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Wavetrains in diverging mixing layers

Yapo, Sylvain Achy, Ph.D.
The University of Arizona, 1991
WAVETRAINS IN DIVERGING MIXING LAYERS

by

SYLVAIN ACHY YAPO

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MECHANICAL ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

1991
As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Sylvain A. Yapo
titled WAVETRAINS IN DIVERGING MIXING LAYERS

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy.

Final approval and acceptance of this dissertation is contingent upon the
candidate's submission of the final copy of the dissertation to the Graduate
College.

I hereby certify that I have read this dissertation prepared under my
direction and recommend that it be accepted as fulfilling the dissertation
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Dissertation Director

Prof. Thomas F. Balsa
STATEMENT BY AUTHOR

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I would like to thank Khalifah Sidik for her devoted love and her complete understanding over the years. Finally, I dedicate this dissertation to my parents for their never ending support and their great sacrifice, I owe it all to them.
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ABSTRACT

It is generally accepted that a linear stability theory, together with a slowly diverging base flow, can describe many of the characteristics of coherent structures in free shears flow.

In this dissertation we model these two-dimensional instability waves as they travel in a slightly inhomogeneous steady and viscous unstable base flow. These unstable and inviscid wave packets are analysed using linear stability theory. The analysis is performed by separating the physical flow in two parts. In the first part, the instability waves are evolving in a parallel mixing layer and their solution serves of initial conditions for the second part of the flow. The parallel flow analysis leads to the receptivity of the flow to both pulse-type and periodic excitations. This part of the study is done by solving the initial-value problem completely and studying its long-time behaviour, which is a wave packet. We then repeat the same analysis with some modifications and arrive at the receptivity of the flow for sinusoidal excitations. We find that a shear layer is very receptive to high-frequency disturbances that are generated near the center line of the layer.

The second part of the solution is concerned with the evolution of the wave packets on longer space-time scales which are associated with non-parallel effects arising from the spreading of the mixing layer. The solution in this part of the physical flow is handled by extending Whitham's kinematic wave theory, and the ray equations for instability waves are derived for physical and propagation spaces using a WKBJ expansion. Our high-frequency ansatz also leads us to the derivation of a very simple complex amplitude equation. While the rays obtained represent characteristics in the complex plane along which the complex frequency of our disturbances is conserved (steady base flow), the amplitude equation expresses the conservation of the volume integrals of a complex wave action density subject to a certain flux and a source term. The amplitude equation was rendered easily tractable due to a transformation of our dependent variables and their practical projections on the cross and propagation spaces.
Different methods, (steepest descent, ray-tracing, and fully numerical solution) are used to solve the ray equations, and comparisons are made among them. The results presented are obtained for the piecewise linear profile of Rayleigh and the general \( \tanh \) profile. The very good agreement among all the methods of solution reveals the validity of the method of characteristics in the complex plane, (ie complex rays). Finally we perform some calculations for spatially varying shear layers and and study their implications in the development of spatial instability modes. We discover that when starting with a convectively unstable base velocity profile it is possible to interrupt the development of spatial instability modes by allowing the base velocity profile to vary slowly and become absolutely unstable. However the reverse is not true. That is to say that in a base flow that is initially absolutely unstable, one does not observe spatial modes, even after the base flow is permitted to assume slowly a convectively unstable profile.
CHAPTER 1

INTRODUCTION

Linear stability theory has been very successful in describing the structures of the instabilities in a multitude of flows. In the majority of these studies, the stability analysis is performed using a normal mode analysis. This strategy usually involves choosing between temporal theory and spatial theory. In temporal stability analysis, the wavenumber \( k \) is taken to be real and the objective of the theory is to determine the complex frequency \( \omega \) in the form of a dispersion relation \( \omega = \Omega(k) \) connecting \( \omega \) to \( k \). This method is almost exclusively used in convective stability investigations, like Taylor-Couette flow and Rayleigh-Benard convection. By contrast, in the study of free shear flows, especially experimental studies, investigators introduce the instability waves by forcing the flow with a single frequency. These experimental results seem to be in better agreement with the predictions of spatial stability theory, where the frequency \( \omega \) is real and the wavenumber \( k \) is complex. This implies a growth in the streamwise direction.

However, when a localized disturbance is introduced into a typical unstable flow, it will quickly disperse into an instability wave packet (Gaster and Grant, 1975). For this reason in recent years, the focus has shifted to the study of wave packets from single normal mode analysis.

1.1. Historical Background

Gaster is considered the pioneer in the study of wave packets. In his initial study, Gaster (1965) observed the simultaneous evolution of many unstable modes and found that they interacted with each other so that their major contributions at a precise location in the flow and at a specific time arose from a dominant complex wavenumber \( k(x,t) \). One might think of this as wave interference although the waves under consideration are unstable; it is for this reason that the dominant
wavenumber is complex. In another paper Gaster (1975), modeled a wave packet in a flat plate boundary layer. The model conditions were chosen to conform to an experimental investigation where a wave packet was artificially excited by a localized pulse perturbation. The theoretical model for the wave packet was built with a linear combination of all spatially growing modes over all wavenumbers and frequencies. Comparisons of the model and experimental data revealed that the linear stability analysis predictions were in very close agreement with the experimental results, especially at stations close to the source of the disturbance.

More recently the subject of wave packets was revisited by Huerre and Monkewitz (1985). The objective of their investigation was to clarify issues relating to convective and absolute instability. They accomplished this task by examining the locations in the complex Ω-plane of the singularities of the complex dispersion relation for a family of free shear layers. The authors concluded that spatial instability modes can only evolve in mixing layers with a small amount of reverse flow. In this instance the flow is said to be convectively unstable. On the other hand, when the external streams are counter-flowing with a substantial amount of reversed flow, the flow is absolutely unstable. In this case spatial instability waves cannot evolve.

Shortly after the study by Huerre and Monkewitz, Balsa published his work on the receptivity of free shear layers for two and three-dimensional excitations (Balsa 1986; 1989). The novel idea in his paper was the representation of a spatial instability mode as a superposition of wave packets. The functional relationship between the mode amplitude and the initial disturbance is called the receptivity of the flow. Balsa arrived at such a relation by solving the initial-value problem completely and analyzing its long-time behavior, which is a wave packet. This wave packet was generated for an impulse type excitation. However, by superposing a series of wave packets, Balsa was able to generate a spatial instability mode for a convectively unstable flow, and describe its receptivity for a single input frequency. The author concluded that a shear layer is very receptive to high-frequency disturbances that are introduced near the centerline of the layer.
Although parallel flow stability theory is quite complete, it cannot predict the variation of the mode characteristics (phase speed, wavenumber, etc.) with position (radial or axial) in a more realistic base flow, such as a slowly diverging mixing layer. Crighton and Gaster (1976), showed that the parameters of an instability wave in a slowly varying mean flow are not identical with those in a parallel flow coincident with the local flow at each station. There is a "history effect" which is lost in the locally parallel analysis. The appropriate way to resolve this effect is to incorporate a slow axial variation of the mean flow (due, for example, to viscous or turbulent stresses) into the analysis. This is usually accomplished by the method of multiple scales or WKBJ expansion or kinematic wave theory.

Indeed both of these methods have been applied extensively in the literature. One of the more elegant application of the method of multiple scales was performed by Bouthier (1972). He derived an evolution equation for the amplitude of a wave packet in a Blasius type boundary layer.

Soon after, Eagles and Weissman (1975) applied the WKBJ method to study the stability of the divergent channel flow. Their results clearly demonstrated the dependence of the growth rate, wavenumber, neutral curves, etc., on the cross stream variable \( y \) and on the flow quantity under consideration (e.g. a pressure disturbance has a slightly different growth rate than a \( z \)-velocity disturbance). Their final determination was that the unstable region (stability curve) was considerably widened and the critical Reynolds number was correspondingly reduced; they also concluded that waves of all frequencies eventually decayed as they evolved downstream. Multiple scales was also employed by Crighton and Gaster (1976) to model the linear stability of a slowly divergent jet flow. In their analysis they selected the base flow inhomogeneity parameter \( \epsilon \) as the ratio of streamwise wavenumber to the streamwise length scale of the flow. With a specific value of \( \epsilon = 0.03 \), they proceeded to make some numerical computations and found them to compare fairly closely with the measurements by Crow and Champagne (1971). Like Eagles and Weissman, their predictions showed that substantial variations are obtained in
such quantities as the phase speed and growth rate, according to the flow disturbances (velocity, pressure, stream function, etc.) considered, and these variations are dependent on the axial as well as the cross-stream position.

Kinematic wave theory has been used very successfully to study the propagation of conservative waves in slightly inhomogeneous base flows, Whitham (1973), Hayes (1970a,b). Several attempts have been made to apply the theory to instability wave systems with relatively less success. It turns out that for parallel shear flows, this application is quite complete and satisfactory, Landahl (1972). In his study he used the theory to determine the conditions required for breakdown of a steady or unsteady laminar flow into high-frequency oscillations. His predictions led him to conclude that the mechanism for breakdown was three-fold: first, the real group velocity of the secondary wave must equal the primary wave velocity; second, temporal and spatial focusing (vanishing ray tube area) must occur; and third, nonlinear effects on the secondary wave play an important role on the breakdown mechanism. One fundamental assumption made by the author was the disparity of length scales between primary and secondary wavelike disturbances, so that he could consider the secondary waves riding on the primary ones. Another assumption was that the inhomogeneity due to the primary disturbance was slowly varying, so that kinematic wave theory applied. Landahl's work concentrated on shear flows, but it could well be generalized for all continuum systems, be they solid, liquid or gaseous, whenever wavelike disturbances of disparate length scales can occur.

Wave packets in inhomogeneous base flows were also treated by Itoh (1981) and Russel (1986). The first author derived an amplitude equation for a wave packet in the form of a lengthy expression involving the integral of the modes, their complex conjugates, and several of their partial derivatives with respect to the slowly varying coordinates, as well as the wavenumber. Itoh then proceeded to make a local analysis in order to see more details of the amplitude variation in the central part of the wave packet. Finally he applied his theory to the Blasius flow and made some comparisons with the experimental results of Gaster and Grant (1975). His predictions showed good qualitative agreement with the experimental results. This author's approach was similar to the one by Nayfeh (1980) and is
suitable for numerical calculations but it does not shed any light on the physics of the problem. For example, it does not present the amplitude equation in the form of a conservation of a certain quantity (wave action density), by a balance between a flux term and a source (or sink) one.

The difficulty in the overwhelming algebra is partly due to the fact that the usual Eulerian variables, velocity $u$ and pressure $p$ are not the best ones to derive a simple amplitude evolution equation. Russel (1986) realized this and introduced a new set of dependent variables $(d, p)$ representing a fluid particle displacement and the fluid pressure respectively. This transformation allowed him to derive a conservation of bilinear wave action density by a method combining those of Jimenez and Whitham (1976) and Hayes (1970a). In his paper, Russel also presented a discussion concerning the distinctions between the observed square amplitude of an amplified wavetrain and its wave action density. Finally, he commented on three types of algebraic focussing; the far-field 'caustics', the near-field 'movable' singularities, and a focussing mechanism similar to Landahl's (1972).

A very restrictive assumptions made by Russel was that he only considered shear flows whose base velocities are inviscid. In most of the practical shear flows of interest (mixing layers, jets, and wakes), the base flow nonuniformity is the result of viscous and turbulent stresses present in the flow. Balsa (1989) built on this idea and derived several versions of the conservation of complex wave action density equation of an inviscid wave packet travelling on a slightly inhomogeneous, unsteady and viscous unstable base flow. His amplitude equation was very simple and involved the conservation of volume integrals of a complex wave action density in propagation space, subject to fluxes carried by the group velocity, and a source term which is a function of the base flow acceleration. The simplicity of the results was made possible because of a transformation of the dependent variables as in Russel's (1986) work, and their decomposition in cross- and propagation spaces.
1.2. Objectives

The main thrust of our present investigation involves the study of the evolution of a two-dimensional wave packet in a slowly diverging mixing layer. This task is accomplished using linear stability theory after splitting the whole flow into two parts: a near-flow which is essentially parallel and a diverging far-flow. In the first part of the physical flow (near-flow), we derive response of a general (tanh) shear layer to an impulse input perturbation by solving the initial-value problem completely. This is done in chapter 2 using the method of Fourier-Laplace transforms. This approach results in a definite integral representing the exact solution of the problem. The exact solution is valid for any value of space and time. However we are more interested in the structures of the solution for large space-time variables, (order of a few wavelengths and periods of oscillations). On these space and time scales, the solution is a wave packet evolving in a parallel flow (see Figure 1b). In chapter 2, we also describe the response of the mixing layer to a single sinusoidal forcing frequency, which leads us to the receptivity of the tanh free shear layer for spatial modes. Finally we close chapter 2 by presenting the Fourier integrals method as it applies to linear dispersive waves in a slightly nonuniform medium, after a brief discussion of the method in uniform media.

The diverging far-flow is handled by a WKBJ expansion procedure. Slow scale variables $\xi = ex$, and $\tau = et$ are introduced in order to treat a solution valid on length and time scales that are even larger than the scales associated with the asymptotic solution in the parallel flow, or the wave packet. In Chapter 3, we draw freely on the ideas of Balsa (1989) and derive a system of equations governing the evolution of the different flow parameters, (wavenumber, frequency, phase, etc.). The amplitude of the wave packet is extracted from the solution of a simple equation representing the conservation of its complex wave action density. We arrive at such a simple equation by introducing a new set of Eulerian variables discussed earlier, and their projections on cross-and propagation spaces.

In the next chapter, (chapter 4), we propose two different methods of solution of the governing equations: ray-tracing method and numerical method, in addition
the method of steepest descent is used to determine the asymptotic approximation of the Fourier integrals of chapter 2. The results from the Fourier integrals method and the numerical method are used to verify the results obtained by the ray-tracing method. With the steepest descent approach we are only able to determine a few characteristics of the disturbance, (growth rate, Doppler-shifted frequency), but with the other two methods, we can determine the amplitude of the disturbance as well. The ray-tracing method is the novel entity of this study and a key issue to be resolved is the use of complex rays and the analytic continuation it implies.

Finally in chapters 5 and 6, we comment on the results of the different methods of solution and their merit, we also explain the importance of the complex amplitude function in diverging shear flows. We then experiment on the spatial variations of mixing layers and the implications that such variations have on the evolution of spatial instability modes.
Our principal objective in this chapter is the derivation of the asymptotic (i.e. large space-time) solution of a wave packet which is generated by a localized and impulsive disturbance in a parallel mixing layer. This solution provides the initial conditions for the evolution of this packet on even longer space-time scales which are associated with non-parallel effects arising from the spreading of the mixing layer.

Our goal is attained in several steps. First, the linearized Euler equations are solved with null initial conditions and a suitable \( y \)-force which generates the perturbations in the mixing layer. The method of Laplace-Fourier transforms is used. Second, the large space-time structure of this solution is determined by the methods of steepest descent; this structure is essentially a wave-train of finite extent in a parallel mixing layer. Very roughly, the initial localized disturbance which is used to perturb the mixing layer disperses into a wave packet on a time scale which is on the order of a few periods of oscillation associated with the most unstable mode of the mixing layer. On this time scale the disturbance is convected downstream a distance which is much shorter than the length scale associated with the spreading of the mixing layer (caused by viscosity or possibly small scale turbulent mixing). Therefore, the wave packet is essentially evolving in a parallel flow. This analysis is repeated, with suitable modifications, to obtain the far-field (essentially a spatial instability mode) generated by a compact \( y \)-force which is oscillating harmonically in time at radian frequency \( \tilde{\omega} \).

The analysis in this chapter provides not only the initial conditions for a packet evolving on a slowly diverging mixing layer, but also gives informations on the receptivity of the flow to point \( y \)-forces. In this regard, the work herein is the continuation of the work of Balsa (1988) which was restricted to piecewise linear
base velocity profiles; the present analysis deals with arbitrary profiles although the numerical results are restricted to the so-called tanh family.

As we mentioned above, the study of the receptivity of flows involves the solution of the initial value problem. Thus our approach will be to solve the initial value problem for a general shape base velocity profile. We will limit ourself to linear theory which is widely accepted to be quite appropriate for a certain range of perturbation levels. In fact, a linear stability analysis coupled with a slowly diverging mean flow has been used to predict reasonably accurately the so-called coherent structures in free shear flows. (Gaster, Kit and Wygnanski 1985, Crighton and Gaster 1976).

2.1. Statement of The Problem

Consider a parallel inviscid and incompressible shear flow in a three-dimensional cartesian coordinate system \( x = (x, y, z) \), and the variable \( t \) representing time. The basic shear flow is at constant pressure and its velocity components are \( [U(y), 0, 0] \). Our base flow is perturbed by a two-dimensional point force acting in the \( y \) direction and the perturbation velocity components and pressure disturbances represented by \( \mathbf{u} = [u(x, t), v(x, t)] \) and \( p(x, t) \) satisfy the linearized Euler equations of motion

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1.1a}
\]

\[
\frac{\partial u}{\partial t} + U' \frac{\partial u}{\partial x} + U' v + \frac{\partial p}{\partial x} = 0, \tag{2.1.1b}
\]

\[
\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \delta(x)\delta(y - y_0)\delta(t), \tag{2.1.1c}
\]

where, \( U' \) indicates the derivative of \( U(y) \) with respect to \( y \), and \( \delta \) represents the Dirac delta function. In the governing equations that we just stated, \( x \) and \( \mathbf{u} \) were nondimensionalized using a characteristic length \( L_c \) and a characteristic speed \( U_c \) associated with the base flow. \( L_c \) and \( U_c \) are not unique and they vary considerably in the literature. From these two scales, we can infer a characteristic time \( L_c/U_c \) and a characteristic pressure \( \rho U_c^2 \), where \( \rho \) is the density of the fluid; time and pressure
are normalized by these respectively. The disturbances in the flow are generated by
a two-dimensional $y$-force located at the point $x = 0$ and $y = y_a$. The mathematical
representation of this force is indicated by the right-hand side of equation (2.1.1c).
The location of the source term is general and its strength (in a nondimensional
sense) is set to unity without any loss of generality because the problem is linear.

We also introduce a new dependent variable $d(x, t) = [\alpha(x, t), \beta(x, t)]$. This
variable which represents a displacement, combined with the pressure variable
$p(x, t)$, will permit an amplitude equation, to be derived in a later section, to take
a very simple form. The components of the perturbation displacement variable are
defined as

$$u = \frac{\partial \alpha}{\partial t} + U \frac{\partial \alpha}{\partial x} - U' \beta, \quad (2.1.1d)$$

$$v = \frac{\partial \beta}{\partial t} + U \frac{\partial \beta}{\partial x}. \quad (2.1.1e)$$

Physically, one can think of $d(x, t)$ as the distance that a particular fluid particle
has moved due to the perturbations in the base flow. Equations (2.1.1a – c) are
solved with null initial conditions; specifically $u = p = 0$ at $t = 0^-$. Note that, our governing equations (2.1.1) were derived from the full Euler
equations after using three fundamental assumptions of: incompressible, inviscid
and two-dimensional disturbances. Although these assumptions are fairly com-
mon in the literature, some remarks are desirable nonetheless. The perturbations
are considered incompressible because we are only considering low Mach number
flows in this study. Typically for ($M < 0.3$) it is common knowledge that the
flow is effectively at constant density; therefore the velocity field is divergence free.
For simplicity, we restrict our attention to two-dimensional disturbances; three-
dimensional disturbances can be handled in a similar fashion as shown by Balsa
(1989). Furthermore, because two-dimensional disturbances are more unstable than
three-dimensional ones, it is natural to examine initially what happens in the for-
mer case. The inviscid disturbances are justified because for the type of inflectional
base velocity profile that we will consider (see Figure 1a), the dynamics of the flow
is practically entirely dominated by the vorticity interactions.
The equations of motion (2.1.1a, b, c) are solved by Fourier transform in the 
x direction and Laplace transform in time \( t \). We introduce the Fourier transform 
pairs by

\[
\hat{\phi} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi e^{-ikx} dx, \quad (2.1.2a)
\]

\[
\phi = \frac{1}{\sqrt{2\pi}} \int_{L} (\hat{\phi}) e^{+ikx} dk, \quad (2.1.2b)
\]

where \( k \) is the streamwise or axial wavenumber and is in general complex, and \( L \) 
is a suitably chosen contour. The equations of motion will also be subject to the 
Laplace transform pairs

\[
\hat{\phi} \equiv \int_{0}^{+\infty} (\hat{\phi}) e^{-\Omega t} dt, \quad (2.1.3a)
\]

\[
\hat{\phi} = \frac{1}{2\pi i} \int_{Br} (\hat{\phi}) e^{+\Omega t} d\Omega, \quad (2.1.3b)
\]

where \( Br \) is the Bromwich contour in the complex \( \Omega \)-plane (see Figure 2a).

The application of the Fourier and Laplace transforms on equations (2.1.1) 
yields a new set of governing equations in \( y \) and \((k, \Omega)\). These are ordinary differ­
ential equations.

\[
ik\hat{u} + \frac{\partial \hat{v}}{\partial y} = 0, \quad (2.1.4a)
\]

\[
\Omega\hat{u} + ikU\hat{u} + U'\hat{v} + ik\hat{p} = 0, \quad (2.1.4b)
\]

\[
\Omega \hat{v} + ikU\hat{v} + \frac{\partial \hat{p}}{\partial y} = \frac{1}{\sqrt{2\pi}} \delta(y - y_0). \quad (2.1.4c)
\]

We multiply equation (2.1.4c) by \( ik \), and differentiate equation (2.1.4b) with 
respect to \( y \) and subtract one from the other to eliminate the variable \( \hat{p} \). We then 
eliminate \( \hat{u} \) using the continuity equation (2.1.4a) and arrive at a single equation in 
term of the transverse velocity \( \hat{v} \)

\[
N[\hat{v}] \equiv \left[ \frac{\partial^2}{\partial y^2} - k^2 - \frac{U''(y)}{U(y) - i\frac{\Omega}{k}} \right] \hat{v} = \frac{ik}{\sqrt{2\pi}} \frac{\delta(y - y_0)}{[U(y) - i\frac{\Omega}{k}]},
\]

The equation above can be written in universal form if we introduce a canonical 
velocity profile \( f = f(y) \) where

\[
U(y) = U_m + \frac{\Delta U}{2} f(y),
\]
and \( U_m = \frac{1}{2}(U_1 + U_2) \) and \( \Delta U = U_1 - U_2 \) represent the average velocity and the velocity difference respectively of the mixing layer. \( U_1 \) and \( U_2 \) are the limiting external velocities as \( y \to \pm \infty \), and we require \( f(\pm \infty) = \pm 1 \). We obtain after some trivial algebra and the above introduction of \( f(y) \)

\[
\mathcal{L}[\bar{v}] \equiv \left[ \frac{\partial^2}{\partial y^2} - k^2 - \frac{f''(y)}{f(y) - \hat{C}} \right] \bar{v} = \frac{2ik}{\sqrt{2\pi \Delta U}} \frac{\delta(y - y_0)}{[f(y) - \hat{C}]}.
\]

(2.1.5a)

\[
\hat{C} = \frac{i\Omega - U_m}{\Delta U/2}.
\]

(2.1.5b)

Note that

\[
\kappa = (k^2)^{1/2} = \begin{cases} 
+k & \text{for } k_R > 0, \\
-k & \text{for } k_R < 0,
\end{cases}
\]

where \( k_R = Re(k) \) stands for the real part of \( k \). Equation (2.1.5a) is a second-order ordinary differential equation with varying coefficients and an impulse right-hand side forcing function. It is identical to the Rayleigh inviscid stability equation except that \( k \) and \( \hat{C} \) are not related but are two independent variables in the complex \((k, \hat{C})\) plane. The equation is complete only after we specify its boundary conditions, namely \( \bar{v}(y \to \pm \infty) = 0 \).

We next write down the solution to (2.1.5a). Let \( v_1(\kappa, y, \hat{C}) \) and \( v_2(\kappa, y, \hat{C}) \) be solutions to the homogeneous version of this equation with the requirement that they vanish at \( y = \pm \infty \), respectively. At the source location, \( y = y_0 \), \( \bar{v} \) is continuous and \( \partial \bar{v}/\partial y \) jumps by a certain amount which is obtained by integrating the right-hand-side of (2.1.5a) across \( y = y_0 \). The final result is,

\[
\bar{v}(k, y, \Omega; y_0) = \frac{2ik}{\sqrt{2\pi \Delta U}} \frac{H(\kappa, y, \hat{C}; y_0)}{[f_o - \hat{C}]W(\kappa, \hat{C})}
\]

(2.1.6a)

\[
H(\kappa, y, \hat{C}; y_0) = \begin{cases} 
v_1(\kappa, y, \hat{C})v_2(\kappa, y_0, \hat{C}) & (y > y_0), \\
v_1(\kappa, y_0, \hat{C})v_2(\kappa, y, \hat{C}) & (y < y_0),
\end{cases}
\]

(2.1.6b)

where \( f_o = f(y_0) \) and \( W(\kappa, \hat{C}) = v_1v_2' - v_1'v_2 \). Note that the Wronskian is independent of \( y \) and the prime denotes differentiation with respect to \( y \).
To find the solution \( v(x, y, t; y_0) \), we need to invert both the Fourier and Laplace transform solution, \( \bar{v}(k, y, \Omega; y_0) \), given by equations (2.1.6). Using the inverse Laplace transform of (2.1.3b) yields the formal solution

\[
\hat{v}(k, y, t; y_0) = \frac{1}{2\pi i} \int_{Br} \bar{v}(k, y, \Omega; y_0)e^{+\Omega t}d\Omega
\]  

(2.1.7)

where as we have indicated before \( Br \) stands for the Bromwich contour. We can then rewrite equation (2.1.7) as

\[
\hat{v}(k, y, t; y_0) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{v}(k, y, \Omega; y_0)e^{+\Omega t}d\Omega,
\]  

(2.1.8)

and \( \gamma \) is chosen so that the Bromwich contour of integration in the complex \( \Omega \)-plane is to the right of all singularities of the integrand (see Figure 2a). Since the Wronskian is a function of \( \kappa \) and \( \hat{C} \), it is quite advantageous to invert the Laplace transform in the complex \( \hat{C} \)-plane rather than the complex \( \Omega \)-plane. The change of variable \( \hat{C} \) for \( \Omega \) transforms the integral in (2.1.8) into

\[
\hat{v}(k, y, t; y_0) = \left( \frac{1}{2\pi i} \right) ik \left( \frac{\Delta U}{2} \right) e^{-ikUm} \int_{-\infty}^{+\infty} i\gamma \bar{v}(k, y, \Omega; y_0)e^{-ik(\Delta\Omega^2)\hat{C}}d\hat{C}.
\]  

(2.1.9)

In the above integral \( \gamma \) is chosen so that all singularities of the solution \( \bar{v}(k, y, \Omega; y_0) \), lie below the line \( \hat{C}_I = \text{const} \), (see Figure 2b), for \( k > 0 \). For \( k < 0 \) the singularities lie above this line. In order to evaluate the integral (2.1.9) by the method of residues, we need to discuss the singularities of the integrand. First, we have a simple pole at \( \hat{C} = \hat{C}_3 = f_o \), arising from the factor \([f_o - \hat{C}] \) in the denominator. The other singularities are poles arising from the zeros of \( W(\kappa, \hat{C}) \). For a free shear layer with the type of inflectional velocity profile that we will consider, there are two such poles, \( \hat{C}_1(\kappa) \) and \( \hat{C}_2(\kappa) \), and they are complex conjugate of each other for real \( \kappa \). \( \hat{C}_1(\kappa) \) and \( \hat{C}_2(\kappa) \) are related to the eigenvalues of the Rayleigh stability equation via (2.1.5b); they can also be interpreted as complex dispersion relations connecting the phase speed \( \hat{C} \) (associated with the canonical base velocity \( f(y) \)) to \( \kappa \) or the complex frequency \( \Omega_1 \) or \( \Omega_2 \) to the wavenumber \( k \). Applying the theorem of residues
to evaluate integral (2.1.9) in the usual manner by closing in the bottom-half $\hat{C}$-plane, we obtain for $k > 0$

$$\hat{v}(k, y, t; y_0) = e^{-ikU_{m}t} \sum_{j=1}^{3} F_j(k, y; y_0)e^{-ik(\Delta U)\hat{c}_j t},$$

or

$$\hat{v}(k, y, t; y_0) = \sum_{j=1}^{3} F_j(k, y; y_0)e^{\Omega_j t} \quad (2.1.10)$$

The various terms in (2.1.10) are the following:

$$F_j(k, y; y_0) = \frac{k^2}{\sqrt{2\pi}} \frac{H(k, y, \hat{C}_j; y_0)}{[f_0 - \hat{C}_j] \frac{aw}{\partial \hat{c}}(k, \hat{C}_j)} \quad \text{for } j = 1, 2 \quad (2.1.11a)$$

$$F_3(k, y; y_0) = \frac{k^2}{\sqrt{2\pi}} \frac{H(k, y, f_0; y_0)}{W(k, f_0)}, \quad (2.1.11b)$$

$$\Omega_j(k) = -ik \left[ U_m + \frac{\Delta U}{2} \hat{c}_j(k) \right] \quad (2.1.11c)$$

The reader will notice that in the equations above, we have replaced the variable $\kappa$ by $k$ because we were considering the case $k > 0$. Thus before we proceed, we must repeat the analysis for $k < 0$. In fact for negative $k$, $\kappa = -k$, and $\Omega_j(-k) = \bar{\Omega}_j(k)$, where the tilde denote complex conjugation. In addition $v_{1,2}(-k, y, \hat{C}_j) = \bar{v}_{1,2}(k, y, \hat{C}_j)$. Thus, combining the above properties we conclude that $F_j(-k, y, \hat{C}_j) = \bar{F}_j(k, y, \hat{C}_j)$. Consequently, for $k < 0$, the solution equivalent to the one in (2.1.10) yields

$$\bar{v}(k, y, t; y_0) = \sum_{j=1}^{3} \bar{F}_j(k, y; y_0)e^{\bar{\Omega}_j t}$$

Now all that is left is to apply the inverse Fourier transform (2.1.26) to equation (2.1.10) and get

$$v(x, y, t; y_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{v}(k, y, t; y_0)e^{+ikx} dk,$$
or, after replacing \( \dot{v}(k, y, t; y_0) \) by its value in (2.1.10), and noting that the integral over negative values of \( k \) is the complex conjugate of its counterpart for positive values of \( k \), we can write

\[
\begin{align*}
v(x, y, t; y_0) &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left\{ \mathcal{F}_1(k, y; y_0)e^{ik[x-C_1t]} \\
&+ \mathcal{F}_2(k, y; y_0)e^{ik[x-C_2t]} + \mathcal{F}_3(k, y; y_0)e^{ik[x-U_0t]} \right\} dk + \text{c.c.} \quad (2.1.12)
\end{align*}
\]

where \( U_0 = U(y_0), \quad C_j = C_j(k) = U_m + \frac{\Delta U}{2} \hat{C}_j(k) \).

The expression c.c. represents the complex conjugate of the whole quantity on the right-hand side of the equation preceding the + sign. The physical meaning of the quantity \( C(k) = i\Omega(k)/k \) is that it represents the phase speed of the disturbance for each Rayleigh mode in the actual base flow \( U(y) \).

The integral solution (2.1.12) represents an exact solution of the differential equation (2.1.5). It contains three different terms. The last term is a convected mode and the first two terms are two Rayleigh modes with dispersion relation \( \Omega = \Omega_{1,2}(k) \), of which one will be stable and the other unstable for real values \( k \), \((-\infty < k < +\infty)\). The complex amplitudes of the different modes \( \mathcal{F}_j(k, y; y_0) \) \((j = 1, 2, 3)\) are directly related to the receptivity of the flow. They depend on the transverse location of the source \( y_0 \) but are independent of the average velocity \( U_m \) and the velocity difference \( \Delta U \). However, the impulse response as a whole, is a function of the shear layer geometry, \((U_m, \Delta U)\). Indeed, as is evident in equation (2.1.12), the frequency and the growth rate of the disturbance are related to \( U_m \) and \( \Delta U \), through the term \( \exp \left[ ik \left( x - \left( U_m + \frac{\Delta U}{2} \hat{C}_j \right) t \right) \right] \). The relation between the receptivity of the flow and the mean velocity profile will be more discernable when we present some numerical results in a later chapter.

### 2.2. Far Field Solution—Wave Packet

The solution quoted in the previous section in (2.1.12) is valid for all time. However, in this section, we are interested in the long time behavior of the solution,
as it will serve as the initial conditions for the evolution of the wave packet in an inhomogeneous mixing layer. The significance of the various space and time scales was discussed at the begining of this chapter. We will concentrate on mixing layers which are convectively unstable. In this type of flow, the disturbance is advected away from the source as it is amplified. The flow returns eventually to the basic steady flow. The convective nature of such a shear layer leads us to consider a moving coordinate system in order to follow the evolution of the perturbation. We then introduce

\[ G = \frac{g - U_2}{\Delta U} = \frac{x/t - U_2}{\Delta U}, \]

where \( g = x/t \) represent our reference frame velocity, it turns out that for parallel shears \( g \) is also the group velocity of the perturbation which in this case is real (Gaster 1968). With this new coordinate system our exact solution (2.1.12) takes the form:

\[
v(x, y; t; y_o) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{3} \int_0^{+\infty} F_j(k, y; y_o) e^{r h_j} dk + c.c., \tag{2.2.2a}
\]

where \( \tau = \Delta U t > 0 \), and

\[
h_j = h_j(k, G) = ik \left[ G + \frac{U_2 - C_j(k)}{\Delta U} \right], \quad \text{for } j = 1, 2 \tag{2.2.2b}
\]

\[
h_3 = h_3(k, G) = ik \left[ G + \frac{U_2 - U_0}{\Delta U} \right]. \tag{2.2.2c}
\]

Recall \( \Delta U = U_1 - U_2 > 0 \). The term \( c.c. \) in (2.2.2a) stands for the complex conjugate of the whole complex expression to the left of the + sign preceding it.

Balsa (1988) has shown that in the limit as \( t \to +\infty \), the convected mode part of the solution is of order unity. For the Rayleigh modes, one (say \( \Omega_1(k) \)) grows exponentially and the other (say \( \Omega_2(k) \)) decays as \( O(1/t) \) as \( t \to +\infty \), provided that \( k \) is in the unstable range of temporal stability. Hence for our purpose, we will neglect the convected mode and the stable Rayleigh mode with dispersion relation \( \Omega_2(k) \), and only consider \( \omega(k) = \Omega_1(k) \). The leading term behavior of the solution will be given by the first term of equation (2.1.12) and will be evaluated using
the steepest descent method as described by Gaster (1968). The method involves deforming the initial contour through a saddle point in the following manner (see Figure 2c). From the origin \( k = 0 \), we move along the level curve \( \text{Re}(h_1) = 0 \) in the first quadrant until we intersect the curve of constant phase (\( \text{Im}(h_1) = \text{const} \)) through the saddle point. We then switch to this latter curve and pass through the saddle point where \( \text{Re}(h_1) \) reaches a local maximum (positive). Finally, this final saddle-point curve always intersects the real \( k \) axis, and we can integrate out to infinity. Before giving the final solution, we mention that the deformed contour described above was for a saddle point in the first quadrant, a similar technique could be used for saddle wavenumbers located elsewhere in the plane. The major contribution to integral (2.2.2a) comes from a saddle point \( k_0 = k_o(G) \) defined by

\[
h_k(k_o, G) = 0,
\]

(2.2.3)

where \( h_k = \partial h / \partial k \). The saddle point \( k_0 = k_o(G) \) itself is in general complex, and it is defined for every real velocity \( G \). Thus the large time solution arising from the saddle point wavenumber \( k_0 \) becomes

\[
v(x, y, t; y_0) = \frac{1}{\sqrt{-\tau h_o^2(G)}} \mathcal{F}_1(k_o, y; y_0)e^{\tau h_o(G)} + \text{c.c.,} \quad (2.2.4a)
\]

where

\[
\tau = \Delta U t,
\]

(2.2.4b)

\[
h_o(G) = h(k_o, G),
\]

(2.2.4c)

\[
h_o^2(G) = \frac{\partial^2 h}{\partial k^2}(k_o, G).
\]

(2.2.4d)

The long-time solution in (2.2.4) has the form of a dispersive and rapidly growing wavetrain. The asymptotic fundamental solution describes the general structure of a wave packet and also the relation between the perturbations in the flow and the initial disturbances.
2.3. Spatial Stability

For spatial modes we take a real value of the frequency $\omega$ and seek the complex wavenumber $k$. Consequently the spatial mode evolve like $\exp(ikx) = \exp(-k_l x)\exp(i(k_R x + \omega t))$ and so they grow or decay exponentially with $x$, and also oscillate in $x$ with wavelength $2\pi/k_R$. These spatial modes describe very well the forced oscillation due to a source of frequency $\omega$, for convectively unstable flows.

This forced oscillation of real frequency $\omega$ can be represented mathematically by a $\cos \omega t$ term on the right-hand side of equations (2.1.1) instead of the $\delta(t)$ impulse function. Proceeding as we did in section 2.1., we reduce the equations of motion to a single ordinary differential equation,

$$L[\ddot{v}] = \frac{2ik}{\sqrt{2\pi} \Delta U} \frac{\delta(y - y_0)}{[f(y) - \tilde{C}] [\Omega^2 + \tilde{\omega}^2]} \Omega.$$

The extra factor on the right-hand-side of (2.3.1) as it compares to (2.1.5a), represents the Fourier and Laplace transform of a sinusoidal input of frequency $\omega$, at $x = 0$, $y = y_o$, and switched on at $t = 0$.

Solving (2.3.1) exactly as we did in section 2.1., we obtain the following equation

$$\ddot{v}(k, y, \Omega; y_o) = -\frac{2ik H(\kappa, y, \tilde{C}; y_o)}{\sqrt{2\pi} \Delta U} \frac{f_o - \tilde{C}}{[f_0 - \tilde{C}]W(\kappa, \tilde{C}) [\Omega^2 + \tilde{\omega}^2]} \Omega,$$

where just like in previous sections, $f_o = f(y_o)$ and $W(\kappa, \tilde{C})$ represents the Wronskian of the independent solutions of (2.1.5).

The evaluation of $\ddot{v}(k, y, t; y_o)$ by the inverse Laplace transform in the $\tilde{C}$-plane leads us to analyse different aspects of the integral solution

$$\ddot{v}(k, y, t; y_o) = \frac{e^{-ikU_m t}}{2\pi i} \int_{-\infty + i\gamma}^{+\infty + i\gamma} \left\{ \frac{k^2 H(\kappa, y, \tilde{C}; y_o)}{\sqrt{2\pi} [f_0 - \tilde{C}]W(\kappa, \tilde{C})} \right\} e^{-ik(\Delta U/C)\tilde{C}} d\tilde{C}.$$  

The response of the problem is directly related to the five poles of the integrand function in (2.3.3). The transient response of the problem is provided by the poles
\( \mathcal{C}_j (j = 1, 2, 3) \) associated with the singularities of \( \ddot{v} \). The 'steady-state' response arises from the two additional poles at \( \mathcal{C}_\pm(k) = \frac{\pm\hat{\omega}}{\Delta U} \left[ \pm \frac{\omega}{k} - U_m \right] \). If the flow is absolutely unstable, i.e. counter-flowing streams, the transient contribution will eventually contaminate the whole flow and dominate the steady-state part of the signal at all points in the flow, and thus making it impossible to extract the response of the problem at the forcing frequency. However, if the flow is convectively unstable, i.e. co-flowing streams, the transient portions of the response are convected further and further away from the source as time evolves, revealing quite clearly the steady state response due to the forcing frequency \( \hat{\omega} \). In this section, we will assume that our particular mixing layer is convectively unstable \( (U_1 > U_2 > 0) \) and restrict ourselves to the steady state response of the forcing signal.

Thus the contribution to integral (2.3.3) arising from the residues at the poles \( \mathcal{C}_\pm(k) \) yields for \( k > 0 \),

\[
\ddot{v}(k, y, t; y_0) = \frac{1}{2} \left\{ \mathcal{G}_1(k, y; y_0)e^{+i\hat{\omega}t} - \mathcal{G}_2(k, y; y_0)e^{-i\hat{\omega}t} \right\}, \quad (2.3.4a)
\]

where

\[
\mathcal{G}_1(k, y; y_0) = \frac{k^2}{2\pi} \frac{H(k, y, \mathcal{C}_-; y_0)}{[f_0 - \mathcal{C}_-] W(k, \mathcal{C}_-)}, \quad (2.3.4b)
\]

\[
\mathcal{G}_2(k, y; y_0) = \frac{k^2}{2\pi} \frac{H(k, y, \mathcal{C}_+; y_0)}{[f_0 - \mathcal{C}_+] W(k, \mathcal{C}_+)}, \quad (2.3.4c)
\]

\[
\mathcal{C}_\pm = \mathcal{C}_\pm(k) = \frac{2}{\Delta U} \left[ \pm \frac{\hat{\omega}}{k} - U_m \right]. \quad (2.3.4d)
\]

We can now perform the inverse Fourier transform (2.1.2b) by contour integration in the \( k \)-plane and obtain the following integral,

\[
v(x, y, t; y_0) = \frac{i}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\{ \mathcal{G}_1(k, y; y_0)e^{+i\hat{\omega}t} - \mathcal{G}_2(k, y; y_0)e^{-i\hat{\omega}t} \right\} e^{+ikx} dk.
\]

The limits of integration can be reduced to include positive values of \( k \) only \((< 0 < +\infty)\), if we realize that \( \mathcal{C}_+(-k) = \mathcal{C}_-(k) \) and \( \mathcal{C}_-(k) = -\mathcal{C}_+(k) \). Thus, for negative
values of $k$, the integrand functions in the above integral are the conjugate of the ones for positive real $k$. We then obtain

$$v(x, y, t; y_0) = \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} \left\{ G_1(k, y; y_0)e^{+i\omega t} - G_2(k, y; y_0)e^{-i\omega t} \right\} e^{+ikx} dk + c.c..$$

(2.3.5)

In addition, $G_2(k, y; y_0) = \tilde{G}_1(k, y; y_0)$ and vice versa, provided of course we replace $\omega$ by $-\omega$ first. Hence integral (2.3.5) can be simplified even further to,

$$v(x, y, t; y_0) = \frac{ie^{i\omega t}}{2\sqrt{2\pi}} \int_0^{+\infty} \left\{ G_1(k, y; y_0)e^{+ikx} + CC(\omega) \right\} dk + c.c.,$$

(2.3.6)

where $CC(\omega)$ denotes the complex conjugate of the term immediately preceding it, provided that we first replace $\omega$ by $(-\omega)$.

The pole of (2.3.6) is given by the zero of $W(k, \tilde{C}_-)$. It represents the spatial eigenvalue, i.e. spatial dispersion relation $k(\omega)$. The final result is given by the residue due to the pole discussed above,

$$v(x, y, t; y_0) = i(\pi/2)^{\frac{1}{2}} \left\{ e^{i\omega t} \frac{k^2}{\sqrt{2\pi}} [f_0 - \tilde{C}_-] \frac{H(k, y, \tilde{C}_-; y_0)}{\frac{\partial W}{\partial k}(k, \tilde{C}_-)} e^{ikx} + CC(\omega) \right\} + c.c.,$$

(2.3.7a)

where, to repeat $k = k(\omega)$ is defined by the spatial dispersion relation, or

$$\nu = -k[U_m + \frac{\Delta U}{2}(\tilde{C}_-(k))] .$$

(2.3.7b)

The canonical phase speed $\tilde{C}_-$ is obtained by solving the Rayleigh stability equation as we did for the impulse response case and selecting the unstable branch. For this particular branch, the solution of the signaling problem has a frequency of the form $\omega(k) = +i\omega$ which is purely imaginary as it should, and note that for $k$ to lie in the right-hand side of the complex wavenumber space (unstable range), $\omega$ must be negative. In the interest of consistency with the previous section, the final solution (2.3.7a) can be rewritten (with $\tilde{C}_-$ written as $\tilde{C}$) as

$$v(x, y, t; y_0) \approx i(\pi/2)^{\frac{1}{2}} e^{i\omega t} \frac{E_1(k, y; y_0)}{\frac{\partial W}{\partial k}(k, \tilde{C})} e^{ik(\omega)x} + c.c.,$$

(2.3.8a)
where,

\[ \mathcal{E}_1(k,y;y_0) = \frac{k^2 H(k,y_0)}{\sqrt{2\pi}} \frac{H(k,y_0)}{|f_0 - \hat{C}|}. \tag{2.3.8b} \]

In observing the expression in equation (2.3.8a), it is clear that it represents a spatial instability mode as it is proportional to \( \exp(i\omega t)\exp(ikx) \) with \( \omega \) a real forcing frequency and \( k \) a complex wavenumber. The spatial growth rate is given by \( (k_I x) \), where \( k_I \) is the imaginary part of \( k \). The final result (2.3.8) also provides information on the spatial receptivity of the flow in terms of the quantity \( \mathcal{E}_1 \). Obviously \( \mathcal{E}_1 \) is a function of the source location \( y_0 \) but independent of the velocity difference \( \Delta U \) and the average velocity \( U_m \) of the free shear layer. We will define a more specific receptivity factor in a later section.

Finally, we remind the reader that our results for spatial instability of the flow are only valid for convectively unstable flows; i.e. when the two streams are co-flowing. For counter-flowing streams a spatial instability mode cannot be derived from the initial-value problem see (Balsa 1988) for details.

2.4. General Solution by Fourier Integrals

Although this chapter mainly treats the parallel flow case, we feel it is appropriate to include the general Fourier integrals solution as it applies to an inhomogeneous base flow because of the similarity of the final solution to the expression of a wave packet in section 2.2. Formally this solution from (2.1.12) looks like

\[ v(x,y,t) = \int_{-\infty}^{+\infty} \mathcal{F}(k,y)e^{\omega t + ikx} dk. \]

The solution above is valid for parallel flows. We are interested in extending the parallel flow Fourier analysis to account for medium nonuniformity under the influence of spatial and temporal growths of the mixing layer. The influence of slow development of spatial mean flow especially arising from boundary layer layer growth, has been studied by various authors, (Bouthier 1972, 1973; Gaster 1974; Saric & Nayfeh 1975; Smith 1979a). In many ways, our present approach will be similar to those just cited. The quasi parallel stability analysis for slowly varying flows leads to a
dispersion relation that changes slowly both with time and streamwise location. In addition, for inhomogeneous flows, the mode shape functions evaluated at the local wavenumber \(k\) and frequency \(\omega\), will depend on \(\xi\) and \(\tau\). The variables \(\xi = \varepsilon x\) and \(\tau = \varepsilon t\) are introduced to indicate mathematically the slow spatial and temporal variations respectively of the different flow parameters. The quantity \(\varepsilon\) represents a small positive parameter \((\varepsilon \ll 1)\), that will be described more precisely in chapter 3. In essence it is a measure of the divergence of the base flow.

For our purpose in this section, it is convenient to use the dispersion relation in the form

\[ k = k(\omega, \xi, \tau). \] (2.4.1)

We are now able to consider the effect of base flow inhomogeneity on an isolated periodic wavetrain emanating from some source at \(\xi = 0\) driven with a simple harmonic motion at a frequency \(\omega\). The resulting disturbance can be represented by a series (see Balsa 1989), whose lowest-order for the transverse velocity component is

\[ v(x, y, t) = A(\xi, \tau)v_m(y, \xi, \tau, \omega)\exp \left( \frac{1}{\varepsilon} \left[ \omega \tau + i \int_0^\xi k(\omega, s, \tau) ds \right] \right). \]

The zeroth-order Rayleigh mode function \(v_m(y, \xi, \tau, \omega)\) has the structure of the mode in a parallel flow evaluated at \(\omega\) and having the mean flow of a shear layer at \(\xi\) and \(\tau\). The slowly varying amplitude function \(A(\xi, \tau)\) is defined by a partial differential equation at a higher-order in the expansion. Additional detail, based on different arguments, is presented later.

A pulse of excitation at \(\xi = 0\) will therefore generate a wave packet given by the integral in the frequency domain of all periodic wavetrains

\[ v(x, y, t) = \int_{-\infty}^{+\infty} A(\xi, \tau)v_m(y, \xi, \tau, \omega)\exp [\tau h(\xi, \tau, \omega)]d\omega, \] (2.4.2a)

where,

\[ h(\xi, \tau, \omega) = \frac{1}{\varepsilon} \left[ \omega + \frac{i}{\tau} \int_0^\xi k(\omega, s, \tau) ds \right]. \] (2.4.2b)
The integral in (2.4.2a) can be evaluated for large values of \( \tau \) by the method of steepest descent (or saddle point method). The basic idea in the method of steepest descent is an extension of Laplace's method to integrals in the complex plane. It consists of deforming the contour of integration in such a way that the major contribution to the integral arises from a small portion of the new path of integration, and this major contribution becoming more and more dominant as the parameter of interest grows (\( \tau \to +\infty \)). The dominant contribution comes from a saddle point \( \omega^* \) in the complex \( \omega \)-plane, such that

\[
\frac{\partial h}{\partial \omega} (\xi, \tau, \omega^*) = 0 
\]

or

\[
\frac{1}{\epsilon} \left[ \tau + i \int_0^\xi \frac{\partial k}{\partial \omega} (\omega^*, s, \tau) ds \right] = 0. 
\]

In general, for points on the path near \( \omega^* \), the function \( h(\omega, \xi, \tau) \) will be complex. Since the modulus of \( h \) is maximum at \( \omega^* \), we can expect \( \text{Re}[h] \) to decrease as we move along the integration path and away from \( \omega^* \), and this decrease will be accentuated as \( \tau \) becomes larger. In addition in order not to run the risk of oscillations in the phase of \( h \) cancelling the relative increase in the real part of \( h \) near \( \omega^* \), we need to select a path going through \( \omega^* \) for which \( \text{Im}[h] = \text{const} \), (see curve \( L \) on Figure 2c). A power-series expansion about the saddle frequency of the exponent function \( h \) gives,

\[
h(\xi, \tau, \omega) = h(\xi, \tau, \omega^*) + (\omega - \omega^*)^2 \frac{\partial^2 h}{\partial \omega^2} (\xi, \tau, \omega^*) + \cdots. 
\]

Along the curve \( L \), we can make a local change of variable \( \omega - \omega^* = i \beta \), where \( \beta \) is real and \( i = (-1)^{1/2} \). The integral in (2.4.2a) is approximated by

\[
v(x, y, t) \approx 2A(\xi, \tau)v_m(y, \xi, \tau, \omega^*)e^{\tau h(\xi, \tau, \omega^*)} \int_0^{+\infty} i \exp [-\tau h^* \beta^2] d\beta \tag{2.4.6a}
\]

\[
h^*_{\omega\omega} = \frac{\partial^2 h}{\partial \omega^2} (\xi, \tau, \omega^*), \tag{2.4.6b}
\]
where the integration limits on $\beta$ have been taken as $[0, +\infty]$ because $\beta^2$ is an even function, and where the integrand has been evaluated at $\omega^*$. The integration of (2.4.6a) gives

$$v(x, y, t) \approx \left[ \frac{2\pi}{-\tau h_{\omega^*}^*} \right]^{1/2} A(\xi, \tau)v_m(y, \xi, \tau, \omega^*) \exp[\tau h(\xi, \tau, \omega^*)], \quad (2.4.7)$$

with $h_{\omega^*}^*$ defined in (2.4.6b).

The expression in (2.4.7) represents the leading-order asymptotic solution at a given $\xi$ and $\tau$. The form is strikingly similar to that of equation (2.2.4a) derived for a parallel mean flow. For each value of $\xi$ and $\tau$, there is a specific value $\omega^*$ that satisfies (2.4.4) and provides a point about which an expansion can be generated in the form of equation (2.4.5). It turns out that for the types of base flow profiles used in this study, there is only one root to equation (2.4.4), providing a one-to-one mapping of $\omega^*$ on the real $(\xi, \tau)$-plane. To evaluate (2.4.7), it is clear that we need to determine $A$ and $v_m$ which we have already mentioned requires the solution of a partial differential equation and an eigenvalue problem respectively. Nevertheless, the present analysis allows us to determine the fast scale exponential part of the solution. Clearly even when the dispersion relation is known explicitly in closed form (profile of Rayleigh), an iterative procedure is required and the amount of computation involved in finding even one root is large. To find the solution as a function of time at a given distant downstream, this iterative procedure has to be performed several times, and its description is available in detail in chapter 4.
CHAPTER 3

FAR FLOW SOLUTION

In the far field the relevant length scale of the base flow is the one associated with the streamwise spreading of the mixing layer. This scale is assumed to be large compared to the thickness of the layer. On the other hand, the characteristic wavelength of the packet is on the order of a few shear layer thicknesses, therefore, it is much smaller than the length scale of the far field. Thus we may think of the wave packet as a localized oscillatory and convecting disturbance whose properties (wavelength, amplitude, frequency) change slowly as it propagates downstream over length scales which are comparable to that associated with the streamwise spreading of the layer. In this chapter we quite generally examine how the cumulative effects of a changing base flow affect the evolution of an unstable wave packet.

In chapter 2 it was mentioned that the derivations in the chapter in question, were to serve as initial conditions for the evolution of a wave packet in a diverging mixing layer. Since the initial disturbance which is placed in the mixing layer rapidly (both in time and space) "disperses" into a wave packet of instability waves, on the length scale of the outer flow this entire process of dispersion seems to take place at a fixed streamwise location, ideally at a point. It is in this sense that the material in chapter 2 provides the "initial" condition for the wave packet in the outer diverging shear flow.

Wave propagation in a slowly varying medium has been developed and used at great extent for conservative wave systems, Lewis (1964), Whitham (1965), Bretherton (1966,1968) and Hayes (1970a). However, the ideas are also applicable for nonconservative waves of unstable flows. Bouthier (1972), and also Crighton and Gaster (1976) all applied the multiple scales method to steady and spatially varying shear flows. More recently, Russel (1986) and Balsa (1989) generalized Whitham's law of conservation of wave action density to include the case of inviscid instability waves.
The method of multiple time scales or the WKBJ method may be used to describe wave propagation in a slowly varying medium whose spatial and temporal rates of change are small in comparison to the wavelength and the period of the disturbance. At the lowest order of the expansion, the complex solution of the problem usually looks like

\[ A(\xi, \tau) d_m(y, \xi, \tau) \exp\left\{ \frac{\phi(\xi, \tau)}{\epsilon} \right\}, \]  

where

\[ \xi = \epsilon x, \quad \tau = \epsilon t. \]  

In the above formula, \((x, t)\) and \((\xi, \tau)\) are the 'fast' and 'slow', streamwise and time coordinates respectively, and \(y\) is the transverse coordinate. The small quantity \(\epsilon\) represents a parameter characterizing small variations of the medium, a more complete discussion of \(\epsilon\) will be offered in a later section. We will see momentarily that the derivatives of the complex phase function \(\phi(\xi, \tau)\) will describe the complex frequency and the wavenumber vector of the disturbance that will be connected through a complex dispersion relation. The amplitude function \(A(\xi, \tau)\) on the other hand is determined by the amplitude equation derived by satisfying a solvability condition applied on a higher-order equation. The reader will notice that all three functions, \(A\), \(d\) and \(\phi\) are complex and even though we refer to \(A\) as the amplitude function, in reality, all three contribute to the physical amplitude and physical phase of the disturbance.

At this point let us state that although wave propagation in three space dimensions is of great practical interest, we will restrict ourselves from this point on to two-dimensional disturbances. The relevant space variables are the streamwise variable and the cross-stream variable. For instability waves the cross-stream structure of a disturbance is given by a mode; in essence the mode shape is represented by \(d = d(y, \xi, \tau)\) which now changes slowly along the streamwise direction because of the divergence of the mixing layer. The oscillatory or wave-like behavior takes place in the streamwise coordinate only and time. Therefore, for all practical purposes two-dimensional instability modes are one space dimensional waves. The reader
is referred to Hayes (1970b) for the concept and dimension of propagation space. Therefore the equations to be derived in this chapter will be for a one-dimensional unsteady basic state. However, for the numerical study of this work, the equations will be reduced for corresponding steady basic flow profiles.

3.1. The Diverging Mixing Layer

The basic flows that will be considered throughout this work will be of two types. The first one is the piecewise linear profile of Rayleigh. The stability of this flow was studied by Balsa (1988). The second basic state is the usual free shear layer or the so-called "tanh" profile.

The base flows can be described by the equation

\[ U(y, \xi) = U_m + \frac{\Delta U}{2} f(\eta), \]

\[ \eta = \frac{y}{L(\xi)} \]

where the function \( L(\xi) \) is a measure of the thickness of the mixing layer: the vorticity thickness is \( 2L \). The dependence on the slow streamwise coordinate \( \xi \) is chosen so that the diverging mixing layer is a reasonable representation of experimental facts. The difference in the two types of profiles that we will study (i.e. Rayleigh profile and tanh profile) rests on the definition of the shape function \( f(\eta) \).

\[ f(\eta) = \begin{cases} +1 & (\eta > +1), \\ \eta & (|\eta| \leq 1), \\ -1 & (\eta < -1), \end{cases} \]

(Linear profile) (3.1.2a)

\[ f(\eta) = \tanh \eta, \]

(General profile) (3.1.2b)

where \( U_m = (U_1 + U_2)/2, \Delta U = U_1 - U_2 \) and \( U_1 \) and \( U_2 \) are the external base flow velocities at \( y = +\infty \) and \( y = -\infty \) respectively. The quantities \( U_m \) and \( \Delta U \) refer to the average velocity and the velocity difference of the base flow respectively.

One of the advantage of the piecewise linear profile of Rayleigh, is that the initial value problem in (2.1.5) with \( U(y, \xi) \) given by (3.1.1a) can be solved in closed
form (see Balsa 1988), hence we have the luxury of providing functional relationships among the flow variables. The unstable dispersion relation for example for a two-dimensional disturbance in this case is given by

\[ \Omega(k, \xi) = \omega L = -iKU_m + \frac{\Delta U}{4} \left[ e^{-4K} - (1 - 2K)^2 \right]^{1/2}. \] (3.1.3)

It is because of the simplicity of this dispersion relation, that the piecewise linear profile serves as a useful model for more general free shear flows, specially for three-dimensional disturbances.

Contrary to the previous example, the tanh profile is not as accommodating, in that the dispersion relation cannot be written down explicitly. In fact, one has recourse to a fully numerical strategy consisting of the solution of an eigenvalue problem. The eigenvalue problem can be described in term of the the perturbation pressure \( p \) for two-dimensional disturbances in terms of the Rayleigh equation

\[ \frac{d^2p}{d\xi^2} - \left( \frac{2f'}{f' - \hat{c}} \right) \frac{dp}{d\xi} - K^2p = 0, \] (3.1.4a)

\[ K = kL(\xi), \] (3.1.4b)

\[ c(K) = \frac{i\omega L}{K} = U_m + \frac{\Delta U}{2} \hat{c}(K), \] (3.1.4c)

\[ f' = \frac{df}{d\eta}. \] (3.1.4d)

Note that \( c(K) \) represents the phase speed of the disturbance and \( \hat{c}(K) \) is called the canonical phase speed of the disturbance. To complete the above problem, we introduce the boundary conditions, \( p(\eta = \pm \infty) = 0 \). The numerical approach to this eigenvalue problem is described in details in Appendix B.

### 3.2. The WKBJ Expansion Procedure

The WKBJ method hinges on the separation of our disturbance wavetrain into a rapidly varying phase, and a slowly changing complex amplitude. For our
perturbation displacement variable $d$ and our perturbation pressure $p$, the high-frequency assumption results in

$$d = \hat{d} \exp(\phi/\epsilon) + c.c. \quad (3.2.1a)$$

$$p = \hat{p} \exp(\phi/\epsilon) + c.c. \quad (3.2.1b)$$

Where $\phi = \phi(\xi, \tau)$ is the complex phase, because $\phi$ is complex, the exponential term in (3.2.1) represents a rapidly oscillating disturbance. Such oscillations occur locally in the wave packet. On the other hand, the slowly changing functions $\hat{d}$ and $\hat{p}$ describe the cumulative changes in the wave packet as it propagates through the diverging mixing layer. The expression $c.c.$ stands for complex conjugate. The small parameter $\epsilon$ ($0 < \epsilon \ll 1$) expresses the inhomogeneity of the base flow. Physically, this translates into the fact that $\epsilon$ represents the ratio of a typical wavelength or period of the wavetrain to the length scale or time scale associated with the diverging mixing layer. This is indicated by the dependence of the flow parameters such as the phase $\phi$ on the slow spatial variable $\xi = \epsilon x$, and the slow time $\tau = \epsilon t$. The local complex wavenumber and complex frequency of these oscillations are defined by

$$k = -i \frac{\partial \phi}{\partial \xi} \quad (3.2.2a)$$

$$\omega = \frac{\partial \phi}{\partial \tau}. \quad (3.2.2b)$$

Note that in the formula above and throughout this chapter, the quantity $i$ represents the purely imaginary number: $i = (-1)^{1/2}$.

The functions $\hat{d}$ and $\hat{p}$ in (3.2.1) are expanded as

$$\hat{d} = d^{(0)} + \epsilon d^{(1)} + \cdots \quad (3.2.3a)$$

$$\hat{p} = p^{(0)} + \epsilon p^{(1)} + \cdots, \quad (3.2.3b)$$

where $d^{(0)}, d^{(1)}, p^{(0)}$ and $p^{(1)}$ are all functions of $y, \xi$ and $\tau$. The above expansions are introduced in the equations of motion and we end up with different equations at different orders of $(\epsilon)$; the interested reader is encouraged to consult the work
by Balsa (1989), and also Appendix C for the details. Since instability waves are modal, the flow variables at lowest order may be written

\[ \mathbf{d}^{(0)} = A(\xi, \tau) \mathbf{d}_m(y, \xi, \tau, -i\frac{\partial \phi}{\partial \xi}) \]  

\[ p^{(0)} = A(\xi, \tau) p_m(y, \xi, \tau, -i\frac{\partial \phi}{\partial \xi}), \]  

where \( A \) is a slowly varying complex amplitude that can be evaluated only at the next order (i.e. \( O(\varepsilon) \)). Quantities \((\mathbf{d}_m, p_m)\) are the so-called "modes", indicated by the subscript \( m \); they vanish as \( y \to \pm \infty \), and they are evaluated at \( k = -i\partial \phi / \partial \xi \). These modes are obtained by solving the Rayleigh equation.

A solution to the Rayleigh equation exists only if a dispersion relation connecting the complex frequency \( \omega \) and the complex wavenumber \( k \) is satisfied. The complex dispersion relation reads

\[ \omega = \Omega(k, \xi, \tau). \]  

In our case, it depends explicitly on \( \xi \) and \( \tau \) because of the spatial and temporal inhomogeneities in the base flow.

At the next order in \( \varepsilon \) (i.e. \( O(\varepsilon) \)) \( d^{(1)} \) and \( p^{(1)} \) will satisfy inhomogeneous equations. A necessary condition on the solvability of these equations imposes a constraint on the amplitude \( A \); this constraint is the amplitude equation. The derivation of this equation may be found in Balsa (1989) or in Appendix C; in its simplest form

\[ \frac{\partial A}{\partial \tau} + \frac{\partial}{\partial \xi} (gA) - iA^2 \int Q \frac{\partial F^{(1)}_t}{\partial y} W \, dy = 0. \]  

Where \( A \) represents the complex wave action density of the wavetrain and it is quadratically dependent on the amplitude \( A \),

\[ A = A^2 \int_{-\infty}^{+\infty} \omega_s(Q^2 + W^2) \, dy. \]  

The Doppler-shifted frequency is \( \omega_s = \omega + ikU \) and \( Q, W \) are the cross-stream and streamwise components of the displacement mode: \( \mathbf{d}_m = (W, Q), \) subject to
some arbitrary normalization, the group velocity is denoted by $g$. Equation (3.2.6) is in standard conservation law form in that volume integrals of $A$ are conserved in propagation space contingent on the source term

$$S = iA^2 \int Q \frac{\partial F^{(1)}_t}{\partial y}Wdy,$$

$$F^{(1)}_t = -\nabla P^{(0)}(\xi, \tau) + \nabla^2_nU(y, \xi, \tau)$$

where $\nabla$ represents the gradient operator in longitudinal space and $\nabla^2_n$ is the Laplacian operator in lateral space. In other words $F^{(1)}_t = F^{(1)}_t(y, \xi, \tau)$ is in longitudinal space (indicated by the subscript $t$) and it represents the base flow acceleration. In the expression for $F^{(1)}_t$ above, we have assumed that it is introduced by a combination of the base flow pressure gradient $\nabla P^{(0)}$ and the presence of viscous forces $\nabla^2 U$. For parallel and steady flows, the source term $S$ vanishes and $A$ obeys a purely conservation law (no source term)

$$\frac{\partial A}{\partial \tau} + \frac{\partial (gA)}{\partial \xi} = 0. \quad (3.2.8)$$

Let us now look at the completeness of our system. Once the Rayleigh instability mode is determined, the dispersion relation and the mode $d_m = (W, Q)$ are known. In view of (3.2.5) and (3.2.2), the phase is obtained from

$$\frac{\partial \phi}{\partial \tau} = \Omega(-i\frac{\partial \phi}{\partial \xi}, \xi, \tau) \quad (3.2.9)$$

and as usually the group velocity is

$$g = i\Omega_k(k, \xi, \tau). \quad (3.2.10)$$

For a given $F^{(1)}_t$, the amplitude equation may be solved for the wave action density and (3.2.7) yields the amplitude. Thus for each value of $\xi$ and $\tau$ we can determine, at least in principle, the phase, amplitude and mode. Equations (3.2.4) yield the lowest-order disturbance in the mixing layer.
3.3. Theory of Ray Tracing

The ray tracing method is identical to the method of characteristics for partial differential equations. In our problem we have two pde’s: one for the phase and one for the amplitude. Characteristics or rays are simply propagation paths, along which a certain physical entity is propagated. For example a system of roads can be considered as a system of propagation paths along which vehicles propagate, similarly a system of air routes radiating from an airport describes another family of propagation paths. The previous two examples are applicable to the study of traffic flows, but in our present study we will concern ourselves with waves propagating in fluids. For this particular subject, the rays are paths of wavenumber, frequency, phase and energy propagations. The solution of the first-order partial differential equation for the phase (3.2.9) by the method of characteristics requires the solution of a system of nonlinear ordinary differential equations

\[
\frac{d\tau}{d\sigma} = 1 \tag{3.3.1a}
\]

\[
\frac{d\xi}{d\sigma} = i\Omega_k(k, \xi, \tau) \tag{3.3.1b}
\]

\[
\frac{dk}{d\sigma} = \frac{\partial k}{\partial \tau} + g \frac{\partial k}{\partial \xi} = -i\Omega_\xi(k, \xi, \tau) \tag{3.3.1c}
\]

\[
\frac{d\omega}{d\sigma} = \frac{\partial \omega}{\partial \tau} + g \frac{\partial \omega}{\partial \xi} = \Omega_\tau(k, \xi, \tau) \tag{3.3.1d}
\]

\[
\frac{d\phi}{d\sigma} = \frac{\partial \phi}{\partial \tau} + g \frac{\partial \phi}{\partial \xi} = \Omega(k, \xi, \tau) + ik. \tag{3.3.1e}
\]

We complete the above system by introducing the amplitude equation (3.2.6) in characteristic form

\[
\frac{dA}{d\sigma} = \frac{\partial A}{\partial \tau} + g \frac{\partial A}{\partial \xi} = S - A \frac{\partial g}{\partial \xi} \tag{3.3.1f}
\]

where you will recall the group velocity is defined by \( g = i\Omega_\xi(k, \xi, \tau) \). The parameter \( \sigma \) is measured along each ray and by its definition in equation (3.3.1a), may be taken identical to time \( \tau \). The system of equations (3.3.1) is already in a form suitable for the application of standard computer routines for ordinary differential equations to
obtain their solutions quickly under specific initial conditions that we will discuss in a later section. The first two equations define the loci of propagation paths (called rays) in \((\xi, \tau)\) space. The group velocity is tangent to these rays. The next three equations describe the variation of wavenumber, frequency and phase along the rays. If the base flow is homogeneous \((\Omega_\xi = 0)\) then the wavenumber is constant along the rays. This is Snell's law. In addition, for the homogeneous medium, the rays are straight. Similarly, if the base flow is independent of time \((\Omega_\tau = 0)\), the frequency is constant along the rays. This implies that in a time independent base flow it is possible to consider (time) oscillations at a fixed frequency this is, of course, well known.

Finally the last equation determines the amplitude or "energy" of the disturbance along the rays. Such energy propagates along the rays with the group velocity.

The first five equations are "kinematic" in nature; they only involve the dispersion relation, and therefore only the lowest order solution in \(\epsilon\). The amplitude equation is more involved because it comes from the next higher order terms in the asymptotic expansion. It is interesting to note that one can obtain all the information on the wavenumber, frequency etc without any reference to the amplitude. This is because we are dealing with linear theory; wavenumber and frequency are independent of the amplitude.

Ray tracing is of value not only because it reveals information on the spatial variation of the wavenumber vector \(k\), but also because wave energy is carried along these rays; therefore ray tracing can be used to determine the distribution of wave amplitude. The concept of "ray tube" area is an excellent one for determining the variation of wave amplitude along the rays. We shall derive a modified ray tube equation later.

At this point in time, we digress slightly to draw a parallel with the subject of classical dynamics. In fact, equations (3.3.1) can be recognized as Hamilton's
equations for a conservative dynamical system. Those are written in terms of a Hamiltonian function

\[ H(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \]

which is the system total energy, (i.e. kinetic plus potential) expressed as a function of generalized coordinates \( q_1, q_2, \ldots, q_n \) and associated momenta \( p_1, p_2, \ldots, p_n \); Hamilton’s equations then take the form

\[
\frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{d\tau} = +\frac{\partial H}{\partial p_i},
\]

for \( i = 1, \ldots, n \). Equations (3.3.1b – d) are precisely of this form with \( n = 1 \) and with \( k, \xi \) and \( \omega \) representing \( p_1, q_1 \) and \( H \). The conservation of frequency along the paths is exactly similar to the conservation of total energy in a dynamical system, when the Hamiltonian is independent of time. This analogy is only formal for our problem of instability waves, but becomes exact for conservative wave systems. It is, of course, at the very essence of wave-particle duality.

The integration of the amplitude equation is not at all straightforward because it cannot be directly performed along each ray together with the other ordinary differential equations in (3.3.1) because knowledge of neighboring solutions is required to determine the partial derivatives on the right-hand side of the equation, specifically the divergence of the group velocity vector \( \partial g / \partial \xi \). The derivative \( \partial g / \partial \xi \) in (3.3.1f) is one in the propagation space. We transform this expression to one in the "augmented space" (involving \( k, \xi \) and \( \tau \)) and obtain

\[
\frac{\partial g}{\partial \xi} = i \left( \frac{\partial \xi}{\partial k} \right)^{-1} \Omega_{kk} + i \Omega_{k\xi}.
\]

The details are given in Hayes (1970b). We then apply the operator \( \partial / \partial \xi \) to the third ray equation (3.3.1c) and transform the result to an expression in the augmented space by using (3.3.2). The result of these operations is to obtain an expression for the evolution of \( \partial k / \partial \xi \) along the rays. We then consider the inverse of \( \partial k / \partial \xi \) and arrive at a differential equation for \( \partial \xi / \partial k \)

\[
\frac{d}{d\sigma} \left( \frac{\partial \xi}{\partial k} \right) = i \Omega_{kk} + 2i \Omega_{k\xi} \left( \frac{\partial \xi}{\partial k} \right) + i \Omega_{\xi\xi} \left( \frac{\partial \xi}{\partial k} \right)^2.
\]

(3.3.3)
Equation (3.3.3) is what Hayes (1970b) called derived ray equations. It is clear that \( \partial \xi / \partial k \) can be directly calculated along each ray. This yields, via (3.3.2), the divergence of the group velocity \( \partial g / \partial \xi \) along each ray. Combining equations (3.3.2) and (3.3.3), we can derive an equation describing the evolution of a certain quantity proportional to the square of the amplitude along each ray:

\[
\frac{dB}{d\sigma} = i\Omega_k \xi B + iTB + i\Omega_{\xi \xi} \left( \frac{\partial \xi}{\partial k} \right) B
\]  

(3.3.4a)

where

\[
B = A \left( \frac{\partial \xi}{\partial k} \right)
\]  

(3.3.4b)

\[
\Upsilon = \frac{\int Q(\partial P_{x(1)}/\partial y)Wdy}{\int \omega_0(Q^2 + W^2)dy}.
\]  

(3.3.4c)

Our system of equations is now complete and we should be able to solve for all the flow parameters \( k, \omega, \phi, A \) of a wave packet evolving in a diverging mixing layer, after we have specified the initial data.


We have mentioned previously that if initial values for \( \xi \) and \( k \) are known, then, the ray equations (3.3.1a, b) may be solved to determine a ray: along this ray, we see from (3.3.1c, d, e) that \( k, \omega \) and \( \phi \) may be obtained by integration. In this procedure it is quite clear that initial values for \( \xi, k, \omega, \phi, \partial \xi / \partial k \) and \( B \) are required. We already know that the asymptotic solution of a wave packet generated in a parallel flow (the inner flow), constitutes the initial conditions for the evolution of the packet in a non-parallel mixing layer. For this reason, we need to perform a matching of the two flows in order to specify these initial conditions for the ray tracing in the outer region. The initial time is determined at \( \sigma = \tau = 0 \), and we can assume without loss of generality that \( \xi = 0 \) initially. At this time, the wave packet will be evolving in an effectively parallel shear layer, where the group velocity \( g \) at every point in the wavetrain is real. Consequently, we have

\[
\sigma = \tau = 0; \quad k = k_0(g) = k_0(\xi/\tau).
\]  

(3.4.1)
The initial frequency is obtained directly from the dispersion relation at $k_0$, or $\omega = \Omega(k_0)$. In parallel shear layers, the phase function of the disturbance takes the form

$$\phi = \phi_o = \Omega(k_0) \tau + i k_0 \xi,$$

however, at the beginning, both $\xi$ and $\tau$ are null so we have

$$\sigma = \tau = 0; \quad \phi_o = 0. \quad (3.4.2)$$

From the definition of the real group velocity $g = \xi/\tau = i \Omega_k(k)$, we arrive at the initial expression for the variable $\partial \xi / \partial k = i \tau \Omega_{kk}$ and resulting for $\tau = 0$ to

$$\sigma = \tau = 0; \quad \frac{\partial \xi}{\partial k} = 0. \quad (3.4.3)$$

Finally, for the variable $B$, the matching of the near flow and the far flow is needed. We expand $B$ in a Taylor series in $\sigma$,

$$B = B_0 + i \Omega_k \xi B_0 \sigma + O(\sigma^2), \quad (3.4.4)$$

where $B_0 = (A \partial \xi / \partial k)_0$ is the initial value sought.

The next step is to do the actual matching of flows as $\sigma \to 0$. In working the receptivity of the parallel mixing layer in chapter 2, we obtained an asymptotic solution of a wave packet evolving in a parallel shear layer. We write this solution for the transverse velocity as a function of the outer variables $\xi$ and $\tau$,

$$v(x, y, t) = \left[ \frac{1}{-(\tau/\epsilon) \Omega_{kk}} \right]^{1/2} \mathcal{F}_1(k, y) \exp \left[ \frac{1}{\epsilon} (ik \xi + \omega \tau) \right] + c.c.. \quad (3.4.5)$$

The WKBJ expansion for the treatment of the diverging mixing layer indicates that the disturbance in the far-field is represented by

$$v(x, y, t) = A(\xi, \tau)v_m(y, \xi, \tau, k) \exp [\phi(\xi, \tau)/\epsilon] + c.c..$$

Near $\sigma = \tau = 0$, the above expression can be rewritten after Taylor expanding the phase function $\phi(\xi, \tau)$ and using the definition of the wavenumber $k$ and the frequency $\omega$ as

$$v(x, y, t) = A_0 v_m(y, k) \exp \left[ \frac{1}{\epsilon} (ik \xi + \omega \tau) \right] + c.c.. \quad (3.4.6)$$
The reader will note that the mode $v_m(y, k)$ is subject to a normalization when we solve the Rayleigh stability equation, and it allows us to choose $v_m(y, k) = 1$ at a fixed value of $y$ (here $y = 0$) in (3.4.6) without loss of generality. Matching the two flows by comparing equation (3.4.5) to equation (3.4.6), we deduce that the initial amplitude $A_0$ is defined by

$$\tau \to 0; \quad A^2 \to A_0^2 = -\frac{\mathcal{F}_1^2(k)}{\tau \Omega_{kk}}. \quad (3.4.7)$$

Introducing the definition of the wave action density $A$ into (3.4.7), we arrive at

$$B_0 = -i \mathcal{F}_1^2(k) \int_{-\infty}^{+\infty} \omega_o(Q^2 + W^2) dy, \quad (3.4.8)$$

where $\omega_o$ represents the Doppler-shifted frequency and $W$ and $Q$ are the components of the displacement variable in cross and propagation-space respectively.

In summary, the initial conditions associated with our governing equations can be written as:

$$\sigma = \tau = 0; \quad \xi = 0, \quad k = k_o(\xi/\tau), \quad \omega = \Omega(k_o),$$

$$\phi = 0, \quad \frac{\partial \xi}{\partial k} = 0, \quad B = B_0(k_o), \quad (3.4.9)$$

where $B_0$ is given in (3.4.8).

3.5. Complex Rays and Analytic Continuation

Another difficulty that we have to contend with is the fact that for non-conservative waves the wavenumber, frequency etc are, in general complex. Thus it is not possible to solve (3.3.1) by the method of characteristics in a way that ensures $\xi$ be real everywhere on a ray. This is because for an instability wave the group velocity is complex and even if the rays are real initially (as they indeed are at $\sigma = \tau = 0$) they will become complex because of (3.3.1a). However if initially we allow the right-hand side of equations (3.3.1) to be the analytic continuation from the real variable $\xi$ to the complex variable $\zeta$, then we can show how we can reconcile this complex solution with one that makes sense physically. In other
words all real functions of \( \xi \), such as the base flow velocity, must be continued analytically into the complex \( \zeta \) plane. This approach is similar to the one adopted by Itoh (1981). We let \( \zeta \) to be a complex position coordinate such that \( \zeta = \xi + i\eta \), and only the real axis (\( \eta = 0 \)) on the complex \( \zeta \)-plane corresponds to the physical space, while the imaginary axis has no relation with the actual flow. We rewrite the characteristic equations in the form

\[
\frac{d\zeta}{d\sigma} = i\Omega_\zeta(k, \zeta, \tau) = g(k, \zeta, \tau) \tag{3.5.1a}
\]

\[
\frac{dk}{d\sigma} = -i\Omega_\zeta(k, \zeta, \tau) \tag{3.5.1b}
\]

\[
\frac{d\omega}{d\sigma} = \Omega_\zeta(k, \zeta, \tau) \tag{3.5.1c}
\]

\[
\frac{d\phi}{d\sigma} = \Omega(k, \zeta, \tau) + ikg \tag{3.5.1d}
\]

\[
\frac{d}{d\sigma} \left( \frac{\partial \zeta}{\partial k} \right) = i\Omega_{kk} + 2i\Omega_\zeta \left( \frac{\partial \zeta}{\partial k} \right) + i\Omega_\zeta \zeta \left( \frac{\partial \zeta}{\partial k} \right)^2 \tag{3.5.1e}
\]

\[
\frac{dB}{d\sigma} = i\Upsilon B + i\Omega_{\zeta\zeta} B + i\Omega_\zeta \zeta \left( \frac{\partial \zeta}{\partial k} \right) B \tag{3.5.1f}
\]

where, \( \Upsilon \) is defined in (3.3.4c). The above equations are solved with the initial conditions

\[
\sigma = \tau = 0; \quad \zeta = 0, \quad k = k_0(\zeta/\tau), \quad \omega = \Omega(k_0),
\]

\[
\phi = 0, \quad \frac{\partial \zeta}{\partial k} = 0, \quad B = B_0(k_0), \tag{3.5.2}
\]

The solutions of equations (3.5.1a) constitute complex rays of kinematic wave theory. They are curves in the complex plane that eventually cross the physical space at (\( \xi = \xi_p, \eta = 0 \)) and \( \tau \). The value of \( \xi_p \) depends on the initial value \( k_0 \). Once a particular complex ray is obtained passing through a particular (\( \xi_p, \eta = 0 \)) at a certain time \( \tau \), we may extract the solution of the rest of the characteristic equations by quadrature up to the physical point of interest. This process requires an iterative adjustment of the initial wavenumber \( k_0 \) to determine the particular ray that pierces the physical propagation space at exactly the position coordinate desired. In chapter 4, we dedicate an entire section to the implementation of this iteration process.
CHAPTER 4

NUMERICAL METHODS OF SOLUTION

This chapter is devoted to the description of the numerical methods used for the solution of our governing equations (3.5.1) derived in previous chapters. We propose to solve these equations using three distinct methods: the method of steepest descent, the ray-tracing method and the MacCormack's noncentered scheme for hyperbolic partial differential equations.

In the steepest descent method, we find an asymptotic value of the integral which defines the evolution of a linear wave packet in a slightly inhomogeneous flow. The solution is in effect determined by the evaluation of the integrand in (2.4.2a) at a saddle point in the complex \( \omega \)-plane.

The ray-tracing method is identical to the method of characteristics for the partial differential equations (3.2.9) and (3.2.6). In fact, for this case, the governing equations are solved along curves in the complex \( \zeta \)-plane. For this reason, our equations describing the evolution of the wavenumber, frequency, etc, \((k, \omega, \phi, \ldots)\) will need to be analytically continued from the real \( \xi \)-variable to the complex \( \xi \)-plane. This analytic continuation consists of introducing a complex space coordinate \( \zeta = \xi + i\eta \), with the real axis \( \xi \) corresponding to the real physical space. We will restrict our attention to two-dimensional flows. We then assume that all the flow parameters \((k, \omega, \phi, \ldots)\) can be continued analytically into the complex \( \zeta \)-plane. The solution will be carried out in the complex plane, but we will attribute a physical meaning to it only where and when a particular ray crosses the real physical plane \( \xi \) at \((\eta = 0)\).

The MacCormack's method of solution, by contrast, will be undertaken in the real space right at the outset. The approach will be to solve the governing equations as a system of partial differential equations. The MacCormack's scheme is a well tested method that is very practical for nonlinear hyperbolic equations.
4.1. Steepest Descent

In the absence of the complex amplitude function $A(\xi, \tau)$ which is given by a partial differential at a higher order, with coefficients involving various integrals through the cross-space of the mode, its derivatives, and its adjoint, the steepest descents method will allow us to determine only the growth rate and the Doppler-shifted frequency of a linear wave packet propagating in a diverging mixing layer. Note that $\xi = \varepsilon x$ and $\tau = ct$ represent the slow space and time variables respectively. The base flow is inhomogeneous on the space scale $\xi$ and is independent of $\tau$. For this task, it is sufficient in the case of two-dimensional disturbances to consider the integral expression

$$ I = \int_{L} e^{\tau h(\xi, \tau, \omega)} d\omega, \quad (4.1.1a) $$

where

$$ h(\xi, \tau, \omega) = \frac{1}{\epsilon} \left[ \omega + \frac{i}{\tau} \int_{0}^{\xi} k(\omega, s) ds \right]. \quad (4.1.1b) $$

The contour of integration $L$ of Figure 2c is discussed in detail in chapter 2, and in the above equation as well as the remaining of this chapter, $i = (\text{-}1)^{\frac{1}{2}}$ represent the purely unit imaginary number. The steepest descent evaluation of the above integral yields the formal result

$$ I \approx \left[ \frac{2\pi}{\frac{\partial^{2} h}{\partial \omega^{2}}(\xi, \tau, \omega^{*})} \right]^{1/2} e^{\tau h(\xi, \tau, \omega^{*})}, \quad \text{for } \tau \gg 1 \quad (4.1.2a) $$

where $\omega^{*}$ is the solution of

$$ \frac{1}{\epsilon} \left[ \tau + i \int_{0}^{\xi} \frac{\partial k}{\partial \omega}(\omega, s) ds \right] = 0, \quad (4.1.2b) $$

for each $\xi$ and $\tau$. Here $k(\omega, \xi)$ is the wavenumber of the unstable mode for each value of frequency as a function of $\xi$.

Numerical evaluations of the expressions in (4.1.2a, b) were obtained for the piecewise linear base flow profile. For each streamwise location $\xi$, the wavenumber $k(\omega, \xi)$ can be found for all complex frequencies $\omega$ and thus $\partial k/\partial \omega$ can be readily
obtained. Thus the integral in (4.1.2b) may be evaluated for any ξ and ω and the equality in this equation determines the complex frequency at the saddle point, ω*. All other quantities of interest, such as ∫₀^ξ k(ω*, s)ds may be determined in a straightforward manner. The nonlinear equation (4.1.2b) is solved by the Newton-Raphson method; specifically the iteration consisted of guessing a value ω for ω*, and this value was corrected according to

\[
ω^{(n+1)} = ω^{(n)} - \left[ τ + i \int_0^ξ \frac{∂k}{∂ω} \, ds \right] \left[ i \int_0^ξ \frac{∂^2 k}{∂ω^2} \, ds \right], \quad n = 1, 2, \ldots
\]  

(4.1.3)

We stopped iterating when a predetermined convergence criterion was satisfied. Typically, convergence was obtained within 5 or 6 iterations. The integrals ∫ k dξ, ∫ ∂k/∂ω dξ and ∫ ∂^2 k/∂ω^2 dξ were calculated using a composite Simpson’s rule integration routine derived for a function defined by equispaced values. The composite Simpson’s rule consists of using a Simpson’s \( \frac{3}{8} \) rule on the first three points in the integral limits, and a Simpson’s \( \frac{1}{3} \) rule on the remaining points. Both integration rules have a global error, \( O((Δξ)^2) \). Hence, we can safely say that the steepest descent calculations as described here, are second-order accurate in ξ. Indeed the method inherits the quadratic accuracy of the Newton-Raphson procedure. For each value of time, τ, we computed the values of ω* at 20 different stations ξ across the finite extent of the wave packet and at each station ξ, we computed values of our integrands using a stepsize \( Δξ = 0.05 \).

4.2. Ray-Tracing Method

The ray tracing method is equivalent to the solution of a system of nonlinear first-order ordinary differential equations. Although it is true that the equations are solved in the complex plane in general, (because of the spreading of the mixing layer) we will see that for parallel base flows, the problem can be posed entirely in terms of real variables. At this point we find it appropriate to mention that all the equations in this chapter will be written for the evolution of a two-dimensional wave
packet in a diverging free shear layer, with the understanding that similar equations could be readily developed for three-dimensional wave packets.

4.2.1. Complex Rays for the Diverging Shear Layer

The governing equations were already derived in chapter 3, and we reproduce them here for reference

\[
\frac{d\sigma}{d\tau} = 1 \tag{4.2.1a}
\]
\[
\frac{d\zeta}{d\sigma} = i\Omega_k(k, \zeta, \tau) \tag{4.2.1b}
\]
\[
\frac{dk}{d\sigma} = -i\Omega_\zeta(k, \zeta, \tau) \tag{4.2.1c}
\]
\[
\frac{d\omega}{d\sigma} = \Omega_r(k, \zeta, \tau) \tag{4.2.1d}
\]
\[
\frac{d\phi}{d\sigma} = \Omega - ik\Omega_k \tag{4.2.1e}
\]
\[
\frac{d}{d\sigma} \left( \frac{\partial \zeta}{\partial k} \right) = i\Omega_{kk} + 2i\Omega_{k\zeta} \left( \frac{\partial \zeta}{\partial k} \right) + i\Omega_{\zeta\zeta} \left( \frac{\partial \zeta}{\partial k} \right)^2 \tag{4.2.1f}
\]
\[
\frac{dB}{d\sigma} = i\Upsilon B + i\Omega_{k\zeta} B + i\Omega_{\zeta\zeta} \left( \frac{\partial \zeta}{\partial k} \right) B. \tag{4.2.1g}
\]

\(\tau\) represents time and \(\zeta\) the complex propagation space. The variables \(k, \omega, \) and \(\phi\) indicate axial wavenumber, complex frequency and complex phase respectively. The quantity \(B\) is the product of the complex wave action density \(A\) and the inverse of the gradient of \(k, \partial \zeta / \partial k\) (see equation (3.3.4b)). The expressions for the wave action density \(A\) and the source term \(\Upsilon\) both depend on the base flow acceleration and they are defined explicitly in chapter 3 (see equations (3.2.7) and (3.3.4c)). All the subscripts notation are used to indicate differentiation of \(\Omega\) with respect to the appropriate variable, i.e. \(\Omega_\zeta = \partial \Omega / \partial \zeta\). Note that equation (4.2.1c) evaluates the change of the frequency \(\omega\) along the rays. In general the frequency is represented by the complex dispersion relation \(\omega = \Omega(k, \zeta, \tau)\). However in all the examples studied in this work, \(\Omega\) will not be a function of \(\tau\) explicitly, that is to say that \(\Omega_r = 0\), so \(d\omega/d\sigma = 0\). Thus the frequency \(\omega\) will remain constant on each ray.
The first three equations determine the trajectory of the rays $\tau = \tau(\sigma)$, $\zeta = \zeta(\sigma)$ and the wavenumber $k = k(\sigma)$ along the rays. The real quantity $\sigma$ is a parameter along each ray; equations (4.2.1) are solved subject to initial conditions which are discussed later. We emphasize that $\Omega = \Omega(\zeta, k)$ is the analytic continuation of the dispersion relation into the complex plane $\zeta$; if the base flow were parallel $\Omega$ would be independent of $\zeta$, $\Omega_{\zeta} = 0$, and the complex wavenumber would be constant along each ray (though varying from ray to ray).

The equations in (4.2.1) can be identified as a system of nonlinear first-order ordinary differential equations. Before we propose a method to integrate these equations, we will write them in a form more suitable for our numerical approach. The equations then take the form

$$\frac{dy_i}{d\tau} = f_i(\tau, y_i), \quad i = 1, 2, \ldots 6 \quad (4.2.2a)$$

where

$$y_1 = \zeta, \quad y_2 = k, \quad y_3 = \omega, \quad (4.2.2b)$$
$$y_4 = \phi, \quad y_5 = \frac{\partial \zeta}{\partial k}, \quad y_6 = B, \quad (4.2.2c)$$
$$f_1 = i\Omega_k, \quad f_2 = -i\Omega_{\zeta}, \quad f_3 = \Omega_{r}, \quad f_4 = \Omega - ik\Omega_k, \quad (4.2.2d)$$
$$f_5 = i\Omega_{kk} + 2i\Omega_k \left( \frac{\partial \zeta}{\partial k} \right) + i\Omega_{\zeta} \left( \frac{\partial \zeta}{\partial k} \right)^2, \quad (4.2.2e)$$
$$f_6 = \left[ i\Upsilon + i\Omega_{k\zeta} + i\Omega_{\zeta} \left( \frac{\partial \zeta}{\partial k} \right) \right] B, \quad (4.2.2f)$$

and they are subject to the initial conditions

$$\tau = 0: \quad y_1 = 0, \quad y_2 = k_0, \quad y_3 = \Omega(k_0), \quad y_4 = 0, \quad y_5 = 0, \quad y_6 = B_0, \quad (4.2.3)$$

where we note that we may set $\sigma = \tau$ without loss of generality. The rays originate at $\zeta = 0$ where the wavenumber and frequency are $k_0$ and $\Omega(k_0)$ respectively. These are determined from the asymptotic form of the wave packet in a parallel shear flow. Both the phase and the ray-tube area vanish initially and $B_0$ is also determined from the asymptotic form of the packet in a parallel mixing layer.

The integration along the rays will be performed using a fourth-order Runge-Kutta method. If we denote the time step $\Delta \tau$, and the dependent variables at time $\tau$ and $\tau + \Delta \tau$ as $y_{i,n}$ and $y_{i,n+1}$ respectively, we obtain
The method is indeed fourth-order accurate and the error is of order $O(\Delta \tau^5)$. The stability limit of the Runge-Kutta method is quite large and we had no problem computing the solution at large values of time for reasonably small values of the time step ($\Delta \tau = 0.8, \tau = 160$). The time steps may appear too large at first glance, but after reflection, one realizes that they are relatively small compared to the time scales associated with the disturbance. In fact a typical smallest period corresponding to the most amplified mode is of the order of 30. So with a time step of $\Delta \tau = 0.8$ or even larger, we are able to capture most of the fine behavior of the solution.

The disadvantage of the fourth-order Runge-Kutta method is that it can be rather expensive since it requires the evaluations of the derivative functions $f_i$ four times. This makes it time consuming for problems with complicated derivative calculations. For the piecewise linear base velocity profile, the dispersion relation $\omega = \Omega(k, \zeta)$ is known explicitly as well as its partial derivatives. This property makes the linear profile of Rayleigh a perfect candidate for the fourth-order Runge-Kutta method. However, for the general shear layer profile, (such as the tanh profile), the
determination of the complex dispersion relation demands the numerical solution
of eigenvalue problems for each computation of the derivative functions. For that
reason we opted for the Adams-Moulton multistep method for the integration of the
governing equations of the tanh shear layer. The Adams-Moulton method expressed
in terms of the derivative functions $f_i$ gives

Predictor: $y_{i,n+1} = y_{i,n} + \frac{\Delta \tau}{24} (55f_{i,n} - 59f_{i,n-1} + 37f_{i,n-2} - 9f_{i,n-3})$, (4.2.5a)

Corrector: $y_{i,n+1} = y_{i,n} + \frac{\Delta \tau}{24} (9f_{i,n+1} + 19f_{i,n} - 5f_{i,n-1} + f_{i,n-2})$. (4.2.5b)

In the equations above, $f_{i,n} = f_i(\tau_n, y_{i,n})$. To determine the (pre) initial values of
the variables needed for the Adams-Moulton method, we employ the fourth-order
Runge-Kutta method. With the Adams-Moulton technique, we have the luxury
of using large time steps, in fact when the predicted and corrected values agree
to as many decimals as the desired accuracy, we can save computational effort by
increasing the step size. The efficiency of the Adams-Moulton method is about twice
that of the Runge-Kutta technique, because only two new functions evaluations $f_i$
are needed per step rather than four, and the error terms are similar. Stability
for this method is also attractive and we had no difficulty performing long-time
calculations, (i.e. $\Delta \tau = 1.0$, $\tau = 180$).

4.2.2. Determining the Physical Rays

We have already mentioned that the characteristics curves (rays) lie, in gen-
eral, off the real plane at $\zeta = \xi + \iota \eta$, i.e. ($\eta \neq 0$), (see Figure 3b). The correct
solution sought at a particular physical point and time $(\xi_*, \tau_*)$ will be given by the
ray crossing the real plane at $(\xi_*, \eta = 0, \tau_*)$. This specific characteristic is uniquely
identifiable by its initial complex wavenumber $k_*$. If such uniqueness does not exist,
various rays may intersect, and the solution at a given physical point is determined
by "information carried along" several distinct rays. While this is always a pos-
sibility in a general case, we have found it not to be the case for the base flows
under consideration because we were successful in comparing the solution from the
ray method with solutions obtained by independent methods. Numerically, we have a so called shooting problem that consists of guessing the initial slope of the rays and hitting the target $\xi = \xi_*$ at some time $\tau = \tau_*$ later. In practice, we guess the complex wavenumber $k_o$ which gives the slope of the ray through the group velocity $i\Omega_k(k_o)$, (see 4.2.1b). The iteration process to determine the appropriate ray will be performed with a Newton-Rapshson technique. We introduce a small complex number $\Delta k_o$, ($\Delta k_o \ll 1$) and set the solution of $\zeta$, at time $\tau_*$, with an initial guess of wavenumber $k_o$ to $\zeta_*(k_o)$. Now we define,

$$q(k_o) = \zeta_*(k_o) - \xi_*,$$  \hspace{1cm} (4.2.6a)

where $\xi_*$ is the point in physical space at which the solution is sought at $\tau_*$. The value of $k_o$ is then updated through the expression

$$k_o^{(n+1)} = k_o^{(n)} - \frac{q(k_o^{(n)})}{\frac{dq}{dk_o}(k_o^{(n)})},$$  \hspace{1cm} (4.2.6b)

the derivative $dq/dk_o(k_o^{(n)})$ can be approximated by

$$\frac{dq}{dk_o}(k_o^{(n)}) \approx \frac{q(k_o^{(n)} + \Delta k_o) - q(k_o^{(n)})}{\Delta k_o}.$$  \hspace{1cm} (4.2.6c)

The right ray will be considered discovered when a predetermine convergence criterion is met; for example, when $|k_o^{(n+1)} - k_o^{(n)}|/|k_o^{(n)}| + |k_o^{(n+1)}|$ is small. In these kind of operations, we used $\Delta k_o$ very small, typically $\Delta k_o \sim 10^{-3}k(\xi_*, \tau_*)$. The method proved to be very successful and convergence was always obtained after an average of about 6 iterations.

4.2.3. Real Rays for Parallel Flows

The equations studied in the previous section are valid for a general inhomogeneous medium. In this section we will restrict ourselves to an homogeneous
steady medium. For parallel and steady flows, the dispersion relationship is only a function of the wavenumber, we simply write,

$$\omega = \Omega(k),$$  \hspace{1cm} (4.2.7)

and the ray equations reduce to:

$$\frac{d\tau}{d\sigma} = 1$$  \hspace{1cm} (4.2.8a)

$$\frac{d\xi}{d\sigma} = i\Omega_k(k)$$  \hspace{1cm} (4.2.8b)

$$\frac{dk}{d\sigma} = 0$$  \hspace{1cm} (4.2.8c)

$$\frac{d\omega}{d\sigma} = 0$$  \hspace{1cm} (4.2.8d)

$$\frac{d\phi}{d\sigma} = \Omega(k) - ik\Omega_k(k)$$  \hspace{1cm} (4.2.8e)

$$\frac{d}{d\sigma} \left( \frac{\partial \xi}{\partial k} \right) = i\Omega_{kk}(k)$$  \hspace{1cm} (4.2.8f)

$$\frac{dB}{d\sigma} = 0.$$  \hspace{1cm} (4.2.8g)

Note that we are no longer using the complex variable $\zeta$ as an analytical continuation of the real physical variable $\xi$. This is the case because for homogeneous and parallel base flows, the group velocity $g = i\Omega_k$ is purely real and the rays lie on the real plane $(\xi, \tau)$, (Gaster 1975). Both the wavenumber and the frequency are conserved along these characteristics. In addition, the wave action density $\mathcal{A}$ and the complex amplitude $A$ satisfy a pure conservation law, equation (4.2.8f); indeed the source term $\Upsilon$ is identically zero, because, in steady parallel flows, the base flow acceleration is zero. In this instance, (4.2.8g) can really be interpreted in terms of ray tube areas (Hayes, 1970a). For the parallel free shear layer, since the wavenumber and the frequency are both conserved, the numerical computations are reduced considerably. In fact, initially we solve for $k_o$ by inverting the function $g = i\Omega_k(k_o)$. The group velocity $g$ is real in this case and is equal to $\xi/\tau$, where $\xi$ is a location anywhere in the extent of the wave packet at time $\tau$. Knowing $k_o$, we can solve for $\omega_o = \Omega(k_o)$, using the dispersion relation. Recall now that
both \( k \) and \( \omega \) are constant along each straight characteristic represented by \( \xi = gt \). Therefore with the value for \( g \), we can obtain the wavenumber and the frequency at any location \((\xi, \tau)\) simply by using the values \( k_0 \) and \( \omega_0 \) of the ray that goes through the point \((\xi, \tau)\). With the values of \( k \) and \( \omega \) determined, we then solve for the phase \( \phi = \omega_0 \tau + k\xi \). The only solution by quadrature is performed on equations (4.2.8f) and (4.2.8g) to solve for the amplitude of the wave packet. For these base flow properties, (homogeneous and steady), it is clear by (4.2.8.g) that the quantity \( B \) is conserved along any ray.

4.3. MacCormack’s Scheme for Nonlinear Hyperbolic Equations

We have mentioned several time before that the equations governing the evolution of the different flow parameters are nonlinear first-order partial differential equations. We are not restricted to the method of characteristics for the solution of these equations. In fact, there are a multitude of numerical methods to solve these types of equations. The particular numerical scheme that we have selected was first introduced by MacCormack(1969). MacCormack’s second-order finite-difference scheme has become an increasingly popular method for the numerical solution of the hyperbolic fluid dynamics equations. The method has been used both with the governing equations in conservation-law form and in nonconservative form. The success of the MacCormack’s method stems from the noncentered algorithms used, in general backward or forward difference schemes to approximate spatial derivatives. Noncentered schemes present three distinct advantages over conventional centered schemes: 1) the program logic is simpler, 2) the inclusion of nonhomogeneous terms into the scheme is trivial, and 3) the finite difference can be easily generalized to several space dimensions.
4.3.1. Review of the Governing Flow Equations

The two-dimensional disturbance equations governing the evolution of a wave packet in a diverging mixing layer may be represented by the single equation

\[
\frac{\partial \phi}{\partial \tau} = \Omega(-i \frac{\partial \phi}{\partial \xi}, \xi, \tau),
\]  

(4.3.1a)

where \( k = -i \frac{\partial \phi}{\partial \xi} \) represents the axial wavenumber. Equation (4.3.1a) is a first-order partial differential equation for the complex phase \( \phi \). The solution of the wave packet would not be complete until and unless we also introduce its amplitude evolution equation. The amplitude equation does not appear directly in terms of \( A(\xi, \tau) \), but it is in a hidden form that is a function of a complex wave action density, (see chapter 3 for details). We reproduce here the additional equations necessary to solve for the amplitude equation of the wave packet.

\[
\frac{\partial}{\partial \tau} \left( \frac{\partial \xi}{\partial k} \right) + g \frac{\partial}{\partial \xi} \left( \frac{\partial \xi}{\partial k} \right) = i\Omega_{kk} + 2i\Omega_{\xi \xi} \left( \frac{\partial \xi}{\partial k} \right) + i\Omega_{\xi \xi} \left( \frac{\partial \xi}{\partial k} \right)^2 \]  

(4.3.1b)

\[
\frac{\partial B}{\partial \tau} + g \frac{\partial B}{\partial \xi} = +i\gamma B + i\Omega_{\xi k} B + i\Omega_{\xi \xi} \left( \frac{\partial \xi}{\partial k} \right) B. \]  

(4.3.1c)

We remind the reader that \( \gamma \) is defined by equation (3.5.9c); it represents the source term emanating from the base flow divergence in addition, \( g \) has already been identified as the complex group velocity of the wavetrain. Analytical continuation is not necessary to solve the governing equations as a system of partial differential equations, therefore, the solution will be obtained in the physical plane \( (\xi, \tau) \).

Knowing that a wave packet convects continuously in the flow as time elapses, we need to introduce a moving coordinate system. Following (Balsa 1988), we propose the change of independent variables,

\[
T = \tau
\]  

(4.3.2a)

\[
G = \frac{\xi/\tau - U_2}{\Delta U},
\]  

(4.3.2b)

where \( U_1 \) and \( U_2 \) are the fast and slow base flow velocities respectively, and \( \Delta U = U_1 - U_2 \) represents the velocity difference. Although Balsa has shown in his study
that linear wave packets are similar in parallel shear flows under the transformation of (4.3.2), this is not the case for diverging flows. Thus our only motivation for this transformation is as stated earlier due to the convective nature of the disturbance in the flow. Applying equations (4.3.2a, b) to our governing equations we obtain the following system of equations.

\[
\frac{\partial \phi}{\partial T} + \left( \frac{-\Delta UG - U_2}{\Delta UT} \right) \frac{\partial \phi}{\partial G} - \Omega = 0 \quad (4.3.3a)
\]

\[
\frac{\partial}{\partial T} \left( \frac{\partial \xi}{\partial k} \right) + \left( \frac{g - \Delta UG - U_2}{\Delta UT} \right) \frac{\partial}{\partial G} \left( \frac{\partial \xi}{\partial k} \right)
- i\Omega_{kk} - 2i\Omega_{k\xi} \left( \frac{\partial \xi}{\partial k} \right) - i\Omega_{\xi\xi} \left( \frac{\partial \xi}{\partial k} \right)^2 = 0 \quad (4.3.3b)
\]

\[
\frac{\partial B}{\partial T} + \left( \frac{g - \Delta UG - U_2}{\Delta UT} \right) \frac{\partial B}{\partial G} - i\gamma B - i\Omega_{k\xi} B - i\Omega_{\xi\xi} \left( \frac{\partial \xi}{\partial k} \right) B = 0 \quad (4.3.3c)
\]

Equation (4.3.3a) is a single equation representing the evolution of a wavetrain in a nonuniform medium. Once the complex phase \( \phi(\xi, \tau) \) in the oscillating wavetrain is defined, the local wavenumber \( k \) and local frequency \( \omega \) can be derived through the derivatives \( k = -i\partial \phi/\partial \xi \) and \( \omega = \partial \phi/\partial \tau \). Furthermore, \( k \) and \( \omega \) satisfy the complex dispersion relation \( \omega = \Omega(k, \xi) \).

The other two equations are simply carried along in order to determine the amplitude of the disturbance. Equation (4.3.3c) describes changes in wave energy density in the nonuniform medium in terms of spatial divergence of the group velocity \( \partial g/\partial \xi \) in the physical plane \( (\xi, \tau) \). Changes in amplitude \( A(\xi, \tau) \) follow from equation (3.2.7). The remaining equation (4.3.3b) is added simply to solve for the inverse gradient \( \partial \xi/\partial k \), and this quantity is used to determine the spatial divergence of the rays. The above equations (4.3.3) may be written in vector form

\[
\frac{\partial U}{\partial T} + A \frac{\partial U}{\partial G} + B = 0 \quad (4.3.4)
\]

Where \( U \) and \( B \) are three-dimensional vectors and \( A \) is a \( 3 \times 3 \) matrix defined by

\[
U = \begin{bmatrix} \phi \\ \frac{\partial \xi}{\partial k} \\ B \end{bmatrix}, \quad B = \begin{bmatrix} -\Omega \\ -i\Omega_{kk} - 2i\Omega_{k\xi}(\frac{\partial \xi}{\partial k}) - i\Omega_{\xi\xi}(\frac{\partial \xi}{\partial k})^2 \\ -i\gamma B - i\Omega_{k\xi} B - i\Omega_{\xi\xi} \left( \frac{\partial \xi}{\partial k} \right) B \end{bmatrix},
\]
4.3.2. Discretization and Initialization of the Flow Parameters

The difficulty in finding a solution for our simultaneous set of equations lies in the fact the wave packets convect with the flow and spread with time in their streamwise extent. One needs some how to guess the position of the disturbance in the flow field in order to obtain the solution efficiently. As extracted from the study of the parallel shear, we are confident that the packet will always lie within a wedge region as represented on an (ξ, τ) diagram see (Figure 3a). By applying the change of variables proposed earlier, the wedge region is mapped into a rectangular computational domain more suitable for a finite difference technique see (Figure 3b).

The discretization points in the G direction are distributed equally [ΔG = 1/(NG - 1) = const.] between the trailing edge and the leading edge of the packet, defined by G = 0, and G = 1 respectively for the linear base velocity profile. For the general shear layer, the trailing edge stands at G = 0.125, and the leading edge is located at G = 0.875. NG is the total number of points that span the width of the wave packet. In our numerical results, we selected NG as odd numbers in the range of 26 ≤ NG ≤ 51, or 0.02 ≤ ΔG ≤ 0.05. The ability to succeed with the task at hand depends on the finite difference scheme used. We will use the well tested MacCormack’s second-order noncentered scheme (MacCormak 1971) as it applies to equation (4.3.4). This method is usually used for problems with discontinuities, such as shock capturing problems. However, the method also gives excellent results for smooth solutions problems. In addition, the MacCormack scheme presents several other attractive advantages. The advantages of noncentered methods over the more conventional centered schemes are: programing logic is simpler, nonhomogeneous
terms or source terms (B) are easily included, and the extension to multidimensional problems is quite trivial. The MacCormack's scheme for this problem translates into

\[
U_{i}^{n+1} = U_i^n - \frac{\Delta T}{\Delta G} A_i^n \left[ U_i^n - U_{i-1}^n \right] - \Delta T B_i^n \tag{4.3.5a}
\]
\[
U_{i}^{n+1} = \frac{1}{2} \left[ U_i^n + U_i^{n+1} - \frac{\Delta T}{\Delta G} A_i^{n+1} \left[ U_{i+1}^{n+1} - U_{i-1}^{n+1} \right] - \Delta T B_i^{n+1} \right] \tag{4.3.5b}
\]

where

\[
U_i^n = U(n\Delta T, i\Delta G) \tag{4.3.6a}
\]
\[
A_i^n = A(n\Delta T, i\Delta G) \tag{4.3.6b}
\]
\[
B_i^n = B(n\Delta T, i\Delta G). \tag{4.3.6c}
\]

The step \( n+1 \) is termed the predictor step and the step \( n+1 \) the corrector step. It should be emphasized that the spatial derivative at \( i = 1 \) and \( i = NG \) will be determined by one-sided differences. Although the accuracy of the method decreases by one order by using the one-sided finite differences, it is not of great concern to us as those two points in question \( i = 1, NG \) will always lie outside the exponentially growing part of the packet.

As far as the initial data is concerned, we will use the same conditions as for the ray tracing method, and they have been discussed in detail in section 4.2.1. In addition, it is important to mention that the independent variable transformations in (4.3.2), introduces a singular point at \( T = \tau = 0 \). This singularity can be removed by letting the wave packet travel in a parallel mixing layer initially, say for \( (0 < T < 5) \). Knowing that for parallel flows the lines of constant \( G \) corresponds to the rays along which \( k, \omega \) and \( B \) are conserved, with \( \phi = ik\xi + \omega \tau \), we can start the integration at any finite \( T_0 \neq 0 \), \( (0 < T_0 < 5) \) with the same initial conditions as at \( T = 0 \) for all variables but the phase.

4.3.3. Stability Analysis

It is necessary in numerical method techniques to select a step size small enough to comply with the stability bound, but yet large enough to complete the
computation within a reasonable amount of computer time. To get a bound on the value of the step size, many authors in the literature have recourse to amplification matrix theory because it is easy and quick to implement and also because of its conservative nature in predicting a stability condition. The method is based on a locally linear analysis of the governing partial differential equations coupled with a discrete harmonic analysis of the linear difference scheme. The partial difference equation system (4.3.4) is
\[
\frac{\partial U}{\partial T} + A \frac{\partial U}{\partial G} + B = 0
\]  
(4.3.7)
The amplification matrix theory for the two-dimensional space \((T, G)\) requires that
\[
\frac{\Delta T}{\Delta G} \leq \frac{1}{\sigma_p}
\]  
(4.3.8a)
\[
\sigma_p = |\sigma(A)|_{max}
\]  
(4.3.8b)
where \(\sigma_p\) is defined to be the local maximum modulus of the eigenvalues of the matrix \(A\) for a given point in the field. For our particular problem since the matrix \(A\) is diagonal, the three eigenvalues are the diagonal elements of matrix \(A\),
\[
\sigma_1 = \frac{-\Delta U G - U_2}{\Delta U T}, \quad \sigma_{2,3} = \frac{g - \Delta U G - U_2}{\Delta U T}.
\]  
(4.3.9)
In practice equation (4.3.8a) is replaced by
\[
\frac{\Delta T}{\Delta G} = \frac{\text{const}}{(\sigma_p)},
\]  
(4.3.10)
Where \(\text{const} < 1\) can be varied during the computations and is usually assigned a value of approximately 0.9. When applying (4.3.10) to determine \(\Delta T\), the mesh spacing \(\Delta G\) is given. It is therefore necessary to determine the minimum \(\Delta T\) among all the mesh points. This minimum \(\Delta T\) which is calculated at each step is the one used for the succeeding integration step.
CHAPTER 5

RESULTS AND DISCUSSION

The principal goal of this study has been to investigate the propagation of inviscid instability waves in diverging mixing layers. The analysis is done using kinematic wave theory. Several authors have demonstrated that kinematic wave theory is quite satisfactory to describe linear instability waves in parallel flows. For the non-parallel or diverging case, equations (3.5.1) prove that the theory is quite extendible to analyse the basic flow parameters; of wavenumber, frequency, and phase.

The case of the amplitude propagation of the wavetrain in a nonuniform medium is a more complicated. It involves an analog Whitham's law of conservation of wave action density. The propagation of such a density is represented in (3.2.6).

When it comes to numerical results, one can say that the principal objective of this dissertation has been to determine in what measure one can really feel comfortable with the idea of ray-tracing for instability waves in inhomogeneous media. Several authors have wrestled with the concept of complex rays; Landahl (1982), Itoh (1981), without making comparative numerical calculations to bring a certain degree of confidence to the complex ray-tracing method. In our present work, we verify satisfactorily the results given by the method of characteristics in the complex plane with two other independent methods.

Even though our focus was on the influence of base flow divergence on the propagation of Rayleigh instability waves, our multiple scale analysis directed us also to the analysis of these waves in parallel flows. The wave packet in parallel flow serves as initial condition to our diverging flow. The parallel flow analysis also involves the investigation of the receptivity of the flow. The receptivity of a flow is defined as the relationship between the mode amplitude and the initial disturbance. In this work, we direct our attention at two types of free shear layers; the piecewise
linear profile or profile of Rayleigh, and the general base velocity profile or so-called "tanh" profile. The receptivity of the profile of Rayleigh both in terms of wave packets and spatial instability waves has been studied by (Balsa 1988), so we restrict our receptivity investigation to the tanh profile.

5.1. Wave Packets

The wave packet resulting from the initial disturbance in the flow is represented by equation (2.2.4). In addition to the complex amplitude $F_1(k)$, we need the phase function $h(k, G)$ which involves the dispersion relation. It is clear that the wave packet has the form of a dispersive and exponentially growing wavetrain. In Figure 4, we have plotted the actual transverse velocity disturbance as represented in (2.2.4) for a value of $\tau = \Delta U t = 50$, as a function of the canonical group velocity $G$. The plot clearly indicates the sinusoidal behavior of the packet and the large fractional changes in amplitude over one wavelength. The wave packet has a definite finite extent that is delimited by the exponential growing part of the solution. For the particular numerical example selected, $U_1 = 1.5, U_2 = 0.5, (\Delta U = 1)$ and $\tau = 50$, the width of the packet is about $2.6 \lambda_c$, where $\lambda_c$ is the wavelength at the center of the packet, $(\lambda_c \approx 14.2)$. As time increases, the appearance of the wave packet remains essentially the same, except of course the amplitude becomes larger. Incidentally, the extent of the packet is always bounded by the values $0 \leq G \leq 1$, where the equality signs refer only to the piecewise linear profile case.

The evolution of a wave packet in a mixing layer is determined by its growth rate $\text{Re}[h(k_o)]$ and its Doppler-shifted frequency $\text{Im}[h(k_o)]$ as seen by an observer moving with the canonical group velocity. The variation of the above mentioned quantities with $G$ is shown in Figures 5a,b. The two sets of curves show that there is very little difference between a wave packet in a mixing layer with the base profile of Rayleigh and with the tanh profile. Both the growth rate and the width of the packet are slightly larger in the piecewise linear base flow case, and this is due to the fact that the two limiting external streams $U_1$ and $U_2$ are reached at finite values of $y$. Indeed for the Rayleigh profile with $U_2 = 0$, the trailing edge of the packet...
remains at the origin \( G = 0 \) whereas for the tanh profile it drifts downstream at a small nonzero velocity. The Doppler-shifted frequencies are practically identical throughout a wide portion of the packet; there are slight differences at the edges. The discrepancies at the two edges are a direct result of the discontinuities in the slope of the profile of Rayleigh at the edges of the mixing layer. We note also a symmetry in the growth rate curves about the center of the wavetrain \((G = 0.5)\). This point corresponds to the average velocity \((U_1 + U_2)/2\) which is also a point of antisymmetry for the base velocity profiles. At \( G = 0.5 \), the center of the wave packet grows at the maximum growth rate of temporal stability theory, equivalently the saddle point wavenumber \( k_0 \) is purely real at this location.

The curves of Figures 5a,b exhibit great similarity, but yet the two flows have some fundamental differences. Specifically, for the Rayleigh and tanh profiles the boundary between absolute and convective instability occurs at different values of the velocity ratio. For the profile of Rayleigh, the limiting velocity ratio is \( R_t = 1.0 \), whereas \( R_t = 1.315 \) for the tanh base velocity profile. In other words, the flow is convectively unstable when the velocity ratio is smaller than \( R_t \), otherwise the flow is absolutely unstable. Absolute instability implies that the impulse response to the flow becomes unbounded for large time at all spatial points. On the other hand, when a flow is convectively unstable, its impulse response decays to zero for all points in space at large enough values of time.

To illustrate the receptivity of the shear layers, we have plotted in Figure 6 the actual transverse disturbance velocity as a function of \( G \) and at \( \tau = 50 \). The results are shown for four different values of the source transverse location \( y_0 \). One can observe that both mixing layers display similar responses for the different \( y_0 \) values, notably that as the dipole vortex originally at \( y_0 = 0 \) is moved closer to the upper edge of the layer, the perturbation velocity at the interface continuously decreases. The striking observation is that, while for maximum response one must place the dipole vortex at the center of the layer \((y_0 = 0)\), a shear layer can be substantially excited by a pulse which lies outside the layer \((y_0 = 2)\). It is apparent on Figure 6 that in both examples, the magnitude of the disturbances are of the same order, with the piecewise linear flow response being a little larger probably
due to its larger growth rate. The four graphs in this figure also indicate that the transverse velocity perturbation of the tanh flow is out of phase with the one for the Rayleigh profile. This phase shift can be attributed entirely to the complex amplitude $\mathcal{F}_1(k_0)$ since we have shown earlier that both flows have almost identical Doppler frequencies.

5.2. Spatial Instability Modes

Shear layer instability has been investigated extensively over the years. Experimental studies have shown that the fundamental structures of this instability is best described by spatial modes when the flow is convectively unstable. For two-dimensional disturbances, the $x$-$t$ structure of a spatial mode is represented mathematically by $e^{i\omega t + ikx}$, where $\omega$ is real and $k$ is in general complex; they are both connected by the dispersion relation in (2.3.7b).

The spatially unstable mode was generated by a sinusoidal excitation of frequency $\bar{\omega}$, at $x = 0, y = y_0$ and turned on at $t = 0$. This configuration depicts the conditions present during a laboratory experiment. We approach the solution of this problem by the Fourier-Laplace transforms method. Mathematically then, the spatial instability mode arises as the residue from the pole at $\bar{\omega}$ in the complex frequency plane. We are now ready to define the receptivity of the flow as it relates to spatial instability. We define a receptivity factor $\mathcal{R}$, where

$$\mathcal{R} = \left| \frac{\Delta U}{\Delta x} \right| \left( k_0 \right). \quad (5.2.1)$$

This definition for the receptivity factor gives us a relation between the transverse component of the velocity $v(x,y,t)$ and the forcing frequency $\bar{\omega}$, for parameter values of the dipole location $y_0$, velocity ratio, etc.

The three solid curves in Figure 7 illustrate the receptivity of the flow, just mentioned in the previous paragraph. Concentrating for a moment on the curve for $y_0 = 0$, we deduce that the mixing layer is very receptive to disturbances at high frequencies. In fact, we see that the flow is about ten times more receptive
at \( \dot{\omega}/\Delta U = 0.625 \) than at \( \dot{\omega}/\Delta U = 0.1 \). Incidentally the graph also depicts the receptivity of the piecewise linear shear layer at \( y_0 = 0 \). The only difference between the two types of flows is the fact that the receptivity factor \( R \) of the piecewise linear is consistently larger for all frequencies. Also the sharp increase as we approach the neutral frequency was only visible for \( y_0 = 0 \) in the case of the general base velocity profile.

The three lower curves of Figure 7 also confirm a result obtained for a wave packet; namely that mixing layers are more responsive to disturbances generated at their center lines \( (y_0 = 0) \). This observation is also explainable with the physical argument that a flow is most sensitive to perturbations originated at a point near the region where the phase speed of the disturbance is equal to the fluid speed. Lastly, the divergence of the three curves as the forcing frequency increases, reveals that the receptivity variation as a function of source location is more significant at higher frequencies.

A more visible effect of dipole location on the receptivity of the flow is provided by Figure 8. It shows the variation of \( R \), as a function of \( y_0 \), for approximately the most unstable excitation frequency \( \dot{\omega} = -0.4 \). The most interesting observation is the nonsymmetry of the curves with respect to \( y_0 = 0 \). This phenomenon is indicative of a bias in the receptivity of the spatial instability mode toward the low-speed side of the layer. The conclusion to draw from this is that perturbations at the slow-speed side result in larger disturbances in the flow than those in the high-speed side.

5.3. Diverging Shear Layers

Waves traveling in a slowly diverging inhomogeneous unstable medium can be partially characterized by dispersion relation connecting frequency and wavenumber that reflects this inhomogeneity. These waves are not conservative and the use of classical kinematic wave theory to describe the evolution of wave packets in such flows has long been controversial. Unlike the analysis of a wave packet in parallel flow, the ray-like character of the disturbance occurs in complex space. Following
the study of (Itoh 1980), we use the concept of complex characteristics to compute the trajectories of rays of in instability waves in a diverging free shear layer. In fact, the complex rays lie in complex position space \( \zeta \) and they are attributed physical meaning, only when their paths cross the real axis.

5.3.1. **Piecewise Linear Base Flow Profile**

To demonstrate the validity of this complex ray-tracing solution, comparisons were made with results given by two other independent methods. The first method is discussed in section 2.4. and its implementation is provided in section 4.1. It consisted of reducing the wavetrain to a Fourier integral and finding an asymptotic solution of this integral. The second method in section 4.3. was a fully numerical one, where we solved directly the system of partial differential equations governing the evolution of a wave packet in a nonuniform medium.

Similarly to the parallel shear layer, the evolution of a wave packet in a diverging mixing layer is also governed by the variation of its growth rate and Doppler-shifted frequency as seen by an observer moving with the group velocity \( g \). As a test of our different methods of solution, we have plotted on Figure 9a,b the growth rate and the Doppler frequency as a function of \( g = \xi/\tau \). The results displayed are for a parallel base profile of Rayleigh (\( \epsilon = 0 \)). We have perfect agreement among all three methods, and they also match perfectly with the results of (Balsa 1988).

With this test completed positively, we can proceed with some confidence to perform calculations for the diverging mixing layer. A comparison between the parallel and diverging free shear layers leads us to discover in Figure 10a that a nonhomogeneous mean flow is less unstable than its homogeneous counterpart. Its growth rate is considerably smaller throughout the packet. We also point out that in the case of the diverging mixing layer, the growth rate of the disturbance is reduced as time elapses, while it remains constant for the parallel base flow, (see Figure 11). The Doppler-shifted frequency as displayed by Figure 10b varies more gradually for the non-parallel flow case. Finally, another perhaps more subtle difference between
the two flows is the loss of the symmetry of these curves about the point of average velocity, \( g = (U_1 + U_2)/2 \). A physical explanation for this difference is the fact that the thickness of the mixing layer is significantly different between the trailing half and the leading half of the packet, resulting in the destruction of the antisymmetry of the base flow about the point of average velocity in the wavetrain.

The most interesting trend of our slowly spreading mixing layer is represented on Figure 12. The graphs show the variation of the complex amplitude \( A(\xi, \tau) \) across the packet. Here the parallel flow theory predicts larger values for both the modulus and the phase of \( A \). The amplitude is fairly constant for all wavenumbers in the wave packet, except at its edges. The sharp increase at the leading edge is probably particular to this specific base velocity profile. As was expected, we see on the Figures 12a,b also that the amplitude level decreases for longer time. As a matter of fact, equation (2.2.4a) indicates that the amplitude goes like \( t^{-1/2} \) for parallel flow. Naturally the effect of this decline of the amplitude with respect to time will be more consequential in the inhomogeneous flow, since for this case the wavetrain has a smaller exponential growth.

The previous remarks are made more explicit in Figures 13 and 14. The first figure depicts the transverse velocity \( v(x, 1, t) \) of the total disturbance in a parallel mixing layer as a function of the velocity \( g = \xi/\tau \), at times \( t = 40, 50 \). It is quite obvious that the wave has grown significantly in time. By contrast, in the diverging shear layer (\( \epsilon = 0.05 \)) of Figure 14, we see that the packet has not grown at all in amplitude, and at certain points around the leading edge it has even a lower amplitude. This is due largely to the behavior of the complex amplitude function, because at \( t = 50 \) the disturbance still experiences a positive growth rate. In comparing the two sets of curves it is quite clear that the disturbance in the parallel flow is about sixty times larger than the one in the diverging shear layer. In addition, the wavelength of the wave packet in the parallel shear layer at time \( t = 40 \) is about 13, \( (\lambda_p \approx 13) \). For the diverging base velocity profile however, the wavelength of the packet is almost twice as large as \( \lambda_p \); \( (\lambda_d \approx 25) \). This leads us to conclude that a wave packet evolving in a diverging mixing layer, increases in
wavelength $\lambda$ or decreases in wavenumber $k$ as compared to a packet evolving in a parallel shear layer.

5.3.2. General Base Flow Profile

The whole theory in this dissertation is contingent on the existence of a dispersion relation. For the tanh base velocity profile, we need to obtain such a relation numerically. This exercise is reduced to the solution of the eigenvalue problem of Appendix B.

A result for the eigenvalues is given in Figure 15, where we plot the distribution of wavenumbers for each mode present in the wave packet for a parallel flow ($\epsilon = 0$). We see that the wavenumbers have identical real parts and opposite imaginary parts on each side of the most amplified mode, (the one whose wavenumber is purely real). This behavior implies a symmetry about $G = 0.5$ that is also apparent in Figures 16a,b,c and 17a,b,c. Figures 16 show the moduli of the eigenfunctions of different disturbance quantities ($p, v, Q, W$) situated at three different stations across the packet ($G = 0.3, 0.5, 0.8$), (ie. for different wavenumbers). Figures 17 on the other hand display the corresponding phases of these quantities. The normalization of the mode shapes are performed on the pressure and is $p = 1$ at $\eta = 0$. Upon close examination of these curves, one notices that $p$ and $v$ maintain the same magnitude throughout the packet. The mode $v$ has two peaks that translate from one side to the other of the point $\eta = 0$, depending whether $G$ is greater or less than 0.5. The amplitudes of $Q$ and $W$ are much larger than those of $v$ and $p$. This difference can be explained by pointing out that $Q$ and $W$ represent fluid particle displacements, which can be obtained by dividing the respective perturbation velocity components by $(U - c)$. Finally, the computed phase profiles are shown on Figures 17a,b,c. The most remarkable thing in these plots is that the $p$-phase is practically constant.

In the case of the diverging free shear layer, we see in Figures 18a,b that the symmetry with respect to center of the packet reported for the parallel case is destroyed. Comparing the curves on Figure 18b, we notice that the skewness is more pronounced as the flow inhomogeneity parameter $\epsilon$ is increased. Even on the
plot for the variable $K = kL(\zeta)$, we observe the same distortion (see Figure 18a), and this due to the fact that the layer continuously spreads in a distance from the trailing edge to the leading edge of the packet. As far as the series of curves on Figures 19 and 20 is concern, there is nothing of great interest to report except maybe to point out that the question of nonsymmetry still persists although not very visible on the plots with the scales selected.

The remaining part of this work will focus on the diverging tanh base velocity profile. The results will be presented as a comparative study of the solutions by the ray-tracing method and the numerical method. For this purpose, we have selected the the base flow with velocity ratio $R = 1$, $(U_1 = 1, U_2 = 0)$. Under this configuration, the flow is convectively unstable (Huerre and Monkewitz 1985), and this carries with it the whole concept of spatial instability (as opposed to temporal instability).

The results for a diverging mixing layer with $\epsilon = 0.05$ show remarkable agreement between the two methods; ray-tracing and numerical solution of the pde's. Figures 21, 22 and 23 represent the growth rate and the Doppler-shifted frequency for such a shear layer. Here again we see the reduction of growth rate with time due to the inhomogeneity of the base flow. This reduction is the largest for a finite band of frequencies around the center of the packet. This leads us to conclude that the most amplified modes in the wave packet are the one that are subject to the largest relative amplitude decrease. As far as the doppler frequency is concerned, on the contrary, the most amplified waves see their phases unchanged for longer times.

We compare further results from the two methods in Figures 24, 25 and 26. The transverse velocity of the disturbance $v(x, 0, t)$ at times $t = 40, 50$ show good agreement throughout the length of the wave packet. There are some differences at the two edges of the wavetrain, however. For these locations in the flow, we encountered some difficulties with the numerical method. We believe that these difficulties are inherent to the second-order spatial differencing we have employed. Comparing the solutions at $t = 40$ and $t = 50$, we see that the wave packet has convected a short distance of about 5 in the streamwise direction. At first glance
this seems to indicate that the whole packet translates downstream at a speed equal
equal to the average velocity of the external streams $U_{av} = 0.5$; this observation will
be confirmed momentarily. The whole wavetrain contains about the same number
of waves but its amplitude and its width have increased by a small amount. Figures
25 and 26 verify that the amplitude function plays an important role in containing
the total growth of the disturbance, specially for the most amplified modes. Thus
simply neglecting $A$ results in erroneous conclusions.

The most dramatic result for the wave action density is presented in Figure
28, where we see a rapid decrease of $|A|$ with respect to time. This behavior
can be approximated by $A \sim 1/t$, however this decrease of wave action is partially
counteracted by the exponential growth rate of the disturbance. For that reason, the
maximum value of the transverse velocity component experiences a mild increases
as time elapses. This behavior is actually a function of the amount of inhomogeneity
present in the mean flow, (see Figure 27). As expected for parallel flow ($\epsilon = 0.0$), we
have an exponential growth of the disturbance. By contrast when a small amount
of divergence is introduced, we notice that the total magnitude of the disturbance
is reduced. The lowest curve of the graph shows, that in principle, for higher values
of base flow inhomogeneity ($\epsilon = 0.10$), the perturbations can be made to decay soon
after their introduction in the flow. This decay is entirely due to the behavior of
the amplitude or the wave action density.

Finally, for completeness, we need to mention something about the convec-
tion of the wave packet in the (convectively unstable) flow $U_1 = 1.5$, $U_2 = 0.5$. In
Figure 29, we have traced the trailing edge (T.E.), the leading edge (L.E.) and the
line of maximum growth rate (C.E.) of a wave packet as it evolves in time in this
particular base flow profile. The increase in distance between the two solid lines
gives a good indication of the spreading of the packet. The slope of the trailing
edge (T.E.) of this wave packet is smaller than the one for a packet traveling in
the corresponding shear layer with a piecewise linear base velocity profile where it
would be $t/x = 2$. On the contrary, the slope of the leading edge (L.E.) of the
wave packet in this flow is larger than its value of $t/x = 2/3$ that it would have
in a free shear layer with a base velocity profile of Rayleigh. As a result, a wave
packet spreads more when it evolves in a free shear layer with a piecewise linear base velocity profile than it does when evolving in a tanh mixing layer. We point out that the bounding edges continuously diverge from each other; they enclose the exponentially large part of the wave packet. The fact that the trailing edge and the leading edge move in the same direction has a definite implication on the stability of the flow. The motion of the edges in this particular mixing layer implies that if we emit a wave packet at each instant of time with a continuously oscillating source, the oldest packets (emitted the earliest) are the farthest downstream and have the largest amplitude, and this leads to a spatial variation of the amplitude which is in fact a spatial instability mode.

A good indication of the speed of convection of the packet is the average distance its center travels in a unit amount of time. This speed is simply the inverse of the slope of the dotted curve in Figure 29. Comparing this speed to the real group velocity at the center of the wave packet in Figure 30, we arrive at a representation of the speed of convection of a wave packet in a mixing layer. We point out also in passing that that convective speed of the packet is identical to average fluid velocity between the two external velocities (here $U_{av} = 1$).

5.4. Convective and Absolute Instabilities

The convective and absolute characteristics of inviscid instabilities in a parallel shear flow have already been discussed in great detail in a previous chapter. Also, the reader is referred to the study by Huerre and Monkewitz (1985), where they classify conditions for these two types of instabilities by studying the singularities of the complex dispersion relation. In the interest of recollection, we repeat here briefly the descriptions of convective and absolute instabilities.

In physical terms, the initial disturbance, which is an impulsive excitation, is localized in space. This disturbance disperses into a wave packet of unstable modes. The packet decays rapidly upstream and downstream of its center. For a convectively unstable flow, the peak disturbance of the packet grows exponentially in time, but if measured at any fixed station $x$, it eventually decays in time because
the localized perturbation translates away. In fact at a far away location \( x \), the disturbance may appear to grow first and then die away as the center of the packet passes by. In contrast, for the case of an absolutely unstable flow, the impulse response becomes unbounded (grows exponentially) for large time at any point \( x \).

Intuitively, one might expect a flow to be convectively unstable only for base flows generated by co-flowing streams \((0 < R < 1)\). However according to Huerre and Monkewitz (1985), an hyperbolic tangent (tanh) shear layer is capable of absorbing a small amount of counter-flow before becoming absolutely unstable. The flow will remain convectively unstable as long as the magnitude of the reverse flow \( |U_2| \) is such that \( |U_1/U_2| < 0.136 \), or the velocity ratio \( R < 1.315 \). For the piecewise linear profile, (Balsa 1986) showed that the limiting velocity ratio is \( R_t = 1 \), that is to say that for the velocity profile of Rayleigh, the flow becomes absolutely unstable for \( R \geq 1 \).

5.4.1. Convectively to Absolutely Unstable Flow

In this section we study a spatially evolving mixing layer. We start with a convectively unstable shear layer \((R < 1.315)\), and gradually reach an absolutely unstable base velocity profile \((R > 1.315)\). This particular configuration can be representative of a mixing layer evolving in a region of adverse pressure gradient which leads to a separated and eventually a reversed flow. Experimentally this scenario could be realized by applying a suction on the slow-moving side of the free shear layer, although to our knowledge it has never been tried.

For our calculations, we have selected the following geometry:

\[
U(y) = U_m + \frac{\Delta U}{2} \tanh y
\]  

\( (5.4.1a) \),

where

\[
U_1 = 1.5 \quad U_2 = 0.5 - \varepsilon x.
\]  

\( (5.4.1b) \)

For this particular spatial variation of the shear layer, with a flow inhomogeneity parameter \( \varepsilon = 0.02 \), the point of transition from convective to absolute instability
is $x = 35.2$ as indicated by a vertical line on Figure 31. Upon close examination of Figure 31, the value of $x = 35.2$ seems to represent a vertical asymptote for the trailing edge of our wave packet. This value is larger than the value of $x = 25$ obtained for the piecewise linear profile whose trailing edge traces out the line given by the open circles on the figure. Clearly the dotted curve also indicates that the center of the packet (point of maximum growth) admits an asymptotic value as $t \to +\infty$. This result is significantly different from the case of the two co-flowing streams (Figure 29).

The contrast between the two geometries has some profound implications in the development of spatial instability modes. Earlier in this section, we discussed how we can arrive at a spatial mode by emitting a successive number of wave packets with an oscillating source. With our present configuration, initially the packets will grow in time and space creating then a spatial variation of amplitude with $x$. However for longer and longer time, the packets will start to accumulate at the asymptotic value of $x$ and they will stop convecting downstream. From that point on, we will not be able to observe any spatial modes. Instead the flow will be temporally unstable since the amplitude of the disturbance will continue to vary with time.

Our recent observations are also substantiated by the representation on Figure 32, where we display the speed of the wave packet represented by the real group velocity at the center of the disturbance, or the average speed of the center edge of the packet. It is obvious that the velocity of the wave packet experiences an exponential decay approaching in the limit a stationary state. Again we notice that the real group velocity of the most amplified wave is a perfect representative of the convective speed of the entire wave packet.

5.4.2. Absolute to Convective Instability Velocity Profile

The flow configuration that is of interest in this section is described physically as two counter-flowing streams under the influence of a favorable pressure gradient.
The pressure gradient acts to cancel slowly the region of reverse flow and finally turning the counter-flowing mixing layer into a co-flowing shear layer.

The analytical representation of the base flow that we have just described is;

\[ U(y) = U_m + \frac{\Delta U}{2} \tanh y \quad (5.4.2a), \]

where

\[ U_1 = 1.5 \quad U_2 = -1 + \epsilon x. \quad (5.4.2b) \]

For this particular spatially varying mixing layer, the point of transition from \( R > 1.315 \) to \( R < 1.315 \) is \( x = 39.8 \) for \( \epsilon = 0.02 \). Even though the free shear layer varies from a counter-flowing one to a co-flowing one, the trailing edge and the leading edge of the wave packet continue to travel in opposite directions as time evolves as indicated in Figure 33. Therefore, the base flow remains absolutely unstable since all the perturbation quantities will grow in time without bound at a fixed \( x \) (see Balsa 1988). The trajectory of the center of the packet has a very small constant positive slope, indicating that the center edge translates at a very slow velocity (Figure 34). We notice also on Figure 34 that the real group velocity at the point of maximum growth is no longer a good representation of the speed of the wave packet.
CHAPTER 6

SUMMARY AND CONCLUSION

In this thesis we have studied the evolution of instability wave packets in diverging mixing layers. Recently, this subject has attracted a lot of interest because it is believed that the large scale coherent structures in unstable free shear flows can be predicted by linear stability analysis.

The flow was divided into two parts: a near flow solution and a far flow one (see Figure 1b). The near flow problem was handled using Fourier integral analysis which, in turn, was solved by an asymptotic method. The asymptotic solution described the perturbation field in the form of a wave packet evolving in a parallel shear layer. The near flow analysis provided not only information about the receptivity of the flow to localized disturbances, but it also served as the initial conditions for the far flow problem. The far flow solution described the evolution of a wave packet on a slowly diverging mixing layer and it was analysed using a WKBJ method. This expansion method led us quite naturally to kinematic wave theory which provided a description of the evolution of the flow basic parameters.

The receptivity part of the study was done in two different fashions; Receptivity as far as wave packets were concerned and receptivity with respect to spatial modes. From these two kinds of receptivity investigations, we concluded that mixing layers are receptive to perturbations introduced at their centers, but they are also responsive to a lesser extent to perturbations placed outside their layers. Additionally, we discovered that free shear layers are most receptive to higher forcing frequencies, specially when they are emitted near a region in the flow where the fluid speed is equal to the phase speed of the disturbance.

Concerning diverging mixing layers, the main thrust of our investigation was to test the concept of complex ray-tracing. Indeed the method of characteristics in the complex plane was verified satisfactorily with two other independent methods. These are the method of steepest descent and a fully numerical solution. The most
valuable discovery was the important role played by the amplitude function of a wave packet traveling in diverging mixing layer. It acts to contain the magnitude of the disturbance in a wavetrain and even forces a negative effective growth rate beyond a certain time.

Finally, we remind the reader that most of the results presented in this project were calculated from ray-tracing and fully numerical methods. It is our opinion that each method presents advantages and disadvantages, but the numerical method is preferable due to the fact that it is computationally cheaper. The ray-tracing approach is more robust, particularly for long time calculations. However it is quite expensive and would be even more so for three-dimensional disturbances. Its justification could be very difficult to explain, specially in light of the fact that for the long time solution where it is superior, the validity of the linear theory is questionable. The fully numerical approach is computationally much cheaper than the previous method, but runs into problems of numerical stability at relatively large times. The steepest descent method is only useful in determining certain parameters of the perturbations, (growth rate, doppler-shifted frequency), and its practicality is limited to flows with an explicit dispersion relationship like the piecewise linear profile of Rayleigh.
APPENDIX A

RAYLEIGH PROFILE'S DISPERSION RELATION

We sketch here the derivation of the dispersion relation (3.1.3) starting with a piecewise linear base velocity profile.

The Rayleigh equation for the piecewise linear profile in terms of the Fourier and Laplace transforms of the transverse perturbation velocity $\tilde{v}$ takes the form

\[
\left( U - i\frac{\Omega L}{K} \right) \left( \frac{\partial^2}{\partial \eta^2} - K^2 \right) \tilde{v} - U'' \tilde{v} = 0, \quad (A.1a)
\]

\[
K = kL(\xi), \quad \Omega = \omega L(\xi) \quad (A.1b)
\]

where $K$ is the axial wavenumber, $\omega$ is the frequency, $U = U(y, \xi)$ $U'' = \partial^2 U/\partial \eta^2$ and $\eta = y/L(\xi)$. The base velocity selected is reproduced here from (3.1.2a),

\[
U(y, \xi) = \begin{cases} 
U_1 = \text{const} & (\eta > 1), \\
U_m + \frac{1}{2} \Delta U \eta & (|\eta| \leq 1), \\
U_2 = \text{const} & (\eta < -1),
\end{cases} \quad (A.2)
\]

where $U_m = (U_1 + U_2)/2$, $\Delta U = U_1 - U_2$ and $L = L(\xi)$. The base velocity $U(y, \xi)$ is continuous across the interfaces at $\eta = \pm 1$. The principal task at hand is to solve (A.1) with decaying boundary conditions as $|\eta| \to +\infty$ and suitable matching conditions across the interfaces at $\eta = \pm 1$.

For this configuration, there are three regions of interest, and in each of them, the solution $\tilde{v}$ can be written as a linear combination of $\exp(\pm i K\eta)$. We have then,

\[
\tilde{v} = \begin{cases} 
Ae^{-K\eta} & (\eta > 1), \\
Be^{-K\eta} + Ce^{K\eta} & (|\eta| \leq 1), \\
Dc^{K\eta} & (\eta < -1).
\end{cases} \quad (A.3)
\]

Before we proceed with the matching at the interfaces, we define:

\[
[(\phi)] = \text{jump in } (\phi). \quad (A.4)
\]
So matching the transverse velocity $\tilde{v}$ at the interfaces $\eta = \pm 1$, we write,

$$[\tilde{v}(\eta = \pm 1)] = 0, \quad (A.5)$$

or,

$$(D - C)e^{-K} - Be^{+K} = 0, \quad (A.6a)$$

and,

$$(A - B)e^{-K} - Ce^{+K} = 0. \quad (A.6b)$$

From equations (A.6a, b), we obtain

$$D = C + Be^{2K}, \quad A = B + Ce^{2K} \quad (A.7)$$

With the above values for $A$ and $D$, the solution for $\tilde{v}$ can be expressed simply in terms of $B$ and $C$. So,

$$\tilde{v} = \begin{cases} (B + Ce^{2K})e^{-K\eta} & (\eta > 1), \\ Be^{-K\eta} + Ce^{+K\eta} & (|\eta| \leq 1), \\ (C + Be^{2K})e^{+K\eta} & (\eta < -1). \end{cases} \quad (A.8)$$

The equations for $B$ and $C$ come from the requirement that the Fourier and Laplace transforms of the pressure is continuous across the interfaces. This requirement can be readily satisfied by using the linearized $x$-component of the momentum equations. Algebraically, we write

$$[\tilde{p}(\eta = \pm 1)] = \left[U'\tilde{v} - \left(U - i\frac{\Omega}{K}\right)\frac{\partial \tilde{v}}{\partial \eta}\right](\eta = \pm 1) = 0, \quad (A.9)$$

or,

$$B\Delta U + 4(KU_2 - i\Omega)e^{+K} + C\Delta U e^{-K} = 0, \quad (A.10a)$$

and,

$$B\Delta U e^{-K} + C\Delta U - 4(KU_1 - i\Omega)e^{+K} = 0. \quad (A.10b)$$

In solving equations (A.10a, b) simultaneously, we arrive at the following functional relation between $\Omega$ and $K$:

$$\Omega(k, \xi) = -iKU_m + \frac{\Delta U}{4} \left[e^{-4K} - (1 - 2K^2)^{1/2}\right]. \quad (A.11)$$

Equation (A.11) represents the complex dispersion relationship connecting complex frequency and complex axial wavenumber for disturbances in an inhomogeneous piecewise linear base flow profile.
APPENDIX B

SOLUTION OF THE RAYLEIGH STABILITY EQUATION

The whole theory of this study is really contingent on the existence of a dispersion relation. Numerically, the problem is simply reduced to the solution of the eigenvalue problem of equation (3.1.4) that we reproduce here as

\[ \frac{d^2 p}{d\eta^2} - \frac{2f'}{f - \hat{c}} - K^2 p = 0, \quad (B.1a) \]

where

\[ \eta = y/L(\xi) \quad (B.1b) \]
\[ K = kL(\xi) \quad (B.1c) \]
\[ \hat{c} = c - 2U_m/\Delta U, \quad c = \omega/k \quad (B.1d) \]
\[ f = f(\eta) = \tanh \eta \quad (B.1e) \]
\[ f' = \frac{df}{d\eta}. \quad (B.1f) \]

Note that the base flow velocity is represented by

\[ U(\eta) = U_m + \frac{1}{2} \Delta U f(\eta). \quad (B.2) \]

The function \( L(\xi) \) indicates the inhomogeneity of the base flow. The problem is completed with the following boundary conditions;

\[ p(\eta \to \pm \infty) = 0. \quad (B.3) \]

The numerical solution of this problem consists first on finite differencing equation (B.1a), this will generate a system of homogeneous linear algebraic equations with for unknown the canonical phase velocity \( \hat{c} \), which is also the complex eigenvalue sought.
Before we discretize our governing equation (B.1a), let us first introduce the following second and first-order finite difference derivatives,

\[
\frac{d^2 p}{d\eta^2} = \frac{p_{j+1} - 2p_j + p_{j-1}}{h^2}, \quad (B.4a)
\]

\[
\frac{dp}{d\eta} = \frac{p_{j+1} - p_{j-1}}{2h}, \quad (B.4b)
\]

where \( h \) denotes the grid spacing. With the differentiation rules just specified, our matrix form equation yields,

\[
Dp = \begin{pmatrix}
D_{1,2} & D_{1,3} & \cdots & \cdots \\
D_{2,1} & D_{2,2} & D_{2,3} & \cdots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
\cdots & \cdots & D_{N-1,1} & D_{N-1,2} & D_{N-1,3} \\
\cdots & \cdots & \cdots & D_{N,1} & D_{N,2}
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
\vdots \\
p_{N-1} \\
p_N
\end{pmatrix} = 0 \quad (B.5a)
\]

where,

\[
D_{j,1} = 1 + \frac{f'}{f - c} h, \\
D_{j,2} = -2 - K^2 h^2, \quad 2 \leq j \leq N - 1, \\
D_{j,3} = 1 - \frac{f'}{f - c} h. \quad (B.5b)
\]

On the other hand, for \( j = 1, N \), the boundary conditions at \( \eta \to \pm\infty \) require that

\[
D_{1,2} = 1, \quad D_{1,3} = -e^{-K_1 h} \\
D_{N,1} = -e^{-K_N h}, \quad D_{N,2} = 1. \quad (B.5c)
\]

As mentioned previously, \( h \) is the grid spacing and \( f' = df/d\eta \). Thus all the coefficients in the tridiagonal matrix \( D \) can be calculated explicitly.

Theoretically the eigenvalue is determined by setting the determinant of the coefficient matrix equal to zero. However, for the large system of equations that we have, typically 200 – 300, it is not practical to do so. Therefore, an alternate method will be used to determine the eigenvalues and corresponding eigenvectors.
In compact matrix notation, equation (B.5) gives

$$D(\hat{c})p = 0, \quad (B.6)$$

where $D$ is a tridiagonal coefficient matrix defined above and $p = (p_1, p_2, \ldots, p_N)$. Vector $p$ represents the $N$ unknown values of the pressure at each grid points. To avoid the trivial null solution ($p = 0$), we transform equation (B.6) by adding the same term on both sides of the equation and obtain

$$Dp + \sigma \begin{pmatrix} 0 \\ \vdots \\ p_0 \\ \vdots \\ 0 \end{pmatrix} = 0 + \sigma \begin{pmatrix} 0 \\ \vdots \\ p_0 \\ \vdots \\ 0 \end{pmatrix}, \quad (B.7)$$

where suitable values for $p_0$ and $\sigma$ will be discussed momentarily. The eigenfunction $p = (p_1, p_2, \ldots, p_N)$ is determined to within an arbitrary constant of proportionality. Therefore, $p_0$ can be set to the complex number $(1, 0)$ without loss of generality. We only replace $p_0 = (1, 0)$ on the right-hand side of (B.7). The right-hand side of the equation is now known, but $p_0$ is treated as an unknown on the left-hand side to be solved along with the other values $p_1, p_2, \ldots, p_N$. The eigenvalue $\hat{c}$ is still undetermined, its value will be found by insisting that it takes the value for which the solution of (B.7) with the known right-hand side has indeed $p_0 = (1, 0)$. In a matter of speaking $p_0 = (1, 0)$ is our compatibility condition.

The procedure described above is in a sense an iteration process for the eigenvalue $\hat{c}$. We are solving an inhomogeneous tridiagonal system of equations at each step, and this can be done very accurately and efficiently. The iteration uses a Newton-Raphson step at each adjustment of $\hat{c}$.

**Newton-Raphson Iteration**

We first guess a complex eigenvalue $\hat{c}$, calculate our coefficient matrix $D$ and solve the tridiagonal system of equations. Let the solution be $p$. Now repeat the above calculation after replacing $\hat{c}$ by $(\hat{c} + \Delta \hat{c})$, where $\Delta \hat{c}$ is a fixed, small
complex number. Now what we want to accomplish is to find a solution $p$ such that $p_{jin} = p(j = in) = 1$. In other words we are trying to find through a Newton-Raphson iterative process, the zero of the function

$$f(\hat{c}) = 1 - p_{jin}(\hat{c}) \quad (B.8)$$

which ensures that the compatibility condition is satisfied. This leads to the adjustment of the eigenvalue via the procedure

$$\hat{c} = \hat{c} + \frac{1 - p_{jin}(\hat{c})}{[p_{jin}(\hat{c} + \Delta \hat{c}) - p_{jin}(\hat{c})]/\Delta \hat{c}}. \quad (B.9)$$

Note that the derivative $f'(\hat{c})$, needed in the Newton-Raphson procedure, is determined numerically. We have convergence when $p_{jin} = 1$ within a prescribed a priori tolerance.
C.1. Formulation of The Problem

Let \( x \) denotes a cartesian coordinate system and \( t \) denotes time. Consider an incompressible basic flow velocity \( U = U(x, t) \). The basic flow satisfies the following nondimensional equations of motion:

\[
\frac{DU}{Dt} = F(x, t) \tag{C.1a}
\]

\[
\nabla \cdot U = 0 \tag{C.1b}
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla \tag{C.1c}
\]
denotes the total derivative in the unperturbed fluid motion. In the momentum equation (C.1a), \( F(x, t) \) represents either the fluid acceleration (kinematic interpretation), or more conventionally the applied forces, like the pressure gradient, the shear stresses or the body forces (dynamic interpretation). In any case, for this communication we assume \( U = U(x, t) \) to be given so that \( F(x, t) \) can always be calculated from (C.1a).

Now we introduce some arbitrary perturbations to our base flow in the form of \( u \) and \( p \). Assuming that the perturbation quantities represent inviscid disturbances, they satisfy the linearized Euler’s equations

\[
\frac{Du}{Dt} + u \cdot \nabla U = -\nabla p \tag{C.2a}
\]

\[
\nabla \cdot u = 0. \tag{C.2b}
\]

These instability waves of the base flow may be assumed to be inviscid if the Reynolds number of the base flow is larger than 500 (Betchov and Szewczyk 1963). The equations to be derived in a later section take a much simpler form (specially
the amplitude equation) if we introduce \( d(x, t) \) and \( p(x, t) \) as our new dependent variables (Balsa 1988). They are defined as

\[
u = \frac{Dd}{Dt} - d \cdot \nabla U. \tag{C.3}
\]

With the new variables, the perturbation equations (C.2a, 2.2b) assume the form

\[
\begin{align*}
\frac{D^2 d}{Dt^2} &= -\nabla p + d \cdot \nabla F \\
\nabla \cdot d &= 0
\end{align*} \tag{C.4}
\]

where \( F \) is the acceleration of the base flow and \( p \) is the perturbation pressure. \( d \) represents the distance that a particular fluid particle has moved due to the perturbations in the base flow.

### C.2. Slowly Diverging Base Flows and Assymptotic Analysis

In this section we consider slowly diverging base flows and write the functional dependence of the base velocity profile, \( U \), on the space coordinate, \( x \) and the time \( t \) as

\[
U = U(X, y, \tau), \tag{C.5}
\]

where the two subspaces \( X \) and \( y \) represent the so-called lateral and longitudinal coordinates (Bretherton 1968). The subspaces are mutually perpendicular and they add up to the physical space \( x \). We then assume that the dependence of \( U \) on \( X \) and \( t \) is through a small parameter, i.e. \( \xi = \epsilon X, \tau = \epsilon t, 0 < \epsilon \leq 1 \). The parameter \( \epsilon \) represents the nonhomogeneity of the base flow in the longitudinal space. Thus we write,

\[
U = U(\xi, y, \tau) \tag{C.6}
\]

The separation of the physical space into a lateral and a longitudinal space implies that the spacial differential operator \( \nabla \), becomes

\[
\nabla \longrightarrow \nabla_n + \epsilon \nabla, \tag{C.7}
\]

where $\nabla_n$ and $\nabla$ represent the gradient operator in the lateral and the longitudinal spaces respectively. The two operators are of order unity. Equation (C.6) represents a base flow which is slowly varying in the longitudinal space and in time. The mass conservation of the base flow, (C.1b), suggests that the lateral velocity component of the base velocity will be of order $\epsilon$. Therefore it is natural to rescale the base flow and write

$$U = U + \epsilon V,$$

(C.8)

with $U$ and $V$ representing the velocities of order unity in the longitudinal and the lateral spaces respectively. We are now ready to apply the expansion of equations (C.7) and (C.8) to the equations of motion governing the base flow (C.1a, 2.1b). First we expand the uniform flow acceleration $F$ as follows

$$F = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \cdots.$$  

(C.9a)

Equations (C.1a, 2.1b) yield

at $O(\epsilon^0)$:

$$F^{(0)} = 0$$

(C.9b)

at $O(\epsilon)$:

$$\frac{\partial U}{\partial \tau} + (U \cdot \nabla + V \cdot \nabla_n)U = F^{(1)}$$

(C.9c)

$$\nabla \cdot U + \nabla_n \cdot V = 0$$

(C.9d)

at $O(\epsilon^2)$:

$$\frac{\partial V}{\partial \tau} + (U \cdot \nabla + V \cdot \nabla_n)V = F^{(2)}.$$  

(C.9e)

The equations above indicate that $F^{(1)} = F^{(1)}(\xi, y, \tau) = F^{(1)}_t$ lies in the longitudinal space (indicated by the subscript $t$) and $F^{(2)} = F^{(2)}(\xi, y, \tau) = F^{(2)}_n$ is in the lateral space (indicated by the subscript $n$).

If the force $F$ is given by the negative of the pressure gradient of the mean flow plus the presence of viscous forces, that is

$$F = -\nabla P + \frac{1}{Re} \nabla^2 U.$$  

(C.10a)
When we choose $\epsilon = O(Re^{-1})$, with $Re >> 1$ representing the Reynolds number of the base flow, and after expanding $P$ as follows

$$P = P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \cdots,$$

(C.10b)

then from equations (C.7,2.7b) we obtain

at $O(\epsilon^0)$:

$$\nabla_n P^{(0)} = F^{(0)} = 0$$

(C.10c)

or $P^{(0)} = P^{(0)}(\xi, \tau)$;

at $O(\epsilon)$:

$$F_t^{(1)} = -\nabla P^{(0)}(\xi, \tau) + \nabla_n^2 U(\xi, y, \tau)$$

(C.10d)

where $\nabla_n^2$ denotes the Laplacian operator in the lateral space.

C.3. Linear Analysis

This section is dedicated to finding an approximate solution to equations (C.4a,b), for small values of the inhomogeneity parameter $\epsilon$. Before submitting a high-frequency ansatz and applying a WKBJ expansion, we exploit the lateral-longitudinal space decomposition and directly substitute equations (C.7) and (C.8) into the disturbance equations (C.4a,b). We obtain,

$$\epsilon^2 d_{\tau\tau} + 2\epsilon^2(U \cdot \nabla + V \cdot \nabla_n) d_{\tau} + \epsilon^2(UU: \nabla \nabla + 2UV: \nabla V: \nabla_n \nabla_n) d = -(\nabla_n + \epsilon \nabla) p - F \cdot (\nabla_n + \epsilon \nabla) d + d \cdot (\nabla_n + \epsilon \nabla) F$$

(C.11a)

$$(\nabla_n + \epsilon \nabla) \cdot d = 0.$$

(C.11b)

To repeat again $\tau = \epsilon t$ is a slow time, $\xi = \epsilon X$ is a slow space variable; $y$ is the lateral coordinate and $\nabla_n = \partial/\partial y$ and $\nabla = \partial/\partial \xi$ represent the gradient operators in the lateral and longitudinal spaces, respectively. $U(\xi, y, \tau)$ and $V(\xi, y, \tau)$ are the base flow velocity components in the spaces with the understanding that the base velocity profile is given by $U + \epsilon V$. Lastly we remind the reader that $F = \epsilon F_t^{(1)} + \cdots$ is the acceleration of the base flow.
We now apply the WKBJ expansion or high-frequency ansatz, a method that has been used by several authors (Lewis 1965; Itoh 1981a). The procedure is the decomposition of the disturbances into a fastly varying phase function $\phi(\xi, \tau)$ and a more slowly changing amplitude function $\hat{d}(\xi, y, \tau)$ and $\hat{p}(\xi, y, \tau)$. We write then

$$d = \hat{d}\exp(\phi/\epsilon) + c.c. \quad (C.12a)$$

$$p = \hat{p}\exp(\phi/\epsilon) + c.c., \quad (C.12b)$$

where c.c. represents the complex conjugate of all the terms explicitly written to the left of the plus (+) sign. The next step is to expand the amplitude functions

$$\hat{d} = d^{(0)} + \epsilon d^{(1)} + \cdots \quad (C.13a)$$

$$\hat{p} = p^{(0)} + \epsilon p^{(1)} + \cdots, \quad (C.13b)$$

and substituting the expansion above and equations (C.12) into equations (C.11), we obtain after collecting terms of $O(\epsilon^0)$ and $O(\epsilon)$, at $O(\epsilon^0)$:

$$\omega^2 d^{(0)} + 2i\omega \cdot U d^{(0)} - \nabla n p^{(0)} + \nabla \cdot \hat{d}^{(0)} + i k \hat{p}^{(0)} = 0 \quad (C.14a)$$

$$i k \cdot d^{(0)} + \nabla n \cdot d^{(0)} = 0, \quad (C.14b)$$

where we have defined $\omega = \phi_r = \partial / \partial \tau$ a (complex) frequency and $k = -i \nabla \phi$ a (complex) wavenumber in the propagation space, with $i = (-1)^{1/2}$. If we also introduce

$$\omega_0 = \omega_0(\omega, k) = \omega + ik \cdot U, \quad (C.15)$$

equations (C.14) simplify to

$$\omega_0^2 d^{(0)} + ik \hat{p}^{(0)} + \nabla n p^{(0)} = 0, \quad (C.16a)$$

$$i k \cdot d^{(0)} + \nabla n \cdot d^{(0)} = 0. \quad (C.16b)$$
Defining a linear operator \( \mathcal{L} \) acting on \( d^{(0)} \) and \( p^{(0)} \), we can rewrite (C.16) more compactly as

\[
\mathcal{L}(\phi, -i\nabla \phi)(d^{(0)}, p^{(0)}) = 0. \tag{C.17}
\]

The quantity \( \omega_0 \) is the Doppler-shifted frequency seen by an observer travelling with the base flow velocity \( U \) in the longitudinal space.

At the next order in \( \epsilon \) we obtain an inhomogeneous system of equations for \( (d^{(1)}, p^{(1)}) \), notably at \( O(\epsilon) \):

\[
\mathcal{L}(\phi, -i\nabla \phi)(d^{(1)}, p^{(1)}) = (R, R_c), \tag{C.18a}
\]

where

\[
R = -\frac{D_t \omega_0}{D_T} d^{(0)} - 2\omega_0 \frac{D_t d^{(0)}}{D_T} - (\nabla \cdot \nabla U \cdot \nabla \phi) d^{(0)} - 2\omega_0 \nabla \cdot \nabla d^{(0)} - \nabla p^{(0)} + d^{(0)} \cdot \nabla F^{(1)}, \tag{C.18b}
\]

\[
R_c = -\nabla \cdot d^{(0)}, \tag{C.18c}
\]

and

\[
\frac{D_t}{D_T} = \frac{\partial}{\partial T} + U \cdot \nabla. \tag{C.18d}
\]

The operator \( D_t/D_T \) indicates a substantial derivative in the propagation space. According to operator theory, since the homogeneous equations (C.17) have non-trivial solutions with null boundary conditions at \( y \to \pm \infty \), the operator \( \mathcal{L} \) is singular. With this condition, the inhomogeneous equations (C.18) will have solutions if and only if the right-hand side \( (R, R_c) \), satisfies the so-called solvability condition. This is done in the next section.

### C.4. Modal Analysis and Solvability

Primarily, we are interested in solving the modal version of equation (C.17), namely,

\[
\mathcal{L}(\omega, k)(d_m, p_m) = 0, \tag{C.19}
\]

where \( d_m \) and \( p_m \) vanishing as \( y \to \pm \infty \) and the subscripts \( m \) denoting 'mode'. Equation (C.19) and its boundary conditions constitute an eigenvalue problem with
a non-trivial solution existing if and only if the frequency \( \omega \), and the wavenumber \( k \) are related through a complex dispersion relation

\[
\omega = \omega(k, \xi, \tau), \tag{C.20a}
\]

which depends on \( \xi \) and \( \tau \) because of the inhomogeneity of the base flow. In turn since \( \omega \) depends on \( k \), the modes have the functional form

\[
d_m = d_m(y, \xi, \tau, k), \quad p_m = p_m(y, \xi, \tau, k) \tag{C.20b}
\]

The decomposition of the continuity and momentum equations leads us to also express \( d_m \) explicitly into its components in the cross and propagation spaces; we then write

\[
d_m = Q - iW, \tag{C.21}
\]

where like \( d_m \) and \( p_m \), \( Q = Q(y, \xi, \tau, k) \) and \( W = W(y, \xi, \tau, k) \). \( Q \) and \( W \) are vectors in the cross and propagation spaces respectively.

After substituting (2.21) into (C.19), we extract

\[
p_m = \omega_o^2 \frac{k \cdot W}{k \cdot k}, \tag{C.22a}
\]

\[
Q = -\omega_o^{-2} \nabla_n \left( \frac{\omega_o^2 k \cdot W}{k \cdot k} \right), \tag{C.22b}
\]

\[
W = \frac{k \cdot W}{k \cdot k}, \tag{C.22c}
\]

with

\[
\omega_o = \omega_o(\omega, k) = \omega + ik \cdot U. \tag{C.22d}
\]

In order to determine the quantities in (C.22), we need to solve for the scalar mode \( k \cdot W \) first. In fact, the scalar mode satisfies the Rayleigh stability equation

\[
\mathcal{R}(k \cdot W) = \nabla_n \cdot \left\{ \omega_o^{-2} \nabla_n \left[ \omega_o^2 (k \cdot W) \right] \right\} - k \cdot k (k \cdot W) = 0. \tag{C.23}
\]

We call \( \mathcal{R} \) the Rayleigh operator.
In real practice equation (C.23) associated with the boundary conditions at \( y = \pm \infty \) represents the eigenvalue problem alluded to earlier. Equation (C.23) will have a solution if and only if

\[
\phi_r = \omega (-i \nabla \phi, \xi, \tau). \tag{C.24}
\]

We interpret (C.24) as the eigenvalue of the problem representing the complex dispersion relationship which nothing but a relation connecting wavenumber and complex frequency. Equation (C.24) is a first order partial differential equation for the complex phase, its solution by several different methods is reserved for a later chapter.

The zeroth-order solution must be proportional to \( (d_m, p_m) \);

\[
d^{(0)} = A d_m, \quad p^{(0)} = A p_m, \tag{C.25}
\]

where \( A = A(\xi, \tau) \) is a slowly varying complex amplitude and \( d_m \) and \( p_m \) are the mode shape functions evaluated at \( k = -i \nabla \phi \). The evolution equation for the amplitude \( A \), is determined at the next order from the solution of \( (d^{(1)}, p^{(1)}) \) by enforcing the Fredholm alternative on the inhomogeneous terms \( (R, R_c) \) (see 2.18a). The solvability condition requires that we define an inner product of two scalar complex-value functions, say \( f \), and \( g \), or

\[
<f, g> = \int fg^* dy, \tag{C.26}
\]

where the integration is performed over the cross space and the \( * \) indicates complex conjugation. Before actually enforcing the solvability condition, we first transform the system of equations (C.19) into a single equation for the variable \( (\nabla \phi \cdot d^{(1)}) \), the final result of these operations yield

\[
R(-i \nabla \phi \cdot d^{(1)}) = (\nabla \phi)^2 R_c - \nabla_n \cdot \frac{R_n (\nabla \phi)^2}{\omega^2} + i \nabla_n \cdot [\omega_o^{-2} \nabla_n (R_t \cdot \nabla \phi)], \tag{C.27a}
\]

where we have written the vector \( R \) as

\[
R = R_n + iR_t, \tag{C.27b}
\]
which is its representation in cross space; $R_n$ and in propagation space $R_t$.

The solvability condition dictates that the right-hand side of (C.27a) must be orthogonal to the solutions of the adjoint Rayleigh operator $\mathcal{R}^*$ defined by

$$\mathcal{R}^*(h) = (\omega_o^*)^2 \nabla_n \cdot \left( (\omega_o^*)^{-2} \nabla_n h \right) - k^* \cdot k^* h$$

with

$$h = (\omega_o^*)^2 k^* \cdot W^*.$$  

(C.28a)

Continuing to enforce the solvability condition, the next step is to multiply the right-hand side of (C.27a) with the conjugate of $h$ and integrate over the cross space. Simplifications of the resulting equation are obtained along the way by successive integrations by parts and by invoking the Rayleigh equation, and we finally arrive at the very simple result:

$$\int (R_n \cdot Q - R_t \cdot W - R_{cpm}) dy = 0,$$

where $R_c, R_n, R_t$ have already been defined previously. Equation (C.29) represents a simple (complex) constraint that will be used to determine the complex amplitude $A(\xi, \tau)$ momentarily.

C.5. Amplitude Equation

In order to use equation (C.29) explicitly we need to use $\mathbf{R}$ from section (C.3) and $(d_m^{(0)}, p_m^{(0)})$ from section (C.4). The algebra will be summarized here in the interest saving space, then the different terms in equation (C.29) become

$$R_c = i\nabla A \cdot W + iA\nabla \cdot W$$

$$R_n \cdot Q = -A \frac{\partial \omega_o}{\partial \tau} Q \cdot Q - A U \cdot \nabla \omega_o Q \cdot Q - 2\omega_o \frac{\partial A}{\partial \tau} Q \cdot Q$$

$$- 2\omega_o U \cdot \nabla A Q \cdot Q - 2\omega_o A \frac{\partial Q}{\partial \tau} \cdot Q - 2\omega_o A U \cdot \nabla Q \cdot Q$$

$$- (V \cdot \nabla_n U \cdot \nabla \phi) A Q \cdot Q - 2\omega_o A V \cdot \nabla_n Q \cdot Q,$$
\[ R_t \cdot W = A \frac{\partial \omega_2}{\partial \tau} W \cdot W + A U \cdot \nabla \omega_2 W \cdot W + 2 \omega_2 \frac{\partial A}{\partial \tau} W \cdot W \]
\[ + 2 \omega_2 U \cdot \nabla W \cdot W + (V \cdot \nabla U \cdot \nabla \phi) AW \cdot W + 2 \omega_2 AV \cdot \nabla_n W \cdot W \]
\[ + iA \nabla p_m \cdot W + i p_m \nabla A \cdot W - iA Q \cdot F_{i}^{(1)} \cdot W. \quad (A.2c) \]

All that is left is to use the different terms above into (C.29) and multiply the whole equation by \( A \). The result of this operation after grouping certain terms together is,
\[
\int \left\{ \frac{\partial (\omega_2 A^2)}{\partial \tau} [Q \cdot Q + W \cdot W] + (\omega_2 A^2) \frac{\partial}{\partial \tau} [Q \cdot Q + W \cdot W] \right.
\[ + U \cdot \nabla (\omega_2 A^2 [Q \cdot Q + W \cdot W]) + A^2 (V \cdot \nabla_n U \cdot \nabla \phi) [Q \cdot Q + W \cdot W] \]
\[ + \omega_2 A^2 V \cdot \nabla_n [Q \cdot Q + W \cdot W] + ip_m \nabla A^2 \cdot W + iA^2 \nabla \cdot (p_m W) \]
\[ - iA^2 Q \cdot \nabla_n F_{i}^{(1)} \cdot W \right\} dy = 0. \]

Let us now introduce two complex quantities,
\[ A = A^2 \int \omega_2 (Q \cdot Q + W \cdot W) dy, \quad (C.30a) \]
and
\[ A_f = A^2 \int U \omega_2 (Q \cdot Q + W \cdot W) dy, \quad (C.30b) \]
which are known from \( A \) once the mode are calculated. The subscript \( f \) stands for flux and the only unkown in \( (C.30) \) is the complex amplitude \( A \). Armed with two new terms, \( (C.29) \) transforms into
\[ \frac{\partial A}{\partial \tau} + \nabla \cdot \left[ A_f + iA^2 \int W p_m dy \right] - iA^2 \int Q \cdot \nabla_n F_{i}^{(1)} \cdot W dy = 0. \quad (C.30c) \]

The coefficients of the equation above are also known from modal calculations and the base flow. Before we attempt a physical description of equation \( (C.30) \), let us first offer a different version of the equation. Our goal is to introduce the group velocity, and in order to do this, we differentiate the modal equations \( (C.19) \), with respect to the wavenumber vector \( k \), and dot the resultant equations from the left.
with an arbitrary vector, say \( \mathbf{K} \), in the propagation space. The final result is the following:

\[
\mathcal{L}(\omega, k)(\mathbf{K} \cdot \frac{\partial d_m}{\partial k}, \mathbf{K} \cdot \frac{\partial p_m}{\partial k}) = (R', R'_c), \tag{C.31a}
\]

where

\[
R' = -2\omega K \frac{\partial \omega}{\partial k} d_m - i K p_m, \tag{C.31b}
\]

\[
R'_c = -iK \cdot d_m, \tag{C.31c}
\]

\[
g = i\frac{\partial \omega}{\partial k} (\xi, \tau \text{ constant}). \tag{C.31d}
\]

g is the group velocity. The stability condition applied to (C.31) results after some simplifications to

\[
\int [U\omega(Q \cdot Q + W \cdot W) + iWp_m] dy = g \int \omega(Q \cdot Q + W \cdot W) dy, \tag{C.32}
\]

which is also called the basic identity of Lewis (1965) for local hyperbolic waves.

Comparing equations (C.32) and (C.30c) allows us to eliminate the argument of the divergence. The second version of our amplitude equation thus reads

\[
\frac{\partial A}{\partial \tau} + \nabla \cdot (gA) - iA^2 \int Q \cdot \nabla F_t^{(1)} \cdot W dy = 0. \tag{C.33}
\]

Equation (C.35) is in the form of a standard conservation law. Volume integrals of \( A \) are conserved in propagation space subject to the flux \( (gA) \) and the source term

\[
\sigma = iA^2 \int Q \cdot \nabla F_t^{(1)} \cdot W dy. \tag{C.34}
\]

For perfectly parallel and steady base flows,

\[
F = \frac{DU}{Dt} = 0, \tag{C.35}
\]

and for this case \( A \) obeys a pure conservation law (without a source term),

\[
\frac{\partial A}{\partial \tau} + \nabla \cdot (gA) = 0, \tag{C.36a}
\]

or,

\[
\frac{\partial A^2}{\partial \tau} + \nabla \cdot (gA^2) = 0. \tag{2.36b}
\]

This special result can also be obtained also by the steepest descents method, which shows that the group velocity is real and given by \( x/t \) (Gaster 1975). For inhomogeneous base flows, the group velocity is complex in general.
APPENDIX D

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