# ANALYSIS OF SIMPLY SUPPORTED ELASTIC BEAM COLUMNS WITH LARGE DEFLECTIONS BY THE RITZ METHOD

by

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TABLE OF CONTENTS

## NOMENCLATURE

| | |
|---|---|
| A | Constant |
| B | $\pi/L$ |
| E | Modulus of elasticity |
| F | Potential Energy per unit length, $\frac{1}{2}EI\theta'^2 - P(1-\cos\theta) + V\sin\theta$ |
| I | Moment of inertia of the cross section |
| i, j, k | Integers |
| L | Length of beam column |
| M | Resultant internal moment in the beam |
| m, n, r | Integers |
| P | Horizontal component of internal force in the beam |
| $P_e$ | Euler's Load $EIB^2$ |
| p | Intensity of distributed horizontal load |
| Q | Vertical Load |
| q | Intensity of distributed vertical load |
| s | Distance along elastic axis of the beam column |
| t | Number of intervals beam is divided into |
| U | Total potential energy of the elastic system |
| V | Vertical component of internal force in the beam |
| $\bar{X}$ | Distance between end points of the beam column |
| x | Horizontal displacement of a point s on elastic axis |

| | |
|---|---|
| $y$ | Vertical displacement of a point s on elastic axis |
| $\theta$ | Angle of inclination of elastic axis at a point s |
| $\theta_0$ | Angle of inclination of elastic axis at $s = 0$ |

## LIST OF ILLUSTRATIONS

## LIST OF COMPUTER PROGRAMS

## ABSTRACT

This thesis investigates the simply supported beam column with large displacements and under arbitrary loading. The potential energy equation and constraints for the system are determined. The Ritz method is used to minimize the potential energy equation. A cosine series and a power series are the assumed Ritz solutions. The differential equation for the system is derived by applying the Euler Lagrange minimizing equation from the calculus of variations to the potential energy equation. The differential equation is rewritten in terms of central finite differences. The solution of the central finite difference equations is determined by conversion of the two point boundary value problem to an initial value problem. The Ritz solution is used for initial values for the central finite difference solution. A general set of Fortran computer programs is presented for solving any loading case of the beam column. A sample load case of the beam column is analyzed using the techniques developed in this thesis.

## CHAPTER I

## INTRODUCTION

### 1.1 General

Little has been published treating the case of a simply supported beam column with large displacements. In 1960 a general approach was published by reference 1 for the formulation and solution of the beam column with large displacements. This thesis is concerned with improvements and extensions of reference 1. The objectives of this thesis are:

(a) to develop a rigorous derivation of the cosine series Ritz solution,

(b) to derive a power series Ritz solution,

(c) to develop a finite difference procedure for the solution of the nonlinear differential equation for a beam column,

(d) to develop a set of Fortran computer programs that may be used to determine the shear, moment, slope, and deflection of a simply supported beam column subjected to arbitrary loads,

(e) to analyze a typical case demonstrating the mathematical techniques and procedures developed in this thesis.

## 1.2 Background Material

The general solution of the nonlinear differential equation for a beam column in static equilibrium is not known. With the advent of the high speed digital computer, numerical techniques which were previously impractical because of computational length and complexity can now be used to analyze the beam column.

The system analyzed in this paper will be limited to a simply supported elastic beam column with large displacements. The following assumptions are made:

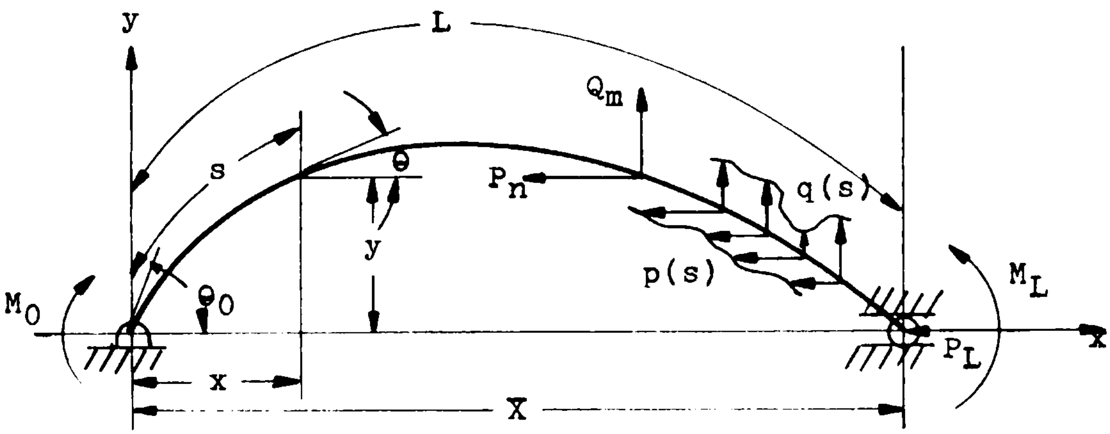
(a) the applied loads and slenderness ratio are restricted so that the induced stresses in the structural member are within the elastic range of the material,

(b) the magnitude and direction of the loads are independent of structural deflections,

(c) the modulus of elasticity of the structure is constant,

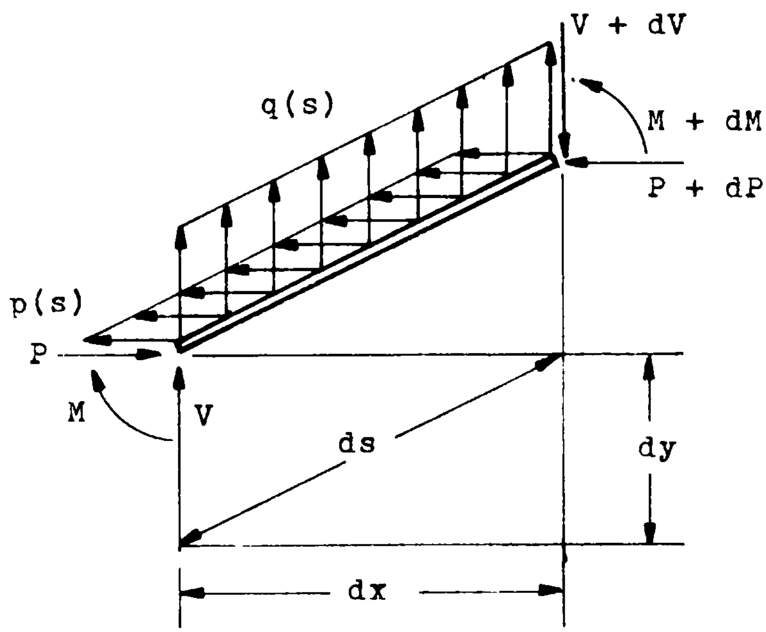
(d) the moment of inertia of the cross section of the structure is constant,

(e) the applied loads are arbitrary as shown in Figure 1.1,




Configuration of Beam Column with Arbitrary Loading

Figure 1.1




Free Body Diagram of an Element

Figure 1.2

(f) all loads, moments, and displacements are in the plane of the beam column, i.e., the x-y plane.

In order to insure single valued functions and for ease of analysis, the s-$\theta$ coordinate system is used. A sketch of the structure with its applied loads and coordinate system is shown in Figure 1.1. The independent variable s is the distance along the elastic axis and the dependent variable $\theta$ is the angle between the x axis and the tangent to the elastic axis. The relationships between x, y, s and $\theta$ are:

$$dy = \sin \theta \, ds$$

$$dx = \cos \theta \, ds \tag{1.1}$$

The above relationships are illustrated graphically in Figure 1.2.

The potential energy of the beam column was formulated for the s-$\theta$ coordinate system by reference 1 and may be expressed as

$$U = \int_0^L F(s, \theta, \theta') \, ds \tag{1.2}$$

where

$U$ = Potential energy,

$F = EI\theta'^2/2 - P\,(1 - \cos \theta) + V \sin \theta$,

$E$ = Modulus of elasticity,

$I$ = Moment of inertia of the cross section,

$\Theta'$ = $d\Theta/ds$,

$$P = P_L + \int_s^L p(s)\, ds + \sum_{n=1}^{k} P_n,$$

$P_L$ = Horizontal concentrated load at $s = L$,

$P_n$ = Concentrated horizontal load,

$p(s)$ = Distributed horizontal load,

$$V = V_0 + \int_0^s q(s)\, ds + \sum_{m=1}^{j} Q_n,$$

$$V_0 = \frac{1}{\bar{X}} \left[ M_L - M_0 + \int_0^L p(s)\, y\, ds - \int_0^L q(s)\, (\bar{X}-x)\, ds + \sum_{n=1}^{k} P_n y_n - \sum_{m=1}^{j} Q_m\, (\bar{X}-x_m) \right]$$

$$\bar{X} = \int_0^L \cos\Theta\, ds$$

$Q_m$ = Concentrated vertical load,

$q(s)$ = Distributed vertical load.

The term, $EI\Theta'^2/2$, is the strain energy per unit length due to bending. The strain energy due to the axial load and to the lateral shear is neglected since it is small for the long slender beam column. The terms, $P(1-\cos\Theta)$ and $V\sin\Theta$, are the potential energy per unit length for the forces P and V shown in Figure 1.2.

Since the beam column is simply supported, there are two constraints for the system. First, the vertical deflections at the end points of the beam column are zero. This condition may be expressed in terms of the following integral constraint

$$y\Big|_0^L = \int_0^L \sin\theta \, ds = 0 \tag{1.3}$$

Second, the applied end moments are finite constraints which may be expressed as

$$EI\theta'\Big|_{s=0} = M_0$$

$$EI\theta'\Big|_{s=L} = M_L \tag{1.4}$$

1.3 Differential Equation

The function $\theta(s)$ which minimizes the potential energy for a given loading is the slope variation for a beam column in stable static equilibrium. From the calculus of variations, a necessary condition for the potential energy integral as expressed by equation (1.2) to be minimized is when the Euler-Lagrange equation is satisfied. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial \theta} - \frac{d}{ds}\left[\frac{\partial F}{\partial \theta'}\right] = 0 \tag{1.5}$$

where F is defined by equation (1.2). Equation (1.5), after substitution for F, is the differential equation for the beam column and may be expressed as

$$EI\theta'' + P \sin \theta - V \cos \theta = 0 \tag{1.6}$$

where $\theta'' = d^2\theta/ds^2$.

The solution of equation (1.6) for $\theta(s)$ must satisfy the constraints on the system as defined by equations (1.3) and (1.4). Equation (1.6) represents a two point boundary value problem where the end moment constraints are the boundary conditions.

## 1.4 General Ritz Method

Except for a few special cases, the nonlinear differential equation (1.6) has no known exact solution. The approximate method of Ritz as explained in reference 2 can be readily used for the minimizing of the potential energy integral, equation (1.2). The application of the Ritz method to the beam column problem consists of assuming the slope variation, $\theta(s)$, to be represented by a finite series of known functions, $G_n(s)$, with undetermined coefficients, $A_0$, $A_1$, .... $A_r$. The assumed Ritz solution may be expressed as

$$\theta(s) = \sum_{n=0}^{r} A_n G_n(s) \tag{1.7}$$

The integral equation (1.2) is reduced to an algebraic equation by the substitution of the assumed solution, equation (1.7), for $\Theta(s)$ in equation (1.2). Thus, the calculus of variations problem is transformed into an ordinary maximum-minimum problem. The minimizing conditions for an ordinary maximum-minimum problem may be expressed as

$$\frac{\partial U}{\partial A_0} = 0, \; \frac{\partial U}{\partial A_1} = 0, \; \ldots \; \frac{\partial U}{\partial A_r} = 0 \tag{1.8}$$

The minimizing conditions for the potential energy of the beam column are derived by applying equations (1.8) to equation (1.2) and may be expressed as

$$\frac{\partial U}{\partial A_m} = \int_0^L \left( \frac{\partial F}{\partial \Theta} \frac{\partial \Theta}{\partial A_m} + \frac{\partial F}{\partial \Theta'} \frac{\partial \Theta'}{\partial A_m} \right) ds = 0 \tag{1.9}$$

where $m = 0, 1, \ldots r$, and $F$ is defined by equation (1.2), and where $\Theta$ is defined by equation (1.7). The partial derivatives in equation (1.9) for $\Theta$ and $\Theta'$ may be expressed as

$$\frac{\partial \Theta}{\partial A_m} = G_m(s)$$

$$\frac{\partial \Theta'}{\partial A_m} = \frac{dG_m(s)}{ds}$$

After integrating the second term by parts and using the definitions for F and the partial derivatives, equation (1.9) may be written as

$$EI\theta' G_m(s) \Big|_0^L - \int_0^L G_m(s) \Big[ EI\theta''$$

$$+ P \sin\theta - V\cos\theta \Big] ds = 0 \qquad (1.10)$$

where $\theta(s)$ is defined by equation (1.7) and where $m = 0, 1, \ldots r$.

Equation (1.10) results in $r + 1$ algebraic equations with $r + 1$ unknowns. The above Ritz procedure reduces the analysis of the beam column to the solution of a set of algebraic equations as compared with the usual procedure of solving the nonlinear differential equation. The coefficients, $A_0$, $A_1$, ... $A_r$, are the coefficients which minimize the potential energy of the structure for the assumed solution.

The first three coefficients are chosen when possible so that the constraints on the potential energy equation are satisfied. This results in $r - 2$ unknown coefficients. Thus, in equations (1.10), $m = 3, 4, \ldots r$ which results in $r - 2$ algebraic equations with $r - 2$ unknowns.

# CHAPTER II

# RITZ SOLUTIONS

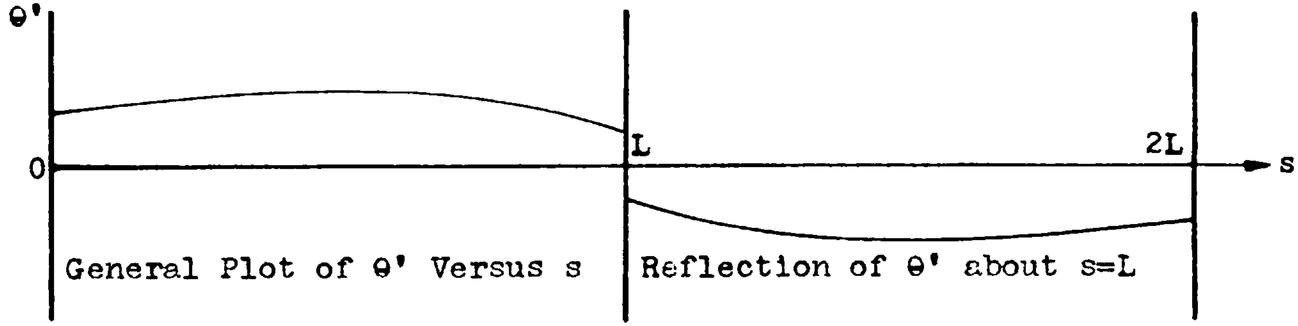
## 2.1 General

The Ritz minimizing procedure gives the best possible coefficients for the assumed solution. Therefore, the accuracy of the approximate Ritz solution is directly dependent on the choice of the functions for the slope variation. The assumed slope variations for this thesis are a cosine series and a power series. In Chapter V these functions will be shown to be excellent solutions with only a few terms for some specific cases. For other cases, especially the beam column with a nonsymmetrical load, a large number of terms results in only a fair solution. For a more accurate solution to a specific problem, the Ritz approximate solution will be used for the starting values of the numerical method described in Chapter III.

## 2.2 Cosine Series

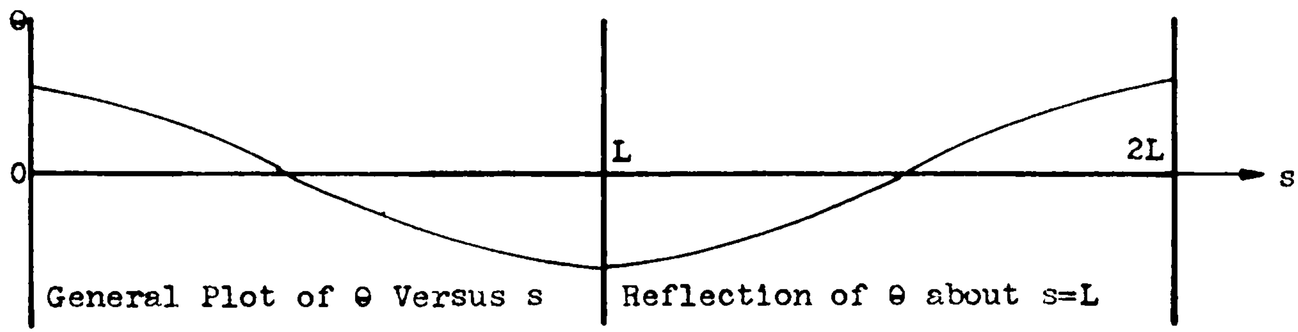
### (a) Nonlinear Case

A typical $\theta(s)$ and $\theta'(s)$ plot for a beam column is shown in Figures 2.1a and 2.1b for s = 0 to




Periodic Representation of θ'

Figure 2.1b




Periodic Representation of θ

Figure 2.1a

$s = L$. If these diagrams are reflected about the end points, the functions $\Theta(s)$ and $\Theta'(s)$ may be considered as periodic in the interval 0 to 2L. The $\Theta(s)$ function has a finite number of maxima and minima in the interval shown in Figure 2.1a. The $\Theta'(s)$ function may be discontinuous at $s = 0$, $s = L$, and $s = 2L$ as shown in Figure 2.1b. The above properties for $\Theta(s)$ satisfy the Dirichle conditions for a Fourier series. Therefore, the Fourier series representation for $\Theta(s)$ will converge to $\Theta(s)$ at all points in the interval 0 to 2L.

A finite cosine series will be used for the assumed Ritz solution and may be expressed as

$$\Theta(s) = \sum_{n=0}^{r} A_n \cos nBs \tag{2.1}$$

where

$$B = \pi / L$$

As the number of terms in equation (2.1) increases, the finite cosine series approaches the cosine Fourier series. Thus, for a sufficient number of terms, equation (2.1) will converge to $\Theta(s)$ at all points in the interval 0 to 2L.

The integral constraint, equation (1.3), after substitution of equation (2.1) for $\Theta(s)$ may be expressed as

$$\int_0^L \sin \left( \sum_{n=0}^{r} A_n \cos nBs \right) ds = 0 \qquad (2.2)$$

The above equation may be expanded and solved for $A_0$ in terms of the other $A_n$'s. Thus, $A_0$ may be expressed as

$$A_0 = \tan^{-1} \left\{ \frac{-\int_0^L \sin \left( \sum_{n=1}^{r} A_n \cos nBs \right) ds}{\int_0^L \cos \left( \sum_{n=1}^{r} A_n \cos nBs \right) ds} \right\} \qquad (2.3)$$

The numerator of the argument of the arctangent in equation (2.3) is the integral of an odd function; therefore, the numerator is zero. Since $\tan^{-1}(0) = 0$, $A_0$ is zero. Thus, equation (2.1) can be rewritten as

$$\theta(s) = \sum_{n=1}^{r} A_n \cos nBs \qquad (2.4)$$

which is the same assumed solution as used in reference 1.

The end moment constraints, equations (1.4), are not satisfied for each term of the assumed solution, equation (2.4), when the end moments $M_0$ and $M_L$ are not zero. This is because $\theta'(s)$ is

$$\theta'(s) = -\sum_{n=1}^{r} nBA_n \sin nBs \qquad (2.5)$$

which is zero when evaluated at $s = 0$ and $s = L$. When the end moments for the beam column are not zero, a discontinuity at the points $s = 0$ and $s = L$ is produced in the first derivative of the periodic representation of $\Theta(s)$ as shown in Figure 2.1b. However, a Fourier series for $\Theta'(s)$ will converge to the average value of the right and left hand limits of $\Theta'(s)$ at the point of discontinuity $s_c$ and it will converge to the value of the right hand limit at $s_c^+$ and to the value of the left hand limit at $s_c^-$. Therefore, if a large number of terms are included in the assumed Ritz cosine series, the moment constraints, equations (1.4), will be approximately satisfied at $0^+$ and $L^-$. The accuracy of the Ritz solution will depend on the magnitude of the applied end moments and on the number of terms used for the cosine series. The smaller the end moments and the larger the number of terms in the series, the more accurate the solution will be.

The $m^{th}$ minimizing condition for the potential energy for the cosine series may be obtained by substituting $\cos(mBs)$ for $G_m(s)$ in equation (1.10) and may be expressed as

$$EI\Theta' \cos mBs \Big|_{0^+}^{L^-} - \int_0^L ( EI\Theta'' + P \sin \Theta - V \cos \Theta) \qquad (2.6)$$

$$\cos mBs \; ds = 0$$

where $\Theta$ is defined by equation (2.4). The nondimensional form of equation (2.6) is determined by evaluating the first term of the equation at $0^+$ and $L^-$ and by integrating the first term under the integral sign. Thus, equation (2.6) may be rewritten as

$$\frac{m^2 A_m}{2} + \frac{(-1)^m M_L - M_O}{LP_e}$$

$$- \int_0^L \left(\frac{P}{P_e} \sin \Theta - \frac{V}{P_e} \cos \Theta\right) \cos mBs \, ds = 0 \qquad (2.7)$$

where $P_e = EIB^2$, $m = 1, 2, \ldots r$ and $\Theta$ is defined by equation (2.4).

(b) Iteration Procedure for the Nonlinear Case

Equation (2.7) usually results in a set of nonlinear algebraic equations which can be solved by an iteration procedure. The iteration form of equation (2.7) may be written as

$$A_m^{k+1} = \frac{2}{m^2} \left[ \int_0^L \left(\frac{P}{P_e} \sin \Theta^k - \frac{V}{P_e} \cos \Theta^k\right) \cos mBs \, ds \right.$$

$$\left. - \frac{(-1)^m M_L - M_O}{LP_e} \right] \qquad (2.8)$$

where k denotes the $k^{th}$ iteration and where $\Theta$ is defined by equation (2.4). The iteration of equation (2.8) converges rapidly because it is of the form $A_m = G$ (sin $A_m$, cos $A_m$). The iteration procedure used in this thesis is block diagrammed in Figure 2.2. An outline of the iteration procedure is as follows:

1. Define the parameter SUM as equal to the sum of all of the coefficients $A_n$'s,

2. Define the superscript j as the number of the iterative trial for $A_1$,

3. Define the superscript k as the number of the iterative trial for $A_2$, $A_3$, ... $A_r$ and SUM,

4. Set $k = 0$,

5. Set $A_1^0$, $A_2^0$, ... $A_r^0$ and $SUM^0$ equal to zero,

6. Set $j = 0$,

7. Solve equation (2.8) for $A_1^{j+1}$ until the iteration converges, i.e., $A_1^{j+1} \approx A_1^j$,

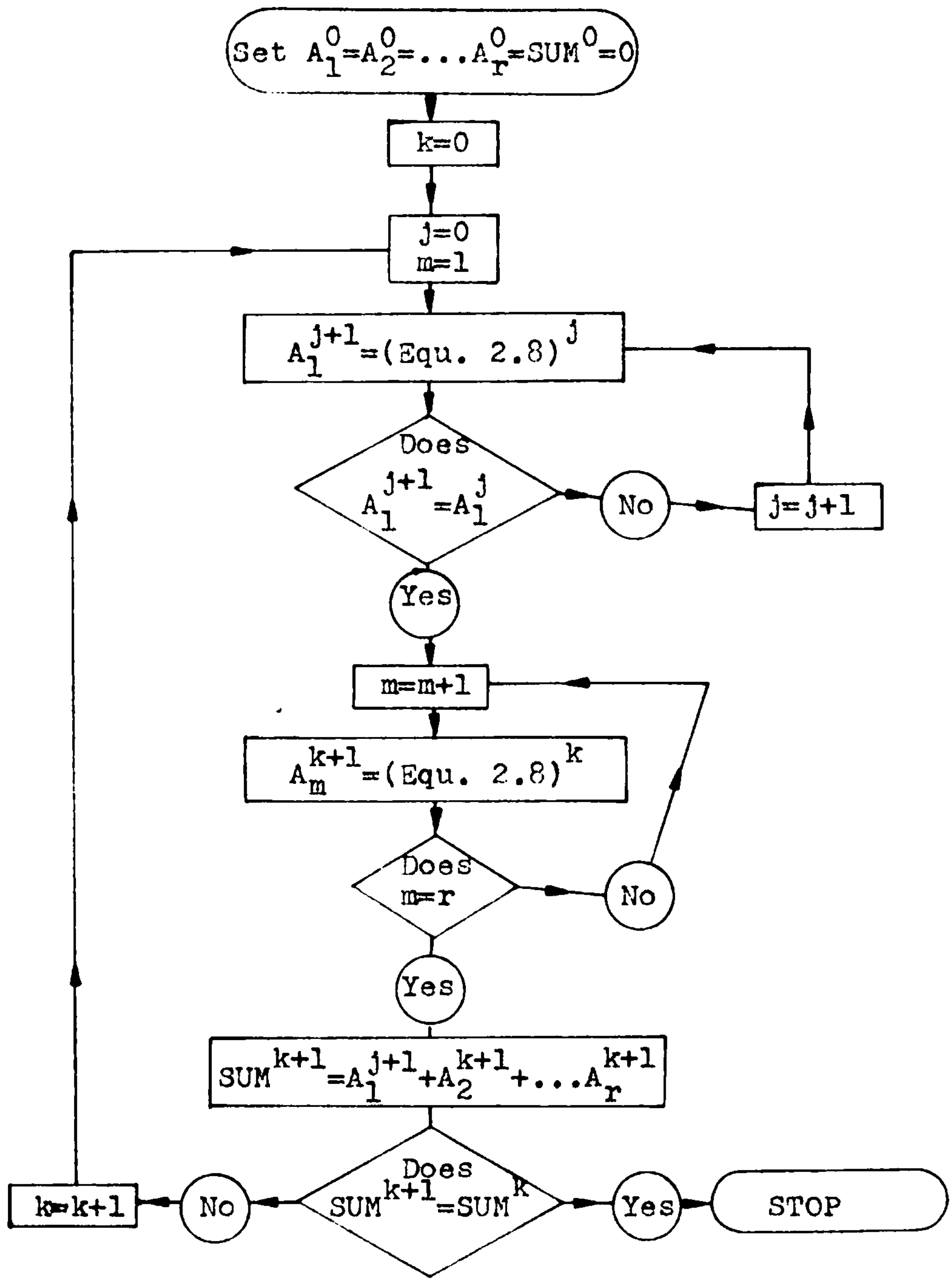
8. Solve equation (2.8) once for $A_2^{k+1}$,

9. Solve equation (2.8) once for $A_3^{k+1}$,

.
.
.

10. Continue solving equation (2.8) for the $A_n^{k+1}$'s until $A_r^{k+1}$ is determined,

11. Set $SUM^{k+1} = A_1^{j+1} + A_2^{k+1} + A_3^{k+1} + \ldots A_r^{k+1}$,




Block Diagram of Iteration Procedure

Figure 2.2

12. If $SUM^{k+1} \neq SUM^{k}$ increment k by 1 and return to step 6, otherwise the problem is solved.

For this particular problem, equation (2.8), testing the parameter SUM for convergence is the same as testing each $A_n$ for convergence. The reason that the iterations for $A_1$ are required to converge for each single iteration for the other $A_m$'s is that $A_1$ is the dominate term in the Ritz cosine series.

For the beam column having a symmetrical loading, only a few terms of equation (2.4) are needed for an accurate solution. If the loading is nonsymmetrical, many terms in equation (2.4) are needed before an accurate answer is possible. For the cases where accuracy is required, it is recommended that only the first few coefficients be calculated and that the Ritz solution then be used for starting values for the numerical method presented in the next chapter.

(c) Linear Cosine Series

The linear form of equation (2.7) is presented in this section and it will be used mainly for finding the starting values for the numerical method presented in the next chapter. If the sine and cosine terms in equation (2.7) are linearized, equation (2.7) may be written as

$$\frac{m^2 A_m}{2} + \frac{(-1)^m M_L - M_0}{L P_e} - \int_0^L \left[ \frac{P}{P_e} \theta - \frac{V}{P_e} \right] \cos mBs \, ds = 0 \qquad (2.9)$$

where $m = 1, 2, \ldots r$ and $\theta$ is defined by equation (2.4). Depending on the functions P and V, equation (2.9) may be integrated analytically giving a set of r linear algebraic equations with r unknowns.

If the functions P and V are of such form that equation (2.9) can not be integrated analytically, then equation (2.7) should be used because it is as easy to integrate numerically as equation (2.9). Also, equation (2.9) is the linear case of equation (2.7) and, therefore, will be less accurate than equation (2.7).

## 2.3 Power Series

### (a) Nonlinear Case

The second form of the Ritz solution for $\theta(s)$ is assumed to be the power series

$$\theta(s) = \sum_{n=0}^{r} A_n s^n \qquad (2.10)$$

The coefficients $A_1$ and $A_2$ are determined by substituting equation (2.10) into equation (1.4) and then solving for the desired coefficients which may be expressed as

$$A_1 = \frac{M_0}{EI} = \frac{M_0}{P_e} B^2$$

$$A_2 = \frac{1}{2}\left[\frac{M_L - M_0}{P_e} B^2 - \sum_{n=3}^{r} n A_n\right] \tag{2.11}$$

where $P_e$ is the Euler load, $EIB^2$. The constraint equation (1.3) may be solved for $A_0$ by substituting equation (2.10) into equation (1.3) and expanding in terms of double angles. Thus, $A_0$ may be expressed as

$$A_0 = \tan^{-1}\left\{\frac{-\int_0^1 \sin g(s)\, ds}{\int_0^1 \cos g(s)\, ds}\right\} \tag{2.12}$$

where

$$g(s) = \frac{M_0 B^2}{P_e}\left(s - \tfrac{1}{2} s^2\right) + \frac{M_L B^2}{2P_e}\left(s^2\right) + \sum_{n=3}^{n} A_n \left(s^n - \frac{ns^2}{2}\right)$$

The assumed Ritz solution, equation (2.10), after substituting equation (2.11) and (2.12) for the coefficients $A_0$, $A_1$ and $A_2$ may be written as

$$\theta(s) = \tan^{-1}\left\{\frac{-\int_0^1 \sin g(s)\, ds}{\int_0^1 \cos g(s)\, ds}\right\} + \frac{M_0}{P_e} B^2 \left(s - \tfrac{1}{2} s^2\right)$$

$$+ \frac{1}{2} \frac{M_L B^2}{P_e} s^2 + \sum_{n=3}^{r} A_n \left(s^n - \tfrac{1}{2} n s^2\right) \tag{2.13}$$

where g(s) is defined by equation (2.12). The assumed solution for $\theta(s)$ now satisfies all of the constraints and the minimizing procedure can be applied.

Instead of using equation (1.10) to derive the minimizing conditions as done with the cosine series, equation (1.9) will be used because the form of the assumed solution for $\theta(s)$ is now

$$\theta(s) = H(A_3, A_4, \ldots A_r) + \sum_{n=3}^{r} A_n G_n(s)$$

which is not the same as equation (1.7). The partial derivative of $\theta(s)$ with respect to $A_m$ is

$$\frac{\partial \theta}{\partial A_m} = \frac{\partial A_0}{\partial A_m} + \left(s^m - \frac{m s^2}{2}\right) \tag{2.14}$$

where

$$\frac{\partial A_0}{\partial A_m} = \frac{\epsilon \int_0^L \sin g(s)\, ds - \eta \int_0^L \cos g(s)\, ds}{\left[\int_0^L \cos g(s)\, ds\right]^2 + \left[\int_0^L \sin g(s)\, ds\right]^2}$$

and where

$$\epsilon = \int_0^L (\cos g(s))\ (s^m - \tfrac{1}{2}\, ms)\ ds$$

$$\eta = \int_0^L \sin g(s)\ (s^m - \tfrac{1}{2}\, ms)\ ds$$

The partial derivative of $\theta'(s)$ with respect to $A_m$ is

$$\frac{\partial \theta'}{\partial A_m} = m\ (s^{m-1} - s) \tag{2.15}$$

The Ritz minimizing condition, equation (1.9) after substituting the expressions for $\theta$, $\theta'$ and their partial derivatives may be written as

$$\frac{M_0 B^2 m}{P_e} \left[\frac{1}{m} - \frac{1}{m+1} - \frac{1}{6}\right] + \frac{M_L B^2}{P_e}\, m \left[\frac{1}{m+1} - \frac{1}{3}\right]$$

$$+ \sum_{n=3}^{r} nmA_n \left[\frac{1}{n+m-1} - \frac{1}{m+1} - \frac{1}{n+1} + \frac{1}{3}\right] \tag{2.16}$$

$$-B^2 \int_0^L \left[\frac{P}{P_e} \sin \theta - \frac{V}{P_e} \cos \theta\right] \left[\frac{\partial A_0}{\partial A_m} + s^m - \tfrac{1}{2}\, ms^2\right] ds = 0$$

where $\partial A_0 / \partial A_m$ is defined by equation (2.14) and $\theta$ is defined by equation (2.13).

(b) Linear Power Series

The Ritz minimizing condition based on the power series, equation (2.16), has one advantage over the minimizing condition based on the cosine series, equation (2.7). The advantage is that the linear case of the power series may be analytically integrated in certain cases where the linear cosine series may not be depending on the functions for P and V.

The end moment constraints for the linear power series are the same for the nonlinear power series, equation (2.11). The linear integral constraint is

$$\int_0^L \sin \theta \, ds = 0 \approx \int_0^L \theta \, ds \tag{2.17}$$

The expression for $A_0$ determined by integrating equation (2.17) is

$$A_0 = - \sum_{n=1}^{r} \frac{A_n}{n+1} \tag{2.18}$$

By substituting for $A_0$, $A_1$ and $A_2$ from equations (2.11) and (2.18) into equation (2.10), the assumed linear function for $\theta$ may be expressed as

$$\theta = \frac{M_0B^2}{P_e} \left(s - \frac{s^2}{2} - \frac{1}{3}\right) + \frac{M_LB^2}{P_e} \left(\frac{s^2}{2} - \frac{1}{3}\right)$$

$$+ \sum_{n=3}^{r} A_n \left(s^n - \frac{ns^2}{2} + \frac{n}{6} - \frac{1}{n+1}\right) \tag{2.19}$$

which satisfies constraint equations (1.3) and (1.4).

The partial derivatives of $\theta$ and $\theta'$ are

$$\frac{\partial\theta}{\partial A_m} = s^m - \tfrac{1}{2}\, ms^2 + m/6 - \frac{1}{m+1}$$

$$\frac{\partial\theta'}{\partial A_m} = m\, s^{m-1} - ms \tag{2.20}$$

The linear minimizing condition for the power series may be determined by substituting equations (2.19) and (2.20) into equation (1.9) and by integrating the $\theta'$ term. The linear minimizing condition is

$$0 = \frac{mM_0B^2}{P_e} \left[\frac{1}{m} - \frac{1}{m+1} - \frac{1}{6}\right] + \frac{mM_L}{P_e} B^2 \left[\frac{1}{m+1} - \frac{1}{3}\right]$$

$$+ \sum_{n=3}^{r} nmA_n \left[\frac{1}{n+m-1} - \frac{1}{n+1} - \frac{1}{m+1} - \frac{1}{3}\right] \tag{2.21}$$

$$- B^2 \int_0^L \left(\theta \frac{P}{P_e} - \frac{V}{P_e}\right) \left(s^m - \frac{ms^2}{2} + \frac{M}{6} - \frac{1}{m+1}\right) ds$$

where $P_e = EIB^2$ and $m = 3, 4, \ldots r$.

Once P and V as functions of s are known, it may be possible to analytically integrate equation (2.21). This would result in r-2 linear algebraic equations with r-2 unknowns.

## 2.4 Use of the Ritz Solutions

(a) Nonlinear Cosine Series

The nonlinear cosine series minimizing conditions, equations (2.7), may be used in solving any simply supported beam column problem. The accuracy will depend on the number of terms in the series and on the magnitude of the applied end moments. In many cases, a one term solution will provide good starting values for the numerical method presented in the next chapter.

(b) Linear Cosine Series

If the value of $\Theta(s)$ is never greater than 1 radian, the linear cosine series minimizing conditions, equations (2.9), will provide good starting values for the numerical method presented in the next chapter.

(c) Nonlinear Power Series

The one term power series minimizing conditions, equations (2.16), will provide good starting values for the numerical method presented in the next chapter. For more than one term, the power series solution becomes difficult to compute.

(d) Linear Power Series

The linear power series minimizing conditions, equations (2.21), may be used when it is not possible to integrate the linear cosine series analytically. The linear power series will provide good starting values for the numerical method presented in the next chapter when $\Theta(s)$ is never greater than 1 radian.

# CHAPTER III

# SOLUTION OF THE DIFFERENTIAL EQUATION BY THE METHOD OF FINITE DIFFERENCES

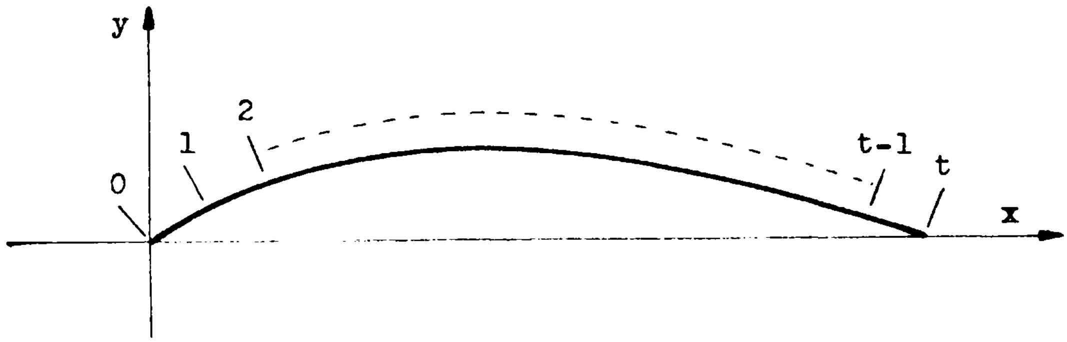
## 3.1 General

When the loading on the beam column is nonsymmetrical, the Ritz solution may contain a large error. Even if the loading is symmetrical, the accuracy of the Ritz solution is unknown. Therefore, a second method which will provide an accurate solution is needed for checking and correcting the Ritz solution. The method of finite differences will be used for the second solution.

## 3.2 Derivation of the Central Difference Equations

The beam column is divided into t equal intervals along the elastic axis as shown in Figure 3.1. The length of each interval is $h = L/t$. The intervals will be indexed so that

$$
\begin{array}{ll}
\text{at}\quad s = 0 & i = 0 \\
s = h = L/t & i = 1 \\
s = 2h = 2L/t & i = 2 \\
\vdots & \vdots
\end{array}
$$




Division of Beam Column into t Intervals

Figure 3.1

$$s = (t-1)h = (t-1)\ L/t \qquad\qquad i = t-1$$

$$s = th = tL/t = L \qquad\qquad i = t$$

The subscript i will be used to indicate at which interval along the beam column a function is to be evaluated, i.e.,

$$\theta_i = \theta(s_i) = \theta(ih) = \theta(iL/t)$$

The differential equation for a beam column, equation (1.6), can be written in finite central difference form as explained by reference 3. The second derivative of $\theta$ expressed in finite central difference form is

$$\theta'' = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} \tag{3.1}$$

The differential equation (1.6) expressed in finite difference form is obtained by substituting equation (3.1) into equation (1.6) and may be written as

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} - h^2B^2 \left[\frac{P_i}{P_e} \sin \theta_i - \frac{V_i}{P_e} \cos \theta_i\right] \qquad (3.2)$$

where i = 0, 1, ... t.

When equation (3.2) is written with i = 0 and t, the terms $\theta_{-1}$ and $\theta_{t+1}$ appear in the expressions. These terms can be determined by using the moment constraints, equations (1.4), at the two end points. The moment constraints in central difference form are

$$\frac{\theta_1 - \theta_{-1}}{2h} = \frac{M_0}{EI} = \frac{M_0}{P_e} B^2 \qquad (3.3)$$

$$\frac{\theta_{t+1} - \theta_{t-1}}{2h} = \frac{M_L}{EI} = \frac{M_L B^2}{P_e}$$

The terms $\theta_{-1}$ and $\theta_{t+1}$ are determined by solving equations (3.3) and may be expressed as

$$\theta_{-1} = \theta_1 - 2hB^2 \frac{M_0}{P_e}$$

$$\theta_{t+1} = \theta_{t-1} + 2hB^2 \frac{M_L}{P_e} \qquad (3.4)$$

Thus, the central difference equations (3.2) for i = 0 may be written as

$$\theta_1 = \theta_0 + hB^2 \frac{M_0}{P_e} - \frac{h^2B^2}{2} \left[\frac{P_0}{P_e} \sin \theta_0 - \frac{V_0}{P_e} \cos \theta_0\right] \qquad (3.5)$$

and for i = t may be expressed as

$$0 = \theta_t - \theta_{t-1} - hB^2 \frac{M_L}{P_e} - \frac{h^2B^2}{2} \left[ \frac{P_t}{P_e} \sin \theta_t - \frac{V_t}{P_e} \cos \theta_t \right] \tag{3.6}$$

Equations (3.2) for i = 1, 2, ... t-1 remain unchanged.

The $P_i$ and $V_i$ used in equations (3.2), (3.5) and (3.6) are

$$P_i = P_L + \int_{s_i}^{L} p(s)\, ds + \sum_{n=j}^{k} P_n$$

$$V_i = V_0 + \int_0^{s_i} q(s)\, ds + \sum_{m=1}^{i} Q_m$$

$$V_0 = \frac{1}{\bar{X}} \left[ M_L - M_0 + \int_0^L p(s)\, y\, ds - \int_0^L q(s)\, (\bar{X}-x)\, ds \right.$$

$$\left. + \sum_{n=1}^{k} P_n y_n - \sum_{n=1}^{j} Q_m\, (\bar{X}-x_m) \right] \tag{3.7}$$

where

$$\bar{X} = \int_0^L \cos \theta\, ds$$

or

$$\bar{X} = \frac{h}{2} \sum_{i=1}^{t} (\cos \theta_{i-1} + \cos \theta_i)$$

## 3.3 Solution of the Finite Difference Equations

There are two apparent methods which may be used to determine the $\theta_i$'s from equations (3.2). One method is to set up the n+1 nonlinear algebraic simultaneous equations and to try to solve them for the $\theta_i$'s. This method is impractical since the solution of a set of nonlinear simultaneous equations is very difficult. The second method is to convert the boundary value problem to an initial value problem. This is done by assuming a value for $\theta_0$ and $\bar{X}$ and then evaluating the $\theta_i$'s across the beam column by using equations (3.2) and (3.5). Once the $\theta_i$'s are known, the boundary conditions at s=L can be checked and a new assumption for $\theta_0$ and $\bar{X}$ made. The value for $\theta_0$ from one of the Ritz solutions in Chapter II is used to start the finite difference procedure. The value of $\bar{X}_{Ritz}$ may be determined from

$$\bar{X}_{Ritz} = \int_0^L \cos\theta \, ds \tag{3.8}$$

where $\theta$ is the assumed Ritz solution.

The values for x and y at each interval will be needed as the $\theta_i$'s are evaluated across the beam. The functions for x and y determined from equations (1.1) are

$$y = \int_0^s \sin \Theta \, ds$$

$$x = \int_0^s \cos \Theta \, ds$$

The integration of the above equations can be completed numerically using the trapezoidal rule as explained in reference 3. Thus, $x_i$ and $y_i$ as functions of $\Theta_i$ are

$$y_i = \frac{h}{2} \sum_{j=1}^{i} (\sin \Theta_{j-1} + \sin \Theta_j)$$

$$x_i = \frac{h}{2} \sum_{j=1}^{i} (\cos \Theta_{j-1} + \cos \Theta_j) \qquad (3.9)$$

It should be noted that $\overline{X}$ must equal $x_t$ and that $y_t$ must be zero for the finial solution.

A procedure is needed for correcting the value of $\Theta_0$ after integrating across the beam column and after checking the constraints at s=L. If $\overline{\Theta}_0$ is defined as the correct value for $\Theta_0$ then

$$\overline{\Theta}_0 = \Theta_0^1 + E \qquad (3.10)$$

where the superscript 1 stands for the first trial value of $\Theta_0$ and E is the total error of $\Theta_0^1$. In this chapter superscripts on a variable will denote the trial number for that variable. The integral constraint equation

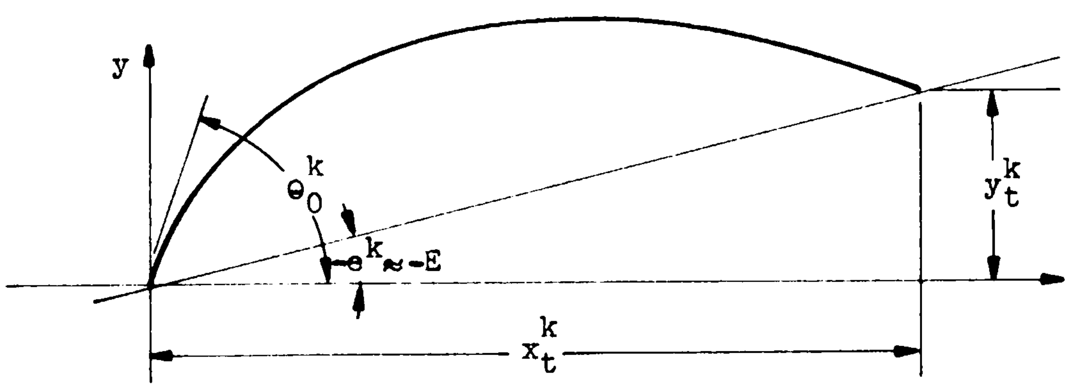
(1.3) states that the value for y at s=L is zero. The error E can be approximated by the angle between the line connecting the end points of the beam column and the x axis as shown in Figure 3.2. Therefore, E can be expressed as

$$E \approx e^1 = -\tan^{-1}\left(y_t^1/x_t^1\right) \tag{3.11}$$

Since E in equation (3.11) is only an approximation, the correction of $\theta_0^1$ will be $\theta_0^2$ with a new error $e^2$ which is

$$e^2 = -\tan^{-1}\left(y_t^2/x_t^2\right) \tag{3.12}$$

The starting value for the solution is the Ritz solution, i.e., $\theta_0^1 = \theta_{0_{Ritz}}$. Thus, $\theta_0^2$ can then be expressed as




Approximation for the Error E

Figure 3.2

$$\theta_0^2 = \theta_{0_{Ritz}} + e^1 \qquad (3.13)$$

and $\theta_0^3$ is

$$\theta_0^3 = \theta_0^2 + e^2 = \theta_{0_{Ritz}} + e^1 + e^2 \qquad (3.14)$$

By generalizing equation (3.14), the error E in equations (3.10) and (3.11) can be written exactly as

$$E = \sum_{k=1}^{\infty} e^k \qquad (3.15)$$

where $e^k = -\tan^{-1} (y_t^k/x_t^k)$ and where the assumption is made that $e^{k+1}$ is less than $e^k$. The solution can now be extrapolated by assuming that $e^k$ is proportional to $e^{k+1}$ by a factor z. Thus, the error E can be estimated by

$$E = \sum_{j=0}^{\infty} e^1 (z)^j \qquad (3.16)$$

where z is raised to the $j^{th}$ power and is defined as

$$z = e^2/e^1$$

The first solution across the beam column will give $e^1$ from equation (3.11). With the $\theta_0^2$ computed from $\theta_0^1$ and $e^1$, $e^2$ can be determined from equation (3.12). Once $e^2$ is known, z can be determined and the total error E estimated. Then $\bar{\theta}_0$ can be approximated with equation (3.10).

The value of z may be used to tell if the solution is converging. The convergence test is

if $z < 1$, then the solution will converge,

if $z \geq 1$, then the solution will not converge.

### 3.4 Numerical Solution Procedure

The following outline is the numerical solution procedure for determining the shear, bending moment, slope and deflection of the beam column at points i along the elastic axis.

1. Determine the starting value for $\theta_0^1$ from one of the Ritz solutions presented in Chapter II.

2. Determine the starting value for $\bar{X}$ from equation (3.8).

3. Determine t, the number of the intervals that the elastic axis is to be divided into.

4. Determine $V_0$, $V_i$ and $P_i$ for $i = 0, 1, \ldots t-1$ from equations (3.7).

5. Solve for $\theta_i$, $x_i$ and $y_i$ using equations (3.2), (3.5) and (3.9) where $i = 1, 2, \ldots t$.

6. Determine $e^1$ from equation (3.11) and then calculate $\theta_0^2$ with equation (3.13).

7. Repeat step 5.

8. Determine $e^2$ from equation (3.12) and then determine E from equation (3.16).

9. Determine $\bar{\theta}_0$ from equation (3.10) and repeat step 5 with $\theta_0 = \bar{\theta}_0$.

10. If $\bar{X} \neq x_t$ set $\bar{X}$ equal to $x_t$ and return to step 4.

11. If $y_t \neq 0$ return to step 4, otherwise the problem is solved.

# CHAPTER IV

## GENERALIZED FORTRAN COMPUTER PROGRAMS

### 4.1 General

The mathematical techniques developed in the preceding chapters are of a numerical nature. The computations involved in solving a particular beam column problem are long, tedious, and involved. For the above reasons, a set of generalized subroutines and programs were written in the Fortran programming language. Only a basic knowledge of Fortran is needed in order to make use of the following programs. A good description of Fortran is given by reference 4. The user must be sure that all of the Fortran statements appearing in the following programs are allowable on the particular digital computer on which they are to be run.

The computer programs require that the unit of length for the system be defined so that the length of the beam column is one unit, i.e., L = 1 unit of length. The user must make sure that all parameters have compatible units. Also, all loads and moments must be entered in terms of their ratio with the Euler Load $P_e$, i.e., $P_L/P_e$.

## 4.2 Mainline Program for the One Term Ritz Solution

Fortran Program 4.1 is the computer one term cosine Ritz solution. No alterations need be made in the program before use. All dimensionless concentrated horizontal loads, vertical loads, and end moments are entered into the program by means of data cards that follow the program cards. The form and order of the data cards are as follows:

Card 1:

| Columns | Data |
|---|---|
| 1-10 | Numerical value of the dimensionless horizontal load at $s = L$, $P_L/P_e$, including decimal point. |
| 11-20 | Numerical value of the dimensionless end moment, $M_L/P_e$, at $s = L$ including the decimal point. |
| 21-30 | Numerical value of the dimensionless end moment, $M_0/P_e$, at $s = 0$ including the decimal point. |

Card 2:

| Columns | Data |
|---|---|
| 1-3 | Three digit number giving the total number of horizontal concentrated loads to be entered, maximum number 999. |

4-6 Three digit number giving the total number of concentrated vertical loads to be entered, maximum number 999.

Cards 3 to N:

| Columns | Data |
|---|---|
| 1-10 | Numerical value of one of the dimensionless concentrated horizontal loads, $P_n/P_e$, including the decimal point. |
| 11-20 | Numerical value of the s coordinate where the horizontal load is located including the decimal point. |

Cards N to M:

| Columns | Data |
|---|---|
| 1-10 | Numerical value of one of the dimensionless vertical loads, $Q_m/P_e$, including the decimal point. |
| 11-20 | Numerical value of the s coordinate where the vertical load is located including the decimal point. |

Once the subroutines discussed in section 4.4 have been added to the beginning of the mainline one term Ritz solution program and the data cards to the end of the mainline program, the Ritz solution program 4.1 is ready to run on the digital computer.

## 4.3 Mainline Program for the Finite Difference Solution

Fortran Program 4.2 is the procedure for the computer finite difference solution. No alterations need be made in the program before use. All dimensionless concentrated horizontal loads, vertical loads, and end moments are entered into the program by means of data cards that follow the program. The data cards for the finite difference program are exactly the same as those used in the Ritz solution program except for one card which contains the $A_1$ term from the Ritz solution. The Ritz coefficient data card goes before all of the other data cards and has the following format:

Ritz Coefficient Card:

| Columns | Data |
|---|---|
| 1-10 | Numerical value of the Ritz coefficient, $A_1$, as found by program 4.1 including the decimal point. |

All of the other data cards follow the Ritz coefficient data card. Once the subroutines described in the next section have been added to the beginning of Program 4.2 and the data cards added to the end, the mainline program for the finite difference solution is ready to run.

## 4.4 Subroutines

The five subroutines described in this section must be used with the Ritz solution Program 4.1 and the finite difference solution Program 4.2. The first subroutine, Program 4.3, is for evaluating the distributed vertical load as a function of s. One addition must be made to Program 4.3 in order for it to be ready for use. The function for the distributed vertical load must be added where indicated in the program. An example is that if the vertical load q(s) for a particular case is $1.25s^2$, then the Fortran statement to be added is

```
FUNQ = 1.25*s**2
```

Program 4.4 is the subroutine for evaluating the distributed horizontal load as a function of s. The one addition necessary for the program to be used is to add the Fortran expression for the distributed horizontal load, p(s), where indicated in the program. Programs 4.5, 4.6, and 4.7 are not altered in any way for use with the mainline Programs 4.1 and 4.2.

Fortran Program 4.1

Mainline One Term Ritz Solution

```
      DIMENSION PM(100), QM(100), X(100), Y(100)
      COMMON (VO,PM,QM,AMO,AML,XB,PL,X,Y)
    1 READ 2,PL,AML,AMO
    2 FORMAT(3F10.0)
C     READ IN NUMBER OF CONCENTRATED LOADS
      READ 3,K,L
    3 FORMAT(2I3)
      DO 4 J=1,100
      QM(J)=0.
    4 PM(J)=0.
C     READ IN CONCENTRATED HORIZONTAL LOADS
      IF(K) 60,60,50
   50 DO 5 J=1,K
      READ 2,PP,S
      I=S*100.
      IF(I) 5,5,55
   55 PM(I)=PP+PM(I)
    5 CONTINUE
C     READ IN CONCENTRATED VERTICAL LOADS
   60 IF(L) 61,61,62
   62 DO 6 J=1,L
      READ 2,PP,S
      I=S*100.
      IF(I) 6,6,66
   66 QM(I)=PP+QM(I)
    6 CONTINUE
      P=3.1415927
      CORR=0.
      A=0.
      DO 7 J=1,100
      S=J
      S=S*.01
      X(J)=S
    7 Y(J)=0.
    8 XB=X(100)
      CALL SHEAR
      AINT=0.
      DO 9 J=1,100
      S=J
      S=S*.01
      SS=S-.01
```

```
 9 AINT=AINT+(P(SS)*SINF(A*COSF(P*SS))-V(SS)*COSF(A*
  1COSF(P*SS)))*COSF(P*SS)+(P(S)*SINF(A*COSF(P*S))-
  2V(S)*COSF(A*COSF(P*S)))*COSF(P*S)
   AINT=AINT*.005
   A=2.*(AINT-AML-AMO)
   IF(.0001-ABSF((CORR-A)/A)) 10,14,14
10 X(1)=(COSF(A)+COSF(A*COSF(P*.01)))*.005
   Y(1)=(SINF(A)+SINF(A*COSF(P*.01)))*.005
   DO 11 J=2,100
   S=J
   S=S*.01
   SS=S-.01
   X(J)=X(J-1)+(COSF(A*COSF(P*SS))+COSF(A*COSF(P*S)))
  1*.005
11 Y(J)=Y(J-1)+(SINF(A*COSF(P*SS))+SINF(A*COSF(P*S)))
  1*.005
   CORR=A
   GO TO 8
14 PRINT 15
15 FORMAT(34H THE ONE TERM RITZ SOLUTION FOR THE,/37H
  1 BEAM COLUMN FOR THE GIVEN LOADING IS-)
   PRINT 16,A
16 FORMAT(/7H THETA=F14.9,14H COS(3.1416 S))
   STOP
   END
```

Fortran Program 4.2

Mainline Finite Difference Solution

```
      DIMENSION PM(100), QM(100), X(100), Y(100),
     1 T(100), SI(100), CO(100)
      COMMON (VO,PM,QM,AMO,AML,XB,PL,X,Y)
 1000 READ 2,A
    1 READ 2,PL,AML,AMO
    2 FORMAT(3F10.0)
C     READ TOTAL NUMBER OF CONCENTRATED LOADS
      READ 3,K,L
    3 FORMAT(2I3)
      DO 4 J=1,100
      QM(J)=0.
    4 PM(J)=0.
C     READ CONCENTRATED HORIZONTAL LOADS
      IF(K) 60,60,50
   50 DO 5 J=1,K
      READ 2,PP,S
      I=S*100.
      IF(I) 5,5,55
   55 PM(I)=PP+PM(I)
    5 CONTINUE
C     READ CONCENTRATED VERTICAL LOADS
   60 IF(L) 61,61,62
   62 DO 6 J=1,L
      READ 2,PP,S
      I=S*100.
      IF(I) 6,6,66
   66 QM(I)=PP+QM(I)
    6 CONTINUE
      P=3.1415927
      X(1)=(COSF(A)+COSF(A*COSF(P*.01)))*.005
      Y(1)=(SINF(A)+SINF(A*COSF(P*.01)))*.005
      DO 11 J=2,100
      S=J
      S=S*.01
      SS=S-.01
      X(J)=X(J-1)+(COSF(A*COSF(P*SS))+COSF(A*COSF(P*S)))
     1 *.005
   11 Y(J)=Y(J-1)+(SINF(A*COSF(P*SS))+SINF(A*COSF(P*S)))
     1 *.005
      KK=0
      XB=X(100)
      C1=.01*P**2
      C2=(.01*P)**2
  100 T(1)=A+C1*AMO-C2*(P(0.)*SINF(A)-V(0.)*COSF(A))*.5
```

```
      SI(1)=SINF(T(1))
      CO(1)=COSF(T(1))
      X(1)=(COSF(T(1))+COSF(A))*.005
      Y(1)=(SINF(T(1))+SINF(A))*.005
      DO 101 J=2,100
      S=J
      S=S*.01-.01
      T(J)=2.*T(J-1)-T(J-1)-C2*(P(S)*SI(J-1)+V(S)*
     1 (-CO(J-1)))
      SI(J)=SINF(T(J))
      CO(J)=COSF(T(J))
      X(J)=X(J-1)+(CO(J-1)+CO(J))*.005
  101 Y(J)=Y(J-1)+(SI(J-1)+SI(J))*.005
      IF(K) 108,106,107
  106 K=1
      E1=-ATANF(Y(100)/X(100))
      A=A+E1
      GO TO 100
  107 K=-1
      E2=-ATANF(Y(100)/X(100))
      C1=E2/E1
      IF(1.-ABSF(E2/E1)) 2000,2000,177
  177 C2=E2
      DO 178 J=1,10
      A=A+C2
  178 C2=C2*C1
      GO TO 100
  108 K=0
      IF(.0001-ABSF(ATANF(Y(100)/X(100))))118,118,888
  888 IF(.0001-ABSF(XB-X(100)))118,118,4000
  118 XB=X(100)
      GO TO 4
 2000 PRINT 2001
 2001 FORMAT(39H ERROR GREATER THAN 1--SOLUTION STOPED.)
      STOP
 4000 PRINT 4002
 4002 FORMAT(/4H    S,12X5HTHETA,15X1HX,19X1HY)
      S=0.
      C2=0.
      C1=0.
      PRINT 4003,S,A,C1,C2
 4003 FORMAT(F6.2,3F20.9)
      DO 4004 J=10,100,10
      S=J
      S=S*.01
 4004 PRINT 4003,S,T(J),X(J),Y(J)
      STOP
      END
```

Fortran Program 4.3

Subroutine for Evaluating Continuous Vertical Load

```
      FUNCTION FUNQ(I)
      S=I
      S=.01*S
      FUNQ= Fortran Expression for the Continuous
            Vertical Load as a Function of s
      RETURN
      END
```

Fortran Program 4.4

Subroutine for Evaluating Continuous Horizontal Load

```
      FUNCTION FUNP(I)
      S=I
      S=.01*S
      FUNP= Fortran Expression for the Continuous
            Horizontal Load as a Function of s
      RETURN
      END
```

Fortran Program 4.5

Subroutine for Evaluating the Total Horizontal Load

```
      FUNCTION P(S)
      COMMON (VO,PM,AMO,AML,XB,PL,X,Y)
      DIMENSION PM(100), QM(100), X(100), Y(100)
      I=S*100.
      P=PL
      IF(I-100)1,3,3
    1 DO 2 J=I,99
    2 P=P+(FUNP(I)+FUNP(I+1))*.005+PM(I)
    3 RETURN
      END
```

Fortran Program 4.6

Subroutine for Evaluating $V_0$

```
      SUBROUTINE SHEAR
      COMMON (VO,PM,QM,AMO,AML,XB,PL,X,Y)
      DIMENSION PM(100), QM(100), X(100), Y(100)
      E=FUNP(1)*Y(1)*.005
      F=(FUNQ(0)*XB+FUNQ(1)*(XB-X(1)))*.005
      G=0.
      H=0.
      DO 1 J=2,100
      E=(FUNP(J-1)*Y(J-1)+FUNP(J)*Y(J))*.005+E
      F=(FUNQ(J-1)*(XB-X(J-1))+FUNQ(J)*(XB-X(J)))*.005+F
      G=G+PM(J)*Y(J)
    1 H=H+QM(J)*(XB-X(J))
      VO=1./XB*(AML-AMO+E-F+G-H)
      RETURN
      END
```

Fortran Program 4.7

Subroutine for Evaluating the Total Vertical Load

```
      FUNCTION V(S)
      COMMON (VO,PM,QM,AMO,AML,XB,PL,X,Y)
      DIMENSION PM(100), QM(100), X(100), Y(100)
      I=S*100.
      V=VO+(FUNQ(1)+FUNQ(0))*.005+QM(1)
      IF(I-1) 3,3,1
    1 DO 2 J=2,I
    2 V=V+(FUNQ(J-1)+FUNQ(J))*.005+QM(J)
    3 RETURN
      END
```

CHAPTER V

# SAMPLE SOLUTION TO BEAM COLUMN PROBLEM

## 5.1 General

In order to illustrate the mathematical techniques discussed in the preceding chapters, an example is presented. All data presented in this chapter were computed by the University of Arizona IBM 7072 digital computer using the general set of Fortran computer programs presented in Chapter IV.

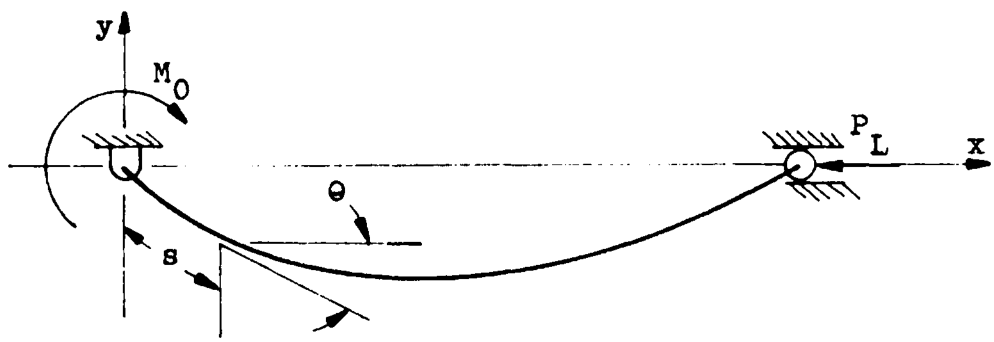
## 5.2 Definition of Sample Problem

The beam column is subjected to an end moment, $M_O$, at $s = 0$ and to an end load $P_L$, at $s = L$ as shown in Figure 5.1. The function for the horizontal load is a constant,

$$P = P_L \tag{5.1}$$

Also, the function for the vertical shear is constant and may be expressed

$$V = V_O = -M_O/\bar{Y} \tag{5.2}$$




Configuration of Beam Column for Sample Problem

Figure 5.1

The length of the beam column is one unit. All other parameters are defined so all units are consistent.

A plot of $P_L/P_e$ versus $\Theta_0$ for a given end moment is defined as a response curve. The term $P_L/P_e$ is proportional to the applied end load on the beam column. The term $\Theta_0$ is the slope of the elastic axis at $s = 0$. The response curves for the Ritz solution and the finite difference solution are plotted and compared in the following section for different values of $M_0/P_e$.

5.3 Ritz Solution

(a) Nonlinear Cosine Series

The Ritz minimizing condition for the sample problem using the cosine series solution is obtained by substituting equations (5.1) and (5.2) into equation

(2.8). Thus, the r algebraic equations to be solved for the coefficients $A_1$, $A_2$, ... $A_r$ in iteration form may be expressed as

$$A_m^{k+1} = \frac{2}{m^2}\left[\int_0^1 \left[\frac{P_L}{P_e}\sin\theta^k + \frac{M_0}{\bar{X}P_e}\cos\theta^k\right]\cos m\pi s \quad ds - \frac{M_0}{P_e}\right] \tag{5.3}$$

where m = 0, 1, ... r and where $\theta$ is defined by equation (2.4). The integral term in equation (5.3) containing $M_0/P_e\bar{X}$ is zero since it is the integral of an odd cosine function. Thus the $m^{th}$ Ritz cosine minimizing condition for the sample is

$$A_m^{k+1} = \frac{2}{m^2}\left[\int_0^1 \left[\frac{P_L}{P_e}\sin\theta^k\right]\cos m\pi s \; ds - \frac{M_0}{P_e}\right] \tag{5.4}$$

(b) Linear Cosine Series

If $M_0/P_e \neq 0$, the beam column is subjected to a nonsymmetrical load. Therefore, the linear cosine solution, equation (2.9), may be used for the purpose of obtaining the starting values $\theta_0$ and $\bar{X}$ for the finite difference procedure. The linear cosine Ritz minimizing conditions, equations (2.9), after substitution of equations (5.1) and (5.2) may be expressed as

$$A_m = 2\frac{M_0}{P_e} \Big/ \left[\frac{P_L}{P_e} - m^2\right] \tag{5.5}$$

where m = 1, 2, ... r.

(c) Linear Power Series

The linear power series could be used to determine a starting value for $\Theta_0$ and $\bar{X}$. For $P_L/P_e = 0.4$ and $M_0/P_e = 0.2$, the linear power series gives $\Theta_0 = -0.65$ for the one term $A_3$, and the linear cosine series gives $\Theta_0 = -.67$ for one term. The cosine and the power series give sufficiently accurate values of $\Theta_0$ and $\bar{X}$ to start the finite difference solution. It is recommended that the linear cosine series solution be used for all cases where equation (2.9) can be analytically integrated. If the cosine series solution requires numerical integration, then the linear power series or the nonlinear cosine series should be used.

5.4 Finite Difference Solution

For the finite difference solution, the terms $P_i$ and $V_i$ for the example problem may be expressed as

$$P_i = P_L$$

$$V_i = V_0 = -M_0/\bar{X}P_e \tag{5.6}$$

The length of the beam column is divided into 100 intervals, i.e., h = 0.01. The finite difference equations (3.2) after substitution of equations (5.6) may be expressed as

$$\theta_1 = \theta_0 + hB^2 \frac{M_0}{P_e} - \frac{h^2B^2}{2} \left[ \frac{P_t}{P_e} \sin \theta_0 + \frac{M_0}{P_e\overline{X}} \cos \theta_0 \right] \quad (5.7a)$$

for i = 0 and

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} - h^2B^2 \left[ \frac{P_t}{P_e} \sin \theta_i + \frac{M_0}{P_e\overline{X}} \cos \theta_i \right] \quad (5.7b)$$

for i = 1, 2, ... 99.

The initial value of $\theta_0$ is determined from the cosine series Ritz solution, equation (5.4). The initial value of $\overline{X}$ is determined by equation (3.8) and may be expressed as

$$\overline{X}_{Ritz} = \int_0^1 \cos \left( \sum_{n=1}^{r} A_n \cos n\pi s \right) ds \quad (5.8)$$

which may be integrated numerically.

The finite difference solution can be determined by using equations (5.6), (5.7), and (5.8) and by following the procedure outlined in section 3.4. The above numerical solution can be assumed as exact since the error is of the order of magnitude of $h^2$. Since h is 0.01, the error in the finite difference solution is

$$\text{ERROR} \approx 0.0001$$

5.5 Numerical Results

The first sample problem is a beam column with a symmetrical load, i.e., with an end load $P_L$ and with $M_0 = 0$.

A plot of the response curves for this case is shown in Figure 5.2 for both the one term, $r = 1$, Ritz cosine solution and the finite difference solution. The finite difference solution gives the same values as the exact solution plotted in reference 1. The response curves for the one term Ritz solution has a maximum error of 3.33% within the range of parameters evaluated.

Ritz solution response curves for $M_0/P_e = 0.05$, 0.10, 0.15, and 0.20 are shown in Figures 5.3, 5.7, 5.9, and 5.11. The three response curves in each figure are for

(a) a one term cosine Ritz solution,

(b) a three term cosine Ritz solution,

(c) a ten term linear cosine Ritz solution.

The response curves for the same values of $M_0/P_e$ as above as determined by the finite difference solution are shown in Figures 5.4, 5.8, 5.10, and 5.12.

A detailed analysis is presented for the case of $M_0 = 0$. There are two distinct regions of solution on the response curves for both the Ritz solution and the finite difference solution as shown in Figures 5.3 and 5.4. The first region is for $\theta_0$ greater than zero and the second is for $\theta_0$ less than zero. The response curves for the Ritz solution show that there is no discontinuity in the solutions for the positive $\theta_0$ region and for the negative $\theta_0$ region. For example, the negative $\theta_0$ region of the Ritz cosine response curves are continuous for $P_L$ from minus infinity to plus infinity as shown in Figure 5.3. As $P_L$ varies from minus infinity to plus infinity, $\bar{\bar{X}}$ varies from 1 to 0 as shown in Figure 5.5a. The finite difference response curve for the negative $\theta_0$ region is discontinuous as shown in Figure 5.4. The finite difference response curve indicates that as $P_L$ varies from minus infinity to some $P_{c1}$, $\bar{\bar{X}}$ varies from 1 to $\bar{\bar{X}}_{c1}$ where $\bar{\bar{X}}_{c1}$ is positive. The finite difference response curve also indicates that for $P_L$ greater than some $P_{c2}$ but less than plus infinity, $\bar{\bar{X}}$ varies from $\bar{\bar{X}}_{c2}$ to -1 where $\bar{\bar{X}}_{c2}$ is negative. The region between $P_{c1}$ and $P_{c2}$ where $\bar{\bar{X}}$ is between $\bar{\bar{X}}_{c1}$ and $\bar{\bar{X}}_{c2}$ is an indeterminate region where no solution could be obtained for the finite difference solution.

If $M_0$ is negative, the two regions of solution for $\theta_0$ are reversed. The curves in Figures 5.8, 5.10, and 5.12 indicate that as $M_0/P_e$ is increased, the indeterminate region between $P_{c1}$ and $P_{c2}$ is increased. When $M_0$ is zero, the indeterminate region does not exist as indicated by Figure 5.2. Figure 5.6 is a plot of the deflection shape of the beam column for the two discontinuous segments of the response curve for both regions of solution in Figure 5.4.

A typical finite difference solution for every tenth interval along the beam column for $P_L = 1.0618\ P_e$ and $M_0 = 0.05\ P_e$ as calculated by the IBM 7072 computer is shown in Table 5.1.

## 5.6 Conclusions

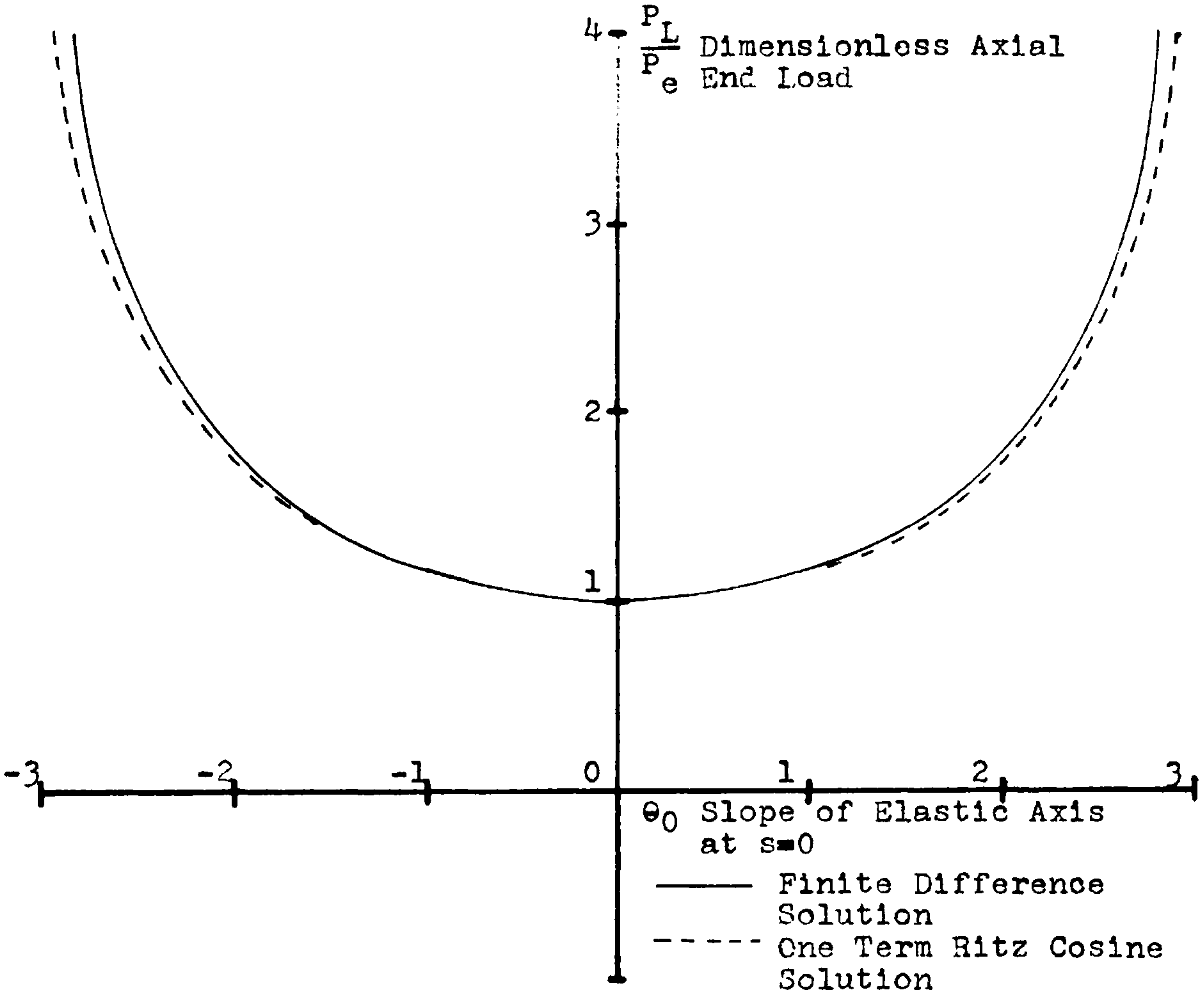
The following conclusions can be drawn from the sample problem:

1. The one term Ritz solution is accurate for a simple column, i.e., $V = 0$ and $P = P_L$.

2. The cosine series Ritz solution provides good values for $\Theta_0$ and $\bar{X}$ which may be used to start the finite difference solution for the sample problem.

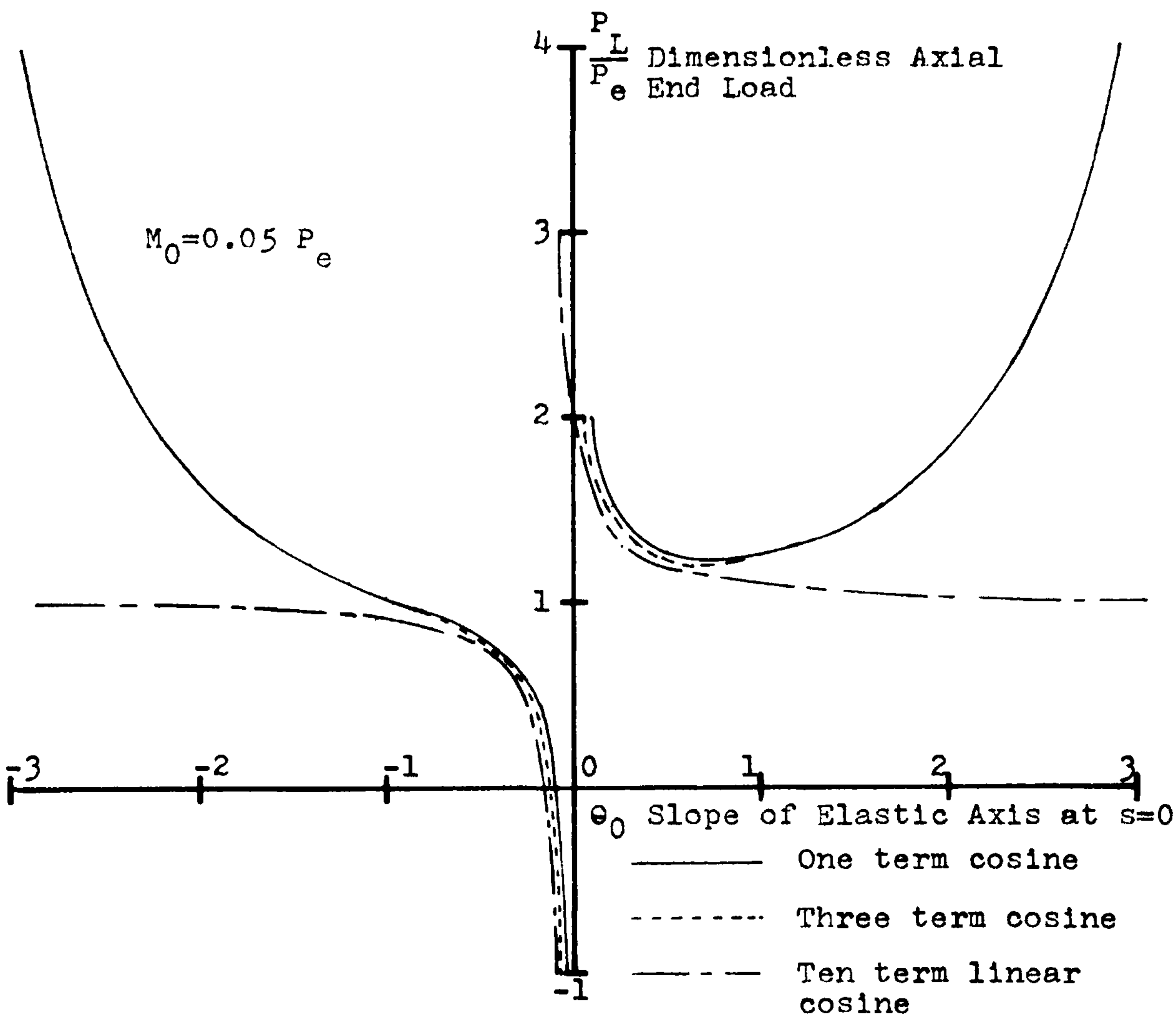
3. For $\Theta_0$ less than one radian, the 10 term linear cosine series solution gives better values for $\Theta_0$ and $\bar{X}$ than the one term nonlinear cosine series and it is easier to use.

4. The finite difference procedure developed in this thesis is practical and accurate. The combined computer time necessary to determine $\Theta_0$ and $\bar{X}$ with a one term Ritz solution and then to determine the finite difference solution for one set of parameters was less than one minute on the IBM 7072 digital computer.




Comparison of Response Curves for Finite Difference Solution and One Term Ritz Cosine Solution

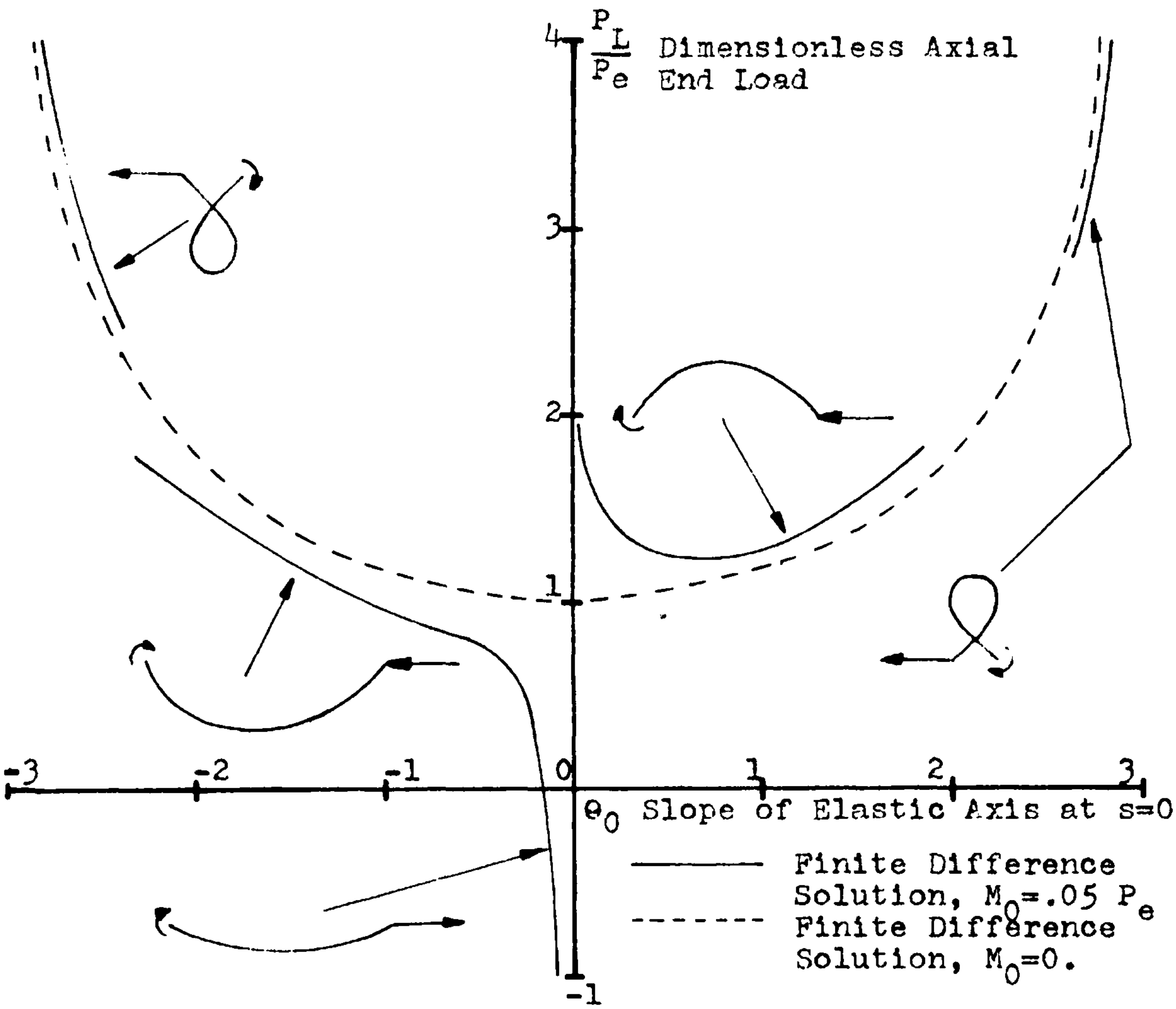
Figure 5.2




Ritz Response Curves for Sample Problem

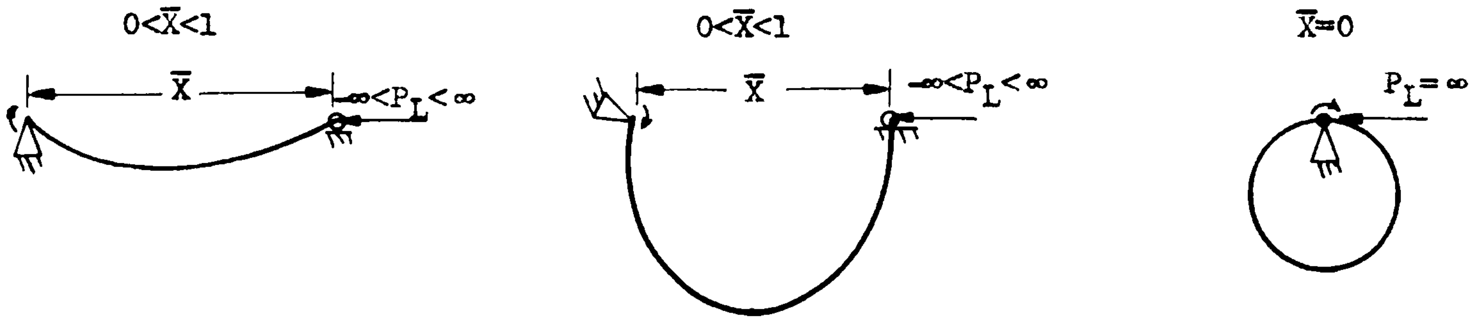
with $M_0$=0.05 $P_e$

Figure 5.3



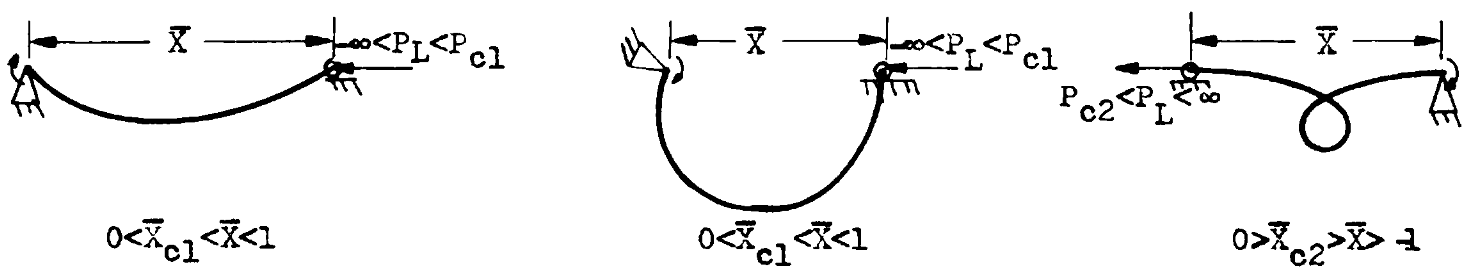

Finite Difference Response Curves for Sample Problem with $M_0$=0.05 $P_e$

Figure 5.4




Ritz Deflection Shapes of the Beam Column for Various End Loads

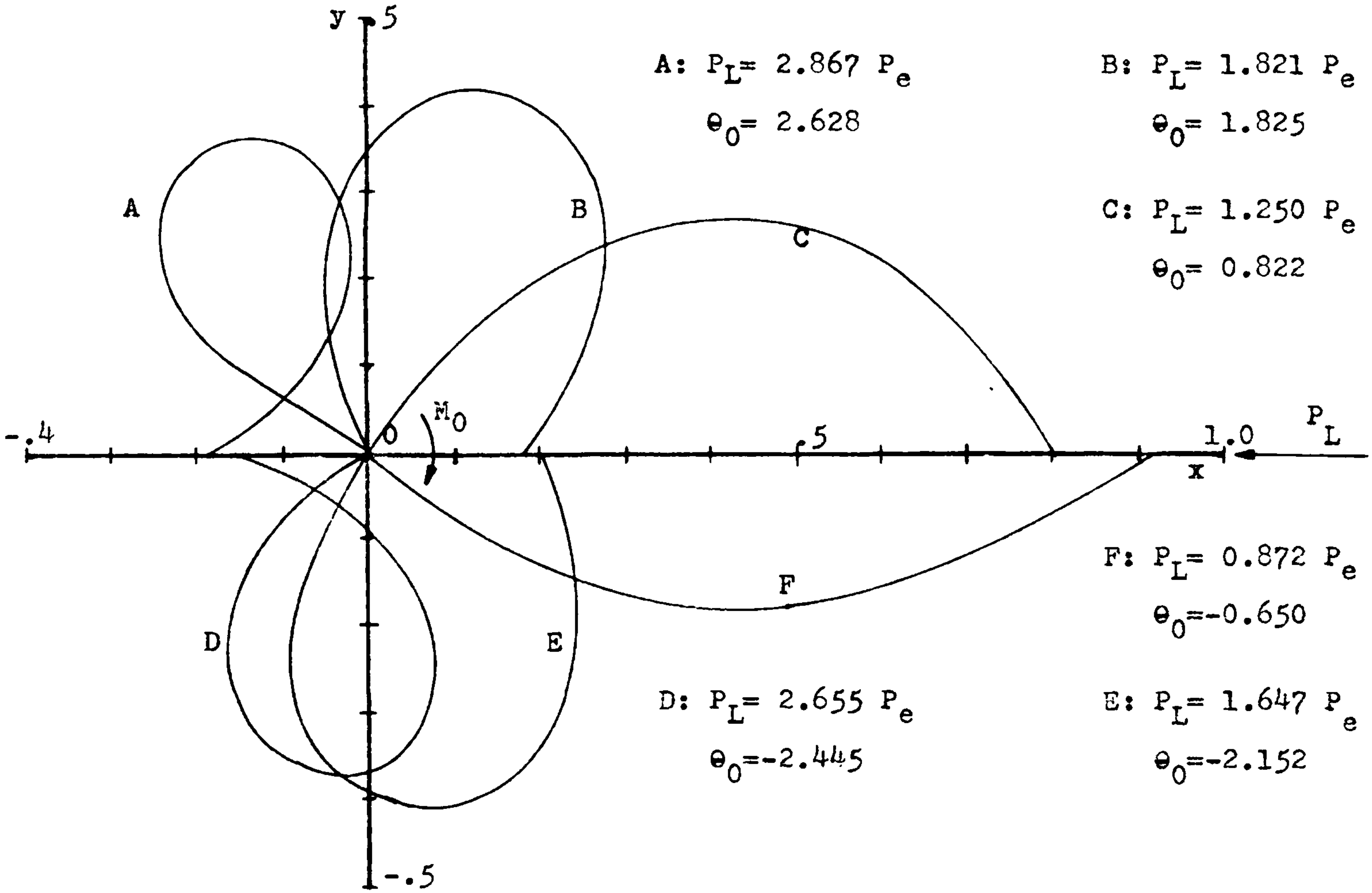
Figure 5.5a




Finite Difference Deflection Shapes of the Beam Column

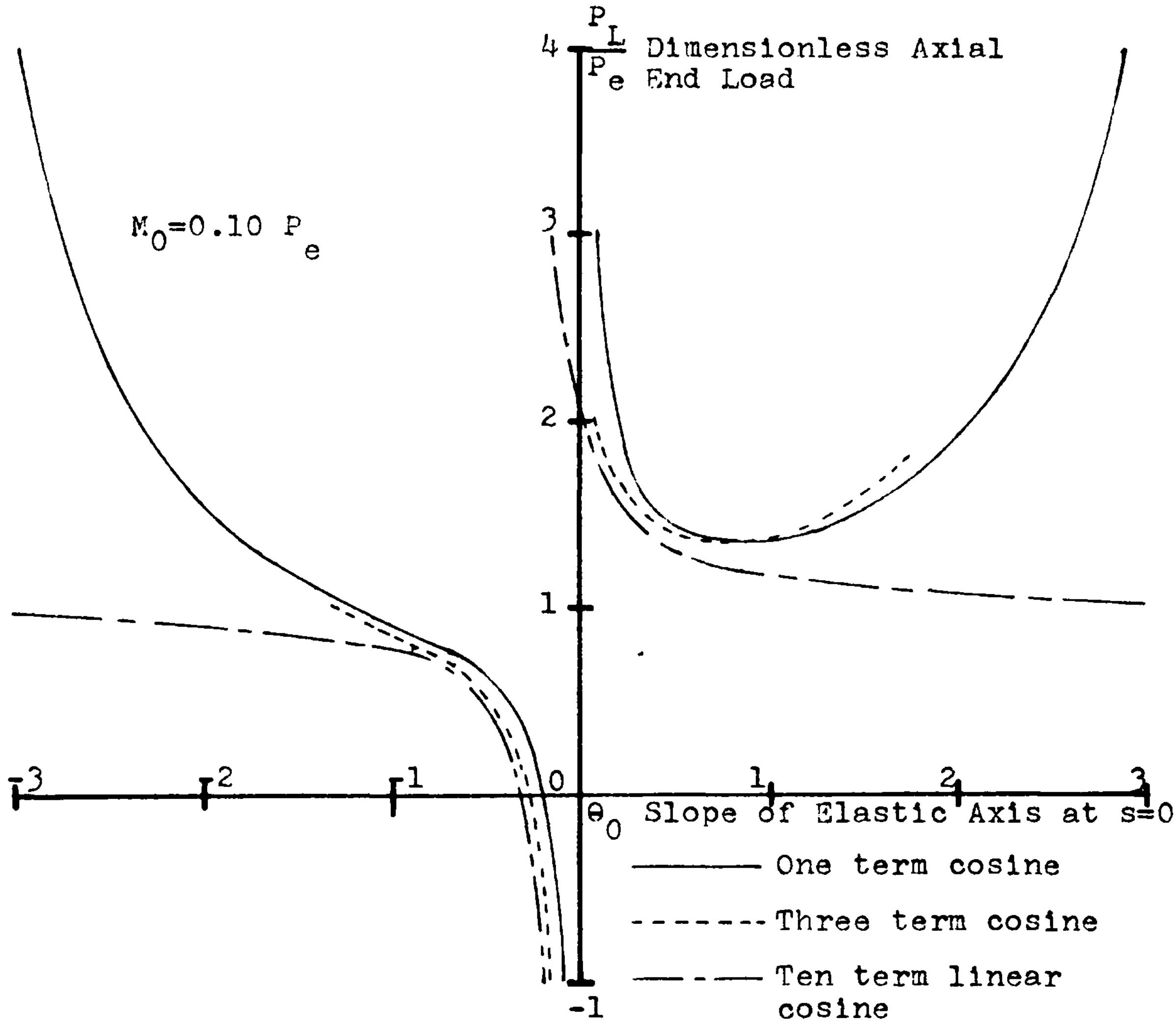
for Various End Loads

Figure 5.5b




Deflection Shape of the Beam Column with $M_0$=0.05 $P_e$ for Various End Loads

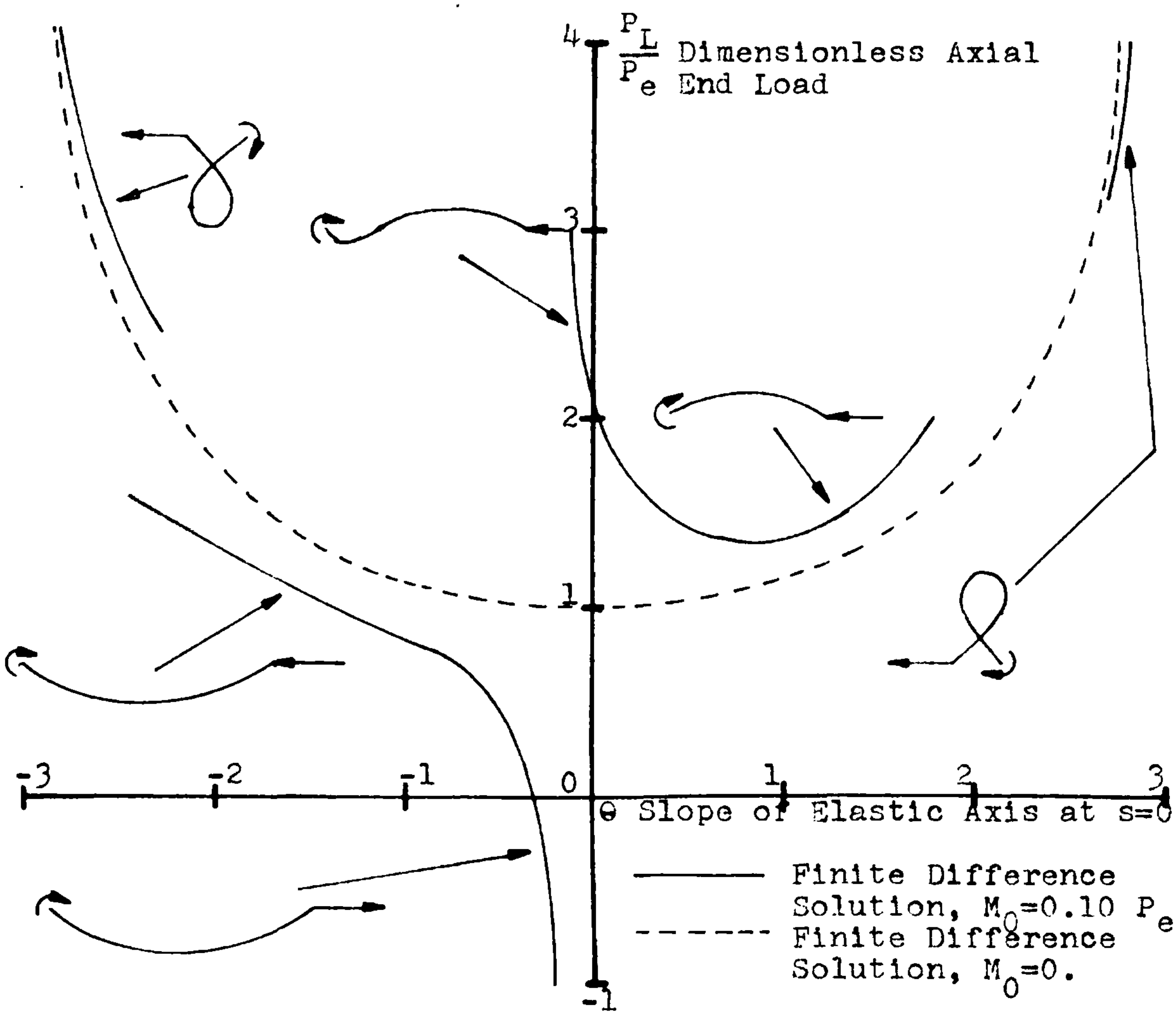
Figure 5.6




Ritz Response Curves for Sample Problem

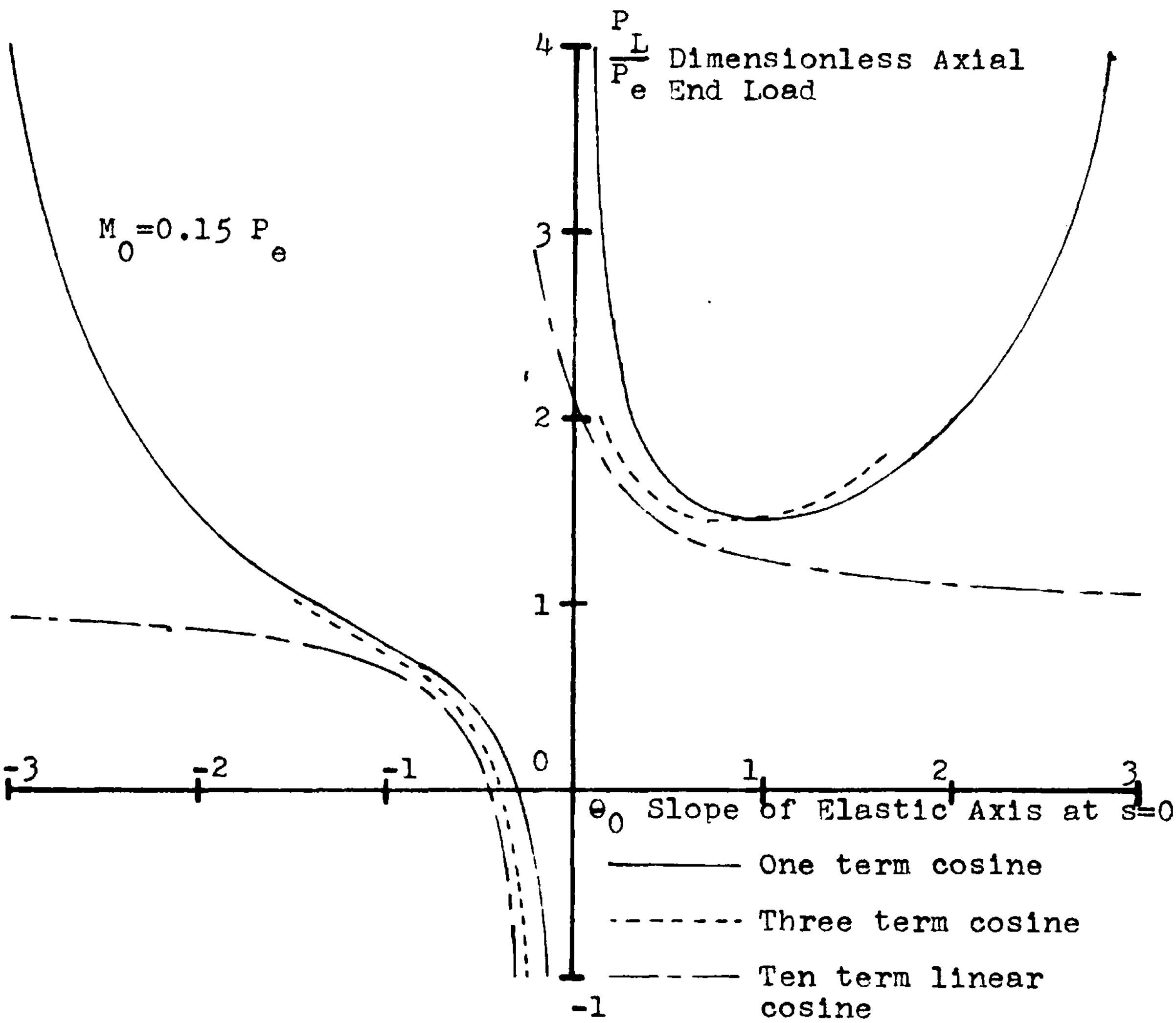
with $M_0$=0.10 $P_e$

Figure 5.7




Finite Difference Response Curves for Sample Problem with $M_0$=0.10 $P_e$

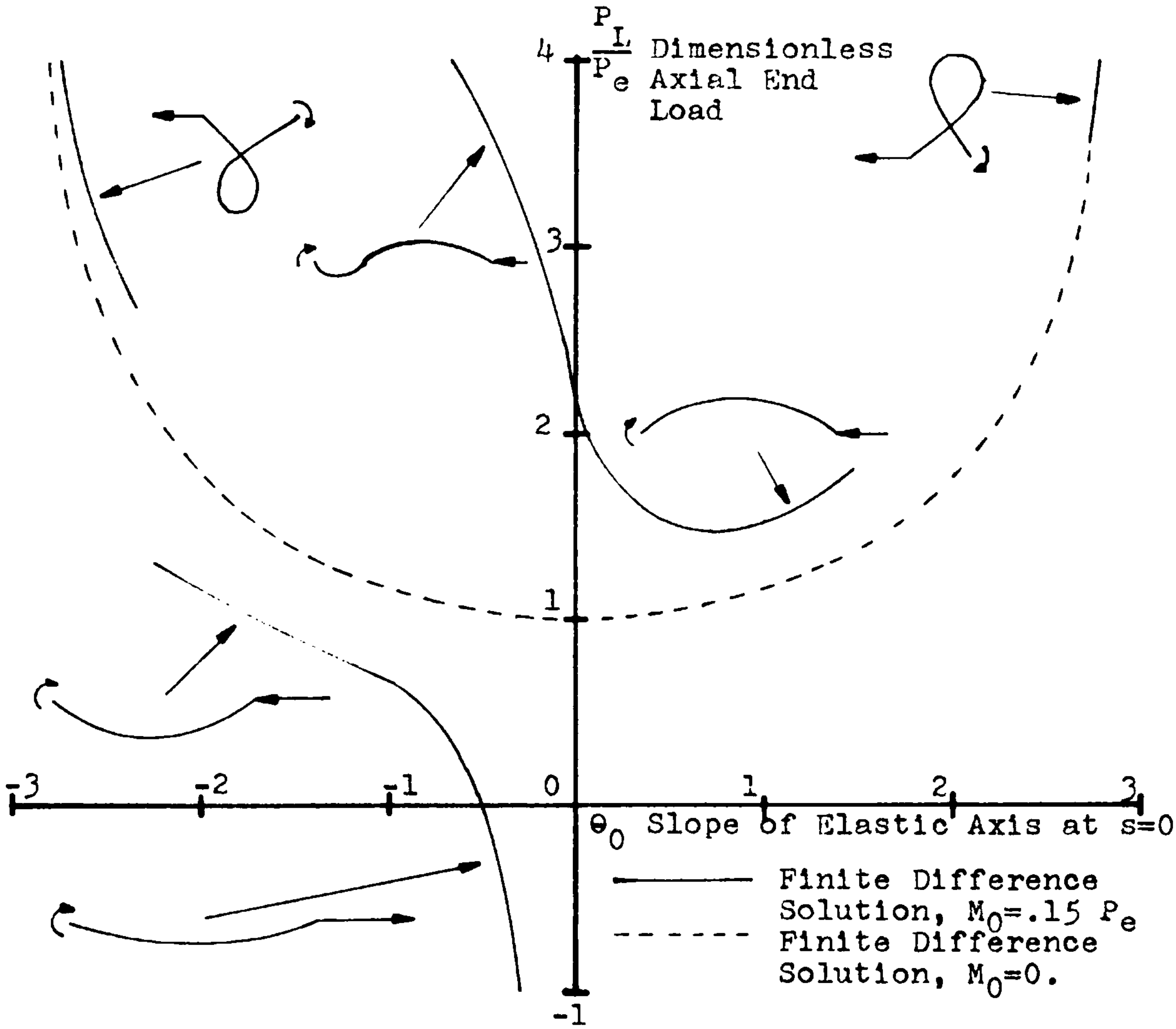
Figure 5.8




Ritz Response Curves for Sample Problem

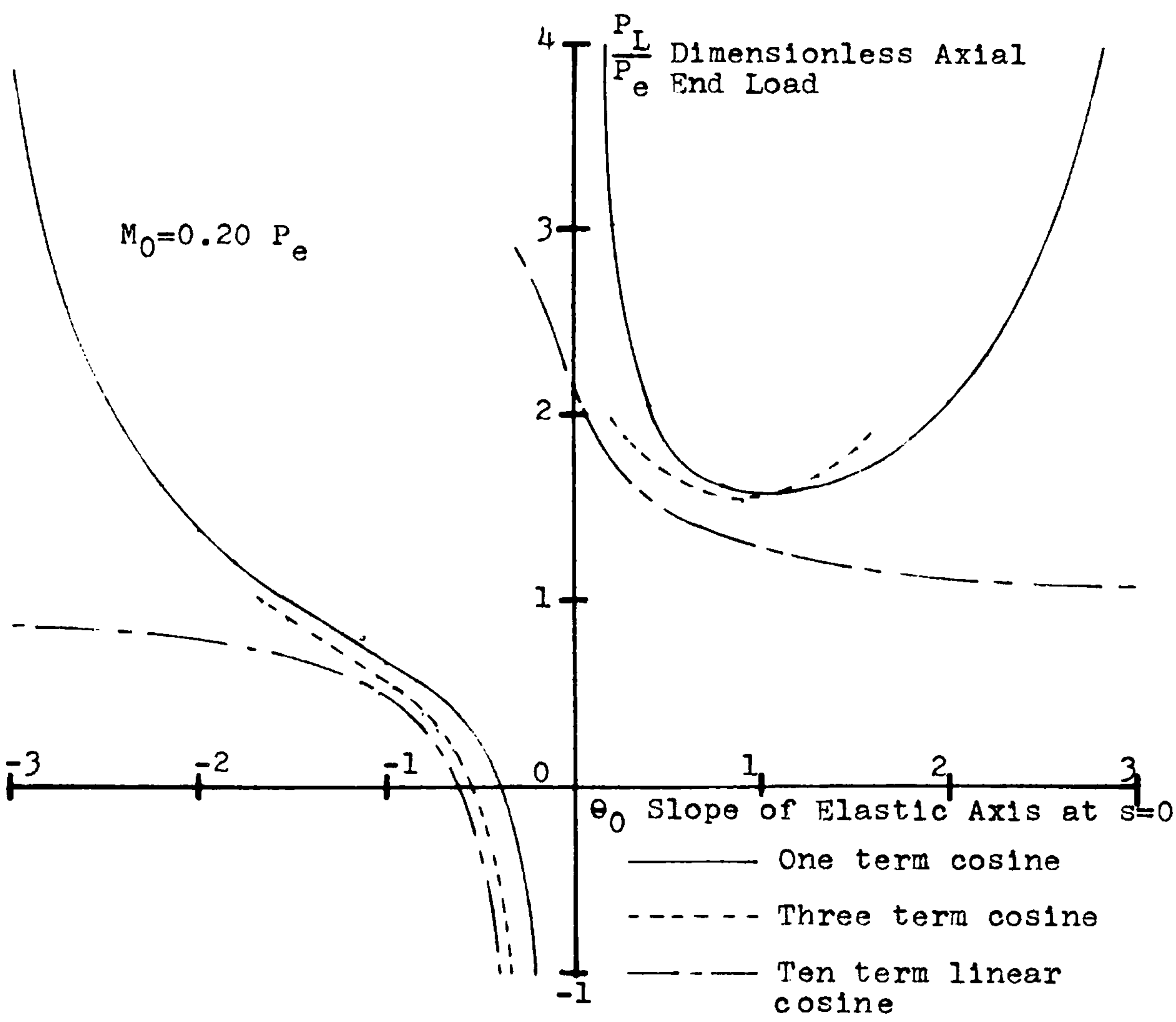
with $M_0$=0.15 $P_e$

Figure 5.9




Finite Difference Response Curves for Sample Problem with $M_0$=0.15 $P_e$

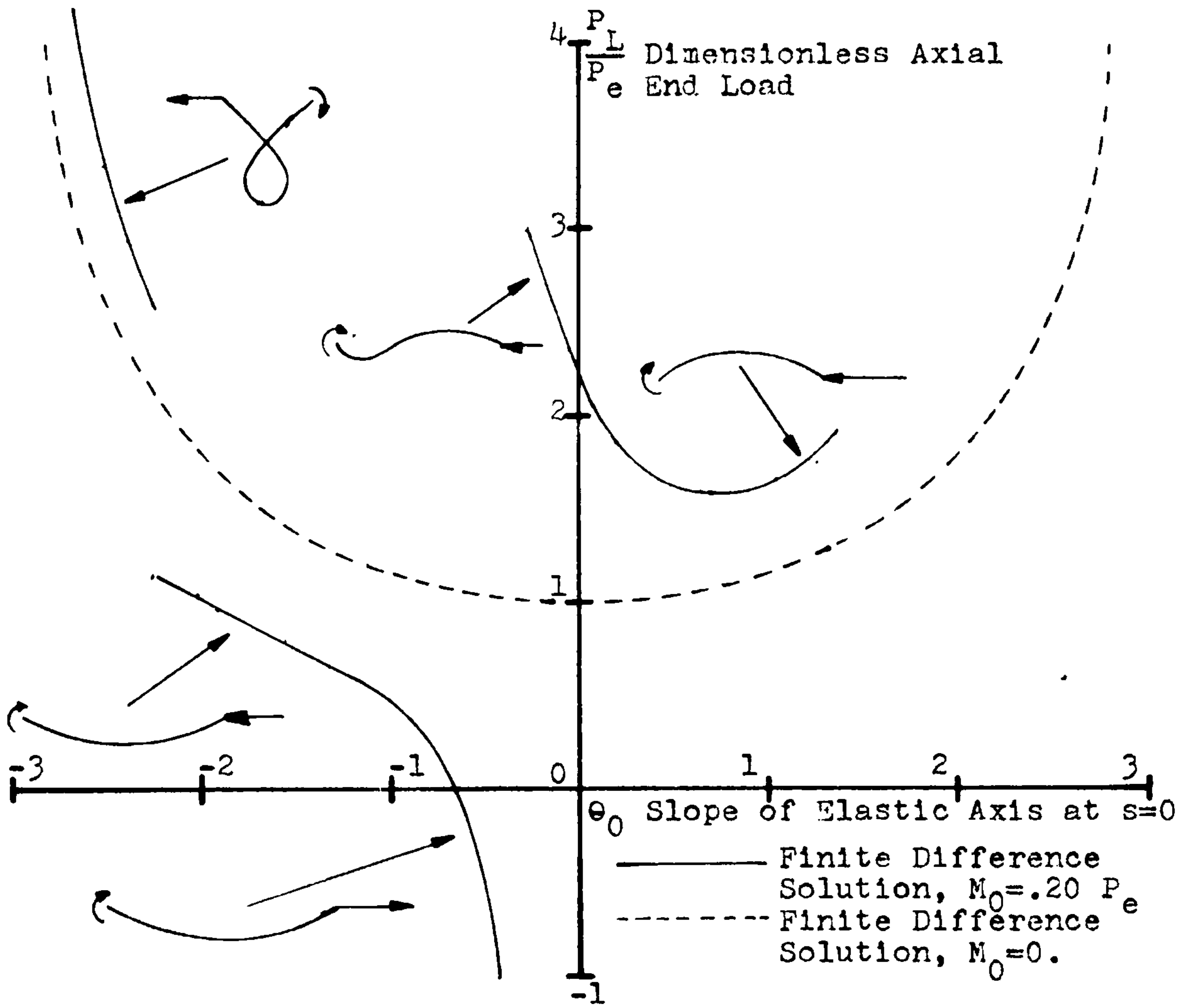
Figure 5.10




Ritz Response Curves for Sample Problem

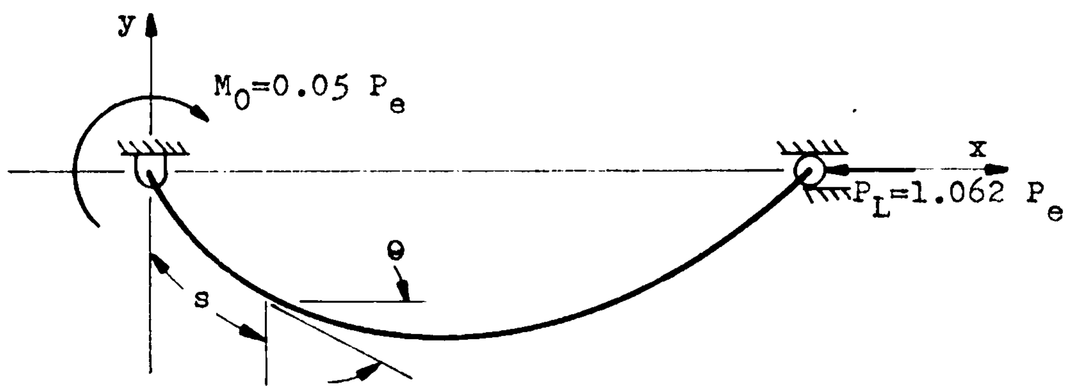
with $M_0$=0.20 $P_e$

Figure 5.11




Finite Difference Response Curves for Sample Problem with $M_0$=0.20 $P_e$

Figure 5.12




| s | θ | dθ/ds | x | y |
|---|---|---|---|---|
| 0.0000 | -1.1445 | 0.4935 | 0.0000 | 0.0000 |
| 0.1000 | -1.0500 | 1.3955 | 0.0436 | -0.0900 |
| 0.2000 | -0.8695 | 2.2153 | 0.0994 | -0.1733 |
| 0.3000 | -0.6150 | 2.8742 | 0.1724 | -0.2426 |
| 0.4000 | -0.3073 | 3.2804 | 0.2624 | -0.2885 |
| 0.5000 | 0.0248 | 3.3615 | 0.3626 | -0.3031 |
| 0.6000 | 0.3480 | 3.1040 | 0.4622 | -0.2838 |
| 0.7000 | 0.6305 | 2.5496 | 0.5509 | -0.2352 |
| 0.8000 | 0.8477 | 1.7941 | 0.6242 | -0.1659 |
| 0.9000 | 0.9835 | 0.9223 | 0.6836 | -0.0850 |
| 1.0000 | 1.0297 | 0.0009 | 0.7352 | 0.0008 |

Finite Difference Solution for a Sample Problem

Table 5.1

## CHAPTER VI

## RECOMMENDATIONS AND CONCLUSIONS

### 6.1 Conclusions

1. The finite central difference method presented in this thesis is a practical and accurate method for determining the shear, bending moment, slope and deflection for a simply supported elastic beam column.

2. The one term cosine Ritz method gave an accurate solution for the example where the loading on the beam column was symmetrical, i.e., the vertical loads and end moments zero and the horizontal load a constant.

3. The linear cosine, the one term cosine, and the three term cosine series Ritz solutions provide good initial values for $\Theta_0$ and $\bar{X}$ for the finite central difference solution for cases where the load on the beam column was nonsymmetrical.

### 6.2 Recommendations

1. It is recommended that a method which will explain the regions where the finite difference solution

of the differential equation becomes indeterminate be developed, i.e., $\bar{X}_{c1}$, $\bar{X}_{c2}$, $P_{c1}$, and $P_{c2}$ as shown in Figure 5.4.

2. It is recommended that the theory developed in this thesis be extended for a varying moment of inertia.

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