APPLICATIONS OF FINITE ELEMENT AND COMPUTER GRAPHICS TECHNIQUES IN AQUIFER ANALYSIS

by

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INTRODUCTION

Aquifer flow systems have been simulated by a variety of techniques. The results of these simulations can be presented in several ways. Of the several approaches in modeling groundwater flow, the finite element method offers an advantage through its ease in approximating various boundary conditions. Graphic methods offer means of simplifying data presentation. The purpose of this paper is to (1) demonstrate the use of a finite element technique in modeling aquifer flow systems, and (2) illustrate a 3-dimensional graphics approach when representing the results of the modeling.

A segment of the aquifer system at the Water Resources Research Center field laboratory in Tucson was modeled using a finite element scheme developed by Pinder and Frind (1972). The response of the modeled system was compared with the actual system by examining the head changes in wells during a two week constant discharge test on a pumping well. In addition to a tabular output of data, a graphic technique was used: The SYMVU Program of Computer Mapping (Laboratory for Computer Graphics and Spatial Analysis, Harvard University) which provides a three dimensional perspective of the head distribution at various times during the simulation of the two week test.

THEORETICAL DEVELOPMENT

The differential ground water flow equations may be transformed into approximate integral equations by Galerkin's procedure. The resulting equations can be efficiently evaluated by a finite element and numerical integration scheme. For our discussion of the Galerkin finite element method applied to the horizontal, two dimensional artesian, nonleaky aquifer, we assume the groundwater flow equation of the form

$$L(h) = \frac{\partial}{\partial x}(T_{xx}\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(T_{yy}\frac{\partial h}{\partial y}) - S\frac{\partial h}{\partial t} - Q = 0 \qquad (1)$$

where $Q = \sum_{i=1}^{n} Q_w(x_i,y_i)\ \delta[(x-x_i)(y-y_i)]$

represents the strength of the sink function and

- h is the hydraulic head,
- Q_W is the discharge from pumping well,
- S is the storage coefficient,
- δ is the dirac function, and
- T is the transmissivity tensor.

To approximate the solution of $L(h)=0$ by Galerkin's method, a trial function is assumed to be of the form $\hat{h}(x,y,t) = \sum_{i=1}^{n} C_i(t)V_i(x,y)$. The V_i's are known polynomials called basis functions. The C_i's are undetermined coefficients. The n basis functions are assumed to be part of a linearly independent, complete set of polynomials, V_i, $(i=1,2,\ldots)$. Essentially, this means that any solution

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of L(h)=0 can be represented as a linear combination of all the basis functions. Since only a part of the set of functions is used, the representation may only be approximate.

If $\hat{h}$ is an approximation, $L(\hat{h})$ is not zero but will be equal to some residual, R(x,y,t). Galerkin's procedure attempts to force this function to zero. This is equivalent to making R orthogonal to all basis functions because the zero function is the only function with this orthogonality condition. Since only n basis functions are used, we only require R be orthogonal to these n functions. Orthogonality of the residual to these functions is mathematically stated as:

$$\iint_D R(x,y,t)V_i(x,y)dxdy = \iint_D L\,[\hat{h}(x,y)]\;V_i(x,y)dxdy$$

$$= \iint_D L\,[\sum_{j=1}^{n} C_j(t)V_j(x,y)]\;V_i\;dx\;dy = 0,\quad i=1,2...n, \tag{2}$$

where D is the two dimensional domain under consideration.

To determine the C_i's, equation (1) is substituted into equation (2). The result is

$$\iint_D \{[\frac{\partial}{\partial x}(T_{xx}\frac{\partial}{\partial x}) + \frac{\partial}{\partial y}(T_{yy}\frac{\partial}{\partial y})]\sum_{j=1}^{n} C_j\;V_j - S\frac{\partial}{\partial t}\sum_{j=1}^{n} C_jV_j - Q\}V_i\;dxdy = 0. \tag{3}$$

Green's theorem is applied to equation (3) to remove the unwieldy second derivatives. The resultant equations can be written in the matrix form

$$[M]\;\{C\} + [N]\;\{\frac{dC}{dt}\} + \{F\} = 0 \tag{4}$$

where $M_{ij} = \iint_D (T_{xx}\frac{\partial V_i}{\partial x}\frac{\partial V_j}{\partial x} + T_{yy}\frac{\partial V_i}{\partial y}\frac{\partial V_j}{\partial y})dxdy$

$N_{ij} = \iint_D SV_iV_jdxdy$

$F_i = \iint_D V_iQ\;dxdy - \int_C V_i \sum_{j=1}^{n} (T_{xx}\frac{\partial V_j}{\partial x} 1_x + T_{yy}\frac{\partial V_j}{\partial y} 1_y)C_i(t)dS$

The last term in the expression for F_i represents the flux across the boundary of the domain and is considered known.

Galerkin's approximate integral equations for groundwater flow can be evaluated by numerical means. The suitability of these equations for computer evaluation is a result of the selection of the basis functions. To select these functions a set of n points or nodes is chosen in the two dimensional domain. The function $V_i(x,y)$ is chosen so that it is unity at the i^{th} node and zero at all other nodes. For convenience, we say V_i is associated with the i^{th} node. The domain is next divided into subdomains or elements such that all nodes lie on the boundaries of these elements. The domain must be divided in such a way that a side of one element meets, at most, one side of an adjacent element. This is necessary to assure continuity of the approximate solution along the interelement boundaries. It is further assumed that V_i, i=1,2 ...,n, is non zero only over the elements containing the i^{th} node. When chosen in this fashion the undetermined coefficient $C_i(t)$ equals the hydraulic head at the i^{th} node, because there are no other non zero contributing factors.

Elements used in the integration scheme introduced by Pinder and Frind (ibid.) are called mixed isoparametric quadrilateral elements (see Figure 1). The basic shape of the element is a quadrilateral, however, the sides can be distorted in certain ways. For this particular type of element there can be up to twelve nodes lying on the boundary of each element.

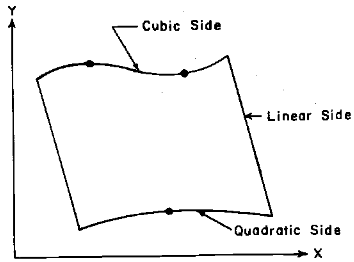


Figure 1. Deformed Mixed Isoparametric Quadrilateral Element. (After Pinder, ibid.)

By the way the domain is divided each node is associated with from one to four elements. A basis function associated with that node, therefore, has from one to four components called shape functions. A shape function which is identically zero outside a single element equals the basis function over that element. Thus the sum of all shape functions for a particular node equals the basis function for the node, and satisfies all requirements of the basis functions.

We let the V_i's in equation 4 represent the shape functions over each element and carry out the integrations over individual elements. The sum of all these "elemental integrations" give the same result as integrating the sum of all basis functions over the whole domain. These integrals are evaluated by the method of Gaussian Quadrature. This numerical procedure provides exact results for polynomial integrands like those appearing in equation 4. With Gaussian Quadrature it is usually necessary to change variables so that an irregular element appears as a square in the new coordinate system. Pinder chose the polynomial shape functions to relate the new and old coordinate systems as well as to satisfy all requirements of basis functions. Linear, quadratic or cubic polynomial shape functions are used with the mixed isoparametric element depending on the shape of the boundary or anticipated shape of the analytical solution of equation 1. A list of these shape elements is given in Pinder and Frind (ibid.).

A finite difference approximation for the time derivative is introduced into equation 4 after the elements of the banded n x n matrices M and N have been evaluated. This converts the system of ordinary differential equations into a system of linear algebraic equations. The system of equations generated is solved for the undetermined coefficients by the particularly efficient algorithm for banded matrices called Cholesky's square root method. The hydraulic head at the nodes is then known, since it equals the coefficients at these nodes.

GRAPHICAL PRESENTATION OF DATA

In recent years significant advances have been made in the use of computer graphics techniques in analysis of problems and in the presentation of results. For example, Yates, et al. (1973) report on the use of interactive graphics in conjunction with time sharing computers for generation of complex structural models by the finite element method. By this technique the model is displayed as it is being constructed on a CRT (cathode ray tube) so that errors in node and element selection or meshing inadequacies are immediately apparent. Some graphics packages permit isometric or orthographic views, which are particularly advantageous in determining errors during the modeling of complex geometries.

Similarly, during the evaluation of a FEM stress analyses of a structure, data contained in hundreds of lines (and possibly hundreds of pages) of computer output, may be reduced to a pictorial display (ibid.). This approach markedly reduces the time required by traditional methods for data reduction, evaluation and presentation of results. Several efficient software programs have been developed for graphic display of structural data (eg. Batdorf and Kapur, 1973).

To date, the use of computer graphics in constructing, calibrating and validating groundwater models has been minimal. Pinder (1973) used computer graphics in conjunction with a finite element model to depict the areal spread of a contaminant in an aquifer system on Long Island, New York. Results were presented on a series of three dimensional plots.

The graphic representation used in our groundwater flow simulations (SYMVU) was developed by the Harvard School of Graduate Design. This method provides three dimensional views of the cone of depression calculated by Galerkin's finite element method. The view of the surface may be rotated 360° horizontally and from 0° to 90° vertically. The surface is generated by interpolating the head values between the nodes in the domain.

RESULTS

A two-week constant discharge test conducted in 1967 on a pumping well at the Water Resources Research Center Field Laboratory at Tucson provided data on the transmissivity and storage coefficient of native aquifer materials. Using drawdown data in observation wells, A, B and C, together with the Jacob modified non-equlibrium method, the calculated transmissivity ranged from 0.055 ft^2/sec to 0.058 ft^2/sec during the test. The average storage coefficient was .023. Wells A, B and C are 15 ft, 120 ft and 260 ft, respectively, from the pumping well. Data from the constant discharge test allowed us the opportunity to evaluate the finite element method of Pinder and Frind (1972) for the geohydrological conditions in the Tucson basin, and concurrently, to gain experience in using the method.

Because the pumping well is partially penetrating and the extent of the aquifer at the site is large, it was assumed that vertical flow was negligible and that the aquifer is artesian. With these assumptions, flow in the aquifer may be adequately described by equation 1.

Analysis of the pump test data using Jacob's method, indicated that the radius of influence of the pumping well during the test was about 1000 ft. Consequently, it was assumed that the boundary of the region to be modeled was at this distance from the pumping well. To simplify the geometry, the boundary was assumed to be rectangular.

In searching for an approximate solution to equation 1 using the finite element-Galerkin approach, several nodal arrangements were considered. Nodal locations were selected to some extent on a trial and error basis. However, it was realized that for any fixed time greater than zero, head values have steep gradients in the vicinity of the pumping well. Consequently, a reasonable approximation to the solution in this region requires that the associated finite elements be of higher order; where the order of the element refers to the order of the non-zero shape functions within that element. In regions further from the well (sink), where the gradient was lower, linear elements were thought to be adequate in approximating the solution. The grid finally selected is shown in Figure 2.

Boundary conditions for the groundwater flow equations are usually stated either as a flux across the boundary, or as a head distribution along the boundary. For a node along the boundary the last condition means that the node has a constant or fixed hydraulic head throughout the simulation. For the section of the aquifer simulated, we assumed all nodes were of the constant head type.

Wells B and C were represented as nodes in the finite element grid. Therefore, the actual and calculated head values were compared at these points. Well A was not represented in the grid since it was in a zone of turbulent flow. The known transmissivity values were assigned to nodes representing wells B and C and the pumping well. For other nodes transmissivity coefficients were assigned estimated values. The transmissivity varied from .045 ft^2/sec along the perimeter to .058 ft^2/sec in the center region. Of the several nodal transmissivity patterns, the one selected was relatively simple and at the same time provided accurate calculated drawdowns for well B and C.

Table 1 gives a comparison of actual drawdown and those calculated by the finite element method at various times during the two weeks. The collapse of the calculated water surface was rapid during the initial 48 hours. The drawdown increased slightly during the remainder of the simulation period. The actual drawdowns also had a rapid increase during the first 48 hours but the rate of increase remained higher than the calculated heads over the remainder of the two week simulation period. Experience indicates that finer adjustment of the transmissivity and storage coefficients will bring the actual and calculated drawdowns into closer agreement during the entire two-week period.

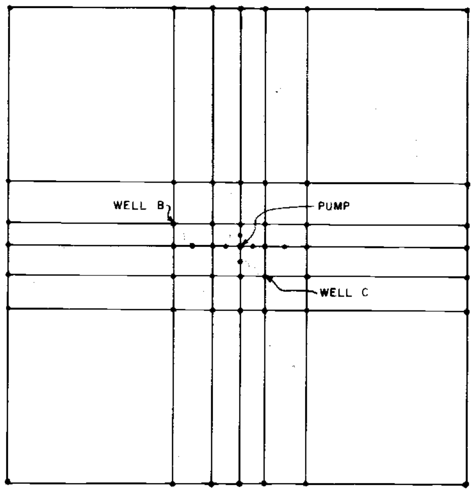
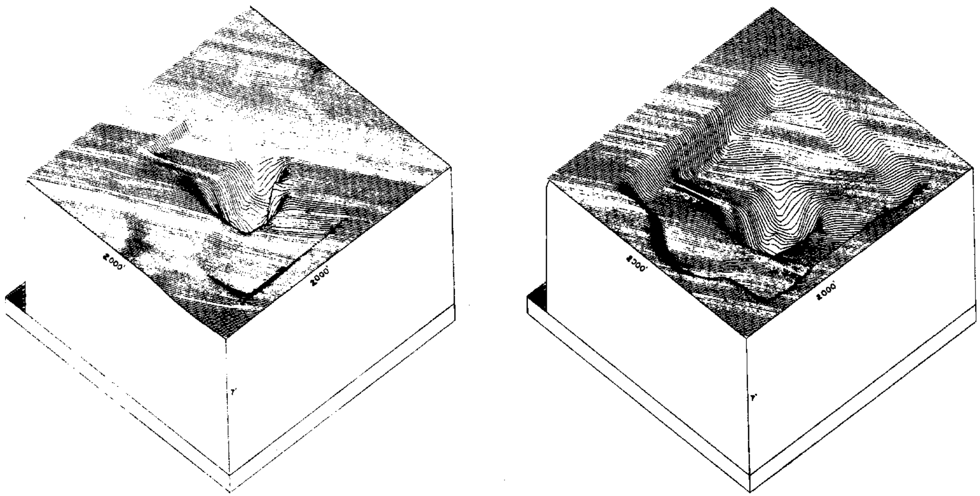


Figure 2. Finite Element Grid for Analysis of two week discharge test.

Table 1. Comparisons of Actual and Calculated Drawdowns.

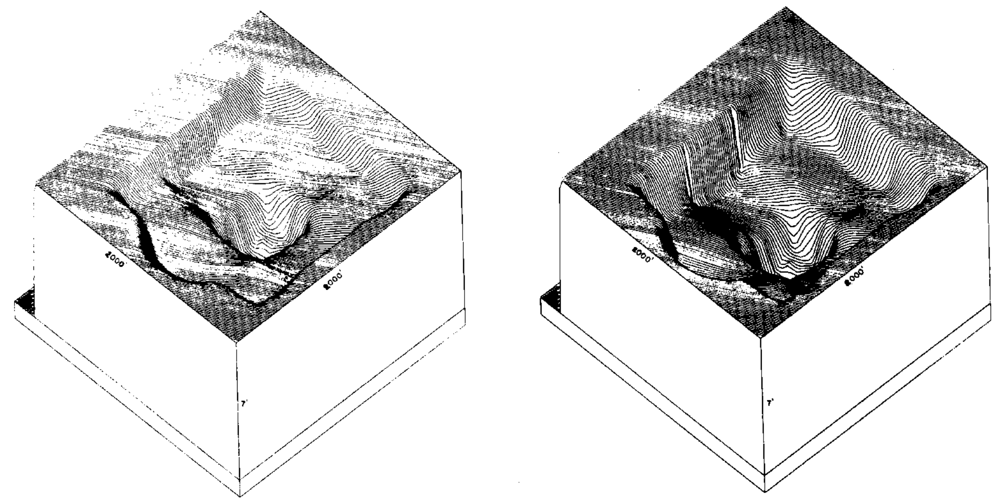
Time (hr)	WELL B Actual	WELL B Calc.	WELL C Actual	WELL C Calc.
6	.5	.4	.87	.95
24	.93	1.23	1.43	1.83
36	1.20	1.53	1.72	2.12
48	1.36	1.71	1.80	2.31
168	2.32	2.03	2.53	2.57
336	2.38	2.07	2.55	2.58

Figures 3 and 4 represent three dimensional perspectives of head distributions in the aquifer for the following four times during the simulation period: 6, 24, 168 and 336 hours. These graphs were constructed using the SYMVU program, described earlier. Note the exaggerated scale; the sides represent the full width and breadth of aquifer modeled and the depth equals the drawdown in the pumping well.



(a) (b)

Figure 3. Simulated Areal Distribution of Drawdown for Constant Discharge Test at (a) 6 hours and (b) one day.



(a) (b)

Figure 4. Simulated Areal Distribution of Drawdown for Constant Discharge Test at (a) one week and (b) two weeks.

These graphs clearly show the rapid collapse of the water surface during the first few hours of pumping and the gradual recession, thereafter. The figures also clearly illustrate that the cone of depression resulting from the modeling process is distinctly rectangular in shape and asymmetric. Furthermore, changes in water surface are abrupt. Intuitively, one would expect the cone of depression to be eliptical and changes in the water to be gradual. Additional work is obviously required to refine transmissivity and/or storativity values. The important point is that the need for such adjustments is clearly visible on the figures; such changes are not so evident in tabular presentation of data. In other words, graphics techniques have the potential of being a valuable aid in dealing with the inverse problem. Furthermore, the coupling of a graphics package with an aquifer model will provide the water manager with a simple tool for visually observing the effect of various pumping patterns on regional water levels.

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REFERENCES CITED

Batdorf, W. J. and Kapur, S. S. 1973. FLING-a FORTRAN language for interactive graphics. In: Numerical and Computer Methods in Structural Mechanics. Edited by: S. J. Fenves, A. R. Robinson, N. Perrone and W. C. Schnobich. Academic Press, New York and London, pp. 513-542.

Pinder, G. F. and Frind, E. O. 1972. Application of Galerkin's Procedure to Aquifer Analysis. Water Resources Research 8(1):108-121.

Pinder, G. F. 1973. A Galerkin Finite Element Simulation of Groundwater Contamination on Long Island, New York. Water Resources Research 9(6):1657-1669.

Yates, D. N., Sable, W. W. and Vinson, T. J. 1973. The DAISY Code. In: Numerical and Computer Methods in Structural Mechanics. Edited by: S. J. Fenves, A. R. Robinson, N. Perrone and W. C. Schnobich. Academic Press, New York and London, pp. 176-210.