A CLOSED LOOP TECHNIQUE
FOR AIRCRAFT PERFORMANCE OPTIMIZATION

BY

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In this thesis a solution technique is developed which allows complex aircraft performance optimization problems to be solved using a simplified mathematical model of the aircraft. This technique is a variation to the normal calculus of variations methodology. The method studied employs a repetitive solution or closed loop technique which forces the simplified aircraft to obey the physical constraints acting on the real aircraft, while allowing the problem to be solved using the control equations obtained from the simplified aircraft.

The technique is applied to the problem of finding the minimum time aircraft trajectory between two points in range-altitude space. It is found that the closed loop technique yields optimal solutions as long as the forces neglected in the simplified model act in the direction of motion and are not functions of either the state or control variables of the aircraft.
CHAPTER I

INTRODUCTION

This thesis deals with some of the difficulties involved in the utilization of calculus of variations (C.O.V.) techniques to solve performance optimization problems. Also presented is a variation to the normal C.O.V. methodology. This variation proves to be a useful tool when solving a certain class of performance optimization problems.

In order to employ C.O.V. techniques, the system whose performance is to be optimized must be described mathematically. When sophisticated systems are the object of performance optimization, applying the theory of the calculus of variations to the system accurately described, leads to analytic difficulties due to the complicated nature of the resulting state and control equations. Solving complicated variational problems becomes unmanageable not only because the state equations themselves are exceedingly complicated but also because Euler-Lagrange equations contain unknown Lagrange multipliers. Solutions using C.O.V. require solving transversality conditions in addition to the Euler-Lagrange equations. It has been shown (See References 1, 2, 3) that the optimal endpoint for a specific performance optimization problem is highly sensitive to the initial values chosen for the Lagrange multipliers appearing in the optimal control equations. As a consequence, when
solving complicated two point boundary value problems using C.O.V.
techniques, it is almost impossible to find the correct initial 
values of the Lagrange multipliers necessary to satisfy all prescribed 
boundary conditions. The use of computerized convergence routines to 
guess these initial values is time consuming and impractical since 
for most problems several multipliers appear in the control equations.

This thesis investigates a technique which allows complex 
variational problems to be solved by using a simplified mathematical 
model of the system to be optimized. It is recognized that simplifying 
the mathematical model of the desired system will not necessarily elimi­
nate the Lagrange multipliers but in most cases it increases the 
tractability of the problem. The simplified model is utilized by incor­
porating a repetitive solution technique which yields the control as 
a function of the state (position, velocity, etc,) of the system (closed 
loop control). The application of normal calculus of variations tech­
niques to solve performance optimization problems yields open loop 
control, with the control a function of time and the initial values 
of the control and state variables.

In this thesis the problem of finding the minimum time aircraft 
trajectory between two points in range altitude space is investigated. 
Performance optimization is carried out on an assumed, real aircraft 
using normal C.O.V. techniques. A simplified model of the assumed, 
real aircraft is then subjected to the repetitive solution optimization 
technique mentioned above. This technique will henceforth be referred 
to as the closed loop technique. The resultant trajectories of the
real and simplified aircrafts are then compared to assess the feasibility of using the closed loop technique to optimize a system by utilizing a simplified model of the system.
CHAPTER II

CALCULUS OF VARIATIONS

In order to discuss the closed loop solution technique, it is first necessary to present, in brief, the methodology normally used in solving a performance optimization problem. The following sections present the formulation of a general optimization problem and the application of the calculus of variations to yield optimal control and endpoint conditions.

Only a summary of the necessary conditions for an optimal solution by the methods of the calculus of variations is presented. Derivation of these conditions are readily found in the literature (see References 4 and 5).

2.1 The Problem of Bolza

A general calculus of variations problem is the problem of Bolza in which it is desired to extremize the following functional form (performance index):

\[ Z(y_1, y_{i_f}, t_1, t_f) + \int_{t_1}^{t_f} F(y_i, u_k, t) dt \]  \hspace{1cm} (2-1)

\[ i = 1, \ldots, n \hspace{1cm} K = 1, \ldots, m \hspace{1cm} m < n \]
subject to the differential equations of constraint

\[ y_i' = \phi_i(y_1, \ldots, y_n, u_1, \ldots, u_k, t) \quad (2-2) \]

and the endpoint conditions

\[ \psi_e(y_{i1}, y_{i2}, t_1, t_f) = 0 \quad (2-3) \]

\[ e = 1, \ldots, \nu < 2(n+1) \]

In this formulation the state variables are represented by the vector \( y_i \) and the control variables by the vector \( u_k \); \( t \) is the independent variable.

In the general problem of Bolza, the performance index may be expressed in terms of both a function \( (Z) \) and an integral \( (F) \). If \( Z = 0 \) the Bolza problem becomes the problem of Lagrange, and if \( F = 0 \) the problem is referred to as the problem of Mayer.

2.2 Optimal Control Conditions

To examine extremal solutions to the functional given by Eq. (2-1) subject to differential equations of constraint and endpoint conditions (Eqs. 2-2, 2-3), a Hamiltonian function, \( H \), is formed in terms of Lagrange multipliers which adjoin the differential equations of constraint (Eq. 2-2) to the integrand, \( F \), in the performance index (Eq. 2-1).

\[ H = \lambda_i \phi_i - F \quad (2-4) \]

\[ i = 1, \ldots, n \]
In Eq. (2-4), \( \lambda_i \) is a vector of Lagrange multipliers which are functions of \( (t) \), the independent variable. In terms of the Hamiltonian function, the calculus of variations requires that, for the performance index to be extremal, the following Euler equations must be satisfied.

**State-Variable Euler Equations:**

\[
\frac{\partial H}{\partial y_i} + \dot{\lambda}_i = 0 \quad , \quad i = 1, \ldots, n
\]  

(2-5)

**Control-Variable Euler Equations:**

\[
\frac{\partial H}{\partial u_k} = 0 \quad , \quad k = 1, \ldots, m
\]  

(2-6)

The state variable and control variable Euler equations have as a first integral:

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t}
\]

(2-7)

The first integral, Eq. (2-7), often provides useful information when solving for the optimal control.

By the manipulation of the Euler Equations (Eqs. 2-5, 2-6), the first integral (Eq. 2-7), and the system's equations of constraint (Eq. 2-2), the control necessary to extremize the performance index (Eq. 2-1) can be determined as a function of the independent variable, \( t \).
2.3 Optimal Endpoint Conditions

The optimal endpoints of an extremal solution are determined by forming a $G$ function which uses Lagrange multipliers to adjoint the endpoint conditions, Eq. (2-3), to the function, $Z$, in the performance index (Eq. 2-1).

$$G(y_{il}, y_{if}, t_1, t_f) = Z(y_{il}, y_{if}, t_1, t_f) + \psi e(y_{il}, y_{if}, t_1, t_f)$$  (2-8)

In Eq. (2-8), $\psi e$ is a vector of Lagrange multipliers which are constants.

Once extremal control is established by the methods presented in the previous section (Section 2.2), the calculus of variations prescribes a set of transversality conditions which must be satisfied at the endpoints of an optimal solution in addition to satisfying the endpoint conditions (Eq. 2-3). In terms of the $G$ function, Eq. (2-8), the transversality conditions are written as follows:

Point 1:

$$\frac{\partial G}{\partial y_{il}} - \lambda_{il} = 0$$  (2-9)

$$\frac{\partial G}{\partial t_1} + \psi e_1 = 0$$  (2-10)

Point f:

$$\frac{\partial G}{\partial y_{if}} + \lambda_{if} = 0$$  (2-11)

$$\frac{\partial G}{\partial t_f} - \psi e_f = 0$$  (2-12)
Manipulation of the transversality and endpoint conditions generally provides relationships which indicate when the optimal endpoint of an optimal solution has been reached.
Attempting to optimize the performance of complicated systems often leads to analytic difficulties as pointed out in Chapter I. Hence, it would appear desirable to make certain assumptions in order to simplify the mathematical model of the system. It should be recognized, however, that simplifying the mathematical model of a system is in effect changing the constraint equation under which the system operates. It is not desirable to simplify a system to the extent that physical constraints are actually eliminated.

To insure an extremal performance index (Eq. 2-1) the calculus of variations dictates that the system must obey both its constraint equation (Eq. 2-2) and the Euler equations (Eqs. 2-5, 2-6). It will now be hypothesized that if a system is mathematically simplified in such a way that the Euler equations of the real system and the simplified system are identical, a repetitive solution (closed loop) technique can be utilized to yield extremal solutions to the real system using the optimal control obtained from the simplified system. This control will be referred to as "suboptimal" with respect to the real system. The closed loop technique is used to force the simplified system to obey the constraints of the real system, thus providing the necessary conditions for an optimal solution to the real system; i.e., both the real system's Euler equations and constraints are now satisfied.
The closed loop technique is implemented by integrating the real system's equations of motion using the "suboptimal" control, while at the same time integrating the simplified system's equation of motion using the identical control. During the integration, the states (velocity, position, etc.) of the real and simplified systems are continuously compared. When a significant difference in the state of the two systems is evident, the value of the control variable is updated to account for the inaccuracies in the simplified mathematical model of the system, from which the control function was derived. This is accomplished by stopping the solution and recalculating the proper instantaneous value (instantaneous initial condition) of the control variable such that the desired endpoint will be reached, using the current values of the real system's state variables as instantaneous initial conditions on the state variables in the simplified system. Figure 3.1 graphically presents the closed loop methodology.
Figure 3.1 Flow Chart of Closed Loop Technique
CHAPTER IV

EXAMPLE PROBLEM FORMULATION

In order to illustrate the closed loop solution technique, a real system will be mathematically modeled and its performance will be optimized with normal calculus of variations techniques. The real system will then be mathematically simplified in accordance with the hypothesis in Chapter III, and the resultant simplified system will be utilized to illustrate the validity of the closed loop methodology.

4.1 Definition of the Problem

The example problem investigated by this thesis is the two point, time optimal flight path problem. The problem is to determine the time optimal aircraft trajectory between two points in range altitude space. The real aircraft (real system) is assumed to operate in a plane under the forces shown in Figure 4.1.

\[\begin{align*} y, \text{ Altitude} \\
L & \quad \text{Flight Path} \\
\gamma & \quad \text{Flight Path Angle} \\
mg & \quad \text{Horizontal Reference} \\
x, \text{ Range} \end{align*}\]

Figure 4.1 Applied Forces Acting on the Assumed Real Aircraft
Where:

\[ T = \text{Thrust force: Assumed to act tangent to the flight path.} \]

For this example the thrust is assumed to be a constant throughout any particular trajectory. This assumption has been shown to be valid for the types of trajectories investigated by this thesis (see Reference 3).

\[ L = \text{Lift force: The lift force is assumed to act perpendicular to the flight path. No bounds are placed on the values of lift necessary to fly any given trajectory. Hence, by assuming the lift to be unbounded, no analytical representation of the lift force will be needed for the analysis.} \]

\[ mg = \text{Weight of the aircraft: The gravitational force will be assumed to act perpendicular to the horizontal reference line. The value of g will be assumed to be constant (32.17 ft/sec}^2\) over the region of space in which the model aircraft operates. The mass \( m \) of the aircraft is also assumed to be constant for any given trajectory.} \]

It is obvious that this model does not represent an actual aircraft since it is assumed to have no drag forces and a constant thrust and mass. However, all that is intended here is to demonstrate a solution technique, for which this model is well suited. All that is needed in order to demonstrate the technique is a mathematical definition of a real system. It does not matter if the real system is correct with respect to the physical world.

Writing the equations of motion for the aircraft in Figure 4.1 in the normal and tangential directions yields:
\[ m \frac{v}{m} \dot{y} = L - mg \cos \gamma \tag{4-1} \]
\[ m \frac{v}{m} \dot{v} = T - mg \sin \gamma \tag{4-2} \]

The following kinematical relationships are also applicable between the range and altitude variables:

\[ \dot{x} = v \cos \gamma \tag{4-3} \]
\[ \dot{y} = v \sin \gamma \tag{4-4} \]

Since no constraints or bounds are imposed on the lift and since the lift force is not contained in Eqs. (4-2), (4-3), and (4-4), Eq. (4-1) is uncoupled from the other equations and may be dropped in the ensuing analysis.

Equations (4-2), (4-3), and (4-4) mathematically describe the real system. These three equations contain four variables: \( x, y, v, \) and \( \gamma \). Hence, one degree of freedom remains for control. In this case \( \gamma \) is the control variable and \( x, y, \) and \( v \) are the state variables of the system. No further constraints are imposed on the aircraft.

For mathematical simplification the real system's equations of motion will be non-dimensionalized. For this purpose a reference velocity is defined as the non-dimensionalizing parameter. The velocity to be used is that of an aircraft in steady, level flight at \( C_L = 1.0 \). Therefore, from the classical definition of lift we obtain:

\[ v^2 = \frac{2mg}{\rho_o A} \tag{4-5} \]
Utilizing Eq. (4-5) the following non-dimensional (N.D.) parameters are defined:

- **N.D. velocity** \( u = v/v_r \)  \hspace{1cm} (4-6)
- **N.D. range** \( \xi = gx/v_r^2 \)  \hspace{1cm} (4-7)
- **N.D. altitude** \( \eta = gy/v_r^2 \)  \hspace{1cm} (4-8)
- **N.D. time** \( \tau = gt/v_r \)  \hspace{1cm} (4-9)
- **N.D. thrust** \( k = T/mg \)  \hspace{1cm} (4-10)

In terms of these non-dimensional parameters, Eqs. (4-2), (4-3), and (4-4) become:

\[
\begin{align*}
\ddot{u} &= k - \sin \gamma \\
\dot{\xi} &= u \cos \gamma \\
\dot{\eta} &= u \sin \gamma 
\end{align*}
\hspace{1cm} (4-11-13)

### 4.2 Solution by Standard Open Loop Calculus of Variations Methods

At this point it is necessary to apply the theory of the calculus of variations to obtain the optimal open loop control for the real system. The trajectory optimization to be performed represents a special case of the problem of Bolza (see Chapter II). In this example the Lagrange formulation of the performance index will be used.

Here, we wish to extremize the performance index,

\[
\int_{t_1}^{t_f} dt \hspace{1cm} (4-14)
\]
subject to the constraints given by Eqs. (4-11), (4-12), and (4-13). The system is also subject to boundary conditions of the form shown in Table 4.1.

Table 4.1

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<td>INITIAL POINT (1)</td>
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</tr>
<tr>
<td>( \xi_1 = 0 )</td>
</tr>
<tr>
<td>( \eta_1 = 0 )</td>
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Applying the variational theory, the Hamiltonian function is formed.

\[
H = \lambda_u (k - \sin \gamma) + \lambda_\xi (u \cos \gamma) + \lambda_\eta (u \sin \gamma) - 1 \tag{4-15}
\]

As discussed in Chapter II for the performance index, Eq. (4-14), to be extremal the Euler equations (Eqs. 2-5, 2-6) must be satisfied. For this example the Euler equations are written as follows:

State Variable Euler Equations:

- \( u \) Euler: \( \lambda_\xi \cos \gamma + \lambda_\eta \sin \gamma + \lambda_u \gamma - 0 \) \tag{4-16}
- \( \xi \) Euler: \( \lambda_\xi = 0 \) \tag{4-17}
- \( \eta \) Euler: \( \lambda_\eta = 0 \) \tag{4-18}
Control Variable Euler Equation:

\[-\lambda_u \cos \gamma - \lambda_\xi u \sin \gamma + \lambda_\eta u \cos \gamma = 0\]  \hspace{1cm} (4-19)

The first integral, Eq. (2-7), also yields useful information.

\[\frac{dH}{dt} = 0\]  \hspace{1cm} (4-20)

Hence,

\[H = \text{constant}\]

In order to carry out the performance optimization it is necessary to write the transversality conditions for the aircraft system; for this purpose the G function is formed.

\[G = \mu_1(\tau_1) + \mu_2(\xi_1) + \mu_3(\eta_1) + \mu_4(u_1 - c_1) + \mu_5(\xi_f - c_2) + \mu_6(\eta_f - c_3)\]

\hspace{1cm} (4-21)

The transversality conditions are written as follows:

**POINT 1:**  
\[\mu_4 - \lambda_{u1} = 0\]  \hspace{1cm} (4-22)  
\[\mu_3 - \lambda_{\xi1} = 0\]  \hspace{1cm} (4-23)  
\[\mu_2 - \lambda_{\eta1} = 0\]  \hspace{1cm} (4-24)  
\[\mu_1 + H_1 = 0\]  \hspace{1cm} (4-25)

**POINT f:**  
\[\lambda_{uf} = 0\]  \hspace{1cm} (4-26)  
\[\mu_5 + \lambda_{\xi_f} = 0\]  \hspace{1cm} (4-27)  
\[\mu_6 + \lambda_{\eta_f} = 0\]  \hspace{1cm} (4-28)  
\[H_f = 0\]  \hspace{1cm} (4-29)
Combining Eqs. (4-20) and (4-29), Eq. (4-15) may be written as

\[ \lambda_u (k - \sin \gamma) + \lambda_\xi (u \cos \gamma) + \lambda_\eta (u \sin \gamma) = 1. \] (4-30)

Equations (4-16) through (4-19) and (4-30) may now be combined into a single differential equation, the solution of which determines the optimal control. To obtain the control, equation (4-19) is solved for the control variable \( \gamma \),

\[ \lambda_\xi \tan \gamma = \lambda_\eta - \frac{\lambda u}{u}, \quad (\cos \gamma \neq 0) \] (4-31)

and differentiated with respect to \( \tau \) (noting from Eq. (4-17) that \( \lambda_\xi' = 0 \)), to yield

\[ \lambda_\xi \sec^2 \gamma \gamma' = \lambda_\eta' - \frac{(u \lambda_\xi' - \lambda_\eta u')}{u^2}. \] (4-32)

The derivatives on the right hand side of Eq. (4-32) are now eliminated by substituting the remaining Euler equations (4-16) and (4-18) and the equation of constraint (4-11) into equation (4-32) to give,

\[ \lambda_\xi \sec^2 \gamma \gamma' = \frac{1}{u^2} \left[ \lambda_\xi u \cos \gamma + \lambda_\eta u \sin \gamma + \lambda_u (k - \sin \gamma) \right]. \] (4-33)

Note that the term in brackets on the right side of Eq. (4-33) is unity according to Eq. (4-30). Equation (4-33) may therefore be written as

\[ \gamma' = \left(\frac{\cos \gamma}{u}\right) \left(\frac{1}{\lambda_\xi}\right); \] (4-34)
This differential control equation is a combination of all of the Euler equations, and its solution in conjunction with the equations of constraint, and the transversality and endpoint conditions will yield a minimum time trajectory between two points in range altitude space $(\xi - \eta)$.

To determine the optimal endpoint of the optimal trajectory, note that Eqs. (4-19) and (4-30) can be combined to yield,

$$\lambda_{u} k = 1 - \lambda_{\xi} u \sec \gamma .$$  \hspace{1cm} (4-35)

Combining Eqs. (4-35) and (4-26) yields a necessary condition for the optimal endpoint.

$$\lambda_{\xi f} = \frac{\cos \gamma_{f}}{u_{f}}$$  \hspace{1cm} (4-36)

Hence, whenever Eq. (4-36) is satisfied the optimal endpoint of the trajectory has been reached.

In summary, optimal open loop performance of the aircraft is obtained by the simultaneous solution of the constraint equations (4-11) - (4-13), and the control equation (4-34) subject to the endpoint condition given by Eq. (4-36).

4.3 Implementation of a Closed Loop Method of Solution

To demonstrate the closed loop technique assume that the problem described in Section 4.1 is to be solved using a simplified mathematical model of the aircraft. Based on the hypothesis presented in Chapter III, the aircraft model must be simplified in such a manner that the
Euler equations (Eqs. 2-5 and 2-6) of the real and simplified aircraft systems are identical. An investigation of the Euler equations indicates that under this hypothesis only forces acting in the direction of motion which are not functions of either the state or control variables may be neglected when simplifying the real system. Hence, to satisfy the hypothesis the thrust force on the aircraft in the example problem will be neglected. The omission of the thrust force is a gross simplification of the real system since it is one of the two prescribed forces acting on the aircraft.

The simplified aircraft is therefore assumed to operate in a plane under the forces shown in Figure 4.2. The definition of the prescribed forces is the same as presented in Section 4.1 for the real aircraft.

![Figure 4.2 Applied Forces Acting on the Simplified Aircraft](image_url)
Proceeding as in Section 4.1 the non-dimensional constraint equations for the simplified aircraft are found to be:

\[ u' = -\sin \gamma \]  
\[ \xi' = u \cos \gamma \]  
\[ \eta' = u \sin \gamma \]

(4-37)  
(4-38)  
(4-39)

At this point it is necessary to apply the theory of the calculus of variations to obtain the optimal open loop control for the simplified system which will be used in the closed loop technique.

The problem is once again to extremize the performance index (Eq. 4-14) subject to the differential equations of constraint (4-37) - (4-39) and the boundary conditions given in Table 4.1.

The Hamiltonian function for the simplified system is formed from Eq. (2-4) yielding,

\[ H = \lambda_u (-\sin \gamma) + \lambda_\xi (u \cos \gamma) + \lambda_\eta (u \sin \gamma) -1. \]  
(4-40)

In terms of Eq. (4-40) the Euler equations for the simplified system become:

State Variable Euler Equations:

- **u Euler:**  
  \[ \lambda_\xi \cos \gamma + \lambda_\eta \sin \gamma + \lambda_u = 0 \]  
  (4-41)

- **\( \xi \) Euler:**  
  \[ \lambda_\xi = 0 \]  
  (4-42)

- **\( \eta \) Euler:**  
  \[ \lambda_\eta = 0 \]  
  (4-43)
Control Variable Euler Equation:

\[-\lambda_u \cos \gamma - \lambda_\xi \, u \sin \gamma + \lambda_\eta \, u \cos \gamma = 0 \]  

A comparison between the Euler equations of the real system (Eqs. 4-16 through 4-19) and those of the simplified system above (Eqs. 4-41 through 4-44) reveals that they are identical, thus satisfying the condition of the hypothesis presented in Chapter III. The \( G \) function and transversality conditions for the simplified system can also be shown to be identical to those of the real system.

To obtain the optimal open-loop control for the simplified system, Eqs. (4-20) and (4-29) are utilized to rewrite equation (4-40) as

\[ \lambda_u (-\sin \gamma) + \lambda_\xi (u \cos \gamma) + \lambda_\eta (u \sin \gamma) = 1, \]  

Now proceeding as before in Section 4.2, Eq. (4-32) is obtained. This equation is now expanded by substituting in the remaining Euler equations (4-41) and (4-43) and the simplified system's equation of constraint (4-37) to give,

\[ \lambda_\xi \sec^2 \gamma \gamma'' = \frac{1}{u^2} \left[ \lambda_\xi \, u \cos \gamma + \lambda_\eta \, u \sin \gamma - \lambda_u \sin \gamma \right] \]  

Now Eq. (4-44) is solved for \( \lambda_\eta \) yielding,

\[ \lambda_\eta = \frac{\lambda_u}{u} + \lambda_\xi \tan \gamma \]  

The substitution of Eq. (4-47) into Eq. (4-46) yields the optimal open-loop control equation for the simplified system,

\[ \gamma'' = \frac{\cos \gamma}{u} \]  

Note that all Lagrange multipliers have been eliminated from the control equation. Hence, the optimal endpoint of the trajectory depends only on the initial value of $\gamma$. Every point along the trajectory is an optimal endpoint. To satisfy the endpoint boundary conditions all that is necessary is to iterate on the initial value of $\gamma$ until a trajectory is found which passes through the desired endpoint. This procedure is several orders of magnitude easier than that encountered in the optimization of the real aircraft where a double iteration must be made to determine not only the initial value of $\gamma$ but also that of $\lambda^*_T$ to reach a desired endpoint (See Eq. 4-34).

At this point all of the information is available that is necessary to implement the closed loop solution. The closed loop methodology presented in Figure 3.1 is to be followed by "flying" the real aircraft (represented by the constraints given in Eqs. 4-11 through 4-13) using the simplified aircraft's control equation (4-48). The initial flight path angle ($\theta_T = 0$) and all update flight path angles are generated by "flying" the simplified aircraft (represented by the constraints given in Eqs. 4-37 through 4-39) using its optimal open loop control, Eq. (4-48), and iterating to find the proper $\gamma_1$ such that the resultant trajectory passes through the desired endpoint.
CHAPTER V

EXAMPLE PROBLEM SOLUTION

Both digital and analog computer solutions were obtained to the example problem presented in Chapter IV. Initially the problem was solved on an Applied Dynamics 2PS-64 hybrid analog computer with digital logic control. These results were later confirmed on the CDC-6400 digital computer. The results presented in this thesis are those obtained on the digital computer.

The problem was solved by first generating the optimal solution of the real aircraft to some desired endpoint \((x_f, y_f)\) using standard calculus of variations techniques as discussed in Section 4.2. The closed loop solution to the same point was then generated using the methodology discussed in Section 4.3, comparing both its path and time-to-fly to those of the known optimal path (the real aircraft's optimal open loop trajectory).

Several cases were solved, obtaining both long and short trajectories. In generating solutions, thrust to weight ratios \((k)\) ranging from 0.3 to 1.0 were used. The interval between control updates for the closed loop solution was regulated by a velocity comparison. Once the proper \(Y_1\) was determined (Step 1 of Figure 3.1) both the real and simplified system's equations of constraint were integrated simultaneously (Step 3 of Figure 3.1) using the initial condition \(Y_1\) and the simplified system's
control equation (4-48). When the velocities of the real and simplified systems varied by some previously chosen value, \( u_{rs} - u_{ss} = \Delta u \), a control update was made (Steps 4, 5 and 1 of Figure 3.1). Physically, this velocity difference indicates that the real system is not behaving as anticipated by the mathematical model (simplified system). The velocity variation was found to be a very good criteria for control update.

All of the cases performed in this study were carried out in terms of non-dimensional parameters. The use of these parameters allows the results to be applied to any particular aircraft with the same general equations of motion independent of the specific physical parameters of that aircraft.

Figures 5.1 through 5.3 present the results of the cases studied. In all cases the initial velocity was chosen to be, \( u_1 = 2.0 \). The results presented show the effect of using various values of the update parameter, \( \Delta u \).

Figure 5.1 presents a long aircraft trajectory flown by an aircraft with a relatively low thrust to weight ratio \((k = 0.3)\). In this case control update was initiated when \( \Delta u \geq 0.1 \). The use of this "fine" update tolerance in this case yields a closed loop trajectory that is essentially identical to the true optimal trajectory.

Figure 5.2 also presents a long aircraft trajectory. In this case however, the aircraft has a relatively high thrust to weight ratio \((k = 1.0)\). Update was initiated in this instance when \( \Delta u \geq 0.3 \). Here the effects of a high thrust to weight ratio and a "coarse" update tolerance yield a closed loop trajectory which deviates somewhat from the optimal trajectory.
Figure 5.3 shows the affect on a relatively short trajectory, flown by an aircraft with $k = 1.0$, of varying the update tolerance, $\Delta u$. This figure illustrates graphically that as the value of $\Delta u$ approaches zero, the closed loop solution approaches the true optimal path (based on Section 4.2).
True optimal and closed loop paths are essentially identical.

Figure 5.1 Closed Loop Comparison, Case 1
### SOLUTION

<table>
<thead>
<tr>
<th></th>
<th>$\tau_f$</th>
<th>$\xi_f$</th>
<th>$\eta_f$</th>
<th>$u_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True Optimal</td>
<td>8.01525245</td>
<td>55.304541130</td>
<td>1.51184164</td>
<td>11.00811820</td>
</tr>
<tr>
<td>Closed Loop ($\Delta u=0.3$)</td>
<td>8.01775291</td>
<td>55.304544023</td>
<td>1.51159292</td>
<td>11.06197804</td>
</tr>
</tbody>
</table>

**Figure 5.2 Closed Loop Comparison, Case 2**

\[
\begin{align*}
k &= 1.0 \\
u_1 &= 2.0 \\
\xi_1 &= 0.0 \\
\eta_1 &= 0.0 \\
\gamma_{1rs} &= -70^\circ \\
\lambda_{\xi_1rs} &= -0.0808653
\end{align*}
\]
Figure 5.3 Closed Loop Comparison, Case 3
CHAPTER VI

CONCLUSIONS

The closed loop technique has been shown to yield optimal solutions to the example problem presented in this thesis. These solutions verify the feasibility of the hypothesis presented in Chapter III. It is interesting that even when the thrust to weight ratio is 1.0, optimal solutions are obtained using the closed loop technique. This indicates that even with a gross simplification of the real system, the closed loop technique yields valid solutions.

The closed loop technique has also been applied to the aircraft problem in which the real aircraft experienced $V^2$ drag. The simplified system in this problem neglected both thrust and drag. In this case the closed loop technique did not in general yield optimal solutions. This indicates that as stated previously in Section 4.3, only forces in the direction of motion which are not functions of the state or control variables may be neglected in the simplified system.

The closed loop technique is therefore, extremely useful in solving variational problems which obey the precepts outlined by the hypothesis presented in Chapter III. Application of the technique to other classes of variational problems appears dubious.
LIST OF SYMBOLS

A - wing planform area
C.O.V. - calculus of variations
$c_1, c_2, c_3$ - constants
F - functional form from the calculus of variations
G - function from the calculus of variations
G - gravitational constant
H - Hamiltonian function (from calculus of variations)
k - thrust to weight ratio
L - lift force
m - mass of the aircraft
N.D. - indicated non-dimensional quantity
rs - refers to real system
ss - refers to simplified system
T - thrust force
t - time
u - non-dimensional velocity
$u_k$ - control variable. (For problem investigated, $\gamma$)
v - velocity
$v_r$ - reference velocity
x - range
y - altitude
$y_i$ - state variable. (For problem investigated, $x, y, v$)
$z$ - functional form from the calculus of variations
$\gamma$ - flight path angle
$\Delta$ - prefix indicating a small increment
$\xi$ - non-dimensional range
$\eta$ - non-dimensional altitude
$\lambda_i$ - Lagrange multipliers (functions of time)
$\mu_e$ - Lagrange multipliers (constants)
$\rho_0$ - reference density
$\tau$ - non-dimensional time
$\phi_i$ - function from the calculus of variations
$\psi_e$ - function from the calculus of variations

**Subscripts**

$1$ - initial value
$f$ - final value

**Superscripts**

$.$ - differentiation with respect to time ($t$)
$'$ - differentiation with respect to N.D. time ($\tau$)

**Operations**

$\int$ - integration
$\partial A/\partial B$ - partial derivative of function $A$ with respect to the variable $B$
$\partial A/\partial B$ - total derivative of function $A$ with respect to the variable $B$
REFERENCES


