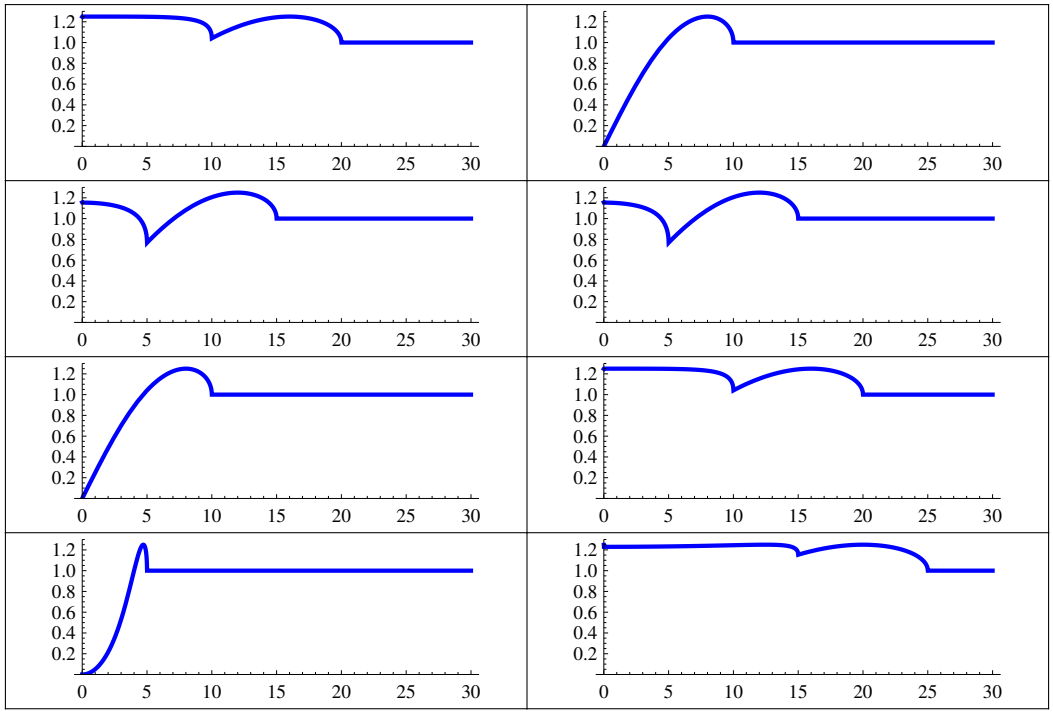

FIGURE 3.8. The diagonal term $g^{(2)}_{zz}$ for a soliton and anti-soliton on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = C_2 = 2$, $\omega_1 = 0$ and $\omega_2 = -10$. The sequence is shown for times $t = -10, -5, 0, 5, 10, 15, 20, 25$ (starting at the top-left corner).

Then Eq. (3.6.16) can be written in the form

$$
\begin{aligned}
g^{(2)}_{\phi\phi} &= \frac{1}{R^2}\frac{(R^4+K^2r^4)^2\sin^2\theta + r^2R^2K^2(R^2-r^2)^2\sin^2(2\theta-\theta_0)}{(R^2+K^2r^2)^2\sin^2\theta + K^2(R^2-r^2)^2\sin^2(\theta-\theta_0)} \\
g^{(2)}_{\phi z} &= -\frac{K(R^2-r^2)\sin\theta}{R^3}\frac{(R^4+K^2r^4)\sin\theta_0 + r^2R^2(1+K^2)\sin(2\theta-\theta_0)}{(R^2+K^2r^2)^2\sin^2\theta + K^2(R^2-r^2)^2\sin^2(\theta-\theta_0)} \\
g^{(2)}_{zz} &= \frac{r^2R^2(1+K^2)^2\sin^2\theta + K^2(R^2-r^2)^2\sin^2\theta_0}{(R^2+K^2r^2)^2\sin^2\theta + K^2(R^2-r^2)^2\sin^2(\theta-\theta_0)} \\
f^{(2)} &= C_0R^2\frac{(R^2+K^2r^2)^2\sin^2\theta + K^2(R^2-r^2)^2\sin^2(\theta-\theta_0)}{(R^2-r^2)^2(R^4+r^4-2R^2r^2\cos 2\theta)\sin^2\theta}
\end{aligned}
\tag{3.6.29}
$$

which is manifestly real-valued.

Figure (3.10) shows this complex two-soliton. Since the behavior of the metric is mostly identical for positive and negative time, we only plot the metric elements

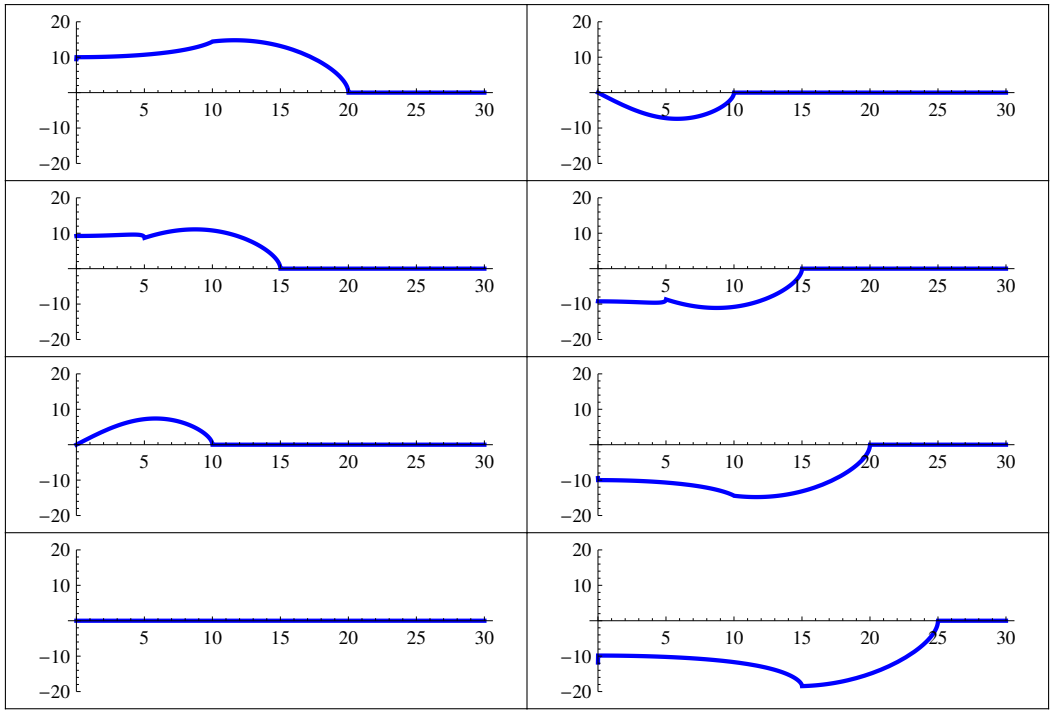

FIGURE 3.9. The non-diagonal term $g^{(2)}_{\phi z}$ for a soliton and anti-soliton on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = C_2 = 2$, $\omega_1 = 0$ and $\omega_2 = -10$. The sequence is shown for times $t = -10, -5, 0, 5, 10, 15, 20, 25$ (starting at the top-left corner).

for positive times $t = 0, 5, 10, 15$. One can see $g^{(2)}_{\phi\phi}$ develops a rapidly growing gravitational wave centered at the z-axis. On the other hand, $g^{(2)}_{zz}$ in figure (3.11) has a destructive interference with the background metric $g^{(0)}_{zz}$. This interference travels to the right for positive time (and to the left for negative time) will maintaining its destructive effect at the origin. The nondiagonal term $g^{(2)}_{\phi z}$ in figure (3.12) clearly develops a rapidly growing solitonic formation. However, this solitonic structure is rather different than the familiar solitons from other fields in physics, as its amplitude increases during its motion to $r \to \infty$.

Series expansion for a diagonal background Although the N-soliton solution can be given in terms of the functions $\mu_s, \Lambda^{(0)}$ and ρ_s, an explicit integration of Eqs. (3.6.8)

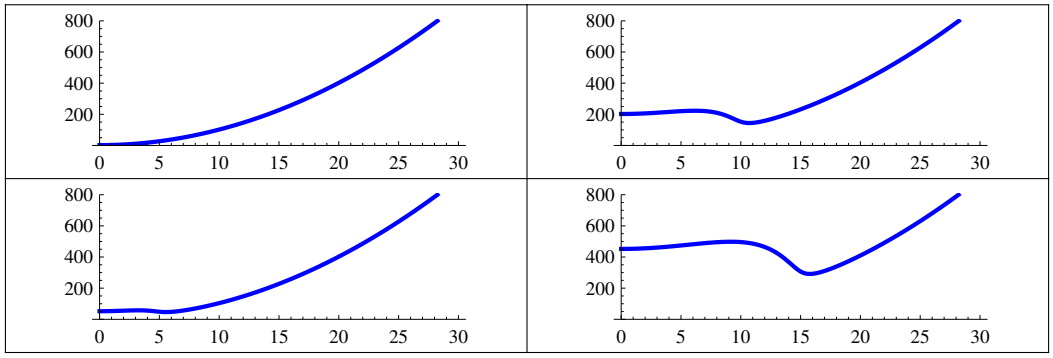

FIGURE 3.10. The diagonal term $g^{(2)}_{\phi\phi}$ for a complex two-soliton on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = \overline{C_2} = 1$, $\omega_1 = \overline{\omega_2} = i$ and the plot is for positive times $t = 0, 5, 10, 15$.

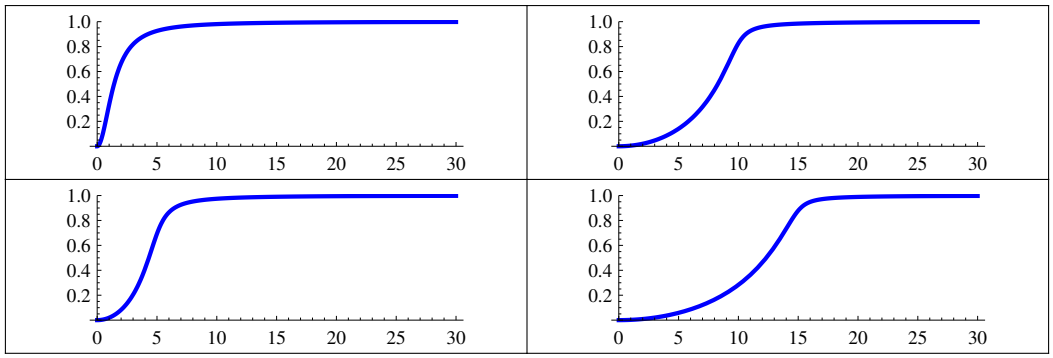

FIGURE 3.11. The diagonal term $g^{(2)}_{zz}$ for a complex two-soliton on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = \overline{C_2} = 1$, $\omega_1 = \overline{\omega_2} = i$ and the plot is for positive times $t = 0, 5, 10, 15$.

might pose a problem for an arbitrary function $\Lambda^{(0)}$. An example for these difficulties arise when one attempts to find solitonic solutions taking as the background metric the famous Einstein-Rosen metric [21].

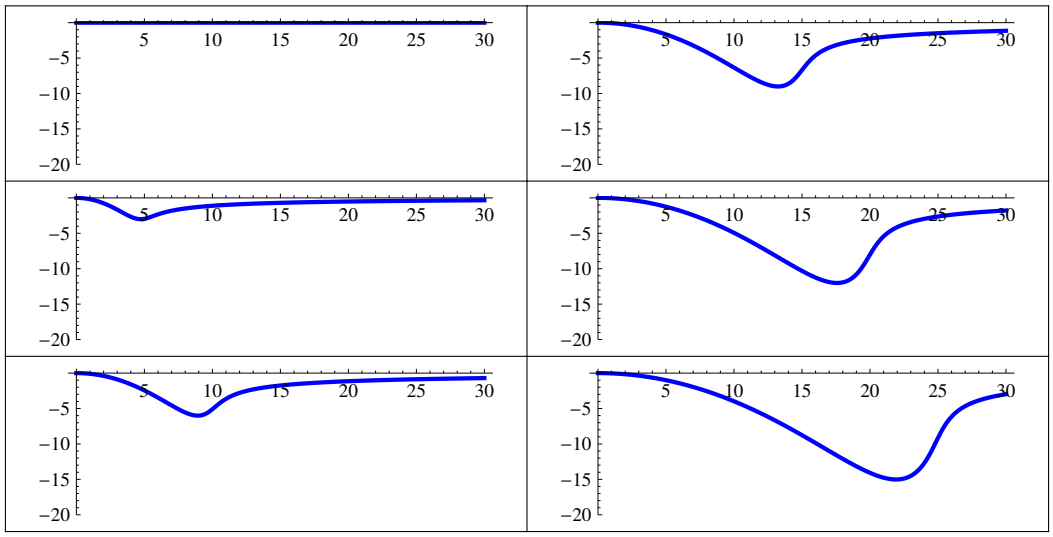

FIGURE 3.12. The non-diagonal term $g^{(2)}_{\phi z}$ for a complex two-soliton on the flat background metric (3.6.19). In this figure, the parameters in Eq. (3.6.28) where chosen to be $C_1 = \overline{C_2} = 1$, $\omega_1 = \overline{\omega_2} = i$ and the plot is for times $t = 0, 5, 10, 15, 20, 25$.

From Eq. (3.6.12) for the functions μ_s, one may write Eqs. (3.6.8) as

$$\begin{aligned} \rho_{s,t} &= \frac{1}{2}\left[-1 \pm \frac{1}{\sqrt{1-\varepsilon_s^2}}\right]\Lambda^{(0)}_{,t} - \frac{\varepsilon_s}{2\sqrt{1-\varepsilon_k^2}}\Lambda^{(0)}_{,r} \\ \rho_{s,r} &= \frac{1}{2}\left[-1 \pm \frac{1}{\sqrt{1-\varepsilon_s^2}}\right]\Lambda^{(0)}_{,r} - \frac{\varepsilon_s}{2\sqrt{1-\varepsilon_s^2}}\Lambda^{(0)}_{,t} \end{aligned} \tag{3.6.30}$$

where

$$\varepsilon_s = \frac{r}{\omega_s - t} \tag{3.6.31}$$

is a function depending on t and r. Here and in the following the plus-minus signs correspond to the choice of sign of the square root of μ_s defined in Eq. (3.6.12). Clearly, $|\varepsilon_s| = 1$ lies on the light-cone $r = |\omega_s + t|$. This parameter ε_s can be interpreted as measuring the ratio between the spatial and temporal distance from the light-cone's vertex. Alternatively, ε_s is just the reciprocal of the slope of the straight line connecting the vertex of the light-cone to a given point on the $r-t$ plane (see figure (3.13)).

We shall consider a series expansion for ρ_s in the parameter ε_s. Expanding the right-hand side of Eqs. (3.6.30) in ε_s gives

$$\rho_{s,t} = \sum_{j=0}^{\infty} Q_{2j}\Lambda^{(0)}_{,t}\varepsilon_s^{2j} + \sum_{j=0}^{\infty} Q_{2j+1}\Lambda^{(0)}_{,r}\varepsilon_s^{2j+1} \quad (3.6.32)$$

$$\rho_{s,r} = \sum_{j=0}^{\infty} Q_{2j}\Lambda^{(0)}_{,r}\varepsilon_s^{2j} + \sum_{j=0}^{\infty} Q_{2j+1}\Lambda^{(0)}_{,t}\varepsilon_s^{2j+1}$$

where the expansion coefficients can be written explicitly as

$$\begin{aligned} Q_0 &= -\frac{1}{2}(1 \pm \frac{1}{2}) \quad (3.6.33)\\ Q_1 &= -\frac{1}{2} \\ Q_{2j} &= \pm\frac{1}{2}(-1)^j \binom{-\frac{1}{2}}{j} \\ Q_{2j+1} &= \frac{(-1)^{j+1}}{2}\binom{-\frac{1}{2}}{j} \end{aligned}$$

and

$$\binom{a}{j} = \frac{a(a-1)(a-2)\cdots(a-j+1)}{j!} \quad (3.6.34)$$

is the generalized binomial coefficients defined for arbitrary a and non-negative integer j. Let

$$\rho_s = \rho_s^{(0)} + \rho_s^{(1)}\varepsilon_s + \rho_s^{(2)}\varepsilon_s^2 + \ldots \quad (3.6.35)$$

be a series expansion for ρ and notice that

$$\varepsilon_{s,t} = -\frac{\varepsilon_s^2}{r} \qquad \varepsilon_{s,r} = \frac{\varepsilon_s}{r} \quad (3.6.36)$$

This means, that the derivatives of ρ are given by the expressions

$$\begin{aligned} \rho_{s,t} &= \rho^{(0)}_{s,t} + \sum_{j=1}^{\infty}\left[\rho^{(j)}_{s,t} - \frac{j-1}{r}\rho_s^{(j-1)}\right](\varepsilon_s)^j \quad (3.6.37)\\ \rho_{s,r} &= \sum_{j=0}^{\infty}\left[\rho^{(j)}_{s,r} + \frac{j}{r}\rho_s^{(j)}\right](\varepsilon_s)^j \end{aligned}$$

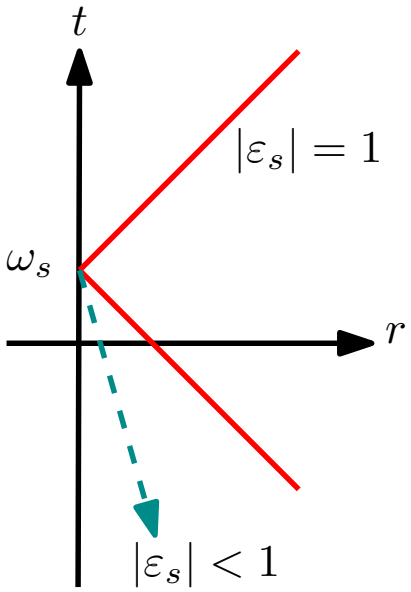

FIGURE 3.13. A spacetime diagram of the ε_s parameter (defined in Eq. (3.6.31)) for real ω_s. The red solid line corresponds to the light-cone $|\varepsilon_s| = 1$, while smaller ε_s's lie inside it, for example on the dashed blue.

Comparing powers of ε_s in Eqs. (3.6.32) and (3.6.37) gives a recursive relation for the coefficients $\rho_s^{(j)}$ as functions of t, r and $\Lambda^{(0)}$. The first term is

$$\rho_s^{(0)} = -\frac{1}{2}(1 \pm \frac{1}{2})\Lambda^{(0)} \tag{3.6.38}$$

The even terms satisfy the equations

$$\begin{aligned} \rho_{s,t}^{(2j)} &= \pm\frac{1}{2}(-1)^j \binom{-\frac{1}{2}}{j} \Lambda_{,t}^{(0)} + \frac{2j-1}{r}\rho_s^{(2j-1)} \\ \rho_{s,r}^{(2j)} &= \pm\frac{1}{2}(-1)^j \binom{-\frac{1}{2}}{j} \Lambda_{,r}^{(0)} - \frac{2j}{r}\rho_s^{(2j)} \end{aligned} \tag{3.6.39}$$

while the odd terms satisfy

$$\begin{aligned} \rho_{s,t}^{(2j+1)} &= \frac{(-1)^{j+1}}{2} \binom{-\frac{1}{2}}{j} \Lambda_{,r}^{(0)} + \frac{2j}{r}\rho_s^{(2j)} \\ \rho_{s,r}^{(2j+1)} &= \frac{(-1)^{j+1}}{2} \binom{-\frac{1}{2}}{j} \Lambda_{,t}^{(0)} - \frac{2j+1}{r}\rho_s^{(2j+1)} \end{aligned} \tag{3.6.40}$$

Solving Eqs. (3.6.39), (3.6.40) gives the expansion coefficients of Eq. (3.6.35) which can be used in Eq. (3.6.10) to find the general N-soliton solution within the light-cones for *any* cylindrical diagonal metric (3.6.1). This technique is demonstrated next by obtaining the one and two gravitational solitons on the Einstein-Rosen background.

Einstein-Rosen solitons The Einstein-Rosen metric [21] has the cylindrical spacetime interval (3.6.1) with

$$\begin{aligned} \Lambda^{(0)}(t,r) &= J_0(r)\sin(t) \\ f(t,r) &= \frac{1}{\sqrt{r}}\exp\{rJ_0(r)J_1(r)\cos(2t) + \\ &\quad +\frac{1}{2}r^2\left[J_0^2(r) - J_0(r)J_2(r) + 2J_1^2(r)\right]\} \end{aligned} \tag{3.6.41}$$

Here J_n denotes the n-th order Bessel function. The Einstein-Rosen solution describes a single polarization gravitational wave propagating normally outwards from the z-axis in the radial direction. It is one of the simplest solutions of Einstein's

field equation describing gravitational waves and historically was the first theoretical evidence for their existence.

In this section, we demonstrate the method developed above to derive gravitational solutions using the Einstein-Rosen metric as a seed. Using the definitions (3.6.41) in Eqs. (3.6.39) and (3.6.40) one immediately sees that the general recursive formula for $\rho_s^{(j)}$ is

$$\begin{aligned}
\rho_s^{(0)} &= -\frac{1}{2}(1 \pm \frac{1}{2})J_0(r)\sin(t) \\
\rho_s^{(1)} &= -\frac{1}{2}J_1(r)\cos(t) \\
\rho_s^{(2j)} &= \pm\frac{1}{2}(-1)^j\binom{-\frac{1}{2}}{j}\Lambda^{(0)} + \frac{2j-1}{r}\int \rho_s^{(2j-1)}dt \\
\rho_s^{(2j+1)} &= \frac{(-1)^{j+1}}{2}\binom{-\frac{1}{2}}{j}\int \Lambda^{(0)}_{,r}dt + \frac{2j}{r}\int \rho_s^{(2j)}dt
\end{aligned} \tag{3.6.42}$$

for $j \geq 1$. All the integrals in this recursive relations are trivial (being integrals of trigonometric functions). For example, up to third order in ε_s, we have

$$\begin{aligned}
\rho_s(r,t) &= -\frac{1}{2}(1 \pm \frac{1}{2})J_0(r)\sin(t) - \frac{1}{2}J_1(r)\cos(t)\varepsilon_s \\
&\quad + \left[\mp\frac{1}{4}J_0(r)\sin(t) - \frac{1}{2r}J_1(r)\sin(t)\right]\varepsilon_s^2 + O(\varepsilon_s^3)
\end{aligned} \tag{3.6.43}$$

where we remind the reader that ε_s was defined in Eq. (3.6.31).

Figures (3.14,3.15,3.16) display the terms $g^{(1)}_{\phi\phi}$, $g^{(1)}_{zz}$ and $g^{(1)}_{\phi z}$ (respectively) for a single gravitational soliton (in solid blue) and gravitational anti-soliton (in dashed red) on the Einstein-Rosen background. Due to the oscillations of the background metric in both space and time it is rather difficult to comprehend the localized nature of the soliton. Nevertheless, the reader is advised to compare these figures with the analogous figures (3.3,3.4,3.5) for which the one soliton / anti-soliton were derived on flat spacetime. One sees immediately that gravitational solitons on flat spacetime and Einstein-Rosen spacetime have very similar structures, as their propagation and shapes are selfsame (up to the oscillatory motion in the case of the oscillatory Einstein-Rosen background).

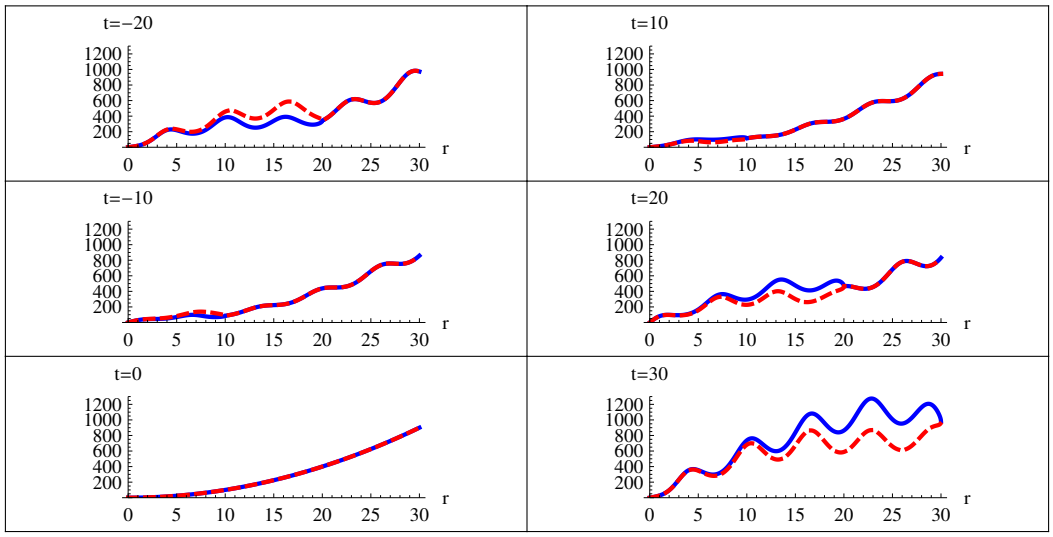

FIGURE 3.14. The diagonal term $g^{(1)}_{\phi\phi}$ for a single soliton or an anti-soliton on the Einstein-Rosen metric (3.6.41). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters where chosen to be $C_1 = 2$, $\omega_1 = 0$. The reader should compare this figure with the analogous gravitational soliton/anti-soliton on flat spacetime in figure (3.3).

This is not a coincidence, as *the same result seems to hold in general for gravitational solitons on arbitrary diagonal backgrounds*. This was also studied numerically by the scheme developed in the next section. Since gravitational solitons on diagonal background have similar structures, the study of two gravitational solitons on Einstein-Rosen background will be omitted (interactions of two gravitational solitons were already studied on flat spacetimes). As for applying the Einstein-Rosen gravitational solitons to study physical phenomena, the reader should note that the Einstein-Rosen soliton-anti-soliton solution does not satisfy the criteria described above ($\Lambda^{(0)} \not\to 0$ as $t \to \pm\infty$). Therefore, Einstein-Rosen solitons cannot describe a compact object.

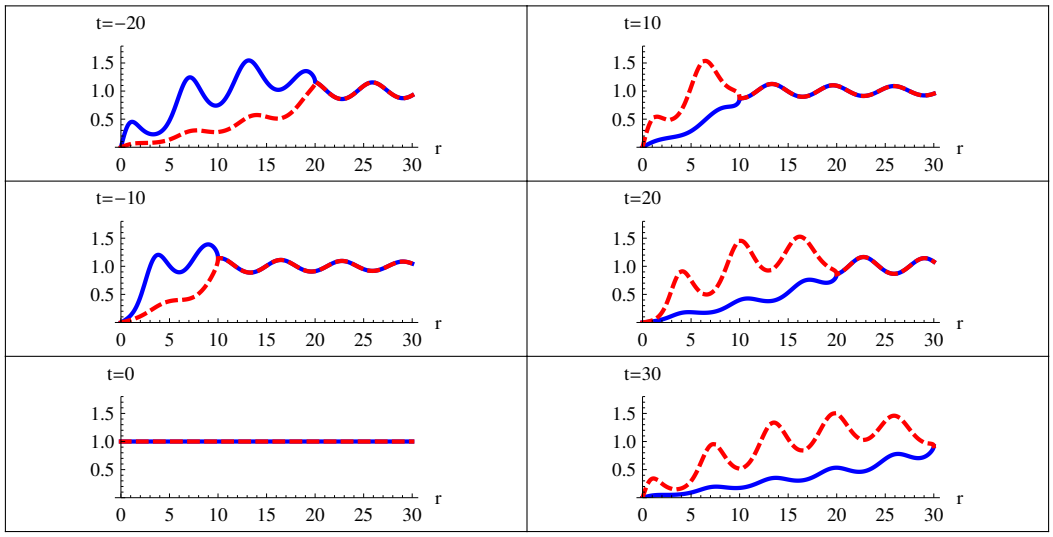

FIGURE 3.15. The diagonal term $g^{(1)}_{zz}$ for a single soliton or an anti-soliton on the Einstein-Rosen metric (3.6.41). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters where chosen to be $C_1 = 2$, $\omega_1 = 0$. The reader should compare this figure with the analogous gravitational soliton/anti-soliton on flat spacetime in figure (3.4).

3.7 Numerical study

To obtain a numerical solution of the reduced Einstein Eq. (3.0.4), one may isolate the second temporal derivative to write it in the form

$$g_{,tt} = g_{,zz} + \frac{1}{\alpha}\left[\alpha_{,z}g_{,z} - \alpha_{,t}g_{,t}\right] + g_{,t}g^{-1}g_{,t} - g_{,z}g^{-1}g_{,z} \tag{3.7.1}$$

Consider a uniform discretization of time and space with intervals Δt and Δz respectively. We use a finite difference approximation for the second and first order derivatives. The first order temporal derivative is approximated using a backward difference, while the first order spatial derivative is approximated using a central difference.

Let us denote $g_{n,m} = g(n\Delta t, m\Delta z)$. The discretization described above allows

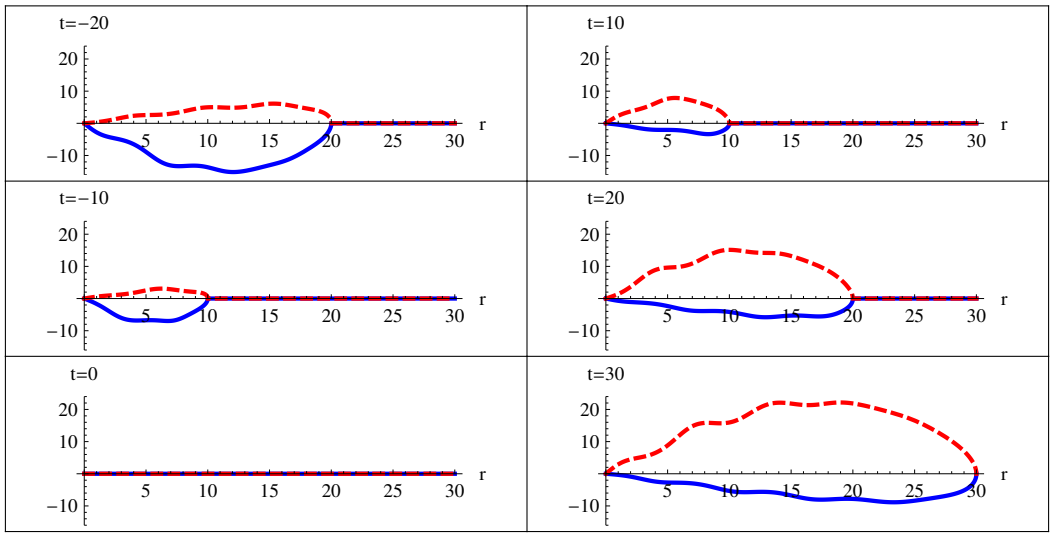

FIGURE 3.16. The non-diagonal term $g^{(1)}_{\phi z}$ for a single soliton or an anti-soliton on the Einstein-Rosen metric (3.6.41). The solid blue line corresponds to a soliton, while the dashed red line is the anti-soliton. In this case the parameters where chosen to be $C_1 = 2$, $\omega_1 = 0$. The reader should compare this figure with the analogous gravitational soliton/anti-soliton on flat spacetime in figure (3.5).

solving Eq. (3.7.1) for the next time step $g_{n+1,m}$,

$$\begin{aligned} g_{n+1,m} &= 2g_{n,m} - g_{n-1,m} + (g_{n,m} - g_{n-1,m})g^{-1}_{n,m}(g_{n,m} - g_{n-1,m}) \\ &\quad -\frac{1}{\alpha}(\alpha_{n,m} - \alpha_{n-1,m})(g_{n,m} - g_{n-1,m}) \\ &\quad + \left(\frac{\Delta t}{\Delta z}\right)^2 \left[g_{n,m+1} - 2g_{n,m} + g_{n,m-1} + \frac{1}{4\alpha}(\alpha_{n,m+1} - \alpha_{n,m-1})(g_{n,m+1} - g_{n,m-1}) \right. \\ &\qquad \left. -\frac{1}{4}(g_{n,m+1} - g_{n,m-1})g^{-1}_{n,m}(g_{n,m+1} - g_{n,m-1}) \right] \end{aligned} \tag{3.7.2}$$

To ensure the accuracy of the scheme, we use fourth-order Runge-Kutta method. The scheme described here has one great merit on the ones typically used in numerical relativity. It is restricted to a specific form of the spacetime metric that depends on only two variables. For this reason, this scheme is extremely fast and can be easily rendered in real-time on any modern computer (no super-computers are needed). This is also the disadvantage of the scheme, as for the same reason it cannot be used

to study more complex gravitational effects that lack the symmetries assumed here, of the existence of two commuting Killing vectors.

It was demonstrated numerically that the numerical solution converges to the analytic solution when $\frac{\Delta t}{\Delta z} \to 0$, as the reader can see from table 3.2 for the case of the Kasner metric [22]. However, one must also impose the condition $\Delta t < \Delta z$ to guarantee the stability of the scheme.

Δt	Δz	maximal error
1	1	525.42
0.5	1	184.51
0.25	1	57.65
0.0125	1	16.50
0.0625	1	4.45
0.03125	1	1.16
0.01	1	0.1222
0.001	1	0.0012
0.0005	1	3.09×10^{-4}
0.00025	1	7.74×10^{-5}
0.000125	1	1.94×10^{-5}
0.00005	1	3.03×10^{-6}

TABLE 3.2. Error in the numerical solutions of the reduced Einstein Eq. (3.0.4)

We will not show here any figures of the numerical solutions of Einstein's equation, as they match the analytical ones shown before rather well. Instead, we would like to focus on the actual error of the numerical solution.

It is a well-known fact in general relativity [5] that when discussing 'weak' gravity, Einstein's field equation reduces to the wave equation. This suggest that the information about the change in the metric propagates at the speed of light. The solitonic solutions studied above are not periodic in space, although periodic boundary conditions were imposed on them for the numerical scheme. Therefore, after a sufficient amount of time the information from outside the mesh (i.e. the fictitious boundary condition) propagates into the mesh and generates a greater error. Thus, the nu-

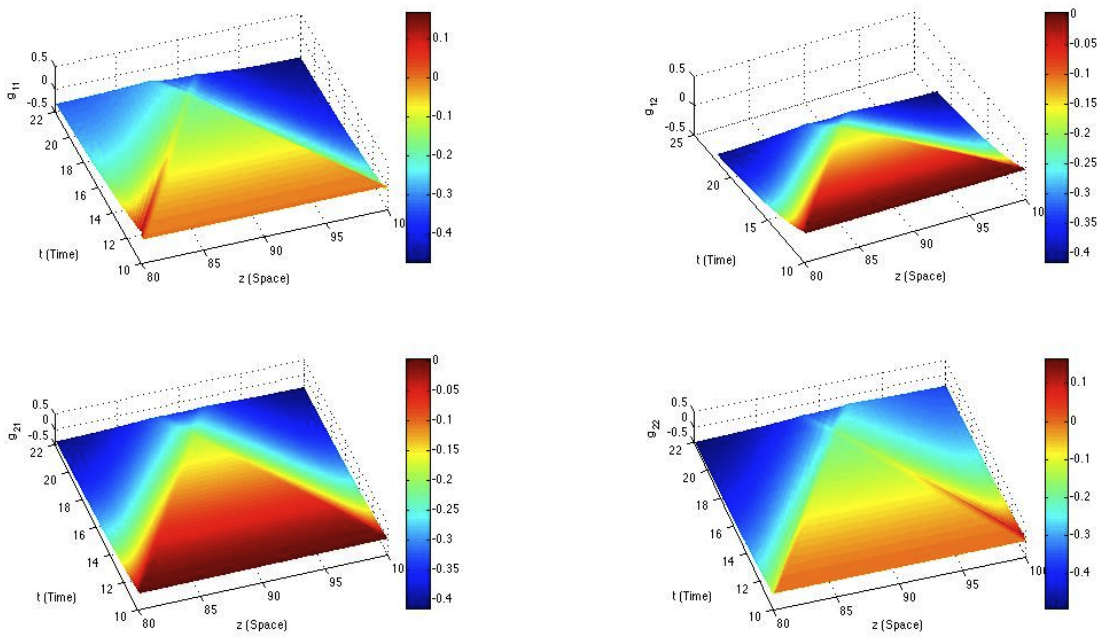

FIGURE 3.17. The error in the numerical solution of the metric elements $g_{11}, g_{12}, g_{21}, g_{22}$ for mesh intervals $\Delta t = 0.005$ and $\Delta z = 0.1$. Here the error is defined to be $(g_{\text{exact}} - g_{\text{numerical}})/g_{\text{exact}}$. The numerical solution is very precise inside the future light-cone (and can be made more precise by decreasing the ratio $\Delta t/\Delta z$). Outside of the future light-cone, the imposed periodic boundary conditions affect the accuracy of the scheme.

merical solution is only valid inside a *triangle of validity* whose bottom side is the mesh at the initial time. This is nothing other than the future light-cone described in section 1.1. Figure (3.17) shows the error (in percentage from the exact solution) on a mesh with $\Delta t = 0.005$ and $\Delta z = 0.01$ of a single soliton on the famous Kasner background [22]. The reader can easily recognize the the triangle of validity in which the numerical solution is nearly identical to the analytical one derived in [17].

The author would like to add that this code was supplemented by a library that visualizes the various types of curvatures of spaces. This was used to verify some of the analytical properties described in the previous chapters and will not be described here.

Chapter 4
The $1+2$ Einstein equation

In this chapter metrics depending on more than two coordinates are studied.

In section 4.1, general diagonal spacetime metrics are considered. These are diagonal spacetime metrics that may depend on all coordinates, and their Lagrangian and field equations are derived explicitly. Section 4.2 limits our attention to such diagonal metrics with one Killing vector. It is shown that in this case the overdetermined Einstein's equations are compatible. A special attention is given to plane gravitational waves, for which simple criteria for asymptotically flatness and compatibility of the field equations are derived. Arguably, the most famous example of such plane waves is the Bondi-Pirani-Robinson waves[1] [20]. It is proven that such waves are *stable* with respect to perturbations that depend also on the second spatial dimension, and the relation to the stability of the Schwarzschild black hole is emphasized. Section 4.3 concentrates on weak gravitational waves, for which an alternative formulation of Einstein's equation in Fourier space is derived and a few applications are stated. Finally, in section 4.4 it is proven that weak gravitational waves satisfy a new system of equations. These new equations are a direct extension of the renowned nonlinear Schrödinger equation, and are therefore quite properly referred to as the *nonlinear Schrödinger-Einstein equations*. A few simple solutions of them are derived, including a generalization of the Bondi-Pirani-Robinson waves.

Throughout this chapter and in this chapter only, Einstein's summation convention is *no longer in use* unless otherwise stated in the text.

[1] Some authors call them 'Robinson waves'. In the Russian literature they are typically referred to as 'Robinson-Bondi waves'.

4.1 General diagonal spacetime metrics

Consider a diagonal spacetime metric

$$g_{\mu\nu} = (H_\mu)^2 \delta_{\mu\nu} \tag{4.1.1}$$

or in matrix form,

$$g_{\mu\nu} = \begin{bmatrix} (H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{bmatrix} \tag{4.1.2}$$

Here initially the metric does not have the proper metric signature $(-+++)$. Instead, it is convenient not to worry about the sign of the metric during the computations and at the end restore the proper metric signature by substituting $H_0 \to iH_0$. Diagonal metrics describe a wide range of physical phenomena, from arbitrary single-polarized gravitational waves [41] to the Schwarzschild black-hole [5]. As shown in the last chapter, a certain class of gravitational solitons are also represented by this class of diagonal metrics. When describing gravitational waves, one may interpret each coefficient H_μ as the factor by which the temporal / spatial dimension x^μ stretches due to the passing gravitational wave.

The inverse metric is

$$g^{\mu\nu} = \frac{1}{(H_\mu)^2} \delta^{\mu\nu} \tag{4.1.3}$$

and the Christoffel symbols are

$$\Gamma^\lambda_{\mu\nu} = 0 \qquad \Gamma^\mu_{\mu\nu} = \partial_\nu (\ln H_\mu) \qquad \Gamma^\nu_{\mu\mu} = -\frac{1}{(H_\nu)^2} H_\mu \partial_\nu H_\mu \tag{4.1.4}$$

where μ, ν, λ are assumed to be mutually exclusive indices ($\mu \neq \nu$, $\mu \neq \lambda$, $\nu \neq \lambda$). Define the rotation coefficients to be

$$Q_{\mu\nu} = \frac{1}{H_\nu} \partial_\nu H_\mu \tag{4.1.5}$$

with which one can write the off-diagonal Ricci curvature tensor as

$$R_{\mu\nu} = -\sum_{\lambda \neq \mu,\nu} \frac{H_\mu}{H_\lambda} \left(\partial_\nu Q_{\lambda\mu} - Q_{\lambda\nu} Q_{\nu\mu} \right) \qquad (\mu \neq \nu) \tag{4.1.6}$$

As for the diagonal elements, the Ricci tensor gives

$$R_{\mu\mu} = -\sum_{\nu\neq\mu} \frac{H_\mu}{H_\nu} E_{\mu\nu} \tag{4.1.7}$$

where

$$E_{\mu\nu} = \partial_\nu Q_{\mu\nu} + \partial_\mu Q_{\nu\mu} + \sum_{\lambda\neq\mu,\nu} Q_{\mu\lambda} Q_{\nu\lambda} \tag{4.1.8}$$

In general, Einstein's vacuum equations $R_{\mu\nu} = 0$ for diagonal metrics give 4 diagonal equations and 6 off-diagonal equations for 4 unknown functions H_μ ($\mu = 0, 1, 2, 3$). This is of course a highly overdetermined system and the conditions for its solvability will be determined in the next section.

The scalar curvature is

$$R = -2\sum_{\mu<\nu} \frac{E_{\mu\nu}}{H_\mu H_\nu} \tag{4.1.9}$$

Since the determinant of the metric is $\det g = (H_0 H_1 H_2 H_3)^2$, the Einstein-Hilbert Lagrangian density from Eq. (1.2.9) is

$$\mathcal{L} = \frac{i}{2} H_0 H_1 H_2 H_3 R = -i \sum_{\mu\neq\nu\neq\lambda\neq\sigma} E_{\mu\nu} H_\lambda H_\sigma \tag{4.1.10}$$

The reader should not be alarmed by the appearance of the imaginary root of unity $i = \sqrt{-1}$. It is there due to the signature of the metric and the transformation $H_0 \to iH_0$, which was mentioned earlier, reveals immediately that this Lagrangian is manifestly real-valued as expected.

In order to simplify this Lagrangian, recall that the field equation obtained from the action is independent of surface integrals. One may thus remove the second order derivatives in $E_{\mu\nu}$ by integration by parts. This gives for all $\mu, \nu, \lambda, \sigma$ mutually different

$$\begin{aligned} E_{\mu\nu} H_\lambda H_\sigma \;\equiv\; & \frac{1}{H_\mu}\left[-H_\lambda(\partial_\mu H_\nu)(\partial_\mu H_\sigma) - H_\sigma(\partial_\mu H_\nu)(\partial_\mu H_\lambda)\right] \\ & + \frac{1}{H_\nu}\left[-H_\lambda(\partial_j H_\mu)(\partial_\nu H_\sigma) - H_\sigma(\partial_\nu H_\mu)(\partial_\nu H_\lambda)\right] \\ & + \frac{1}{H_\lambda}\left[H_\sigma(\partial_\lambda H_\mu)(\partial_\lambda H_\nu)\right] + \frac{1}{H_\sigma}\left[H_\lambda(\partial_\sigma H_\mu)(\partial_\sigma H_\nu)\right] \end{aligned} \tag{4.1.11}$$

where the symbol $\equiv$ denotes equality up to surface terms. Substituting this into Eq. (4.1.10) yields a very concise formula for the Lagrangian density in terms of the metric coefficients only,

$$\mathcal{L} \equiv \sum_{\mu} \frac{i}{H_\mu} \sum_{\nu\neq\lambda\neq\sigma\neq\mu} H_\nu(\partial_\mu H_\lambda)(\partial_\mu H_\sigma) \tag{4.1.12}$$

4.2 Diagonal spacetime metrics in $1+2$ coordinates

4.2.A Introduction

Consider the diagonal metric in Eq. (4.1.1) with the additional assumption that it is independent of x^3. Mathematically, this means that the metric has the Killing vector ∂_3 and depends on only three of the four coordinates: x^0, x^1, x^2. Among many other physical phenomena, such metrics describe gravitational waves with a single polarization, free to vary in two out of the three spatial dimensions.

In this case, the off-diagonal terms of the Ricci curvature tensor (4.1.6) give only three independent equations $R_{01} = R_{02} = R_{12} = 0$. Explicitly, these equations have a very symmetry form,

$$\begin{aligned}
\frac{\partial_0\partial_1 H_2}{H_2} + \frac{\partial_0\partial_1 H_3}{H_3} &= \frac{(\partial_0 H_2)(\partial_1 H_0)}{H_0 H_2} + \frac{(\partial_0 H_3)(\partial_1 H_0)}{H_0 H_3} + \frac{(\partial_0 H_1)(\partial_1 H_2)}{H_1 H_2} \\
&\quad + \frac{(\partial_0 H_1)(\partial_1 H_3)}{H_1 H_3} \\
\frac{\partial_0\partial_2 H_1}{H_1} + \frac{\partial_0\partial_2 H_3}{H_3} &= \frac{(\partial_0 H_1)(\partial_2 H_0)}{H_0 H_1} + \frac{(\partial_0 H_3)(\partial_2 H_0)}{H_0 H_3} + \frac{(\partial_0 H_2)(\partial_2 H_1)}{H_1 H_2} + \frac{(\partial_0 H_2)(\partial_2 H_3)}{H_2 H_3} \\
\frac{\partial_1\partial_2 H_0}{H_0} + \frac{\partial_1\partial_2 H_3}{H_3} &= \frac{(\partial_1 H_2)(\partial_2 H_0)}{H_0 H_2} + \frac{(\partial_1 H_0)(\partial_2 H_1)}{H_0 H_1} + \frac{(\partial_1 H_3)(\partial_2 H_1)}{H_1 H_3} + \frac{(\partial_1 H_2)(\partial_2 H_3)}{H_2 H_3}
\end{aligned} \tag{4.2.1}$$

coupled to the four diagonal equations

$$R_{\mu\mu} = 0 \tag{4.2.2}$$

The reader should recall that the *general* vacuum Einstein field equations are always an overdetermined system (see section 1.2). The symmetric metric $g_{\mu\nu}$ has ten

elements, four of which may be eliminated through the use of gauge transformations. Thus the vacuum Einstein equations $R_{\mu\nu} = 0$ are an overdetermined system of ten equations for six unknowns. In normal circumstance this might raise the question of compatibility. Nevertheless, this is not an issue, as one can prove using the four Bianchi identities that they are indeed compatible [5].

However, in this case the situation is different. The $1+2$ vacuum Einstein equations for diagonal metrics (4.2.1, 4.2.2) form an overdetermined system of seven equations for four unknown functions. Their compatibility must be proven as well separately, as it does not follow from the argument typically used for non-diagonal metrics. The author could not find any evidence for such a result in the literature. Whether the $1+2$ Einstein equations for diagonal metrics are indeed compatible is a very natural question to ask, as such metrics have many applications in cosmology and astronomy, some of which will be described soon. Fortunately, it turns out that the answer is affirmative, as the next theorem proves.

Theorem 4.2.3. *The $1+2$ Einstein equations for diagonal metrics (4.2.1,4.2.2) are compatible.*

Proof. Proving the statement of this theorem using the original degrees of freedom H_0, H_1, H_2 and H_3 is very difficult. Instead, it is much easier to exploit the special role of H_3 as the degree of freedom that corresponds to the Killing vector ∂_3. Define[2],

$$H_0 = e^{-\Lambda}\gamma \qquad H_1 = e^{-\Lambda}\beta \qquad H_2 = e^{-\Lambda}\alpha \qquad H_3 = e^{\Lambda} \tag{4.2.4}$$

[2]The function γ that is used here is *not* γ_s that was defined in Eqs. (2.3.12) and (3.6.13), nor it is the famous Lorentz factor $\gamma = \frac{1}{\sqrt{1-v^2}}$ from special relativity.

Using these new degrees of freedom, the off-diagonal Einstein Eqs. (4.2.1) are

$$\begin{aligned}
\partial_0\partial_1\alpha &= -2\alpha(\partial_0\Lambda)(\partial_1\Lambda) + \frac{(\partial_0\beta)(\partial_1\alpha)}{\beta} + \frac{(\partial_0\alpha)(\partial_1\gamma)}{\gamma} \\
\partial_0\partial_2\beta &= -2\beta(\partial_0\Lambda)(\partial_2\Lambda) + \frac{(\partial_0\alpha)(\partial_2\beta)}{\alpha} + \frac{(\partial_0\beta)(\partial_2\gamma)}{\gamma} \\
\partial_1\partial_2\gamma &= -2\gamma(\partial_1\Lambda)(\partial_2\Lambda) + \frac{(\partial_1\alpha)(\partial_2\gamma)}{\alpha} + \frac{(\partial_1\gamma)(\partial_2\beta)}{\beta}
\end{aligned} \tag{4.2.5}$$

As for the diagonal Eqs. (4.2.2), it is convenient to represent them in an equivalent form through the variational formulation. The Einstein-Hilbert Lagrangian (4.1.12) is now

$$\mathcal{L} = 2\left[\frac{\alpha\beta}{\gamma}(\partial_0\Lambda)^2 - \frac{\alpha\gamma}{\beta}(\partial_1\Lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_2\Lambda)^2 - \frac{(\partial_0\alpha)(\partial_0\beta)}{\gamma} + \frac{(\partial_1\alpha)(\partial_1\gamma)}{\beta} + \frac{(\partial_2\beta)(\partial_2\gamma)}{\alpha}\right] \tag{4.2.6}$$

The variations of the action, $\frac{\delta S}{\delta\alpha} = \frac{\delta S}{\delta\beta} = \frac{\delta S}{\delta\gamma} = 0$, give

$$\begin{aligned}
\beta\partial_0\partial_0\beta - \gamma\partial_1\partial_1\gamma &= -\beta^2(\partial_0\Lambda)^2 + \gamma^2(\partial_1\Lambda)^2 - \frac{\beta^2\gamma^2}{\alpha^2}(\partial_2\Lambda)^2 \\
&\quad + \frac{\beta}{\gamma}(\partial_0\beta)(\partial_0\gamma) - \frac{\gamma}{\beta}(\partial_1\beta)(\partial_1\gamma) + \frac{\beta\gamma}{\alpha^2}(\partial_2\beta)(\partial_2\gamma) \\
\alpha\partial_0\partial_0\alpha - \gamma\partial_2\partial_2\gamma &= -\alpha^2(\partial_0\Lambda)^2 - \frac{\alpha^2\gamma^2}{\beta^2}(\partial_1\Lambda)^2 + \gamma^2(\partial_2\Lambda)^2 \\
&\quad + \frac{\alpha}{\gamma}(\partial_0\alpha)(\partial_0\gamma) + \frac{\alpha\gamma}{\beta^2}(\partial_1\alpha)(\partial_1\gamma) - \frac{\gamma}{\alpha}(\partial_2\alpha)(\partial_2\gamma) \\
\alpha\partial_1\partial_1\alpha + \beta\partial_2\partial_2\beta &= -\frac{\alpha^2\beta^2}{\gamma^2}(\partial_0\Lambda)^2 - \alpha^2(\partial_1\Lambda)^2 - \beta^2(\partial_2\Lambda)^2 \\
&\quad + \frac{\alpha\beta}{\gamma^2}(\partial_0\alpha)(\partial_0\beta) + \frac{\alpha}{\beta}(\partial_1\alpha)(\partial_1\beta) + \frac{\beta}{\alpha}(\partial_2\alpha)(\partial_2\beta)
\end{aligned} \tag{4.2.7}$$

while the last diagonal equation, $\frac{\delta S}{\delta\Lambda} = 0$, is

$$\partial_0\left(\frac{\alpha\beta}{\gamma}\partial_0\Lambda\right) - \partial_1\left(\frac{\alpha\gamma}{\beta}\partial_1\Lambda\right) - \partial_2\left(\frac{\beta\gamma}{\alpha}\partial_2\Lambda\right) = 0 \tag{4.2.8}$$

To prove the statement of the theorem, we differentiate each of Eqs. (4.2.7) with respect to x^2, x^1 and x^0 respectively. This gives three third order equations for α, β and γ. One may now eliminate each of the third order terms using the non-diagonal

Eqs. (4.2.5). After a lengthy algebra, one sees that with the aid of Eqs. (4.2.5) once more, all 38 terms in each equation completely vanish. Therefore the $1+2$ Einstein equations for diagonal metrics are indeed compatible. □

The coordinates used in theorem (4.2.3) will be proven to be very useful. They provide us with an alternative way to study general diagonal spacetime metrics. In these coordinates, the spacetime interval is

$$ds^2 = e^{-2\Lambda}\left[-(\gamma dx^0)^2 + (\beta dx^1)^2 + (\alpha dx^2)^2\right] + e^{2\Lambda}(dx^3)^3 \tag{4.2.9}$$

and should be very familiar. Indeed, by setting $\beta = \gamma$ and defining $f = \gamma^2 e^{-2\Lambda}$ one sees immediately that this spacetime interval is nothing other than

$$ds^2 = f\left[-(dx^0)^2 + (dx^1)^2\right] + \alpha^2 e^{-2\Lambda}(dx^2)^2 + e^{2\Lambda}(dx^3)^3 \tag{4.2.10}$$

When the metric is independent of x^2 this spacetime is precisely the one studied in section 3.6.B. Furthermore, in such a case the Lagrangian is

$$\mathcal{L} = 2\left[\alpha(\partial_0\Lambda)^2 - \alpha(\partial_1\Lambda)^2 - \frac{(\partial_0\alpha)(\partial_0\gamma)}{\gamma} + \frac{(\partial_1\alpha)(\partial_1\gamma)}{\gamma}\right] \tag{4.2.11}$$

and Eqs. (4.2.5,4.2.7,4.2.8) are equivalent to the ones studied in chapter 3. Therefore, this naturally generalizes the results of the previous chapter for diagonal spacetimes.

There is another merit of using the new degrees of freedom, as in the course of the proof of theorem (4.2.3) we just derived a conservation law. This is of course Eq. (4.2.8). Whenever the metric is asymptotically flat, it also yields the integral of motion,

$$P = \int\left[\frac{\alpha\beta}{\gamma}\partial_0\Lambda\right]dx^1dx^2 \tag{4.2.12}$$

which is the conjugate momentum of the function Λ, as can be easily seen from the Lagrangian in Eq. (4.2.6).

4.2.B Plane gravitational waves

Waves come in many forms and shapes. The simplest of which are of course plane waves, whose wavefronts are parallel planes extended ad infinitum. In general relativity, plane gravitational waves are typically studied as a special case of the famous *pp*-waves [41]. The *pp*-class consists of any spacetime metric that can be casted into the form,

$$ds^2 = H(u,x,y)du^2 + 2dudv + dx^2 + dy^2 \tag{4.2.13}$$

For such a metric, Einstein's vacuum equation reduces to Laplace's equation,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 \tag{4.2.14}$$

and is therefore *linear* in H. A *pp*-wave is called a *plane wave* if H can be transformed into

$$H(u,x,y) = a(u)(x^2 - y^2) + 2b(u)xy \tag{4.2.15}$$

where $a(u)$ and $b(u)$ control the waveform of the two possible polarizations.

Diagonal metrics of $1+2$ coordinates with the spacetime interval (4.2.9) may describe *pp* spacetimes as well as many non-*pp* spacetimes (such as the Schwarzschild black hole). Either way, they correspond to a different class of solutions of Einstein's vacuum equations (see figure (4.1)).

The equations considered in the previous section allow investigating diagonal metrics (4.2.9) with

$$\Lambda = \Lambda(\eta) \qquad \alpha = \alpha(\eta) \qquad \beta = \beta(\eta) \qquad \gamma = \gamma(\eta) \tag{4.2.16}$$

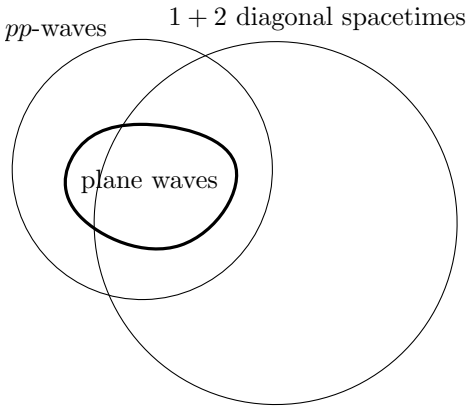

FIGURE 4.1. There are various classes of wave solutions of Einstein's vacuum equation. A diagonal metric is not always a *pp*-spacetime (and vice versa).

where each metric coefficient depends on all three coordinates through

$$\eta = \frac{1}{2}(-x^0 + px^1 + qx^2) \tag{4.2.17}$$

The form of the parameter η corresponds to the naive definition of a plane wave propagating with velocity $v = (p^2 + q^2)^{-1}$, similarly to plane waves studied in other fields of physics. The factor $1/2$ is there because then setting $p = 1$ and $q = 0$ reduces η to its former definition from the previous chapters. But is this 'naive' plane wave consistent with the canonical definition of a plane wave as a subset of the *pp*-class?

To answer this question, consider Einstein's vacuum equations. For the metric coefficients (4.2.16), the diagonal Eq. (4.2.8) gives

$$\left(\frac{\alpha\beta}{\gamma}\Lambda'\right)' = p^2\left(\frac{\alpha\gamma}{\beta}\Lambda'\right)' + q^2\left(\frac{\beta\gamma}{\alpha}\Lambda'\right)' \tag{4.2.18}$$

Integrating it and noticing that Λ' must vanish at some moment of time shows that the integration constant is trivial. Hence the following algebraic relation holds,

$$\alpha^2\beta^2 = p^2\alpha^2\gamma^2 + q^2\beta^2\gamma^2 \tag{4.2.19}$$

To ensure asymptotic flatness, one *may* impose the conditions $\alpha, \beta, \gamma \to 1$ and $\Lambda \to$ constant at spatial infinity. In particular, this guarantees that the metric converges to the Minkowski metric. Taking this limit in Eq. (4.2.19) gives the relation

$$1 = p^2 + q^2 \tag{4.2.20}$$

which has a clear physical interpretation. It shows that the gravitational wave is asymptotically flat if it travels precisely at the speed of light, just like the plane *pp*-waves [41].

Whenever $p, q \neq 0$ the off-diagonal Eqs. (4.2.5) give three nonlinear ordinary

differential equations,

$$\begin{aligned}
\frac{\alpha''}{\alpha} &= -2(\Lambda')^2 + \frac{\alpha'}{\alpha}\left(\frac{\beta'}{\beta} + \frac{\gamma'}{\gamma}\right) \\
\frac{\beta''}{\beta} &= -2(\Lambda')^2 + \frac{\beta'}{\beta}\left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right) \\
\frac{\gamma''}{\gamma} &= -2(\Lambda')^2 + \frac{\gamma'}{\gamma}\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right)
\end{aligned} \tag{4.2.21}$$

where the prime denotes differentiating with respect to the variable η. Similarly, the diagonal Eqs. (4.2.7) are

$$\begin{aligned}
\alpha^2(\beta\beta'' - p^2\gamma\gamma'') &= (-\alpha^2\beta^2 + p^2\alpha^2\gamma^2 - q^2\beta^2\gamma^2)(\Lambda')^2 \\
&\quad + \left(\frac{\alpha^2\beta}{\gamma} - p^2\frac{\alpha^2\gamma}{\beta} + q^2\beta\gamma\right)\beta'\gamma' \\
\beta^2(\alpha\alpha'' - q^2\gamma\gamma'') &= (-\alpha^2\beta^2 - p^2\alpha^2\gamma^2 + q^2\beta^2\gamma^2)(\Lambda')^2 + \left(\frac{\alpha\beta^2}{\gamma} + p^2\alpha\gamma - q^2\frac{\beta^2\gamma}{\alpha}\right)\alpha'\gamma' \\
\gamma^2(p^2\alpha\alpha'' + q^2\beta\beta'') &= (-\alpha^2\beta^2 - p^2\alpha^2\gamma^2 - q^2\beta^2\gamma^2)(\Lambda')^2 + \left(\alpha\beta + p^2\frac{\alpha\gamma^2}{\beta} + q^2\frac{\beta\gamma^2}{\alpha}\right)\alpha'\beta'
\end{aligned} \tag{4.2.22}$$

As in the last section, Einstein's equations in this case are an overdetermined system. They are seven equations for four unknowns α, β, γ and Λ. Unfortunately, theorem (4.2.3) is no longer valid, as the functions sought here are of a very special form, depending on the coordinates x^0, x^1, x^2 through the phase η only. This means that compatibility has to be studied once more.

Replacing then the off-diagonal Eqs. (4.2.21) in the diagonal Eqs. (4.2.22). A lengthy algebra that exploits the relation just derived in Eq. (4.2.19) reveals that the plane wave Eqs. (4.2.21) and (4.2.22) are compatible if and only if,

$$(\alpha^2)'(\beta^2)' = p^2(\alpha^2)'(\gamma^2)' + q^2(\beta^2)'(\gamma^2)' \tag{4.2.23}$$

Indeed, this means that every gravitational wave that is diagonal and *planar* must satisfy the compatibility condition (4.2.23) and propagate at the speed of light (by Eq. (4.2.20)). It will next be proven that one may always take $p = 1$ and $q = 0$.

Thus, without loss of generality it is sufficient to consider a wave propagating along the positive x^1-axis only.

To prove this claim, subtract each pair of consecutive off-diagonal equations (4.2.21) to write them as

$$\begin{aligned}(\ln\frac{\alpha}{\beta})'' &= (\ln\frac{\alpha}{\beta})'(\ln\frac{\gamma}{\alpha\beta})' \\ (\ln\frac{\gamma}{\alpha})'' &= (\ln\frac{\gamma}{\alpha})'(\ln\frac{\beta}{\alpha\gamma})' \\ (\ln\frac{\beta}{\gamma})'' &= (\ln\frac{\beta}{\gamma})'(\ln\frac{\alpha}{\beta\gamma})'\end{aligned} \tag{4.2.24}$$

These equations can be integrated immediately to yield

$$\begin{aligned}(\ln\frac{\alpha}{\beta})' &= C_1\frac{\gamma}{\alpha\beta} \\ (\ln\frac{\gamma}{\alpha})' &= C_2\frac{\beta}{\alpha\gamma} \\ (\ln\frac{\beta}{\gamma})' &= C_3\frac{\alpha}{\beta\gamma}\end{aligned} \tag{4.2.25}$$

where C_1, C_2 and C_3 are arbitrary constants of integration. Adding Eqs. (4.2.25) and multiplying the result by $\alpha\beta\gamma$ gives a compatibility condition

$$C_1\gamma^2 + C_2\beta^2 + C_3\alpha^2 = 0 \tag{4.2.26}$$

Therefore, there are several possibilities.

If $C_1 = 0$ then β is proportional to α. Rescaling the coordinate x^1 by the same proportionality factor shows that such metric is of the form (4.2.9) with $\beta = \alpha$. The second case is when either $C_2 = 0$ or $C_3 = 0$. Assuming without loss of generality that it is the former $C_2 = 0$, show that γ is proportional to α and therefore from Eq. (4.2.23), $\beta = \pm\alpha$ giving again the form (4.2.9) with $\beta = \alpha$. The last case is when α, β and γ are all proportional to one another.

Either way, it is clear that one may always assume that the metric is of the form (4.2.9) with $\alpha = \beta$. From Eq. (4.2.20) there is an angle θ such that $p = \cos\theta$ and

$q = \sin\theta$. This angle determines the direction of propagation of the gravitational wave. But then, a simple rotation of the $x^1 - x^2$ plane by θ shows that without loss of generality, one may always consider the wave to propagate along the positive x^1-axis. This means that $p = 1$ and $q = 0$, and the variable η in Eq. (4.2.17) reduces back to its original definition from section 1.4.

We have therefore shown that without loss of generality, the metric can can *always* be written in the form (4.2.9) with $\alpha = \beta$ such that all of the metric coefficients depend only on $\eta = \frac{1}{2}(x^1 - x^0)$. However, then Eq. (4.2.23) together with Eq. (4.2.20) prove that $\alpha = \gamma$ as well. Further rescaling of η finally yields the metric

$$ds^2 = e^{-2\Lambda}\left[-(dx^0)^2 + (dx^1)^2 + (\alpha dx^2)^2\right] + e^{2\Lambda}(dx^3)^3 \tag{4.2.27}$$

This is equivalent to the spacetime interval (4.2.9) with $\beta = \gamma = 1$, $p = 1$ and $q = 0$. For such a metric, all of Einstein's vacuum Eqs. (4.2.5), (4.2.7) and (4.2.8) yield a single equation,

$$\alpha'' + 2\alpha(\Lambda')^2 = 0 \tag{4.2.28}$$

This is the famous *Bondi-Pirani-Robinson waves* [20]. It is a known fact that it is a plane *pp*-wave [41].

4.2.C Stability

As shown in the last section, by properly choosing the coordinate system used, one may always describe a plane gravitational wave with a diagonal metric as propagating along the positive x^1-axis. This means that all the coefficients of the metric defined in Eq. (4.2.9) are functions of the light-cone coordinate $\eta = \frac{1}{2}(x^1 - x^0)$ alone. Furthermore it was shown that one may further assume that $\beta = \gamma = 1$. These waves satisfy the Bondi-Pirani-Robinson Eq. (4.2.28),

$$0 = \alpha_0'' + 2\alpha_0(\Lambda_0')^2 \tag{4.2.29}$$

where $\alpha_0 = \alpha(\eta)$ and $\Lambda_0 = \Lambda(\eta)$, and the prime denotes differentiation with respect to the variable η.

Eq. (4.2.29) reveals a peculiar situation where the two degrees of freedom satisfy only one equation, which is thus an underdetermined system for α_0 and Λ_0. Therefore one of these functions can be set arbitrary. Physically, this means that such gravitational waves may have any wave profile as determined by Λ_0. As a practical example, consider the solution $\alpha_0 = \tanh\eta$, for which Eq. (4.2.29) implies that $\Lambda_0' = \frac{\pm 1}{\cosh\eta}$ is of a soliton-like form.

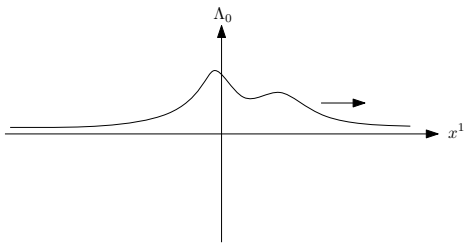

FIGURE 4.2. The profile of the Bondi-Pirani-Robinson gravitational wave can have arbitrary form, with the wave propagating in the positive x^1-direction only.

Since α_0 vanishes at $\eta = 0$, the metric is singular at this point. Nevertheless, this is not a physical singularity. A simple but lengthy calculation shows that all of the components of the Weyl tensor [3] are proportional to either one of the two components,

$$C_{0220} = (\alpha_0)^2 e^{-4\Lambda_0} C_{0330} = -(\alpha_0)^2 e^{-2\Lambda_0}[\Lambda_0'' + 3(\Lambda_0')^2] \tag{4.2.30}$$

and therefore the BPR spacetime is never singular when $\alpha_0 = 0$ (in fact, $\alpha_0 = 0$ corresponds to points where the spacetime is flat). In the following it will be assumed that Λ_0' is of Schwarz class and that α_0 has polynomial behavior at infinity. This assumption guarantees asymptotical flatness of the BPR wave. In particular, under this assumption Eq. (4.2.29) implies that α_0 is asymptotically linear as $|\eta| \to \infty$. By properly choosing the coordinates it can always be assumed that α_0 converges to a constant as $\eta \to -\infty$.

Since the BPR waves are a solution the Einstein vacuum Eq. (1.2.3), the stress-energy tensor $T_{\mu\nu}$ is identically zero and there is no other source of gravitational

[3] The Weyl tensor is another measure of the curvature of spacetime. It is the only part of the curvature of spacetime that exists for vacuum solutions of Einstein's equation [5].

fields. One can therefore think of the field and its stress as in equilibrium under the gravitational effects of the Bondi-Pirani-Robinson gravitational waves themselves. We have an equilibrium, but is it stable? The goal of this section is to explore this fascinating question.

The stability of plane gravitational waves is of great importance. In the past two decades there have been serious attempts to detect gravitational waves, so far unsuccessful. This includes but is not limited to the Laser Interferometer Gravitational-Wave Observatory (LIGO), a monumental project costing several hundred million dollars [42]. If plane gravitational waves are unstable, there is a very good reason they are difficult to detect, as small departures from idealized waves might destroy them.

To study this question of stability, consider a small perturbation of the Bondi-Pirani-Robinson waves that allows them to propagate weakly in the perpendicular x^2-direction and to even return in the negative x^1-direction. In this approximation the equations are linear and it is possible to separate their disturbance into proper modes and find their frequencies, whether real (stability) or imaginary (instability). Therefore we consider metric coefficients of the form

$$\alpha = \alpha_0 + \delta\alpha \qquad \beta = 1 + \delta\beta \qquad \gamma = 1 + \delta\gamma \qquad \Lambda = \Lambda_0 + \delta\Lambda \tag{4.2.31}$$

where $\delta\alpha$, $\delta\beta$, $\delta\gamma$ and $\delta\Lambda$ are small corrections that depend on all variables ζ, η and x^2. Here $\alpha_0 = \alpha_0(\eta)$ and $\Lambda_0 = \Lambda_0(\eta)$ depend on η only are the original BPR coefficients.

Linearizing Eqs. (4.2.5) with respect to the perturbation, and transforming to Fourier modes $\zeta \to \Omega$ and $x^2 \to k$ gives three second order differential equations for

the perturbations of α, β and γ:

$$\begin{aligned}
(\delta\beta)'' - (\delta\gamma)'' &= 4i\Omega\Lambda_0'(\delta\Lambda) + \left[2i\Omega\partial_\eta + \Omega^2 - 2(\Lambda_0')^2\right](\delta\beta) \\
&\quad + \left[2i\Omega\partial_\eta - \Omega^2 + 2(\Lambda_0')^2\right](\delta\gamma) \\
(\delta\alpha)'' &= -4\alpha_0\Lambda_0'(\delta\Lambda)' + \left[2i\Omega\partial_\eta + \Omega^2 - 4(\Lambda_0')^2 - \frac{\alpha_0''}{\alpha_0}\right](\delta\alpha) \\
&\quad + 2\alpha_0(\Lambda_0')^2(\delta\beta) + \left[2\alpha_0'\partial_\eta - \frac{4k^2}{\alpha_0} - 2\alpha_0(\Lambda_0')^2\right](\delta\gamma) \\
(\delta\alpha)'' &= -4\alpha_0\Lambda_0'(\delta\Lambda)' + (-2i\Omega\partial_\eta + \Omega^2 - 4(\Lambda_0')^2 - \frac{\alpha_0''}{\alpha_0})(\delta\alpha) \\
&\quad + \left[2\alpha_0'\partial_\eta + \frac{4k^2}{\alpha_0} - 2\alpha_0(\Lambda_0')^2\right](\delta\beta) + 2\alpha_0(\Lambda_0')^2(\delta\gamma)
\end{aligned} \tag{4.2.32}$$

where the primes denote differentiation with respect to the variable η.

As for the diagonal equations in (4.2.7), the first equation is a second order ordinary differential equations for the function $\delta\alpha$,

$$(\delta\alpha)'' = -4\alpha_0\Lambda_0'(\delta\Lambda)' - \left[\Omega^2 + 2(\Lambda_0')^2\right](\delta\alpha) + \alpha_0'\left[(\delta\beta + \delta\gamma)' - i\Omega(\delta\beta - \delta\gamma)\right] \tag{4.2.33}$$

The second and third equations in (4.2.7) yield two first order equations for the perturbations $\delta\beta$ and $\delta\gamma$ provided that the frequency k is nonzero,

$$\begin{aligned}
(\delta\beta)' &= \left[\frac{\alpha_0'}{\alpha_0} + i\Omega\right](\delta\beta) - 2\Lambda_0'(\delta\Lambda) \\
(\delta\gamma)' &= \left[\frac{\alpha_0'}{\alpha_0} - i\Omega\right](\delta\gamma) - 2\Lambda_0'(\delta\Lambda)
\end{aligned} \tag{4.2.34}$$

and are otherwise automatically satisfied. Last but not least, Eq. (4.2.8) is a first order equation for the difference in perturbations of β and γ,

$$0 = -\left[2i\Omega\alpha_0\partial_\eta + i\Omega\alpha_0' - \frac{2k^2}{\alpha_0}\right](\delta\Lambda) - i\Omega\Lambda_0'(\delta\alpha) + \left[\alpha_0\Lambda_0'(\delta\beta - \delta\gamma)\right]' \tag{4.2.35}$$

Thus, we obtained an overdetermined system of ODEs that includes seven equations for the four unknowns $\delta\alpha$, $\delta\beta$, $\delta\gamma$ and $\delta\Lambda$. Quite remarkably, a very similar situation occurs in what physically seems to be a complete different stability problem in general relativity - the stability of the Schwarzschild singularity. In a famous work

of Regge and Wheeler [43], they showed that the complete set of Einstein's equation gives only seven equations for four unknowns, just like we have here. To analyze the equations herein obtained, it is convenient to distinguish two cases that are physically very different.

The case $k = 0$ corresponds to a metric independent of x^2, namely a wave that propagates along the positive x^1-axis only and may collide head-on with a weak wave propagating in the negative x^1-axis. As was already mentioned in the first chapter, block diagonal metrics depending on η and ζ only may be casted to a very special form (1.5.1). In this case, one may therefore assume without loss of generality that $\delta\beta = \delta\gamma$. This reduces the stability equations to five equations for three unknowns $\delta\alpha$, $\delta\beta$ and $\delta\Lambda$. Subtracting the last two equations in (4.2.32) gives

$$4i\Omega(\delta\alpha)' = 0 \tag{4.2.36}$$

If $\Omega = 0$ then the wave is perturbed in the η-direction only. However, it was shown in the last section that such waves can always be transformed to the BPR form, and therefore $\Omega = 0$ is a trivial perturbation. Otherwise, $\delta\alpha$ is constant. Using this in Eqs. (4.2.32) and (4.2.33) again, shows that in fact $\delta\alpha$ vanishes, $\delta\alpha = 0$.

Since the wave cannot perturb α, Eq. (4.2.35) can be simplified to

$$0 = 2\alpha_0(\delta\Lambda)' + \alpha_0'(\delta\Lambda) \tag{4.2.37}$$

which may be integrated immediately to give $\delta\Lambda = \frac{M}{\sqrt{|\alpha_0|}}$. So far, all of Einstein's equations are satisfied with the exception of the first equation in (4.2.32), which determines $\delta\beta$

$$(\delta\beta)' + \Lambda_0'(\delta\Lambda) = 0 \tag{4.2.38}$$

It can be integrated immediately once Λ_0 is known. In fact, it will be shown in section 4.4.B that when $k = 0$ the nonlinear Schrödinger-Einstein equations are integrable in general.

The case $k \neq 0$ describes a wave that may collide with a transverse wave propagating in the x^2-direction as well as reflect in the negative x^1-direction. Here the stability of the BPR wave and that of the Schwarzschild singularity reveal even greater similarity. For the latter, Regge and Wheeler proved that the seven stability equations are equivalent to three differential equations coupled to one algebraic relation, from which the stability of the Schwarzschild black hole followed [43]. Considering the fact that the two problems are physically quite different, it is surprising and refreshing news that also here one may reduce the overdetermined set of seven equations to a much simpler set of four equations for the four unknowns.

Theorem 4.2.39. *If the frequency* $k \neq 0$ *then the seven stability equations (4.2.32), (4.2.33), (4.2.34) and (4.2.35) are equivalent to four equations, three of which are first order equations for* $\delta\beta$, $\delta\gamma$ *and* $\delta\Lambda$,

$$\begin{aligned}
(\delta\beta)' &= \left[\frac{\alpha_0'}{\alpha_0} + i\Omega\right](\delta\beta) - 2\Lambda_0'(\delta\Lambda) \qquad (4.2.40)\\
(\delta\gamma)' &= \left[\frac{\alpha_0'}{\alpha_0} - i\Omega\right](\delta\gamma) - 2\Lambda_0'(\delta\Lambda)\\
(\delta\Lambda)' &= -\left(\frac{ik^2}{\Omega\alpha_0^2} + \frac{\alpha_0'}{2\alpha_0}\right)(\delta\Lambda)\\
&\quad + \left[-\frac{i}{2\Omega}\Lambda_0'' + \frac{k^2}{2\Omega^2}\frac{\Lambda_0'}{\alpha_0^2} - \frac{3i}{4\Omega}\Lambda_0'\frac{\alpha_0'}{\alpha_0}\right](\delta\beta - \delta\gamma) + \frac{1}{2}\Lambda_0'(\delta\beta + \delta\gamma)
\end{aligned}$$

plus an algebraic relation for $\delta\alpha$,

$$0 = (\delta\alpha) + \left(\frac{i}{2\Omega}\alpha_0' + \frac{k^2}{\Omega^2\alpha_0}\right)(\delta\beta - \delta\gamma) \qquad (4.2.41)$$

Proof. Eqs. (4.2.34) allow completely eliminating any derivatives of $\delta\beta$ and $\delta\gamma$ in each of the stability equation. In fact, differentiating them and plugging their derivatives into the first off-diagonal equation in (4.2.32) gives a trivial result. Therefore the first off-diagonal equation in (4.2.32) is the consequence of two of the diagonal equations.

Similarly, the diagonal Eq. (4.2.33) can be used to eliminate the second derivative of $\delta\alpha$ in the two remaining off-diagonal Eqs. (4.2.32). The result can be simplified

even more by excluding the derivatives of $\delta\beta$ and $\delta\gamma$ again. This gives two first order equations for δa,

$$\begin{aligned} 0 &= 2\Omega(\Omega + i\partial_\eta)(\delta\alpha) \\ &\quad + \left[2\alpha_0(\Lambda_0')^2 - \frac{(\alpha_0')^2}{\alpha_0}\right](\delta\beta) - \left[2i\Omega\alpha_0' + \frac{4k^2}{\alpha_0} + 2\alpha_0(\Lambda_0')^2 - \frac{(\alpha_0')^2}{\alpha_0}\right](\delta\gamma) \\ 0 &= 2\Omega(\Omega - i\partial_\eta)(\delta\alpha) \\ &\quad + \left[2i\Omega\alpha_0' + \frac{4k^2}{\alpha_0} - 2\alpha_0(\Lambda_0')^2 + \frac{(\alpha_0')^2}{\alpha_0}\right](\delta\beta) + \left[2\alpha_0(\Lambda_0')^2 - \frac{(\alpha_0')^2}{\alpha_0}\right](\delta\gamma) \end{aligned} \tag{4.2.42}$$

Adding these equations proves that the algebraic Eq. (4.2.41) holds. It can be used in the last diagonal equation (4.2.35) to eliminate $\delta\alpha$ altogether, yielding an equation for $\delta\Lambda$. This is the last equation in (4.2.40) (the first two equations in (4.2.40) were already derived in Eqs. (4.2.34). This proves the that seven equations imply the four equations (4.2.40) and (4.2.41).

Conversely, it is a straight-forward but an elaborated task to use Eqs. (4.2.40) and (4.2.41) in the seven Eqs. (4.2.32), (4.2.33), (4.2.34) and (4.2.35) and see that they are identically satisfied. □

The fact that Eqs. (4.2.40) are decoupled from $\delta\alpha$, shows that one only needs to focus on these equations. One solved, they can be immedietly used in Eq. (4.2.41) to give the function $\delta\alpha$.

Let $\delta\beta = \frac{1}{2}\alpha_0(B^+ + B^-)$, $\delta\gamma = \frac{1}{2}\alpha_0(B^+ - B^-)$ and $\delta\Lambda = \alpha_0 L$. From Eqs. (4.2.40), the functions B^+, B^- and L satisfy

$$\begin{aligned} (B^+)' &= i\Omega B^- - 4\Lambda_0' L \\ (B^-)' &= i\Omega B^+ \\ L' &= -\left(\frac{ik^2}{\Omega\alpha_0^2} + \frac{3\alpha_0'}{2\alpha_0}\right)L + \left[-\frac{i}{2\Omega}\Lambda_0'' + \frac{k^2}{2\Omega^2}\frac{\Lambda_0'}{\alpha_0^2} - \frac{3i}{4\Omega}\Lambda_0'\frac{\alpha_0'}{\alpha_0}\right]B^- + \frac{1}{2}\Lambda_0' B^+ \end{aligned} \tag{4.2.43}$$

One may eliminate B^+ from the third equation using the second equation, to get a first order equation relating L and B^-. It is a miracle that it can be written in a very

simple form

$$\Psi' + \left(\frac{ik^2}{\Omega \alpha_0^2} + \frac{3}{2} \frac{\alpha_0'}{\alpha_0} \right) \Psi = 0 \tag{4.2.44}$$

where $\Psi = 2i\Omega L - \Lambda_0' B^-$ is a complex-valued 'wave-function'. It may be integrated immediately to obtain an algebraic relation between L and B^-,

$$\Psi \equiv 2i\Omega L - \Lambda_0' B^- = \frac{K}{|\alpha_0|^{3/2}} \exp \left[-\frac{ik^2}{\Omega} \int \frac{d\eta}{\alpha_0^2} \right] \tag{4.2.45}$$

where K is an arbitrary constant of integration.

This allows to obtain a single second order equation for B^- alone. Differentiate the second equation in (4.2.43). The derivative $(B^+)'$ may be eliminated through the first equation in (4.2.43) while the function L can also be excluded using the algebraic relation just derived. This yields a single second order equation for the function B^- only,

$$(B^-)'' + (\Omega^2 + 2\Lambda_0'^2) B^- + 2K \frac{\Lambda_0'}{|\alpha_0|^{3/2}} \exp \left[-\frac{ik^2}{\Omega} \int \frac{d\eta}{\alpha_0^2} \right] = 0 \tag{4.2.46}$$

Once solved, L and B^+ can be easily obtained from Eq. (4.2.45) and the second equation in (4.2.43).

In both cases $k = 0$ and $k \neq 0$ imaginary modes yield a spacetime that is not asymptotically flat and can therefore be disregarded on physical grounds. Consequently, we conclude that there are no unstable solutions for the perturbation, and that the Bondi-Pirani-Robinson wave is stable.

As an example, let $\alpha_0 = \tanh \eta$ and $\Lambda_0' = \frac{\pm 1}{\cosh \eta}$ be the BPR wave that was mentioned earlier. For this wave the integral in Eq. (4.2.45) can be evaluated explicitly,

$$\Psi(\eta, \Omega, k) = \frac{K}{|\tanh \eta|^{3/2}} \exp \left[-\frac{ik^2}{\Omega} (\eta - \coth \eta) \right] \tag{4.2.47}$$

Consider now a collision between the incident wave Ψ (for fixed frequencies k and Ω) and a wave barrier. Normally, such a collision generates a reflected wave. However, here the asymptotic behavior of Ψ is

$$\Psi \sim K \exp \left[-\frac{ik^2}{\Omega} (\eta \mp 1) \right] \text{ as } \eta \to \pm\infty \tag{4.2.48}$$

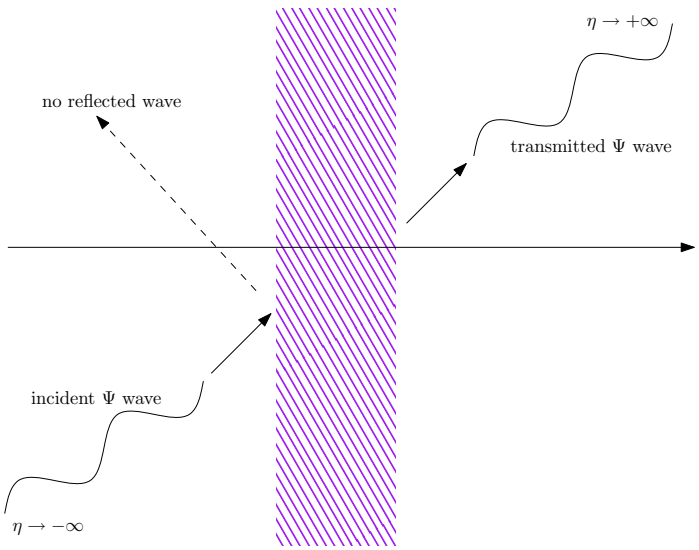

FIGURE 4.3. The diagram shows the 'wave-function' Ψ for $\alpha_0 = \tanh \eta$. In this case, Eq. (4.2.48) shows that Ψ represents a wave traveling to the right both before and after hitting the wave barrier. Therefore there is no reflective wave and the barrier is transparent for Ψ. The only remenant of the interaction is a phase shift.

showing that the wave Ψ maintains the same asymptotic behavior except for a phase shift and is furthermore transparent to the barrier (see figure (4.3)). This is a striking fact and a hint of integrability.

In studying the stability of the Bondi-Pirani-Robinson wave and that of the Schwarzschild black hole, we noticed a remarkable analogy between the problems. It is therefore very reasonable to except that stability follows in more general circumstances. The fact that both the Bondi-Pirani-Robinson wave and the Schwarzschild black hole are a special case of the block diagonal metric (1.5.1) integrable by the inverse scattering transform, makes it tempting to conjecture that perhaps all the solutions of the Einstein vacuum equation belonging to this class of metrics are stable. It is the author's goal to explore this issue in future work.

4.3 Weak gravitational waves

4.3.A The general case

The previous two sections focused on gravitational waves of arbitrary magnitude. The goal of this section is to study *weak* gravitational waves. The main reason to restrict one's attention to weak waves is practical: one may use perturbation theory to derive easier equations for analysis. These new equations can in many cases be solved, and suffice in order to study gravitational waves in our solar system.

The assumption of weak gravitational waves requires the existence of a small parameter with which perturbation theory can be used. In general relativity, there is no such parameter in general. This issue can be overcomed by artificially introducing a small parameter $\varepsilon \ll 1$. Furthermore, it is assumed that the metric is nearly flat, in the sense that it is close to the Minkowski flat spacetime metric. This allows expanding the diagonal coefficients H_i in the metric (4.1.1) in powers of ε,

$$H_\mu = 1 + h_\mu \varepsilon + g_\mu \varepsilon^2 \tag{4.3.1}$$

The two leading terms for the rotation coefficients (4.1.5) are

$$Q_{\mu\nu} = \frac{\partial h_\mu}{\partial x^\nu}\varepsilon + (-h_\nu \frac{\partial h_\mu}{\partial x^\nu} + \frac{\partial g_\mu}{\partial x^\nu})\varepsilon^2 + O(\varepsilon^3) \tag{4.3.2}$$

and similarly, the two leading terms of the coefficients E_{ik} (4.1.8) are

$$\begin{aligned} E_{\mu\nu} &= \left[\frac{\partial^2 h_\mu}{\partial (x^\nu)^2} + \frac{\partial^2 h_\nu}{\partial (x^\mu)^2}\right]\varepsilon \\ &+ \left[\frac{\partial^2 g_\mu}{\partial (x^\nu)^2} + \frac{\partial^2 g_\nu}{\partial (x^\mu)^2} - \frac{\partial}{\partial x^\nu} h_\nu \frac{\partial h_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\mu} h_\mu \frac{\partial h_\nu}{\partial x^\mu} + \sum_{\lambda \neq \mu,\nu} \frac{\partial h_\mu}{\partial x^\lambda}\frac{\partial h_\nu}{\partial x^\lambda}\right]\varepsilon^2 + O(\varepsilon^3) \end{aligned} \tag{4.3.3}$$

The non-diagonal Einstein vacuum equations $R_{\mu\nu} = 0$ are

$$\sum_{\lambda \neq \mu,\nu} \frac{\partial^2 h_\lambda}{\partial x^\mu \partial x^\nu} = 0 \tag{4.3.4}$$

in the leading order, and

$$\sum_{\lambda\neq\mu,\nu}\left[\frac{\partial^2 g_\lambda}{\partial x^\nu\partial x^\mu}+(h_\mu-h_0)\frac{\partial^2 h_\lambda}{\partial x^\nu\partial x^\mu}-\frac{\partial h_\lambda}{\partial x^\nu}\frac{\partial h_\nu}{\partial x^\mu}-\frac{\partial}{\partial x^\nu}h_\mu\frac{\partial h_\lambda}{\partial x^\mu}\right]=0 \tag{4.3.5}$$

in the second order. Similarly, the diagonal Einstein vacuum equations $R_{\mu\mu}=0$ are

$$\sum_{\nu\neq\mu}\left[\frac{\partial^2 h_\mu}{\partial(x^\nu)^2}+\frac{\partial^2 h_\nu}{\partial(x^\mu)^2}\right]=0 \tag{4.3.6}$$

in the leading order, and

$$\sum_{\nu\neq\mu}\left[\frac{\partial^2 g_\mu}{\partial(x^\nu)^2}+\frac{\partial^2 g_\nu}{\partial(x^\mu)^2}-\frac{\partial}{\partial x^\nu}h_\nu\frac{\partial h_\mu}{\partial x^\nu}-\frac{\partial}{\partial x^\mu}h_\mu\frac{\partial h_\nu}{\partial x^\mu}-h_\nu\left(\frac{\partial^2 h_\mu}{\partial(x^\nu)^2}-\frac{\partial^2 h_\nu}{\partial(x^\mu)^2}\right)+\sum_{\lambda\neq\mu,\nu}\frac{\partial h_\mu}{\partial x^\lambda}\frac{\partial h_\nu}{\partial x^\lambda}\right]=0 \tag{4.3.7}$$

4.3.B Waves in $1+2$ coordinates

Assume now that the spacetime metric depends on three of the four spacetime variables, namely $H_i=H_i(x^0,x^1,x^2)$ as in section 4.2. This reduces the off-diagonal Eqs. (4.3.4) to

$$\begin{aligned}\frac{\partial^2}{\partial x^0\partial x^1}(h_2+h_3) &= 0\\ \frac{\partial^2}{\partial x^0\partial x^2}(h_1+h_3) &= 0\\ \frac{\partial^2}{\partial x^1\partial x^2}(h_0+h_3) &= 0\end{aligned} \tag{4.3.8}$$

which can be easily solved. Since these are three equations for four unknowns h_0,h_1,h_2,h_3 it is convenient to denote by $h\equiv h_3$ the auxiliary degree of freedom. Then,

$$\begin{aligned}h_0 &= -h+e_{02}(x^0,x^2)+e_{01}(x^0,x^1)\\ h_1 &= -h+e_{10}(x^0,x^1)+e_{12}(x^1,x^2)\\ h_2 &= -h+e_{20}(x^0,x^2)+e_{21}(x^1,x^2)\end{aligned} \tag{4.3.9}$$

where the $e_{\mu\nu}$'s are arbitrary functions of two variables and h is an arbitrary function of x^0, x^1, x^2.

Consider the case $e_{\mu\nu} = 0$ for all μ, ν in which $h_0 = h_1 = h_2 = -h$. This means that all the degrees of freedom of the leading order equation are contained in the function h. So far only the leading order off-diagonal Eq. (4.3.4) is satisfied. In this case where $e_{ij} = 0$, all of the diagonal equations $R_{\mu\mu} = 0$ give the same equation in the first order. Specificaly, Eq. (4.3.6) reduces to the wave equation

$$\Box h = 0 \tag{4.3.10}$$

where $\Box = -\frac{\partial^2}{\partial(x^0)^2} + \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2}$ is the d'Alembert operator in three dimensions. This is a known result [5], which demonstrates once more the fact that the vacuum Einstein equation reduces to Laplace's equation in the non-relativistic limit.

Consider next the general case, when one does not impose that the functions e_{ij} vanish. In this case, the diagonal equation $R_{33} = 0$ gives again the wave Eq. (4.3.10) in the leading order, and the equations $R_{00} = R_{11} = R_{22} = 0$ are in the leading order,

$$\begin{aligned}
-\Box h + \frac{\partial^2 e_{02}}{\partial(x^2)^2} + \frac{\partial^2 e_{01}}{\partial(x^1)^2} - \frac{\partial^2 e_{10}}{\partial(x^0)^2} - \frac{\partial^2 e_{20}}{\partial(x^0)^2} &= 0 \\
-\Box h + \frac{\partial^2 e_{12}}{\partial(x^2)^2} + \frac{\partial^2 e_{01}}{\partial(x^1)^2} - \frac{\partial^2 e_{10}}{\partial(x^0)^2} + \frac{\partial^2 e_{21}}{\partial(x^1)^2} &= 0 \\
-\Box h + \frac{\partial^2 e_{12}}{\partial(x^2)^2} + \frac{\partial^2 e_{21}}{\partial(x^1)^2} - \frac{\partial^2 e_{20}}{\partial(x^0)^2} - \frac{\partial^2 e_{02}}{\partial(x^2)^2} &= 0
\end{aligned} \tag{4.3.11}$$

But the wave Eq. (4.3.10) shows that h is eliminated from these equations. Note that these equations do have non-trivial solutions, for example

$$\begin{aligned}
e_{20} = e_{02} &= (x^0)^2 + (x^2)^2 \\
e_{01} = e_{10} &= (x^0)^2 + (x^1)^2 \\
e_{12} = e_{21} &= (x^1)^2 - (x^2)^2
\end{aligned} \tag{4.3.12}$$

In fact, one can add any linear term and even scale them to obtain more solutions. To

find the general solution, we notice that Eqs. (4.3.12) are equivalent to the equations,

$$\underbrace{\frac{\partial^2 e_{01}}{\partial(x^1)^2} - \frac{\partial^2 e_{10}}{\partial(x^0)^2}}_{\text{function of } x^0,x^1} = \underbrace{\frac{\partial^2 e_{02}}{\partial(x^2)^2} - \frac{\partial^2 e_{20}}{\partial(x^0)^2}}_{\text{function of } x^0,x^2} = \underbrace{\frac{\partial^2 e_{12}}{\partial(x^2)^2} + \frac{\partial^2 e_{21}}{\partial(x^1)^2}}_{\text{function of } x^1,x^2} \tag{4.3.13}$$

Note that each expression (separated by equal signs) is a function of two variables only, and that the intersection of all sets of pairs of such variables is empty. This means that each of the expressions is a constant, and therefore

$$\begin{aligned} \frac{\partial^2 e_{01}}{\partial(x^1)^2} - \frac{\partial^2 e_{10}}{\partial(x^0)^2} &= K \\ \frac{\partial^2 e_{02}}{\partial(x^2)^2} - \frac{\partial^2 e_{20}}{\partial(x^0)^2} &= K \\ \frac{\partial^2 e_{12}}{\partial(x^2)^2} + \frac{\partial^2 e_{21}}{\partial(x^1)^2} &= K \end{aligned} \tag{4.3.14}$$

where K is an arbitrary constant. These are three equations for six unknowns, giving us an underdetermined system of equations. By fixing e_{10}, e_{20} and e_{21}, these equations can be easily solved for e_{01}, e_{02} and e_{12}.

It is time to consider the next order correction. This will be done for $e_{ij} = 0$ only. The non-diagonal Eqs. (4.3.5) yield

$$\begin{aligned} \frac{\partial^2}{\partial x^1 \partial x^2}(g_0 + g_3) &= 2h\frac{\partial^2 h}{\partial x^1 \partial x^2} \\ \frac{\partial^2}{\partial x^0 \partial x^2}(g_1 + g_3) &= 2h\frac{\partial^2 h}{\partial x^0 \partial x^2} \\ \frac{\partial^2}{\partial x^0 \partial x^1}(g_2 + g_3) &= 2h\frac{\partial^2 h}{\partial x^0 \partial x^1} \end{aligned} \tag{4.3.15}$$

The correction to the diagonal equation $R_{33} = 0$ is

$$\Box g_3 = (\nabla h)^2 \tag{4.3.16}$$

and the other diagonal Eqs. (4.3.7) give

$$\begin{aligned} \Box g_0 + \frac{\partial^2}{\partial(x^0)^2}(+g_0 - g_1 - g_2 - g_3) &= -2h\frac{\partial^2 h}{\partial(x^0)^2} + (\nabla h)^2 \\ \Box g_1 - \frac{\partial^2}{\partial(x^1)^2}(-g_0 + g_1 - g_2 - g_3) &= 2h\frac{\partial^2 h}{\partial(x^1)^2} + (\nabla h)^2 \\ \Box g_2 - \frac{\partial^2}{\partial(x^2)^2}(-g_0 - g_1 + g_2 - g_3) &= 2h\frac{\partial^2 h}{\partial(x^2)^2} + (\nabla h)^2 \end{aligned} \tag{4.3.17}$$

where $(\nabla h)^2 = -\left(\frac{\partial h}{\partial x^0}\right)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2$. Note that once h is known, the linear equation for g_3 can be solved using standard techniques. In turn, this gives a coupled set of linear equations for g_0, g_1, g_2. This means that up to order ε^2, the function h comprises *all the information* about the gravitational wave except the initial / boundary conditions.

Surprisingly, a similar result holds true up to the fifth order in ε. Up to fifth order in ε, the behavior of a gravitational wave is completely determined by the amplitude of the gravitational field in the direction of the Killing vector ∂_3. This result is the topic of the next section.

4.3.C Higher order analysis

As was just shown, the metric coefficients

$$H_0 = 1 - h\varepsilon + g_0\varepsilon^2 \qquad H_1 = 1 - h\varepsilon + g_1\varepsilon^2 \qquad H_2 = 1 - h\varepsilon + g_2\varepsilon^2 \tag{4.3.18}$$

describe a general perturbation of the Minkowski spacetime up to second order in a small parameter ε. In the last section it was assumed that g_0, g_1, g_2 and h are all independent of ε. One may consider an alternative approach.

Instead of treating Eqs. (4.3.18) as a perturbation expansion, the four degrees of freedom g_0, g_1, g_2 and h can be considered as new degrees of freedom replacing the original H_0, H_1, H_2 and H_3. The only caveat is that now, the new degrees of freedom are functions of the spacetime variables x_0, x_1, x_2 together with the auxiliary parameter ε. In particular the fourth metric coefficient is now

$$H_3 = 1 + h\varepsilon \tag{4.3.19}$$

instead of $H_3 = 1 + h\varepsilon + g_3\varepsilon^2$ as it was before.

It will prove very convenient, to use once more the functions α, β, γ and Λ introduced in the proof of theorem (4.2.3). The perturbation of Minkowski spacetime

described earlier means that Λ is of the form

$$\Lambda = \varepsilon h \tag{4.3.20}$$

where $h = h(x^0, x^1, x^2; \varepsilon)$. It is easy to see from the Lagrangian (4.2.6) that when expanding in powers of ε, this automatically takes care of the first order correction in ε, as

$$H_0 = e^{-\Lambda}\gamma = (1 - h\varepsilon)(1 + O(\varepsilon)) \tag{4.3.21}$$

and we have already seen that up to the first order $H_0 = 1 - h\varepsilon$. Due to the symmetry in the x^0, x^1, x^2 coordinates, the same result holds true for H_1, H_2. This means that the leading order correction in α, β, γ from a Minkowski spacetime is of second order in ε. In general then, one has

$$\alpha = 1 + \varepsilon^2 a \qquad \beta = 1 + \varepsilon^2 b \qquad \gamma = 1 + \varepsilon^2 c \tag{4.3.22}$$

where a, b, c are functions of x^0, x^1, x^2 and the auxiliary parameter ε. Expanding the Einstein-Hilbert Lagrangian (4.2.6) in ε allows us to write it as

$$\frac{1}{2\varepsilon^2}\mathcal{L} = \mathcal{L}_{\text{linear}} + \varepsilon^2 \mathcal{L}_{\text{nonlinear}} \tag{4.3.23}$$

where the linear term is just the Lagrangian of the wave equation,

$$\mathcal{L}_{\text{linear}} = \frac{1}{2}\left[(\partial_0 h)^2 - (\partial_1 h)^2 - (\partial_2 h)^2\right] \tag{4.3.24}$$

and the nonlinear term is

$$\begin{aligned} \mathcal{L}_{\text{nonlinear}} &= \big[(a+b-c)(\partial_0 h)^2 - (a-b+c)(\partial_1 h)^2 - (-a+b+c)(\partial_2 h)^2 \\ &\quad -(\partial_0 a)(\partial_0 b) + (\partial_1 a)(\partial_1 c) + (\partial_2 b)(\partial_2 c)\big] + O(\varepsilon^6) \end{aligned} \tag{4.3.25}$$

The diagonal Einstein equations $\frac{\delta S}{\delta a} = \frac{\delta S}{\delta b} = \frac{\delta S}{\delta c} = 0$ are now

$$\begin{aligned} 0 &= \partial_0\partial_0 b - \partial_1\partial_1 c + (\partial_0 h)^2 - (\partial_1 h)^2 + (\partial_2 h)^2 \\ 0 &= \partial_0\partial_0 a - \partial_2\partial_2 c + (\partial_0 h)^2 + (\partial_1 h)^2 - (\partial_2 h)^2 \\ 0 &= \partial_1\partial_1 a + \partial_2\partial_2 b + (\partial_0 h)^2 + (\partial_1 h)^2 + (\partial_2 h)^2 \end{aligned} \tag{4.3.26}$$

and the non-diagonal equations are

$$\begin{aligned} \partial_0\partial_1 a &= -2\,(\partial_0 h)\,(\partial_1 h) \\ \partial_0\partial_2 b &= -2\,(\partial_0 h)\,(\partial_2 h) \\ \partial_1\partial_2 c &= -2\,(\partial_1 h)\,(\partial_2 h) \end{aligned} \tag{4.3.27}$$

This shows that the functions a, b, c are completely determined by the function h. Moreover, their form can be easily derived by simple integration of Eqs. (4.3.27) once h is known. We are thus left to study the diagonal equation $\delta S/\delta h = 0$, which gives a nonlinear evolution equation for the function h,

$$\Box h = -\varepsilon^2\left[\partial_0\left((a+b-c)\partial_0 h\right) - \partial_1\left((a-b+c)\partial_1 h\right) - \partial_2\left((-a+b+c)\partial_2 h\right)\right] \tag{4.3.28}$$

To decouple the last equation from the equations for a, b, c, consider the Fourier transform $(x^0, x^1, x^2) \xrightarrow{\mathcal{F}} (\omega, p, q)$ of Eqs. (4.3.27). After symmetrization, these are just

$$\begin{aligned} a(\vec{k}) &= -\int \frac{\omega_1 p_2 + \omega_2 p_1}{(\omega_1+\omega_2)(p_1+p_2)} h(\vec{k}_1)h(\vec{k}_2)\delta(\vec{k}-\vec{k}_1-\vec{k}_2)d\vec{k}_1 d\vec{k}_2 \\ b(\vec{k}) &= -\int \frac{\omega_1 q_2 + \omega_2 q_1}{(\omega_1+\omega_2)(q_1+q_2)} h(\vec{k}_1)h(\vec{k}_2)\delta(\vec{k}-\vec{k}_1-\vec{k}_2)d\vec{k}_1 d\vec{k}_2 \\ c(\vec{k}) &= -\int \frac{p_1 q_2 + p_2 q_1}{(p_1+p_2)(q_1+q_2)} h(\vec{k}_1)h(\vec{k}_2)\delta(\vec{k}-\vec{k}_1-\vec{k}_2)d\vec{k}_1 d\vec{k}_2 \end{aligned} \tag{4.3.29}$$

where the wave vector is $\vec{k} = (\omega, p, q)$ and the integration is performed over a six dimensional Fourier space. This means that our interaction terms in the Lagrangian are very simple. They are

$$\begin{aligned} (a+b-c)(\partial_0 h)^2 &= 2\int \frac{(\omega_1 p_2 q_2 + \omega_2 p_1 q_1)\omega_3\omega_4}{(\omega_1+\omega_2)(p_1+p_2)(q_1+q_2)} \times \\ &\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \\ (a-b+c)(\partial_1 h)^2 &= 2\int \frac{(p_1\omega_2 q_2 + p_2\omega_1 q_1)p_3 p_4}{(\omega_1+\omega_2)(p_1+p_2)(q_1+q_2)} \times \\ &\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \\ (-a+b+c)(\partial_2 h)^2 &= 2\int \frac{(q_1\omega_2 p_2 + q_2\omega_1 p_1)q_3 q_4}{(\omega_1+\omega_2)(p_1+p_2)(q_1+q_2)} \times \\ &\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \end{aligned} \tag{4.3.30}$$

and

$$
\begin{aligned}
(\partial_0 a)(\partial_0 b) &= -\int \frac{(\omega_1 p_2+\omega_2 p_1)(\omega_3 q_4+\omega_4 q_3)}{(p_1+p_2)(q_3+q_4)} \times \\
&\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \\
(\partial_1 a)(\partial_1 c) &= -\int \frac{(\omega_1 p_2+\omega_2 p_1)(p_3 q_4+p_4 q_3)}{(\omega_1+\omega_2)(q_3+q_4)} \times \\
&\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \\
(\partial_2 b)(\partial_2 c) &= -\int \frac{(\omega_1 q_2+\omega_2 q_1)(p_3 q_4+p_4 q_3)}{(\omega_1+\omega_2)(p_3+p_4)} \times \\
&\quad h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4
\end{aligned}
\tag{4.3.31}
$$

Thus one may completely eliminate a, b and c from the Lagrangian (4.3.23) and write it in terms of the function h only. The nonlinear interaction term (4.3.25) can now be written in the Zakharov-Schulman form [44], as

$$
\mathcal{L}_{\text{nonlinear}} = \int T_{1234} h(\vec{k}_1)h(\vec{k}_2)h(\vec{k}_3)h(\vec{k}_4)\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \tag{4.3.32}
$$

where

$$
\begin{aligned}
T_{1234} &= \frac{(\omega_1 p_2+\omega_2 p_1)(\omega_3 q_4+\omega_4 q_3)}{(p_1+p_2)(q_3+q_4)} - \frac{(\omega_1 p_2+\omega_2 p_1)(p_3 q_4+p_4 q_3)}{(\omega_1+\omega_2)(q_3+q_4)} \\
&\quad - \frac{(\omega_1 q_2+\omega_2 q_1)(p_3 q_4+p_4 q_3)}{(\omega_1+\omega_2)(p_3+p_4)} \\
&\quad + 2\frac{(\omega_1 p_2 q_2+\omega_2 p_1 q_1)\omega_3\omega_4 - (p_1\omega_2 q_2+p_2\omega_1 q_1)p_3 p_4 - (q_1\omega_2 p_2+q_2\omega_1 p_1)q_3 q_4}{(\omega_1+\omega_2)(p_1+p_2)(q_1+q_2)}
\end{aligned}
\tag{4.3.33}
$$

The last analysis culminated in a Lagrangian for the function h only. This Lagrangian is correct up to the fifth order in ε, and yields a nonlinear evolution equation for h. One may proceed in a similar fashion to include higher order terms, but it will not be done here. This formulation is very powerful and can be used to obtain a lot of insight about Einstein's equations. Specifically, one may use it to write the Lagrangian and Hamiltonian in their canonical form.

Moreover, by applying quantum language to this classical problem, this analysis can be used to examine the realization of various processes. In particular, one may prove immediately that certain interactions between weak gravitational waves can never take place in Nature.

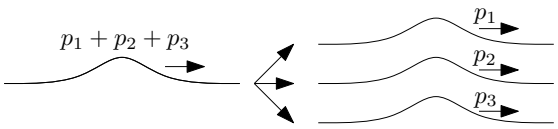

FIGURE 4.4. The collapse of one gravitational wave to three gravitational waves is a forbidden process in $1D$. Due to the time-reversal symmetry, the opposite process is also impossible.

As an example, consider the special case of waves with momentum mostly in the x^1-direction. To model this, we let $q_\mu = \delta p_\mu$ ($\mu = 1, 2, 3, 4$) and take the limit $\delta \to 0$ in Eq. (4.3.33) to get

$$\begin{aligned} T_{1234} &= \frac{(\omega_1 p_2 + \omega_2 p_1)(\omega_3 p_4 + \omega_4 p_3)}{(p_1 + p_2)(p_3 + p_4)} - \frac{2(\omega_1 p_2 + \omega_2 p_1) p_3 p_4}{(\omega_1 + \omega_2)(p_3 + p_4)} \\ &\quad + 2\frac{(\omega_1 p_2^2 + \omega_2 p_1^2)\omega_3\omega_4 - (\omega_1 + \omega_2) p_1 p_2 p_3 p_4}{(\omega_1 + \omega_2)(p_1 + p_2)^2} \end{aligned} \tag{4.3.34}$$

Since the integral defining the nonlinear Lagrangian $\mathcal{L}_{\text{nonlinear}}$ is performed over a symmetric space in $\vec{k}_1, \vec{k}_2, \vec{k}_3$ and $\vec{k}_4$, only its symmetric part give a non-trivial contribution to the integral. Let $S = \text{Symm}(T_{1234})$ be the symmetrization of T_{1234}. Here $S = S(\omega_1, \omega_2, \omega_3, \omega_4; p_1, p_2, p_3, p_4)$ is independent of q_i due to the initial assumption. A direct but tedious computation shows that

$$S(-|p_1 + p_2 + p_3|, |p_1|, |p_2|, |p_3|; -(p_1 + p_2 + p_3), p_1, p_2, p_3) = 0 \tag{4.3.35}$$

and therefore a collapse of a single one-dimensional wave to three waves does not happen in the weak field approximation employed here. This is not a surprising result considering the fact that the Bondi-Pirani-Robinson waves were shown to have arbitrary profile in section 4.2.C. Similarly,

$$S(-|p|, -|p|, |p|, |p|; -p, -p, p, p) = 0 \tag{4.3.36}$$

and therefore there is no two-wave to two-wave interaction (for waves with equal momentum p).

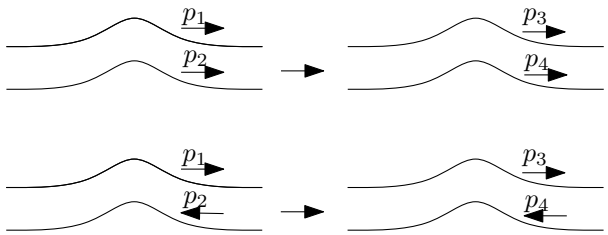

FIGURE 4.5. Two gravitational waves that travel along a single axis cannot interact and change momenta. Even if all waves share the same momentum p the process is forbidden.

As a last example, notice that on the resonant surface $|p_1| + |p_2| = |p_3| + |p_4|$. If one assumes that either $p_1, p_2, p_3, p_4 > 0$ or $p_1, p_3 > 0$, $p_2, p_4 < 0$ then again

$$S = 0 \tag{4.3.37}$$

4.3.D Axial symmetry

One may use a similar treatment to the one done before to include weak waves with other symmetries, such as weakly interacting waves with axial symmetry. This will be done here in the leading order in ε only. Weak waves with axial symmetry are most easily represented in cylindrical coordinates $x^\mu = (t, r, z, \theta)$ by a metric that is independent of θ. They can describe the asymptotic behavior of numerous compact objects located and moving along the same axis, and even the head-on collision of two Schwarzschild black holes traveling along the z-axis. Let

$$H_0 = i(1 + \varepsilon h_0) \qquad H_1 = 1 + \varepsilon h_1 \qquad H_2 = 1 + \varepsilon h_2 \qquad H_3 = r(1 + \varepsilon h_3) \tag{4.3.38}$$

be a perturbation of the Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + dz^2 + (rd\theta)^2 \tag{4.3.39}$$

In the leading order in ε, the (non-trivial) non-diagonal elements of the Ricci curvature tensor are

$$\begin{aligned} R_{01} &= \varepsilon\frac{\partial}{\partial t}\left[\frac{1}{r}(h_1-h_3)-\frac{\partial(h_2+h_3)}{\partial r}\right] \\ R_{02} &= -\varepsilon\left[\frac{\partial^2(h_1+h_3)}{\partial t\partial z}\right] \\ R_{12} &= \varepsilon\frac{\partial}{\partial z}\left[\frac{1}{r}(h_1-h_3)-\frac{\partial(h_0+h_3)}{\partial r}\right] \end{aligned} \tag{4.3.40}$$

Denoting $h=h_3$ as before, the simplest solution of the the second equation is $h_1=-h_3$. This means that the equations $R_{01}=R_{12}=0$ reduce to

$$\begin{aligned} \frac{\partial^2 h_2}{\partial t\partial r} &= -\frac{\partial^2 h}{\partial t\partial r}-\frac{2}{r}\frac{\partial h}{\partial t} \\ \frac{\partial^2 h_0}{\partial z\partial r} &= -\frac{\partial^2 h}{\partial z\partial r}-\frac{2}{r}\frac{\partial h}{\partial z} \end{aligned} \tag{4.3.41}$$

which are easily solvable once the function h is known. Take a solution for which h_0 and h_2 are equal, and denote $h_0=h_2=f$. Then f satisfies

$$\frac{\partial f}{\partial r}=-\frac{\partial h}{\partial r}-\frac{2}{r}h \tag{4.3.42}$$

To determine the function h, we consider the diagonal equation $R_{33}=0$, giving

$$\begin{aligned} R_{33} &= \varepsilon r^2\left[\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial r^2}\right)h-\frac{1}{r}\frac{\partial(h_0-h_1+h_2+2h_3)}{\partial r}\right] \\ &= \varepsilon r^2\left[-\Box h+\frac{4}{r^2}h\right] \end{aligned} \tag{4.3.43}$$

where in the last line we used the non-diagonal equations, and the d'Alembertian in cylindrical coordinates is $\Box=-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$. This means that the function h satisfies a variant of the cylindrical wave equation

$$\Box h=\frac{4}{r^2}h \tag{4.3.44}$$

There are also the other diagonal equations $R_{00}=R_{11}=R_{22}=0$, but one can easily verify that they only provide one new equation,

$$R_{00}+R_{22}=2\varepsilon\left[\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)f+\frac{\partial^2 h}{\partial r^2}+\frac{3}{r}\frac{\partial h}{\partial r}\right] \tag{4.3.45}$$

Therefore the entire set of Einstein's equations reduces to only three equations,

$$\begin{aligned} 0 &= \Box h - \frac{4}{r^2}h \\ 0 &= \frac{\partial f}{\partial r} + \frac{\partial h}{\partial r} + \frac{2}{r}h \\ 0 &= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) f + \frac{\partial^2 h}{\partial r^2} + \frac{3}{r}\frac{\partial h}{\partial r} \end{aligned} \tag{4.3.46}$$

for two unknowns f and h, an overdetermined system as before. Nevertheless, by differentiating the third equation with respect to r and using the second equation the result reduces back to the first equation for h. Therefore this overdetermined system is in fact compatible.

Since the first wave equation in (4.3.46) is decoupled from the rest, one may use it to find h, which can be used to obtain the function f. Thus, here like in the last two sections, the gravitational wave is fully determined by the value of the function h.

4.4 The nonlinear Schrödinger-Einstein equations

The nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + |\psi|^2 \psi \tag{4.4.1}$$

has been playing an ever growing role in nonlinear physics. It is considered to be universal [45] in the sense that it can be derived in general circumstances, and is therefore ubiquitous in applications, ranging from optics to water waves. The goal of the present section is to show that also *gravitational waves* fall under the wide scope of the NLS.

To accomplish this important task, certain assumptions on the $1+2$ Einstein equations must be imposed beyond the weak field approximation. In fact, there are different assumptions on the spacetime structure that may reduce the Einstein's equation to NLS-like forms. However, the only case that is studied here is the one

that best serves as a natural extension of the Bondi-Pirani-Robinson metric studied earlier. The assumptions that will be made next are in no way the most general ones that can be considered, and the author hopes to extend them in the future.

Throughout this section the coordinates (ζ, η, x^2) are used instead of (x^0, x^1, x^2) (see the beginning of section 4.2.C).

4.4.A Derivation

Consider a perturbation of the diagonal metric (4.2.9) of the form

$$\begin{aligned} \Lambda &= \frac{\varepsilon}{2}(\psi e^{i\eta} + \psi^* e^{-i\eta}) \\ \beta &= 1 + \varepsilon h + \varepsilon^2 \beta_2 \\ \gamma &= 1 + \varepsilon h + \varepsilon^2 \gamma_2 \end{aligned} \tag{4.4.2}$$

where all functions $\alpha, h, \beta_2, \gamma_2$ and ψ are assumed to be $O(1)$ in ε. It will also be assumed that the change of gravitational wave in the negative x^1-direction is slower than in the positive x^1-direction and the x^2-direction. To model this, rescale ζ differently than η and x^2, through

$$u = \varepsilon\eta \qquad v = \varepsilon^2\zeta \qquad w = \varepsilon x^2 \tag{4.4.3}$$

and assume that all the unknown functions $\alpha, h, \beta_2, \gamma_2$ and ψ are functions of u, v, w [4]. The function $\psi(u, v, w)$ takes on a special role, as it is the only *complex-valued function* and determines the envelope of the gravitational wave as manifested by the function Λ. Since ψ depends on v and w as well as u, the amplitude of the wave may vary weakly in directions different than the original Bondi-Pirani-Robinson wave studied in section 4.2.C (that depends only on u / η).

Using these assumptions in the Lagrangian (4.2.6) allows expanding it in ε. This

[4] These are not the same u and v used in section 3.5 to describe synchronous metrics.

yields

$$\begin{aligned}\mathcal{L} &= \varepsilon^4\big\{i\alpha\left[(\partial_v\psi)\psi^* - \psi(\partial_v\psi^*)\right] + \alpha\delta|\psi|^2 - \frac{2}{\alpha}|\partial_w\psi|^2 + \frac{4}{\alpha}(\partial_w h)^2 \\ &\quad +2(\partial_u\alpha)(\partial_v h) + 2(\partial_v\alpha)(\partial_u h) - (\partial_u\alpha)(\partial_u\delta)\big\}\end{aligned} \tag{4.4.4}$$

in the leading order, where $\delta = \beta_2 - \gamma_2$. Taking the variation of the action S with respect to h, δ, α and ψ^* gives the four *nonlinear Schrödinger-Einstein equations for diagonal metrics*:

$$\begin{aligned}0 &= \partial_w(\frac{1}{\alpha}\partial_w h) + \frac{1}{2}\partial_u\partial_v\alpha \\ 0 &= \partial_u\partial_u\alpha + \alpha|\psi|^2 \\ 0 &= i\left[(\partial_v\psi)\psi^* - \psi\partial_v(\psi^*)\right] + \delta|\psi|^2 + \frac{2}{\alpha^2}|\partial_w\psi|^2 - \frac{4}{\alpha^2}(\partial_w h)^2 - 4\partial_u\partial_v h + \partial_u\partial_u\delta \\ 0 &= 2i\alpha\partial_v\psi + i(\partial_v\alpha)\psi + \alpha\delta\psi + 2\partial_w(\frac{1}{\alpha}\partial_w\psi)\end{aligned} \tag{4.4.5}$$

But as before, they must be coupled to the non-diagonal equations as well. The last two equations in (4.2.5) are equivalent in the leading order ε^3. The first equation, however, is non-trivial in both order ε^2 and ε^3. In the order ε^2 it gives again the second equation in Eqs. (4.4.4). Therefore up to order ε^3 there are two more (non-trivial) nonlinear Schrödinger-Einstein equations:

$$\begin{aligned}0 &= i\alpha\left[(\partial_u\psi)\psi^* - \psi(\partial_u\psi^*)\right] + 2(\partial_u\alpha)(\partial_u h) \\ 0 &= i\alpha\left[(\partial_w\psi)\psi^* - \psi(\partial_w\psi^*)\right] + 2(\partial_u\alpha)(\partial_u w) - 2\alpha\partial_u\partial_w h\end{aligned} \tag{4.4.6}$$

The derivation used here uses similar scaling (4.4.3) as in the derivation of the usual NLS Eq. (4.4.1) from the wave equation. Nevertheless, here the situation is rather different, as the nonlinear Schrödinger-Einstein equations are a system of six equations for four unknowns ψ, α, δ and h (compared to a single NLS for one unknown ψ). The connecting link between these two systems is the last equation in (4.4.5). If one assumes it is a *single* equation for the wave envelope ψ with known coefficients $\alpha = 1$, $\delta = -2|\psi|^2$, the NLS Eq. (4.4.1) is obtained immediately with coordinates v, w

instead of t, x respectively. However, this reduction is *only valid* if the functions α and δ are given *independently* of the other nonlinear Schrödinger-Einstein equations (in fact, from the second Eq. (4.4.5) one sees that the assumption $\alpha = 1$ is only possible for the trivial case $\psi = 0$). Therefore, the reader should only consider this as a heuristic argument to show that the title 'nonlinear Schrödinger-Einstein' equations is indeed appropriate here.

Since the envelope ψ is complex it has two real degrees of freedom. It is convenient to separate them by defining $\psi = Ae^{i\phi}$ where A and ϕ are real-valued. In terms of the real-valued functions the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \varepsilon^4 \Big\{ -2\alpha A^2 \partial_v \phi - \frac{2}{\alpha}(\partial_w A)^2 - \frac{2}{\alpha} A^2 (\partial_w \phi)^2 + \alpha \delta A^2 - (\partial_u \alpha)(\partial_u \delta) \\ &\qquad + 2(\partial_v \alpha)(\partial_u h) + 2(\partial_u \alpha)(\partial_v h) + \frac{4}{\alpha}(\partial_w h)^2 \Big\}\end{aligned} \tag{4.4.7}$$

giving the *real-valued nonlinear Schrödinger-Einstein equations for diagonal metrics*

$$\begin{aligned}
0 &= \partial_w(\frac{1}{\alpha}\partial_w h) + \frac{1}{2}\partial_u \partial_v \alpha \qquad (4.4.8)\\
0 &= \partial_u \partial_u \alpha + \alpha A^2 \\
0 &= -2(\partial_v \phi)A^2 + \delta A^2 + \frac{2}{\alpha^2}(\partial_w A)^2 + \frac{2}{\alpha^2}(\partial_w \phi)^2 A^2 - \frac{4}{\alpha^2}(\partial_w h)^2 - 4\partial_u \partial_v h + \partial_u \partial_u \delta \\
0 &= \alpha A \partial_v \phi + \frac{A}{\alpha}(\partial_w \phi)^2 - \partial_w(\frac{1}{\alpha}\partial_w A) - \frac{A}{2}\alpha\delta \\
0 &= \partial_v \left(\frac{1}{2}\alpha A^2\right) + \partial_w \left(\frac{1}{\alpha} A^2 \partial_w \phi\right)
\end{aligned}$$

while the off-diagonal equations are

$$\begin{aligned}
0 &= -\alpha(\partial_u \phi)A^2 + (\partial_u \alpha)(\partial_u h) \qquad (4.4.9)\\
0 &= -\alpha(\partial_w \phi)A^2 + (\partial_u \alpha)(\partial_w h) - \alpha \partial_u \partial_w h
\end{aligned}$$

This reveals two important features. The first and the last nonlinear Schrödinger-Einstein equation in Eq. (4.4.8) are conservation laws. The latter may be interpreted similarly to the continuity equation in the case of the usual Schrödinger equation.

Conveniently, all but the third and fourth equations do not contain δ. This means that one may consider the other five equations as an overdetermined system for four unknowns h, α, A, ϕ. It is easy to see that this system is indeed compatible, by differentiating the first two equations in u and v (respectively) and subtracting the results.

4.4.B Solutions

The Bondi-Pirani-Robinson waves were derived by assuming that the metric depends only on η. In the new rescaled coordinates this is equivalent to assuming that the metric depends on u only. By gauge invariance one may assume that $h = \delta = 0$, showing immediately that the wave phase $\phi =$ constant, while all of the diagonal nonlinear Schrödinger-Einstein equations reduce to a single equation,

$$0 = \partial_u \partial_u \alpha + \alpha A^2 \tag{4.4.10}$$

It is equivalent to Eq. (4.2.29), with the only differences being a constant factor and the lack of derivatives on A (appearing due to the form of the function Λ in Eq. (4.4.2)). The goal is to generalize this solution next and obtain new solutions.

Generalized $1D$ *Bondi-Pirani-Robinson waves* can be achieved by considering waves that are free to move in both the negative and positive x^1 directions. This means that the BPR wave travels just as before in the positive x^1-direction, but its interaction with a fellow wave traveling in the opposite direction is allowed. Now the metric coefficients depend on u and v but not on w.

As was mentioned in section 1.5, in this case a simple change of coordinates allows taking the metric coefficients in (4.2.9) to be $\beta = \gamma$. By Eqs. (4.4.2), these means that $\delta = 0$. Then the fourth equation in (4.4.8) gives a non-trivial wave if

$$\partial_v \phi = 0 \tag{4.4.11}$$

Therefore the first and third equations in (4.4.8) are just the wave equation for α and h respectively,

$$\partial_u \partial_v \alpha = 0 \qquad \partial_u \partial_v h = 0 \tag{4.4.12}$$

with d'Alembert solutions $\alpha(u,v) = \alpha_1(u) + \alpha_2(v)$ and $h(u,v) = h_1(u) + h_2(v)$. Using this result in the second equation in (4.4.8) gives the amplitude

$$A^2(u,v) = -\frac{\alpha_1''(u)}{\alpha_1(u) + \alpha_2(v)} \tag{4.4.13}$$

which automatically satisfies the fifth equation in (4.4.8). We are therefore left with the two off-diagonal Eqs. (4.4.9), the second of which is trivial. The first equation in (4.4.9) gives the phase of the wave ϕ,

$$\phi(u) = -\int \frac{\alpha_1' h_1'}{\alpha_1''} du \tag{4.4.14}$$

and it can be easily verified that it is indeed compatible with Eq. (4.4.11).

Therefore this $1D$ wave depends on four arbitrary functions α_1, α_2, h_1 and h_2.

In the special case where the wave is restricted to move along the positive x^1-direction only, $\alpha_2(v) = h_2(v) = 0$ and one may take $h_1(u) = 0$ by a proper choice of coordinates. This leaves us with only one arbitrary function α_1, which is the case of the BPR waves considered before.

'Stationary' solutions are gravitational waves that are independent of v. This is an appropriate language since here the coordinate v plays the role of the time coordinate t in the NLS Eq. (4.4.1). This assumption is equivalent to studying the gravitational wave in a reference frame traveling in the negative x^1-direction at the speed of light with respect to the original frame. One may even weaken this assumption by requiring that the derivative of only three out of the five unknown functions vanishes, namely that

$$\partial_v \alpha = \partial_v A = \partial_v \delta = 0 \tag{4.4.15}$$

and it is further assumed that ϕ is a linear function in v,

$$\phi = \frac{1}{2}\lambda(u,w)v \tag{4.4.16}$$

and that

$$h = 0 \tag{4.4.17}$$

This is yet another generalization of the BPR waves. By allowing δ to vary and the other coefficients to change with w, this corresponds to a BPR wave colliding with a transverse wave moving in the x^2-direction.

The assumption $h = 0$ simplifies the non-diagonal Eqs. (4.4.9) greatly, which now give

$$\partial_u\phi = \partial_w\phi = 0 \tag{4.4.18}$$

and therefore $\lambda(u,w) =$ constant. Therefore one is left to solve the diagonal nonlinear Schrödinger-Einstein Eqs. (4.4.8).

To do so, it is convenient to use the Lagrangian formalism once more. From the assumptions made together with the result obtained from the non-diagonal equations, the Lagrangian (4.4.7) reduces to

$$\mathcal{L} \;=\; \varepsilon^4\left\{\alpha A^2(\delta-\lambda) - \frac{2}{\alpha}(\partial_w A)^2 - (\partial_u\alpha)(\partial_u\delta)\right\} \tag{4.4.19}$$

Varying the action with respect to α, A and δ (respectively) gives three diagonal equations

$$\begin{aligned} 0 &= \partial_u\partial_u\delta + A^2(\delta-\lambda) + \frac{2}{\alpha^2}(\partial_w A)^2 \\ 0 &= \partial_w(\frac{2}{\alpha}\partial_w A) + \alpha A(\delta-\lambda) \\ 0 &= \partial_u\partial_u\alpha + \alpha A^2 \end{aligned} \tag{4.4.20}$$

If δ is also independent of w, $\partial_w\delta = 0$, then it is a function of u only. We may therefore differentiate the first equation by w to eliminate δ. This gives precisely the second equation. Therefore, under this assumption $\partial_w\delta = 0$, we obtain an overdetermined

system of two independent equations for three unknowns α, δ and A. Consider it in the form

$$\begin{aligned} 0 &= q(u) + p(u)A^2 + \frac{2}{\alpha^2}(\partial_w A)^2 \\ 0 &= \partial_u \partial_u \alpha + \alpha A^2 \end{aligned} \tag{4.4.21}$$

where $q = \partial_u \partial_u \delta$ and $p = \delta - \lambda$ will be assumed to be the *known* functions.

If the first of the two functions vanishes $q(u) \equiv 0$ while the other is negative $p(u) = -2k^2(u)$ then the first equation may be solved easily for A to yield

$$A(u, w) = \exp\left(F(u, w)\right) \tag{4.4.22}$$

where the function $F(u, w)$ satisfies

$$\partial_w F = \alpha k \tag{4.4.23}$$

Substituting this result in the other equation gives an ordinary differential equation for F,

$$\partial_u \partial_u \left(\frac{\partial_w F}{k}\right) + \frac{\partial_w F}{k} e^{2F} = 0 \tag{4.4.24}$$

which can be integrated immediately in w to yield

$$k(u)\partial_u \partial_u \left(\frac{F}{k}\right) + \frac{1}{2} e^{2F} = \mu(u) \tag{4.4.25}$$

where $\mu(u)$ is an arbitrary function of u. The last equation is a second order differential equation in u and therefore depends on two constants of integration. Since the function F may depend on w as well, the constants of integration can be functions of w. Namely, the solution is of the form

$$F = F(u, c_1(w), c_2(w)) \tag{4.4.26}$$

Thus, the above analysis shows that there are four arbitrary quantities $k(u)$, $\mu(u)$, $c_1(w)$ and $c_2(w)$ defining the solution.

Eq. (4.4.25) has many interesting solutions. As a simple example, consider the case where $k \equiv 1$ and $\mu = \text{constant}$. Eq. (4.4.25) may now be written in the form

$$\frac{1}{2}(\partial_u F)^2 + V(F) = E(w) \tag{4.4.27}$$

where the potential $V(F)$ is

$$V(F) = \frac{1}{4}e^{2F} - \mu F \tag{4.4.28}$$

For each fixed value of w, Eq. (4.4.27) describes the (classical) motion of a material body with an arbitrary energy $E(w)$ under the potential $V(F)$ defined by Eq. (4.4.28). This potential has only one equilibrium point $F_0 = \frac{1}{2}\ln(2\mu)$ for which its value is $V(F_0) = \frac{1}{2}\mu(1 - \ln(2\mu))$. By writing $F = F_0 + \Delta F$ and approximating the potential (4.4.28) about its equilibrium point F_0, Eq. (4.4.27) is

$$\frac{1}{2}(\partial_u \Delta F)^2 + \mu(\Delta F)^2 = E - V(f_0) \tag{4.4.29}$$

Whenever $\mu > 0$, one may denote $\mu = \frac{1}{2}C^2$ to obtain oscillatory solutions

$$\Delta F(u, w) = \frac{\sqrt{E - V(F_0)}}{C(w)} \cos(C(w)u - \varphi(w)) \tag{4.4.30}$$

Appendix A
Notations and conventions

In this work we use the 'general relativistic' convention, in which all spacetime metrics have signature $(-+++)$. In the first three chapters, Einstein's summation convention is used unless stated otherwise. The Belinski-Zakharov transform assumes spacetime metrics that depend on only two variables, and therefore one has to be able to distinguish these two coordinates from the other two. For this purpose Einstein's summation convention is used in three different ways:

- Lower-case Greek indices $\mu, \nu, \ldots$ will be summed over all spacetime coordinates $\mu, \nu, \cdots = 0, 1, 2, 3$.
- Lower-case Latin indices $i, j, k, \ldots$ will run over $i, j, k, \cdots = 0, 3$, while other lower-case Latin indices a, b, c, d, e run over $a, b, c, d, e = 1, 2$.

There will be one letter of special significance throughout this text. The letter s will always denote a '$s$oliton', and thus the symbols s and $\tilde{s}$ will always run over the number of solitons N. For this symbols we *do not* use the summation convention. Note however, that there is no reason to confuse the index s with the spacetime interval ds^2.

Similarly, the symbol η will be used to denote to different things. $\eta_{\mu\nu}$ denotes the four-dimensional Minkowski metric

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.0.1}$$

It should not be confused with η_{ij} in which the lower-case Latin indices indicate that this tensor operators on a lower-dimensional subspace. Namely,

$$\eta_{ij} = \begin{bmatrix} -e & 0 \\ 0 & 1 \end{bmatrix} \tag{A.0.2}$$

where $e = \pm 1$ denoting the relevant subspace. Whenever it appears without indices, η denotes a light-cone coordinate.

As for the units, in all sections but section 1.1, we use units in which the speed of light $c = 1$.

Appendix B

Curvature tensors for the block diagonal metric

Consider a spacetime interval of the block diagonal form of Eq. (1.5.1),

$$ds^2 = g_{ij}dx^i dx^j + g_{ab}dx^a dx^b \tag{B.0.1}$$

where we use the summation convention on $i, j, k, \cdots = 0, 3$; $a, b, \cdots = 1, 2$; and assume g_{ij} and g_{ab} depend only on the coordinates x^k. We will denote the determinants of each of the 2×2 blocks by,

$$\begin{aligned} \det g_{ij} &= -ef^2 \\ \det g_{ab} &= e\alpha^2 \end{aligned} \tag{B.0.2}$$

Their product is the determinant of the full metric tensor,

$$\det g_{\mu\nu} = -f^2\alpha^2 \tag{B.0.3}$$

Computing the Christoffel symbols through Eq. (1.2.8) gives

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}) \\ \Gamma^k_{ab} &= -\frac{1}{2}g^{kl}\partial_l g_{ab} \\ \Gamma^c_{ib} &= \frac{1}{2}g^{cd}\partial_i g_{bd} \end{aligned} \tag{B.0.4}$$

where the rest of the Christoffel symbols can be obtained through the symmetry $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, or otherwise vanish identically. The trace of the Christoffel symbols is

$$\Gamma^\lambda_{i\lambda} = \partial_i(\ln \alpha f) \qquad \Gamma^\lambda_{a\lambda} = 0 \tag{B.0.5}$$

The Christoffel symbols and their trace can be used in Eq. (1.2.7) to give the Ricci curvature tensor,

$$
\begin{aligned}
R_{ij} &= \frac{1}{2}\partial_k\left[g^{kl}(\partial_i g_{lj}+\partial_j g_{il}-\partial_l g_{ij})\right]-\partial_j\partial_i\ln(f\alpha) \\
&\quad +\frac{1}{2}g^{kl}(\partial_i g_{lj}+\partial_j g_{il}-\partial_l g_{ij})\partial_k\ln(f\alpha) \\
&\quad -\frac{1}{4}g^{lm}g^{kn}(\partial_i g_{mk}+\partial_k g_{im}-\partial_m g_{ik})(\partial_j g_{ln}+\partial_l g_{jn}-\partial_n g_{jl}) \\
&\quad -\frac{1}{4}g^{bd}g^{ac}(\partial_i g_{ad})(\partial_j g_{bc}) \\
R_{ib} &= 0 \\
R_{ab} &= -\frac{1}{2\alpha f}\partial_k\left(\alpha f g^{kl}g^{cd}\partial_l g_{ac}\right)g_{db}
\end{aligned}
\tag{B.0.6}
$$

Appendix C

The integrability condition for A_i

Recall that in Eq. (1.5.14) we defined the matrices A_i to be

$$A_i = e\alpha g_{,i} g^{-1} \tag{C.0.1}$$

The purpose of this appendix is to find a condition on the matrix A_i to allow writing it in this form of a 'logarithmic derivative'.

Differentiate A_i with respect to x^j and multiply the result by the anti-symmetric matrix ε^{ij} defined in Eq. (1.5.19). This gives

$$\varepsilon^{ij} A_{i,j} = \varepsilon^{ij} \left[\frac{\alpha_{,j}}{\alpha} A_i - \frac{e}{\alpha} A_i A_j \right] \tag{C.0.2}$$

where the term $\varepsilon^{ij} g_{,ij}$ was canceled due to the symmetry of g and anti-symmetry of ε. The last equation can be written in the homogeneous form

$$\varepsilon^{ij} \left[A_{i,j} - \frac{\alpha_{,j}}{\alpha} A_i + \frac{e}{\alpha} A_i A_j \right] = 0 \tag{C.0.3}$$

giving the sought condition.

Appendix D

The commutation relation for D_i

The matrix ε^{ij} defined in Eq. (1.5.19) gives us an easy tensorial way to compute commutators through the relation

$$[D_0, D_3] = \varepsilon^{ij} D_i D_j \tag{D.0.1}$$

This form is extremely convenient, because the anti-symmetry of ε^{ij} means that tracing it with every symmetric tensor gives zero.

In our case the operators have the form $D_i = \partial_i + P_i \partial_\lambda$ and most terms in the product $D_i D_j$ are symmetric and cancel out to give

$$\varepsilon^{ij} D_i D_j = \varepsilon^{ij} \left[\partial_i P_j + P_i \partial_\lambda P_j\right] \partial_\lambda \tag{D.0.2}$$

Let's compute each term separately. Eq. (1.5.21) defined the functions P_i to be

$$P_i = -\frac{2\lambda(\lambda\beta_{,i} + e\alpha\alpha_{,i})}{\lambda^2 - e\alpha^2} \tag{D.0.3}$$

When differentiating them with respect to x^j the numerator gives a symmetric term and therefore cancels out upon multiplication by ε^{ij}. The only non-trivial contribution is from the derivative of the denominator, yielding

$$\varepsilon^{ij} \partial_i P_j = -\frac{4e\alpha\lambda^2}{(\lambda^2 - e\alpha^2)^2} \varepsilon^{ij} \alpha_{,i} \beta_{,j} \tag{D.0.4}$$

For the second term $\varepsilon^{ij} P_i \partial_\lambda P_j$ the non-trivial contribution comes from a different source. In this case differentiating the denominator with respect to λ leaves the numerator of P_j intact, and thus we end with a symmetric term (together with the numerator of P_i) which cancels out. The only contribution is from the derivative of the numerator then, giving

$$\varepsilon^{ij} P_i \partial_\lambda P_j = \frac{4e\alpha\lambda^2}{(\lambda^2 - e\alpha^2)^2} \varepsilon^{ij} \alpha_{,i} \beta_{,j} \tag{D.0.5}$$

Therefore, it is now clear that Eq. (D.0.2) cancels out to yield a big fat zero,

$$\varepsilon^{ij} D_i D_j = 0 \tag{D.0.6}$$

Namely, the operators D_i and D_j commute. Note that this result actually extends for any i, j as if $i = j$ the commutator vanishes identically.

Appendix E

Solving the pole trajectory equations

Consider the *pole trajectory* Eqs. (1.6.2)

$$\partial_i \mu_s + \frac{2\mu_s(\mu_s \partial_i \beta + e\alpha \partial_i \alpha)}{\mu_s^2 - e\alpha^2} = 0 \tag{E.0.1}$$

Substituting $\mu_s = \alpha\tilde{\mu}_s$ and simplifying the expression gives

$$\tilde{\mu}_s \partial_i \alpha + \alpha \partial_i \tilde{\mu}_s + e\frac{\tilde{\mu}_s \partial_i \alpha - \alpha \partial_i \tilde{\mu}_s}{\tilde{\mu}_s^2} + 2\partial_i \beta = 0 \tag{E.0.2}$$

The left-hand side of these equations is a full derivative with respect to x^i. Its integral is

$$\alpha\tilde{\mu}_s + e\frac{\alpha}{\tilde{\mu}_s} + 2\beta = 2\omega_s \tag{E.0.3}$$

where ω_s is a complex integration constant. When restoring the original pole $\mu_s = \alpha\tilde{\mu}_s$ this is equivalent to Eq. (1.6.3), namely the poles of the dressing matrix are given by the quadratic equation

$$\mu_s^2 + 2(\beta - \omega_s)\mu_s + e\alpha^2 = 0 \tag{E.0.4}$$

Appendix F

Derivatives with respect to the metric

In order to calculate the variation of the Lagrangian (3.1.2), the derivatives of α, $\Delta^{(t)}$ and $\Delta^{(z)}$ with respect to the matrix g_{ab} must be calculated. They are summarized in table F.1.

$\frac{\partial \alpha}{\partial g_{ab}} = \frac{1}{2}\alpha g^{ab}$	$\frac{\partial \dot{\alpha}}{\partial \dot{g}_{ab}} = \frac{1}{2}\alpha g^{ab}$	$\frac{\partial \Delta^{(t)}}{\partial g_{ab}} = 0$
$\frac{\partial \alpha}{\partial \dot{g}_{ab}} = 0$	$\frac{\partial \dot{\alpha}}{\partial g_{ab}} = \left(\frac{1}{2}\alpha g^{ab}\right)_{,t}$	$\frac{\partial \Delta^{(t)}}{\partial \dot{g}_{ab}} = \left(\alpha^2 g^{ab}\right)_{,t}$

Table F.1. Derivatives with respect to the metric coefficients g_{ab}

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