SOME GENERAL PROPERTIES OF DESIGN MODULES

by

Romeo Iguico Mercado

A Thesis Submitted to the Faculty of the

COMMITTEE ON OPTICAL SCIENCES

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

1 9 7 1

STATEMENT BY AUTHOR

This thesis has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: Romeo I. Mercado

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

O. N. Stavroudis — 11 January 1971

O. N. Stavroudis
Professor of Optical Sciences

Date

ACKNOWLEDGMENTS

The author wishes to express his profound gratitude to his thesis director, Dr. Orestes N. Stavroudis, for his guidance and forbearance during the preparation of this thesis, for his assistance in writing the computer programs used in this study and for the countless enlightening discussions regarding the design modules.

The author would like to thank Dr. R. H. Noble and Dr. R. V. Shack for their helpful suggestions concerning this thesis.

The author also would like to extend his sincere appreciation to the Perkin-Elmer Corporation, the Northrop Corporation and the Fulbright-Hays Foundation for their fellowship support which made it possible for him to pursue his graduate studies.

This work was supported in part by project THEMIS administered by the U. S. Air Force Office of Scientific Research.

TABLE OF CONTENTS

		Page
	LIST OF TABLES	vi
	LIST OF ILLUSTRATIONS	vii
	ABSTRACT	viii
1.	INTRODUCTION	1
	Background	1
	Symbols, Definitions and Conventions	3
2.	THE DELANO $Y,\bar{Y}$ DIAGRAM	6
	The Lagrange Invariant	7
	The $\Omega,\bar{\Omega}$ Parameters	9
	Basic Relationships	10
3.	TWO-SURFACE OPTICAL SYSTEM	12
	Focal Lengths and Focal Distances	12
	Thickness and Power of System	16
	Conjugate Points and Planes	17
	The Cardinal Points	20
	Focal Points	20
	Principal Points	21
	Nodal Points	24
	Entrance and Exit Pupil Planes	25
	Normalization	27
	Two-Mirror System	28
4.	THIRD-ORDER ABERRATIONS AND THE MODULE	30
	Auxiliary Quantities	31
	Third-Order Aberration Coefficients	32
	The Module	35
	Canonical Optical Parameters	37
	Critical Values of Free Parameters	40
	Parameter k	40
	Parameter θ	41
	Parameter Bounds	45

TABLE OF CONTENTS--Continued

Page

5. ADDITIONAL PROPERTIES OF MODULES 46

Modules with Zero Coma 46
Modules with Zero Astigmatism 49
Both Coma and Astigmatism Equal to Zero 51
Modules with Zero Petzval Curvature 51
Real Zeros of the Quartic 53
Case I 54
Case II 54
Case III 55
Two-Mirror Case 55
Both Coma and Petzval Sum Equal to Zero 56
Both Astigmatism and Petzval Sum Equal to Zero 57
Zero Coma, Astigmatism and Petzval Sum 57
Modules with Zero Distortion 58

6. TWO-MODULE SYSTEMS 60

Two-Module System at Finite Conjugates 60
Zero Coma 61
Zero Astigmatism 65
Zero Coma and Astigmatism 66
Application 68
Two-Module System at Infinite Conjugates 69
Zero Astigmatism 74
Zero Coma 75

7. CONCLUSION 78

APPENDIX: NUMERICAL EXAMPLES OF MODULES 82

REFERENCES 95

LIST OF TABLES

Table		Page
I.	Values of Canonical Optical Parameters and Equivalent $y, \bar{y}$ Diagram Parameters Corresponding to the Critical Values of k . .	43
II.	Values of Canonical Optical Parameters and Equivalent $y, \bar{y}$ Diagram Parameters Corresponding to the Critical Values of θ . .	44

LIST OF ILLUSTRATIONS

Figure		Page
1.	Two-Surface Optical Systems with Image Plane Located at Infinity	13
2.	$Y, \bar{Y}$ Diagram for a Two-Surface Optical System with Finite Conjugate Points	14
3.	Normalized $y, \bar{y}$ Diagrams for a Two-Surface Optical System with One Conjugate Point at Infinity	29
4.	Nomenclature Used in the Analysis of Two-Module System at Finite Conjugates	62
5.	Typical Example of Two-Module System at Finite Conjugates	63
6.	Two Common Examples of Afocal Systems	71
7.	Normalized $y, \bar{y}$ Diagrams of Afocal Systems	72
8.	Typical Example of Two-Module System at Infinite Conjugates	73
A-1.	Values of q, c_2, Ω_2 and Φ_2 as Functions of k . .	84
A-2.	Values of t_1 and y_1 as Functions of k	85
A-3.	Values of c_1 and Φ_1 as Functions of k	86
A-4.	Values of t_2 and τ_2^* as Functions of k	87
A-5.	Values of $\bar{y}_1$ and $\bar{y}_2$ as Functions of k for Modules with Zero Astigmatism	88
A-6.	Values of $\bar{\Omega}_1$ and $\bar{\Omega}_2$ as Functions of k for Modules with Zero Astigmatism	89
A-7.	Examples of Modules ($k_o < k_\infty < k^* > 1$)	90
A-8.	Examples of Modules ($k_o < k^* < k_\infty < 1$)	91
A-9.	Constructible Modules and Their $y, \bar{y}$ Diagrams . .	92

ABSTRACT

In this paper the class of two-surface optical systems designated as modules, which possess zero third-order spherical aberration relative to a pair of conjugate planes one of which is at infinity, has been further analyzed using the parameters of the Delano $y,\bar{y}$ diagram. For a given set of three indices of refraction n_1, n_2, and n_3, functional relationship among the $y,\bar{y}$ diagram parameters which eliminate simultaneously other Seidel aberrations are derived. Expressions for zero coma, astigmatism and Petzval curvature are also given. Criteria for selecting the non-optical parameter k which defines the desired properties of modules are described. A one-to-one correspondence between the canonical optical parameters defined in previous studies of modules and certain quantities derivable from the $y,\bar{y}$ diagram representation is shown. Critical values of the free parameters of modules for both the real and the imaginary cases are derived and defined relative to the $y,\bar{y}$ diagram parameters. The problem of coupling two modules for both finite and infinite conjugates are explored and the properties of two-module systems are analyzed for the two cases. Possible applications of these results to optical design are discussed.

CHAPTER 1

INTRODUCTION

Background

Certain two-surface optical systems with fixed focal lengths and having the property that, relative to a pair of conjugate planes, one finite and the other infinite, the third-order spherical aberration is zero, have been described and analyzed by Stavroudis (1967, 1969a, 1969b). It is thought that such systems might find an application in the early stages of the process of optical design. If this class of systems could be arranged so that the rear and front foci of adjacent systems coincide, then the resulting optical system would also have zero third-order spherical aberration. For this reason, these two-surface optical systems which possess zero third-order spherical aberration with respect to a pair of conjugate planes, one of which is at infinity, are designated in this thesis as modules.

Using conventional optical parameters of curvatures and axial separations, Stavroudis (1967, 1969a, 1969b) analyzed modules which possess either refracting or reflecting spherical surfaces. To obtain a one-parameter family of lenses meeting the required conditions for modules, he defined a non-optical parameter and expressed his four conventional optical parameters as functions of this new parameter. He defined aperture planes relative to which third-order astigmatism is zero and he derived an expression for coma. He also indicated a means of defining the domains of his free parameters for constructible modules.

Powell (1970) analyzed the two-surface systems first described by Stavroudis (1969b) using the first-order parameters of the $y,\bar{y}$ diagram which was introduced by Delano (1963) and used by Pegis et al. (1967). The module analyzed by Powell (1970) in terms of the $y,\bar{y}$ diagram parameters was a normalized two-surface refracting system which has its object plane at infinity and has fixed focal length. Powell (1970) expressed the third-order coefficients of spherical aberration, astigmatism and coma of modules in terms of the first-order $y,\bar{y}$ diagram parameters. He also defined a parameter to obtain a functional relationship of the free parameters of two interconnected modules which varies the axial separation between the two systems.

This thesis amplifies the $y,\bar{y}$ diagram analysis initiated by Powell (1970) and extends the work already published on the general properties of modules.

In this thesis, the third-order aberrations of modules have been further analyzed to include Petzval contribution and distortion. Functional relationships among the $y,\bar{y}$ diagram parameters which provide conditions for modules to eliminate other Seidel aberrations are derived and analyzed. A comparison of the canonical optical parameters defined by Stavroudis (1969b) and certain quantities derivable from the $y,\bar{y}$ diagram was made and the relationship between his parameter f and the Lagrange invariant (Ж) is established. The critical values of the free parameters for both the real and the pure imaginary cases are derived. Numerical examples of modules are incorporated to illustrate its properties. The properties of two-module systems are discussed and analyzed for both the finite and infinite conjugates cases.

Symbols, Definitions and Conventions

The terms and symbols used in this thesis are defined as they appear in the text. In general, the nomenclature, definitions and conventions used follow those given in the Military Standardization Handbook of Optical Design (MIL-HDBK-141, 1962), with the following exceptions and modifications: the Smith-Helmholtz-Lagrange invariant is simply called the Lagrange invariant and is denoted by the Russian letter Ж (zhe), t_j will be the axial thickness of the space between the j-1 and jth surface, and n_j shall be the refractive index of the space between the j-1 and the jth surface.

With the above exceptions and changes, light is considered traveling from left to right through the system. The optical system is regarded as a series of surfaces starting with an object surface and ending with an image surface. The surfaces are numbered consecutively, in the order in which light is incident on them, starting with zero for the object surface and ending with k+1 for the image surface. A general surface will be called the jth surface. All quantities between surfaces will be given the number of the immediately succeeding surface. The radius of the jth surface is r_j and its curvature is c_j, the reciprocal of r_j. The quantities c_j and r_j, are considered positive when the center of curvature lies to the right of the surface. The thickness t_j is positive if the jth surface physically lies to the right of the (j-1)th surface and is negative if it lies to the left. The refractive index n_j is positive if the physical ray travels from left to right. Otherwise it is negative. The right-handed cartesian coordinate system is used with the optical axis coincident with the Z-axis. Light travels initially toward larger values of z. Lower case letters, y_j and $\bar{y}_j$, have been used to represent the paraxial heights of the marginal and principal rays at the jth surface, respectively. The slope angles of the marginal and principal rays in the space between the j-1 and the jth surfaces are denoted by u_j and $\bar{u}_j$, respectively, where u_j is equal to $(y_j-y_{j-1})/t_j$ and $\bar{u}_j$ is equal to $(\bar{y}_j-\bar{y}_{j-1})/t_j$.

Some changes in notations were made for the $y, \bar{y}$ diagram parameters as used by Delano (1963). The nomenclature used for these parameters are defined as they appear in the text. The figures in the text show the symbols and quantities used in this thesis.

CHAPTER 2

THE DELANO $Y,\bar{Y}$ DIAGRAM

This chapter consists of a brief general discussion of the $y,\bar{y}$ diagram first introduced by Delano (1963) to represent graphically the first-order properties of any axially symmetric optical system which has refracting or reflecting surfaces.

The $y,\bar{y}$ diagram is a two-dimensional plot in cartesian coordinates of the paraxial marginal ray height y versus the paraxial principal ray height $\bar{y}$ for each surface throughout an axially symmetric optical system. The $\bar{y}$-axis is the axis of abscissas and the y-axis is the axis of ordinates. Generally, a point $P_j(\bar{y}_j, y_j)$ in the $y,\bar{y}$ plane is defined by the paraxial principal ray height $\bar{y}_j$ and the paraxial marginal ray height y_j at the jth surface.

The $y,\bar{y}$ diagram for an optical system is in general, a polygonal figure on the $y,\bar{y}$ plane. The vertices correspond to either refracting or reflecting surfaces. Extensions of the first and last segments of the diagram intersect the $\bar{y}$ and y axes at the object or image planes and the pupil planes, respectively. Therefore, the point $(\bar{y},0)$ represents either an object or image plane and the point $(0,y)$ denotes the pupil plane in the optical system layout. The distance between two points on the same line in the $y,\bar{y}$ diagram corresponds to a transfer between corresponding planes or surfaces in the optical system while a point common to two different line segments corresponds to either a reflection or a refraction. The layout of the optical system can be constructed readily from the data obtained from its $y,\bar{y}$ diagram. The separations between points in the $y,\bar{y}$ plane are proportional to the distances between corresponding planes or surfaces in the system layout.

The Lagrange Invariant

In an axially symmetric optical system the paraxial ray trace equations at the (j+1)th surface are given by the following refraction equations:

$$\omega_{j+1} = \omega_j - \phi_j y_j \qquad \text{for marginal ray} \qquad (1)$$

$$\bar{\omega}_{j+1} = \bar{\omega}_j - \phi_j \bar{y}_j \qquad \text{for principal ray} \qquad (2)$$

The transfer equations are as follows:

$$y_{j+1} = y_j + \tau_{j+1}\omega_{j+1} \qquad \text{for marginal ray} \qquad (3)$$

$$\bar{y}_{j+1} = \bar{y}_j + \tau_{j+1}\bar{\omega}_{j+1} \qquad \text{for principal ray} \qquad (4)$$

where

$$\omega_j = n_j u_j \qquad \text{reduced marginal ray angle} \qquad (5)$$

$$\bar{\omega}_j = n_j \bar{u}_j \qquad \text{reduced principal ray angle} \qquad (6)$$

$$\phi_j = c_j(n_{j+1} - n_j) \text{ power of jth surface} \tag{7}$$

$$\tau_{j+1} = t_{j+1}/n_{j+1}. \text{ reduced thickness} \tag{8}$$

The curvature c_j is the curvature of the jth surface and n_j, t_j, u_j and $\bar{u}_j$ are the index of refraction, axial thickness and slope angles of the marginal and principal rays, respectively, for the preceeding space.

Eliminating ϕ_j from equations (1) and (2) and rearranging terms the invariant on refraction is obtained as

$$\mathcal{K}_j = \bar{\omega}_j y_j - \omega_j \bar{y}_j = \bar{\omega}_{j+1} y_j - \omega_{j+1} \bar{y}_j. \tag{9}$$

Similarly, by eliminating τ_{j+1} from equations (3) and (4) and rearranging terms yields the invariant on transfer given by

$$\mathcal{K}_{j+1} = \bar{\omega}_{j+1} y_{j+1} - \omega_{j+1} \bar{y}_{j+1} = \bar{\omega}_{j+1} y_j - \omega_{j+1} \bar{y}_j. \tag{10}$$

Comparing equations (9) and (10) we have

$$\mathcal{K}_j = \mathcal{K}_{j+1} = \mathcal{K}, \tag{11}$$

where $\mathcal{K}$ is the Lagrange invariant defined by

$$\mathcal{K} = \bar{\omega} y - \omega \bar{y}. \tag{12}$$

Therefore the Lagrange invariant is constant throughout the whole optical system. The Lagrange invariant together with y_j, $\bar{y}_j$ and z_j, the axial coordinate of the jth surface with respect to an axial origin located at the object or pupil plane, at every surface of an axially symmetric optical system forms a set of independent parameters which completely define the paths of two paraxial rays through the system and from which the conventional optical parameters of radii of curvature and axial separations could be determined.

The $\Omega,\bar{\Omega}$ Parameters

The Ω and $\bar{\Omega}$ parameters are defined by the following equations:

$$\Omega = \omega/\mathcal{K} \qquad \bar{\Omega} = \bar{\omega}/\mathcal{K} \tag{13}$$

Introducing the Ω,Ω parameters in equation (12) results to

$$\Omega\bar{y} - \bar{\Omega}y + 1 = 0. \tag{14}$$

Equation (14) is an equation of a straight line in the $y,\bar{y}$ plane with $\bar{y}$-intercept equal to $-1/\Omega$ and whose y-intercept is $1/\bar{\Omega}$. It has a slope equal to $\Omega/\bar{\Omega}$.

In general, fixed values of the $\Omega,\bar{\Omega}$ parameters imply that the line in the $y,\bar{y}$ diagram refers to a space of fixed index of refraction. Hence, every line segment in the $y,\bar{y}$ plane is associated with a medium of refractive index n. The coordinates of each point on a line represents a plane normal to the optical axis in the actual system layout whereby the coordinates are the heights of the principal and marginal rays at this plane.

Basic Relationships

Delano (1963) has shown that given the value of the Lagrange invariant, $\mathcal{H}$, the $y_j, \bar{y}_j$ parameters and the axial coordinate z_j at every surface of an axially symmetric optical system the following related quantities may be derived.

$$t_j = z_j - z_{j-1} \qquad \text{axial thickness} \qquad (15)$$

$$n_j = t_j/\tau_j \qquad \text{refractive index} \qquad (16)$$

$$\omega_j = (y_j - y_{j-1})/\tau_j \qquad \text{reduced marginal ray angle} \qquad (17)$$

$$\bar{\omega}_j = (\bar{y}_j - \bar{y}_{j-1})/\tau_j \qquad \text{reduced principal ray angle} \qquad (18)$$

$$r_j = (n_{j+1} - n_j)/\phi_j \qquad \text{radius of curvature} \qquad (19)$$

$$\tau_j = (y_{j-1}\bar{y}_j - y_j\bar{y}_{j-1})/\mathcal{H} \qquad \text{reduced axial thickness} \qquad (20)$$

$$\phi_j = (\omega_j\bar{\omega}_{j+1} - \omega_{j+1}\bar{\omega}_j)/\mathcal{H} \qquad \text{surface power} \qquad (21)$$

If equations (17), (18) and (21) are divided by the Lagrange invariant, the resulting equations are as follows:

$$\Omega_j = \omega_j/\mathcal{H} = (y_j - y_{j-1})/\mathcal{H}\tau_j \qquad (22)$$

$$\bar{\Omega}_j = \bar{\omega}_j/\mathcal{H} = (\bar{y}_j - \bar{y}_{j-1})/\mathcal{H}\tau_j \qquad (23)$$

$$\Phi_j = \phi_j/\mathcal{H} = \Omega_j\bar{\Omega}_{j+1} - \Omega_{j+1}\bar{\Omega}_j \qquad (24)$$

Therefore, given the Lagrange invariant and the $\Omega, \bar{\Omega}$ parameters at each surface the optical system is defined by the following set of equations written for the jth surface:

$$\phi_j = (\Omega_j\bar{\Omega}_{j+1} - \Omega_{j+1}\bar{\Omega}_j)\text{Ж} \qquad \text{power of surface} \qquad (25)$$

$$y_j = (\Omega_j - \Omega_{j+1})/\Phi_j \qquad \text{marginal ray height} \qquad (26)$$

$$\bar{y}_j = (\bar{\Omega}_j - \bar{\Omega}_{j+1})/\Phi_j \qquad \text{principal ray height} \qquad (27)$$

$$\tau_j = (y_{j-1}\bar{y}_j - y_j\bar{y}_{j-1})/\text{Ж} \qquad \text{reduced thickness} \qquad (20)$$

Equations (20), (22), (23), (24), (25), (26) and (27) form the set of tools to be used in the analysis of the general properties of modules.

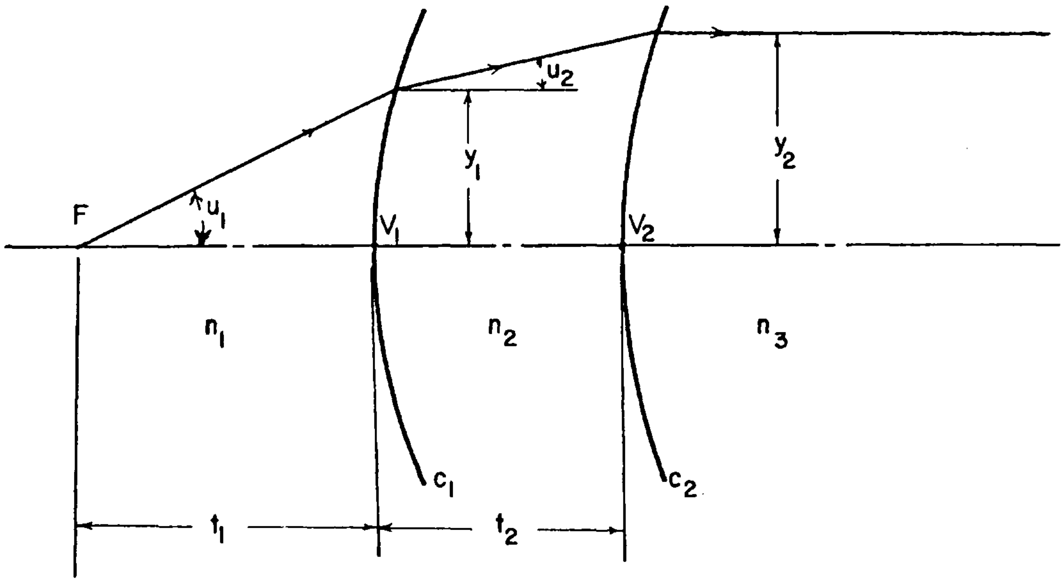
CHAPTER 3

TWO-SURFACE OPTICAL SYSTEMS

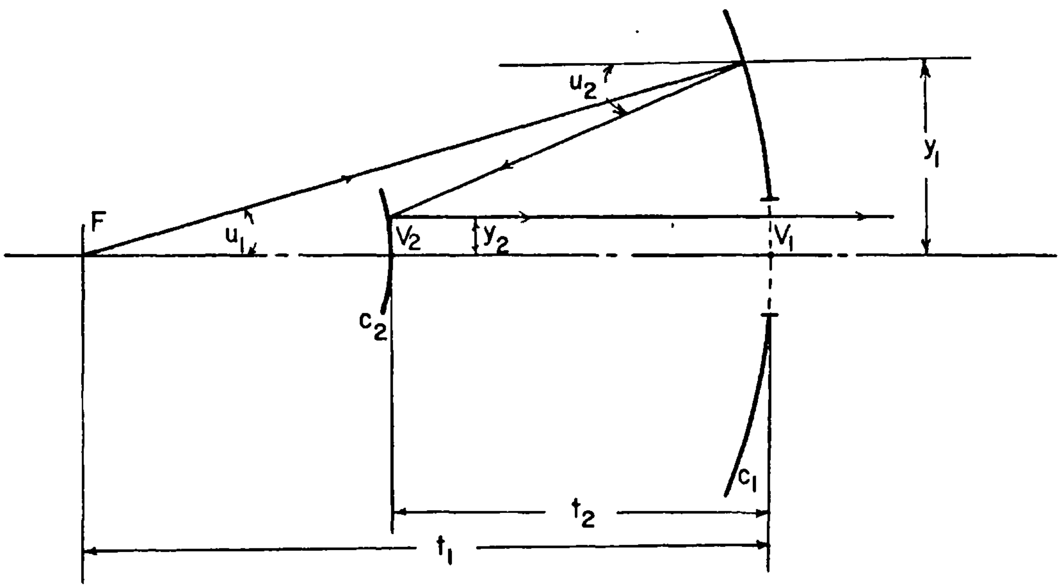
In this chapter is discussed the first-order properties of two-surface optical systems which possess either refracting or reflecting spherical surfaces. The optical systems described herein could be either a two-mirror system or a two-surface lens with fixed focal lengths. Rotational symmetry is assumed. The equations derived are general and applicable to the two-surface systems at any locations of conjugate planes. The class of two-surface systems discussed is shown schematicall in Fig. 1. The $y,\bar{y}$ diagram representation of two-surface systems at finite conjugates is shown in Fig. 2.

Focal Lengths and Focal Distances

The focal lengths of an optical system are defined as the axial distances separating the focal and principal planes in the object and image spaces. The axial distance from the primary principal plane to the front focal plane is called the front focal length while that separating the secondary principal plane to the rear focal plane is called the rear or back focal length. The focal planes in the actual layout of the two-surface system are represented by the points $F(\bar{y}_F, y_F)$ and $F'(\bar{y}_{F'}, y_{F'})$ in the $y,\bar{y}$ diagram. The principal



(a) Refracting System



(b) Reflecting System

Fig. 1. Two-Surface Optical Systems with Image Plane Located at Infinity

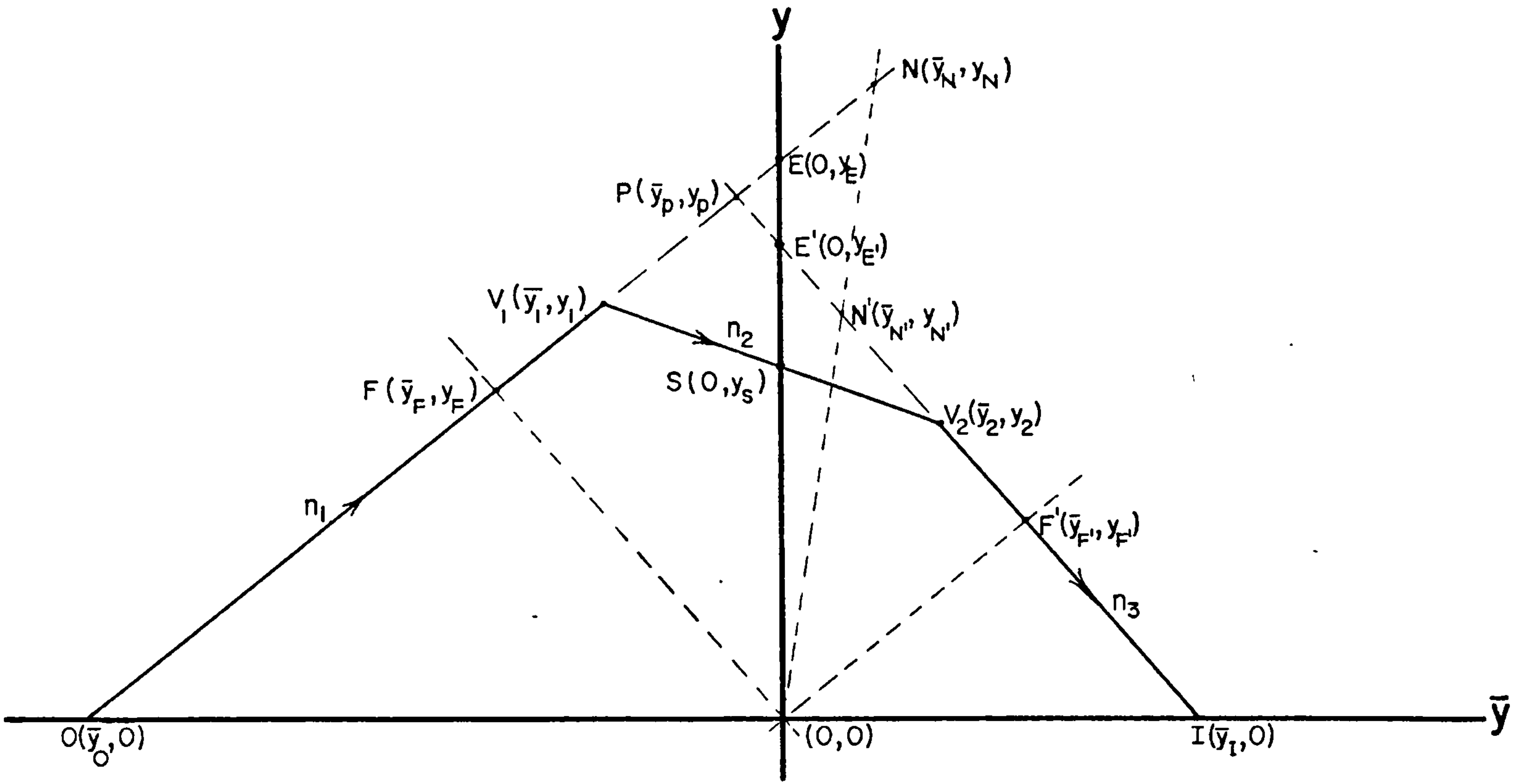


Fig. 2. Y,$\bar{Y}$ Diagram for a Two-Surface Optical System with Finite Conjugate Points

planes correspond to the points $P(\bar{y}_p, y_p)$ and $P'(\bar{y}_{p'}, y_{p'})$ which are coincident points in the $y, \bar{y}$ plane and simply denoted by the point P. The primed coordinates denote points in image space. Hence, F and P are the front focal point and primary principal point, respectively, while F' is the rear focal point and P' is the secondary principal point.

Using equation (20) the front focal length is given by

$$f = PF = n_1(y_p \bar{y}_F - \bar{y}_p y_F)/\mathcal{K}, \tag{28}$$

and the rear focal length is

$$f' = P'F' = n_3(y_{p'} \bar{y}_{F'} - \bar{y}_{p'} y_{F'})/\mathcal{K}. \tag{29}$$

The focal distances are defined as the axial distances separating the focal planes and the lens surfaces in the object and image spaces. The axial distance from the first surface $V_1(\bar{y}_1, y_1)$ to the front focal plane F is called the front focal distance and that from the last surface $V_2(\bar{y}_2, y_2)$ of the two-surface system to its rear focal plane F' is the back focal distance. Applying equation (20), the front focal distance is

$$FFD = V_1F = n_1(y_1 \bar{y}_F - \bar{y}_1 y_F)/\mathcal{K}, \tag{30}$$

and the back focal distance is

$$BFD = V_2F' = n_3(y_2 \bar{y}_{F'} - \bar{y}_2 y_{F'})/\mathcal{K}. \tag{31}$$

Thickness and Power of System

The thickness of an optical system is the axial distance from the first surface to the last surface. Using equation (20), the thickness of the two-surface system is given by

$$t_2 = V_1V_2 = n_2(y_1\bar{y}_2 - \bar{y}_1y_2)/\text{※}. \qquad (32)$$

The reduced thickness, τ_2, is equal to t_2/n_2.

From elementary analytic geometry, twice the area of a triangle with vertices at $(\bar{y}_1,y_1)$, $(\bar{y}_2,y_2)$ and the origin, (0,0), is given by

$$2A = (y_1\bar{y}_2 - \bar{y}_1y_2).$$

Comparing the above relationship with those of equations (28), (29), (30), (31) and (32), it is clear that the product of the reduced axial distance and the Lagrange invariant is equal to twice the area of the triangle formed by the points representing the axial separations and the origin in the $y,\bar{y}$ diagram.

The power of a thick lens in terms of the axial thickness and power of individual surfaces is given by

$$\phi = \phi_1 + \phi_2 - \tau_2\phi_1\phi_2, \qquad (33)$$

where ϕ_1 is the power of the first surface, ϕ_2 is the power of the second surface and τ_2 is the reduced axial thickness. The power of the two-surface system is also equal to the reciprocal of the reduced focal length. Hence, ϕ is equal to $-n_1/f$ or n_3/f'.

Dividing equation (33) by the Lagrange invariant results to

$$\Phi = \phi/\mathcal{K} = \Phi_1 + \Phi_2 - \tau_2 \mathcal{K} \Phi_1 \Phi_2. \tag{34}$$

Using equations (24), (26) and (27), equation (34) is reduced to

$$\Phi = \Omega_1 \bar{\Omega}_3 - \Omega_3 \bar{\Omega}_1. \tag{35}$$

Equation (35) gives the power of the two-surface system, normalized to $\mathcal{K}$, as function of the $\Omega, \bar{\Omega}$ parameters of the object and image spaces.

Conjugate Points and Planes

The object line of a two-surface optical system in the $y, \bar{y}$ diagram is given by

$$\Omega_1 \bar{y} - \bar{\Omega}_1 y + 1 = 0 \tag{36}$$

and its image line is

$$\Omega_3 \bar{y} - \bar{\Omega}_3 y + 1 = 0, \tag{37}$$

where Ω_1 and $\bar{\Omega}_1$ are the $\Omega, \bar{\Omega}$ parameters in object space and Ω_3 and $\bar{\Omega}_3$ are the $\Omega, \bar{\Omega}$ parameters in image space.

Any pair of object and image points related by a transverse magnification m_T are conjugate points and the corresponding planes in the optical layout defined by these points in the $y, \bar{y}$ diagram are the conjugate planes. The conjugate points are represented in the $y, \bar{y}$ plane by the points of intersection of a line through the origin with the object and image lines. This line intersecting the origin and the conjugate points is called the conjugate line and is given by

$$y = m_c\bar{y}, \tag{38}$$

where m_c is the slope of the conjugate line.

For two conjugate points $A(y_a,y_a)$ and $A'(y_{a'},y_{a'})$ in object and image spaces, repectively, the slope of the conjugate line is

$$m_c = y_a/\bar{y}_a = y_{a'}/\bar{y}_{a'} \ . \tag{39}$$

The transverse magnification at the conjugate points A and A' is given by

$$m_T = y_{a'}/y_a = \bar{y}_{a'}/\bar{y}_a. \tag{40}$$

Using equations (40) and (37) in (39) and the term (-1) in the resulting equation is replaced by $(\Omega_1\bar{y}_a - \bar{\Omega}_1 y_a)$, the slope of the conjugate line becomes

$$m_c = y_a m_T \Omega_3/(\bar{\Omega}_3 y_{a'} + \Omega_1\bar{y}_a - \bar{\Omega}_1 y_a). \tag{41}$$

Dividing equation (41) by y_a and solving for the slope we obtain

$$m_c = (m_T\Omega_3 - \Omega_1)/(m_T\bar{\Omega}_3 - \bar{\Omega}_1). \tag{42}$$

Equation (42) gives the slope of the conjugate line as function of the transverse magnification at the conjugate planes and the $\Omega,\bar{\Omega}$ parameters.

The coordinates of the conjugate points in object and image spaces can be determined by solving simultaneously the equation of the conjugate line and that of the object and image spaces. When equations (38) and (36) are solved simultaneously and equations (42) and (35) are substituted, the coordinates of the conjugate point in object space are given by the following:

$$\bar{y} = -(\bar{\Omega}_3 m_T - \bar{\Omega}_1)/m_T \Phi \tag{43}$$

$$y = -(m_T \Omega_3 - \Omega_1)/m_T \Phi \tag{44}$$

The coordinates of the conjugate point in image space is obtained in the same way by solving equations (37) and (38) simultaneously, resulting to

$$\bar{y}' = -(m_T \bar{\Omega}_3 - \bar{\Omega}_1)/\Phi \tag{45}$$

$$y' = -(m_T \Omega_3 - \Omega_1)/\Phi. \tag{46}$$

Equations (43), (44), (45) and (46) express the coordinates of the pair of conjugate points in the $y,\bar{y}$ diagram in terms of their transverse magnification, the $\Omega,\bar{\Omega}$ parameters in image and object spaces and the power of the two-surface system normalized with respect to the Lagrange invariant.

The Cardinal Points

Focal Points

The coordinates of the front focal point is defined by the point of intersection of the object line with a conjugate line parallel to the image line since the point in image space conjugate to the front focal point is located at infinity. Hence, the slope of this conjugate line at the front focal point is equal to the slope of the image line and is given by

$$m_3 = \Omega_3/\bar{\Omega}_3, \tag{47}$$

and the equation of the corresponding conjugate line is

$$y = m_3\bar{y}. \tag{48}$$

Solving equations (48) and (36) simultaneously, gives the $y,\bar{y}$ coordinates of the front focal point in terms of the $\Omega,\bar{\Omega}$ parameters in object and image spaces. The resulting equations are as follows:

$$\bar{y}_F = - \bar{\Omega}_3/(\Omega_1\bar{\Omega}_3 - \Omega_3\bar{\Omega}_1) = -\bar{\Omega}_3/\Phi \tag{49}$$

$$y_F = - \Omega_3/(\Omega_1\bar{\Omega}_3 - \Omega_3\bar{\Omega}_1) = -\Omega_3/\Phi \tag{50}$$

The rear focal point is defined by the point of intersection of the image line and a conjugate line with slope $m_1 = \Omega_1/\bar{\Omega}_1$ which is equal to that of the object line.

The equation of the conjugate line intersecting the image line at the rear focal point is

$$y = m_1\bar{y}. \tag{51}$$

The coordinates of the rear focal point is obtained by solving equations (51) and (38) simultaneously which results to

$$\bar{y}_{F'} = \bar{\Omega}_1/(\Omega_1\bar{\Omega}_3 - \bar{\Omega}_1\ \Omega_3) = \bar{\Omega}_1/\Phi \tag{52}$$

$$y_{F'} = \Omega_1/(\Omega_1\bar{\Omega}_3 - \bar{\Omega}_1\ \Omega_3) = \Omega_1/\Phi. \tag{53}$$

Substituting equations (49), (50), (52) and (53) in equations (43), (44), (45) and (46), the coordinates of any conjugate point can be expressed in terms of the coordinates of the focal points and the transverse magnification at the conjugate planes. The relationships are given by the following:

$$\begin{aligned} \bar{y} &= \bar{y}_F + (1/m_T)\bar{y}_{F'} \\ y &= y_F + (1/m_T)y_{F'} \\ \bar{y}' &= m_T\bar{y}_F + \bar{y}_{F'} \\ y' &= m_T y_F + y_{F'} \end{aligned} \tag{54}$$

Principal Points

The principal points of the two-surface system are axial conjugate points of unit transverse magnification and are represented in the $y,\bar{y}$ diagram by the point of intersection of the object and image lines. The principal points are denoted by $P(\bar{y}_p, y_p)$ and $P'(\bar{y}_{p'}, y_{p'})$ which are coincident points in the $y,\bar{y}$ diagram. The $y,\bar{y}$ heights at these points are determined by solving simultaneously equations (36) and (37) or by applying equations (43), (44), (45) and (46) with m_T set equal to unity. The results are

$$\bar{y}_p = \bar{y}_{p'} = (\bar{\Omega}_1 - \bar{\Omega}_3)/\Phi \tag{55}$$

$$y_p = y_{p'} = (\Omega_1 - \Omega_3)/\Phi. \tag{56}$$

Substituting equations (49), (50), (52) and (53) in the above equations, the relationships between the coordinates of the principal points and the focal points are obtained and they are as follows:

$$\bar{y}_p = \bar{y}_{p'} = \bar{y}_F + \bar{y}_{F'} \tag{57}$$

$$y_p = y_{p'} = y_F + y_{F'} \tag{58}$$

The locations of the principal points or planes with respect to the surfaces of the two-surface system can be determined by specifying their axial separations. The distance from the first surface to the primary principal plane, V_1P, is obtained by employing equation (20) which results to

$$V_1P = n_1(y_1\bar{y}_p - \bar{y}_1 y_p)/\Psi. \tag{59}$$

Similarly, the distance separating the second surface from the secondary principal point is

$$V_2P' = n_3(y_2\bar{y}_{P'} - \bar{y}_2 y_{P'})/\text{Ж}. \tag{60}$$

Using equations (58), (57), (56), (55), (36), (37) and applying equations (22) and (23), the expressions for the separations between surfaces and principal points are reduced to the following:

$$V_1P = n_1\tau_2\Phi_2/\Phi = n_1t_2\phi_2/n_2\phi \tag{61}$$

$$V_2P' = -n_3\tau_2\Phi_1/\Phi = -n_3t_2\phi_1/n_2\phi \tag{62}$$

Dividing equation (61) by (62) we obtain

$$V_1P = -(n_1\phi_2/n_3\phi_1)(V_2P'). \tag{63}$$

The relationships between the coordinates of the principal points and the focal points given by equations (57) and (58) make it possible to express the focal lengths of the system given by equations (28) and (29) in terms of the coordinates of the focal points. Substituting equations (57) and (58) in (28) and (29) result to

$$f = n_1(y_{F'}\bar{y}_F - \bar{y}_{F'}y_F)/\text{Ж} \tag{64}$$

$$f' = n_3(y_F\bar{y}_{F'} - \bar{y}_F y_{F'})/\text{Ж}. \tag{65}$$

Substituting equations (49), (50), (52) and (53), equations (64) and (65) are reduced to

$$f = -n_1/\phi$$
$$f' = n_3/\phi. \qquad (66)$$

Nodal Points

The nodal points of the two-surface system are axial points where the angular magnification is unity. The nodal points are denoted by the points $N(\bar{y}_N, y_N)$ and $N'(\bar{y}_{N'}, y_{N'})$ in the $y, \bar{y}$ diagram. Points N and N' are conjugate to each other.

The angular magnification m_A between conjugate points when expressed in terms of the transverse magnification is given by

$$m_A = n_1/n_3 m_T. \qquad (67)$$

At the nodal points, m_A is unity and therefore

$$m_T = n_1/n_3 \qquad (68)$$

at this pair of conjugate points.

The coordinates of the nodal points in terms of the coordinates of the focal points are obtained by substituting equation (68) in equations (54). The results give the coordinates of the primary nodal point as

$$\bar{y}_N = \bar{y}_F + (n_3/n_1)\bar{y}_{F'}$$
$$y_N = y_F + (n_3/n_1)y_{F'} . \qquad (69)$$

The coordinates of the secondary nodal point are as follows:

$$\bar{y}_{N'} = (n_1/n_3)\bar{y}_F + \bar{y}_{F'} \qquad (70)$$
$$y_{N'} = (n_1/n_3)y_F + y_{F'}$$

Entrance and Exit Pupil Planes

In Fig. 2, the point of intersection of the y-axis and the object line defines the entrance pupil of the two-surface system. The entrance pupil point is denoted by $E(0,y_E)$ in the $y,\bar{y}$ diagram, where y_E is the height of the entrance pupil from the optical axis. The point $E'(0,y_{E'})$ defines the exit pupil of the system. The exit pupil point is the point of intersection of the image line with the y-axis. The ordinate $y_{E'}$ represents the exit pupil height or radius from the optical axis.

The entrance and exit pupil heights are the y-intercepts of the equations of the object and image lines and their values as a function of the $\Omega,\bar{\Omega}$ parameters can be determined by equating the $\bar{y}$ variable in equations (36) and (37) to zero which result to

$$y_E = 1/\bar{\Omega}_1 \qquad (71)$$

$$y_{E'} = 1/\bar{\Omega}_3. \qquad (72)$$

Hence, the entrance pupil height is the reciprocal of the $\bar{\Omega}$ parameter in object space and the exit pupil height is the reciprocal of the $\bar{\Omega}$ parameter in the image space.

Dividing equation (71) by (72) gives

$$y_E = (\bar{\Omega}_3/\bar{\Omega}_1)y_E', \tag{73}$$

the relationship between the entrance and exit pupil heights in terms of the $\bar{\Omega}$ parameters. When expressed in terms of the coordinates of the focal points, equation (73) becomes

$$y_E = -(\bar{y}_F/\bar{y}_{F'})y_{E'}. \tag{74}$$

The axial distance FE from the front focal point to the entrance pupil point is the entrance pupil distance. Applying equation (20), we obtain

$$FE = -n_1\bar{y}_F y_E/\text{Ж}. \tag{75}$$

The exit pupil distance, F'E', is similarly defined as the axial distance from the rear focal point to the exit pupil point and is equal to

$$F'E' = -n_3\bar{y}_{F'}y_{E'}/\text{Ж}. \tag{76}$$

Substituting equations (71), (72), (49) and (52) in (75) and (76) result to

$$\begin{aligned} FE &= n_1\bar{\Omega}_3/\text{Ж}\Phi\bar{\Omega}_1 \\ F'E' &= -n_3\bar{\Omega}_1/\text{Ж}\Phi\bar{\Omega}_3. \end{aligned} \tag{77}$$

From (77), the reduced pupil distances are related by

$$FE/n_1 = -(\bar{\Omega}_3/\bar{\Omega}_1)^2(F'E'/n_3), \tag{78}$$

which when expressed in terms of $\bar{y}$ coordinates of the focal points yields

$$FE/n_1 = -(y_F/y_{F'})^2(F'E'/n_3). \quad (79)$$

By eliminating the $\Omega,\bar{\Omega}$ parameters from equations (78) and (77), the relationship between the reduced pupil distances becomes

$$FE/n_1 = -(1/\phi)^2(n_3/F'E'). \quad (80)$$

Using equation (66), it can be shown that the reduced pupil distances are related by the square of the reduced focal lengths of the system. This relationship is given by

$$FE/n_1 = -(f/n_1)^2(n_3/F'E') = -(f'/n_3)^2(n_3/F'E'). \quad (81)$$

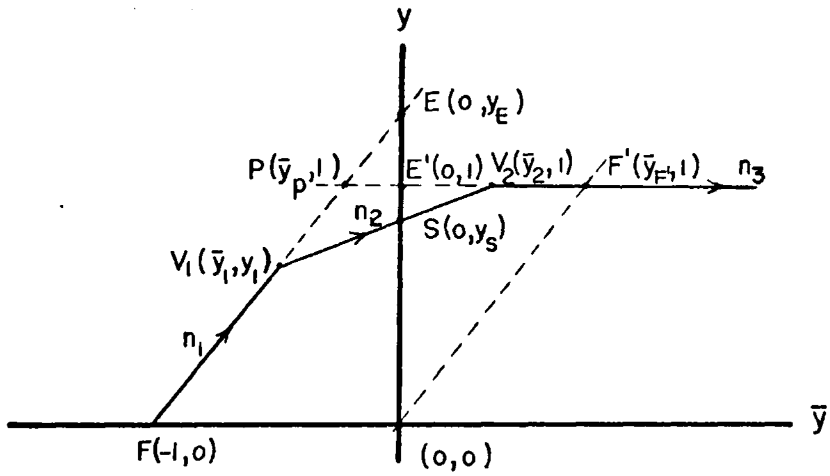
Equation (80) implies that the reduced entrance pupil distance is proportional to the negative reciprocal of the product of the square of the system power and the reduced exit pupil distance.

Normalization

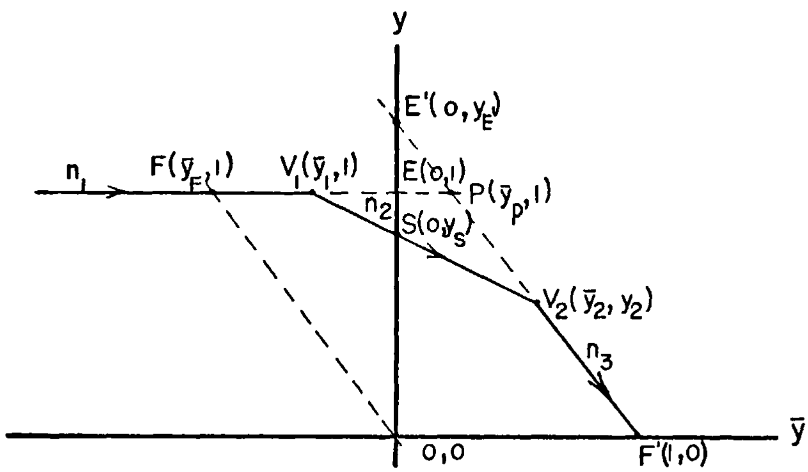
So far, all the equations presented are general and apply to any pair of conjugate planes. Relative to a pair of conjugate planes where one is finite and the other is infinite, like the case of modules, all the equations derived in this chapter are greatly simplified if the $y,\bar{y}$ parameters are normalized. For this particular case where one of the conjugate planes is at infinity, it is convenient to normalize the $y,\bar{y}$ parameters with respect to the object or image height and one of the pupil heights. This requires the image or object height and one of the pupil heights to be set equal to unity in the $y,\bar{y}$ diagram. If the object is located at infinity, the y heights in the $y,\bar{y}$ diagram are normalized with respect to the actual entrance pupil height and the $\bar{y}$ heights are normalized relative to the actual image height. For the case where the image is at infinity, the normalization of the marginal ray heights is with respect to the actual exit pupil height while the principal ray heights are normalized relative to the actual object height. The $y,\bar{y}$ diagrams for these two cases are illustrated in Fig. 3. Henceforth, the $y,\bar{y}$ parameters where one of the conjugate points is located at infinity, shall mean normalized heights. The above normalization scheme follows that one proposed by Lopez-Lopez (1970).

Two-Mirror System

All the equations derived in this chapter are for the general case of two-surface systems but biased to the refracting systems. To apply the equations presented to the case of a two-mirror system in air, all that is required in the formulas is to set $n_1 = n_3 = 1$ and $n_2 = -1$.



(a) Image Point at Infinity



(b) Object Point at Infinity

Fig. 3. Normalized $y, \bar{y}$ Diagrams for a Two-Surface Optical System with One Conjugate Point at Infinity

CHAPTER 4

THIRD-ORDER ABERRATIONS AND THE MODULE

In this chapter the third-order aberrations of the two-surface systems discussed in chapter 3 are analyzed in terms of the $y,\bar{y}$ diagram parameters. The two-surface system analyzed has fixed focal length and with one of the conjugate planes located at infinity.

In the foregoing discussion and derivation of equations, rotational symmetry is assumed and the indices of refraction n_1, n_2 and n_3 are fixed. It is also assumed that the two-surface system consists of spherical surfaces and the image plane is at infinity. All $y,\bar{y}$ coordinates and $\Omega,\bar{\Omega}$ parameters used in this chapter are referred to the normalized $y,\bar{y}$ diagram for a two-surface system shown in Fig. 3.

The design module is defined in this chapter and the critical values of its free parameters are derived. A comparison of the canonical optical parameters defined by Stavroudis (1969b) and those used in this analysis is presented. The first-order properties of the two-surface system discussed in the preceding chapter apply to modules as well.

Auxiliary Quantities

To compute the coefficients of the third-order aberrations in an optical system, several auxiliary quantities should first be determined.

The paraxial angles of incidence at the jth surface are given by

$$i_j = u_j + c_j y_j \quad \text{for the marginal ray} \tag{82}$$

and

$$\bar{i}_j = \bar{u}_j + c_j \bar{y}_j \quad \text{for the principal ray.} \tag{83}$$

Applying equations (19), (25), (22), (23), (26) and (27), the paraxial angles of incidence can be written in terms of the Lagrange invariant, the refractive indices and the $\Omega, \bar{\Omega}$ parameters as

$$i_j = \frac{\text{Ж}}{n_j(n_{j+1}-n_j)}(\Omega_j n_{j+1} - \Omega_{j+1} n_j) \tag{84}$$

$$\bar{i}_j = \frac{\text{Ж}}{n_j(n_{j+1}-n_j)}(\bar{\Omega}_j n_{j+1} - \bar{\Omega}_{j+1} n_j). \tag{85}$$

The other auxiliary quantity associated with the paraxial marginal ray is defined for the jth surface by

$$S_j = \frac{- n_j y_j (u_{j+1}+i_j)(n_{j+1}-n_j)}{2\text{Ж} n_{j+1}} . \tag{86}$$

The same quantity defined for the paraxial principal ray at the jth surface is

$$\bar{S}_j = \frac{-\,n_j\bar{y}_j(\bar{u}_{j+1}+\bar{i}_j)(n_{j+1}-n_j)}{2\pi n_{j+1}} \,. \tag{87}$$

Substituting equations (84) and (22) in (86) and equations (85) and (23) in (87) result to

$$S_j = \tfrac{1}{2}(y_j/n_{j+1}^2)(\Omega_{j+1}n_j^2 - \Omega_j n_{j+1}^2) \tag{88}$$

$$\bar{S}_j = \tfrac{1}{2}(\bar{y}_j/n_{j+1}^2)(\bar{\Omega}_{j+1}n_j^2 - \bar{\Omega}_j n_{j+1}^2). \tag{89}$$

Third-Order Aberration Coefficients

The Seidel aberrations of an optical system may be computed from paraxial ray data which provide information to calculate the third-order aberration coefficients. These Seidel aberration coefficients represent the algebraic sum of the third-order surface contributions throughout the optical system. The third-order aberration coefficients for an optical system of k surfaces are given by the following:

Spherical $$B = \sum_{j=1}^{k} S_j i_j^2 \tag{90}$$

Coma $$F = \sum_{j=1}^{k} S_j i_j \bar{i}_j \tag{91}$$

Astigmatism $$C = \sum_{j=1}^{k} S_j \bar{i}_j^2 \tag{92}$$

Distortion $$E = \sum_{j=1}^{k} \left\{ \bar{S}_j i_j \bar{i}_j + \mathcal{H}(\bar{u}_j^2 - \bar{u}_{j+1}^2) \right\} \tag{93}$$

Petzval Curvature $$P = \sum_{j=1}^{k} c_j (n_j - n_{j+1})/n_j n_{j+1} \tag{94}$$

Using equations (84), (85), (88), (89), (22), (23), (19) and (25), the aberration coefficients could be written in terms of the $y, \bar{y}$ diagram parameters as

$$B = \tfrac{1}{2}\mathcal{H}^2 \sum_{j=1}^{k} \frac{(\Omega_{j+1} n_j^2 - \Omega_j n_{j+1}^2)(\Omega_j n_{j+1} - \Omega_{j+1} n_j)^2 y_j}{n_j^2 n_{j+1}^2 (n_{j+1} - n_j)^2} \tag{95}$$

$$F = \tfrac{1}{2}\mathcal{H}^2 \sum_{j=1}^{k} \frac{(\Omega_{j+1} n_j^2 - \Omega_j n_{j+1}^2)(\Omega_j n_{j+1} - \Omega_{j+1} n_j) y_j \alpha_j}{n_j^2 n_{j+1}^2 (n_{j+1} - n_j)^2} \tag{96}$$

$$C = \tfrac{1}{2}\mathcal{H}^2 \sum_{j=1}^{k} \frac{(\Omega_{j+1} n_j^2 - \Omega_j n_{j+1}^2)(\bar{\Omega}_j n_{j+1} - \bar{\Omega}_{j+1} n_j)^2 y_j}{n_j^2 n_{j+1}^2 (n_{j+1} - n_j)^2} \tag{97}$$

$$E = \tfrac{1}{2}\mathcal{H}^2 \sum_{j=1}^{k} \frac{(\bar{\Omega}_{j+1} n_j^2 - \bar{\Omega}_j n_{j+1}^2)(\Omega_j n_{j+1} - \Omega_{j+1} n_j) \bar{y}_j \alpha_j}{n_j^2 n_{j+1}^2 (n_{j+1} - n_j)^2}$$

$$+ \mathcal{H}^3 \sum_{j=1}^{k} \left\{ \frac{n_{j+1}^2 \bar{\Omega}_j^2 - \bar{\Omega}_{j+1}^2 n_j^2}{n_j^2 n_{j+1}^2} \right\} \tag{98}$$

$$P = - \sum_{j=1}^{k} \left\{ \frac{\Omega_j \bar{\Omega}_{j+1} - \Omega_{j+1} \bar{\Omega}_j}{n_j n_{j+1}} \right\} \tag{99}$$

where $\alpha_j = \bar{\Omega}_j n_{j+1} - \bar{\Omega}_{j+1} n_j$.

For a two-surface system with image plane located at infinity described in chapter 3 and whose normalized $y,\bar{y}$ diagram is shown in Fig. 3, the above third-order aberration coefficients become

$$B = \beta\left\{(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)^2 (n_3 - n_2)^2 y_1 - \Omega_2^3 n_3^2 n_1^2 (n_2 - n_1)^2\right\} \tag{100}$$

$$F = \beta\left\{(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(n_3 - n_2)^2 y_1 - n_1^2 n_3 (\bar{\Omega}_2 n_3 - n_2)(n_2 - n_1)^2 \Omega_2^2\right\} \tag{101}$$

$$C = \beta\left\{(\Omega_2 n_1^2 - n_2^2)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)^2 (n_3 - n_2)^2 y_1 - \Omega_2 n_1^2 (\bar{\Omega}_2 n_3 - n_2)^2 (n_2 - n_1)^2\right\} \tag{102}$$

$$E = \beta\left\{(\bar{\Omega}_2 n_1^2 - \bar{\Omega}_1 n_2^2)(n_2 - \Omega_2 n_1)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(n_3 - n_2)^2 \bar{y}_1 + (n_2^2 - \bar{\Omega}_2 n_3^2)(n_2 - n_1)^2 \Omega_2 \bar{\Omega}_2 n_1^2 \bar{y}_2\right\} + \frac{\mathcal{K}^3}{n_1^2 n_2^2 n_3^2}\left\{(\bar{\Omega}_1^2 n_2^2 - \bar{\Omega}_2^2 n_1^2) n_3^2 + n_1^2 (\bar{\Omega}_2^2 n_3^2 - n_2^2)\right\} \tag{103}$$

$$P = - \left\{ \frac{n_3(\bar{\Omega}_2 - \Omega_2 \bar{\Omega}_1) + n_1 \Omega_2}{n_1 n_2 n_3} \right\} \tag{104}$$

where $\beta = \frac{1}{2} \mathcal{K}^2 / n_1^2 n_2^2 (n_2 - n_1)^2 (n_3 - n_2)^2$.

The Module

The class of two-surface systems to be analyzed are those which possess zero third-order spherical aberration relative to a pair of conjugate planes one of which is at infinity. Such two-surface systems, with one of its conjugate planes at infinity and with the third-order spherical aberration with respect to this pair of conjugate planes zero, shall be designated as modules.

Modules are then defined by equating equation (110) to zero, which yields

$$(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)^2 (n_3 - n_2)^2 y_1 - n_1^2 \Omega_2^3 (n_2 - n_1)^2 = 0. \qquad (105)$$

Solving for y_1, gives

$$y_1 = \frac{n_1^2 n_3^2 (n_2 - n_1)^2 \Omega_2^3}{(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)^2 (n_3 - n_2)^2} \, . \qquad (106)$$

Equation (106) gives the normalized value of the marginal ray height at the first surface of the module in terms of the Ω_2 parameter and the indices of refraction. This functional relationship is the condition that the two-surface system be a module.

If equation (105) is expanded and the terms are rearranged, it becomes a cubic of the form

$$a_1 \Omega_2^3 + b_1 \Omega_2^2 + c_1 \Omega_2 + d_1 = 0, \qquad (107)$$

where

$$a_1 = n_1^2 \left\{ n_1^2 (n_3 - n_2)^2 y_1 - n_3^2 (n_2 - n_1)^2 \right\}$$

$$b_1 = - n_1^2 n_2 (2n_1 + n_2)(n_3 - n_2)^2 y_1 \qquad (108)$$

$$c_1 = n_1 n_2^2 (n_1 + 2n_2)(n_3 - n_2)^2 y_1$$

$$d_1 = - n_2^4 (n_3 - n_2)^2 y_1 .$$

Stavroudis (1969b) solved this cubic expressed in terms of the front focal distance, a parameter related to the focal length of the system and another parameter which he called q. Applying the general method of solving cubic polynomials, attributed to Cardan (1545), he solved this cubic by introducing a new non-optical parameter k. Following the procedure he used, the three roots of equation (107) are

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1 + 2n_2 - (n_2 - n_1)\left[\left\{ \frac{k+1}{k-1} \right\}^{1/3} w^r + \left\{ \frac{k-1}{k+1} \right\}^{1/3} w^{-r} \right]} \qquad (109)$$

$(r = 0,1,2)$,

where $w = \exp(2\pi i/3)$, a complex cube root of unity. The value of y_1 obtained in terms of the parameter k is

$$y_1 = \frac{27 n_2^2 n_3^2 (k^2 - 1)}{4 n_1 (n_3 - n_2)^2 (n_2 - n_1)} . \qquad (110)$$

Only those values of k which yield real values of Ω_2 and y_1 are of interest. From equation (110) real values of y_1 occur only if k is either real or pure imaginary. Stavroudis (1969b) has shown that for real values of k, the solution to the cubic can be real only when r is equal to zero. Hence for real k,

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1+2n_2 - (n_2-n_1)\left[\left\{\frac{k+1}{k-1}\right\}^{1/3} + \left\{\frac{k-1}{k+1}\right\}^{1/3}\right]} . \tag{111}$$

For pure imaginary values of k, Stavroudis (1969b) introduced another free parameter θ defined by

$$k = i \tan \theta . \tag{112}$$

Substituting equation (112) in equations (110) and (109) we obtain

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1+2n_2 + 2(n_2-n_1)\cos \frac{2}{3}(\theta + \pi r)} \quad (r=0,1,2) \tag{113}$$

and

$$y_1 = \frac{-27n_2^2n_3^2 \sec^2\theta}{4n_1(n_3-n_2)^2(n_2-n_1)} . \tag{114}$$

Canonical Optical Parameters

Stavroudis (1969b), in his analysis of modules, transformed the conventional optical parameters he used into a more convenient form which he called canonical optical parameters. He defined the canonical optical parameters, written using my notations, as follows:

$$\left.\begin{aligned} C_1 &= (n_2 - n_1)c_1 f \\ C_2 &= (n_3 - n_2)c_2 f \\ T_1 &= t_1/n_1 f \\ T_2 &= t_2/n_2 f \end{aligned}\right\} \quad (115)$$

where $f = \left\{n_2t_1+n_1t_2-(n_2-n_1)c_1t_1t_2\right\}/n_1n_2$, c_1 and c_2 are the curvatures of the module's first and second surfaces, t_1 is the negative value of the front focal distance and t_2 is the axial thickness of the module. He also defined

$$Q_r = q_r/n_1, \quad (116)$$

$$\text{where } q_r = \frac{3n_2^{\;2}}{(2n_2+n_1) - (n_2-n_1)\left[\left\{\frac{k+1}{k-1}\right\}^{1/3}w^r + \left\{\frac{k-1}{k+1}\right\}^{1/3}w^{-r}\right]}$$

and obtained T_1 in terms of the free parameter k given by

$$T_1 = \frac{27n_3^2\, n_2^2\, (k^2 - 1)}{4n_1(n_3-n_2)^2(n_2-n_1)}\,. \quad (117)$$

Comparing the above canonical optical parameters and the equations with the $y,\bar{y}$ diagram parameters, we note that equations (116) and (109) are identical as are (117) and (110). Therefore, the canonical optical parameter T_1 is identical to y_1 and the parameter Q is identical to Ω_2.

The equivalence of the remaining canonical optical parameters and equations with the $y,\bar{y}$ diagram parameters used in this analysis of modules is easily established when the relationship between the parameter f and the Lagrange invariant is obtained. To show the relationship existing between f and Ж, T_1 is equated to y_1, which gives

$$y_1 = t_1/n_1 f. \qquad (118)$$

From the $y,\bar{y}$ diagram, the value of t_1 is obtained as

$$t_1 = (n_1/Ж)\begin{vmatrix} 0 & -1 \\ y_1 & \bar{y}_1 \end{vmatrix} = n_1 y_1/Ж. \qquad (119)$$

Substituting equation (119) in (118), we obtain

$$f = 1/Ж. \qquad (120)$$

Therefore, the parameter f defined and used by Stavroudis (1969b) is the reciprocal of the Lagrange invariant.

From the relationship between the parameter f and the Lagrange invariant given by (120), the equivalence between the remaining canonical optical parameters and the $y,\bar{y}$ diagram parameters are now established and they are as follows:

$$C_1 = \Phi_1 \qquad C_2 = \Phi_2 \qquad T_2 = \tau_2 Ж \qquad (121)$$

In terms of the $y, \bar{y}$ and the $\Omega, \bar{\Omega}$ parameters, equation (121) could be written as

$$\left.\begin{aligned} C_1 &= \bar{\Omega}_2 - \Omega_2\bar{\Omega}_1 = (1 - \Omega_2)/y_1 \\ C_2 &= \Omega_2 \\ T_2 &= (y_1\bar{y}_2 - \bar{y}_1) = (1 - y_1)/\Omega_2. \end{aligned}\right\} \quad (122)$$

Critical Values of Free Parameters

Parameter k

The first critical value, k_o, is the value of the free parameter k when $\Omega_2 = 1$. For Ω_2 equal to unity, equation (111) becomes

$$\left\{\frac{k+1}{k-1}\right\}^{1/3} + \left\{\frac{k-1}{k+1}\right\}^{1/3} = -(n_1 + 3n_2)/n_1, \quad (123)$$

which when solved for k gives

$$k_o = \frac{(3n_2+2n_1)\sqrt{3n_2-n_1}}{3\sqrt{3}\, n_2\sqrt{n_1+n_2}} = \sqrt{1 - \frac{4n_1^3}{27\, n_2(n_1+n_2)}}\,. \quad (124)$$

This value of k results in zero values for the parameters c_1, C_1 and Φ_1.

The second critical value, k_∞, of the parameter k is its value when Ω_2 is infinite. For this value of Ω_2, equation (111) is reduced to

$$\left\{\frac{k+1}{k-1}\right\}^{1/3} + \left\{\frac{k-1}{k+1}\right\}^{1/3} = \frac{n_1+2n_2}{n_2-n_1} , \tag{125}$$

which when solved for k gives

$$k_\infty = \frac{(n_2+2n_1)\sqrt{4n_2-n_1}}{3\sqrt{3}\ \ n_2\sqrt{n_1}} = \sqrt{1 + \frac{4(n_2-n_1)^3}{27n_2^{\ 2}n_1}} . \tag{126}$$

At $k = k_\infty$, the values of the parameters c_1, C_1, Φ_1, t_2, T_2 and τ_2* vanish.

The third critical value, k^*, is the value of k when y_1 is unity. Substituting $y_1=1$ in equation (110) and solving for k, we obtain

$$k^* = \sqrt{1 + \frac{4n_1(n_3-n_2)^2(n_2-n_1)}{27n_2^{\ 2}n_3^{\ 2}}} . \tag{127}$$

This value of k makes t_2, T_2 and τ_2* to become zero.

Table I gives the corresponding values of the canonical optical parameters and their equivalent $y,\bar{y}$ diagram parameters for the three critical values of k including the parameters values for k = 1, 0 and infinity.

Parameter θ

For the case where k is pure imaginary, the free parameter θ defined by equation (112) is used to evaluate the $y,\bar{y}$ diagram parameters. The first critical value, θ_o, of the free parameter θ is defined as its value when Ω_2 is unity. For $\Omega_2 = 1$, equation (113) becomes

$$(n_1+2n_2) + 2(n_2-n_1) \cos \tfrac{2}{3}(\theta + \pi r) = 3n_2{}^2/n_1, \tag{128}$$

which when solved for θ, gives

$$\theta_o = -\pi r + \tfrac{3}{2} \text{ arc cos } \left\{\frac{n_1+3n_2}{2n_1}\right\}. \tag{129}$$

At this value of θ, the parameters c_1, C_1 and Φ_1 vanish.

The second critical value, θ_∞, is the value of θ when Ω_2 is infinite. Substituting infinity for Ω_2 in equation (113) and solving for θ, gives

$$\theta_\infty = -\pi r + \tfrac{3}{2} \text{ arc cos } \left\{\frac{n_1+2n_2}{2(n_1-n_2)}\right\} . \tag{130}$$

For this value of θ, the parameters c_1, C_1 and Φ_1 become infinite while t_2, T_2 and $\tau_2{}^*$ vanish.

The third critical value of θ is θ^*, which is the value of the free parameter when y_1 is equal to unity. For $y_1 = 1$ in equation (114),

$$\theta^* = -\pi r + \text{arc cos } \left\{\frac{3n_1n_3}{2(n_3 - n_2)}\sqrt{\frac{3}{n_1(n_1-n_2)}}\right\} . \tag{131}$$

For $\theta = \theta^*$, the parameters t_2, T_2 and $\tau_2{}^*$ become all equal to zero.

Table II gives the corresponding values of the canonical optical parameters and their equivalent $y,\bar{y}$ diagram parameters for the three critical values of θ.

Table I. Values of Canonical Optical Parameters and Equivalent $y,\bar{y}$ Diagram Parameters Corresponding to the Critical Values of k

k	0	1	∞	k_o	k_∞	k^*
Ω_2 Φ_2 C_2 Q	$\frac{3n_2^2}{n_1(4n_2-n_1)}$	0	$\frac{n_2^2}{n_1^2}$	1	∞	--
y_1 T_1	$\frac{-27n_2^2n_3^2}{4n_1(n_3-n_2)^2(n_2-n_1)}$	0	∞	$\frac{-n_1^2n_3^2}{(n_2^2-n_1^2)(n_3-n_2)^2}$	$\frac{n_3^2(n_2-n_1)^2}{n_1^2(n_3-n_2)^2}$	1
Φ_1 C_1	$\frac{4(n_3-n_2)^2(n_2-n_1)^2(3n_2-n_1)}{27n_2^2n_3^2(4n_2-n_1)}$	∞	0	0	$-\infty$	--
τ_2* T_2	$\frac{n_1(4n_2-n_1)}{\;}\left[1+\frac{27n_2^2n_3^2}{4n_1(n_3-n_2)^2(n_2-n_1)}\right]$	∞	$-\infty$	$1+\frac{n_1^2n_3^2}{(n_2^2-n_1^2)(n_3-n_2)^2}$	0	0

Table II. Values of Canonical Optical Parameters and Equivalent $y,\bar{y}$ Diagram Parameters Corresponding to the Critical Values of θ

θ	θ_o	θ_∞	θ^*
Ω_2 Φ_2 C_2 Q	1	∞	--
y_1 T_1	$\dfrac{-n_1^2 n_3^2}{(n_2^2-n_1^2)(n_3-n_2)^2}$	$\dfrac{n_3^2(n_2-n_1)^2}{n_1^2(n_3-n_2)^2}$	1
Φ_1 C_1	0	$-\infty$	--
τ_2^* T_2	$1+\dfrac{n_1^2 n_3^2}{(n_2^2-n_1^2)(n_3-n_2)^2}$	0	0

Parameter Bounds

Values of Ω_2 and y_1 as functions of k are defined by equations (110) and (111) and their corresponding values as functions of θ are defined by (113) and (114). Since the canonical optical parameters and the equivalent $y,\bar{y}$ diagram parameters could be expressed in terms of Ω_2 and y_1, functional relationships exist between the parameter k or θ and the canonical optical parameters or equivalent $y,\bar{y}$ diagram parameters. Bounds on the free parameters k and θ are equivalent to bounds placed on the canonical optical parameters or the $y,\bar{y}$ diagram parameters.

The critical values of the free parameters k and θ are functions of the indices of refraction n_1, n_2 and n_3. Therefore, the order in which the critical values occur depends upon the relative values of these refractive indices. Stavroudis (1969b) made a thorough analysis of all possible orderings of the critical values and the corresponding conditions on the relative values of the set of three indices of refraction.

CHAPTER 5

ADDITIONAL PROPERTIES OF MODULES

This chapter presents further analysis of the remaining third-order aberrations of the module. Conditions for modules to simultaneously eliminate third-order spherical and other Seidel aberrations are derived. Limitations in the choice of values of the free parameters and the $y,\bar{y}$ diagram parameters are discussed for modules free of additional third-order aberrations. Numerical examples of modules are given in the appendix. All equations derived in this chapter conform with the assumptions made in the preceding chapter.

Modules with Zero Coma

The two-surface system with image plane at infinity can be made free from coma if equation (101) is set equal to zero. That is

$$(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(n_3 - n_2)^2 y_1 - (\bar{\Omega}_2 n_3 - n_2)(n_2 - n_1)^2 n_1^2 n_3 \Omega_2^2 = 0. \qquad (132)$$

Equation (132) when solved for y_1, becomes

$$y_1 = \frac{n_1^2 n_3 (n_2 - n_1)^2 (\bar{\Omega}_2 n_3 - n_2)\Omega_2^2}{(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(n_3 - n_2)^2} . \qquad (133)$$

Equation (133) gives the value of y_1 as a function of the $\Omega, \bar{\Omega}$ parameters and the refractive indices of the three media for a two-surface system with zero third-order coma.

Equating (133) and (106), the condition for zero spherical aberration, results in a condition for the modules to be free of coma. The condition is given by

$$(\bar{\Omega}_2 n_3 - n_2)(n_2 - \Omega_2 n_1) - n_3 \Omega_2 (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1) = 0. \tag{134}$$

Stavroudis (1969b) has shown that the two-surface module can be free from coma only when $k = 0$. Hence, for zero coma, Ω_2 in equation (111) becomes a function of only the refractive indices. That is

$$\Omega_2 = 3n_2^2 / n_1(4n_2 - n_1), \tag{135}$$

and the height of the marginal ray at the first surface is

$$y_1 = \frac{-27 n_2^2 n_3^2}{4 n_1 (n_3 - n_2)^2 (n_2 - n_1)} . \tag{136}$$

When equation (26) is applied at the first surface of the module, we obtain

$$y_1 = (1 - \Omega_2)/(\bar{\Omega}_2 - \Omega_2 \bar{\Omega}_1). \tag{137}$$

Equation (137) when solved for $\bar{\Omega}_2$, yields

$$\bar{\Omega}_2 = (\Omega_2 \bar{\Omega}_1 y_1 + 1 - \Omega_2)/\Omega_2 y_1 . \tag{138}$$

Substituting equation (135) and equation (136) in (138) results to

$$\bar{\Omega}_2 = \bar{\Omega}_1 + A_1, \tag{139}$$

where

$$A_1 = \frac{4\{3n_2^2 + n_1(4n_2-n_1)\}n_1(n_3-n_2)^2(n_2-n_1)}{81\, n_2^4\, n_3^2} . \tag{140}$$

If equation (134) is expanded and solved for $\bar{\Omega}_2$, we obtain

$$\bar{\Omega}_2 = \{n_2 + \Omega_2(n_3\bar{\Omega}_1 - n_1)\}/ n_3 . \tag{141}$$

Substituting equation (135) into (141), gives

$$\bar{\Omega}_2 = n_2\{n_1(n_2-n_1) + 3n_2n_3\bar{\Omega}_1\}/n_1n_3(4n_2-n_1). \tag{142}$$

Solving equations (139) and (142), simultaneously, will give the values of $\bar{\Omega}_1$ and $\bar{\Omega}_2$ as functions of the refractive indices.

The location of the stop $S(0,y_s)$ in the $y,\bar{y}$ diagram is defined by the point of intersection of the line associated with n_2 which is given by

$$\Omega_2\bar{y} - \bar{\Omega}_2 y + 1 = 0 \tag{143}$$

and the y-axis. Hence

$$y_s = 1/ \bar{\Omega}_2 . \tag{144}$$

Therefore, the value of $\bar{\Omega}_2$ obtained by solving simultaneously equations (139) and (142), locates the stop position of the module with zero coma while the obtained value of $\bar{\Omega}_1$ determines the height of the entrance pupil since $y_E = 1/\bar{\Omega}_1$.

Modules with Zero Astigmatism

The condition for a two-surface system with image plane located at infinity, to be free from astigmatism is obtained when equation (102) is set equsl to zero. Hence,

$$(\Omega_2 n_1^2 - n_2^2)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)^2 (n_3 - n_2)^2 y_1 - n_1{}^2 \Omega_2 (\bar{\Omega}_2 n_3 - n_2)^2 (n_2 - n_1)^2 = 0. \qquad (145)$$

Solving for y_1, gives

$$y_1 = \frac{\Omega_2(\bar{\Omega}_2 n_3 - n_2)^2 (n_2 - n_1)^2 n_1{}^2}{(\Omega_2 n_1^2 - n_2^2)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)^2 (n_3 - n_2)^2} . \qquad (146)$$

Equation (146) gives the value of y_1 as function of the $\Omega, \bar{\Omega}$ parameters and the indices of refraction for the two-surface system with zero astigmatism.

If equation (146) is equated to (106), the condition for zero spherical aberration, the result provides the condition for modules to be free from astigmatism,

$$(\bar{\Omega}_2 n_3 - n_2)^2 (n_2 - \Omega_2 n_1)^2 - (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)^2 n_3{}^2 \Omega_2{}^2 = 0. \qquad (147)$$

Being a difference of two squares, the above equation is factored.

$$\left\{(\bar{\Omega}_2 n_3 - n_2)(n_2 - \Omega_2 n_1) - n_3 \Omega_2 (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)\right\} \cdot \left\{(\bar{\Omega}_2 n_3 - n_2)(n_2 - \Omega_2 n_1) + n_3 \Omega_2 (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)\right\} = 0 \qquad (148)$$

If the module has zero coma, then equation (134) is satisfied. Since the left factor of equation (134) is identical to the first factor of equation (148), then astigmatism is also zero when (134) vanishes, which occurs only when $k = 0$.

The second factor of equation (148) will also eliminate astigmatism if it vanishes. This leads to the equation

$$(\bar{\Omega}_2 n_3 - n_2)(n_2 - \Omega_2 n_1) + n_3 \Omega_2 (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1) = 0. \quad (149)$$

Substituting equation (138) in (149) and solving for $\bar{\Omega}_2$, gives

$$\bar{\Omega}_2 = \frac{n_2 \{ y_1 (n_2 - n_1 \Omega_2) - n_3 (\Omega_2 - 1) \}}{2 n_3 y_1 (n_2 - n_1 \Omega_2)} . \quad (150)$$

Since Ω_2 and y_1 are functions of the free parameter k, the solution of equation (150) gives the value of $\bar{\Omega}_2$ for zero astigmatism indirectly in terms of k. Equation (150) also defines the location of the aperture stop in the $y,\bar{y}$ diagram for the module to be free from astigmatism since y_s is the reciprocal of $\bar{\Omega}_2$. Once the value of $\bar{\Omega}_2$ is obtained for zero astigmatism, the height of the entrance pupil could be determined by solving equation (138) for $\bar{\Omega}_1$ and taking its reciprocal.

Both Coma and Astigmatism Equal to Zero

Modules can be free simultaneously of coma and astigmatism only if either the condition for zero coma is satisfied or the first factor of the condition for zero astigmatism vanishes. However, equation (134) vanishes only for concentric systems as shown by Stavroudis (1969b) and Powell (1970). Therefore, modules with zero coma and astigmatism simultaneously must be concentric two-surface systems.

The $y, \bar{y}$ diagram parameters of modules with both coma and astigmatism equal to zero are defined by equations (135), (136), (139) and (142). The parameters are all functions of the indices of refraction of the three media. The aperture stop location of the module is defined by the value of $\bar{\Omega}_2$, obtained by solving simultaneously equations (139) and (142). Its entrance pupil height is equal to the reciprocal of $\bar{\Omega}_1$.

Modules with Zero Petzval Curvature

The condition for a two-surface system with image plane at infinity to have zero Petzval contribution is obtained if equation (104) is set equal to zero,

$$n_3(\bar{\Omega}_2 - \Omega_2\bar{\Omega}_1) + n_1\Omega_2 = 0, \tag{151}$$

which using equation (137), can be written as

$$n_3(1 - \Omega_2) + n_1 y_1 \Omega_2 = 0. \tag{152}$$

When solved for y_1, equation (152) becomes

$$y_1 = - n_3(1 - \Omega_2)/n_1\Omega_2 \ . \tag{153}$$

Equation (153) gives the value of y_1 as function of Ω_2 and the indices of refraction of the object and image spaces for a two-surface system with zero Petzval curvature.

Equating (153) with (106) and rearranging terms results to a quartic equation given by

$$a_2\Omega_2^4 + b_2\Omega_2^3 + c_2\Omega_2^2 + d_2\Omega_2 + e_2 = 0, \tag{154}$$

where

$$\left.\begin{aligned} a_2 &= n_1^3 n_3(n_2^2 - n_1 n_3)(n_3 - n_1) \\ b_2 &= n_1^2 n_3(n_3 - n_2)^2(n_1 + n_2)^2 \\ c_2 &= - 2n_1 n_2 n_3(n_3 - n_2)^2(n_1^2 + n_1 n_2 + n_2^2) \\ d_2 &= n_2^2 n_3(n_3 - n_2)^2(n_1 + n_2)^2 \\ e_2 &= - n_2^4 n_3(n_3 - n_2)^2 \ . \end{aligned}\right\} \tag{155}$$

The quartic equation (154) is the condition for a module to be free from Petzval curvature. The quartic has at most four real roots, therefore, there are at most four possible modules which will be free from Petzval contributions for a given set of three indices of refraction.

Real Zeros of the Quartic

The fundamental theorem of algebra states that such polynomial equation (154) has at least one root which could be either real or complex. We also know that such polynomial has at most four zeros. Since we are interested only on the real values of Ω_2, let us analyze the nature of the real zeros of the given quartic. To do this we make use of an important theorem in algebra known as the Decartes' rule of signs. Every standard text in algebra or theory of equations such as Dickson (1939, pp. 76-80) discusses this theorem.

From the equations of the coefficients of the quartic, we note that the relative values of the three indices of refraction determine the variations in sign of the successive terms of the given polynomial. We also note that the quartic can have zero roots only when $n_3 = n_2$, which is an impossible case for modules or any other two-surface system.

For a given set of three positive refractive indices n_1, n_2 and n_3, we observe that only the first term of the quartic could possibly change in sign. The remaining terms have the same algebraic signs regardless of the relative values of the three indices of refraction. Let equation (154) be denoted by $G(\Omega_2) = 0$. The change in the number of variations in sign of the terms in the given polynomial could be grouped in the following three cases:

Case I: (a) $n_2^2 - n_1n_3 > 0$, $n_3 - n_1 > 0$.

(b) $n_2^2 - n_1n_3 < 0$, $n_3 - n_1 < 0$.

Case II: (a) $n_2^2 - n_1n_3 < 0$, $n_3 - n_1 > 0$.

(b) $n_2^2 - n_1n_3 > 0$, $n_3 - n_1 < 0$.

Case III: (a) $n_2^2 - n_1n_3 = 0$.

(b) $n_3 - n_1 = 0$.

An analysis of the above three cases might prove useful in the selection of glasses for modules with zero Petzval sum.

Case I. Case I(a) implies that $n_2 > n_1$ and I(b) implies $n_1 > n_2$. For both, the number of variations in sign in $G(\Omega_2)$ is three, hence, $G(\Omega_2) = 0$ has at most three positive real roots or at least one. The given quartic has at most only one real negative root, since the number of variations in sign in $G(-\Omega_2)$ is one. Therefore, for this case there are at most three positive values of Ω_2 and at most one real negative value which shall make the module eliminate Petzval contribution.

Case II. The number of variations in sign in $G(\Omega_2)$ for this case is four, hence $G(\Omega_2) = 0$, has at most four positive real zeros. The given quartic could also have two positive real roots or none at all. Since there is no variations in sign in $G(-\Omega_2)$, the quartic cannot have any real negative root. This implies that Ω_2 can have only positive real values for this particular case. Case II(a) implies that $n_3 > n_2$ while II(b) implies $n_2 > n_3$.

<u>Case III</u>. For this case, the quartic reduces to a cubic. The variations in sign in the resultant cubic is three which implies that $G(\Omega_2) = 0$, can have at most three positive real roots or at least one. There is no variation in sign in $G(-\Omega_2)$, therefore, the given polynomial can never have any real negative zero. Case III(b) applies to the case of modules in air or any medium common to the object and image spaces.

Two-Mirror Case

The quartic equation (154) degenerates into a simple quadratic equation for the case of a two-mirror module in air. The resulting equation is

$$2\Omega_2^{\,2} - 1 = 0, \qquad (156)$$

which when solved gives values of $\pm \frac{1}{2}\sqrt{2}$ for Ω_2. These two real values imply the existence of two modules in air consisting of spherical reflecting surfaces with zero Petzval curvature. Equation (111), applied to a two-mirror module in air could be written as

$$\left\{\frac{k+1}{k-1}\right\}^{1/3} + \left\{\frac{k-1}{k+1}\right\}^{1/3} = (\Omega_2+3)/2\Omega_2 \,, \qquad (157)$$

which when solved gives $k = \pm 1.221032$ when $\Omega_2 = \frac{1}{2}\sqrt{2}$. When $\Omega_2 = -\frac{1}{2}\sqrt{2}$, equation (157) yields $k = \pm i\, 1.364291$. Using equation (112), these pure imaginary values of k correspond to $\theta = \pi r \pm 0.9383$ radians, where $r = 0,1,2$. Therefore, zero Petzval sum in a two-mirror module in air limits us to the above two values of the Ω_2 parameter which implicitly impose a bound in the choice of values of the free parameters k and θ. Only the four calculated values of the free parameters satisfy the condition for this additional third-order property of the two-mirror module in air.

Both Coma and Petzval Sum Equal to Zero

The height of the marginal ray at the first surface of the module with zero coma is given by equation (136) and the marginal ray height at the same surface for a two-surface system which has zero Petzval curvature is given by (153). Equating (136) to (153) results in an equation which provides the condition for a module to simultaneously have zero coma and Petzval sum. The resultant equation when solved for Ω_2, gives

$$\Omega_2 = \frac{4n_1(n_3 - n_2)^2(n_2 - n_1)}{27n_2{}^2n_3 + 4n_1(n_3 - n_2)^2(n_2 - n_1)} \qquad (158)$$

Therefore, for a set of three indices of refraction, there is only one module which can simultaneously possess zero coma and Petzval curvature.

Both Astigmatism and Petzval Sum Equal to Zero

The marginal ray height at the first surface of the module with zero astigmatism is obtained if equation (150) is solved for y_1 resulting in

$$y_1 = \frac{-\,n_2 n_3(\Omega_2-1)}{(n_2-n_1\Omega_2)(2n_3\bar{\Omega}_2-n_2)} \,. \tag{159}$$

If equation (159) is set equal to (153) and the resultant expression is solved for $\bar{\Omega}_2$, we obtain

$$\bar{\Omega}_2 = \frac{n_2(2n_1\Omega_2-n_2)}{2n_3(n_1\Omega_2-n_2)} \,. \tag{160}$$

Equation (160) provides the condition for a module with zero astigmatism, in which the Petzval contribution is zero. Equation (160) also defines the location of the aperture stop in the $y,\bar{y}$ diagram as a function of the Ω_2 parameter and the three indices of refraction.

Zero Coma, Astigmatism and Petzval Sum

The condition for modules to eliminate both coma and astigmatism is the vanishing of the expressions in equation (134) or the left factor of equation (148). This implies that a module with zero coma because it satisfies equation (134) also has zero astigmatism. Hence, the condition for a module to have simultaneously zero coma and zero Petzval contribution which is given by equation (158), is identical to the requirements for modules to be free simultaneously of coma, astigmatism and Petzval curvature since (158) was derived with the assumption that (134) vanishes.

If equation (158) is substituted in equation (141), we obtain

$$\bar{\Omega}_2 = n_2/n_3 + A'_2(n_3\bar{\Omega}_1 - n_1), \tag{161}$$

where

$$A'_2 = \frac{4n_1(n_3-n_2)^2(n_2-n_1)}{n_3\{27n_2{}^2n_3 + 4n_1(n_3-n_2)^2(n_2-n_1)\}} .$$

Equation (161) gives the functional relationship between $\bar{\Omega}_2$ and $\bar{\Omega}_1$. It also implies the relative locations of the stop and the entrance pupil of the module to be free simultaneously of coma, astigmatism and Petzval curvature.

Modules with Zero Distortion

The two-surface system with image plane at infinity, shall eliminate distortion if equation (103) vanishes. That is

$$(\bar{\Omega}_2n_1^2-\bar{\Omega}_1n_2^2)(n_2-\Omega_2n_1)(\bar{\Omega}_1n_2-\bar{\Omega}_2n_1)(n_3-n_2)^2\bar{y}_1$$

$$+ n_1^2\Omega_2\bar{\Omega}_2\ \bar{y}_2(n_2 - n_1)^2(n_2{}^2- \bar{\Omega}_2n_3{}^2)$$

$$+\frac{*}{n_3^2}(n_2-n_1)^2(n_3-n_2)^2\{n_3^2(\bar{\Omega}_1^2n_2^2-\bar{\Omega}_2^2n_1^2)+(\bar{\Omega}_2^2n_3^2-n_2^2)n_1\} = 0. \tag{162}$$

Equation (27) applied to the first surface, yields

$$\bar{y}_1 = (\bar{\Omega}_1-\bar{\Omega}_2)/(\bar{\Omega}_2-\Omega_2\bar{\Omega}_1). \tag{163}$$

By eliminating the quantities in the denominator in (163) by by substituting equation (137), the relationship between y_1 and $\bar{y}_1$ is obtained.

$$\bar{y}_1 = (\bar{\Omega}_1 - \bar{\Omega}_2) y_1 / (1 - \Omega_2) \tag{164}$$

When equation (27) is applied to the second surface,

$$\bar{y}_2 = (\bar{\Omega}_2 - 1)/\Omega_2. \tag{165}$$

Substituting equations (164) and (165) in (162), we obtain

$$\begin{aligned} & n_3^2(\bar{\Omega}_2 n_1^2 - \bar{\Omega}_1 n_2^2)(n_2 - \Omega_2 n_1)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(n_3 - n_2)^2 (\bar{\Omega}_1 - \bar{\Omega}_2) y_1 \\ & + n_3^2 (n_2^2 - \bar{\Omega}_2 n_3^2)(n_2 - n_1)^2 (\bar{\Omega}_2 - 1)(1 - \Omega_2) n_1{}^2 \bar{\Omega}_2 \\ & + \mathtt{x}(n_2 - n_1)^2 (n_3 - n_2)^2 (1 - \Omega_2) \left\{ n_3^2 (\bar{\Omega}_1^2 n_2^2 - \bar{\Omega}_2^2 n_1^2) + (\bar{\Omega}_2^2 n_3^2 - n_2^2) n_1 \right\} = 0. \end{aligned} \tag{166}$$

When equation (166) is solved for y_1 and the resulting equation is set equal to (106), the condition for zero spherical aberration, we obtain the condition for the module to have zero distortion.

$$\begin{aligned} & n_1^2\, n_3{}^4 \Omega_2{}^3 (\bar{\Omega}_2 n_1^2 - \bar{\Omega}_1 n_2^2)(\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1)(\bar{\Omega}_1 - \bar{\Omega}_2) \\ & + (1 - \Omega_2)(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1) \left\{ n_3^2 (n_2^2 - \bar{\Omega}_2 n_3^2)(\bar{\Omega}_2 - 1)\, n_1{}^2 \bar{\Omega}_2 \right. \\ & \left. + \mathtt{x}(n_3 - n_2)^2 \left[n_3^2 (\bar{\Omega}_1^2 n_2^2 - \bar{\Omega}_2^2 n_1^2) + n_1^2 (\bar{\Omega}_2^2 n_3^2 - n_2^2) \right] \right\} = 0. \end{aligned} \tag{167}$$

The above condition or equation (167) involves all the unknown $\Omega, \bar{\Omega}$ parameters of the module and seems very difficult to satisfy in practice.

CHAPTER 6

TWO-MODULE SYSTEMS

This chapter is concerned with the problem of coupling or forming arrays of modules to produce systems which have zero third-order spherical aberration. The properties of modules coupled are divided into two cases. The first case deals with the properties of two-module systems at finite conjugates and the other pertains to two modules coupled at infinite conjugates. A slightly different normalization scheme has been used for each case to simplify their $y,\bar{y}$ diagram representations and the accompanying algebraic formulations. Astigmatism and coma were made to vanish under certain conditions. Expressions for zero coma and astigmatism for the two cases of the two-module system are derived.

Two-Module System at Finite Conjugates

The nomenclature used in the analysis of two-module systems at finite conjugates is illustrated in Fig. 4(a). The normalized $y,\bar{y}$ diagram of the system is shown in Fig. 4(b). The y heights are normalized with respect to the exit pupil of A or the entrance pupil of B. The $\bar{y}$ heights are normalized relative to the height of the image of the system. Typical example of a two-module system at finite conjugates and the corresponding $y,\bar{y}$ diagram representation is shown in Fig. 5.

Modules A and B are coupled in such a manner that the object is at the front focal plane of A and its image located at the rear focal plane of B.

Let k_a and k_b be the free parameters for modules A and B, respectively. For module A,

$$y_1 = \frac{27 n_2^2 n_3^2 (k_a^2 - 1)}{4 n_1 (n_3 - n_2)^2 (n_2 - n_1)}$$

and (168)

$$\Omega_2 = \frac{3\, n_2^{\,2}/n_1}{(n_1 + 2n_2) - (n_2 - n_1)\left[\left\{\frac{k_a+1}{k_a-1}\right\}^{1/3} + \left\{\frac{k_a-1}{k_a+1}\right\}^{1/3}\right]} .$$

For B,

$$y_1 = \frac{-\,27 n_3^2 n_4^2 (k_b^2 - 1)}{4 n_5 (n_4 - n_3)^2 (n_5 - n_4)}$$

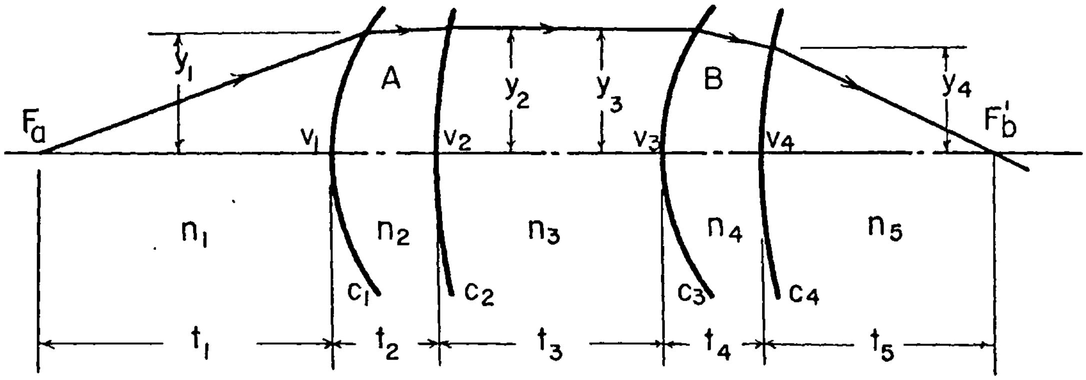
(169)

$$\Omega_4 = \frac{-\,3 n_4^2 / n_5}{(2n_4 + n_5) + (n_5 - n_4)\left[\left\{\frac{k_b+1}{k_b-1}\right\}^{1/3} + \left\{\frac{k_b-1}{k_b+1}\right\}^{1/3}\right]} .$$

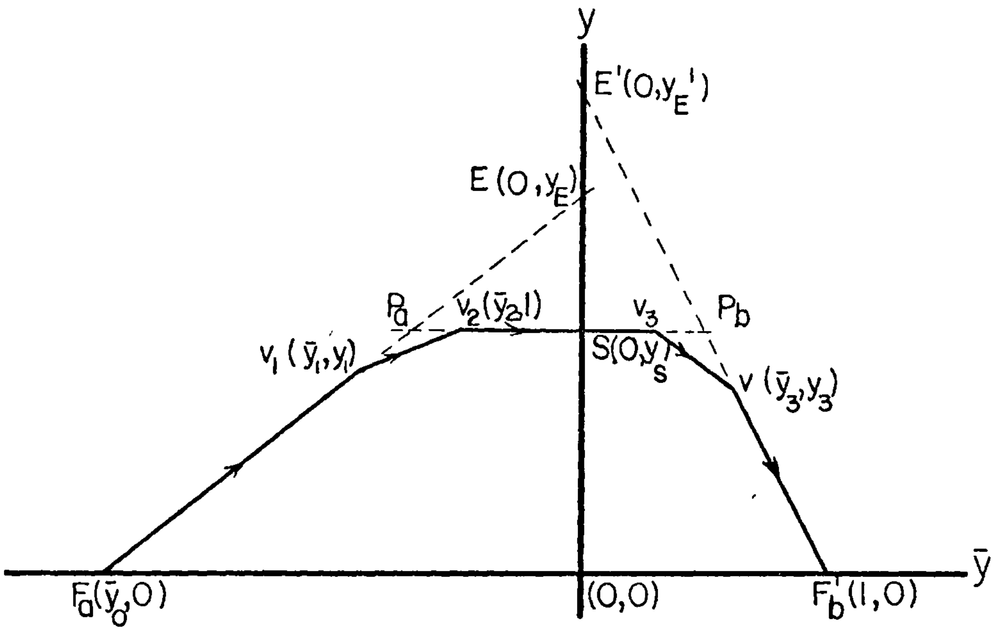
The two-module system coupled in such manner has zero third-order spherical aberration.

Zero Coma

To determine the condition for which the two-module system shall be free of coma, equation (96) is applied to the four surfaces of the system and set equal to zero. The result yields the following equation.

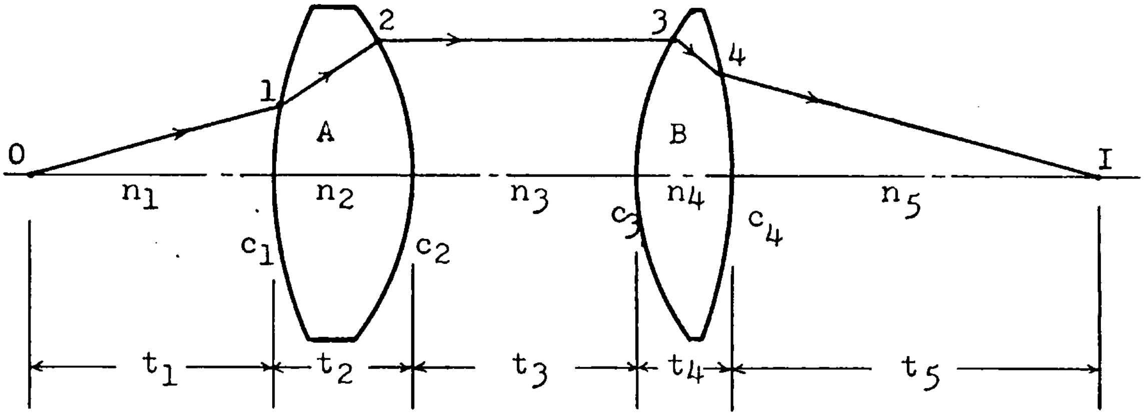


(a) Optical System Layout



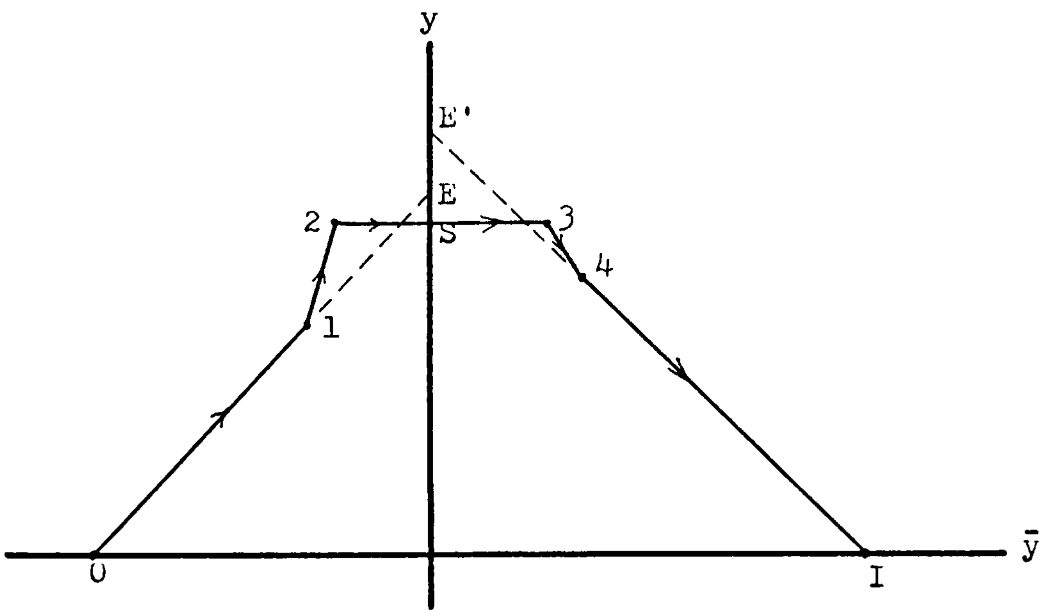
(b) Normalized y,ȳ Diagram

Fig. 4. Nomenclature Used in the Analysis of Two-Module System at Finite Conjugates



$c_1 = 0.184135$	$t_1 = 16.0374$	$n_1 = n_5 = 1.9525$
$c_2 = -0.274616$	$t_2 = 1.78539$	$n_2 = n_4 = 1.5731$
$c_3 = 0.267107$	$t_4 = 1.13447$	$n_3 = 1.0$
$c_4 = -0.157263$	$t_5 = 17.3695$	$k_a = 0.994$
		$k_b = 0.9935$

(a) Layout of System



(b) $y, \bar{y}$ Diagram

Fig. 5. Typical Example of Two-Module System at Finite Conjugates

$$an_3y_1(\Omega_2n_1^2-\Omega_1n_2^2)(\Omega_1n_2-\Omega_2n_1)(\bar{\Omega}_1n_2-\bar{\Omega}_2n_1)$$

$$- b\Omega_2^2(\bar{\Omega}_2n_3-n_2) - c\Omega_4^2(n_4-\bar{\Omega}_4n_3)$$

$$- dn_3y_4(n_4^2+\Omega_4n_5^2)(\Omega_4n_5+n_4)(\bar{\Omega}_4n_5-\bar{\Omega}_5n_4) = 0, \tag{170}$$

where

$$\left.\begin{aligned}
a &= n_4^2n_5^2(n_3-n_2)^2(n_4-n_3)^2(n_5-n_4)^2 \\
b &= n_1{}^2n_3{}^2n_4{}^2n_5{}^2(n_2-n_1)^2(n_4-n_3)^2(n_5-n_4)^2 \\
c &= n_1{}^2n_2{}^2n_3{}^2n_5{}^2(n_2-n_1)^2(n_3-n_2)^2(n_5-n_4)^2 \\
d &= n_1{}^2n_2{}^2(n_2-n_1)^2(n_3-n_2)^2(n_4-n_3)^2.
\end{aligned}\right\} \tag{171}$$

Equation (26) applied to the first and fourth surfaces results to

$$\bar{\Omega}_2 = [(y_1\bar{\Omega}_1-1)\Omega_2 + \Omega_1]/y_1\Omega_1 \tag{172}$$

$$\bar{\Omega}_4 = [(1-y_4\bar{\Omega}_5)\Omega_4 + 1]/y_4. \tag{173}$$

Substituting equations (172) and (173) in (170) and collecting like terms, we obtain

$$A_2\bar{\Omega}_1 - B_2\bar{\Omega}_5 + C_2 = 0, \tag{174}$$

where

$$\left.\begin{aligned}
A_2 &= n_3y_1y_4[ay_1(\Omega_2n_1^2-\Omega_1n_2^2)(\Omega_1n_2-\Omega_2n_1)^2 - b\Omega_2{}^3] \\
B_2 &= n_3\Omega_1y_1y_4[c\Omega_4{}^3 - dy_4(n_4{}^2+ \Omega_4n_5^2)(\Omega_4n_5+n_4)^2]
\end{aligned}\right\} \tag{175}$$

$$C_2 = an_1n_3y_1y_4(\Omega_2n_1^2-\Omega_1n_2^2)(\Omega_1n_2-\Omega_2n_1)(\Omega_2-\Omega_1)$$

$$- b\Omega_2{}^2y_4(n_3\Omega_1-n_3\Omega_2-n_2\Omega_1y_1) - c\Omega_1\Omega_4{}^2y_1(n_4y_4-n_3\Omega_4-n_3)$$

$$- dn_3n_5\Omega_1y_1y_4(n_4{}^2+\Omega_4n_5{}^2)(\Omega_4n_5+n_4)(\Omega_4+ 1). \qquad (175)'$$

Equation (174) gives indirectly the functional relationship between the locations of the entrance and exit pupil planes of the two-module system since $y_E=1/\bar{\Omega}_1$ and $y_{E'}= 1/\bar{\Omega}_5$. Equation (174) is the condition that should be satisfied to eliminate coma for the entire system. A similar functional relationship between $\bar{\Omega}_2$ and $\bar{\Omega}_4$ is obtained, had equation (172) is solved for $\bar{\Omega}_1$ and equation (173) is solved for $\bar{\Omega}_5$, before substitution to (170).

Zero Astigmatism

When equation (97) is applied to the system and set equal to zero, the condition for zero astigmatism is obtained, and is given by

$$an_3{}^2y_1(\Omega_2n_1{}^2 - \Omega_1n_2{}^2)(\bar{\Omega}_1n_2 - \bar{\Omega}_2n_1)^2$$

$$- b\Omega_2(\bar{\Omega}_2n_3 - n_2)^2 + c\Omega_4(n_4 - \bar{\Omega}_4n_3)^2$$

$$- dn_3{}^2y_4(n_4{}^2 + \Omega_4n_5{}^2)(\bar{\Omega}_4n_5 - \bar{\Omega}_5n_4)^2 = 0. \qquad (176)$$

Substitution of equations (172) and (173) in (176), results to the following after expanding and collecting like terms.

$$A_3\bar{\Omega}_1^2 + B_3\bar{\Omega}_1 + C_3\bar{\Omega}_5{}^2 + D_3\bar{\Omega}_5 + E_3 = 0, \qquad (177)$$

where

$$
\left.
\begin{aligned}
A_3 &= n_3^{\,2}y_1^{\,2}y_4^{\,2}[ay_1(\Omega_2 n_1^{\,2}-\Omega_1 n_2^{\,2})(n_2\Omega_1-n_1\Omega_2)^2 - b\Omega_2^{\,3}] \\
B_3 &= 2n_3y_1y_4^{\,2}[an_1n_3y_1(\Omega_2 n_1^{\,2}-\Omega_1 n_2^{\,2})(n_2\Omega_1-n_1\Omega_2)(\Omega_2-\Omega_1) \\
&\quad - b\Omega_2^{\,2}(\Omega_1 n_3-n_2y_1\Omega_1-n_3\Omega_2)] \\
C_3 &= n_3^{\,2}\Omega_1^{\,2}y_1^{\,2}y_4[c\Omega_4^{\,3}y_4 - d(n_4^2+\Omega_4 n_5^2)(n_4+n_5\Omega_4 y_4)^2] \\
D_3 &= 2n_3\Omega_1^{\,2}y_1^{\,2}y_4[c\Omega_4^{\,2}(n_4y_4-n_3-\Omega_4 n_3) \\
&\quad + dn_3n_5(n_4^2+\Omega_4 n_5^{\,2})(\Omega_4+1)(n_4+n_5\Omega_4 y_4)] \\
E_3 &= an_1^{\,2}n_3^{\,2}y_1y_4^{\,2}(\Omega_2 n_1^{\,2}-\Omega_1 n_2^{\,2})(\Omega_2-\Omega_1)^2 \\
&\quad - b\Omega_2 y_4^{\,2}(\Omega_1 n_3-n_2\Omega_1 y_1-n_3\Omega_2)^2 + c\Omega_1^{\,2}\Omega_4 y_1^{\,2}(n_4y_4-n_3-\Omega_4 n_3)^2 \\
&\quad - dn_3^{\,2}n_5^{\,2}\Omega_1^{\,2}y_1^{\,2}y_4(n_4^{\,2}+\Omega_4 n_5^{\,2})(\Omega_4+1)^2 .
\end{aligned}
\right\}
\quad (178)
$$

Equation (177) gives the functional relationship between the entrance and exit pupil planes for the two-module system at finite conjugates to eliminate astigmatism.

Zero Coma and Astigmatism

The two-module system at finite conjugates can be made free simultaneously of coma and astigmatism if equations (174) and (177) are both satisfied at the same time. Solving equations (174) and (177) simultaneously results to two quadratic equations given by the following:

$$X_1\bar{\Omega}_1^2 + Y_1\bar{\Omega}_1 + Z_1 = 0 \tag{179}$$

and

$$X_1\bar{\Omega}_5^2 + Y_2\bar{\Omega}_5 + Z_2 = 0, \tag{180}$$

where

$$\left.\begin{aligned}
X_1 &= A_3{B_2}^2 + {A_2}^2C_3 \\
Y_1 &= B_3{B_2}^2 + 2C_3A_2C_2 + D_3A_2B_2 \\
Z_1 &= C_3{C_2}^2 + D_3B_2C_2 + {B_2}^2E_3 \\
Y_2 &= B_3A_2B_2 - 2A_3B_2C_2 + D_3{A_2}^2 \\
Z_2 &= A_3{C_2}^2 - B_3A_2C_2 + {A_2}^2E_3.
\end{aligned}\right\} \tag{181}$$

Equations (179) and (180) give implicitly at most two pairs of entrance-exit pupil heights for which the two-module system has both coma and astigmatism equal to zero. Since only real values of $\bar{\Omega}_1$ and $\bar{\Omega}_5$ are of interest, we impose the following conditions on the discriminants of equations (179) and (180).

$$\left.\begin{aligned}
{Y_1}^2 - 4X_1Z_1 \geq 0. \\
{Y_2}^2 - 4X_1Z_2 \geq 0.
\end{aligned}\right\} \tag{182}$$

The above set of equations could be used as a guide in the selection of modules A and B, with free parameters k_a and k_b, which may be coupled at finite conjugates to produce a system with both coma and astigmatism equal to zero.

Application

To apply the theory presented on the two-module system at finite conjugates, the following system parameters commonly used to specify an optical system should be given:

f' = system focal length

N = f-number of system

θ = $2\bar{u}_1$, field angle

If the f-number of the system is defined as $N = f'/2y_E$, then the Lagrange invariant of the system is

$$\text{Ж} = n_1\bar{u}_1 f'/2N = n_1\theta f'/4N. \tag{183}$$

Equation (183) shall enable us to determine the conventional optical parameters of the individual modules.

The axial separation between the two modules is given by

$$t_{23} = t_3 = n_3(\bar{y}_3 - \bar{y}_2)/\text{Ж}. \tag{184}$$

The value of t_3 may be chosen arbitrarily or may be specified in addition to the system parameters. A given value of t_3 produce an additional constrained equation when substituted in (184) which may be used to determine explicitly the pair of pupil planes to make the system eliminate either coma or astigmatism.

Two-Module System at Infinite Conjugates

When two modules A and B defined by k_a and k_b, respectively, are combined in a manner such that the second conjugate point of A coincides with the first conjugate point of B, the resulting two-module system has zero third-order spherical aberration for objects located at infinity. Such two-module system is afocal or telescopic. The axial separation between the second surface of A and the first surface of B is equal to the sum of the back focal distance of A and the front focal distance of B. Two common examples of afocal systems are schematically shown in Fig. 6. For such systems, it is convenient to normalize the $\bar{y}$ heights of the corresponding $y,\bar{y}$ diagram relative to the image height of A. The y heights are normalized with respect to the height of the entrance pupil of the system which follows the normalization scheme proposed by Lopez-Lopez (1970). The normalized $y,\bar{y}$ diagrams corresponding to the optical system layouts shown in Fig. 6, are illustrated in Fig. 7. The first case (a), applies to Keplerian telescopes while the second (b) is for the Galilean type. A typical example of two-module system at infinite conjugates is shown in Fig. 8.

To define a functional relationship between the free parameters k_a and k_b for the two-module system at infinite conjugates, Powell (1970) introduced the parameter g by letting $y_3 = gy_2$. For module A, we have the following:

$$y_2 = \frac{-27n_1^2 n_2^2(k_a^2-1)}{4n_3(n_2-n_1)^2(n_3-n_2)}$$

$$\Omega_2 = \frac{-3n_2^2/n_3}{(2n_2+n_3) + (n_3-n_2)\left[\left\{\frac{k_a+1}{k_a-1}\right\}^{1/3} + \left\{\frac{k_a-1}{k_a+1}\right\}^{1/3}\right]} \tag{185}$$

For module B,

$$y_3 = \frac{27n_4^2 n_5^2(k_b^2-1)}{4n_3(n_5-n_4)^2(n_4-n_3)}$$

and (186)

$$\Omega_4 = \frac{3n_4^2/n_3}{(n_3+2n_4) - (n_4-n_3)\left[\left\{\frac{k_b+1}{k_b-1}\right\}^{1/3} + \left\{\frac{k_b-1}{k_b+1}\right\}^{1/3}\right]} .$$

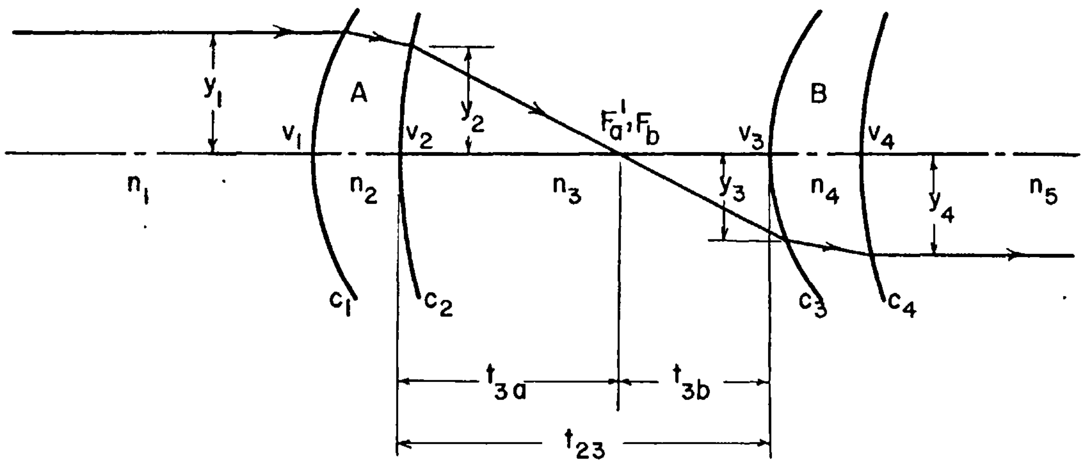
Solving for g, Powell (1970) obtained

$$g = \frac{-n_4^2 n_5^2(n_2-n_1)^2(n_3-n_2)(k_b^2-1)}{n_1^2 n_2^2(n_5-n_4)^2(n_4-n_3)(k_a^2-1)} , \tag{187}$$

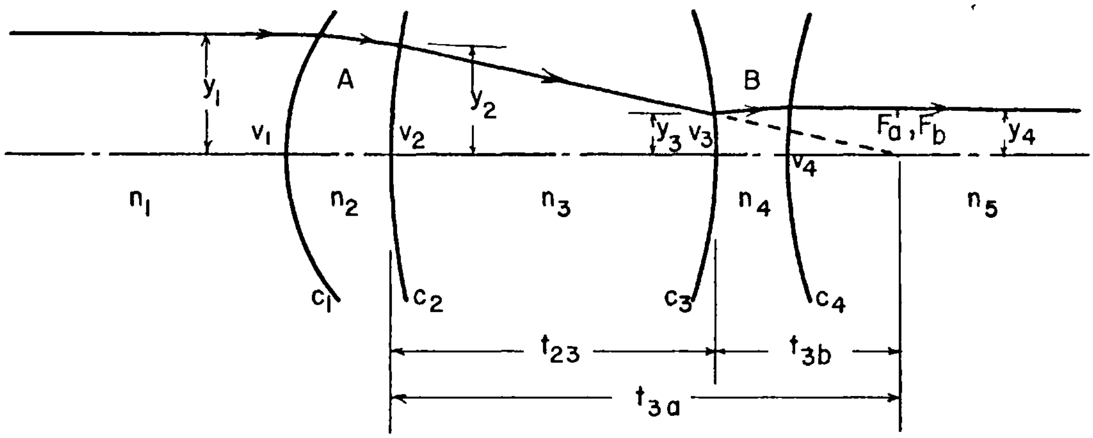
which defines g as function of k_a and k_b. For the case where the free parameters are pure imaginary, equation (186) becomes

$$g = \frac{-n_4^2 n_5^2(n_2-n_1)^2(n_3-n_2)\sec^2\theta_b}{n_1^2 n_2^2(n_5-n_4)^2(n_4-n_3)\sec^2\theta_a}, \tag{188}$$

where θ_a and θ_b are the free parameters of modules A and B, respectively, and are defined individually by equation (112). The parameter g could also define the functional relationship between Ω_2 and Ω_4 of the two-module system, but such relationship is algebraically not feasible.

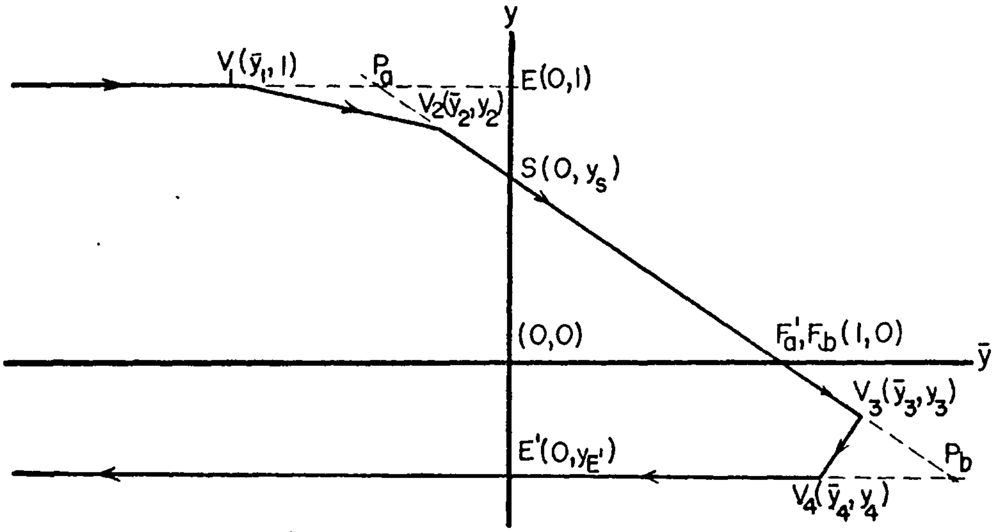


(a) Keplerian

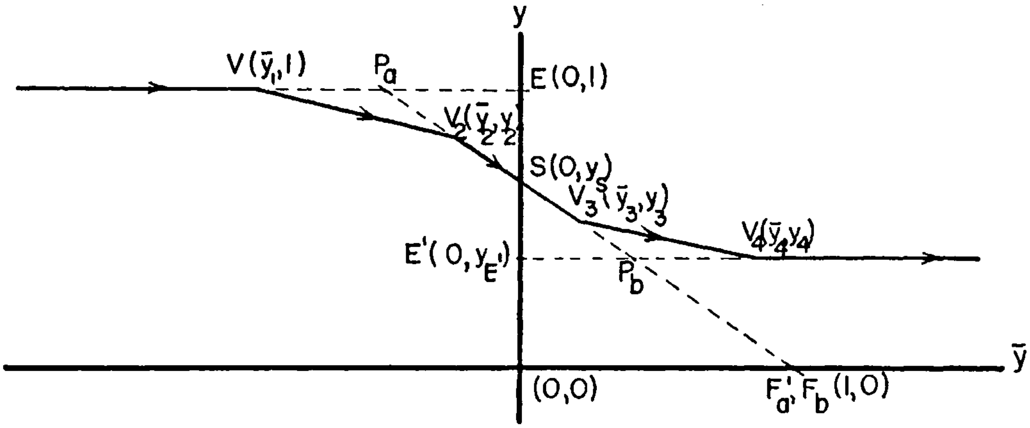


(b) Galilean

Fig. 6. Two Common Examples of Afocal Systems

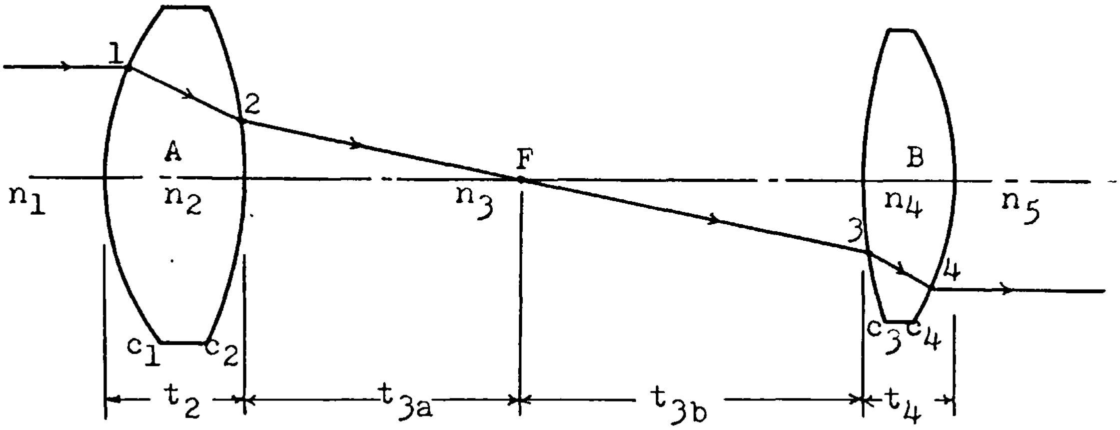


(a)



(b)

Fig. 7. Normalized $y, \bar{y}$ Diagrams for Afocal Systems



$c_1 = 0.274616$

$c_2 = -0.184135$

$c_3 = 0.157263$

$c_4 = -0.267107$

$t_2 = 1.78539$

$t_{3a} = 16.0374$

$t_{3b} = 17.3695$

$t_4 = 1.13447$

$n_1 = n_5 = 1.0$

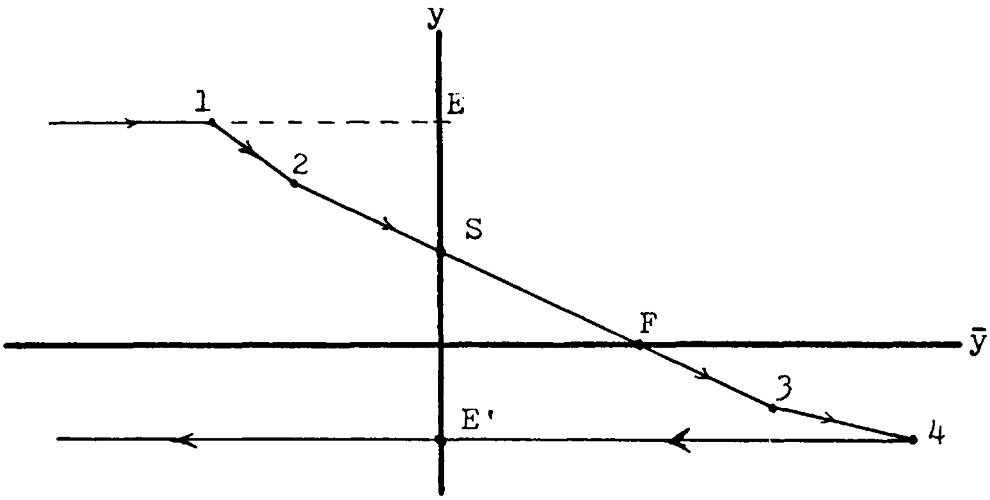
$n_2 = n_4 = 1.5731$

$n_3 = 1.9525$

$k_a = 0.994$

$k_b = 0.9935$

(a) Layout of System



(b) $y, \bar{y}$ Diagram

Fig. 8. Typical Example of Two-Module System at Infinite Conjugates

Zero Astigmatism

The condition for zero astigmatism is obtained if equation (97) is applied to the system and the resulting expressions set equal to zero. Hence,

$$A\Omega_2(n_2-\bar{\Omega}_2 n_1)^2/n_1^{\,2} - B(n_2^2+\Omega_2 n_3^2)(\bar{\Omega}_2 n_3-\bar{\Omega}_3 n_2)^2 y_2 + C(\Omega_4 n_3^2+n_4^2)(\bar{\Omega}_3 n_4-\bar{\Omega}_4 n_3)^2 y_3 - D\Omega_4(\bar{\Omega}_4 n_5-\bar{\Omega}_5 n_4)^2 y_4/n_5^2 = 0, \qquad (188)$$

where

$$\left.\begin{aligned} A &= [n_1 n_3 n_4 (n_3-n_2)(n_4-n_3)(n_5-n_4)]^2 \\ B &= [n_4(n_2-n_1)(n_4-n_3)(n_5-n_4)]^2 \\ C &= [n_2(n_2-n_1)(n_3-n_2)(n_5-n_4)]^2 \\ D &= [n_2 n_3 n_5 (n_2-n_1)(n_3-n_2)(n_4-n_3)]^2. \end{aligned}\right\} \qquad (189)$$

Equation (26) applied to the second and third surfaces of the two-module system, gives

$$\left.\begin{aligned} \bar{\Omega}_2 &= [\Omega_2(1-\bar{\Omega}_3 y_2) + 1]/y_2 \\ \bar{\Omega}_4 &= [\Omega_4(1-\bar{\Omega}_3 y_4) + 1]/y_4. \end{aligned}\right\} \qquad (190)$$

Replacing $\bar{\Omega}_5$ by $1/y_4$, and substituting (190) in equation (188), results to

$$A_4\bar{\Omega}_3 + B_4 = 0, \qquad (191)$$

where

$$\left.\begin{aligned} \mathcal{A} &= 2n_1n_5y_2y_4[A\Omega_2^2n_5y_4(n_2y_2-n_1-\Omega_2n_1) \\ &\quad + B(n_2^2+\Omega_2n_3^2)(\Omega_2+1)(n_3\Omega_2+n_2)n_1n_3n_5y_2y_4 \\ &\quad - C(\Omega_4n_3^2+n_4^2)(1+\Omega_4)(n_4+\Omega_4n_3)n_1n_3n_5y_2y_3 \\ &\quad + D\Omega_4^2(n_5\Omega_4+n_5-n_4)n_1y_2y_4] \\ \mathcal{B} &= [A\Omega_2n_5^2y_4^2(n_2y_2-n_1-\Omega_2n_1)^2 \\ &\quad - B(n_2^2+\Omega_2n_3^2)(\Omega_2+1)^2n_1^2n_3^2n_5^2y_2y_4{}^2 \\ &\quad + C(\Omega_4n_3^2+n_4^2)(1+\Omega_4)^2n_1^2n_3^2n_5^2y_2^2y_3 \\ &\quad - D\Omega_4(n_5\Omega_4+n_5-n_4)^2n_1^2y_2^2y_4]. \end{aligned}\right\} \quad (192)$$

Since $y_s = 1/\bar{\Omega}_3$, equation (191) when solved for $\bar{\Omega}_3$, gives the location of the aperture stop of the two-module system in order to eliminate astigmatism.

Zero Coma

When equation (96) is applied to the two-module system at infinite conjugates and the result set equal to zero, we obtain

$$\begin{aligned} &[A\Omega_2^2(n_2-\bar{\Omega}_2n_1)/n_1] + B(n_2^2+\Omega_2n_3^2)(\Omega_2n_3+n_2)(\bar{\Omega}_2n_3-\bar{\Omega}_3n_2)y_2 \\ &\quad + C(\Omega_4n_3^2+n_4^2)(n_4+\Omega_4n_3)(\bar{\Omega}_3n_4-\bar{\Omega}_4n_3)y_3 \\ &\quad + [D\Omega_4^2(\bar{\Omega}_4n_5-\bar{\Omega}_5n_4)y_4/n_5] = 0, \end{aligned} \quad (193)$$

the condition for the system to have zero coma. If $\bar{\Omega}_5 = 1/y_4$ and equation (190) are substituted in (193), the resulting equation after expansion and grouping of like terms, becomes

$$M\bar{\Omega}_3 + N = 0, \tag{194}$$

where

$$M = n_1 n_5 y_2 y_4 [A\Omega_2{}^3 - By_2(n_3\Omega_2+n_2)^2(n_2^2+\Omega_2 n_3^2)$$

$$+ Cy_3(n_4+\Omega_4 n_3)^2(\Omega_4 n_3^2+n_4^2) - Dy_4\Omega_4{}^3] \tag{195}$$

$$N = [An_5 y_4 \Omega_2^2(n_2 y_2 - n_1 - n_1\Omega_2)$$

$$+ Bn_1 n_3 n_5 y_2 y_4(\Omega_2+1)(n_2^2+ \Omega_2 n_3^2)(\Omega_2 n_3+ n_2)$$

$$- Cn_1 n_3 n_5 y_2 y_3(1+\Omega_4)(\Omega_4 n_3^2+n_4^2)(n_4+\Omega_4 n_3)$$

$$+ Dn_1 y_2 y_4 \Omega_4{}^2(\Omega_4 n_5+n_5-n_4). \tag{196}$$

Equation (95) when applied to the two-module system at infinite conjugates, gives

$$A\Omega_2{}^3 - By_2(n_3\Omega_2+n_2)^2(n_2^2+\Omega_2 n_3^2)$$

$$+ Cy_3(n_4+\Omega_4 n_3)^2(\Omega_4 n_3^2+n_4^2) - Dy_4\Omega_4{}^3 = 0. \tag{197}$$

The left side of equation (197) is identical to the quantities inside the bracket, [ ], in (195). Hence, M in equation (195) also vanishes. Since M is equal to zero, then N should also vanish to satisfy equation (194). Equation (196) when set equal to zero and solved for y_4, yields

$$y_4 = R/S, \tag{198}$$

where

$$R = Cn_1n_3n_5y_2y_3(1+\Omega_4)(\Omega_4n_3^2+n_4^2)(n_4+\Omega_4n_3) \tag{199}$$

$$S = An_5\Omega_2^2(n_2y_2-n_1-n_1\Omega_2)$$

$$+ Bn_1n_3n_5y_2(\Omega_2+1)(n_2^2+\Omega_2n_3^2)(\Omega_2n_3+n_2)$$

$$+ Dn_1y_2\Omega_4^{\ 2}(\Omega_4n_5+n_5-n_4). \tag{200}$$

Equation (198) determines the exit pupil for which the two-module system at infinite conjugates can have zero coma. The exit pupil height, y_4, is implicitly a function of the free parameters k_a and k_b of the individual modules. If equation (197) is solved for y_4 and the resulting expression is set equal to equation (198), a functional relationship be- k_a and k_b will be obtained which shall provide a criterion in the choice of the free parameters of the individual modules to be coupled to insure zero coma. In general, functional relationship between the free parameters of the coupled modules is algebraically complicated. A computer solution of this problem is suitable.

CHAPTER 7

CONCLUSION

It has been shown that parameters derivable from the Delano $y,\bar{y}$ diagram form a convenient set of independent parameters which completely describe and define some of the general properties of design modules. Such representations were found to yield insights in the analysis of modules which cannot readily be obtained using better known methods. Its only drawback is that the constructibility of modules is not at once visualized from these parameters without transforming to the conventional optical parameters of curvatures and axial separation. Such transformations are straightforward however.

The canonical optical parameters defined by equation (115) were introduced by Stavroudis (1969b) as a convenient form of describing modules. These are comparable to the parameters associated with the $y,\bar{y}$ diagram when the parameter f is equated to $1/\mathfrak{X}$, the reciprocal of the Lagrange invariant.

Critical values of the free non-optical parameters, k and θ, were defined in terms of the $y,\bar{y}$ diagram parameters and the three indices of refraction n_1, n_2 and n_3. Values of the canonical optical parameters and equivalent $y,\bar{y}$ diagram parameters, which correspond to the critical values of k and θ, are given in Tables I and II.

Conditions for modules to eliminate simultaneously third-order spherical and other Seidel aberrations were obtained. The limitations or constraints in the choice of values of the free parameters and the $y,\bar{y}$ diagram parameters were analyzed. The proper location of aperture stop defined by equation (150) eliminates third-order astigmatism. For a given set of three indices of refraction n_1, n_2 and n_3 the module was found to eliminate Petzval curvature if Ω_2 is chosen so that it satisfies the quartic equation (154). Application of Decartes' rule of sign in this quartic determines the number of possible values of Ω_2 which provide values of the free parameter k for zero Petzval contribution. For a two-mirror module in air, the quartic degenerates into a quadratic which implies the existence of two possible modules with zero Petzval sum. It was also shown that for a two-mirror module in air, there are exactly two real and two pure imaginary values of the free parameter k which provide zero Petzval curvature.

It was shown that modules with zero coma and Petzval curvature may be defined algebraically by equation (158). Their constructibility depends on the relative values of the three refractive indices n_1, n_2 and n_3. In like manner modules with zero astigmatism and Petzval curvature, modules with zero coma, astigmatism and Petzval sum and modules with zero distortion have been defined. Conditions for such desirable combinations seem very difficult to satisfy for the case of zero distortion but the others may be feasible defending on the choice of parameters. Numerical examples of modules generated with the aid of the General Electric time sharing computer yield promising data for possible applications of these results in the process of optical design.

Two-module systems with zero third-order spherical aberration were defined for both finite and infinite conjugates. Functional relationships among the $y, \bar{y}$ diagram parameters of the individual modules were obtained which insure zero third-order spherical aberration of the coupled systems. Location of the aperture stop of the two-module systems were specified to eliminate third-order astigmatism. Functional relationships between the free parameters of the individual modules coupled to eliminate simultaneously coma and astigmatism were not feasible. For a given set of five indices of refraction n_1, n_2, n_3, n_4 and n_5 it is believed that a computer solution could be obtained such that for a choice of k_a for module A, the corresponding value of k_b for module B could be determined for a constructible two-module system with zero or minimum amount of coma and astigmatism.

The method of applying the results of this study of the properties of modules to the design of multi-element refracting systems is clear. Arrays of modules could be arranged such that the rear and front foci of successive modules coincide which will yield to such required systems with initially zero third-order spherical aberration. The parameters associated with the individual modules could be varied to optimize the design.

APPENDIX

NUMERICAL EXAMPLES OF MODULES

To illustrate what have been discussed about modules, two values of refractive indices, $n_1 = 1.7335$ and $n_2=1.51823$, are randomly picked from the glass catalog. For $n_3 = 1$, the calculated critical values of the free parameter k are as follows:

$$k_o = 0.947121$$

$$k_\infty = 0.999815$$

$$k^* = 0.996774$$

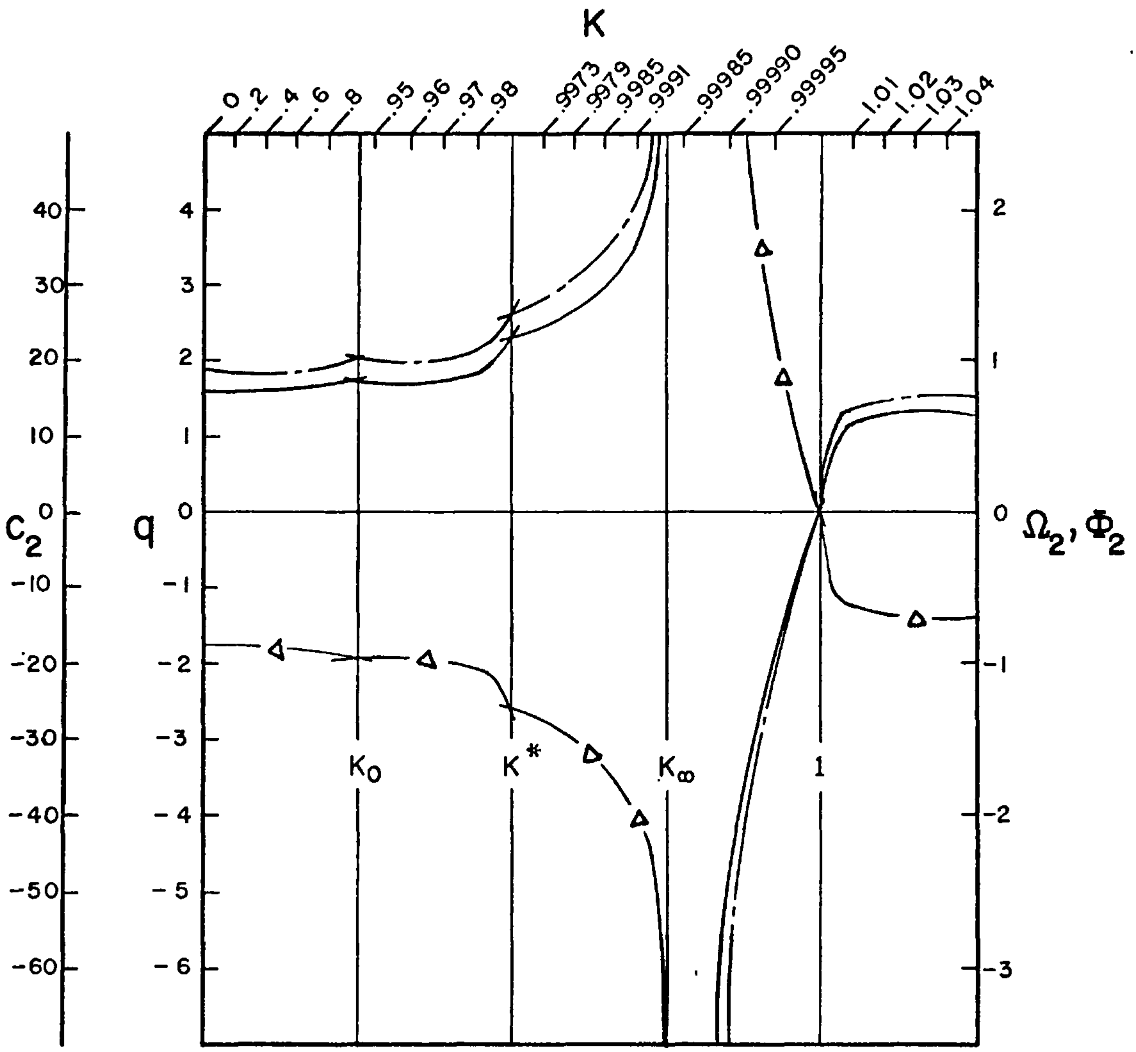
The above critical values of k were computed with the aid of the General Electric time sharing service, Mark I computer, using the program "MØDULE" in BASIC language. The computer program, "MØDULE", generates tables of conventional optical parameters for modules, real case, for three input values of refractive indices. In addition to the critical values, tables calculated at five values between critical values were generated. An option for including additional values of the free parameters was provided in the program.

Another computer program, "MØD-BAR", which is a modification of the "MØDULE" has been written to generate tables of the $y,\bar{y}$ diagram parameters, real case of k, for three input values of refractive indices. The data generated from the "MØDULE" and the "MØD-BAR" using the same set of three input values of refractive indices and an f-value (1/☒) of 0.1 are shown plotted as functions of k in figures A-1, A-2, A-3 and A-4.

Using the same set of three indices of refraction, the fourth degree equation (154) was solved with the aid of the General Electric Mark II time sharing computer. A system subroutine called "ZØRP***", in FORTRAN, was employed to determine the zeros of the polynomial. The quartic equation yields two real roots of Ω_2 (2.0138932 and 0.55637065) which imply the existence of two modules with zero Petzval sum for the given set of three indices of refraction.

The program "MØDULE" was also modified to generate tables of $\bar{\Omega}$ and $\bar{y}$ parameters for modules with zero astigmatism. Graphs showing the values of these parameters as functions of k, using the same set of three refractive indices and f-value, are plotted in figures A-5 and A-6.

Some examples of constructible modules are shown in figures A-7 and A-8. The corresponding $y,\bar{y}$ diagrams of two typical modules are illustrated in Fig. A-9. Examples of ray trace data generated using the program "THRAYA" are given on pages 93 and 94 which show the relative values of the aberrations.



n_1 = 1.73350
n_2 = 1.51823
n_3 = 1.0

Fig. A-1. Values of q, c_2, Ω_2 and Φ_2 as Functions of k

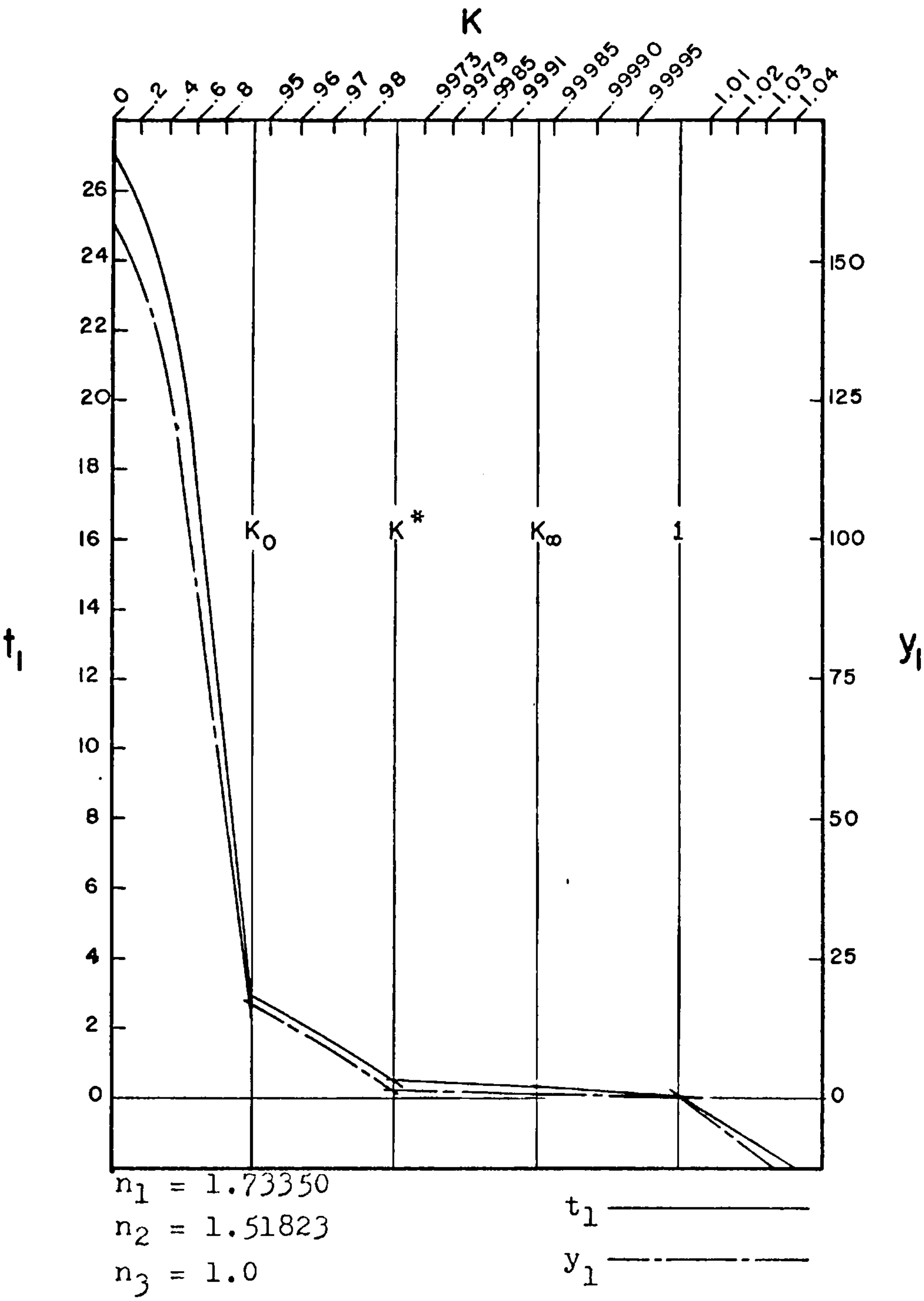
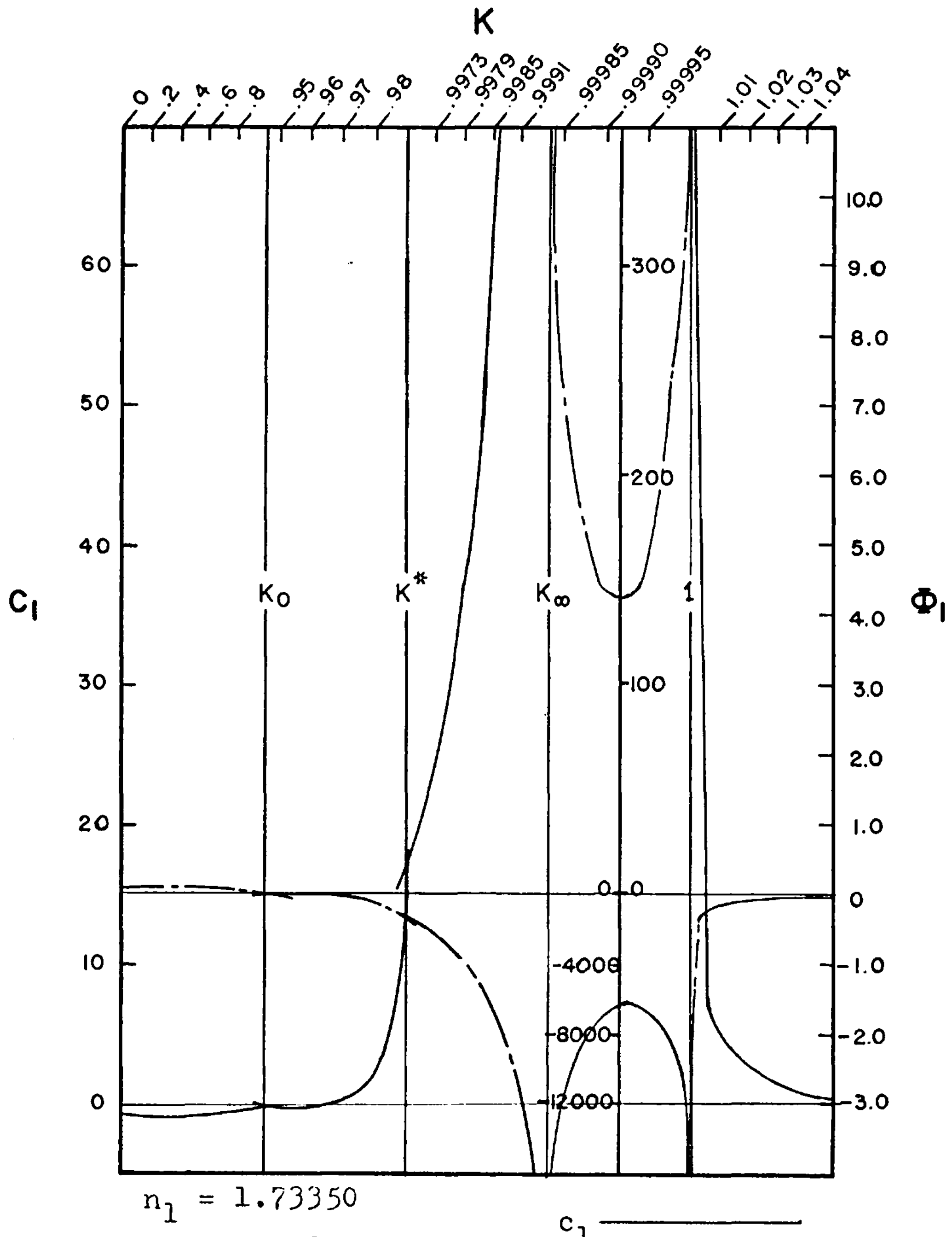


Fig. A-2. Values of t_1 and y_1 as Functions of k



$n_1 = 1.73350$

$n_2 = 1.51823$

$n_3 = 1.0$

Fig. A-3. Values of c_1 and Φ_1 as Functions of k

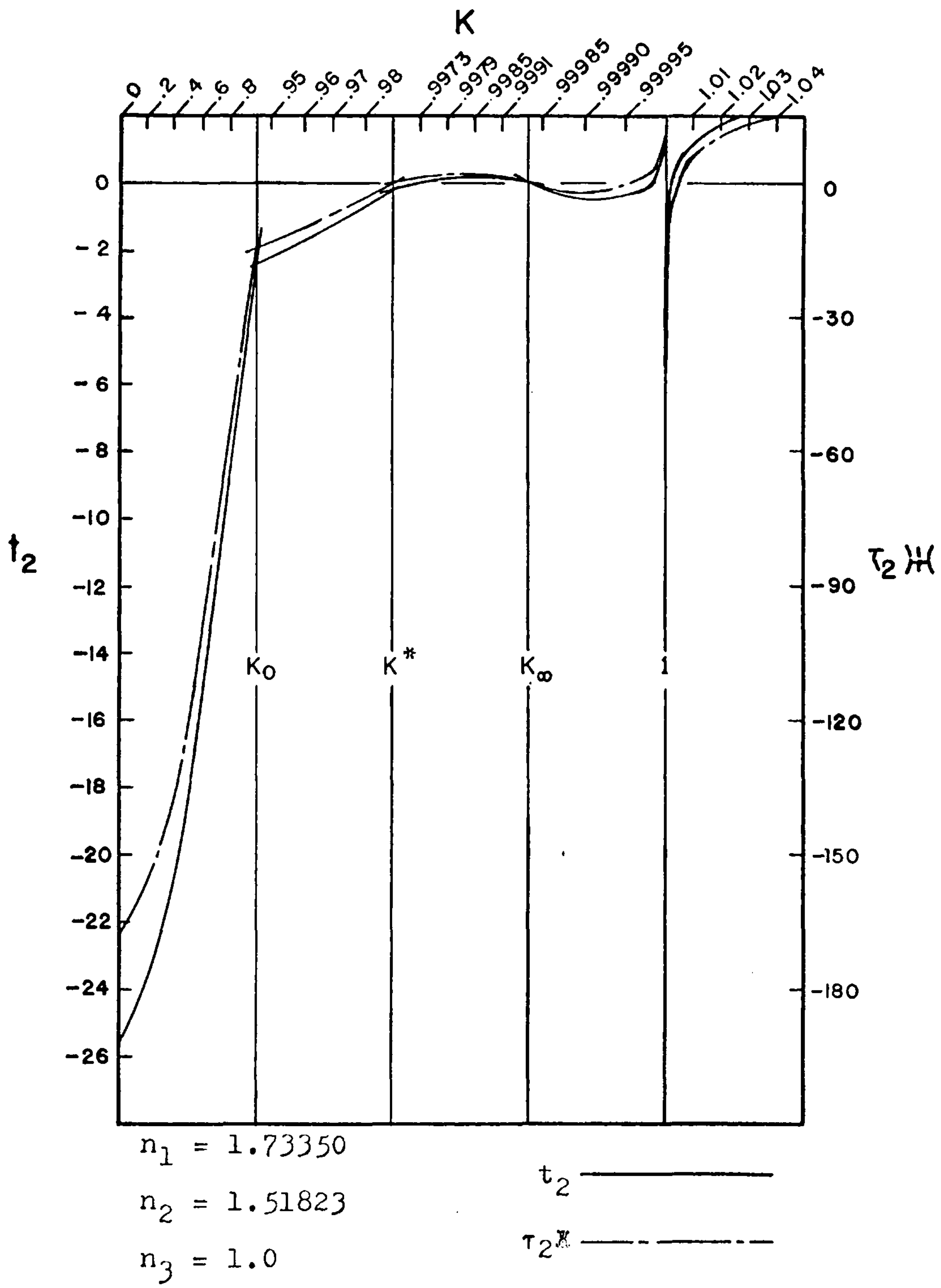
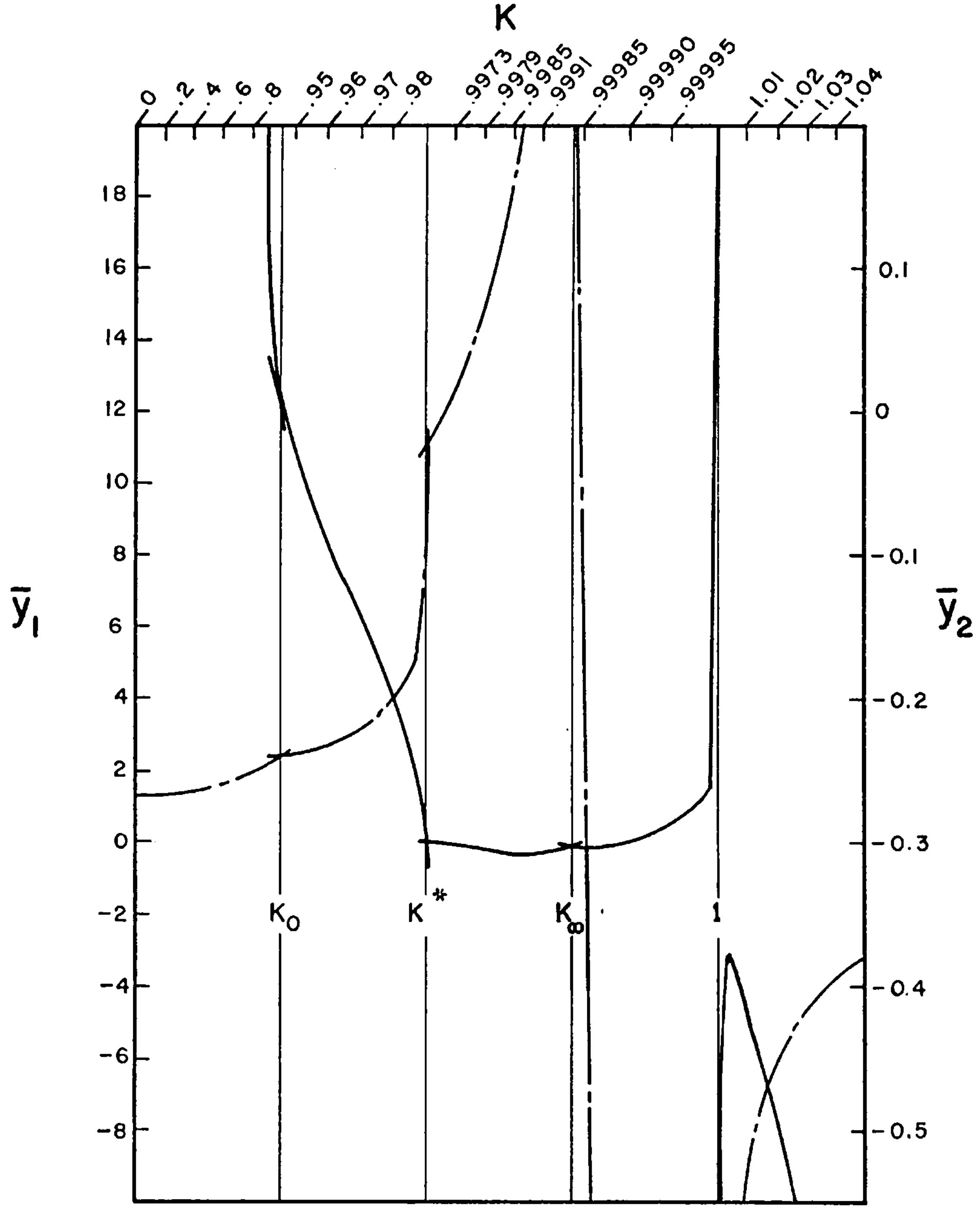


Fig. A-4. Values of t_2 and τ_2* as Functions of k



$n_1 = 1.73350$

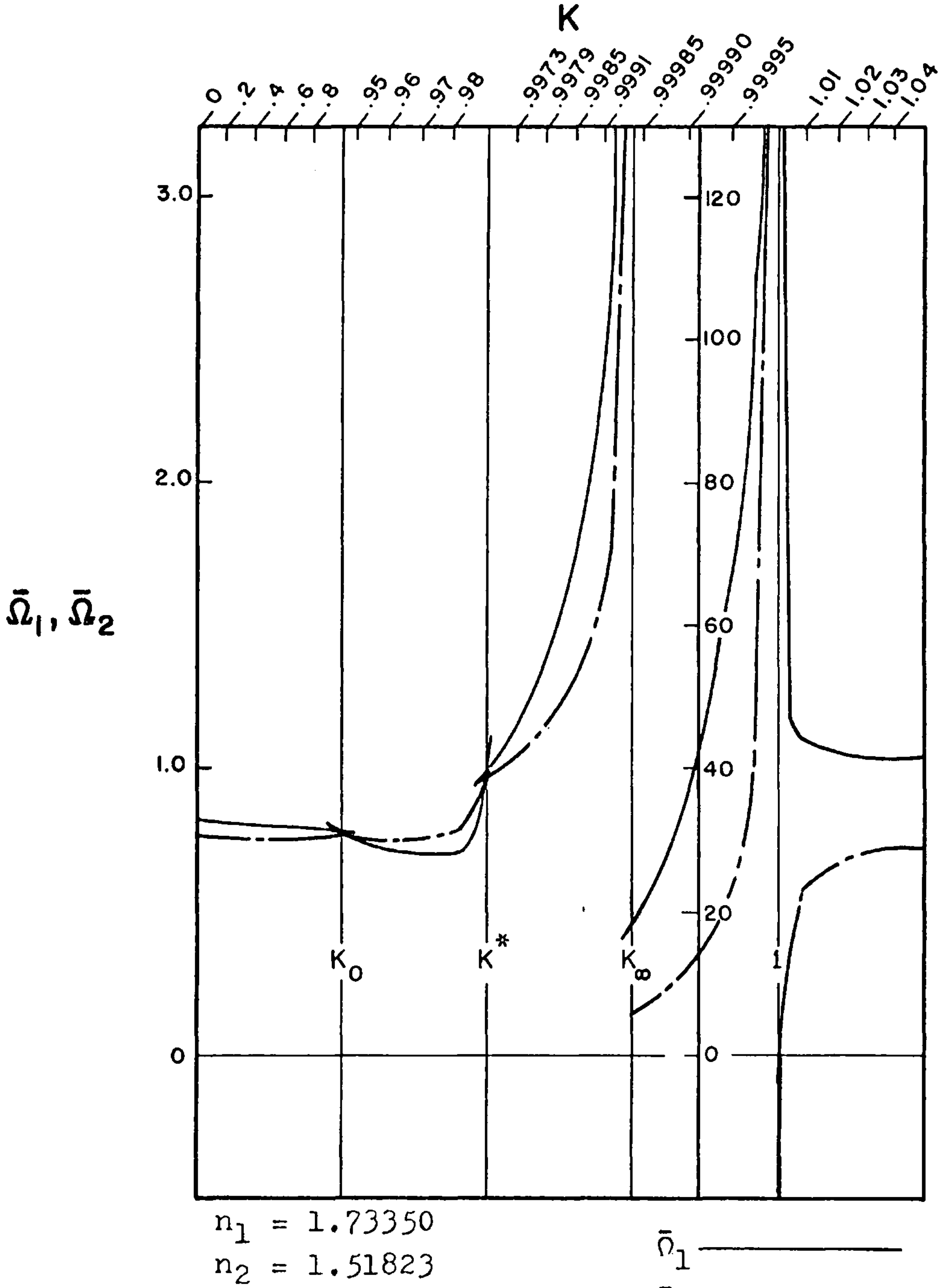
$n_2 = 1.51823$

$n_3 = 1.0$

$\bar{y}_1$ ————————

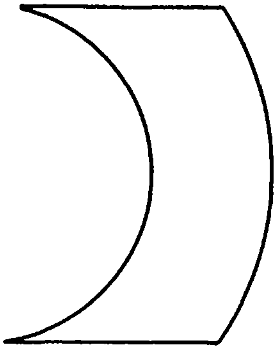
$\bar{y}_2$ ——·——·——

Fig. A-5. Values of $\bar{y}_1$ and $\bar{y}_2$ as Functions of k for Modules with Zero Astigmatism



$n_1 = 1.73350$
$n_2 = 1.51823$
$n_3 = 1.0$

Fig. A-6. Values of $\bar{\Omega}_1$ and $\bar{\Omega}_2$ as Functions of k for Modules with Zero Astigmatism



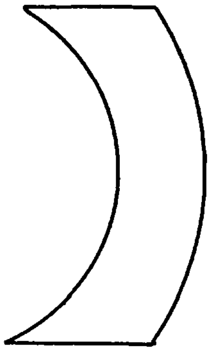
$c_1 = -0.443581$

$c_2 = -0.248473$

$t_1 = 12.775$

$t_2 = 1.55025$

$k = 1.008508$



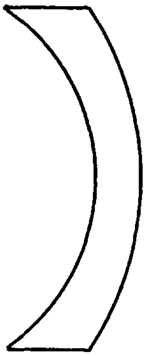
$c_1 = -0.409394$

$c_2 = -0.245106$

$t_1 = 13.517$

$t_2 = 1.17703$

$k = 1.0090$



$c_1 = -0.362926$

$c_2 = -0.240297$

$t_1 = 14.7244$

$t_2 = 0.545859$

$k = 1.00980$

$c_1 = -0.347866$

$c_2 = -0.238673$

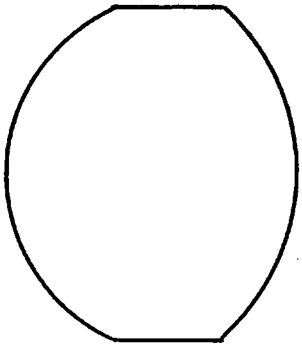
$t_1 = 15.1774$

$t_2 = 0.302244$

$k = 1.0101$

$n_1 = 1.5731 \quad n_2 = 1.9525 \quad n_3 = 1.0 \quad 1/X = 10$

Fig. A-7. Examples of Modules ($k_o < k_\infty < k^* > 1$)



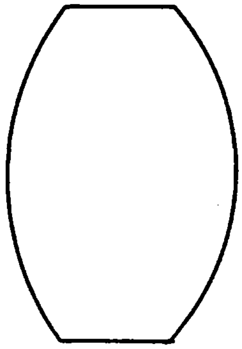
$c_1 = 0.414699$

$c_2 = -0.324973$

$t_1 = 10.7023$

$t_2 = 3.81669$

$k = 0.9960$



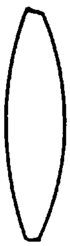
$c_1 = 0.264356$

$c_2 = -0.294339$

$t_1 = 13.3712$

$t_2 = 2.93919$

$k = 0.9950$



$c_1 = 0.146009$

$c_2 = -0.263775$

$t_1 = 18.0353$

$t_2 = 0.793953$

$k = 0.993250$

$c_1 = 0.135933$

$c_2 = -0.260681$

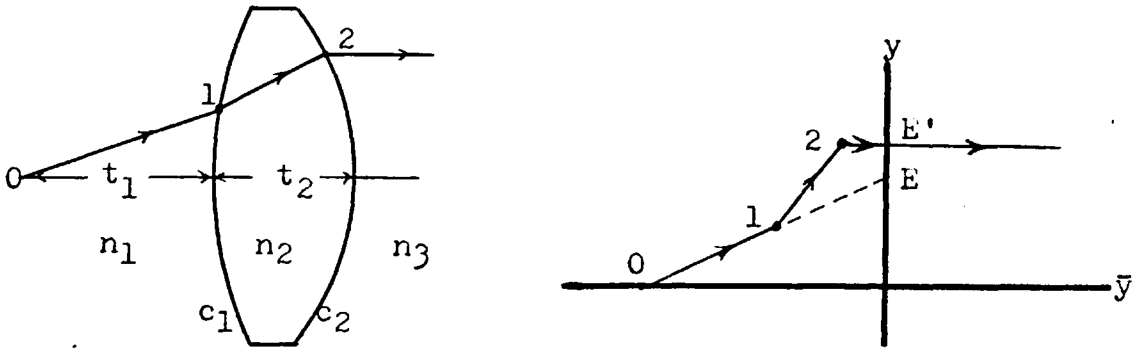
$t_1 = 18.7009$

$t_2 = 0.444406$

$k = 0.9930$

$n_1 = 1.9525 \quad n_2 = 1.5731 \quad n_3 = 1.0 \quad 1/F = 10$

Fig. A-8. Examples of Modules ($k_o < k^* < k_\infty < 1$)

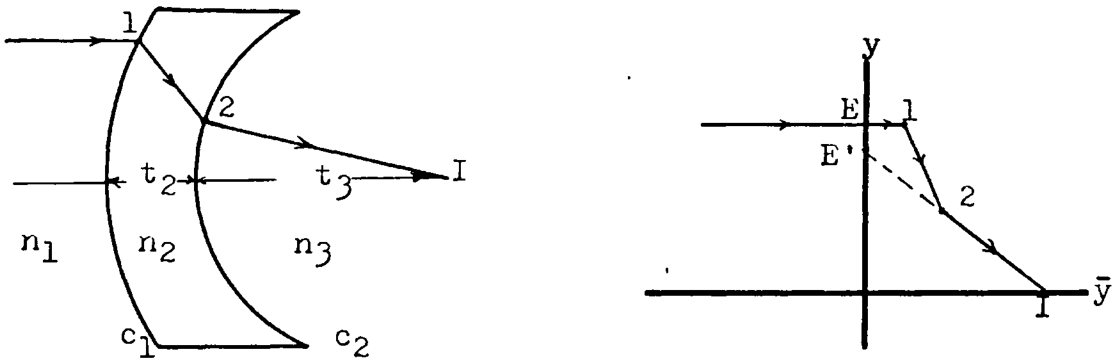


n_1 = 1.9525 t_1 = 16.0374 c_1 = 0.184135

n_2 = 1.5731 t_2 = 1.78539 c_2 = -0.274616

n_3 = 1.0 k = 0.994

(a)



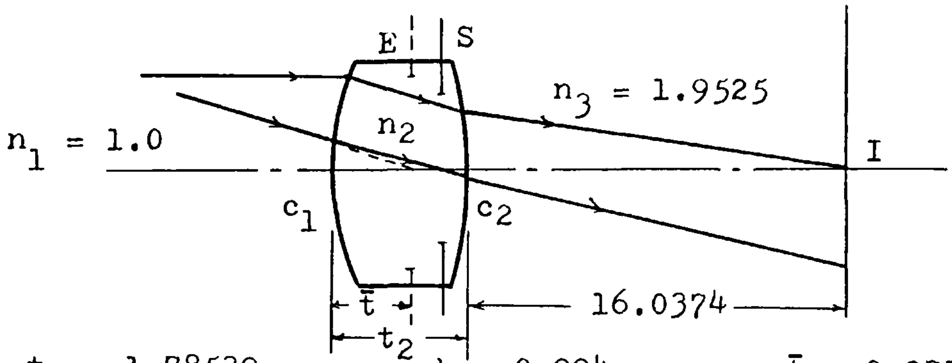
n_1 = 1.0 t_2 = 1.17703 c_1 = 0.245106

n_2 = 1.9525 t_3 = 13.517 c_2 = 0.409394

n_3 = 1.5731 k = 1.009

(b)

Fig. A-9. Constructible Modules and Their $y,\bar{y}$ Diagrams



$t_2 = 1.78539$ $k = 0.994$ $\bar{t} = 0.971702$

$n_2 = 1.5731$

	MARGINAL	PRINCIPAL
Y =	.10000	.00000
U =	.00000	.20000

SURFACE 1: $c_1 = 0.274616$

SPH =	.000012	COMA =	-.000064
AST =	.000342	CURV =	.001342
DST =	-.007167	PETZ =	-.100046
LONG COL =	-.000023	LAT COL =	.000125

I =	.02746	-.14663
Y =	.10000	.19434
U =	.01000	.14658

SURFACE 2: $c_2 = -0.184135$

SPH =	-.000012	COMA =	-.000064
AST =	-.000342	CURV =	-.000569
DST =	-.003040	PETZ =	.022745
LONG COL =	.000050	LAT COL =	.000267

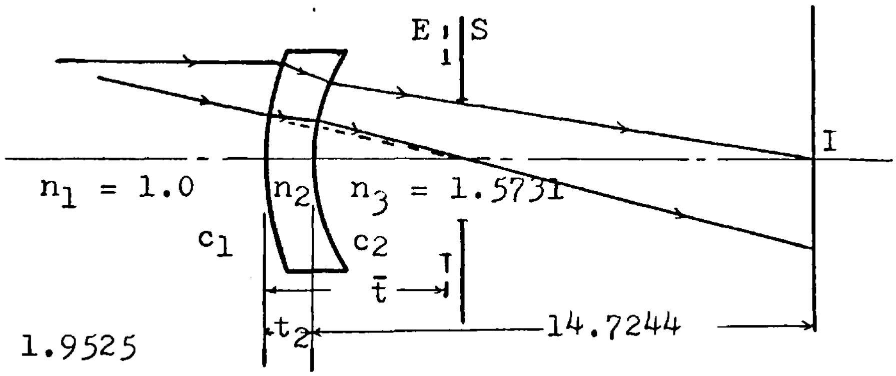
I =	-.02513	-.13418
Y =	.08214	-.06736
U =	.00512	.12051

IMAGE PLANE

Y =	-.00000	-2.00000
U =	.00512	.12051

SUMS OF ABERRATIONS

SPH =	.000000	COMA =	-.000128
AST =	-.000000	CURV =	.000773
DST =	-.010207	PETZ =	-.077301
LONG COL =	.000027	LAT COL =	.000392



$n_2 = 1.9525$

$t_2 = 0.545859$ $k = 1.0098$ $\bar{t} = 3.45984$

	MARGINAL	PRINCIPAL
Y =	.10000	.00000
U =	.00000	.20000

SURFACE 1: $c_1 = 0.240297$

SPH =	.000009	COMA =	-.000012
AST =	.000017	CURV =	.001189
DST =	-.001669	PETZ =	-.117226
LØNG CØL =	-.000058	LAT CØL =	.000081

I =	.02403	-.03372
Y =	.10000	.69197
U =	.01172	.18355

SURFACE 2: $c_2 = 0.362926$

SPH =	-.000009	COMA =	-.000012
AST =	-.000017	CURV =	-.000465
DST =	-.000653	PETZ =	.044830
LØNG CØL =	.000063	LAT CØL =	.000088

I =	.02225	.03122
Y =	.09360	.59178
U =	.00636	.17602

IMAGE PLANE

Y =	.00000	-2.00000
U =	.00636	.17602

SUMS OF ABERRATIONS

SPH =	-.000000	COMA =	-.000024
AST =	-.000000	CURV =	.000724
DST =	-.002322	PETZ =	-.072396
LØNG CØL =	.000005	LAT CØL =	.000169

REFERENCES

Delano, Edwin. "First-Order Design and the $y, \bar{y}$ Diagram". Applied Optics. 2(12): 1251-1256, 1963.

Dickson, L. E. New First Course in the Theory of Equations. New York: John Wiley & Sons, 1939, pp. 42-45.

Lopez-Lopez, F. "Normalization of the Delano Diagram". Optical Sciences Technical Report 57. University of Arizona, 1970.

Military Standardization Handbook - Optical Design. United States Department of Defense. MIL-HDBK-141, 1962.

Pegis, R. J., T. P. Vogl, A. K. Rigler and R. Walters. "Semiautomatic Generation of Optical Prototypes". Applied Optics. 6(5): 969-973, 1967.

Powell, F. M. "$Y, \bar{Y}$ Diagram Analysis of Two-Surface Optical Systems with Zero Third-Order Spherical Aberration". M. S. Thesis. University of Arizona, 1970.

Stavroudis, O. N. " Two-Mirror Systems with Spherical Refracting Surfaces". Journal of the Optical Society of America. 57(6): 741-748, 1967.

Stavroudis, O. N. "Two-Surface Refracting Systems with Zero Third-Order Spherical Aberration". Journal of the Optical Society of America. 59(3): 288-293, 1969a.

Stavroudis, O. N. "Two-Surface Optical Systems with Zero Third-Order Spherical Aberration". Optical Sciences Technical Report 37. University of Arizona, 1969b.