DEADBEAT CONTROL OF LINEAR SAMPLED-DATA CONTROL SYSTEMS USING MULTIPLE FEEDBACK PATHS

by

James Henry Gibbons

A Thesis Submitted to the Faculty of the

DEPARTMENT OF ELECTRICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

1 9 6 5

STATEMENT BY AUTHOR

This thesis has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: James Henry Gibbons

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

James L. Melsa
James L. Melsa
Assistant Professor of Electrical Engineering

May 5, 1965
Date

ACKNOWLEDGEMENTS

The author expresses his appreciation to Dr. James L. Melsa for his patience, encouragement and assistance in the preparation of this thesis.

LIST OF ILLUSTRATIONS

ABSTRACT

A control system is presented that provides deadbeat response of linear sampled-data control systems. The classical digital controller designed for deadbeat response does not consider non-zero initial conditions of the system state-variables. This can lead to non-deadbeat performance. The control system developed in this thesis combines classical z-transformation techniques with the state-space approach to provide an improved control method.

Deadbeat response for an arbitrary step function input is obtained with the Multiple Feedback Path System. This system uses each state-variable fed back through a fixed gain element. The same system is used with an "input" digital controller to achieve deadbeat response to higher order inputs. A "partial" digital controller is developed to assist the Multiple Feedback Path System when all the state-variables are not accessible. The "partial" and "input" digital controllers are then combined for higher order inputs with varying numbers of measurable state-variables. Three methods for finding the element values of the basic system are given.

The system presented is an improvement over the classical method. There are fewer restrictions on initial conditions and a wider class of systems can be compensated.

CHAPTER 1

INTRODUCTION

The classical z-transformation technique for control of sampled-data systems is well known. However, in recent years emphasis has been placed on time domain study using state-space methods. This thesis uses both approaches, and combinations of the two, to develop a control system that provides deadbeat response of linear sampled-data control systems with a specified input. The combination of the two approaches gives a decidedly improved control method.

Deadbeat response for an arbitrary step function input is obtained with the Multiple Feedback Path System. This system uses each state-variable fed back through a fixed gain element. The same system is used with an "input" digital controller to achieve deadbeat response to higher order inputs. A "partial" digital controller is developed to assist the Multiple Feedback Path System when all the state-variables are not accessible. The "partial" and "input" digital controllers are then combined for higher order inputs with varying numbers of measurable state-variables.

The classical method of designing digital controllers for deadbeat response does not consider non-zero initial conditions of the system state-variables. It is shown by example how such a system

TABLE OF CONTENTS

gives unexpected and undesirable results for non-zero initial conditions. In addition, the classical digital controller designed for deadbeat response requires excessive settling time and, in some cases, causes overshoot for a special class of control systems.

Both of these problems are overcome with the Multiple Feedback Path System.

In this thesis, deadbeat response means achieving the following performance for a specific input:

1. Zero steady-state error.
2. No overshoot.
3. Minimum settling time.
4. No intersampling ripple.

The systems discussed are restricted to those with unity numerators.

In Chapter 2 a brief discussion of time domain calculations using state-variable techniques is given. Also, an analysis of the classical digital controller is made.

The Multiple Feedback Path System for step function inputs is studied in Chapter 3. Three methods for determining the element values are presented. A procedure is developed for finding a "partial" digital controller which augments the feedback system when all of the state-variables cannot be measured.

Chapter 4 treats the same feedback system with higher order inputs. The modification required for these inputs is combined with the "partial" digital controller.

Various numerical examples are used throughout the thesis to illustrate procedures and to demonstrate their plausibility.

It is assumed that the reader has a general background knowledge of sampled-data control systems. Suitable treatment may be found in books such as Kuo (1963) or Ragazzini and Franklin (1958).

CHAPTER 2

CLASSICAL DIGITAL CONTROLLER

2.1 INTRODUCTION

The terms of the classical digital controller have a particular meaning with respect to the overall system and the input. The origin and purpose of these constituents are discussed in detail to gain insight into the system behavior. The knowledge gained is used in later developments to connect the classical technique with the time domain approach.

An example is presented to illustrate how initial conditions can cause systems compensated by digital controllers to give other than deadbeat performance. In addition, the state-variable diagram and state-transition equation are given to provide a method for systematic calculations in the time domain.

2.2 SYSTEM REPRESENTATION

A linear sampled-data control system can be represented either graphically by a state-variable diagram consisting of integrators, amplifiers and summing devices or by a state-transition equation at the sampling instants (Tou 1964). Figure 2.1 shows the overall representation of the closed loop sampled-data system

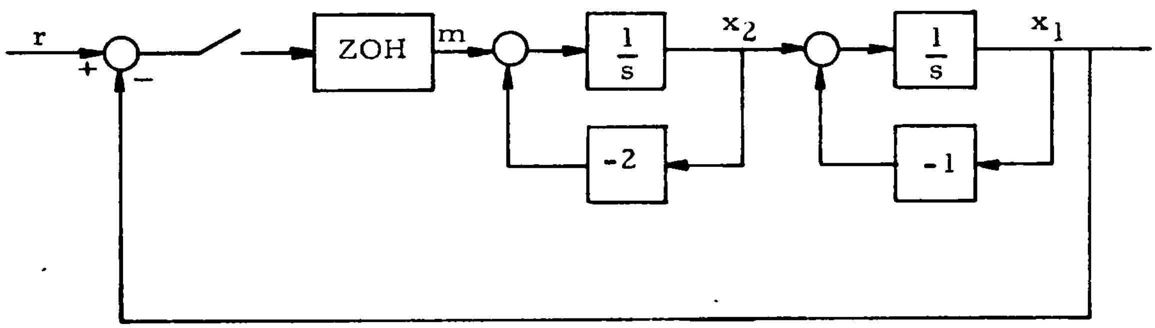

Figure 2.1 State Variable Diagram of System $\frac{1}{(s+1)(s+2)}$ with a ZOH

$\frac{1}{(s+1)(s+2)}$ with a zero order hold, ZOH. The state-variables are x_1 and x_2, m is the control signal input, and r is the system input. The output is normally designated x_1. The state-variables need not be arranged in any special order, except for inputs of order higher than a step function. This restriction affects only the state-variables nearest the output which is discussed in Chapter 4.

The transition equation is

$$\underline{x}(\overline{k+1}T) = \Phi(T)\underline{x}(kT) + m(kT)\underline{f}(T) \qquad (2.1)$$

where:

$k = 0, 1, 2, \ldots$

$\Phi(T)$ is the nxn state-transition matrix,

m(kT) is the control sequence input,

$\underline{f}(T) = \int_0^T \Phi(T-\tau)D\,d\tau$,

D is an nxl vector $= \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$,

and n is the order of the system.

The state-transition matrix, $\Phi(T)$, may be found readily by two different ways. The φ_{ij} term of the nxn matrix is the response at the output of the ith state-variable when a unit impulse is applied at the input of the jth state-variable. Another method is to describe the differential equations by their Laplace transformations in matrix form. By suitable manipulation, the $\Phi(T)$ matrix can be found.

Because $\Phi(T)$ is exponential in nature, it possesses two properties that are useful in the development of this thesis. They are:

1. $\Phi(kT) = (\Phi(T))^k$
2. $\Phi(-kT) = (\Phi(kT))^{-1}$

2.3 DIGITAL CONTROLLER TERMS

The digital controllers, D(z), discussed in this chapter, are found by classical z-transformation techniques such as presented, for example, in Kuo (1963). The location of the D(z) can be seen in Figure 2.2. The criteria used for design is deadbeat response to a step function input, unless otherwise specified.

The D(z) for an nth order system normally takes the form

$$D(z) = \frac{K(1-g_1 z^{-1})(1-g_2 z^{-1}) \ldots (1-g_n z^{-1})}{(1-z^{-1})(1+b_1 z^{-1}+b_2 z^{-2}+ \ldots b_{n-1} z^{-n+1})} \quad (2.2)$$

The terms are discussed below.

1. The term K is a constant gain factor that is unique for each system.
2. The numerator factors are the denominator of the system z-transformation. The polynomial that results from these factors is the control sequence input, m(kT), that the system requires for deadbeat response. It may be altered as discussed below.
3. The b_i's are the discrete error signals for a unit step input.

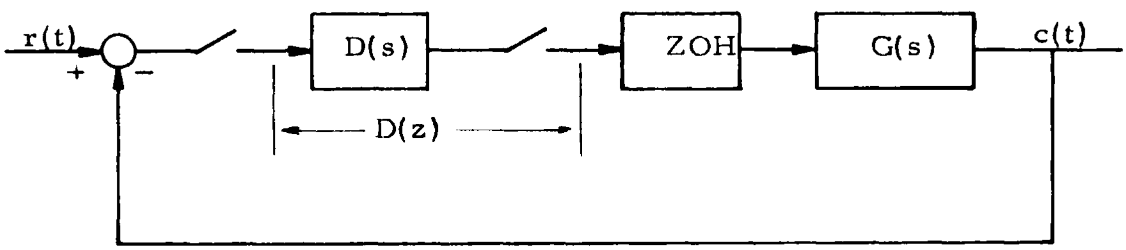

Figure 2.2 Typical Digital Controller for Linear Sampled Data System

4. The $(1-z^{-1})$ term may or may not appear in the denominator. When it does not appear, the system has one integration in its transfer function. In this case a similar term appears as a numerator factor which cancels it. When the term does appear, it represents an integrator that holds the control input to the plant at a constant value after n sampling periods. To get the true control sequence, a $(1-z^{-1})$ term must be factored from the numerator polynomial and cancelled with the denominator term (Tou and Vadhanaphuti 1961).
5. The system reaches equilibrium at the end of the nth sampling period.

The D(z) discussed above is modified in the following manner when designed for a ramp input:

1. The value of the K term changes.
2. Another factor is added in the numerator that is not a direct function of a state-variable.
3. An additional integration term is placed in the denominator and the b_i's are changed and are no longer recognizable error signals.
4. The settling time is $n + 1$ sampling periods. In general, for each order increase of the input requires one additional sampling period to reach equilibrium.

If the number of integrators that appear in the system is higher than the order of the input, the form of D(z) is changed. The excess $(1-z^{-1})$ terms may not be placed in the numerator of D(z) as discussed above because of stability considerations. Systems of this type require additional sampling periods to settle and overshoot occurs in some cases. The Multiple Feedback Path System gives deadbeat response for this class of systems. However, the "partial" digital controller may have the same problem. A discussion of how this affects the "partial" digital controller will be given in Chapter 3.

2.4 INITIAL CONDITION EFFECTS

The classical design of a digital controller does not consider non-zero initial conditions on the state-variables. This is a problem that can lead to highly undesirable results. It is possible that the specified input cannot drive the system to deadbeat response when its state-variables are in a broad range of values. An example is given to illustrate the severity of this problem.

Example 2.1 shows how a system takes excessive sampling periods to settle as well as giving overshoot until it settles. The system is compensated with a classically designed D(z). A difference equation is used to find the control sequence input and the transition equation is used to calculate the values of the state-variables at the sampling instants.

Example 2.1

Given the system whose plant transfer function is $G(s) = \frac{1}{s(s+1)}$, determine the time response of the system if it is compensated with a D(z) designed for deadbeat response to a unit step input. Let $x_1(0) = 0$, $x_2(0) = 1$, and $T = 1$ second. (See Figure 2.3.) The D(z) is calculated to be:

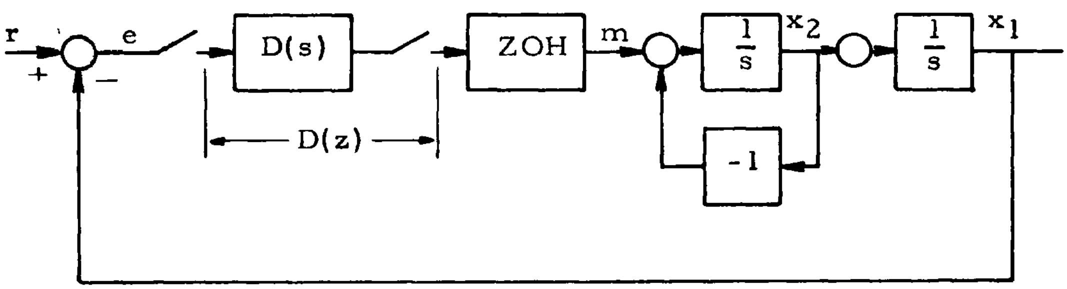

Figure 2.3 System $\frac{1}{s(s+1)}$ Compensated with a Classical D(z)

$$D(z) = \frac{1.582(1-.368z^{-1})}{1+.418z^{-1}} = \frac{1.582-.582z^{-1}}{1+.418z^{-1}} .$$

Rewriting this in terms of the control signal and the error signal, $r-x_1$, it becomes

$$\frac{m(kT)}{e(kT)} = \frac{1.582-.582z^{-1}}{1+.418z^{-1}} .$$

The time valued control signal is

$$m(kT) = 1.582e(kT) -.582e(\overline{k-1}T) -.418m(\overline{k-1}T).$$

The transition matrix is found to be:

$$\phi(T) = \begin{bmatrix} 1 & .632 \\ 0 & .368 \end{bmatrix} \text{ and } \underline{f}(T) = \begin{bmatrix} .368 \\ .632 \end{bmatrix} .$$

Determining the control input at k=0 and calculating the value of the state-variables at the end of the first sampling period yields,

For k=0

$m(0) = 1.582$ and

$$x_1(T) = x_1(0) + .632x_2(0) + .368m(0)$$

$$= 1.214$$

$$x_2(T) = .368x_2(0) + .632m(0)$$

$$= 1.368$$

The system has 21.4% overshoot at the end of the first sampling period.

The same calculations are made for $k = 1$, giving

$$e(T) = -.214 \text{ and}$$

$$m(T) = -1.582.$$

Using this in the transition equation gives:

$$x_1(2T) = x_1(T) + .632x_2(T) + .368m(T)$$

$$= 1.497$$

$$x_2(2T) = .368x_2(T) + .632m(T)$$

$$= -.497$$

The system now has almost 50% overshoot. Subsequent calculations show the system slowly settles.

Successive time interval calculations using $m(kT)$ and the transition equation yields

$m(2T) = 0$	$x_1(3T) = 1.183$	$x_2(3T) = -.183$
$m(3T) = 0$	$x_1(4T) = 1.067$	$x_2(4T) = -.067$

It is seen that this system takes longer than n sampling periods to settle reasonably close to the input, and at the same time it gives overshoot at each sampling instant.

2.5 SUMMARY

In this chapter the transition equation and state-variable diagram are presented in order that they may be used for calculation of examples. A study is made of the classical $D(z)$. Portions of the $D(z)$ are used to provide a method for determining the element values of the Multiple Feedback Path System presented in the next chapter. An example is used to show how initial conditions can cause other than deadbeat response for systems compensated with a classical D(z). The next chapter is devoted to presenting a feedback control system using all the state-variables that solves this problem.

CHAPTER 3

THE MULTIPLE FEEDBACK PATH SYSTEM

3.1 INTRODUCTION

It is shown in the previous chapter how initial conditions can become serious problems using the classical D(z). This chapter presents a control system that completely eliminates the initial condition problem while achieving deadbeat response for step function inputs. The system feeds back all of the state-variables through time invariant gain elements. Three methods are presented to find these element values.

In practical systems all of the state-variables may not be measurable. A "partial" digital controller is developed to assist the feedback control system in achieving deadbeat response when varying numbers of the state-variables are not measurable.

3.2 MULTIPLE FEEDBACK PATH SYSTEM

Kalman and Bertram (1958) have shown that deadbeat response for arbitrary step inputs can be achieved by feeding back each state-variable through a fixed gain element which is unique for the system. A slight modification of the system developed in that paper is used in this thesis and called a Multiple Feedback Path System, MFPS. (See Figure 3.1.) The basic difference is that in

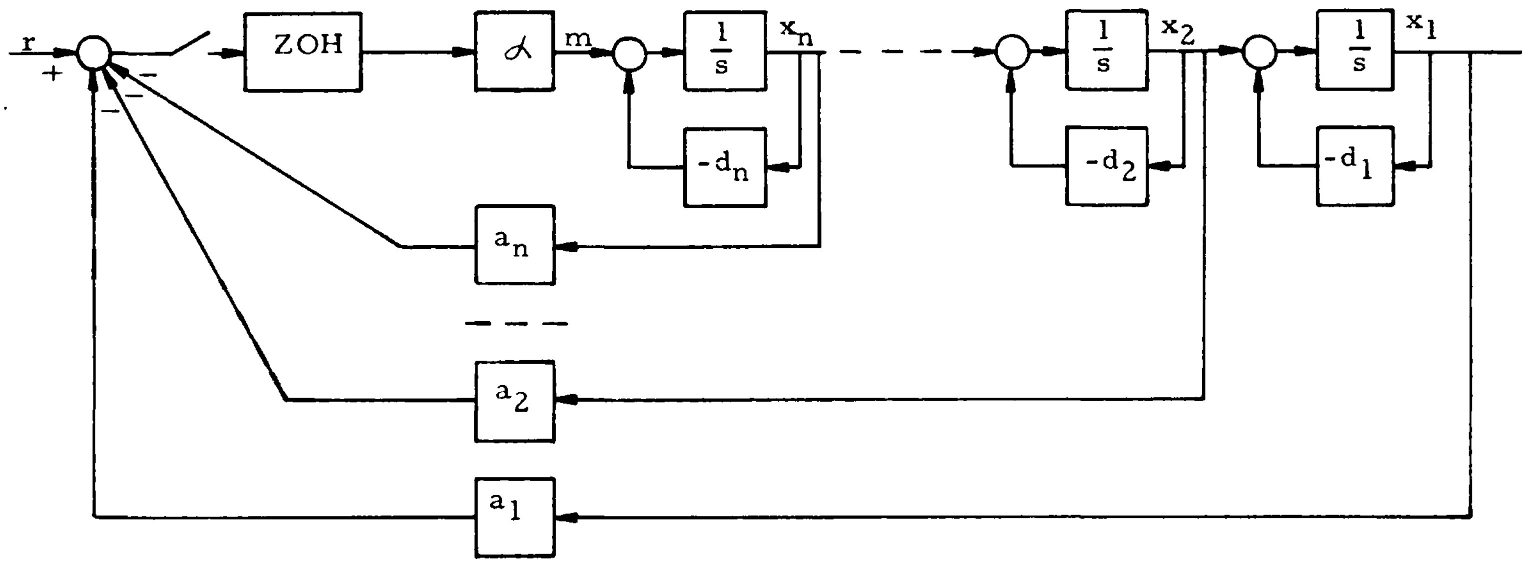

Figure 3.1 Multiple Feedback Path System for nth Order System

the MFPS all the state-variables fed back are summed at a single point in the system.

It is seen from the diagram that there are n+1 unknowns for an nth order system with a step input. There are a number of methods for determining these unknowns; however, they are all derived from the fact that the system reaches equilibrium after n sampling periods for step inputs, thereby making available n equations from the transition equation and one from the final control input, m(kT), for $kT \geq nT$.

3.3 ELEMENT VALUE DETERMINATION

Three disparate methods are presented to determine the n+1 values required in the MFPS.

3.3.1 Direct Method

The direct method is discussed first. It follows most naturally from the solution of the n+1 equations. Also, it is the first method found by the author. It proceeds as follows:

1. Draw a state-variable diagram.
2. Determine $\phi(T)$ and $\underline{f}(T)$.
3. Determine the equilibrium values of the state-variables and the control input. The latter is the product of all the d_i's. (See Figure 3.1.)
4. Determine $\underline{x}(nT)$ in terms of the unknown elements and use the equilibrium values from 3 above to obtain n equations.

5. Use the equilibrium value of the control input to write the (n+1)st equation.

6. Solve the equations simultaneously for α, a_1, a_2, ... a_n.

Two examples are given to illustrate the procedure just presented. In addition, the corresponding system's classical D(z) is given in order to point out similar terms.

Example 3.1

Given a system whose plant transfer function is $G(s) = \frac{1}{(s+1)(s+2)}$, design an MFPS to produce deadbeat response to a unit step input. Determine its time response at the sampling instants. Assume T=1 second and the initial conditions are zero.

The state-variable diagram is shown in Figure 3.2. The state-transition matrix is found to be:

$$\phi(T) = \begin{bmatrix} e^{-T} & e^{-T}-e^{-2T} \\ 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} .368 & .233 \\ 0 & .1353 \end{bmatrix},$$

$$\underline{f}(T) = 1/2 \begin{bmatrix} 1-2e^{-T}+e^{-2T} \\ 1-e^{-2T} \end{bmatrix} \doteq \begin{bmatrix} .1998 \\ .432 \end{bmatrix}, \text{ and}$$

$$m(kT) = \alpha(1-a_1x_1(kT)-a_2x_2(kT))$$

Inspection of the diagram shows the equilibrium values to be:

$$x_1(2T) = 1$$
$$x_2(2T) = 1$$
$$m(2T) = 2$$

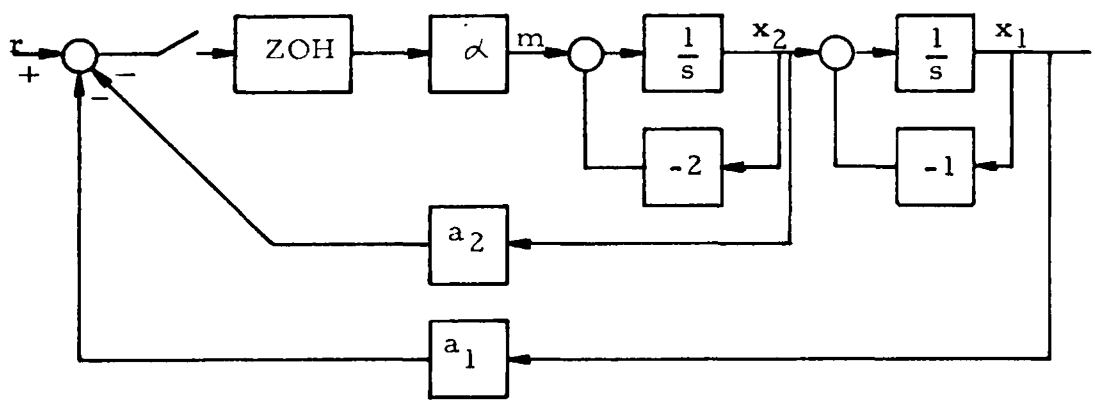

Figure 3.2 MFPS for System $\frac{1}{(s+1)(s+2)}$

Using the transition equation to find the successive time values gives the following:

For $k = 0$

$$m(0) = \alpha$$

$$x_1(T) = .368x_1(0) + .233x_2(0) + .1998m(0)$$

$$= .1998\alpha$$

$$x_2(T) = .1353x_2(0) + .432m(0)$$

$$= .432\alpha$$

For $k = 1$

$$m(T) = \alpha(1 - .1998a_1\alpha - .432a_2\alpha)$$

$$x_1(2T) = (.368)(.1998\alpha) + (.233)(.432\alpha) + .1998m(T)$$

$$x_2(2T) = (.1353)(.432\alpha) + .432m(T)$$

Using the equilibrium values, the three equations become:

$$x_1(2T) = .374\alpha - .0399a_1\alpha^2 - .0864a_2\alpha^2 = 1$$

$$x_2(2T) = .491\alpha - .0864a_1\alpha^2 - .1869a_2\alpha^2 = 1$$

$$m(2T) = \alpha - a_1\alpha - a_2\alpha = 2$$

Simultaneous solution of the first two equations yields $\alpha = 3.66$. Using this value, the remaining unknowns are found to be:

$$a_1 = .252$$

$$a_2 = .202$$

With this data the value of x_1, x_2, and m at the sampling instants are:

$x_1(T) = .731 \qquad x_2(T) = 1.582 \qquad m(T) = 1.818$

$x_1(2T) = 1.0 \qquad x_2(2T) = 1.0 \qquad m(2T) = 2.0$

The classical D(z) is computed to be:

$$D(z) = \frac{3.66 + 1.818z^{-1} + 2.0z^{-2} + 2.0z^{-3} \quad \ldots}{1 + .269z^{-1}}$$

It is seen that the numerator corresponds to m(kT).

Example 3.2

Given a system whose plant transfer function is $G(s) = \frac{1}{s(s+1)}$, design an MFPS to produce deadbeat response to a unit step input. Determine its time response at the sampling instants. Assume T = 1 second and the initial conditions are zero.

The state-variable diagram is shown in Figure 3.3. The state-transition matrix is:

$$\phi(T) = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} = \begin{bmatrix} 1 & .632 \\ 0 & .368 \end{bmatrix},$$

$$\underline{f}(T) = \begin{bmatrix} T-1+e^{-T} \\ 1-e^{-T} \end{bmatrix} = \begin{bmatrix} .368 \\ .632 \end{bmatrix}, \text{ and}$$

$$m(kT) = \alpha(1 - a_1x_1(kT) - a_2x_2(kT)).$$

From the diagram the equilibrium values are determined to be:

$$x_1(2T) = 1$$

$$x_2(2T) = 0$$

$$m(2T) = 0$$

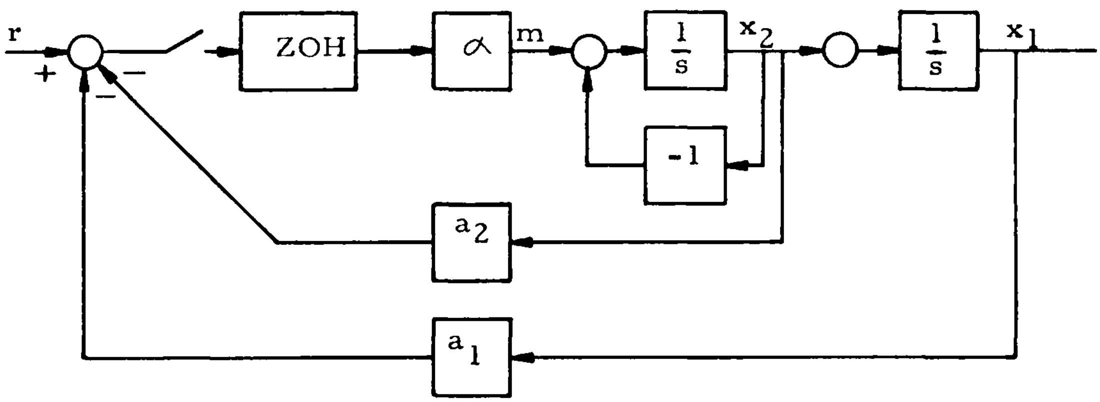

Figure 3.3 MFPS for System $\frac{1}{s(s+1)}$

Using the transition equation, the final equations are found to be:

$$x_1(2T) = 1.135\,\alpha - .1353a_1\,\alpha^2 - .233a_2\,\alpha^2 = 1$$

$$x_2(2T) = .865\,\alpha - .233a_1\,\alpha^2 - .399a_2\,\alpha^2 = 0$$

$$m(2T) = \alpha(1-a_1) = 0$$

The last equation yields $a_1 = 1$. Using this value the first two equations solved simultaneously give:

$$\alpha = 1.582 \text{ and } a_2 = .786$$

Using these values the following time values are:

$$x_1(T) = .582 \qquad x_2(T) = 1.0 \qquad m(T) = -.582$$

$$x_1(2T) = 1.0 \qquad x_2(2T) = 0 \qquad m(2T) = 0$$

The classical D(z) was calculated to be

$$D(z) = \frac{1.582(1-.368z^{-1})}{1 + .418z^{-1}} = \frac{1.582 - .582z^{-1}}{1 + .418z^{-1}} .$$

Based on the information from the foregoing examples and the knowledge gained in the previous chapter, the following facts are summarized:

1. The gain constant, K, associated with D(z) is the same as α.
2. The control sequence is the same for both methods of compensation; therefore, the response for all time will be the same, assuming that the initial conditions are zero.

3. For $kT \geq nT$; m, x_2, x_3, ... x_n have an equilibrium value of zero when the system has one or more integrations which requires a_1 to be unity.

4. The n equations from the transition equation are nonlinear with respect to the element values. As the order of the system increases, the complexity of the solution by simultaneous means becomes staggering. It is mainly for this reason that other methods for finding the values of the elements are sought.

3.3.2 D(z) Information

The control input sequence found in D(z) may be used as another method to calculate the element values of MFPS. Knowledge of the values of m(kT) leads to n linear equations which involve all the feedback elements. These equations can be solved simultaneously, by use of matrix manipulation, or with a digital computer.

The procedure outlined below is straightforward; however, it requires that the value of each state-variable be calculated for all time:

1. Determine $\phi(T)$ and $\underline{f}(T)$ for the system.
2. The control input is given by

$$m(kT) = \alpha(1 - a_1x_1(kT) - a_2x_2(kT) - \ldots a_nx_n(kT)). \quad (3.1)$$

3. The values of α and m(kT) are taken from D(z). Using these values and the transition equation, determine the successive values for $\underline{x}(kT)$, assuming $\underline{x}(0) = 0$.

4. Use the information from 2 and 3 above to write n linear equations. Solve.

Example 3.3

Given the system whose plant transfer function is $G(s) = \frac{1}{s(s+1)(s+2)}$, calculate the elements of the MFPS using D(z). (See Figure 3.4.) Assume T = 1 second, the input is a unit step function and the initial conditions are zero. The classical D(z) is computed to be:

$$D(z) = \frac{3.66(1-.368z^{-1})(1-.1353z^{-1})}{1 + .692z^{-1} + .0691z^{-2}}$$

or

$$D(z) = \frac{3.66 - 1.841z^{-1} + .1822z^{-2}}{1 + .692z^{-1} + .0691z^{-2}}$$

$$\Phi(T) = \begin{bmatrix} 1 & 1-e^{-1} & 1/2(1-2e^{-1}+e^{-2}) \\ 0 & e^{-1} & e^{-1}-e^{-2} \\ 0 & 0 & e^{-2} \end{bmatrix} = \begin{bmatrix} 1 & .632 & .1998 \\ 0 & .368 & .233 \\ 0 & 0 & .1353 \end{bmatrix}$$

$$\underline{f}(T) = \begin{bmatrix} 1/4(2-3+4e^{-1}-e^{-2}) \\ 1/2(1-2e^{-1}+e^{-2}) \\ 1/2(1-e^{-2}) \end{bmatrix} = \begin{bmatrix} .0840 \\ .1998 \\ .432 \end{bmatrix}$$

Inspection of D(z) gives the following values:

$\alpha = 3.66$

$m(0) = 3.66$

$m(T) = -1.841$

$m(2T) = .1822$

$m(3T) = 0$

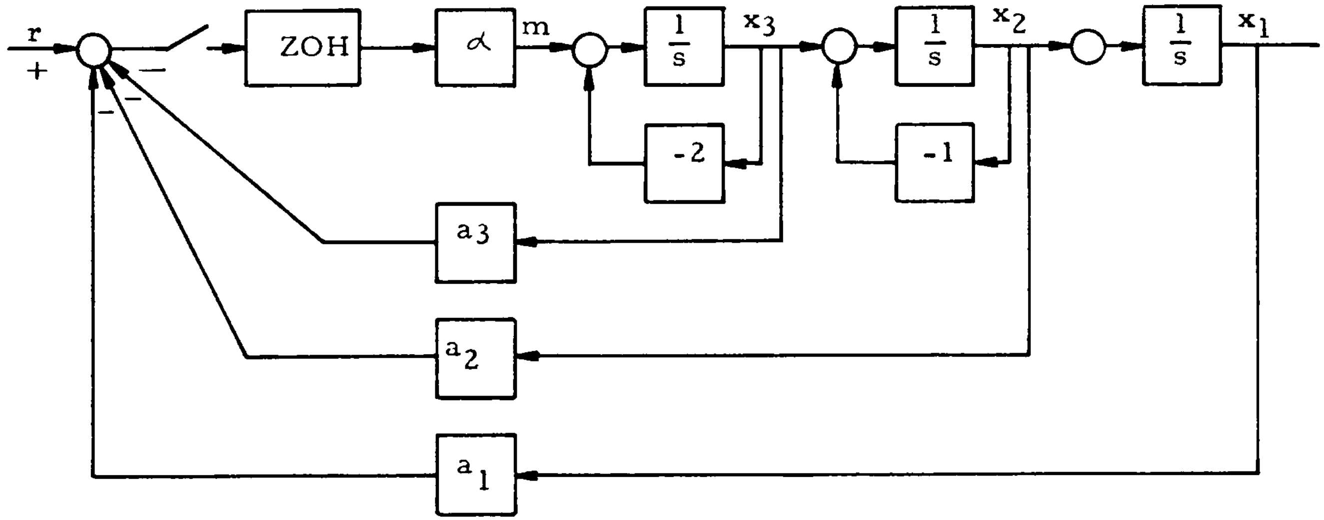

Figure 3.4 MFPS for System $\frac{1}{s(s+1)(s+2)}$

The following time values are computed to be:

$x_1(T) = .308 \qquad x_2(T) = .731 \qquad x_3(T) = 1.582$

$x_1(2T) = .931 \qquad x_2(2T) = .269 \qquad x_3(2T) = -.582$

$x_1(3T) = 1.0 \qquad x_2(3T) = 0 \qquad x_3(3T) = 0$

The simultaneous equations for m(kT), k = 1, 2, 3 are:

$$m(T) = 3.66\,(1-.308a_1-.731a_2-1.582a_3) = -1.841$$

$$m(2T) = 3.66\,(1-.931a_1-.269a_2-.582a_3) = .1822$$

$$m(3T) = 3.66\,(1-a_1) = 0$$

The simultaneous solution of these equations reveals that

$a_1 = 1$, $a_2 = .854$, and $a_3 = .361$.

This method is straightforward and easy to accomplish if the classical D(z) is known. It provides a direct connection between the classical and state-variable approaches, and shows some of the inner relationships. Once either the MFPS or the D(z) is known, the other can be found.

3.3.3 Iterative Design Procedure

Tou (1964) has given another method for determining the elements of the MFPS which is called the iterative process. The approach is different from that of the two previous methods.

A linear sampled-data control system with a step function input may be viewed differently in state-space. A change of variable makes it look like the same system with no input, but with non-zero initial conditions. This system is now allowed to reach equilibrium by settling to the origin in n sampling periods. By a linear transformation it is shown that $\underline{x}(nT)$ is a null vector and the control sequence may be written as a function of the system state-variables at the sampling instants (Kalman and Bertram 1958).

Since the input is zero, the control input is of the form

$$m(kT) = -\alpha(a_1 x_1(kT) + a_2 x_2(kT) + \dots a_n x_n(kT)). \qquad (3.2)$$

The transition equation is used to write $\underline{x}(nT)$ as a function of $\underline{x}(0)$ and the control sequence. Setting $\underline{x}(nT) = 0$, n linear equations result in terms of $\underline{x}(0)$, $m(0)$, ... $m(\overline{n-1}T)$, which can be solved.

The procedure is as follows:

1. Determine $\Phi(T)$, $\Phi(2T)$, ... $\Phi(nT)$ and $\underline{f}(T)$.
2. Using the transition equation, successive iteration gives:

   For k = 0

   $$\underline{x}(T) = \Phi(T)\underline{x}(0) + m(0)\underline{f}(T) \qquad (3.3)$$

   For k = 1

   $$\underline{x}(2T) = \Phi(T)\underline{x}(T) + m(T)\underline{f}(T) \qquad (3.4)$$

   Rewriting this upon substitution of the value above for $\underline{x}(T)$ gives:

   $$\underline{x}(2T) = \Phi(2T)\underline{x}(0) + m(0)\,\Phi(T)\underline{f}(T) + m(T)\underline{f}(T). \qquad (3.5)$$

   This is continued and each successive iteration is written as

   $$\underline{x}(kT) = F\left[\underline{x}(0),\ m(0),\ m(T),\ \dots m(\overline{k-1}T)\right].$$

3. Set $\underline{x}(nT) = 0$ and solve the n linear simultaneous equations for m(0). The solution will be of the form

$$m(0) = -\alpha(a_1 x_1(0) + a_2 x_2(0) + \dots a_n x_n(0)). \quad (3.2)$$

An example will be used to clarify the above procedure.

Example 3.4

Given the system whose plant transfer function is $G(s) = \frac{1}{s(s+1)(s+2)}$, determine the MFPS element values by the iterative design procedure. T = 1 second.

$$\phi(T) = \begin{bmatrix} 1 & 1-e^{-T} & 1/2(1-2e^{-T} \quad e^{-2T} \\ 0 & e^{-T} & e^{-T}-e^{-2T} \\ 0 & 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} 1 & .632 & .1998 \\ 0 & .368 & .233 \\ 0 & 0 & .1353 \end{bmatrix}$$

$$\phi(2T) = \begin{bmatrix} 1 & .865 & .374 \\ 0 & .1353 & .1170 \\ 0 & 0 & .01832 \end{bmatrix}, \quad \phi(3T) = \begin{bmatrix} 1 & .950 & .451 \\ 0 & .0498 & .0473 \\ 0 & 0 & .00248 \end{bmatrix}$$

$$\underline{f}(T) = \begin{bmatrix} .0840 \\ .1998 \\ .432 \end{bmatrix}$$

Using the transition equation gives:

For k = 0

$$\underline{x}(T) = \phi(T)\underline{x}(0) + m(0)\underline{f}(T)$$

For k = 1

$$\underline{x}(2T) = \phi(T)\underline{x}(T) + m(T)\underline{f}(T)$$

which can be written as

$$\underline{x}(2T) = \Phi(2T)\underline{x}(0) + m(0)\,\Phi(T)\underline{f}(T) + m(T)\underline{f}(T)$$

For $k = 2$

$$\underline{x}(3T) = \Phi(T)\underline{x}(2T) + m(2T)\underline{f}(T)$$

which, upon rewriting, is

$$\underline{x}(3T) = \Phi(3T)\underline{x}(0) + m(0)\,\Phi(2T)\underline{f}(T) + m(T)\,\Phi(T)\underline{f}(T) + m(2T)\underline{f}(T)$$

Rewriting $\underline{x}(3T)$ as three equations yields:

$$x_1(3T) = 0 =$$

$$x_1(0) + .950x_2(0) + .451x_3(0) + .418m(0) + .297m(T) + .0841m(2T)$$

$$x_2(3T) = 0 =$$

$$.0498x_2(0) + .0473x_3(0) + .0776m(0) + .1740m(T) + .1998m(2T)$$

$$x_3(3T) = 0 =$$

$$.00248x_3(0) + .00792m(0) + .0585m(T) + .432m(2T)$$

Solving these simultaneously for m(0) gives:

$$m(0) = -3.66x_1(0) - 3.13x_2(0) - 1.322x_3(0)$$

Observe that if x_1 comes from an integrator, a_1 is unity and α is the coefficient of the $x_1(0)$ term. The remaining element values can be found by factoring this quantity from the respective coefficients. If, however, x_1 is not an integrator, then α must be found by some other method such as the constant term in D(z) or by the direct method.

Using this information:

$$\alpha = 3.66$$

$$a_1 = 1$$

$$a_2 = .854$$

$$a_3 = .361$$

As expected, these are the same results obtained in example 3.3. This procedure reduces to the solution of n linear equations which is especially well suited for a digital computer.

3.4 "PARTIAL" DIGITAL CONTROLLER

The previous development has assumed that all of the state-variables are available for use in the MFPS. In some practical systems this might not be the case. Kalman and Bertram (1958) discussed in general two methods that could be used to alleviate this problem. One of these is to make a model or analog of the plant and subject it to the same system input signal. An extension of this idea is given below by finding a "partial" digital controller, PD(z), to be placed in the forward path of the system that will "assist" the MFPS to insure deadbeat response when all of the state-variables are not measurable. A method is developed for determining the required PD(z) when varying numbers of the state-variables are measurable.

The procedure for determining PD(z) is outlined below:

1. Arrange the state-variables so that the j measurable ones correspond to $x_1, x_2, \ldots x_j$ in the MFPS.
2. Assume all of the state-variables are available and determine the element values for the MFPS. It is to

be noted that the element values will change depending upon the arrangement of the state-variables; however, they can always be found.

3. Draw or visualize an equivalent system as shown in Figure 3.5.

4. Consider that portion of the system in the broken-lined box as the required PD(z). It may be thought of as a closed loop sampled system whose transfer function is:

$$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1+\overline{GH}^*(s)} \tag{3.6}$$

PD(z) is readily determined to be:

$$PD(z) = \frac{G(z)}{1+\overline{GH}(z)} \tag{3.7}$$

where

$$G(z) = \alpha$$

$$GH(z) = \mathcal{Z}\left[\frac{(1-e^{-sT})}{s} \sum_{i=j+1}^{n} a_i \prod_{p=i}^{n} (s+d_p)^{-1}\right], \tag{3.8}$$

and

$\mathcal{Z}[\;]$ represents the z-transformation of $[\;]$.

An example will be used to illustrate the procedure.

Example 3.5

Given the system whose plant transfer function is $G(s) = \frac{1}{s(s+1)(s+2)}$, determine the PD(z) required to give

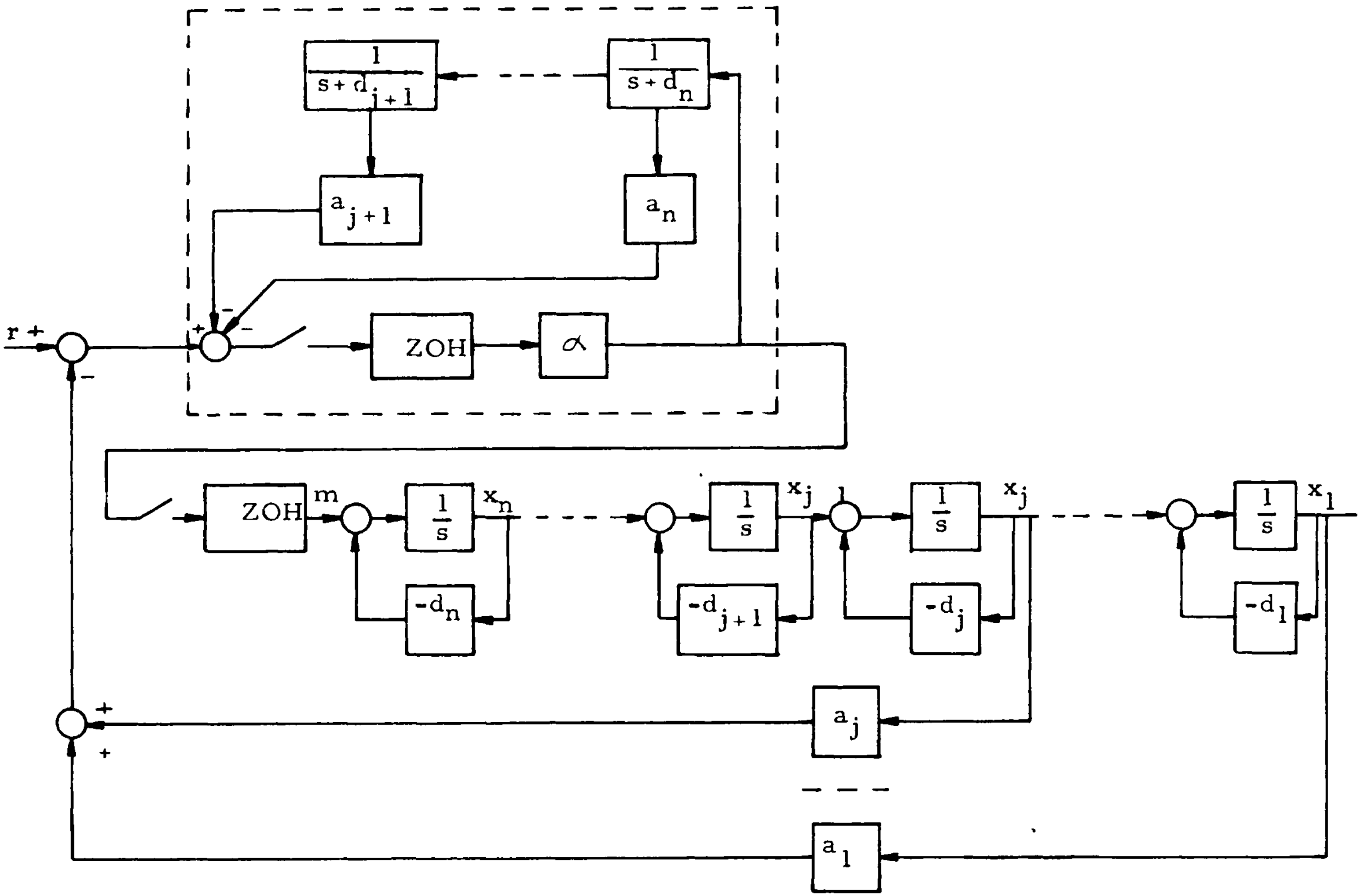

Figure 3.5 General nth Order System with j State-Variables Measurable

deadbeat response for a unit step input when:

1. x_3 is not measurable, and
2. x_2 and x_3 are not measurable.

Assume T = 1 second and the initial conditions are zero. (See Figure 3.6.)

From Example 3.3

$$\alpha = 3.66 \qquad a_2 = .854$$

$$a_1 = 1 \qquad a_3 = .361$$

Forming GH(s), transforming it into GH(z) and applying equation 3.7, gives:

1. $$GH(s) = \frac{1.322\,(1-e^{-sT})}{s(s+2)}$$

$$GH(z) = \frac{.572z^{-1}}{1-.1353z^{-1}}$$

$$PD(z) = \frac{3.66 - .496z^{-1}}{1 + .436z^{-1}}$$

2. $$GH(s) = \frac{1.215\,\alpha(1-e^{-sT})\,(1+.297s)}{s(s+1)\,(s+2)}$$

$$GH(z) = \frac{1.196z^{-1} + .01907z^{-2}}{(1-.368z^{-1})\,(1-.1353z^{-1})}$$

$$PD(z) = \frac{3.66\,(1-.368z^{-1})\,(1-.1353z^{-1})}{1 + .692z^{-1} + .0691z^{-2}}$$

As is seen, a PD(z) can be found regardless of the number of state-variables measurable. The method is straightforward and not difficult to accomplish when only a few state-variables are inaccessible. In the limit, when only the output is measurable,

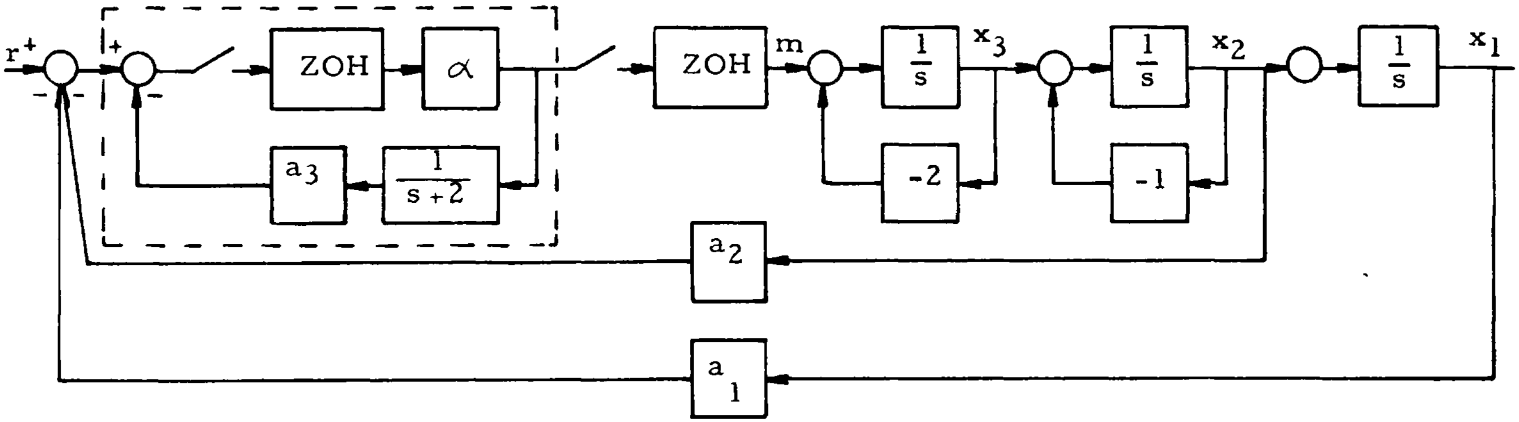

Figure 3.6(a) System $\frac{1}{s(s+1)(s+2)}$ with x_3 Inaccessible

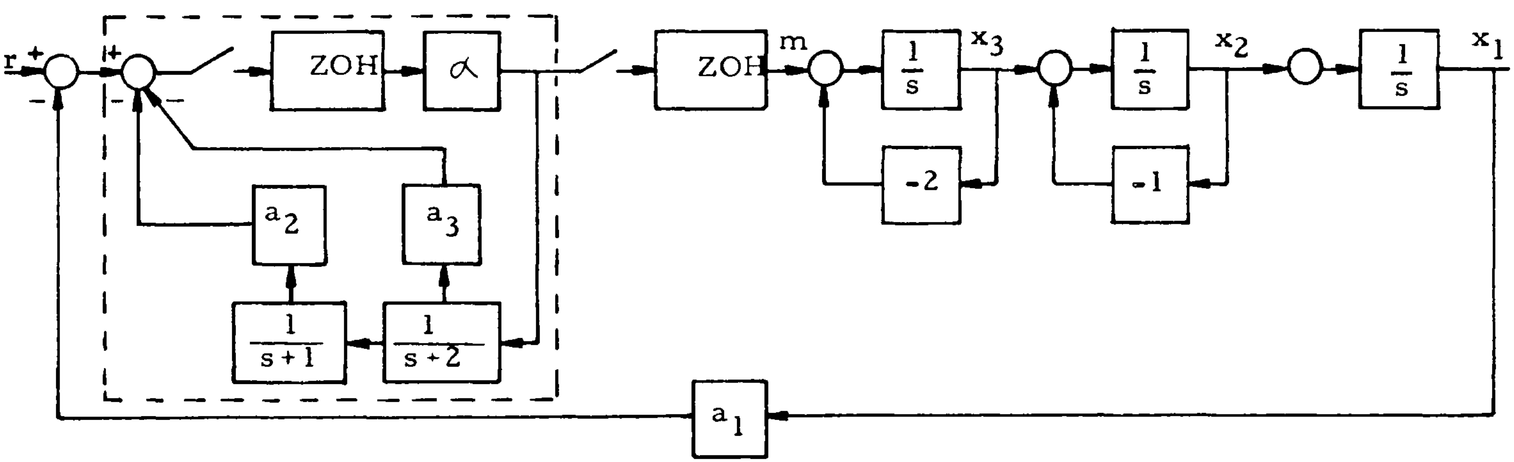

Figure 3.6(b) System $\frac{1}{s(s+1)(s+2)}$ with x_2 and x_3 Inaccessible

$$PD(z) = D(z).$$

This is a direct method for calculating D(z) if the MFPS elements are known.

The restriction is placed on the PD(z) that it not contain an integrator as one of the unmeasurable state-variables. If this is done, the system becomes unstable for the reason stated in Section 2.3 above. There is no restriction, however, on other state-variables included in PD(z).

It is pointed out that if any of the state-variables are inaccessible, then there are initial conditions that will cause the system not to have deadbeat response.

3.5 SUMMARY

The use of a classical digital controller to attain deadbeat response to a step input raises two problems. The first, which applies to all systems, is that deadbeat performance is dependent upon the initial conditions of the state-variables. The second pertains to those systems that have more than one integrator in their transfer function. The settling time for this type system is increased one sampling period for each integrator above one and at times overshoot results.

The Multiple Feedback Path System is presented which overcomes these problems and insures deadbeat response. A "partial" digital controller replaces the unmeasurable state-variables and helps the MFPS achieve deadbeat performance. A method is developed to find the PD(z). The restriction that the PD(z) not replace an integrator is imposed because of stability problems.

Three methods are given for determining the element values of the MFPS. They are:

1. A direct method which uses the most direct approach to the solution. However, for $n > 2$, the solution equations become non-linear and as n increases the solution is formidable.
2. A method using information from the classical D(z). The value for each state-variable for all time must be computed to find the element values. This method gives a direct link between the classical and state-variable approaches.
3. An iteration process which uses the concept of coordinate transformation in state-space. It is the best of the three methods given for high order systems, since the procedure is particularly well suited for programming on a digital computer.

The next chapter extends this system to one that provides deadbeat response for higher order inputs.

CHAPTER 4

EXTENSION OF THE MULTIPLE FEEDBACK PATH SYSTEM

4.1 INTRODUCTION

It has been seen that information from all of the "states" is used in the MFPS for step inputs. Additional knowledge in the form of successive input derivatives is needed if the system is to follow ramp or higher order inputs with deadbeat response. By placing a device in front of the MFPS, deadbeat response to these types of inputs can be achieved.

A procedure is given to find the "input" elements for a ramp input and is then extended to higher order inputs. A method is developed to transfer this "input" element to the usual location of a classical D(z) and combine it with the "partial" digital controller. This provides considerable flexibility since the combination can be adapted for different type inputs with varying numbers of measurable state-variables while using the same basic MFPS.

4.2 "INPUT" DIGITAL CONTROLLER

It is assumed that for a ramp input the system has at least one integrator and that it produces x_1. A parabolic input requires two integrators producing x_1 and x_2. An additional integrator is needed for every increase in order of the input.

Kalman and Bertram (1958) have shown that if the continuous derivative of the input is known, the system will respond with deadbeat performance. The required settling time is n where n is the order of the system. This can best be seen if a ramp input is considered first. It is shown in the previous chapter how x_2 settled to zero for a system of this type with a step input. The equilibrium value of x_2 must now be the slope of the input if x_1 is to give zero steady-state error. Since x_2 is non-zero, an additional input is required to cancel its effect. This reasoning is immediately extended to higher order inputs.

In most sampled-data systems the input is only available in sampled form. For this reason a discrete differentiator rather than the one mentioned above is used to obtain the derivative of the input. This adds two features to the system which are similar to those with a D(z). The first is that the settling time increases one sampling period for each order increase of the input since the derivative at k = 0 is unknown. The second restricts the input to that for which the system was designed; otherwise deadbeat response is not attained.

Assume first that the input is a ramp. A differentiator and constant gain element are placed ahead of the MFPS as shown in Figure 4.1(a). These elements can be combined to make an "input" digital controller, ID(z) (Figure 4.1(b)). The control input is:

$$m(kT) = \alpha(PkT - a_1x_1(kT) - a_2x_2(kT) - \ldots a_nx_n(kT) + Pa_0) \quad (4.1)$$

where P is the slope of the input ramp.

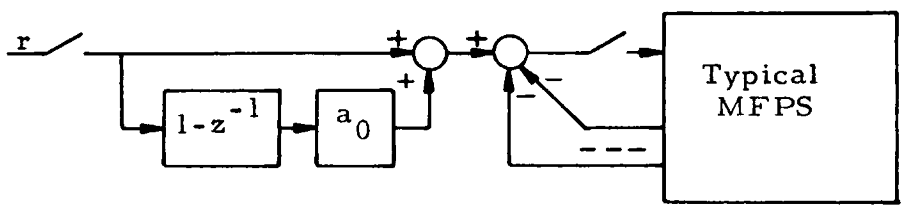

Figure 4.1(a) Discrete Differentiator and Gain Element, a_0, for Ramp Input

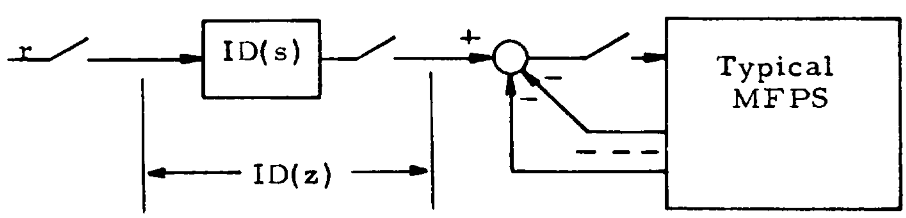

Figure 4.1(b) "Input" Digital Controller

Since all of the element values of the MFPS are known or can be found, a_0 is calculated from equation 4.1 at equilibrium. The procedure for finding a_0 is:

1. Find the element values of the MFPS assuming a step input and all the state-variables are measurable.
2. Determine the equilibrium values of the system for the prescribed input.
3. Using these values, solve equation 4.1 for a_0.

Note that the value of a_0 is independent of the slope of the ramp. This is because x_2 settles to the slope of x_1 and the equilibrium values of the remaining state-variables plus m(kT) are either functions of the slope or zero.

The method used above can be extended to higher order inputs by taking successive derivatives of the input in a similar manner. The required gain elements are found by using the above procedure to successively find a_0, a_{01}, a_{02}, ...

4.3 "COMBINED" DIGITAL CONTROLLER

It may be desirable to transfer the "input" digital controller into the forward path and merge it with PD(z). Such an arrangement will be called the "combined" digital controller, CD(z). This will make the system more compact and easier to analyze.

Transferring ID(z) can be accomplished by block diagram manipulation and pulse transform techniques (Kuo 1963). Consider the two networks shown in Figure 4.2 (a) and (b). The intention is to transform (a) into (b). For convenience the multiple feedback

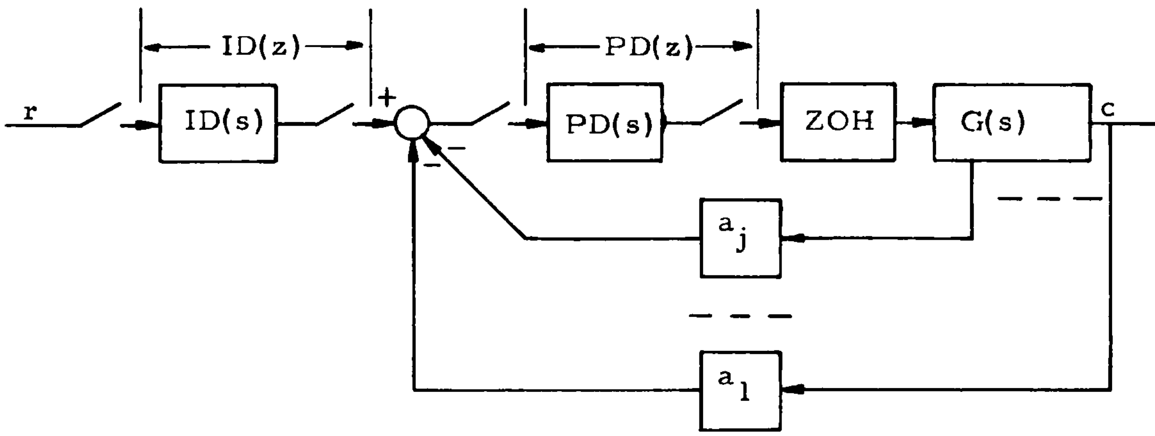

Figure 4.2(a) Block Diagram Representation of "Input" D(z) and "Partial" D(z)

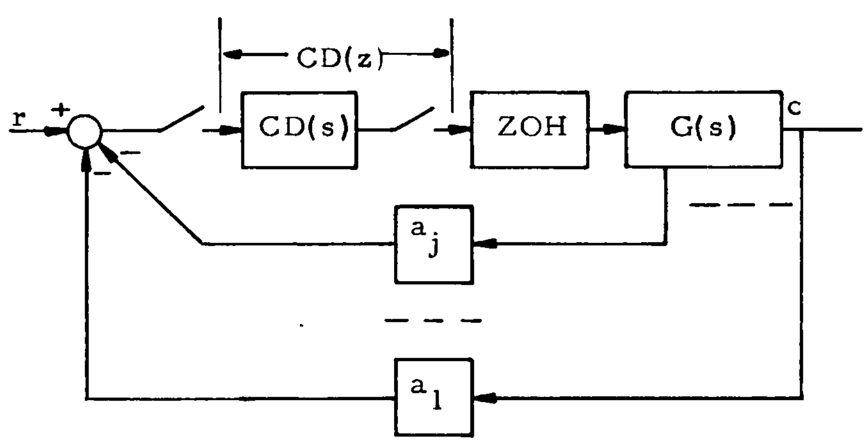

Figure 4.2(b) Block Diagram Representation of "Combined" D(z)

portion of Figure 4.2 may be redrawn into the block diagram shown in Figure 4.3. Here G(s) represents the system transfer function with ZOH. The H(s) is the mathematical combination of the state-variables and their feedback elements that are fed back. Using this representation, the closed loop transfer function of Figure 4.2 (a) and (b) is written in pulse transform notation as equations 4.2 and 4.3, respectively.

$$\frac{C^*(s)}{R^*(s)} = \frac{\overline{PD}^*(s)\, G^*(s)\, \overline{ID}^*(s)}{1 + \overline{PD}^*(s)\, \overline{GH}^*(s)} \tag{4.2}$$

$$\frac{C^*(s)}{R^*(s)} = \frac{\overline{CD}^*(s)\, G^*(s)}{1 + \overline{CD}^*(s)\, \overline{GH}^*(s)} \tag{4.3}$$

Setting these two equal and solving for $\overline{CD}^*(s)$ yields:

$$\overline{CD}^*(s) = \frac{\overline{ID}^*(s)\, \overline{PD}^*(s)}{1 + \overline{GH}^*(s)\, \overline{PD}^*(s)\, [\, 1 - \overline{ID}^*(s)\,]} \tag{4.4}$$

Taking the z-transformation of equation 4.4 gives the required conversion to a "combined" digital controller in terms of the ID(z), PD(z), and GH(z), defined in equation 4.6 below. It is

$$CD(z) = \frac{ID(z)\, PD(z)}{1 + GH(z)\, PD(z)\, [1 - ID(z)]} \tag{4.5}$$

When j state-variables are measurable, GH(z) in equations 4.2-4.5 is:

$$GH(z) = \mathcal{Z}\left[\frac{(1 - e^{-sT})}{s} \sum_{q=1}^{j} a_q \prod_{p=q}^{n} (s + d_p)^{-1} \right] \tag{4.6}$$

where d_i is defined in Figure 3.1.

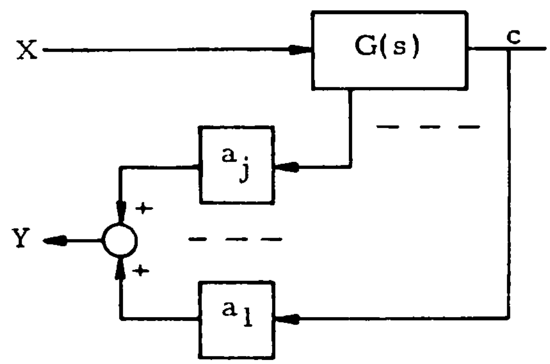

Figure 4.3(a) Diagram of Feedback Portion of MFPS

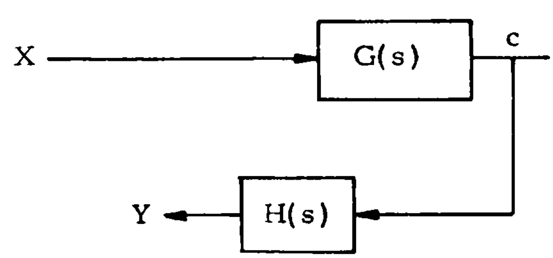

Figure 4.3(b) Block Diagram Representation of Feedback Portion of MFPS

The procedure to find CD(z) is then:

1. Find PD(z).

2. Find ID(z).

3. Determine GH(z) by equation 4.6.

4. Calculate CD(z) by equation 4.5.

Each of these steps may be accomplished for various numbers of measurable state-variables.

4.4 SUMMARY

The Multiple Feedback Path System is used as the basic system for providing deadbeat response for ramp and higher order inputs. A method for finding the value of the "input" D(z) is established based on the element values of the MFPS. This procedure is independent of the number of measurable state-variables. If they are all accessible the system will respond properly because all initial conditions will settle in n sampling periods.

The "input" D(z) was transferred to the forward path of the system and joined with the "partial" D(z) to make a "combined" digital controller. The terms of CD(z) will change appreciably according to the number of measurable state-variables. However, it can be found for any number available.

CHAPTER 5

CONCLUSIONS

The Multiple Feedback Path System provides a better method for attaining deadbeat response of linear sampled-data control systems than the classical digital controller. It assures proper response for any initial conditions and for any linear, unity numerator system. The control elements of the two methods can be found with about the same relative ease for simple systems with step function inputs. However, for complex systems the MFPS can be more easily obtained by programming the iterative process on a digital computer. Finding the elements required for higher order inputs is simple once the basic MFPS is known. The classical approach requires a complete recalculation of all terms of D(z) except the system z-transformation.

The MFPS with the "partial" digital controller provides deadbeat response for those systems that have inaccessible state-variables. It is possible, however, for the unmeasurable state-variables to have initial conditions that will cause non-deadbeat performance. The overall system performance is still better than with the classical D(z) for the following reasons. The initial condition of the accessible state-variables is immediately fed back, and the unknown "state" information is available sooner since the value of each state-variable is a function of the previous ones.

The MFPS provides another method for determining the classical D(z). This is a direct connection between the classical and state-space approaches. The "partial" D(z) and the "combined" D(z) present systems where both the old and new techniques are combined.

There are two possible areas for additional research that have come to the attention of the author. First, extension of the ideas and methods presented in this thesis to systems whose transfer functions are non-unity numerators. Second, the development of a similar sequence for continuous systems.

In conclusion, a control system for sampled systems has been presented that combines the classical z-transformation technique with the state-space approach. It was shown that this control system is a definite improvement on the classical method.

LIST OF REFERENCES

KALMAN, R. E., and BERTRAM, J. E.: General Synthesis Procedure for Computer Control of Single-Loop and Multiloop Linear Systems, Trans. AIEE Vol. 77, pt. II, pp. 602-609, 1958.

KUO, B. C.: Analysis and Synthesis of Sampled-Data Control Systems, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963.

RAGAZINNI, J. R., and FRANKLIN, G. F.: Sampled-Data Control Systems, McGraw Hill Book Co., Inc., New York, N. Y., 1958.

TOU, J. T.: Modern Control Theory, McGraw Hill Book Co., Inc., New York, N. Y., 1964.

TOU, J. T., and VADHANAPHUTI, B.: Optimum Control of Nonlinear Discrete-Data Systems, Trans. AIEE Vol. 80, pt. II, pp. 166-171, 1961.