FIELD OF A MAGNETIC LINE SOURCE IN THE PRESENCE OF A SEMIELLIPTICAL BOSS

by

Willard Parker Webster

A Thesis Submitted to the Faculty of the

DEPARTMENT OF ELECTRICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

1966

STATEMENT BY AUTHOR

This thesis has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: Willard Parker Webster

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

George Tyras
GEORGE TYRAS
Associate Professor of Electrical Engineering

5/13/66
Date

ACKNOWLEDGMENT

Sincere thanks must be given to Dr. George Tyras, first for giving the author the opportunity to work on the problem and secondly for his helpful guidance, confidence and impetus throughout the project. The work on this project was supported by the National Science Foundation, Grant GP-2307.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure | Page

ABSTRACT

The field of a magnetic line source above and parallel to an infinite semielliptical cylindrical boss on a conducting plane of infinite exent has been obtained using the Green's function technique. Two cases are investigated, a surface impedance on the boss alone and a surface impedance on both the boss and the conducting plane. It is found in each case that the surface impedance must be small and it must also exhibit a specific functional dependence upon the coordinates of the surface. Some plots of the magnetic field are given for a perfectly conducting boss and for a boss with a surface impedance. In general the main effect of the surface impedance is to cause a small attenuation of the field.

1. INTRODUCTION

Relatively little investigation has been made of electromagnetic waves incident upon semielliptical cylindrical bosses on infinite ground planes. Thus far, the major contribution in this area has been made by Wait and Murphy (1957, 1958) in two papers; one concerned with a plane wave incident upon a perfectly conducting semielliptical cylindrical boss on a perfectly conducting ground plane and the other paper deals with a semicircular cylindrical boss under similar restraints. The bosses were used as models to study the influence of a ridge on a low frequency ground wave.

Since the earth is not a perfect conductor, a more realistic model of a ridge would be obtained by allowing a small surface impedance to exist on the boss and the plane. This thesis investigates two such models. The first model imposes a surface impedance on the boss alone and the second model imposes a surface impedance on both the boss and the plane. For each model, the surface impedance is a function of the coordinates of the surface. In Wait's papers, the source of the incident electromagnetic waves is at infinity. For this investigation, the source is considered to be located at a point in the upper half space above the boss and the plane.

The first case, a surface impedance on the boss alone, should be a good model of an island surrounded by deep water. If the water around the island were shallow, then the second case, a variable surface impedance on both the boss and the plane, would be a good model. The modeling aspects of the bosses are discussed in more detail at the end of the thesis.

The functional dependence of the surface impedances is chosen such that homogeneous boundary conditions exist on the surface of the boss and the conducting plane. Due to their geometries, the problems are essentially two dimensional boundary value problems in elliptical coordinates with delta function sources and homogeneous boundary conditions. Such problems lend themselves to ready solution by the Green's function technique; hence, that is the method of solution which shall be employed.

Because of the elliptical geometry, the solutions of both problems are most easily written in terms of the Mathieu functions and the modified Mathieu functions. As an aid to the reader, a summary of the pertinent Mathieu function relations is given in the Appendix. For a more comprehensive discussion, the reader is referred to the references cited in the Appendix.

2. EVALUATION OF THE FIELD FOR AN IMPEDANCE ON THE BOSS ALONE

2.1 Development of the Problem

The first problem to be investigated shall be the determination of the field of a magnetic current line source in the presence of and parallel to an imperfectly conducting semielliptical boss on a perfectly conducting ground plane. The geometrical configuration is shown in Figure 1, where the primed coordinates denote source coordinates and the unprimed coordinates denote field coordinates. MKS units will be used throughout the thesis and the time dependence is taken as $e^{-i\omega t}$ (henceforth suppressed).

Figure 2 shows how the coordinates of a point P are related in the three different coordinate systems, elliptic cylinder, cylindrical and cartesian. Note that all three coordinate systems are concentric about the origin. The coordinates are expressed as

$$\begin{aligned} \text{cartesian:} &\quad (x,\ y,\ z) \\ \text{cylindrical:} &\quad (\rho,\ \varphi,\ z) \\ \text{elliptic cylinder:} &\quad (u,\ v,\ z) \end{aligned}$$

and the relationships between the coordinates are given by

$$x = c \cosh u \cos v$$

$$y = c \sinh u \sin v$$

$$\rho = (x^2 + y^2)^{\frac{1}{2}}$$

$$\varphi = \arctan (y/x)$$

$$z = z,$$

where c is the semifocal distance of the ellipse, i.e., c is the distance from the origin to one of the foci of the ellipse. For a and b defined as shown in Figure 2,

$$c = (b^2 - a^2)^{\frac{1}{2}}.$$

In terms of the Hertzian magnetic vector potential $\vec{\pi}^{(m)}$ and the magnetic current density $\vec{J}^{(m)}$, the vector wave equation and the electric and magnetic fields are given by

$$(\nabla^2 + k_o^2)\vec{\pi}^{(m)} = \frac{-i}{\omega\mu_o}\vec{J}^{(m)} \tag{1}$$

$$\vec{E} = i\omega\mu_o \nabla \times \vec{\pi}^{(m)} \tag{2}$$

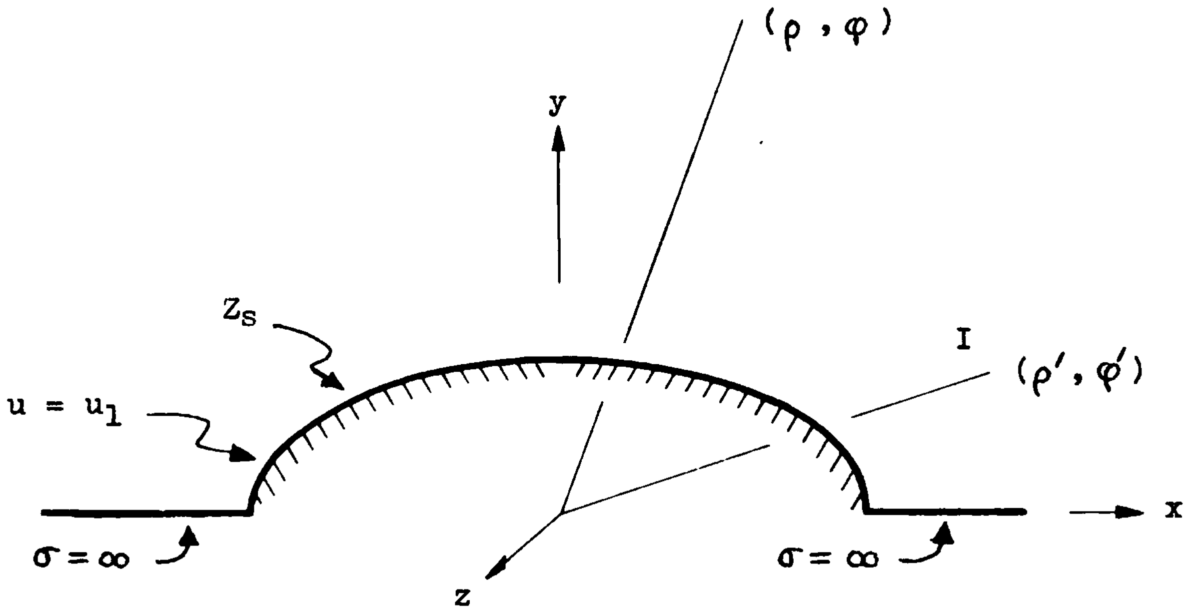
$$\vec{H} = k_o^2 \vec{\pi}^{(m)} + \nabla(\nabla \cdot \vec{\pi}^{(m)}) \tag{3}$$

For an axial magnetic line current, the magnetic current density may be expressed as

$$\vec{J}^{(m)} = I \ \delta(x - x') \ \delta(y - y') \ \hat{a}_z$$

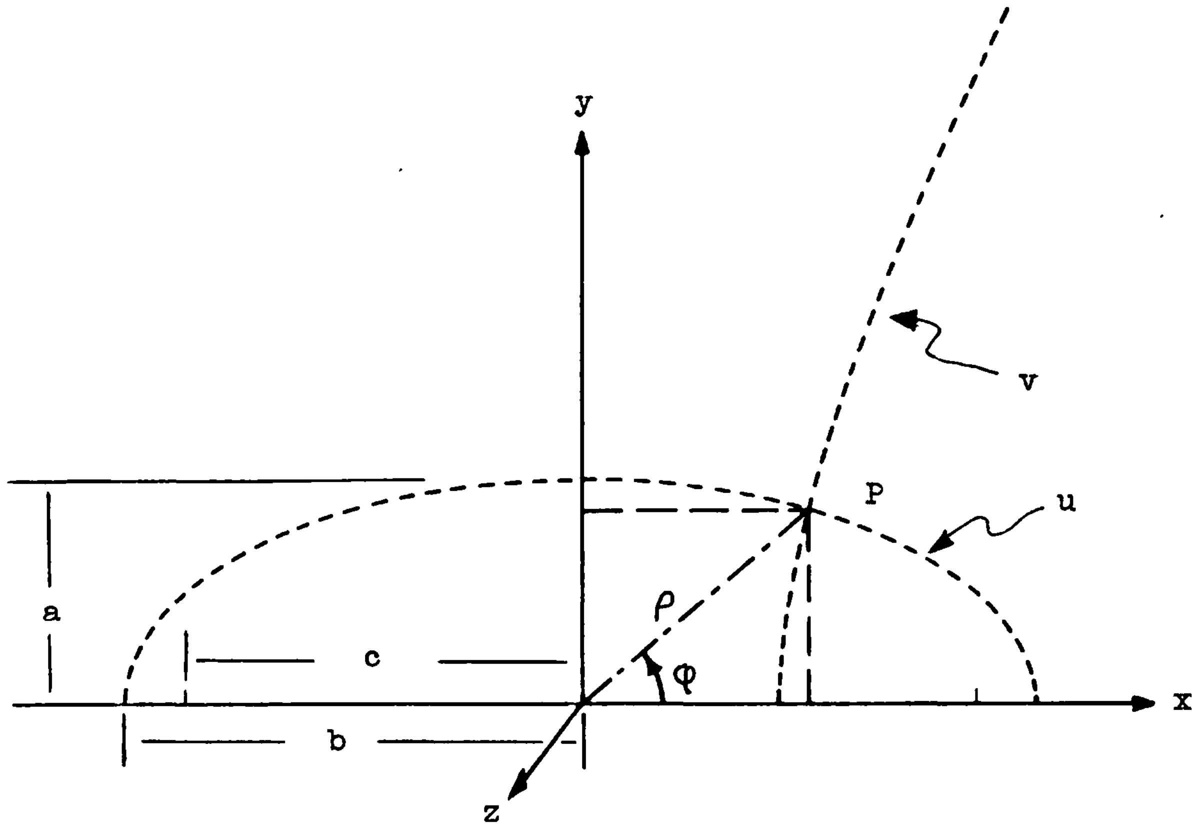
An infinitesimal element of volume $d\tau$ in cartesian and in elliptical cylinder coordinates is given by

$$d\tau = dx\,dy\,dz = c^2(\cosh^2 u - \cos^2 v)\ du\,dv\,dz .$$



$$Z_s = \frac{bZ_0Z}{(b^4 - c^2x^2)^{\frac{1}{2}}}$$

Fig. 1.--Infinite imperfectly conducting semielliptical boss on a perfectly conducting ground plane of infinite extent



At point P the following relations between the coordinate systems exist.

$$x = c \cosh u \cos v$$

$$y = c \sinh u \sin v$$

$$\rho = (x^2+y^2)^{\frac{1}{2}}$$

$$\varphi = \arctan(y/x)$$

$$z = z$$

$$c = (b^2-a^2)^{\frac{1}{2}} = \text{semifocal distance}$$

Fig. 2.--Relationship between the coordinate systems

Hence the relationship between the delta functions will be

$$\delta(x-x')\,\delta(y-y') = \frac{\delta(u-u')\,\delta(v-v')}{c^2(\cosh^2 u' - \cos^2 v')}$$

since the integral of the delta function over all space must be equal to 1. For convenience in writing, define Δ as

$$\Delta = c(\cosh^2 u - \cos^2 v)^{\frac{1}{2}}$$

Thus in terms of the given current density, in elliptic cylinder coordinates the wave equation and the fields are given by

$$(\nabla^2 + k_o^2)\,\pi_z^{(m)} = -\frac{iI}{\omega\mu_o\Delta^2}\;\delta(u-u')\,\delta(v-v') \tag{4}$$

$$\vec{E} = i\omega\mu_o\,\nabla \times \pi_z^{(m)}\,\hat{a}_z$$

$$\vec{H} = k_o^2\,\pi_z^{(m)}\hat{a}_z + \nabla(\nabla\cdot\pi_z^{(m)}\hat{a}_z),$$

where

$$\nabla \times \pi_z^{(m)}\hat{a}_z = \frac{1}{\Delta}\left[\frac{\partial}{\partial v}\pi_z^{(m)}\hat{a}_u - \frac{\partial}{\partial u}\pi_z^{(m)}\hat{a}_v\right]$$

$$\nabla^2\pi_z^{(m)} = \frac{1}{\Delta^2}\left[\frac{\partial^2}{\partial u^2}\pi_z^{(m)} + \frac{\partial^2}{\partial v^2}\pi_z^{(m)}\right] \tag{5}$$

and

$$\nabla(\nabla\cdot\pi_z^{(m)}\hat{a}_z) = 0.$$

Hence the fields can now be written as

$$E_u = \frac{i\omega\mu_0}{\Delta} \frac{\partial}{\partial v} \Pi_z^{(m)}$$

$$E_v = \frac{-i\omega\mu_0}{\Delta} \frac{\partial}{\partial u} \Pi_z^{(m)}$$

$$H_z = k_0^2 \Pi_z^{(m)}$$

The Green's function for the given geometric configuration satisfies

$$(\nabla^2 + k_0^2)\, G = \frac{-1}{\Delta^2}\, \delta(u-u')\, \delta(v-v') \tag{6}$$

Comparison of (4) and (6) indicates

$$\Pi_z^{(m)} = \frac{iI}{\omega\mu_0}\, G$$

Thus the field quantities may now be rewritten in terms of the Green's function as

$$E_u = \frac{-I}{\Delta} \frac{\partial}{\partial v} G \tag{7}$$

$$E_v = \frac{I}{\Delta} \frac{\partial}{\partial u} G \tag{8}$$

$$H_z = i\omega\epsilon_0 I\, G\ . \tag{9}$$

2.2 The Leontovitch Boundary Conditions

Next the boundary conditions must be considered. Assuming that the frequency is such that the radius of curvature of the semielliptical boss is large compared to the wavelength of the incident wave and that the surface impedance on the boss is small, then the so called Leontovitch (impedance) boundary conditions can be assumed to exist (Brekhovskikh 1960, Jones 1964). Hence, a small surface impedance Z_s will be assumed to exist on the boss. At the surface of the perfectly conducting plane the tangential electric field must be zero. The boundary conditions can then be expressed as

$$E_u = 0 \qquad \text{at} \quad v = 0, \pi \quad ; \quad u > u_1$$

$$\frac{E_v}{H_z} = -Z_s \qquad \text{at} \quad u = u_1 \quad ; \quad 0 \leq v \leq \pi$$

where $u = u_1$ describes the surface of the boss in elliptic cylinder coordinates. In terms of the Green's function, the boundary conditions become

$$\frac{\partial G}{\partial v} = 0 \qquad \text{at} \quad v = 0, \pi \tag{10}$$

$$\frac{\frac{\partial G}{\partial u}}{G} = -i\omega\epsilon_0 \Delta Z_s \qquad \text{at} \quad u = u_1$$

If the surface impedance is assumed to be of the form

$$Z_s = \frac{Z_o Z}{c(\cosh^2 u_1 - \cos^2 v)^{1/2}}$$

with Z_o as the free space impedance and Z as a small arbitrary constant, then the second boundary condition reduces to

$$\frac{\frac{\partial G}{\partial u}}{G} = -i k_o Z \qquad \text{at} \quad u = u_1 \tag{11}$$

Hence with the assumed surface impedance, the boundary conditions are homogeneous. If a different surface impedance is assumed, the boundary conditions are not homogeneous and the problem cannot be solved using the Green's function technique.

From the relationships between the coordinate systems given in Figure 2, one can easily show that the expression for the surface impedance reduces to

$$Z_s = \frac{Z_o Z}{(b^2 - c^2 \cos^2 v)^{1/2}} = \frac{b Z_o Z}{(b^4 - c^2 x^2)^{1/2}} \ .$$

Both expressions are graphed in Figure 3 and Figure 4, respectively, for the special case b = 3.0 and $1.0 \leq c \leq 3.0$. Figure 4 gives a clearer representation of the surface impedance since one is usually more familiar with cartesian coordinates than elliptical coordinates.

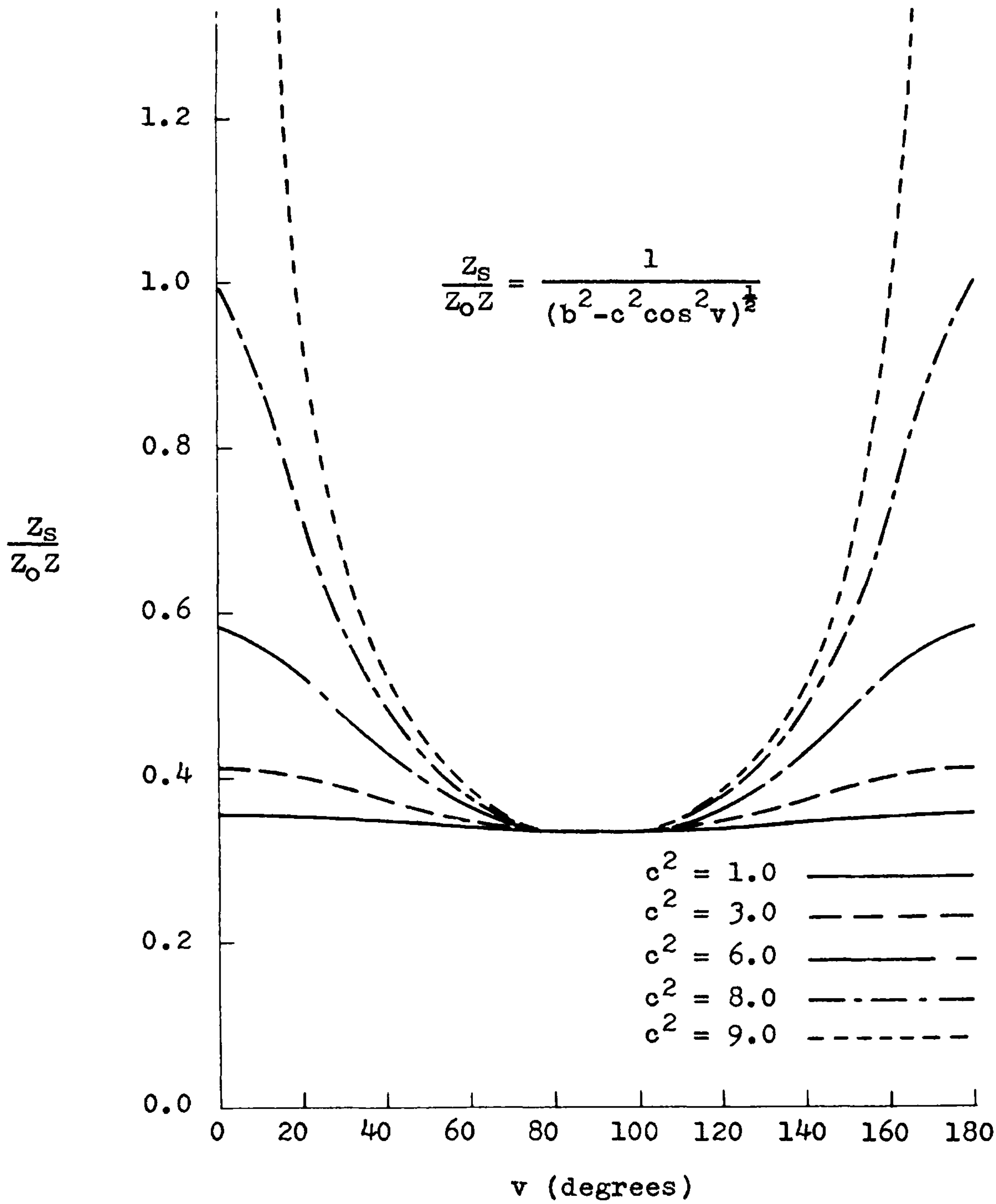


Fig. 3.--Graph of the normalized surface impedance on the boss, $Z_s/Z_o Z$, versus the angular elliptical coordinate v, for the special case b = 3.0

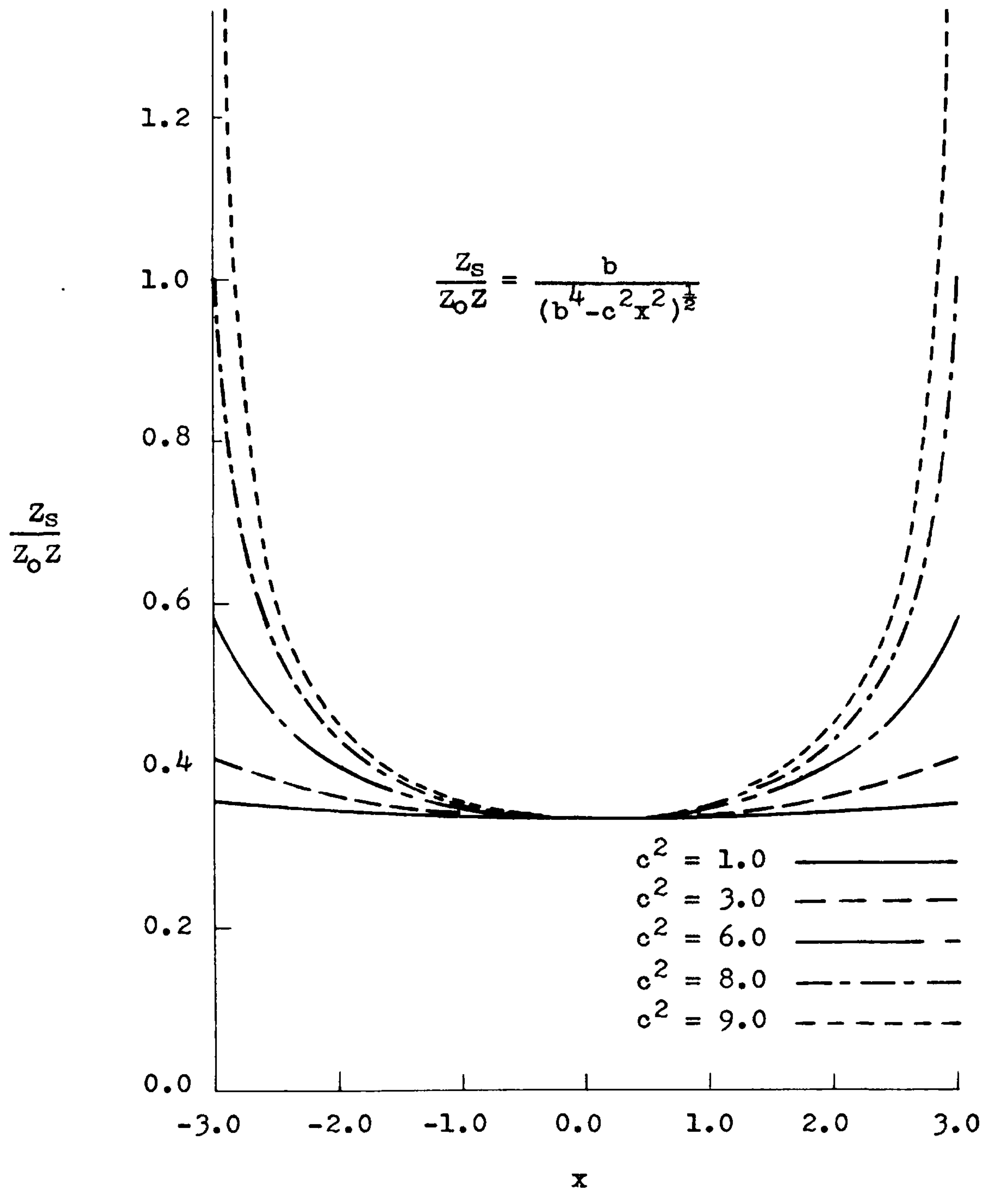


Fig. 4.--Graph of the normalized surface impedance on the boss, $Z_s/Z_o Z$, versus the cartesian coordinate x, for the special case b = 3.0

2.3 Determination of the Green's Function and the Field

Equations (6), (10) and (11) give sufficient information to determine the Green's function. From (5) and (6) it is seen that G must satisfy

$$\frac{\partial^2}{\partial u^2} G + \frac{\partial^2}{\partial v^2} G + k_o^2 \Delta^2 G = - \delta(u-u') \ \delta(v-v') \ . \qquad (12)$$

Equation (12) is a function of two variables. It must be reduced to a function of one variable before it can be solved. With that end in mind assume a solution of (12) of the form

$$G = \sum_{m=0}^{\infty} Se_m(k_o^2c^2, v)\, g_m(u) \qquad (13)$$

where $Se_m(k_o^2c^2,v)$ is the even Mathieu function (Appendix) which is the solution of

$$\frac{d^2}{dv^2} Se_m + (b - k_o^2c^2\cos^2 v)\, Se_m = 0 \ . \qquad (14)$$

Substitution of (13) and (14) into (12) gives

$$\sum_{m=0}^{\infty} \left\{ Se_m(k_o^2c^2) \left[\frac{\partial}{\partial u^2} g_m(u) - (b - k_o^2c^2\cosh^2 u)\, g_m(u) \right] \right\} = - \delta(u-u')\, \delta(v-v')$$

Multiplying both sides of the above equation by $Se_n(k_o^2c^2,v)$ and then integrating the resulting equation over the interval $(0, 2\pi)$ reduces the wave equation to an equation of one variable as given below.

$$\frac{d^2}{du^2} g_m(u) - (b - k_o^2c^2\cosh^2 u)\, g_m(u) = - \frac{Se_m(k_o^2c^2, v')}{N_m}\, \delta(u-u') \qquad (15)$$

where N_m is the normalization constant for $Se_m\ (k_o^2 c^2, v)$ (Appendix). Let

$$\tilde{g} = \frac{N_m}{Se_m(k_o^2 c^2, v')}\ g_m(u)$$

then (15) reduces to

$$\frac{d^2}{du^2}\,\tilde{g} - (b - k_o^2 c^2 \cosh^2 u)\,\tilde{g} = -\ \delta(u-u') \tag{16}$$

The solution of the homogeneous form of (16) is the modified Mathieu function (Appendix).

Seeking a solution of (16) in the usual manner of the Green's function technique it is found that the appropriate combination of modified Mathieu functions for $\tilde{g}$ will be

$$\begin{aligned} \tilde{g}_< &= A\, Je_m(k_o^2 c^2, u) + B\, He_m^{(1)}(k_o^2 c^2, u) \qquad && u < u' \\ \tilde{g}_> &= C\, He_m^{(1)}(k_o^2 c^2, u) && u > u' \end{aligned} \tag{17}$$

where A, B and C are constants and $Je_m(k_o^2 c^2, u)$ and $He_m^{(1)}(k_o^2 c^2, u)$ represent the modified Mathieu functions (Appendix). Substitution of (17) into (11) gives

$$\frac{B}{A} = -\ \frac{k_o Z\, Je_m(k_o^2 c^2, u_1) - i\, Je_m'(k_o^2 c^2, u_1)}{k_o Z\, He_m^{(1)}(k_o^2 c^2, u_1) - i\, He_m^{(1)\prime}(k_o^2 c^2, u_1)}$$

Two more independent equations are needed in order to solve for the constants A, B and C. The needed equations are obtained from the requirements on the Green's function solution at the source.

$$\left.\begin{array}{l} \tilde{g}_< = \tilde{g}_> \\ \tilde{g}'_> - \tilde{g}'_< = \dfrac{-1}{P(u)} \end{array}\right\} \text{ at } u = u'$$

The value of P(u) is determined from the homogeneous form of (16) after it has been rearranged into the standard Sturm-Liouville form. The Sturm-Liouville equation (Sagan, 1963) is given below.

$$\frac{d}{du}\left[P(u)\,\frac{d\tilde{g}}{du}\right] + \left[Q(u) + \lambda R(u)\right]\tilde{g} = 0$$

It is easily seen that for the problem at hand P(u) = 1.0. In terms of the modified Mathieu functions, the Green's function conditions reduce to

$$A\, Je_m(k_o^2c^2, u') + (B-C)\, He_m^{(1)}(k_o^2c^2, u') = 0 \tag{18}$$

$$A\, Je'_m(k_o^2c^2, u') + (B-C)\, He_m^{(1)\prime}(k_o^2c^2, u') = -1 \quad . \tag{19}$$

Solving for the constants A, B and C from equations (17), (18) and (19) gives

$$A = \frac{-1}{W}\, He_m^{(1)}(k_o^2c^2, u')$$

$$B = \frac{1}{W}\left[\frac{k_o Z\, Je_m(k_o^2c^2,u_1) - i\, Je_m'(k_o^2c^2,u_1)}{k_o Z\, He_m^{(1)}(k_o^2c^2,u_1) - i\, He_m^{(1)\prime}(k_o^2c^2,u_1)}\right] He_m^{(1)}(k_o^2c^2,u')$$

$$C = \frac{-1}{W}\left\{Je_m(k_o^2c^2,u') - \left[\frac{k_o Z\, Je_m(k_o^2c^2,u_1) - i\, Je_m'(k_o^2c^2,u_1)}{k_o Z\, He_m^{(1)}(k_o^2c^2,u_1) - i\, He_m^{(1)\prime}(k_o^2c^2,u_1)}\right] He_m^{(1)}(k_o^2c^2,u')\right\}$$

where W is the Wronskian of the modified Mathieu functions $Je_m(k_o^2c^2,u)$ and $He_m^{(1)}(k_o^2c^2,u)$ (Appendix).

$$W = Je_m(k_o^2c^2,u)\, He_m^{(1)\prime}(k_o^2c^2,u) - Je_m'(k_o^2c^2,u)\, He_m^{(1)}(k_o^2c^2,u) = \frac{2i}{\pi}$$

The total solution has now been determined and can be written as

$$G_> = \frac{i\pi}{2}\sum_{m=0}^{\infty}\frac{1}{N_m}\, Se_m(k_o^2c^2,v)\, Se_m(k_o^2c^2,v')\, He_m^{(1)}(k_o^2c^2,u)$$

$$\left\{Je_m(k_o^2c^2,u') - \left[\frac{k_o Z\, Je_m(k_o^2c^2,u_1) - i\, Je_m'(k_o^2c^2,u_1)}{k_o Z\, He_m^{(1)}(k_o^2c^2,u_1) - i\, He_m^{(1)\prime}(k_o^2c^2,u_1)}\right] He_m^{(1)}(k_o^2c^2,u')\right\}$$

$G_<$ can be obtained from $G_>$ by interchanging u and u′.

The expressions for the electric and magnetic fields can now be obtained by substituting the expressions for $G_>$ and $G_<$ into (7), (8) and (9).

2.4 Green's Function for a Semicircular Cylindrical Boss

It should be noted at this point that if the elliptical boss is permitted to degenerate into a semicircular cylindrical boss the surface impedance reduces to

$$Z_s = \frac{Z_0 Z}{b}$$

where b is the radius of the cylinder (Fig. 3, Fig 4). The Green's function reduces to

$$G_> = \frac{i}{2} \sum_{m=0}^{\infty} \epsilon_m \cos m\varphi \cos m\varphi' \, H_m^{(1)}(k_0\rho)$$

$$\cdot \left\{ J_m(k_0\rho') - \left[\frac{Z J_m(k_0 b) - i b J_m'(k_0 b)}{Z H_m^{(1)}(k_0 b) - i b H_m^{(1)\prime}(k_0 b)} \right] H_m^{(1)}(k_0\rho') \right\}$$

where ϵ_m is Neumann's constant

$$\epsilon_m = \begin{cases} 1 & \text{for} \quad m = 0 \\ 2 & \text{for} \quad m \neq 0 \end{cases}$$

and $J_m(x)$ and $H_m^{(1)}(x)$ are Bessel functions. The prime denotes the derivative with respect to the argument. The above results are identical to those obtained by solving the semicircular cylindrical boss problem directly (Webster and Tyras, 1966).

3. EVALUATION OF THE FIELD FOR AN IMPEDANCE ON BOTH THE BOSS AND THE PLANE

3.1 Development of the Problem

The problem that is to be solved is identical to the previous problem but with the added boundary condition that a surface impedance exists on the conducting plane (Fig. 5). The boundary conditions then become

$$\frac{E_v}{H_z} = -Z_{sb} \qquad \text{at} \quad u = u_1 \; ; \quad 0 \le v \le \pi$$

$$\frac{E_u}{H_z} = \pm Z_{sp} \qquad \text{at} \quad v = 0, \pi \; ; \quad u > u_1 \tag{20}$$

where Z_{sb} is the surface impedance on the boss and Z_{sp} is the surface impedance on the conducting plane. If surface impedances of the form

$$Z_{sb} = \frac{Z_0 Z_1}{c(\cosh^2 u_1 - \cos^2 v)^{1/2}} \; ; \qquad Z_{sp} = \frac{Z_0 Z_2}{c(\cosh^2 u - 1.0)^{1/2}}$$

are assumed to exist on the boundaries, where Z_1 and Z_2 are small arbitrary constants and Z_0 is the free space impedance, then using (7), (8) and (9) the boundary conditions reduce to

$$\frac{\frac{\partial G}{\partial u}}{G} = -i k_o Z_1 \qquad \text{at} \quad u = u_1 \tag{21}$$

$$\frac{\frac{\partial G}{\partial v}}{G} = \mp i k_o Z_2 \qquad \text{at} \quad v = 0, \pi . \tag{22}$$

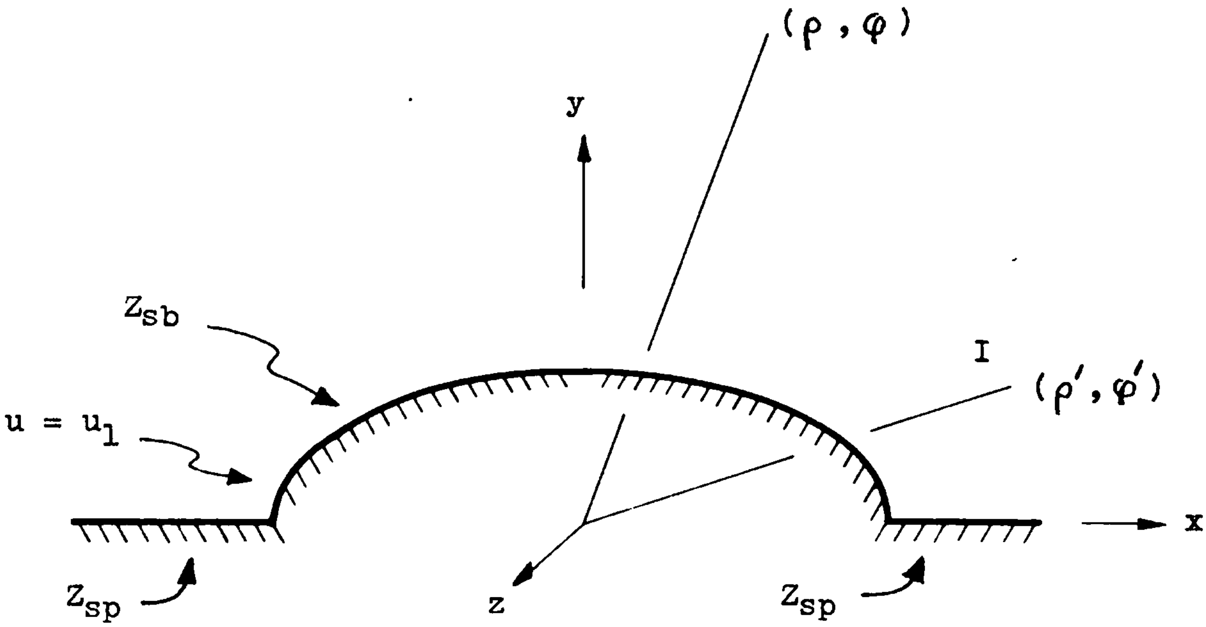
As given above, the surface impedances are not written in their most easily understood form. Using the relations given in Figure 2, one can reduce the surface impedance expressions to

$$Z_{sb} = \frac{Z_o Z_1}{(b^2 - c^2 \cos^2 v)^{1/2}} = \frac{b Z_o Z_1}{(b^4 - c^2 x^2)^{1/2}}$$

and

$$Z_{sp} = \frac{Z_o Z_2}{(x^2 - c^2)^{1/2}} \quad .$$

These expressions are plotted in Figure 3, Figure 4, and Figure 6 for b = 3.0 and 1.0 ≤ c ≤ 3.0. Note that for either large distance from the boss or for a near cylindrical boss, the surface impedance on the plane varies inversely with the distance from the center of the boss. The surface impedance on the boss is identical with that obtained in the first problem.



$$Z_{sp} = \frac{Z_o Z_2}{(x^2-c^2)^{\frac{1}{2}}} \qquad Z_{sb} = \frac{bZ_o Z_1}{(b^4-c^2x^2)^{\frac{1}{2}}}$$

Fig. 5.--An infinite imperfectly conducting semi-elliptical boss on an imperfectly conducting ground plane of infinite extent

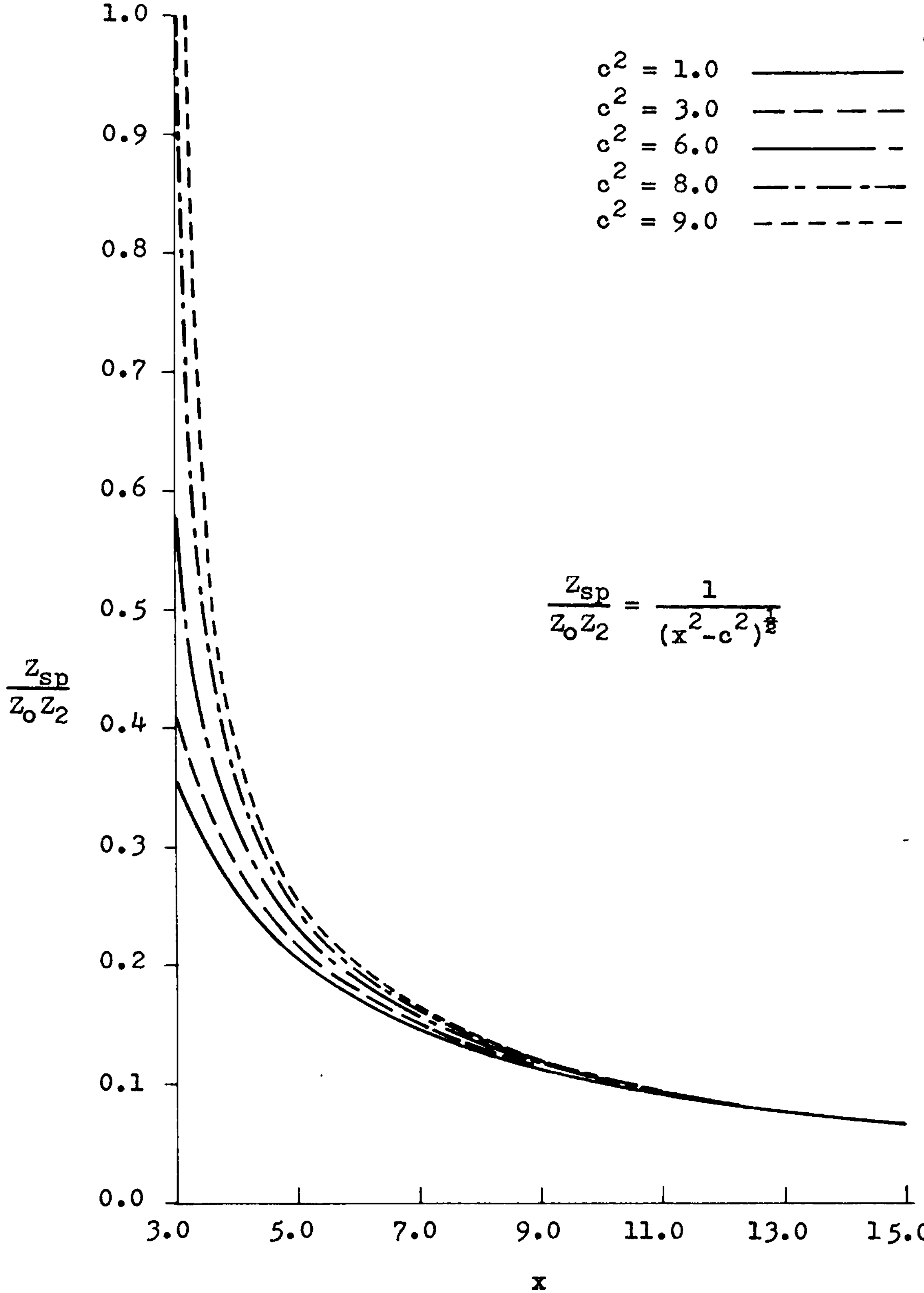


Fig. 6.--Graph of the normalized surface impedance on the conducting plane, Z_{sp}/Z_oZ_2, versus the cartesian coordinate x, for the special case b = 3.0

3.2 Determination of the Green's Function and the Field

Equations (6), (21) and (22) give sufficient information to determine the Green's function. Assume a solution of (6) of the form

$$G = \sum_{m=0}^{\infty} \left[D_\nu \, So_\nu(k_o^2c^2, v) + Se_\nu(k_o^2c^2, v) \right] \, g_\nu(u) \tag{23}$$

where ν is a function of the index m and the coefficient D_ν can depend on any of the problem's parameters except the coordinates. $Se_\nu(k_o^2c^2, v)$ is the even Mathieu function and $So_\nu(k_o^2c^2, v)$ is the odd Mathieu function (Appendix). Both $Se_\nu(k_o^2c^2, v)$ and $So_\nu(k_o^2c^2, v)$ satisfy (14) but for different values of separation constant. Any linear combination of the Mathieu functions will also satisfy (14). Substituting (23) into (6) gives

$$\sum_{m=0}^{\infty} \left\{ \left[D_\nu \, So_\nu(k_o^2c^2, v) + Se_\nu(k_o^2c^2, v) \right] \cdot \left[\frac{d^2}{du^2} g_\nu(u) - (d - k_o^2c^2 \cosh^2 u) \, g_\nu(u) \right] \right\} = -\delta(u-u') \, \delta(v-v') \tag{24}$$

where d is the appropriate separation constant.

Multiplying both sides of (24) by

$$D_\mu \, So_\mu(k_o^2c^2, v) + Se_\mu(k_o^2c^2, v)$$

and intergrating with respect to v over the interval $(0, \pi)$ gives

$$\frac{d^2}{du^2} g_\nu(u) - (d - k_o^2 c^2 \cosh^2 u)\, g_\nu(u) = -\frac{\delta(u-u')}{CON} \tag{25}$$

Note that in the derivation of (25) the orthogonality of $So_\nu(k_o^2c^2, v)$ and $Se_\nu(k_o^2c^2, v)$ for the given boundary conditions must be used. The normalization constant CON is given by

$$CON = \frac{\int_0^\pi \left[D_\nu So_\nu(k_o^2c^2, v) + Se_\nu(k_o^2c^2, v)\right]^2 dv}{D_\nu So_\nu(k_o^2c^2, v') + Se_\nu(k_o^2c^2, v')}$$

Solving (25) analogous to the solution of (15) gives

$$G_> = \frac{i\pi}{2} \sum_{m=0}^{\infty} \frac{D_\nu So_\nu(k_o^2c^2, v) + Se_\nu(k_o^2c^2, v)}{CON} \cdot He_\nu^{(1)}(k_o^2c^2, u)$$

$$\cdot \left\{ Je_\nu(k_o^2c^2, u') - \left[\frac{k_o Z_1 Je_\nu(k_o^2c^2, u_1) - i Je_\nu'(k_o^2c^2, u_1)}{k_o Z_1 He_\nu^{(1)}(k_o^2c^2, u_1) - i He_\nu^{(1)\prime}(k_o^2c^2, u_1)}\right] He_\nu^{(1)}(k_o^2c^2, u') \right\} \tag{26}$$

where the prime denotes the derivative with respect to the argument.

In order to complete the solution the values of D_ν, ν and CON must be determined. First D_ν shall be found. From the boundary condition given by (22) and the expression for G given by (23) it is found that

$$D_\nu = - \left. \frac{\frac{\partial}{\partial v} Se_\nu(k_o^2c^2, v) \pm i k_o Z_2 Se_\nu(k_o^2c^2, v)}{\frac{\partial}{\partial v} So_\nu(k_o^2c^2, v) \pm i k_o Z_2 So_\nu(k_o^2c^2, v)} \right|_{v=0,\pi} . \tag{27}$$

It can be shown (Appendix) that

$$\left. \frac{\partial}{\partial v} Se_\nu(k_o^2c^2, v) \right|_{v=0} = \left. So_\nu(k_o^2c^2, v) \right|_{v=0} = 0$$

and

$$\left. \frac{\partial}{\partial v} So_\nu(k_o^2c^2, v) \right|_{v=0} = \left. Se_\nu(k_o^2c^2, v) \right|_{v=0} = 1 .$$

Thus (27) reduces to

$$D_\nu = -i k_o Z_2 . \tag{28}$$

Next the value of ν must be determined. After substituting (28) into (27) for $v=\pi$, it is found (Appendix) that (27) becomes

$$-ik_oZ_2 = \frac{-\alpha \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \mu \sin \mu\pi - ik_oZ_2 \alpha \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \cos \mu\pi}{\beta \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \mu \cos \mu\pi - ik_oZ_2 \beta \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \sin \mu\pi} \tag{29}$$

where the parameter μ is given by

$$\mu = 2m + p + \delta \quad .$$

The expression given in (29) can be reduced to

$$\tan(2m+p+\delta)\pi = \frac{-ik_0 Z_2\left(2m+p+\delta+\frac{\alpha}{\beta}\right)}{\frac{\alpha}{\beta}(2m+p+\delta)+k_0^2 Z_2^2} \quad . \qquad (30)$$

From (30), it is seen that δ is the solution of a transcendental equation. Determination of δ cannot be accomplished directly but it can be somewhat simplified. In the beginning of the problem, $k_o Z_2$ was restricted to be small to guarantee existence of the Leontovitch boundary conditions. If $k_o Z_2$ is taken as zero then (30) reduces to

$$\tan(2m+p+\delta)\pi = 0$$

which implies

$$\delta = 0 \quad .$$

Thus for small $k_o Z_2$, δ would be expected to be small. The expression given by (30) now reduces to

$$\delta\pi = \frac{-ik_0 Z_2\left(2m+p+\delta+\frac{\alpha}{\beta}\right)}{\frac{\alpha}{\beta}(2m+p+\delta)+k_0^2 Z_2^2} \quad . \qquad (31)$$

The expressions for α and β (Appendix) are given by

$$\alpha = \left[\sum_{m=-\infty}^{\infty} c^{\nu}_{2m+p} \right]^{-1} ; \qquad \beta = \left[\sum_{m=-\infty}^{\infty} (2m+p+\delta)\, c^{\nu}_{2m+p} \right]^{-1} .$$

If a new parameter ξ is defined as

$$\xi = \left[\sum_{m=-\infty}^{\infty} (2m+p)\, c^{\nu}_{2m+p} \right]^{-1}$$

then it is readily seen that

$$\frac{\alpha}{\beta} = \delta + \frac{\alpha}{\xi}$$

and the expression for δ now becomes

$$\delta\pi = \frac{-ik_0 Z_2 \left(2m+p+2\delta+\frac{\alpha}{\xi}\right)}{\left(\delta+\frac{\alpha}{\xi}\right)\left(2m+p+\delta\right)+k_0^2 Z_2^2} . \tag{32}$$

The order of the Mathieu functions, ν , can now be expressed as

$$\nu = m + \delta \tag{33}$$

where m is the index of summation given in (26).

Equation (32) cannot be solved in a straightforward manner since the Floquet coefficients, c^{ν}_{2m+p}, in the summations for α and ξ are functions of δ. A solution can be obtained by using an iterative process to solve (32). A method for solving for the Floquet coefficients is given by Tamir (1962). However, as it is given, the method calculates the Floquet coefficients of integer order only. As one might suspect, the method can be extended to calculation of the complex order coefficients.

Assuming that the method used for the calculation of the Floquet coefficients is at least as involved as the method given by Tamir, and that this method is then used to calculate the coefficients for each iteration in the solution of (32), the author has come to the conclusion that numerical evaluation of this problem is impractical. As was found for a semicircular boss (Webster and Tyras, 1966), the addition of a small surface impedance on the plane would be expected to have only a small effect upon the fields.

The last term which must be evaluated to complete the solution of the Green's function is the normalization constant CON. The numerator of CON is given by

$$NUM = \int_0^{\pi} \left[-ik_o Z_2 \, So_{\nu}(k_o^2 c^2, v) + Se_{\nu}(k_o^2 c^2, v) \right]^2 dv .$$

If the above term is evaluated using the Mathieu functions as given in the Appendix, the numerator of CON can be written in the following form

$$
\begin{aligned}
NUM = \sum_{m=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \zeta(m,r) \frac{c^{\nu}_{2m+p}\, c^{\nu}_{2r+p}}{\nu_m^2 - \nu_r^2} \Big\{ & (\alpha \nu_m - k_o^2 Z_2^2 \beta^2 \nu_r) \\
& (\sin \nu_m \pi \cos \nu_r \pi) - (\alpha^2 \nu_r - k_o^2 Z_2^2 \beta^2 \nu_m)(\sin \nu_r \pi \cos \nu_m \pi) \\
& - i\, 2\alpha\beta k_o Z_2 (\nu_m - \nu_m \cos \nu_m \pi \cos \nu_r \pi - \nu_r \sin \nu_m \pi \sin \nu_r \pi) \Big\} \\
+ \sum_{m=-\infty}^{\infty} (c^{\nu}_{2m+p})^2 \Big\{ & \frac{\pi}{2} (\alpha^2 - k_o^2 Z_2^2 \beta^2) + (\alpha^2 + k_o^2 Z_2^2 \beta^2) \frac{\sin 2\nu_m \pi}{4\nu_m} \\
& - i\alpha\beta k_o Z_2 \frac{\sin^2 \nu_m \pi}{\nu_m} \Big\}
\end{aligned}
\tag{34}
$$

where

$$
\zeta(m,r) = \begin{cases} 0 & m = r \\ 1 & m \neq r \end{cases}
$$

and

$$
\nu_m = 2m + p + \delta \ .
$$

Now the solution is complete. Substitution of (34) into the expression for CON and then using (28) and the solution to (29) in the expression for the Green's function given by (26) gives the desired solution. The electric and magnetic fields can be obtained by substituting $G_>$ and $G_<$ into (7), (8) and (9). As a check on the validity of the solution one should note that if Z_2 is set to zero, the Green's function reduces to the Green's function which was determined for the case of an impedance on the boss alone.

3.3 Green's Function for a Semicircular Cylindrical Boss

As was done in the first problem the elliptical boss is permitted to degenerate into a semicircular cylindrical boss. The surface impedance reduces to

$$Z_{surface} = \begin{cases} \dfrac{Z_0 Z_1}{b} & \text{on the boss} \\[2ex] \dfrac{Z_0 Z_2}{\rho} & \text{on the plane} \end{cases}$$

and the expressions for ν reduce to

$$\tan \nu\pi = \frac{-i 2 \nu k_0 Z_2}{\nu^2 + k_0^2 Z_2^2}$$

and $\nu = m + \delta$, where

$$\delta \sim \begin{cases} \sqrt{\dfrac{-i 2 k_0 Z_2}{\pi}} & m = 0 \\[2ex] \dfrac{-i 2 k_0 Z_2}{m} & m \neq 0 \end{cases}$$

The expression for $G_>$ reduces to

$$G_> = i\,2\pi k_o Z_2 \sum_{m=0}^{\infty} \frac{\nu^2(\nu \cos\nu\varphi' - ik_oZ_2 \sin\nu\varphi')(\nu\cos\nu\varphi - ik_oZ_2\sin\nu\varphi)}{(\nu^2 - k_o^2 Z_2^2)\left[2\pi\nu^2 k_o Z_2 + i(\nu^2 - k_o^2Z_2^2)\sin^2\nu\pi\right]}$$

$$\left\{ J_\nu(k_o\rho') - \left[\frac{Z_1 J_\nu(k_ob) - ib\,J_\nu'(k_ob)}{Z_1 H_\nu^{(1)}(k_ob) - ib\,H_\nu^{(1)\prime}(k_ob)}\right] H_\nu^{(1)}(k_o\rho')\right\} H_\nu^{(1)}(k_o\rho)$$

where J_ν (x) and $H_\nu^{(1)}(x)$ are Bessel functions and the prime denotes the derivative with respect to the argument.

These results are identical to those obtained by solving for the Green's function of the semicircular cylindrical boss directly (Webster and Tyras, 1966).

4. SUMMARY OF RESULTS

4.1 The Green's Functions and the Fields

For the case of a surface impedance on the boss alone, the Green's function is

$$G_7 = \frac{i\pi}{2} \sum_{m=0}^{\infty} \frac{1}{N_m} Se_m(k_o^2c^2, v)\, Se_m(k_o^2c^2, v')\, He_m^{(1)}(k_o^2c^2, u)$$

$$\cdot \left\{ Je_m(k_o^2c^2, u') - \left[\frac{k_o Z\, Je_m(k_o^2c^2, u_1) - i Je_m'(k_o^2c^2, u_1)}{k_o Z\, He_m^{(1)}(k_o^2c^2, u_1) - i He_m^{(1)\prime}(k_o^2c^2, u_1)} \right] He_m^{(1)}(k_o^2c^2, u') \right\} \tag{35}$$

N_m is the normalization constant given in the Appendix and $u = u_1$ describes the surface of the boss. The surface impedance must be of the form (Fig. 3 and Fig. 4)

$$Z_s = \frac{Z_o Z}{c(\cosh^2 u_1 - \cos^2 v)^{1/2}}$$

in order to have homogeneous boundary conditions. If a surface impedance is assumed to exist on both the boss and the conducting plane then the Green's function is

$$G_> = \frac{i\pi}{2} \sum_{m=0}^{\infty} \frac{1}{NUM} \left[Se_\nu(k_o^2c^2, v) - ik_o Z_2\, So_\nu(k_o^2c^2, v)\right]$$

$$\left[Se_\nu(k_o^2c^2, v') - ik_o Z_2\, So_\nu(k_o^2c^2, v')\right] \quad He_\nu^{(1)}(k_o^2c^2, u)$$

$$\left\{ Je_\nu(k_o^2c^2, u') - \left[\frac{k_o Z_1 Je_\nu(k_o^2c^2, u_1) - iJe_\nu'(k_o^2c^2, u_1)}{k_o Z_1 He_\nu^{(1)}(k_o^2c^2, u_1) - i\,He_\nu^{(1)\prime}(k_o^2c^2, u_1)}\right] He_\nu^{(1)}(k_o^2c^2, u') \right\} \tag{36}$$

where NUM is given by (34) and ν is determined by (32) and (33). The surface impedances must be of the form (Fig. 3, Fig. 4, and Fig. 6)

$$Z_{sb} = \frac{Z_o Z_1}{c(\cosh^2 u_1 - \cos^2 v)^{1/2}} \qquad Z_{sp} = \frac{Z_o Z_2}{c(\cosh^2 u - 1.0)^{1/2}}$$

where u_1, u and v are coordinates of the surface and Z_1 and Z_2 are small arbitrary constants. It is easily seen that is Z_1 is set to zero the Green's function (36) reduces to the Green's function (35).

The electric and magnetic fields can be determined from each of the Green's functions above from the expressions given below.

$$E_{uz} = - \frac{I}{c(\cosh^2 u - \cos^2 v)^{1/2}} \frac{\partial}{\partial v} G_z .$$

$$E_{vz} = \frac{I}{c(\cosh^2 u - \cos^2 v)^{1/2}} \frac{\partial}{\partial u} G_z$$

$$H_{zz} = i\omega\epsilon_o I G_z$$

4.2 Graphs of the Magnetic Field

The normalized magnetic field for the problem of a surface impedance existing on the boss alone, equation (35), has been evaluated for a particular set of parameters. The values of the parameters which were used in the evaluation are

1. $k_o b = 3.0$
2. $k_o^2 c^2 = 1.0,\ 3.0,\ 6.0,\ 8.0,\ 9.0$
3. $k_o \rho = 3.0$
4. $\varphi = 0^o$-- 180^o in 10^o steps
5. $k_o \rho' = 5.0$
6. $\varphi' = 0^o,\ 30^o,\ 60^o,\ 90^o$
7. $k_o Z = 0.0,\ 0.2$-10.09.

If one replaces the parameters a, b, c, ρ, ρ' and Z in Figures 1, 3, and 4 with $k_o a$, $k_o b$, $k_o c$, $k_o \rho$, $k_o \rho'$ and $k_o Z$

respectively, then the figures will represent the correct geometrical configuration and the correct surface impedance for the evaluated problem.

In the following figures, the magnitude of the normalized magnetic field, $|H_z| / \omega\epsilon_0 I$, is plotted for the source point alternately located at $k_o\rho' = 5.0$ with $\varphi' = 0°$, $30°$, $60°$ and $90°$ while the field point is rotated through $180°$ with $k_o\rho = 3.0$. The boss is varied from a near semi-circular cylindrical boss to a flat strip on the plane by varying the value of the parameter $k_o^2c^2$ from 1.0 to 9.0.

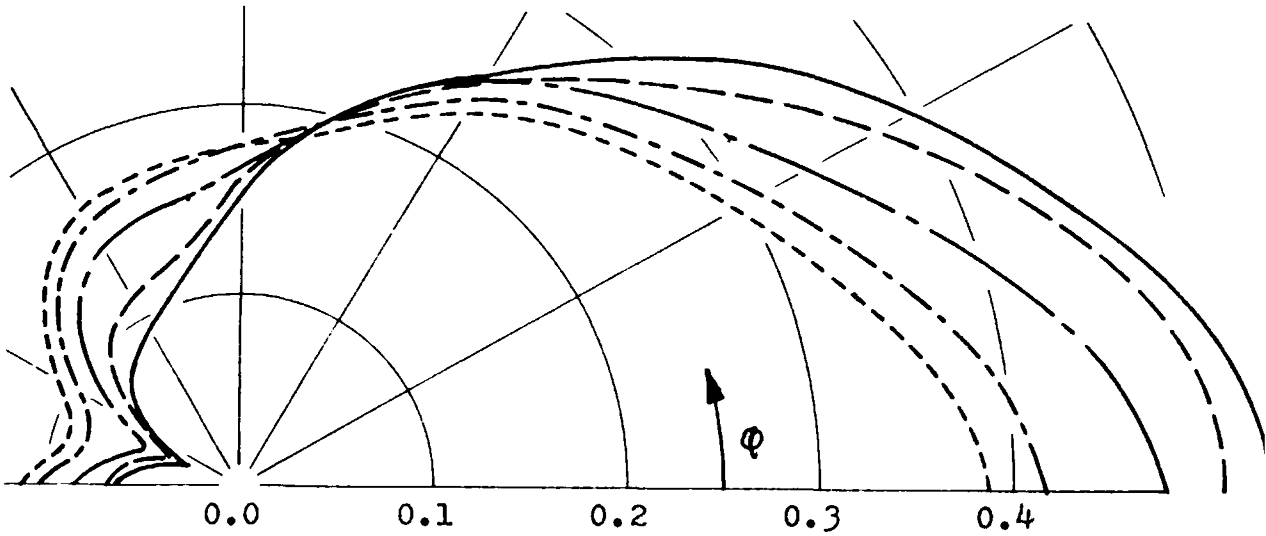


Fig. 7.--Graph of the magnetic field for $\varphi' = 0^{\circ}$ and $k_o Z = 0.0$

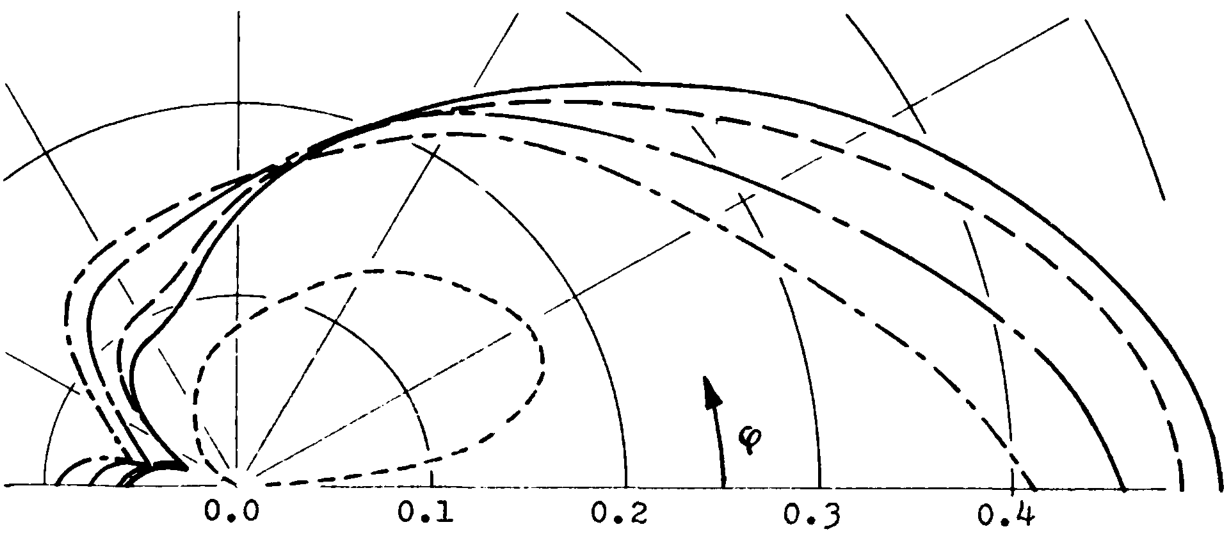


Fig. 8.--Graph of the magnetic field for $\varphi' = 0^{\circ}$ and $k_o Z = 0.2\text{-}10.09$

$k_o b = 3.0$

$k_o \rho = 3.0$

$k_o \rho' = 5.0$

$k_o^2 c^2 = 1.0$ ————————

$k_o^2 c^2 = 3.0$ — — — — — —

$k_o^2 c^2 = 6.0$ ———— - ———— -

$k_o^2 c^2 = 8.0$ —·—·—·—

$k_o^2 c^2 = 9.0$ - - - - - - - -

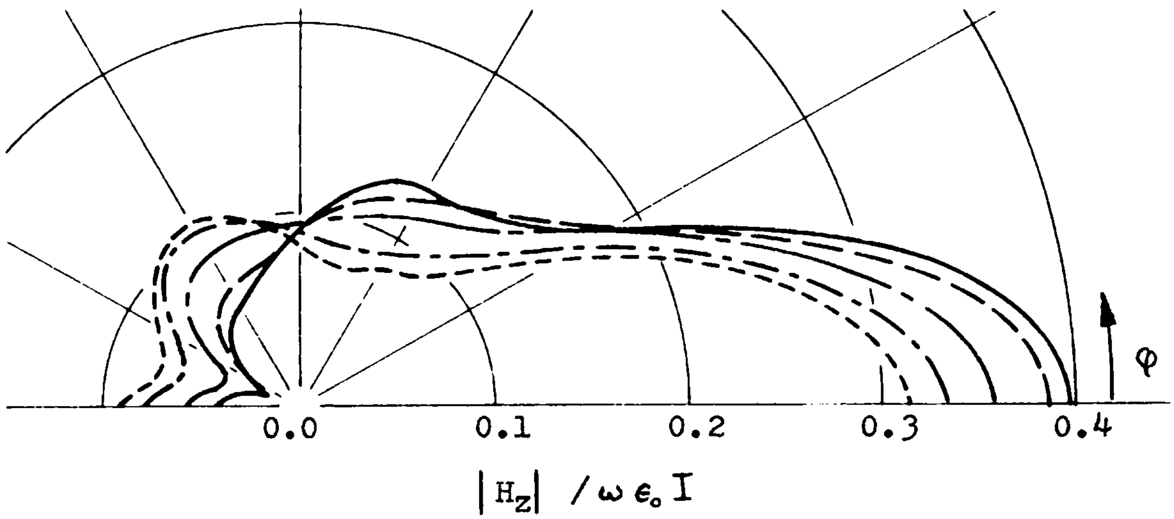


Fig. 9.--Graph of the magnetic field for $\varphi' = 30^{\circ}$ and $k_o Z = 0.0$

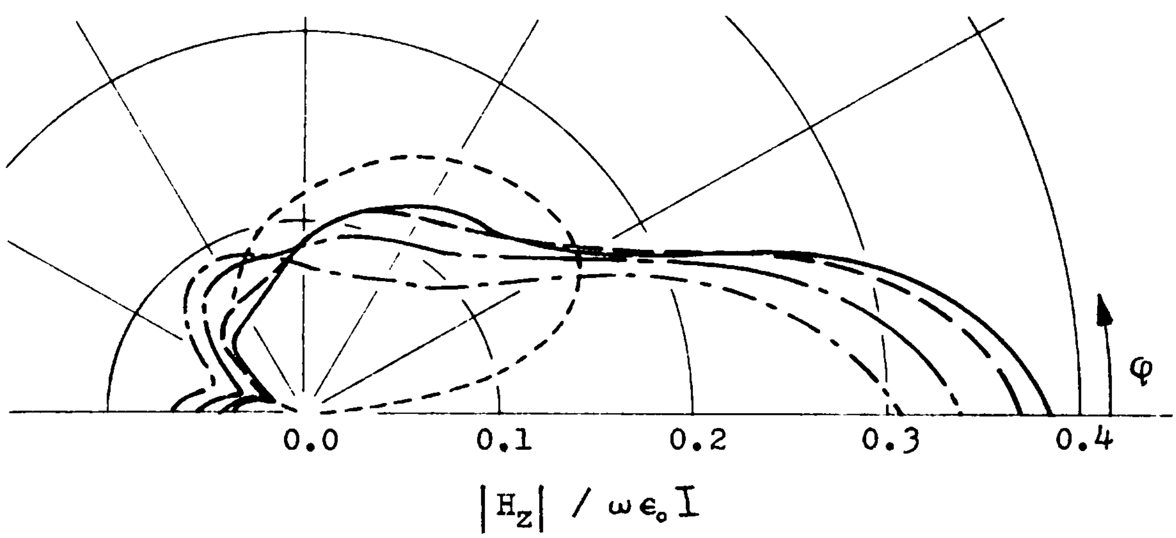


Fig. 10.--Graph of the magnetic field for $\varphi' = 30^{\circ}$ and $k_o Z = 0.2$-10.09

$k_o b = 3.0$

$k_o \rho = 3.0$

$k_o \rho' = 5.0$

$k_o^2 c^2 = 1.0$ ————————

$k_o^2 c^2 = 3.0$ — — — — — —

$k_o^2 c^2 = 6.0$ ——— - ———

$k_o^2 c^2 = 8.0$ — - — - — —

$k_o^2 c^2 = 9.0$ - - - - - - - -

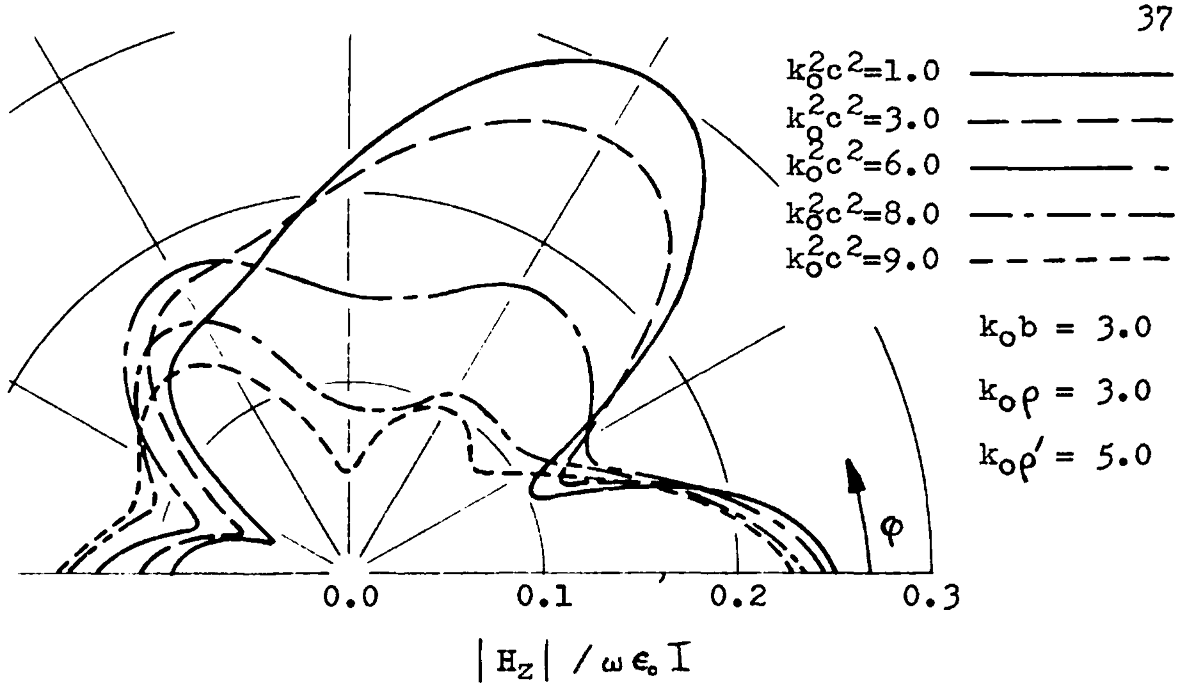


Fig. 11.--Graph of the magnetic field for $\varphi' = 60^o$ and $k_oZ = 0.0$

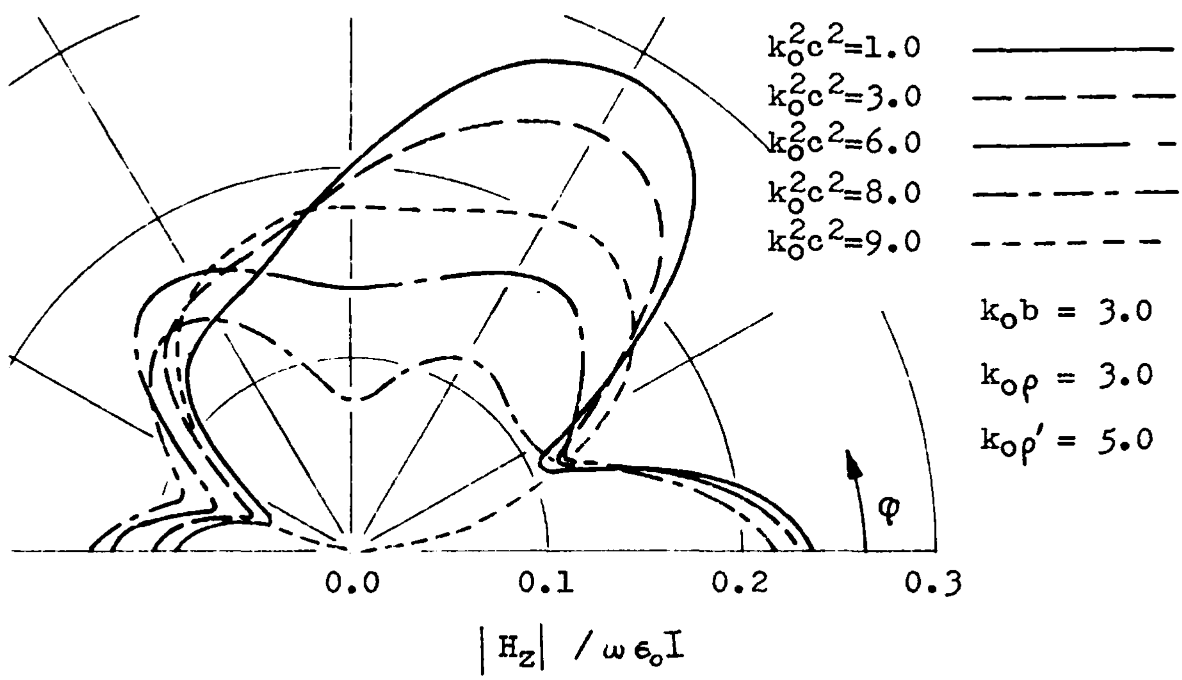


Fig. 12.--Graph of the magnetic field for $\varphi' = 60^o$ and $k_oZ = 0.2-i0.09$

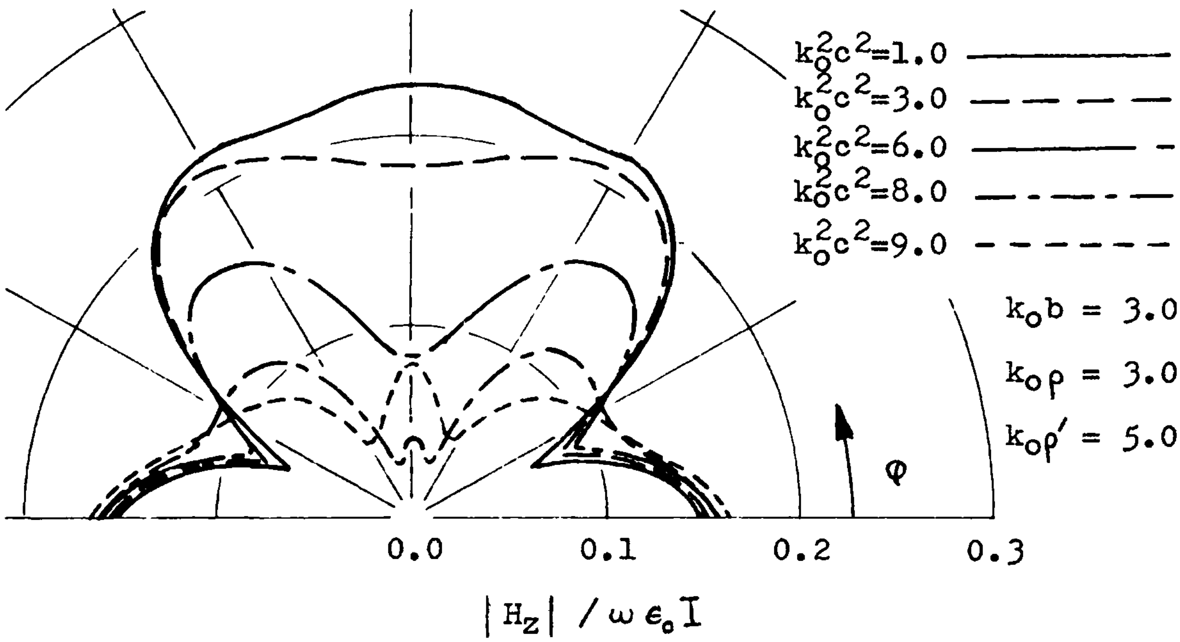


Fig. 13.--Graph of the magnetic field for $\varphi' = 90^{\circ}$ and $k_oZ = 0.0$

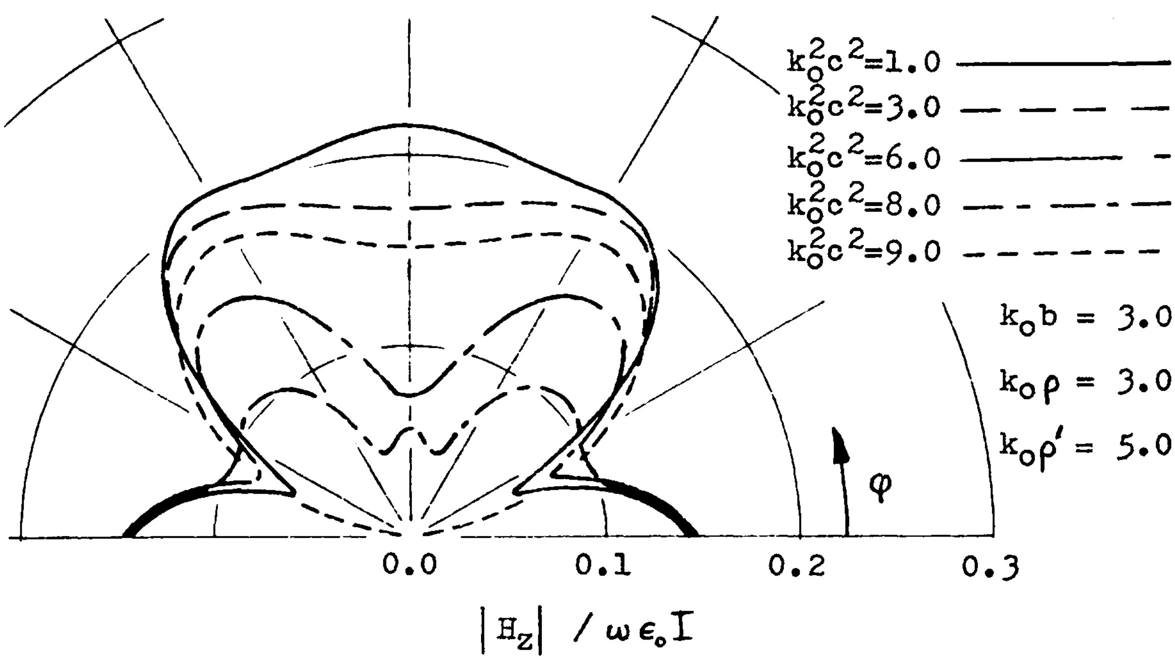


Fig. 14.--Graph of the magnetic field for $\varphi' = 90^{\circ}$ and k_oZ = 0.2-10.09

4.3 Discussion of the Graphical Results

From the graphs of the magnetic field in the previous section, it is seen that as the eccentricity of the elliptical boss is increased ($k_o^2c^2$ increased), the field on the same side of the boss as the source decreases while the field on the opposite side of the boss increases. This result is to be expected since as the eccentricity increases, the boss takes on a flatter profile; thus decreasing the backward scatter and increasing the forward scatter.

Although there seems to be a dependence between the angle of the maximum field and the angle of the source, one should be cautioned not to make a quick generalization. The total field is maximum in the direction of the source only if there exists constructive interference of the incident field and the reflected field in that direction. If destructive interference exists in the direction of the source, then the field will have a maximum in some other direction (Fig. 12 with $k_o^2c^2 = 6.0$).

For all values of $k_o^2c^2$ except $k_o^2c^2 = 9.0$, the effect of the surface impedance is to cause a small overall attenuation of the field. There is a rather drastic attenuation of the field for $k_o^2c^2 = 9.0$. This radical change can be explained upon investigation of the nature of the surface impedance on the boss (Fig. 3 and Fig. 4). As $k_o^2c^2$ approaches 9.0, the surface impedance increases without bound at the junction of the boss and the conducting plane ($v = 0, \pi$). There is a question as to the validity of the calculated field for this case since one of the requirements for the existence of the Leontovitch boundary conditions is that the surface impedance be small. Indeed this requirement is not satisfied near the edges of the boss. The requirement on the radius of curvature is satisfied however.

One could also question the validity of the other field plots since although the surface impedance is small, the radius of curvature is on the order of half a wavelength. Again, the Leontovitch boundary requirements are not satisfied. In spite of this apparent violation of the requirements, it is the opinion of the author that the plots are still valid since a small surface impedance would be expected to change the field only slightly from the field which exists for a perfectly conducting boss.

Although the parameters which were chosen in the evaluation of the magnetic field did not satisfy the radius of curvature requirement, it should be noted that it is possible to choose parameters which do satisfy the requirement. This corresponds to increasing the frequency of the source.

4.4 Some Concluding Remarks

In the solution of both of the problems considered in this thesis, the surface impedance on the boss was found to exhibit a specific functional dependence upon the coordinates of the surface of the boss. Now it is natural to ask, is this particular surface impedance physically realizable? The answer is affirmative. Westman (1961) gives the conductivity of rich agricultural land as approximately 1×10^{-2} mho/meter and the conductivity of rocky land and steep hills as approximately 2×10^{-3} mho/meter. Hence, if we postulate an island with an elliptical shape, rocky cliffs at the shoreline and a gradual transition from the rocky cliffs to rich agricultural land at the center of the island, then the desired surface impedance profile will be obtained. Other suitable islands may be postulated.

Next one should attempt to postulate a physical problem which could be represented by the problem with the surface impedance on the conducting plane. To that end, consider the problem of an electromagnetic wave impinging upon an island which is surrounded by a shallow submerged land shelf which increases in depth with increased distance from the island. The shelf with the sea above is essentially a two-layer earth problem in which the upper layer is a much better conductor than the lower layer (Watt, 1963). Watt has shown that for a thin upper layer ($\hat{\gamma}_1 h \ll 1$), a vertically incident wave encounters an apparent surface impedance, Z_a, given by

$$Z_a \sim \frac{Z_1}{\gamma_1 h}$$

where Z_1 is the intrinsic impedance of the sea, γ_1 is the propagation constant of an electromagnetic wave in the sea and h is the depth of the shelf below the surface of the sea.

Assume that the depth of the shelf exhibits a functional dependence upon the horizontal distance from the center of the island given by

$$h = d(x^2-c^2)^{\frac{1}{2}} \tag{37}$$

where d is a constant and c is the semifocal distance of the island. The apparent surface impedance then becomes

$$Z_a \sim \frac{Z_1}{\gamma_1 d\,(x^2-c^2)^{1/2}}$$

The constant d is assumed to be small so that the depth of the shelf is small. Next let

$$Z_0 Z_2 = \frac{Z_1}{\gamma_1 d}$$

then the apparent surface impedance becomes identical with the surface impedance of the therorectical problem.

$$Z_a \sim \frac{Z_0 Z_2}{(x^2-c^2)^{1/2}} \quad .$$

Since Z_a is assumed to be small, the above expression is a valid approximation if

$$|Z_0 Z_2| \ll (\chi^2 - c^2)^{1/2} .$$

Thus Z_a is small and the requirements for the existence of the Leontovitch boundary conditions are again satisfied. It would therefore seem that the problem with the surface impedance on both the boss and the conducting plane would be a good model of an island surrounded by a shallow submerged land shelf which increases in depth with increased distance from the island according to equation (37).

In this thesis only two problems have been considered, a surface impedance on the boss only and a surface impedance on both the boss and the conducting plane. As a third problem, a surface impedance existing on the conducting plane alone could have been investigated. This is not necessary however, since if one sets Z_1 to zero in (36) the desired solution is obtained.

APPENDIX

MATHIEU FUNCTIONS

Mathieu functions are solutions of the Mathieu differential equation

$$\frac{d^2 P}{d v^2} + (b - k_o^2 c^2 \cos^2 v) P = 0$$

where P represents the even periodic function $Se_\nu(k_o^2c^2, v)$, or the odd periodic function $So_\nu(k_o^2c^2, v)$. The parameters b and $k_o^2c^2$ are constants. For integer order the Mathieu functions are given by

$$Se_{2r+p}(k_o^2c^2, v) = \sum_{m=0}^{\infty} De_{2m+p}^{2r+p} \cos(2m+p)v$$

$$So_{2r+p}(k_o^2c^2, v) = \sum_{m=0}^{\infty} Do_{2m+p}^{2r+p} \sin(2m+p)v$$

where p = 0, 1 (Blanch, 1951). When p = 0 the functions are of period π and for p = 1 the functions are of period 2π. The functions are orthogonal on the interval 2π, thus for the even periodic functions the following relationship holds.

$$\int_0^{2\pi} Se_r(k_o^2c^2, v)\; Se_m(k_o^2c^2, v)\, dv = \begin{cases} 0 & \text{for } r \neq m \\ N_r & \text{for } r = m \end{cases}$$

where the normalization constant, N_r, is given by

$$N_r = \begin{cases} \pi \left[2(De_0^r)^2 + (De_2^r)^2 + \cdots \right] \\ \pi \left[(De_1^r)^2 + (De_3^r)^2 + \cdots \right] \end{cases}$$

The coefficients, De_{2m+p}^{2r+p} and Do_{2m+p}^{2r+p}, and the normalization constant, N_r, are tabulated (Blanch, 1951).

The modified Mathieu functions are solutions of the modified Mathieu differential equation,

$$\frac{d^2P}{d^2u} - (b - k_o^2c^2 \cosh^2 u)\; P = 0$$

where b and $k_o^2c^2$ are again constants. The modified functions which are used in this thesis are given by

$$Je_{2r+p}(k_o^2c^2, u) = (-1)^r \sum_{m=0}^{\infty} (-1)^m De_{2m+p}^{2r+p}\, J_{2m+p}(k_oc \cosh u)$$

$$He_{2r+p}^{(1)}(k_o^2c^2, u) = (-1)^r \sum_{m=0}^{\infty} (-1)^m De_{2m+p}^{2r+p}\, H_{2m+p}^{(1)}(k_oc \cosh u)$$

where $J_n(x)$ and H_n (x) are Bessel functions (Blanch, 1951).

For non-integer order, the Mathieu functions are defined as follows

$$Se_\nu(k_o^2c^2,x) = \alpha \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \cos(2m+p+\delta)\, x$$

$$So_\nu(k_o^2c^2,x) = \beta \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \sin(2m+p+\delta)\, x$$

$$Je_\nu(k_o^2c^2,x) = \alpha \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu}\, J_{2m+p+\delta}(k_o c \cosh x)$$

$$He_\nu^{(1)}(k_o^2c^2,x) = \alpha \sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu}\, H_{2m+p+\delta}^{(1)}(k_o c \cosh x)$$

where the coefficients, c_{2m+p}^{ν}, are the Floquet coefficients (Blanch, 1965). The normalization constants α and β are determined by the following expressions

$$\alpha = \left[\sum_{m=-\infty}^{\infty} c_{2m+p}^{\nu} \right]^{-1}$$

$$\beta = \left[\sum_{m=-\infty}^{\infty} (2m+p+\delta)\, c_{2m+p}^{\nu} \right]^{-1} .$$

If the integer m is the closest integer to the real part of ν, then the parameter δ is obtained from

$$\nu = m + \delta .$$

When m is an even integer, then p = 0, and when m is an odd integer, p = 1.

It can be shown that the Wronskian of the Mathieu functions is a constant (McLachlan, 1964). The modified Mathieu functions used in this thesis have been normalized such that for large argument they degenerate into the appropriate Bessel functions,

$$Je_{2r+p}(k_o^2c^2, x) \longrightarrow J_{2r+p}(k_o\rho)$$

$$He^{(1)}_{2r+p}(k_o^2c^2, x) \longrightarrow H^{(1)}_{2r+p}(k_o\rho)$$

$$\frac{d}{dx}\; Je_{2r+p}(k_o^2c^2, x) \longrightarrow k_o\rho\; J'_{2r+p}(k_o\rho)$$

$$\frac{d}{dx}\; He^{(1)}_{2r+p}(k_o^2c^2, x) \longrightarrow k_o\rho\; H^{(1)\prime}_{2r+p}(k_o\rho)$$

where $k_o\rho = k_oc \cosh x$ and the prime denotes the derivative with respect to the argument of the function. Hence it is easily seen that the Wronskian is given by

$$W\left\{ Je_{2r+p}(k_o^2c^2, x)\,;\, He^{(1)}_{2r+p}(k_o^2c^2, x)\right\} = \frac{2i}{\pi} \,.$$

LIST OF REFERENCES

Blanch, G. (1951), Tables Relating to Mathieu Functions, edited by A. N. Lowan, Columbia University Press.

Blanch, G. (1965), Handbook of Mathematical Functions, NBS Appl. Math. Series #55, edited by M. Abramowitz and I. Stegun, U.S. Gov't Printing Office, pp. 721-750.

Brekhovskiskh, L.M. (1960), Waves in Layered Media, Academic Press Inc., pp. 14-15.

Jones, D.S. (1964), The Theory of Electromagnetism, The Macmillan Co., p. 325.

McLachlan, N.W. (1964), Theory and Application of Mathieu Functions, Dover Pub. Inc.

Sagan, H. (1961), Boundary and Eigenvalue Problems in Mathematical Physics, John Wiley and Sons, Inc., p. 163.

Tamir, T. (1962), "Characteristic Exponents of Mathieu Functions," Math. Comp. Vol. 16, #77, pp. 100-105.

Wait, J.R. and Murphy, A. (1957), "Influence of a Ridge on the Low-Frequency Ground Wave," NBS Jnl. Research Vol. 58, #1, pp. 1-5.

Wait, J.R. and Murphy, A. (1958), "Further Studies of the Influence of a Ridge on the Low-Frequency Ground Wave," NBS Jnl. Research Vol. 61, #1, pp. 57-60.

Watt, A.D., et. al. (1963), "Some Electrical Characteristics of the Earth's Crust," Proc. IEEE, Vol. 51, pp. 897-910.

Webster, W.P. and Tyras, G. (1966), "Field of a Magnetic Line Source in the Presence of a Semicylindrical or Semielliptical Boss," 1966 IEEE Region Six Conference Record, (to be published).

Westman, H.P. (1961), Reference Data for Radio Engineers, American Book-Stratford Press Inc., p. 714.