ANALYSIS OF STRUCTURAL SYSTEMS 
HAVING LARGE DEFLECTIONS

by
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STATEMENT BY AUTHOR

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ABSTRACT

A method for analyzing planar structural systems having nonlinear load-displacement properties due to displacements which exceed the limitations of linear structural theory is presented herein. Structures which exhibit nonlinear material or connection properties as well as nonlinear geometric properties are also considered. The method of displacements and a general force-deformation relation are used in the analysis. Both an iterative and a numerical integration scheme are presented as methods of solution of the equations characteristic of nonlinear systems.

In the iterative scheme, the linear solution is obtained as a first approximation. The displacements thus found are used to describe a new displaced configuration of the structure which is then used to obtain a second approximation. This procedure is repeated until convergence is achieved.

In the numerical integration scheme, the equations are transformed into a set of simultaneous nonlinear first-order, ordinary differential equations. The solution by quadrature of these differential equations for a
given set of initial conditions and loading on the structure yields the complete displacement configuration for the structure from which the internal load distribution in the structure may be easily determined. The differential equations are solved by the fourth-order Runge-Kutta numerical integration scheme.

Examples are considered and difficulties which arise in the analysis are discussed.
CHAPTER 1

INTRODUCTION

Nearly all structures built up to this time (1965) have been designed using the classical theory of structures which presupposes that the material and structure behavior follow linear laws. Linear theory has been sufficiently precise because most structures, in order to fulfill their design objectives, deflect very little. However, some structures that do not obey linear laws, such as guyed towers and draped cable roofs, have been satisfactorily constructed. In these systems the deflections are not large compared with the structure size and design usually implemented by first order extensions of linear theory.

Present-day demands tend towards a more efficient use of material and an architectural appearance which is free of massive structural framework. Furthermore, modern aerospace requirements tend increasingly towards lighter and more efficient structural systems. These structures are inherently more flexible, and a precise analysis of them will require methods which consider both the nonlinearities of the materials from which they are constructed and changes in the geometry of the system which results from large displacements.
CHAPTER 2

GENERAL STATE OF THE ART

In recent years considerable effort has been directed toward the development of methods of analysis of structural systems that are subject to large deflections and have members stressed beyond the proportional limit of the material. It should be noted that in order to obtain an analytical formulation of the equations characteristic of nonlinear systems, all relations needed to describe the physical system must be expressed in mathematical terms. Appropriate equations or functions, either algebraic or differential, must be found to relate all the variables of the system. Plotted empirical relations derived from experimental measurements (such as stress-strain curves) must be represented in the form of mathematical equations.

In order to give accurate analytic form to stress-strain curves while at the same time keeping the mathematical expressions involved reasonably simple, Ramberg and Osgood (1943) developed a method for representing stress-strain curves by three parameters. This formulation has been widely used to analyze axially stressed elements and truss systems especially in the aerospace industry.
For example, Wilder (1953), Denke (1956) and Wilson (1960) studied the behavior of trusses having elements stressed beyond the proportional limit of the material using this formulation. Wilder and Denke used iterative procedures to solve the resulting nonlinear simultaneous algebraic equations and a considerable amount of mathematical manipulation of a trial and error character was found to be required to obtain a solution for the specific illustrations presented. Wilson used an incremental method and thereby replaced the analysis of the nonlinear truss by the sum of a series of linear structures, each of which was subjected to an increment of the applied load and thus avoided convergence problems. He also extended the method to framed systems by using a Ramberg-Osgood polynomial to represent moment-curvature relationships.

Richard (1961) and Goldberg and Richard (1963) used an expression to represent force-deflection and moment-rotation relationships involving three parameters which is essentially the inverse of the Ramberg-Osgood polynomial to analyze nonlinear truss and frame systems. This method was based on a differential point of view and the equations characteristic of systems having material nonlinearities were shown to be a system of simultaneous first order ordinary differential equations. The solution of these equations for a given set of initial conditions and loading
gave the complete displacement pattern of the structure from which the internal load distribution could be easily determined. The feasibility of the method was illustrated by the solution of several structures of practical size and interest.

Horne (1962) investigated the effects of finite deformations on the elastic stability of plane frames. The stability of plane frames is usually discussed with the assumption that there is no deformation of the type associated with the buckling mode. This assumes that the direct axial deformations are negligible in comparison with the flexural deformations. Horne's investigation of the stability of frames accounted for axial deformations that are produced in the structure by the loads prior to buckling. However, his method is based on the assumption that deflections remain very small. He made no attempt to analyze frames in which the deflections effectively change the geometry and as a result he assumed geometric linearity inherent in small deflection theory.

Gjelsvik and Bodner (1962) used energy methods to analyze snap buckling of shallow arches. They were able to predict the buckled position of the arch and verify their results with experimental data for a shallow arch with fixed supports.
Saafan (1962) in his earlier work investigated the nonlinear behavior caused by finite deflections and neglected the effects of axial loads and differences between arc length and chord length of members. In his later work (1963) he included these effects in his analysis; however, his assumption on the axial deformation is valid only if the members do not rotate in going from their unloaded position to their loaded position. This assumption is approximately true if the rotation is small. However, in his paper he presents an example in which the members rotate through angles as much as twenty-five degrees with the result that the total axial deformation in the example used is in error by as much as 35 percent.

Chu (1959) also investigated the effect of the difference between arc length and chord length. Johnson (1961) investigated the elastic stability of rigid frame structures subject to sidesway. He used third degree polynomials to describe the deformation of each member. The coefficients in the polynomial were in terms of the end displacements and slopes. The energies and moments related to each member were, in turn, related to the coefficients of the polynomial. By a combination of an energy method and moment distribution he was able to arrive at a solution assuming small deflections. Masur (1954) investigated
the effect of the difference between chord length and arc length on the buckling load in statically indeterminate trusses.

The effect of strain hardening was included in a study of elasto-plastic analysis of continuous frames and beams by Johnson and Sawyer (1960). In this study it was assumed the stress-strain relation for steel could be idealized into three straight lines representing the elastic range, the plastic range and the strain hardening range respectively. Limit design methods were used for a solution in which strength of the structure was determined using moment-curvature relations as well as the ultimate moment.

Ang (1961) developed an analysis for frames subjected to lateral deformations. The nonlinear characteristics of both the members and the connections were considered. He derived relationships for the individual members and moment-rotation curves for the nonrigid connections. After a compatible set of resisting moments was obtained, the corresponding set of loads required to produce the particular joint displacements were computed. By solving a set of such problems, load-joint displacement relationships could be obtained for a range of loads, or conversely, for a range of displacements. Ang assumed in his study that the moment-curvature relationship
corresponded to the case of pure flexure. Effects of axial forces, shearing forces, buckling, and residual stresses on the relationship were neglected. It should be noted that Ang's procedure solves for the load necessary to produce a specified displacement, the inverse of the usual procedure of determining displacements from a known set of loads. Studies concerning uniqueness of solution and convergence criteria were not included for this method in the paper.

Kron (1956) simplified the method developed by Langefors (1951, 1952) by following the basic formulation of Wilder (1953) and Hoff (1944). However, while their formulas required the simultaneous formulation of all of the nonlinear equations describing the entire structure, Kron established and solved the nonlinear equations in stages by tearing the structure apart into smaller structures or into individual components. He then set up the nonlinear deformation-force equations of each small structure separately and interconnected the whole system by tensorial methods. The relationship used to express the nonlinear material properties of the members was essentially the Ramberg-Osgood polynomial.

King (1956) and Vickery (1961) investigated the effect of distortion on the plastic limit load. These investigations considered first order effects only.
Ojalvo (1956) and Lu (1961) solve the problem of the frame with members stressed into the plastic range by determining a compatible moment and rotation at a joint by the intersection of two moment versus end-rotation curves of the adjoining members. They include the effects of residual stresses and instability of the beam-columns in the analysis.

Rogers and Lee (1962) found a solution to the finite deflection of a linearly viscoelastic cantilever clamped horizontally at one end and loaded vertically at the other by replacing the elastic constant with a visco-elastic operator which described in integral form the nonlinear stress-strain and creep properties of the cantilever. This method led to an equation that was not readily interpretable and a solution had to be obtained by iteration of an equivalent nonlinear equation.

All of these methods either assumed small displacements or attempted solution for large displacements by only first order extensions of linear theory. The purpose of this thesis is to present a method for the analysis of structural systems that are flexible and deflect well beyond the linear range. This analysis includes consideration of nonlinear internal force-deformation relationships arising from nonlinear stress-strain properties and/or connections having nonlinear properties. Only axial
and flexural forces are considered. Buckling and shear deformations are not considered. In general, value of the fully plastic moment for a given element is affected by the axial and shear forces; however, these effects are neglected in this study. It is assumed that the loads are proportional and that they are applied to the structure in such a manner that time rate of loading need not be considered in determining the nonlinear parameters.

The force-deformation relationship which is used is due to Richard (1961) and is of a general nature. This relationship may be used to represent the elastic and the elastic, perfectly plastic cases, and also can include strain hardening effects. It is assumed the material exhibits the same properties in tension and compression although this assumption need not be made if the direction of the internal forces is known at the outset so that the respective properties may be included as initial properties. If internal force reversal does occur, it is assumed that the force-deformation relationship does not change. It is also assumed that the difference between the actual length and the chord length of flexural members is negligible.
CHAPTER 3

MATHEMATICAL FORMULATION

3.1 Introduction. The system coordinates are used to describe the structure configuration throughout this presentation since their use greatly facilitates defining the large displacements involved both within the mathematical formulation of the problem and in the computer programs. Positive coordinates and rotations are defined in Figure 3.1. Positive forces and displacements are in the direction of the positive coordinates.

![Figure 3.1 Definition of Positive Coordinates](image)

All necessary relationships are derived for a single structure segment such as shown in Figure 3.2. Articulated structures of arbitrary size and configuration may then be generated by connecting individual segments in the pattern
Figure 3.2 Definition of Coordinates and Displacements
of the desired structure and requiring compatibility and equilibrium at the connecting node points of the segments. Support conditions are specified by requiring the proper displacements to be zero. Loads are applied at the node points only with positive directions defined by the coordinates in Figure 3.1.

It is assumed that the element is free of intermediate loading, hence the algebraic minimum and maximum moments occur at the node points. It is also assumed that all flexural nonlinear material behavior occurs at the node points and that the member itself remains linearly elastic under flexural loads. It has been shown in literature (Hodge, 1959) that there is generally no appreciable error due to such an assumption, and that the plastic hinge is generally confined to a small portion of the member. Nonlinear axial behavior is assumed evenly distributed over the length of the member.

3.2 Derivation of the Compatibility Relationships. For all derivations in this section, refer to Figure 3.2. In this figure the final length of the segment chord $l'$ is defined as:

$$l' = ( (x_4' - x_1')^2 + (x_5' - x_2')^2)^{1/2}$$  \hspace{1cm} (3.1)

Squaring both sides, dividing through by $l'$ and noting that
**Equation 3.2**

\[
\cos \alpha' = \frac{x_4' - x_1'}{l'} \\
\sin \alpha' = \frac{x_5' - x_2'}{l'}
\]

Then

\[
l' = (x_4' - x_1') \cos \alpha' + (x_5' - x_2') \sin \alpha' \tag{3.3}
\]

**However, using the relationships**

\[
x_1' = x_1 + \Delta x_1 \tag{3.4a}
\]
\[
x_2' = x_2 + \Delta x_2 \tag{3.4b}
\]
\[
x_4' = x_4 + \Delta x_4 \tag{3.4c}
\]
\[
x_5' = x_5 + \Delta x_5 \tag{3.4d}
\]

Equation 3.3 may be written

\[
l' = (x_4' - x_1') \cos \alpha' + (x_5' - x_2') \sin \alpha'
\]
\[
+ (\Delta x_4' - \Delta x_1') \cos \alpha' + (\Delta x_5' - \Delta x_2') \sin \alpha' \tag{3.5}
\]

**But**

\[
(x_4' - x_1') = l_0 \cos \alpha_o \tag{3.6a}
\]
\[
(x_5' - x_2') = l_0 \sin \alpha_o \tag{3.6b}
\]
Hence

\[ l' = l_0 (\cos \alpha_0 \cos \alpha' + \sin \alpha_0 \sin \alpha') \]

\[ + (\Delta x_4 - \Delta x_1) \cos \alpha' + (\Delta x_5 - \Delta x_2) \sin \alpha' \]  \hspace{1cm} (3.7)

or

\[ l' = l_0 \cos(\alpha_0 - \alpha') + (\Delta x_4 - \Delta x_1) \cos \alpha' \]

\[ + (\Delta x_5 - \Delta x_2) \sin \alpha' \]  \hspace{1cm} (3.8)

The change in length \( s_1 \) of the chord is

\[ s_1 = l' - l_0 \]  \hspace{1cm} (3.9)

Substituting Equation 3.8 for \( l' \) the resulting expression for axial deformation of the chord becomes

\[ s_1 = l_0 (\cos(\alpha_0 - \alpha') - 1) + (\Delta x_4 - \Delta x_1) \cos \alpha' \]

\[ + (\Delta x_5 - \Delta x_2) \sin \alpha' \]  \hspace{1cm} (3.10a)

From Figure 3.2 it is seen that the rotations between chord and tangent at the points, \( s_2 \) and \( s_3 \), may be written as

\[ s_2 = \Delta x_5 + (\alpha_0 - \alpha') \]  \hspace{1cm} (3.10b)

\[ s_3 = \Delta x_6 + (\alpha_0 - \alpha') \]  \hspace{1cm} (3.10c)
Defining the internal deformation matrix as

\[
\mathbf{s} = \begin{pmatrix}
\mathbf{s}_1 \\
\mathbf{s}_2 \\
\mathbf{s}_3
\end{pmatrix}
\]  
(3.11)

and the displacement matrix as

\[
\mathbf{\Delta x} = \begin{pmatrix}
\mathbf{\Delta x}_1 \\
\mathbf{\Delta x}_2 \\
\mathbf{\Delta x}_3 \\
\mathbf{\Delta x}_4 \\
\mathbf{\Delta x}_5 \\
\mathbf{\Delta x}_6
\end{pmatrix}
\]  
(3.12)

the compatibility relationship may be written in matrix form as

\[
\mathbf{s} = \mathbf{p} + \mathbf{b} \mathbf{\Delta x}
\]  
(3.13)

where

\[
\mathbf{p} = \begin{pmatrix}
1_0 (\cos(\alpha - \alpha') - 1) \\
\alpha - \alpha' \\
\alpha - \alpha'
\end{pmatrix}
\]  
(3.14a)

and

\[
\mathbf{b} = \begin{bmatrix}
-\cos\alpha' & -\sin\alpha' & 0 & \cos\alpha & \sin\alpha & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  
(3.14b)
The matrix $b$ is defined here as the primary compatibility matrix and the matrix $p$ is defined as the secondary compatibility matrix.

That these relationships can be degenerated to the small deflection case may be demonstrated by reference to Figure 3.3.

![Figure 3.3 Degeneration to Small Deflection Case](image)

For small displacements $l' \approx l_0$, \[ \frac{\alpha' - \alpha_0}{l_0} \approx \frac{\delta}{l_0} \]

and $\cos(\alpha_0 - \alpha') \approx 1$.

Hence

\[ s = \begin{pmatrix} 0 \\ \delta/l_0 \end{pmatrix} + \begin{bmatrix} -\cos\alpha_0 & -\sin\alpha_0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta x_1 - \Delta x_4 \end{pmatrix} \]

(3.15)
But

\[ \delta = \Delta x_1 \sin \alpha - \Delta x_2 \cos \alpha - \Delta x_4 \sin \alpha - \Delta x_5 \cos \alpha \]  

(3.16)

Hence

\[
\begin{bmatrix}
- \cos \alpha & - \sin \alpha & 0 & \cos \alpha & \sin \alpha & 0 \\
\frac{\sin \alpha}{1_0} & \frac{\cos \alpha}{1_0} & 1 & \frac{\sin \alpha}{1_0} & - \frac{\cos \alpha}{1_0} & 0 \\
\frac{- \sin \alpha}{1_0} & \frac{- \cos \alpha}{1_0} & 0 & \frac{- \sin \alpha}{1_0} & \frac{- \cos \alpha}{1_0} & 1
\end{bmatrix} \Delta x \quad (3.17)
\]

With the exception of the signs of the individual elements which are peculiar to the particular coordinate system used this matrix is the compatibility matrix commonly derived in small deflection theory (Goldberg and Richard, 1963).

3.3 Derivation of the Equilibrium Relationships.

The equilibrium relationships for large geometric changes are identical to the same relationships for small deflection theory except the displaced geometry of the element must be used instead of the original geometry of the element. This is reasonable since equilibrium is only a function of the forces existing in the deflected position, not of the path followed to reach that position.
Figure 3.4 shows a typical displaced element in equilibrium. $S_2$ and $R_3$ are shown in (a) as being in the same direction since from the viewpoint of the element these forces are identical. The same is true of $S_3$ and $R_6$. In (b), $S_2$ and $S_3$ are resolved into a couple acting at the ends of the element and perpendicular to it. By requiring equilibrium at the ends, the following equations result:

$$R_1 = - S_1 \cos \alpha' - (S_2 + S_3) \frac{\sin \alpha'}{l} \tag{3.18a}$$

$$R_2 = - S_1 \sin \alpha' + (S_2 + S_3) \frac{\cos \alpha'}{l} \tag{3.18b}$$

$$R_3 = S_2 \tag{3.18c}$$

$$R_4 = S_1 \cos \alpha' + (S_2 + S_3) \frac{\sin \alpha'}{l} \tag{3.18d}$$

$$R_5 = S_1 \sin \alpha' - (S_2 + S_3) \frac{\cos \alpha'}{l} \tag{3.18e}$$

$$R_6 = S_3 \tag{3.18f}$$

Let the internal force matrix be

$$S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{3.19}$$
Figure 3.4 Element Forces in Equilibrium
and the external load matrix be

\[
R = \begin{pmatrix}
R_1 \\
R_2 \\
R_3 \\
R_4 \\
R_5 \\
R_6
\end{pmatrix}
\]

(3.20)

Then the equilibrium relationship may be written in matrix form as

\[
R = aS
\]

(3.21)

where

\[
a = \begin{pmatrix}
-cos\alpha' - \frac{sin\alpha'}{l'} - \frac{sin\alpha'}{l} \\
-sin\alpha' - \frac{cos\alpha'}{l'} - \frac{cos\alpha'}{l} \\
0 & 1 & 0 \\
\frac{cos\alpha'}{l'} & \frac{sin\alpha'}{l'} & \frac{sin\alpha'}{l} \\
\frac{sin\alpha'}{l'} & -\frac{cos\alpha'}{l'} - \frac{cos\alpha'}{l} \\
0 & 0 & 1
\end{pmatrix}
\]

(3.22)

The matrix a is defined as the equilibrium matrix.
3.4 Derivation of the Internal Force-Deformation Relationships. Unless the structure is extremely flexible, the large geometric distortions necessary to require consideration of the effects of these distortions will produce yielding in some of the members or connections. Yielding of this type produces a force-deformation relationship that is nonlinear, hence requiring a mathematical relationship which expresses this nonlinearity. A relationship which produces a wide range of force-deformation relationships and which is very amenable to computer programming was developed by Richard (1961). This relationship is

\[
\frac{T}{T_0} = \frac{e}{e_0} \left( 1 + \left| \frac{e}{e_0} \right|^n \right)^{\frac{1}{n}} \tag{3.23}
\]

in which \( T \) is the stress (force or moment), \( e \) is the strain (displacement or rotation), \( T_0 \) is the maximum stress (force or moment), \( e_0 \) is the normalized strain (displacement or rotation) and \( n \) is some positive exponent. Adapting this equation to the case of axial and flexural forces in a manner identical to that of Goldberg and Richard (1963), the following matrix formulation of the force-deformation relationship is obtained:

\[
S = ks \tag{3.24}
\]
where

\[
k = \begin{bmatrix}
k_{11} & 0 & 0 \\
0 & k_{22} & \frac{1}{2}k_{22} \\
0 & \frac{1}{2}k_{33} & k_{33}
\end{bmatrix}
\]  \hspace{1cm} (3.25)

The individual matrix elements are

\[
k_{11} = \frac{AE}{l_0} \left( 1 + \frac{AE}{l_0} s_1 \frac{1}{n} \right) \hspace{1cm} (3.26a)
\]

\[
k_{22} = \frac{4EI}{l_0} \left( 1 + \frac{4EI}{l_0} s_2 + \frac{2EI}{l_0} s_3 \frac{1}{n} \right) \hspace{1cm} (3.26b)
\]

\[
k_{33} = \frac{4EI}{l_0} \left( 1 + \frac{2EI}{l_0} s_2 + \frac{4EI}{l_0} s_3 \frac{1}{n} \right) \hspace{1cm} (3.26c)
\]

\(A_R\) is the maximum axial force, and \(F_R\) is the maximum moment, both of which must be determined for the particular structural element being considered. The value of these maximum forces will generally depend on the cross section of the member and the moment-rotation character-
istics of the connections, as well as the shape of the stress-strain curve. It should be noted, however, that the value of these forces is the value to which the resulting force-deformation relationship becomes asymptotic. A judicious choice of these forces and the exponent $n$ allow a wide choice of curves. By choosing these reference factors very high compared with the forces acting within the structure, a linear relationship results.

![Nondimensional Stress-Strain Relationship](image)

**Figure 3.5** Nondimensional Stress-Strain Relationship

Figure 3.5 is a reproduction of the curve represented by Equation 3.23 as given by Goldberg and Richard (1963, p. 338). By choosing the appropriate values of $n$
it is possible to represent linear stress-strain; nonlinear stress-strain; and the case of elastic, perfectly-plastic stress-strain. Since the function is always increasing, some types of strain hardening may be represented by proper choice of parameters; however, this will not be considered here.

3.5 Summary of Equations. The relationships describing the action of an individual element may be summarized as follows:

Compatibility \[ s = p + b \Delta x \]  (3.13)
Equilibrium \[ R = aS \]  (3.21)
Force-deformation \[ S = ks \]  (3.24)

Substituting Equation 3.24 into Equation 3.21
\[ R = aks \]  (3.27)

Substituting Equation 3.13 into Equation 3.27
\[ R = akp + akb \Delta x \]  (3.28)

Defining the primary stiffness matrix as
\[ K = akb \]  (3.29a)
and the secondary stiffness matrix as
\[ K_a = akp \]  (3.29b)

Equation 3.28 may be written
\[ R = K_a + K \Delta x \]  (3.30)

Solution of these equations may not be obtained directly since both K and K_a are functions of the displaced
geometry which is not known. Two methods of solution are available. Either of the equations may be iterated by one of several numerical procedures until convergence is achieved or they may be differentiated and then integrated numerically. Advantages and disadvantages of both methods will be presented and discussed in succeeding chapters.

The general procedure used is to treat the secondary stiffness matrix $K_a$ as an additional set of loads. Hence, Equation 3.30 may be solved for the displacements as

$$\Delta x = K^{-1}(R - K_a)$$  \hspace{1cm} (3.31)

Because of the complexity of the computations involved for even small structural systems, the solution to Equation 3.31 was obtained through use of an IBM 1401-7072 digital computer system. Essentially the same procedure was used in the computer programs for both the iteration procedure and the integration procedure. The control criteria, member and material information, structure geometry, reactions and load were programmed as input data. All necessary matrices were then formed internal to the program for the entire structure. The secondary stiffness matrix was formed by matrix multiplication and subtracted from the loads. The primary stiffness matrix was then formed and columns and rows corresponding to zero displacements and reactions respectively were lined out.
In its full form as given by Equation 3.29, \( K \) is singular and may not be inverted. The above procedure removes the singularity of \( K \).

In the procedure used here, \( K \) is not inverted directly but is instead reduced to a lower triangle matrix by the Gauss elimination method. In this method, all row operations performed on \( K \) are performed on the load vector \( R - K_a \). The displacements are then found by back substitution into the lower triangle matrix. When the correct value of the displacements is found it is substituted back in Equations 3.13, 3.21 and 3.24 to find the internal deformations and internal forces and to recompute the loads as a check on the accuracy of the solution. A copy of the programs and flow diagrams may be found in the Appendix.

### 3.6 Reduction to Axial Case

Equations 3.13, 3.21 and 3.24 may be reduced to the case of axial forces only by reducing the matrices involved to include only those terms which relate to axial forces. Hence:

\[
\begin{align*}
    s &= \begin{bmatrix} s_1 \end{bmatrix} \\
    p &= \begin{bmatrix} l_0 \cos(\alpha_0 - \alpha') - 1 \end{bmatrix} \\
    a^T &= \begin{bmatrix} -\cos \alpha' & -\sin \alpha' & \cos \alpha' & \sin \alpha' \end{bmatrix}
\end{align*}
\]  

(3.32a)  
(3.32b)  
(3.32c)
\[ \Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_4 \\ \Delta x_5 \end{bmatrix} \]  

\[ R = \begin{bmatrix} R_1 \\ R_2 \\ R_4 \\ R_5 \end{bmatrix} \]  

\[ b = a^T \]  

\[ s = \begin{bmatrix} s_1 \end{bmatrix} \]  

\[ k = \begin{bmatrix} k_{11} \end{bmatrix} \]  

By using Equations 3.32, the Equations 3.29, 3.30 and 3.31 apply to the axial case. A similar reduction may be made for the case of pure flexure.
CHAPTER 4

SOLUTION BY ITERATION

4.1 Method. In the iterative procedure used, the initial geometry was first assumed to be the undeformed geometry as in linear theory. In Equation 3.29b, $K_a$ is given as

$$K_a = akp$$  \hspace{1cm} (3.29b)

By referring to Equation 3.14a, it may be seen that for the first iteration $p$ is identically zero. Hence $K_a$ is identically zero also and Equation 3.31 becomes

$$\Delta x = K^{-1}R$$  \hspace{1cm} (4.1)

Displacements computed from this equation were then added to the initial geometry giving an approximate displaced geometry. The new values were then substituted into Equation 3.31 resulting in a new set of displacements more closely representing the true displacements. This procedure was repeated until convergence was obtained. No convergence problems were found even with problems involving great geometric nonlinearities except in regions of geometric instability. This is demonstrated in the following example problems.
4.2 Example Problem 4.1. To demonstrate the applicability of the equations to systems undergoing extremely large movements, a simple three bar truss was solved using the iterative procedure. It was not intended that this example should be interpreted as being of practical interest since the very large changes in length of the members and the exclusion of buckling of compression members is not generally realistic for common structural materials. However, the problem does demonstrate the complete versatility of the equations just derived.

The truss system solved is shown in Figure 4.1. Loads, $R$, and coordinates, $x$, are shown on the figure. Elastic properties of the materials were assumed to be linear. For all members, $AE = 200$ and $l_0 = 100$. 

![Figure 4.1 Example Truss](image)
The truss was loaded downward from its apex with a single steadily increasing load, \( R_4 \). Inspection of the truss system under this type of loading indicated that a point would be reached when the truss would "snap through" into its mirror position. In Figure 4.2, three separate loading conditions are shown. The unloaded position of the truss is shown dashed in for reference. The mirror image of the truss is shown in (c). The internal deformations and forces are given in Table 4.1 for these same three load conditions. Note that the internal forces are \( AE/l_0 \) times the internal deformations and that the internal deformations are exactly the difference between the original length of 100 and the final lengths as shown in Figure 4.2 (a,b,c). In Figure 4.2 (d,e,f) the internal forces are resolved into their horizontal and vertical components at the end of each member. These results clearly demonstrate that equilibrium was achieved. Hence compatibility, equilibrium and the force-deformation equations are all satisfied and the solutions are, therefore, the correct ones.
Figure 4.2 Example Problem 4.1, Displacements and Equilibrium
Table 4.1

Example Problem 4.1, Internal Deformations and Forces

<table>
<thead>
<tr>
<th>Load</th>
<th>Internal Deformations</th>
<th>Internal Forces</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>S(_1) -9.2 S(_2) -9.2 S(_3) 5.3</td>
<td>S(_1) -18.4 S(_2) -18.4 S(_3) 10.6</td>
<td>3</td>
</tr>
<tr>
<td>61</td>
<td>-26.9 -26.9 22.2</td>
<td>-53.9 -53.9 44.4</td>
<td>15</td>
</tr>
<tr>
<td>62</td>
<td>16.9 16.9 -6.7</td>
<td>33.8 33.8 -13.4</td>
<td>17</td>
</tr>
</tbody>
</table>

These solutions were obtained for each loading using the iterative procedure described. In Figure 4.3 the complete load-deflection curve is given for this loading. The iterative procedure used readily produced the points given on this plot up to the point of "snap through." At loads greater than the "snap through" load the iterative procedure passed directly to the new stable configuration. The "snap through" portion of the curve is a region of instability. The portion from E to G was found by realizing that this part of the curve is the inverse of the portion A to C. The unstable part of the curve shown was only estimated.

Accuracy of the solution was checked by back substitution. The number of iterations required for convergence to a change of angle of orientation of any member between each iteration of less than .001 radians is given in Table 4.1. The number of iterations required for
Figure 4.3 Example Problem 4.1, Load-Displacement Curve
convergence to this accuracy did not increase above four up to a load of 50. Beyond this point a significant increase in the number of iterations was required as indicated in Table 4.1.

4.3 Example Problem 4.2. In this example the three bar truss shown in Figure 4.1 was loaded with an increasing horizontal load, \( R_3 \). Investigation of this loading condition indicated that horizontal bar should increase in tensile load for a time then decrease to zero load as the apex of the truss moved directly above the roller support. As the load increased beyond this point this bar should become a compression member. The force-displacement curve for coordinate \( x_5 \) in Figure 4.5 plainly shows that this was indeed the case.

Table 4.2

Example Problem 4.2, Internal Deformations and Forces

<table>
<thead>
<tr>
<th>Load</th>
<th>Internal Deformations</th>
<th>Internal Forces</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s_1 ) ( s_2 ) ( s_3 )</td>
<td>( s_1 ) ( s_2 ) ( s_3 )</td>
<td></td>
</tr>
<tr>
<td>( R_3 )</td>
<td>30</td>
<td>17.0</td>
<td>-12.7</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>29.1</td>
<td>-18.4</td>
</tr>
<tr>
<td></td>
<td>54</td>
<td>43.4</td>
<td>-23.3</td>
</tr>
</tbody>
</table>

The internal deformations and forces are given in Table 4.2 for three load conditions. The three displacement
configurations corresponding to each load are shown in Figure 4.4 (a,b,c). In Figure 4.4 (d,e,f) the internal forces are resolved into their horizontal and vertical components at the end of each member. Again it is demonstrated by these figures and Table 4.2 that compatibility, equilibrium and force-deformation requirements are met. Hence the solutions given are the correct solutions under the assumptions made. There is some error in the equilibrium check in Figure 4.4f but this is near the point of instability where insufficient significant figures as used by the computer precluded a more exact solution.
Figure 4.4 Example Problem 4.2, Displacements and Equilibrium
Figure 4.5 Example Problem 4.2, Load-Displacement Curves
CHAPTER 5

SOLUTION BY NUMERICAL INTEGRATION

5.1 Method. The equations derived in Chapter 3 are the large displacement equations of the displacement method. In the displacement method the displacements are treated as the unknown quantities and the loads as the known quantities. Equation 3.30 represents in matrix form a set of simultaneous algebraic equations. In linear systems these equations are linear. In nonlinear systems the matrices $K$ and $K_a$ are functions of the displaced geometry. The displaced geometry is the original geometry plus the displacements. Hence, since the matrices $K$ and $K_a$ contain products of the unknown displacements, the set of equations resulting from Equation 3.30 are nonlinear when geometry changes are considered. If a differential point of view is taken, ordinary nonlinear differential equations result. This reduces the problem to an initial value problem which can be satisfactorily integrated using the Runge-Kutta fourth order procedure. The Runge-Kutta procedure is a self starting single step procedure. This means it does not require a past history and that the interval of integration may be easily changed. In the
fourth-order process, four evaluations of the first derivatives are made. This method is presented rigorously in Ince (1926) and others and, therefore, will not be presented here.

5.2 Derivative of the Compatibility Equations.

Noting that $l' = s_1 + l_0$ and using Equations 3.4, Equation 3.1 may be written

$$ (s_1 + l_0)^2 = (x_4 - x_1 + \Delta x_4 - \Delta x_1)^2 + (x_5 - x_2 + \Delta x_5 - \Delta x_2)^2 \tag{5.1} $$

In differential form this becomes

$$ (s_1 + l_0)ds_1 = (x_4 - x_1 + \Delta x_4 - \Delta x_1)(d\Delta x_4 - d\Delta x_1) + (x_5 - x_2 + \Delta x_5 - \Delta x_2)(d\Delta x_5 - d\Delta x_2) \tag{5.2} $$

or

$$ ds_1 = \frac{(x_4' - x_1')}{l'} (d\Delta x_4 - d\Delta x_1) + \frac{(x_5' - x_2')}{l'} (d\Delta x_5 - d\Delta x_2) \tag{5.3} $$

Using Equations 3.2

$$ ds_1 = \cos \alpha' (d\Delta x_4 - d\Delta x_1) + \sin \alpha' (d\Delta x_5 - d\Delta x_2) \tag{5.4a} $$
The differential form of Equations 3.10b and 3.10c is

\[ ds_2 = d \Delta x_3 - d \alpha' \]  \hspace{1cm} (5.4b)

\[ ds_3 = d \Delta x_6 - d \alpha' \]  \hspace{1cm} (5.4c)

But from Figure 3.2

\[ \alpha' = \arctan \left( \frac{x_5' - x_2'}{x_4' - x_1'} \right) \]  \hspace{1cm} (5.5)

The differential form of \( \alpha' \) is

\[ d \alpha' = \cos \alpha' \left( d \Delta x_5 - d \Delta x_2 \right) - \sin \alpha' \left( d \Delta x_4 - d \Delta x_1 \right) \]  \hspace{1cm} (5.6)

Therefore, the differential form of the compatibility equations becomes

\[
\begin{bmatrix}
-\cos \alpha' & -\sin \alpha' & 0 & \cos \alpha' & \sin \alpha' & 0 \\
-\sin \alpha' & \cos \alpha' & 1 & \sin \alpha' & -\cos \alpha' & 0 \\
\sin \alpha' & \cos \alpha' & 0 & \sin \alpha' & -\cos \alpha' & 1
\end{bmatrix}
\begin{bmatrix}
d \Delta x
\end{bmatrix}
\]

\hspace{1cm} (5.7)

Comparison with the equilibrium matrix as given on page 20 shows that the differential form of the compatibility relationships is the transpose of the equilibrium matrix. That is,

\[ ds = a^T d \Delta x \]  \hspace{1cm} (5.8)
5.3 Derivative of the Equilibrium Equations. To establish this derivative, certain relationships are needed. By noting that \( ds_1 = dl' \), Equation 5.4a may be used where \( dl' \) is needed. From Equations 3.2 and 3.4

\[
\cos \alpha' = \frac{x_4 - x_1 + \Delta x_4 - \Delta x_1}{l'}
\]  

(5.9)

Hence

\[
d\cos \alpha' = \frac{l'(d \Delta x_4 - d \Delta x_1) - (x_4' - x_1') dl'}{l'^2}
\]  

(5.10)

Substituting Equation 5.4a for \( dl' \) and using Equation 3.2b

\[
d\cos \alpha' = \frac{\sin^2 \alpha'}{1'} (d \Delta x_4 - d \Delta x_1) - \frac{\sin^2 \alpha'}{2_1'} (d \Delta x_5 - d \Delta x_2)
\]  

(5.11)

In matrix form

\[
d\cos \alpha' = \begin{bmatrix}
-\frac{\sin^2 \alpha'}{1'} & \frac{\sin^2 \alpha'}{2_1'} & 0 & \frac{\sin^2 \alpha'}{1'} & -\frac{\sin^2 \alpha'}{2_1'} & 0
\end{bmatrix} d\Delta x
\]  

(5.12a)

Similarly,

\[
d\sin \alpha' = \begin{bmatrix}
\frac{\sin^2 \alpha'}{2_1'} & -\cos^2 \alpha' & 0 & -\sin^2 \alpha' & \cos^2 \alpha' & 0
\end{bmatrix} d\Delta x
\]  

(5.12b)

\[
d\cos \alpha' = \begin{bmatrix}
\frac{\cos^2 \alpha'}{1'} & \frac{\sin^2 \alpha'}{12} & 0 & -\cos^2 \alpha' & -\sin^2 \alpha' & 0
\end{bmatrix} d\Delta x
\]  

(5.12c)
\[
\frac{d\sin\alpha'}{l'} = \begin{bmatrix}
\frac{\sin^2\alpha'}{l'^2} & -\frac{\cos^2\alpha'}{l'^2} & 0 & -\frac{\sin^2\alpha'}{l'^2} & \frac{\cos^2\alpha'}{l'^2} & 0
\end{bmatrix} d\Delta x
\] (5.12d)

These equations may be written

\[
d\cos\alpha' = A\ d\Delta x \quad (5.13a)
\]

\[
d\sin\alpha' = B\ d\Delta x \quad (5.13b)
\]

\[
d\frac{\cos\alpha'}{l'} = C\ d\Delta x \quad (5.13c)
\]

\[
d\frac{\sin\alpha'}{l'} = D\ d\Delta x \quad (5.13d)
\]

where the matrices A, B, C and D are given in Equations 5.12.

The differential form of Equation 3.18a is

\[
dR_1 = -\cos\alpha'\ dS_1 - d\cos\alpha'\ S_1 - \frac{\sin\alpha'}{l'}\ (dS_2 + dS_3) \\
- \frac{d\sin\alpha'}{l'}\ (S_2 + S_3) \quad (5.14)
\]

Substituting Equations 5.13 into this expression gives

\[
dR_1 = -\cos\alpha'\ dS_1 - \frac{\sin\alpha'}{l'}\ (dS_2 + dS_3) - Ad\Delta xS_1 \\
- Dd\Delta x(S_2 + S_3) \quad (5.15a)
\]

Similarly,

\[
dR_2 = -\sin\alpha'\ dS_1 + \frac{\cos\alpha'}{l'}\ (dS_2 + dS_3) - Bd\Delta xS_1 \\
+ Cd\Delta x(S_2 + S_3) \quad (5.15b)
\]
Therefore, the differential form of the equilibrium equations becomes

\[ \text{d}R = a \text{d}S + \text{d}aS \]  

(5.16)

where \( a \) is defined in Equations 3.22 and

\[
\text{da} = \begin{bmatrix}
-A \text{dA}x & -D \text{dA}x & -D \text{dA}x \\
-B \text{dA}x & C \text{dA}x & C \text{dA}x \\
0 & 0 & 0 \\
A \text{dA}x & D \text{dA}x & D \text{dA}x \\
B \text{dA}x & -C \text{dA}x & -C \text{dA}x \\
0 & 0 & 0 
\end{bmatrix}
\]  

(5.17)

5.4 Derivative of the Force-Deformation Equations.

The first equation from the expanded form of Equations 3.24 is

\[ S_1 = k_{11s1} \]  

(5.18)

In differential form

\[ \text{d}S_1 = \text{dk}_{11s1} + k_{11}\text{ds}_1 \]  

(5.19)
But $k_{11}$ is a function of $s_1$. Hence

\[ dS_1 = \left( s_1 \frac{dk_{11}}{ds_1} + k_{11} \right) ds_1 \]  

(5.20)

Differentiating Equation 3.26a

\[ \frac{dk_{11}}{ds_1} = - \left( \left( \frac{AE}{l_0} \right)^2 \left( \frac{AE}{l_0} A_R \right) s_1 \right)^{n-1} \left( 1 + \left( \frac{AE}{l_0} A_R \right) s_1 \right)^n \]  

(5.21)

Substituting Equations 3.26a and 5.21 into Equation 5.20

\[ dS_1 = k_{11}^* ds_1 \]  

(5.22a)

Similarly,

\[ dS_2 = k_{22}^* ds_2 + \frac{1}{2} k_{22}^* ds_3 \]  

(5.22b)

\[ dS_3 = \frac{1}{2} k_{33}^* ds_2 + k_{33}^* ds_3 \]  

(5.22c)

Where

\[ k_{11}^* = \frac{AE}{l_0} \frac{1}{\left( 1 + \left( \frac{AE}{l_0} A_R \right) s_1 \right)^n} \]  

(5.23a)

\[ k_{22}^* = \frac{4EI}{l_0} \frac{1}{\left( 1 + \left( \frac{4EI}{l_0} s_2 + \frac{2EI}{l_0} s_3 \right) \frac{1}{FR} \right)^n} \]  

(5.23b)
\[ k_{33}^* = \frac{\frac{4EI}{I_o}}{1 + \left( \frac{2EI}{I_o} \frac{s_2}{s_3} \frac{F_R}{F_{R}} \right)^n \left( \frac{1}{n+1} \right)} \]  

(5.23c)

In matrix form

\[
dS = \begin{bmatrix}
  k_{11}^* & 0 & 0 \\
  0 & k_{22}^* & \frac{1}{2}k_{22}^* \\
  0 & \frac{1}{2}k_{33}^* & k_{33}^*
\end{bmatrix} ds
\]

(5.24)

Or

\[ dS = dk \, ds \]  

(5.25)

5.5 Summary of Equations. The differential form of 3.13, 3.21 and 3.24 may then be written as

Compatibility \[ ds = a^T \, d\Delta x \]  

(5.8)

Equilibrium \[ dR = adS + daS \]  

(5.16)

Force-Deformation \[ dS = dk \, ds \]  

(5.25)

Substituting Equation 5.8 into Equation 5.25 gives

\[ dS = dk \, a^T \, d\Delta x \]  

(5.26)

Substituting Equations 3.24 and 5.26 into Equation 5.16 yields

\[ dR = a \, dk \, a^T \, d\Delta x + da \, k \, s \]  

(5.27)

where

\[ s = p + b \, \Delta x \]  

(3.13)
The loads are assumed to be proportional and may be represented in the form

\[ R = \lambda P \]  \hspace{1cm} (5.28)

where \( \lambda \) are the constants of proportionality and \( P \) is the load factor. Taking the derivative with respect to \( P \) yields

\[ \frac{dR}{dP} = \lambda \]  \hspace{1cm} (5.29)

Hence Equation 5.27 may be written

\[ \lambda = a \, \text{dk} \, a^T \, \dot{x} + da \, ks \]  \hspace{1cm} (5.30)

where

\[ \dot{x} = \frac{d \Delta x}{dP} \]  \hspace{1cm} (5.31)

and \( dP \) is accounted for in the right-hand term by changing the terms \( d \Delta x \) to \( x \) in Equations 5.17.

Letting

\[ K^* = a \, \text{dk} \, a^T \]  \hspace{1cm} (5.32a)

\[ K_a^* = da \, ks \]  \hspace{1cm} (5.32b)

Equation 5.30 may be written

\[ \lambda = K^* \, \dot{x} + K_a^* \]  \hspace{1cm} (5.33)

Solving this set of equations for \( x \) gives

\[ \dot{x} = K_a^{*-1} \lambda - K_a^{*-1} K^* \]  \hspace{1cm} (5.34)
5.6 Discussion of Solution. The equations symbolized by the matrix Equation 5.34 are a set of simultaneous ordinary first order nonlinear differential equations. However, the solution may not be obtained by direct integration of Equation 5.34 because the secondary stiffness matrix $K_a^s$ is a function of the differential equilibrium matrix $da$ which in turn is a function of the independent variable $x$ as shown by Equations 5.32b, 5.17 and 5.13. The vector $x$ is not separable from $da$. As a further complication, the equilibrium matrix $a$, the secondary compatibility matrix $p$, and the primary compatibility matrix $b$ are all functions of the displaced geometry and, hence, indirectly functions of the independent variable $x$. However, if the last term of Equation 5.34 is small compared with the first term, solution may still be obtained by holding the displaced geometry constant during each increment of integration. After completing the Runge-Kutta integration procedure, a first approximation for $x$ is thus obtained. This approximation may then be substituted into Equations 5.13. At this point, two choices are available. If the nonlinearities are not too great and the interval small enough, the integration may proceed to the next step using $x$ from the interval just completed. If the nonlinearities are too great for this procedure, the integration may be repeated for the interval just completed using the $x$ obtained as a first approximation
to the displaced geometry thus finding a second approxima-
tion to \( x \). This may in turn be substituted into Equation
5.13 and a third approximation calculated. This iterative
procedure may be repeated within the integration interval
until convergence to the correct values of \( x \) is achieved.

5.7 Reduction to Axial Case. Equations 5.8, 5.16 and
5.25 may be reduced to the case of axial forces only
by the procedure followed in paragraph 3.6, that is, elim-
inating elements not pertaining to axial forces and defor-
mations.

\[
\begin{align*}
\text{ds} &= \{ \text{ds}_1 \} \\
\mathbf{a}^T &= \begin{bmatrix}
-\cos \alpha' & -\sin \alpha' & \cos \alpha' & \sin \alpha'
\end{bmatrix} \\
b &= \mathbf{a}^T
\end{align*}
\]

\[
\dot{x} = \begin{bmatrix}
\frac{d \Delta x_1}{dP} \\
\frac{d \Delta x_2}{dP} \\
\frac{d \Delta x_4}{dP} \\
\frac{d \Delta x_5}{dP}
\end{bmatrix}
\]

\[
ds = \text{ds}_1
\]

\[
(5.35a, 5.35b, 5.35c, 5.35d, 5.35e)
\]
\[ S = \left\{ S_1 \right\} \]  

\[ \lambda = \left\{ \begin{array}{c} \frac{dR_1}{dP} \\ \frac{dR_2}{dP} \\ \frac{dR_4}{dP} \\ \frac{dR_5}{dP} \end{array} \right\} \]  

\[ dk = \left[ \begin{array}{c} k^* \\ 11 \end{array} \right] \]  

\[ da = \left[ \begin{array}{c} -A \chi \\ -B \chi \\ A \chi \\ B \chi \end{array} \right] \]  

\[ A = \left[ \begin{array}{cccc} -\sin^2 \alpha' & \sin 2 \alpha' \sin \alpha' & \sin 2 \alpha' & -\sin 2 \alpha' \sin \alpha' \\ \frac{\sin 2 \alpha'}{21} & \frac{\sin 2 \alpha'}{21} & \frac{\sin 2 \alpha'}{21} & \frac{\sin 2 \alpha'}{21} \end{array} \right] \]  

\[ B = \left[ \begin{array}{cccc} \sin^2 \alpha' - \cos^2 \alpha' & -\sin \alpha' & \sin 2 \alpha' \cos \alpha' & \cos \alpha' \end{array} \right] \]  

Using Equations 5.35, Equations 5.32, 5.33 and 5.34 apply to the axial case. A similar reduction may be made for the case of pure flexure.
5.8 Example Problem 5.1. To indicate the effect of iterating within the integration interval, Example Problem 4.1 was solved using the procedure just described. Table 5.1 gives results of 1, 3 and 5 iterations within the integration interval as compared with the results obtained by the iteration procedure used in Example Problem 4.1. These results are plotted in Figure 5.1. All results were checked by back substituting the calculated displacements back into the equations to find the applied loads. These computed loads were then compared with the input loads. These computed loads are also included in Table 5.1 to indicate the accuracy of the solution obtained using the various parameters and methods. It is interesting to note that in the most nonlinear area of the results, i.e., at the higher loads, the iteration procedure of Chapter 4 gave six significant figures while the numerical integration procedure gave only three. The apparent error in the integration procedure was probably the result of the large number of matrix inversions required. In the Runge-Kutta procedure this error accumulates from interval to interval while in the iterative procedure of Chapter 4 the error is eliminated at each interval of load by iterating until the correct result is achieved.
### Table 5.1
Comparison of Results, Iteration Procedure and Numerical Integration

<table>
<thead>
<tr>
<th>Load ( R_x )</th>
<th>1 Iter</th>
<th>Calc Load</th>
<th>3 Iter</th>
<th>Calc Load</th>
<th>5 Iter</th>
<th>Calc Load</th>
<th>Iteration Procedure</th>
<th>Calc Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.894</td>
<td>4.9810</td>
<td>1.903</td>
<td>4.9999</td>
<td>1.903</td>
<td>4.9999</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>14.982</td>
<td>34.6053</td>
<td>15.187</td>
<td>34.9954</td>
<td>15.189</td>
<td>34.9976</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>17.662</td>
<td>39.4825</td>
<td>17.960</td>
<td>39.9915</td>
<td>17.964</td>
<td>39.9963</td>
<td>17.8490</td>
<td>40.0000</td>
</tr>
<tr>
<td>45</td>
<td>20.611</td>
<td>44.3132</td>
<td>21.061</td>
<td>44.9826</td>
<td>21.071</td>
<td>44.9940</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>23.941</td>
<td>49.0610</td>
<td>24.675</td>
<td>49.9599</td>
<td>24.704</td>
<td>49.9881</td>
<td>24.5039</td>
<td>50.0000</td>
</tr>
<tr>
<td>55</td>
<td>27.861</td>
<td>53.6424</td>
<td>29.232</td>
<td>54.8844</td>
<td>29.348</td>
<td>54.9650</td>
<td>29.0556</td>
<td>55.0000</td>
</tr>
<tr>
<td>60</td>
<td>32.818</td>
<td>57.8200</td>
<td>36.305</td>
<td>59.4385</td>
<td>37.260</td>
<td>59.6741</td>
<td>37.3309</td>
<td>60.0000</td>
</tr>
</tbody>
</table>
Figure 5.1 Comparative Results of Iteration Procedure and Numerical Integration
5.9 Example Problem 5.2. In this problem two linear helical tension springs were supported by pinned connections as shown in Figure 5.2. The springs were connected at joint 2 to form a pinned connection. A vertical load was applied at joint 2.

![Figure 5.2 Example Problem 5.2, Spring System](image)

Stiffness factors for the springs were computed by plotting a load-elongation curve using an Instron Engineering Corporation Model TT.C Universal Testing Instrument. The springs were found to be linear throughout the range of loading used in this example. The load-elongation curves obtained are plotted in Figure 5.3. The stiffness factors for each of the springs is the slope of the appropriate curve. Hence

\[ K_A = \frac{AE}{I_0} A = 1.622 \text{ lbs/in} \]

\[ K_B = \frac{AE}{I_0} B = 2.280 \text{ lbs/in} \]
Figure 5.3 Example Problem 5.2, Load-Elongation Curves
The system was preloaded to the configuration shown in Figure 5.2 to eliminate initial stability problems and to separate the coils of the springs in order to insure linear action. Loads were applied using weights available in the laboratory in increments up to a maximum load of 20 pounds. Downward movement of the load was measured with reference to the framework carrying the two supports.

The problem was solved in the computer to obtain the theoretical results. Both a ten interval and a twenty interval solution were completed. Both results were identical to four significant figures indicating the result was accurate. Back substitution gave a check to three significant figures. The experimental and theoretical results are plotted in Figure 5.4. Comparison of the points plotted in this figure indicates excellent correlation of results.

The linear case was also solved in the computer and is plotted on Figure 5.4 for comparison with the nonlinear case. Note that the linear case is approximately fifty percent in error at two inches displacement and one hundred percent in error at three inches displacement.

As the load increased, the system springs approached the vertical and the system stiffness approached the sum of the individual spring stiffnesses. The slope of the curve in Figure 5.4 will approach this system stiffness as the
Figure 5.4 Example Problem 5.2, Load-Displacement Curve
load continues to increase. In fact, the curve will approach a tangent at the slope given by this system stiffness as an asymptote as shown in Figure 5.4.

5.10 Example Problem 5.3. In this example, a flexible simply supported beam was loaded as shown in Figure 5.5a. These loads might represent the weight of the beam idealized into concentrated loads at the node points with an additional load at joint 2. Practically, this beam system might be thought of as a draped roof system supporting an equipment load at joint 2. Because the supports are pinned, axial forces become increasingly important as the beam deflects and flexural forces decrease in importance.

The displacements and internal forces which are most significant are identified schematically in Figure 5.5b. The beam properties were assumed to be as follows:

\[
\begin{align*}
AE & = 100,000 \\
EI & = 100,000 \\
A_R & = 5,000 \text{ with } n = 10 \\
F_R & = 1,000 \text{ with } n = 10 \\
P_{\max} & = 500
\end{align*}
\]

No dimensions are given for these values since they are irrelevant; however, if dimensions were assumed it would be necessary that they be compatible. The values given are fictitious and are chosen only to demonstrate the action of a beam in which large deflections are important.
Figure 5.5 Example Problem 5.3, Flexible Beam
A diagram of the beam at $P/P_{\text{max}} = .3$, which is about the time the yield hinge forms at joint 2, and at $P/P_{\text{max}} = 1.0$ is given in Figure 5.5c. Figure 5.6 gives the deflections both when geometric changes are considered and when they are ignored. Figure 5.7 shows the moments for the above cases. The axial load is determined only for the case where geometry changes were considered since it is a function of the displaced geometry.

As the load on the beam increased, a yield hinge began to form at joint 2. The effect of this hinge forming is reflected by the irregularity of the deflection curves in Figure 5.6 and the axial load curve in Figure 5.7. As the hinge continued to develop, the beam became "kinked" at joint 2. The effect of this hinge was a reduction in moment at joints 3 and 4 and an increase in the axial load. In effect, the beam began to straighten out and approach the condition of a pin joint at joint 2 restrained in its rotation by the yield moment $F_R$. Hence the load carrying ability of the system changed from a beam acting only in flexure to a truss type action in which the main load carrying capacity was axial. Moment reversals were encountered at joints 3 and 4 and if the peak moment at these joints essentially equaled the yield moment, a non-conservative analysis would be necessary. This eventuality will not be considered here. If the load continued
Figure 5.6 Example Problem 5.3, Load-Deflection Curves
Figure 5.7 Example Problem 5.3, Load-Internal Force Curves
to increase until the maximum axial load was reached the beam would, of course, fail. The axial load for only one segment is given since for the relative small displacements encountered, the variation in axial load in the system is small.

Considerable error in linear theory is evident by comparison of the curves given in Figures 5.6 and 5.7; however, these errors only begin to become significant after a load of $P/P_{\text{max}} = 0.1$. Beyond this point the deflections exceed what would normally be expected by linear theory.
CHAPTER 6

DISCUSSION

The studies of structural systems included in this thesis and others cited in this thesis demonstrated several worthwhile points about very flexible structures that are not readily observed in linear analysis.

Flexible structures must be studied carefully to determine the relative magnitudes of the axial and flexural forces expected. In studies of cantilever beams it was found that axial stiffness values of $AE$ ten times greater than the flexural stiffness values of $EI$ were sufficient to produce numerical singularities in the stiffness matrix. Theoretically, these singularities did not exist but for a computer carrying only eight significant figures the relative magnitudes of the stiffness matrix elements were sufficiently different to cause a certain loss of significant figures. As a result, the stiffness matrix was singular. To demonstrate the sensitivity of the system to the relative magnitudes of $AE$ and $EI$, consider a typical cantilever as given in Figure 6.1. The expanded form of the stiffness matrix $K^*$ is given in Equation 5.32a. This matrix, with appropriate rows and columns lined out to
account for the reactions and zero displacements, is given in Figure 6.2.

Before this matrix may be inverted, its rank must be the same as its order, i.e., the smallest nonzero determinant must equal the order of the matrix. Theoretically, the rank of this matrix as given in Figure 6.2 is equal to its order. However, if the flexural stiffness elements are very small compared with the axial stiffness elements and the computations are such that not enough significant figures are carried to retain the effect of these flexural stiffness elements, they become, in effect, zero and the rank of the matrix reduces to unity. Hence, when flexural stiffness is very much smaller than axial stiffness, the matrix inversion process presents difficulties. In like manner, the matrix becomes numerically singular when the flexural stiffness is much greater than the axial stiffness.

Figure 6.1 A Typical Cantilever Beam
This point is further demonstrated by considering the method of analysis applied to the cantilever in Figure 6.1. In this method, the flexure segment is replaced by its chord as shown in Figure 3.2. If the axial stiffness is large numerically when compared with the flexural stiffness, the chord is inextensible and the deflected position of the node points becomes a function only of the angle of the chord $\alpha$ and the chord length $l$. If the chord is assumed inextensible, the system then reduces to a one degree of freedom system for each segment. Solution using the standard displacement method procedure would result in three degrees of freedom, $x_4$, $x_5$ and $x_6$. Hence, the governing equations which result from the expansion of Equation 3.31 involving these three degrees of freedom as independent variables must then be dependent for this case and, therefore, the stiffness matrix $K$ in Equation 3.29a is singular and cannot be inverted. This is precisely the conclusion drawn by considering the rank of the matrix.

Hence, structures which support their load mainly through axial action should be treated as axial systems and structures which carry their load essentially through beam action of the members should be treated as frame systems. If both axial and flexural forces are important in the system, they may be considered together provided the precision of the computations is such that enough
Figure 6.2 Differential Stiffness Matrix for Cantilever Beam

\[ K^* = \begin{bmatrix}
\frac{AE}{l_o} \cos^2 \alpha' + \frac{12EI}{l_o l_1'^2} \sin^2 \alpha' & \left(\frac{AE}{l_o} - \frac{12EI}{l_o l_1'^2}\right) \sin \alpha' \cos \alpha' & \frac{6EI}{l_o l_1'} \sin \alpha' \\
\left(\frac{AE}{l_o} - \frac{12EI}{l_o l_1'^2}\right) \sin \alpha' \cos \alpha' & \frac{AE}{l_o} \sin^2 \alpha' + \frac{12EI}{l_o l_1'^2} \cos^2 \alpha' - \frac{6EI}{l_o l_1'} \cos \alpha' & \frac{4EI}{l_o} \\
\frac{6EI}{l_o l_1'} \sin \alpha' & -\frac{6EI}{l_o l_1'} \cos \alpha' & \frac{4EI}{l_o}
\end{bmatrix} \]
significant figures are retained to insure accuracy of solution. The accuracy of solution for the theory used may always be checked by back substitution. However, this is not a check on the accuracy of the equations used for the solution. For example, back substitution for the linear case in Figure 5.4 checked throughout the range considered but the correct results clearly indicate the extent of the error. It should be remembered that the mathematical model may no longer represent the physical system outside the limitations placed on it by the assumptions used in its derivation. These limitations must be carefully considered in the method presented here.

Stability of the system must also be considered. If the supports in the spring system of Example Problem 5.2 are moved farther apart such that the center joint lies on the line of the supports, the system has no initial resistance to the loading applied and is, therefore, unstable. In experimenting with this system, it was found that the initial center deflection had to be at least four inches before enough significant figures could be carried by the computer to arrive at a solution. Use of dual precision routines would, of course, improve this situation.

Unstable conditions are easily determined if the load-deflection history is carefully plotted. Instability
in the system is generally indicated in the loading history when the structure no longer resists additional load. This means the structure has zero stiffness, or, inversely, infinite flexibility. Hence, when a load-deflection curve such as given in Figure 4.3, 4.5, or 5.4, approaches zero slope at any point, this implies that the structure is unstable under that particular load.
CHAPTER 7

CONCLUSIONS

A method has been presented for the analysis of structural systems which undergo large deflections. The approach used in the solution of the characteristic equations is a numerical procedure and the credibility of the results generally depends upon comparison of the numerical results with known results. Where these comparisons were made, the results indicated a high degree of reliability for the method. However, experimental results were only obtained for the case of axial loads and no comparison was made for systems in which flexure is important. Flexural systems are highly complex and little has been done in the analysis of such systems in which large deflections are developed. Consequently, experimental results are not generally available. An experimental program designed to acquire such results is beyond the scope of this thesis.

The iterative procedure used is a simple procedure mathematically and yields good results in a reasonable number of iterations. In all of the examples presented, the method converged to the correct result even though
some displaced configurations of the structural models were extreme. The numerical integration scheme involved a highly complex set of matrices and matrix manipulations. The method was further complicated because of the form of the equations and iteration within each interval of integration was required to obtain a solution. However, despite this complication, the method produced results identical with the iterative procedure. Both methods of solution were checked by back substitution of the computed displacements into the equations to produce the values of the member loads. These computed loads were then compared with the applied loads. Further checks were made in the integration procedure by decreasing the interval of integration and comparing results. In all cases, the solutions were checked in this manner and found to be accurate.

Other methods of solution may be applicable and, in fact, desirable depending on the availability of computer facilities and the type and complexity of the problem being solved. It is unlikely that any method will be free of the necessity of inverting the stiffness matrix since this is a basic procedure in solving for the displacements. Inversion procedures for digital computers are generally susceptible to truncation errors for ill-conditioned equations, and for large systems correcting procedures or double precision routines may be required.
It was demonstrated that numerical singularities may exist in this system of equations and a general study may be needed to determine a procedure for analyzing them. Instability and buckling are complex problems of great concern in large deflection problems. Instability was only briefly discussed but it is apparent from this discussion that some concept of the physical action of the structure is necessary before a complete understanding of the stability problem may be had. Nonlinear properties of structural shapes and connections and the interaction effects of axial, flexural and shear forces upon yield hinges which rotate through large angles needs investigating. Time rate of loading may be an important factor when the structure is forced to move through large displacements.

Physical systems which displace enough to require an analysis by the method presented here are not common at this time, and indeed may never be; however, with the rapid advance of technology and the new structural requirements which will accompany that advance, it is not unforeseeable that highly flexible structural systems may soon be required. It is felt that the method of analysis presented here, and the further elaborations which may be made upon it, will contribute to the solution of these systems.
APPENDIX

COMPUTER PROGRAM

Form $R - K^*$ and solve for $x$

Perform Runge-Kutta integration procedure

Compute new geometry

Is Runge-Kutta completed?

Yes

Increment Coordinates and compute final geometry

No

Back substitute to find $s, S$ and compute loads for check

Go to B

Print Headings and original coordinates, new coordinates and displacements

Print Headings and member information

Print Headings and $s, S$, computed $R$ and applied $R$

Are Integration Increments complete?

Yes

Go to A

No

Go to B

Go to D
*** HANSEN

*

**COMPILE FORTRAN, EXECUTE FORTRAN**

C NONLINEAR GEOMETRY-NONLINEAR MATERIAL
C RUNGE-KUTTA INTEGRATION
C AXIAL AND FLEXURAL FORCES

DIMENSION COOR(20, 16), DATA(20, 14), ZEROX(20), R(30, 2)
DIMENSION A(30, 30), BIGK(30, 30), C(30, 30), XK(30, 30)
DIMENSION BIGP(30), PSTAR(3C), XRK(30), DELPT(30)

C READ IN COORDINATES OF JOINTS AS JOINT NUMBER,
C X COORD., Y COORD.

C DATA....STORAGE OF MEMBER INFORMATION (M, 14)
C READ IN MEMBER INFORMATION AS MEM NUM, LAG JOINT,
C LEAD JOINT, A, E, I, AXIAL REF AND N, FLEX REF AND N
C ZEROX....ZERO DISPLACEMENT MATRIX (MX)
C R........EXTERNAL LOAD SUBSCRIPT AND VALUE MATRIX (MR, 2)
C IDENT....ANY IDENTIFYING NUMBER
C ICODE....PUNCH COL. 20

C 1 FOR RESULTS OF EACH INCREMENT ONLY
C 2 FOR MATRICES FORMED INTERIOR TO RUNGE-KUTTA, RESULTS
C 3 FOR ALL MATRICES FORMED, RESULTS
C 4 FOR FINAL MATRICES FORMED IN BACK SUB., RESULTS

C M........NUMBER OF MEMBERS
C MJ........NUMBER OF JOINTS
C MX........NUMBER OF ZERO DISPLACEMENTS
C MR........NUMBER OF KNOWN LOADS
C INC........NUMBER OF LOAD INCREMENTS
C ITERL....NUMBER OF ITERATIONS ON RUNGE-KUTTA
C VER........VERTICAL PROJECTION OF MEMBER
C HOR........HORIZONTAL PROJECTION OF MEMBER

6 READ 505, IDENT, ICODE, M, MJ, MX, MR, INC, ITEPL

505 FORMAT(8I10)

MM = 3*M
LL = 3*MJ
XINC=INC
IRK=0
ITER=0

D017I=1,MJ
D017J=1,16

17 COOR(I, J) = 0.
D019I=1,M
D019J=1,14

19 DATA(I, J) = 0.
D025I=1, LL
DELPT(I) = 0.

25 XPK(I) = 0.

C READ IN MEMBER INFORMATION
D010I=1,M
10 READ 509, (DATA(I,J), J=1,6), (DATA(I,J), J=11,14)
      509 FORMAT(3F3.0,F11.0,6F10.0)
C READ IN COORDINATES OF JOINTS
DO211=1,MJ
      READ 508, (COOR(I,J), J=1,3)
      508 FORMAT(3F10.0)
C READ IN ZERO DISPLACEMENT SUBSCRIPTS
      READ506*(ZER0X(I), I=1,MX)
      506 FORMAT(8F10.0)
C READ IN KNOWN LOAD SUBSCRIPTS AND LOADS
      READ 507, ((R(I,J), J=1,2), I=1,MR)
      507 FORMAT(8F10.0)
D0221=1,MR
22 R(1,2)=R(I,2)/XINC
C COMPUTE LENGTHS AND ANGLES
DO341=1,M
      J1=DATA(1,2)
      J2=DATA(1,3)
      HOR=COOR(J2,2)-COOR(J1,2)
      VER=COOR(J2,3)-COOR(J1,3)
      DATA(I,7)=SQRTF(HOR**2+VER**2)
      DATA(I,8)=DATA(I,7)
      IF(HOR)18*36*18
      18 TEMP2 = 3.14159/2.
      GO TO 39
      18 TEMP1=ABSF(VER/HOR)
      TEMP2=ATANF(TEMP1)
      39 IF(HOR)7,7,8
      7 IF(VER)9,9,5
      9 DATA(I,9)=TEMP2
      GO TO 34
      5 DATA(I,9)=6.28318-TEMP2
      GO TO 34
      8 IF(VER)15,29,29
      15 DATA(I,9)=3.14159-TEMP2
      GO TO 34
      29 DATA(I,9)=3.14159+TEMP2
      34 DATA(I,10)=DATA(I,9)
      4 ITER=ITER+1
      DO 72 IJK=1, ITERL
      2 IRK = IRK + 1
C FORM B TRANSPOSE IN COMPATIBILITY
DO113I=1,LL
DO113J=1,MM
113 A(I,J)=0.
      DO 13J=1,M
      J1=DATA(J,2)
      J2=DATA(J,3)
\[ I = 3*J \]
\[ I_1 = 3*J_1 \]
\[ I_2 = 3*J_2 \]
\[ A(I_1-2, I-2) = - \left( COGR(J_2,15) - COGR(J_1,15) \right) / DATA(J,P) \]
\[ A(I_1-1, I-2) = - \left( COGR(J_2,16) - COGR(J_1,16) \right) / DATA(J,8) \]
\[ A(I_1, I-1) = 1 \]
\[ A(I_2-2, I-2) = A(I_1-2, I-2) \]
\[ A(I_2-1, I-2) = A(I_1-1, I-2) \]
\[ A(I_2, I) = 1 \]

GO TO TC (13,115,115,33), ICODE

115 PRINT 616, IRK, ITER, IDENT

616 FORMAT(25H RUNGE-KUTTA ITERATION..., 15/
       25H RUNGE-KUTTA INTERVAL....15/
       20H IDENTIFICATION.....110)

PRINT 607

607 FORMAT(24H COMPATIBILITY MATRIX, 8/)

DC 14N=1,MM

14 PRINT 605, (A(L,N), L=1,LL)

604 FORMAT(5H ROW)

605 FORMAT(1P6E20.8)

C FORM B**XRK

33 DO119 I=1,MM

BIGP(I) = 0.

DC119 K=1,LL

119 BIGP(I) = BIGP(I) + A(K,I) * XRK(K)

GO TO (152,152,153,152), ICODE

153 PRINT 639

639 FORMAT(9H B**XRK/)

PRINT 605, (BIGP(N), N=1,MM)

C FORM P MATRIX

152 DO 125 J=1,M

I = 3*J

A(I,J) = DATA(J,9) - DATA(J,10)

A(I-1,J) = A(I,1)

125 A(I-2,J) = DATA(J,7) * (COSF(A(I,1)) - 1.)

GO TO (126,127,127,126), ICODE

127 PRINT 632

632 FORMAT(9H P MATRIX/)

PRINT 605, (A(N,1), N=1,MM)

C FORM P + B**XRK

126 DO 128 I=1,MM

126 BIGP(I) = BIGP(I) + A(I,1)

GO TO (111,111,110,111), ICODE

130 PRINT 633

633 FORMAT(8H P+B**XRK/)

PRINT 605, (BIGP(N), N=1,MM)

C FORM STIFFNESS MATRIX, K

C NOTE...CHECKS FOR EXPONENT UNDERFLOW ARE INCLUDED.

C THESE ARE NECESSARY FOR LARGE EXPONENTS

111 DO 100 I=1,MM

100 J=1,MM
100 XK(I,J) = 0.
DO 101 J=1,M
   I = 3*J
   EXP = DATA(J,12)
   TEMP1 = DATA(J,4)*DATA(J,5)/DATA(J,7)
   TEMP3 = ABSF(TEMP1/DATA(J,11)*BIGP(I-2))
   IF(TEMP3 - .1**(45./EXP))2C,81,81
   TEMP2 = 1.
   GO TO 88
81 TEMP2 = 1. + TEMP3**EXP
88 XK(I-2,I-2) = TEMP1/TEMP2**(1./EXP)
   EXP = DATA(J,14)
   TEMP1 = 4.*DATA(J,5)*DATA(J,6)/DATA(J,7)
   TEMP3 = ABSF((TEMP1*BIGP(I-1) + .5*EXP1*BIGP(I))
   1/DATA(J,13))
   IF(TEMP3 - .1**(45./EXP))104,114,114
   TEMP2 = 1.
   GO TO 118
104 TEMP2 = 1. + TEMP3**EXP
118 XK(I-1,I-1) = TEMP1/TEMP2**(1./EXP)
   XK(I-1,I ) = XK(I-1,I-1)/2.
   TEMP3 = ABSF(TEMP1*.5*BIGP(I-1) + TEMP1*BIGP(I))
   1/DATA(J,13))
   IF(TEMP3 - .1**(45./EXP))124,133,133
   TEMP2 = 1.
   GO TO 135
133 TEMP2 = 1. + TEMP3**EXP
135 XK(I ,I ) = TEMP1/TEMP2**(1./EXP)
101 XK(I ,I-1) = XK(I,I)/2.
   GO TO (27,99,99,27),ICODE
99 PRINT 608
608 FORMAT(20H0STIFFNESS MATRIX, K/
   DO16N=1,MM
   PRINT 604, N
   16 PRINT 605, (XK(N,L),L=1,MM)
C FORM THE DERIVATIVE OF EQUILIBRIUM, ADOT
27 DO105I=1,LL
   DO105J=1,MM
105 A(I,J) = 0.
   DO106J=1,M
       J1=DATA(J,2)
       J2=DATA(J,3)
       I = J*3
       I1=3*J1
       I2=3*J2
       EXP = DATA(J,8)
       TCOS=(COOR(J2,15)-COOR(J1,15))/EXP
       TSIN=(COOR(J2,16)-COOR(J1,16))/EXP
       TEMP1=TSIN**2/EXP
       TEMP2=TSIN*TCOS/EXP
       TEMP3=TCOS**2/EXP
       A(I1-2,I2-2) = TEMP1*COOR(J1,9) - TEMP2*COOR(J1,10)
1 \[ -\text{TEMP1} \times \text{COOR}(J2, 9) + \text{TEMP2} \times \text{COOR}(J2, 10) \]
1 \[ A(11-1,1-2) = -\text{TEMP2} \times \text{COOR}(J1, 9) + \text{TEMP3} \times \text{COOR}(J1, 10) \]
1 \[ +\text{TEMP2} \times \text{COOR}(J2, 9) - \text{TEMP3} \times \text{COOR}(J2, 10) \]
A (12-2,1-2) = -\text{A}(11-2,1-2)
A (12-1,1-2) = -\text{A}(11-1,1-2)
\text{TEMP1} = (\text{TCOS}^2 - \text{TSIN}^2) / \text{EXP}^2
\text{TEMP2} = 2 \times \text{TEMP2} / \text{EXP}
A (11-2,1-1) = -\text{TEMP2} \times \text{COOR}(J1, 9) + \text{TEMP1} \times \text{COOR}(J1, 10)
1 +\text{TEMP2} \times \text{COOR}(J2, 9) - \text{TEMP1} \times \text{COOR}(J2, 10)
A (11-1,1-1) = \text{TEMP1} \times \text{COOR}(J1, 9) + \text{TEMP2} \times \text{COOR}(J1, 10)
1 -\text{TEMP1} \times \text{COOR}(J2, 9) - \text{TEMP2} \times \text{COOR}(J2, 10)
A (12-2,1-1) = -\text{A}(11-2,1-1)
A (12-1,1-1) = -\text{A}(11-1,1-1)
106 \text{A}(11-2,1) = \text{A}(11-2,1-1)
A (11-1,1) = \text{A}(11-1,1-1)
A (12-2,1) = \text{A}(12-2,1-1)
107 \text{A}(12-1,1) = \text{A}(12-1,1-1)
\text{GO TO} (107, 108, 108, 107), \text{ICODE}
108 \text{PRINT} 630
630 \text{FORMAT}(13H\text{ADOT\ MATRIX/})
\text{DO} 109 \text{N}=1, \text{LL}
\text{PRINT} 604, \text{N}
109 \text{PRINT} 605, ( \text{A}(N,L)*, L=1, \text{MM})
\text{C FORM A DOT*K}
107 \text{DO} 1100 \text{I}=1, \text{LL}
\text{DO} 1100 \text{J}=1, \text{MM}
\text{C(I,J)=0.}
\text{DO} 1100 \text{K}=1, \text{MM}
110 \text{C(I,J)=C(I,J) + A(I,K) * XK(K,J)}
\text{GO TO} (129, 129, 112, 129), \text{ICODE}
112 \text{PRINT} 609
609 \text{FORMAT}(17H\text{PRODUCT, A DOT*K/)}
\text{DO} 23 \text{N}=1, \text{LL}
\text{PRINT} 604, \text{N}
23 \text{PRINT} 605, ( \text{C}(N,L)*, L=1, \text{MM})
\text{C FORM ADOT*K*(P+B*XRK) = BIGK(A)}
129 \text{DO} 1160 \text{I}=1, \text{LL}
\text{XRK(I) = 0.}
\text{DO} 1160 \text{K}=1, \text{MM}
116 \text{XRK(I) = XRK(I) + C(I,K) * BIGP(K)}
\text{GO TO} (154, 154, 155, 154), \text{ICODE}
155 \text{PRINT} 640
640 \text{FORMAT}(26H\text{A DOT*K*(P+J*XRK) = BIGK(A/)})
\text{PRINT} 605, ( \text{XRK}(N)*, N=1, \text{LL})
\text{C FORM EQUILIBRIUM MATRIX}
154 \text{DO} 1340 \text{I}=1, \text{LL}
\text{DO} 1340 \text{J}=1, \text{MM}
134 \text{A(I,J) = 0.}
\text{DO} 24 \text{J}=1, \text{M}
\text{J1=DATA(J,2)}
\text{J2=DATA(J,3)}
\text{I = 3*J}
I1 = 3 * J1
I2 = 3 * J2
EXP = DATA(J, 8)
A(I1 - 2, I - 2) = -(COOR(J2, 15) - COOR(J1, 15)) / EXP
A(I1 - 1, I - 2) = -(COOR(J2, 16) - COOR(J1, 16)) / EXP
A(I2 - 2, I - 2) = -A(I1 - 2, I - 2)
A(I2 - 1, I - 2) = -A(I1 - 1, I - 2)
A(I1 - 2, I - 1) = A(I1 - 1, I - 2) / EXP
A(I1 - 1, I - 1) = -A(I1 - 2, I - 2) / EXP
A(I1, I - 1) = 1.
A(I2 - 2, I - 1) = -A(I1 - 2, I - 1)
A(I2 - 1, I - 1) = -A(I1 - 1, I - 1)
A(I1 - 2, I) = A(I1 - 2, I - 1)
A(I1 - 1, I) = A(I1 - 1, I - 1)
A(I2 - 2, I) = A(I2 - 2, I - 1)
A(I2 - 1, I) = A(I2 - 1, I - 1)

24 A(I2, I) = 1.
GO TO (156, 157, 158, 159), ICODE

157 PRINT 641
641 FORMAT(22HOEQUILIBRIUM MATRIX, A/)
DO 158 N = 1, L
PRINT 604, N
158 PRINT 605, (A(N, L), L = 1, M)

C FORM DERIVATIVE OF STIFFNESS MATRIX, K(DOT)

156 DO 121 J = 1, M
I = 3 * J
EXP = DATA(J, 12)
TEMP1 = DATA(J, 4) * DATA(J, 5) / DATA(J, 7)
TEMP3 = ABSF(TEMP1 / DATA(J, 11) * BIGP(I - 2))
IF(TEMP3 - .1**(45. / EXP))96, 97, 97
96 TEMP2 = 1.
GO TO 54
97 TEMP2 = 1. + TEMP3**EXP
54 XK(I - 2, I - 2) = TEMP1 / TEMP2**(1. + 1. / EXP)
EXP = DATA(J, 14)
TEMP1 = 4.*DATA(J, 5) * DATA(J, 6) / DATA(J, 7)
TEMP3 = ABSF((TEMP1 * BIGP(I - 1) + .5*TEMP1*BIGP(I))
1/DATA(J, 13))
IF(TEMP3 - .1**(45. / EXP))136, 137, 137
136 TEMP2 = 1.
GO TO 138
137 TEMP2 = 1. + TEMP3**EXP
138 XK(I - 1, I - 1) = TEMP1 / TEMP2**(1. + 1. / EXP)
XK(I - 1, I) = XK(I - 1, I - 1) / 2.
TEMP3 = ABSF((TEMP1 * .5*BIGP(I - 1) + TEMP1*BIGP(I))
1/DATA(J, 13))
IF(TEMP3 - .1**(45. / EXP))139, 140, 140
139 TEMP2 = 1.
GO TO 141
140 TEMP2 = 1. + TEMP3**EXP
141 XK(I, I) = TEMP1 / TEMP2**(1. + 1. / EXP)
121 XK(I, I - 1) = XK(I, I) / 2.
GO TO (122,123,123,122),ICODE
123 PRINT 638
638 FORMAT(13H0K DOT MATRIX/)
DO131N=1,MM
PRINT 604,N
131 PRINT 605,(XK(N,L),L=1,MM)
C FORM A*K DOT
122 DO132 I=1,LL
DO132 J=1,MM
C(I,J) = 0.
DO 132 K=1,MM
132 C(I,J) = C(I,J) + A(I,K) * XK(K,J)
GO TO (159,159,160,15y),ICODD
160 PRINT 642
642 FORMAT(7HOA*KDOT/)
DC161N=1 ,LL
PRINT 604*N
161 PRINT 605 , ( C(N,L),L=1,MM)
C FORM BIGK = A*KDOT*A(TRANSPOSE)
159 DO1431=1,LL
DO143 J=1,LL
BIGK(1,J) = 0.
DO143 K=1,MM
143 BIGK(I,J) = BIGK(I,J) + C(I,K) * A(J,K)
GO TO (117,117,145,117),ICODE
145 PRINT 636
636 FORMAT(43H0STIFFNESS MATRIX, BIGK=A*KDOT*A(TRANSPOSE)/)
DO146N=1,LL
PRINT604, N
146 PRINT 605 , ( BIGK(N,L),L=1,LL)
C ELIMINATE COLUMNS IN BIGK CORRESPONDING TO ZERO DISPL.
117 K=0
N=0
DO31J=1,LL
IF(MX-N+1)26,3,3
3 XJ=J
IF(ZEROX(N+1)-XJ)2d,26,28
26 N=N+1
K=K+1
GO TO 31
28 JJ=J-K
DO301=1,LL
BIGK(I, JJ)=BIGK(I,J)
30 CONTINUE
31 CONTINUE
C BIGK IS NOW (LL, LL-MX)
C FORM STAR MATRICES, C IS REUSED AS BIGK STAR
N =0
IC=0
MX1=LL-MX
DO351=1,LL
IF(MR-N+1)35,163,163
163 XI=I
   IF(R(N+1,1)-XI)=35,36,35
36 IC=IC+1
   PSTART(I) = XRTK(I)
   DO53J=1,MX1
37 C(IC+J)=EIGK(I,J)
   N=N+1
35 CONTINUE
   GO TO(46,80,80,46),ICODE
80 PRINT 612
612 FORMAT(1H0STAR/)
   DO70I=1,MX1
   PRINT 604,1
70 PRINT 605, (C(I,J),J=1,MX1)
   PRINT 637
637 FORMAT(7H0P STAR//)
   PRINT 605, (PSTART(I),I=1,MX1)
C
SOLVE FOR X STAR BY ELIMINATION
    MX1+1
C
PLACE R STAR - P STAR BEHIND BIGK STAR
   DO40I=1,MX1
40 C(I,N) = R(I,2) - PSTART(I)
   NN=MX1-1
   DO42I=1,NN
42 C(I,N) = C(I,N) - C(I,N)*D
   C(I,N) = C(I,N)/C(1,1)
   C(1,1) = 1.
   DO43I=1,NN
43 C(I,N) = C(I,N) - C(I,N)*D
   GO TO (186,187,187,186),ICODE
187 DO1651=1,MX1
   PRINT 604,1
185 PRINT 605, (C(I,J),J=1,N)
186 K=0
   N=N
C
ADD ZEROX(I)=C* INTO X STAR
   DO55I=1,LL
55 C(I,J)=C(J,J)*(I=I+1
56 C(I,1)=0.
\[ N = N + 1 \]
\[ K = K + 1 \]
\[ C(1,1) = C(K, M + 1) \]

55 CONTINUE

GO TO (120, 11, 11, 120) ICODE

11 PRINT 611

611 FORMAT ('OH OX DOT IN RUNGE-KUTTA/')

PRINT 605, (C(N,1), N=1,LL)

C

PERFORM RUNGE-KUTTA INTEGRATION PROCEDURE

120 D052 I = 1, MJ

N = 3*I

GO TO (84, 85, 86, 87), IRK

84 XRK(N-2) = COOR(1, 6) + C(N-2,1)/2.
XRK(N-1) = COOR(1,7) + C(N-1,1)/2.
XRK(N) = COOR(1,8) + C(N,1)/2.
COOR(I,15) = COOR(I,4) + C(N-2,1)/2.
COOR(I,16) = COOR(I,5) + C(N-1,1)/2.
COOR(I,12) = C(N-2,1)/6.
COOR(I,13) = C(N-1,1)/6.
COOR(I,14) = C(N,1)/6.

GO TO 52

85 XRK(N-2) = COOR(1, 6) + C(N-2,1)/2.
XRK(N-1) = COOR(1,7) + C(N-1,1)/2.
XRK(N) = COOR(1,8) + C(N,1)/2.
COOR(I,15) = COOR(I,4) + C(N-2,1)/2.
COOR(I,16) = COOR(I,5) + C(N-1,1)/2.
COOR(I,12) = COOR(I,12) + C(N-2,1)/3.
COOR(I,13) = COOR(I,13) + C(N-1,1)/3.
COOR(I,14) = COOR(I,14) + C(N,1)/3.

GO TO 52

86 XRK(N-2) = COOR(1, 6) + C(N-2,1).
XRK(N-1) = COOR(1,7) + C(N-1,1).
XRK(N) = COOR(1,8) + C(N,1).
COOR(I,15) = COOR(I,4) + C(N-2,1).
COOR(I,16) = COOR(I,5) + C(N-1,1).
COOR(I,12) = COOR(I,12) + C(N-2,1)/3.
COOR(I,13) = COOR(I,13) + C(N-1,1)/3.
COOR(I,14) = COOR(I,14) + C(N,1)/3.

GO TO 52

87 COOR(I,12) = COOR(I,12) + C(N-2,1)/6.
COOR(I,13) = COOR(I,13) + C(N-1,1)/6.
COOR(I,14) = COOR(I,14) + C(N,1)/6.
COOR(I,15) = COOR(I,4).
COOR(I,16) = COOR(I,5).
COOR(I,9) = COOR(I,12).
COOR(I,10) = COOR(I,13).
COOR(I,11) = COOR(I,14).
XRK(N-2) = COOR(I,6).
XRK(N-1) = COOR(I,7).
XRK(N) = COOR(I,8).

52 CONTINUE
GO TO (162, 32, 32, 162), ICODE
32 PRINT 643
643 FORMAT(24H0DELTA X FOR RUNGE-KUTTA/) PRINT 635, (XRK(N), N=1,LL) PRINT 644
644 FORMAT(31HOCOCR MATRIX FOLLOWING STMT. 52/) DO164N=1,MJ
164 PRINT 605, (COOR(N,L), L=1,16) PRINT 645
645 FORMAT(38HOLENGTHS AND ANGLES FOLLOWING STMT. 52/) DO165N=1,M
165 PRINT 605, (DATA(N,L), L=7,10)
C COMPUTE NEW LENGTHS AND ANGLES
162 DO121=1,M
J1=DATA(1,2) J2=DATA(1,3) HOR=COOR(J2,15) - COOR(J1,15) VER=COOR(J2,16) - COOR(J1,16) DATA(1,8)=SQRT(HGR**2+VER**2)
IF(HOR)53,62,53
62 TEMP2 = 3.14159/2.
GO TO 63
53 TEMP1=ABSF(VER/HOR) TEMP2=ATANF(TEMP1) 63 IF(HOR)64,64,65
64 IF(VER)66,66,67
66 TEMP3=TEMP2 GO TO 12
67 TEMP3=6.28318-TEMP2 GO TO 12
65 IF(VER)69,73,73
69 TEMP3=3.14159-TEMP2 GO TO 12
73 TEMP3=3.14159+TEMP2 GO TO 12
12 DATA(1,10)=TEMP3 GO TO (2, 2, 2, 72), IRK
72 IRK=O
D0711=1,MJ N = 3*I COOR(1,4)=COOR(1,4)+COOR(I,12) COOR(1,5)=COOR(1,5)+COOR(I,13) COOR(1,6)=COOR(1,6)+COOR(I,12) COOR(1,7)=COOR(1,7)+COOR(I,13) COOR(1,8)=COOR(1,8)+COOR(I,14) COOR(1,15) = COOR(1,4) COOR(1,16) = COOR(1,5) XRK(N-2) = COOR(I,6) XRK(N-1) = COOR(I,7) 71 XRK(N ) = COOR(I,8)
C COMPUTE FINAL LENGTHS AND ANGLES
DO891=1,M
J1=DATA(1,2)
J2=DATA(1,3)
HGR=COOR(J2,4) - COOR(J1,4)
VER=COOR(J2,5) - COOR(J1,5)
DATA(1,8)=SQRTF(HGR**2+VER**2)
IF(HOR)68,68
74 TEMP2 = 3.14159/2.
GO TO 77
68 TEMP1=ASSF(VER/HOR)
TEMP2=ATANF(TEMP1)
77 IF(HOR)90,90,91
90 IF(VER)92,92,93
92 TEMP3=TEMP2
GO TO 89
93 TEMPL3=6.28318-TEMP2
GO TO 89
91 IF (VER)94,94,95
94 TEMP3=3.14159-TEMP2
GO TO 89
95 TEMP3=3.14159+TEMP2
89 DATA(1,10)=TEMP3
GO TO (166,166,167,166),ICODE
167 PRINT 646
646 FORMAT(31HCOR MATRIX FOLLOWING STMT. 76/)
DO168N=1,MJ
PRINT 604,N
168 PRINT 605, (CGR(N,L),L=1,16)
PRINT 647
647 FORMAT(36HLENGTHS AND ANGLES FOLLOWING STMT. 76/)
DO169N=1,M
PRINT 604,N
169 PRINT 605, (DATA(N,L),L=7,1C)
C ZERO RESULT STORAGE SPACES
166 DO831=1,LL
DO83J=1,5
83 C(I,J)=0.*
C PLACE DISPLACEMENTS IN C(I,1)
DO581=1,MJ
J=3*I
C(J-2,1)=COOR(I,6)
C(J-1,1)=COOR(I,7)
58 C(J,1)=COOR(I,8)
C FORM B TRANSPOSE IN COMPATIBILITY
DO 441=1,LL
DO 44J=1,MM
44 A(I,J)=0.*
DO 47J=1,M
J1=DATA(J,2)
J2=DATA(J,3)
I = 3*J
11=J1*3
I2=J2*3
EXP = DATA(J,8)
A(11-2,1-2) = -(COOR(J2,4) - COOR(J1,4))/EXP
A(11-1,1-2) = -(COOR(J2,5) - COOR(J1,5))/EXP
A(11,1-1) = 1.*
A(I2,1,1) = A(I2,2,1)
A(I2,1-2) = A(I2-1,1-2)

GO TO (188,188,188,189),ICODE
189 PRINT 65?, ITER, IDENT
652 FORMAT(25H1RUNG-KUTTA INTERVAL....15/
1 20H IDENTIFICATION....110)

GO TO (170,170,171,171),ICODE
171 PRINT 648
648 FORMAT(30HFINAL COMPATIBILITY MATRIX, B/)
D0172N=1,MM
PRINT 604*, N
172 PRINT 605, (A(I,2)*L=1,LL)
C COMPUTE INTERNAL DEFORMATIONS
170 DO591=1,MM
C(I,2) = 0.
DO59K=1,LL
59 C(I,2) = C(I,2) + A(K,1)*C(K,1)
C FORM P MATRIX
DO 48J=1,M
I = 3*J
A(I,1) = DATA(J,9) - DATA(J,10)
A(I-1,1) = A(I,1)
48 A(I-2,1) = DATA(J,7) * (COSF(A(I,1)) - 1.*
GO TO (173,173,174,174),ICODE
174 PRINT 649
649 FORMAT(15HFINAL P MATRIX/)
PRINT 605, (A(L,1)*L=1,MM)
173 DO601=1,MM
60 C(I,2) = C(I,2) + A(I,1)
C FORM STIFFNESS MATRIX, K
DO 98 J=1,M
I = 3*J
EXP = DATA(J,12)
TEMP1 = DATA(J,4)*DATA(J,5)/DATA(J,7)
TEMP3 = ABSF(TEMP1/DATA(J,11)*C(I-2,2))
IF(TEMP3 - .1**(45./EXP))102,133,103
102 TEMP2 = 1.
GO TO 149
103 TEMP2 = 1.* + TEMP3**EXP
149 XK(I-2,1-2) = TEMP1/TEMP2**((1./EXP)
EXP = DATA(J,14)
TEMP1 = 4.*DATA(J,5)*DATA(J,6)/DATA(J,7)
TEMP3 = ABSF(TEMP1*C(I-1,2) + .5*TEMP1*C(I,2))
1/DATA(J,13)
IF(TEMP3 - .1**(45./EXP))150,142,142
150 TEMP2 = 1.*
Go To 151

142 TEMP2 = 1. + TEMP3**EXP
151 XK(I-1, I-1) = TEMPl/TEMP2**(1./EXP)
   XK(I-1, I ) = XK(I-1, I-1)/2.
   TEMP3 = ABSF((TEMPl*.5*0(I-1,2) + TEMPl*0(I,2) )
   1/UATA(J,13))
   IF TEMP3 = .1**(45./EXP))144,147,148
144 TEMP2 = 1.
   Go To 148
147 TEMP2 = 1. + TEMP3**EXP
148 XK(I, I ) = TEMPl/TEMP2**(1./EXP)
98 XK(I, I-1) = XK(I, I)/2.
   Go To (175, 175, 176, 176), ICODE
176 PRINT 650
650 FORMAT(26H0FINAL STIFFNESS MATRIX, K/) 
   DO177 N=1,MM 
   PRINT 604,N
177 PRINT 605, (XK(N,L), L=1,MM) 
C
   COMPUTE INTERNAL LOADING
175 DO61 I=1,MM 
   C(I,3)=0.
   DO61 K=1,MM 
61 C(I,3)=C(I,3)+XK(I,K)*C(K,2)
C
   FORM EQUILIBRIUM MATRIX
   DO 491 = 1 , LL 
   DO 49 J=1 , MM 
49 A(I, J)=0.
   DO 51 J=1,M 
   J1=DATA(J,2)
   J2=DATA(J,3)
   I = 3*J
   I1=J1*3
   I2=J2*3
   EXP = DATA(J,8)
   A(I-I-2, I-2)=- (COOR(J2,4) - COOR(J1,4))/EXP
   A(I-I-1, I-2)=- (COOR(J2,5) - COOR(J1,5))/EXP
   A(I-2-2, I-2) = -A(I-I-2, I-2)
   A(I-2-1, I-2) = -A(I-I-1, I-2)
   A(I-I-2, I-1) = A(I-I-1, I-2)/EXP
   A(I-I-1, I-1) = -A(I-I-2, I-2)/EXP
   A(I1 , I-1) = 1.
   A(I-2, I-1) = -A(I1-2, I-1)
   A(I-2, I-1) = -A(I1-1, I-1)
   A(I1-2, I ) = A(I1-2, I-1)
   A(I1-1, I ) = A(I1-1, I-1)
   A(I-2, I ) = A(I-2, I-1)
   A(I-2, I ) = A(I-2, I-1)
   A(I1 , I ) = 1.
   Go To (178, 178, 179, 179), ICODE
179 PRINT 651
651 FORMAT(28H0FINAL EQUILIBRIUM MATRIX, A/) 
   DO180 N=1,LL
PRINT 604, N
180 PRINT 605, (A(N,L),L=1,MM)
C COMPUTE EXTERNAL LOADING
178 DO50I=1,LL
C(I,4)=0.
DO50K=1,MM
5C C(I,4)=C(I,4)+A(I,K)*C(K,3)
C ARRANGE INPUT LOADS FOR PRINTING
DO76I=1,MR
76 DELPT(1)=DELPT(1)+K(1,2)
K=0
N=0
DO75I=1,LL
IF(MX-N+1)182,183,183
183 XI=1
184 C(I,5)=0.
N=N+1
GO TO 75
182 K=K+1
C(I,5) = DELPT(K)
75 CONTINUE
PRINT 613, ITER
613 FORMAT(41H NONLINEAR GEOMETRY, NONLINEAR MATERIAL / 126H AXIAL AND FLEXURAL FORCES/24H RUNGE-KUTTA INTEGRATION/ 226H INCREMENTS COMPLETED......15)
PRINT 615, IDENT
615 FORMAT(20H IDENTIFICATION......!!!/)
PRINT 620
620 FORMAT(40H UCINITNT , INITIAL , 140H FINAL , DISP, , 240H LACEMENTS , JNT)
PRINT 621
621 FORMAT(40H NUM COORDINATES , , 140H COORDINATES , , 240H ROTATIONS , )
PRINT 622
622 FORMAT(40H X , Y , 140H X , Y , , 240H Y ,)
DO78N=1,MJ
TEMP1 = CCOR(N,6)*57.2958
78 PRINT 623,(CCOR(N,J),J=1,7),TEMP1
623 FORMAT(F5.0,7F15.5)
PRINT 624
624 FORMAT(40H UCOREM , LAG LEAD AREA , E , 140H INERTIA L , L ALPH A , 240H ALPH A , AXIAL FLEXURAL )
PRINT 625
625 FORMAT(40H NUM JNT JNT , 140H ZERO , PRIME , ZERO , , 240H PRIME , REF N , REF N /)
DC79N=1,N
TEMP1 = DATA(N,9)*57.2958
TEMP2 = DATA(N,10)*57.2958
79 PRINT 626, (DATA(N,J),J=1,8),TEMP1,TEMP2,
1(DATA(N,J),J=11,14)
PRINT 627
627 FORMAT(4OH INPUT,40H INTERNAL,140H INTERNAL)
PRINT 628
628 FORMAT(4OH LOADS,140H DEFORMATIONS,140H LOADS)
1D081=1,LL
82 PRINT629, I, C(I,2), C(I,3), C(I,4),C(I,5)
629 FORMAT(I5,F15.5,F15.5,F15.5)
IF(ITER-INC 14,6,6
END
REFERENCES


