DESIGN OF SAMPLED-DATA CONTROL SYSTEMS USING THE SECOND METHOD OF LIAPUNOV

by

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ABSTRACT

This investigation is concerned with the design of sampled-data control systems.

The design criterion is primarily based on the minimization of a performance index by using the Second Method of Liapunov.

The performance index is of the form

$$I = \sum_{k=1}^{\infty} \left[\underline{x}^t(k)Q\underline{x}(k) + \alpha^2 u^2(k-1) \right]$$

where Q is an nxn positive definite matrix, $\underline{x}(k)$ is the state vector of the element to be controlled, u(k) is the control input and α^2 is a weighting factor indicative of the "cost of control." By using the Liapunov functions it is shown that Q may also be positive semidefinite. This is useful in order to interpret the performance index in terms of response to a differential equation model.

In using the Second Method, difficulties arise in the analytic evaluation and minimization of the performance index. An alternative formulation of the problem based on dynamic programming is given, and the results are illustrated with a number of examples.

CHAPTER I

INTRODUCTION

During the past few years the analysis and design of control systems by the state variable approach has been worked out by numerous investigators. In sampled-data control systems this new technique offers some advantages over the classical z transform methods. The principal advantages of the state variable approach are

1. It provides a direct way of investigating system behavior at instants of time other than the sampling points.
2. It gives a direct way of knowing the behavior at points in the system other than the actual output or position state.
3. It provides the form of system representation necessary for the design of optimum systems using either the Second Method of Liapunov or Dynamic Programming.

This thesis is concerned with the design of sampled-data control systems in the time domain. The representation of the dynamical system is then in state variables.

Gibbons (1965) has shown that by feeding back all the state variables of a plant, it is possible to achieve deadbeat response. In a book by Tou (1964) it has been proved that in order to obtain an optimum system, given a quadratic type performance index, it is also necessary to feedback all the state variables of the plant. A basic assumption of this thesis is that the optimum design is to be achieved by feeding back all of the state variables through constant gain elements.

The systems discussed are restricted to the ones which assure asymptotic stability.

In Chapter II the state variable representation of the dynamical system is given with emphasis on phase variables.

Performance indices of quadratic form are studied in Chapter III. The performance criterion is associated with the differential equation which describes the model or ideal system. A method of obtaining beforehand the maximum deviation of the actual response with respect to the model is presented.

In Chapter IV a discussion on the Second Method of Liapunov is given. Two attempts to evaluate and minimize analitically the performance index by using the Second Method are developed. In the first approach the performance index is not directly related to the model and is only good for open loop stable plants. In the second procedure, the performance index is referred to the model and is useful for plants which are open loop stable or unstable. For the second case the optimum system is obtained by differentiating I with respect to each parameter and setting the partial derivatives equal to zero. This is done in an example for a second order system with only one adjustable parameter. Difficulties in finding the parameters for the optimum system arise for plants of order higher than two, since non-linear equations with respect to the feedback parameters are obtained. An alternative formulation based on dynamic programming is used to avoid this difficulty. A relationship between the optimum Liapunov function and the optimum return by using dynamic programming is obtained in Chapter V.

Chapter V also offers design examples performing the optimization of the system by means of dynamic programming. Systems of order two and three with and without zeroes are studied in the three examples. The performance indices are referred to models based on first and second order differential equations.

Finally in Chapter VI suggestions for other investigations are offered.

CHAPTER II

SYSTEM DESCRIPTION

2.1 Introduction

This chapter is concerned with the time-domain representation of an autonomous sampled-data control system. This time-domain description is particularly well suited to the study of optimal control.

The primary representation of the dynamical system is the state variable approach. Among the references available on states variables are the books by Lindorff (1965), Freeman (1965) and Tou (1964). The fundamental advantage of the time-domain approach over the classical z transform is that it provides a total description of the dynamic state of the plant, in contrast to the output/input relationship which is obtained by the z transform technique.

2.2 System Representation

Before deriving the state equations for a sampled-data system, it is convenient to define the terminology concerning the elements of this system.

The plant, $G_p(s)$, (see Fig. 2-1) is the element to be controlled. It is assumed to be described by an nth order linear differential equation with constant coefficients and might represent a chemical process, a missile,

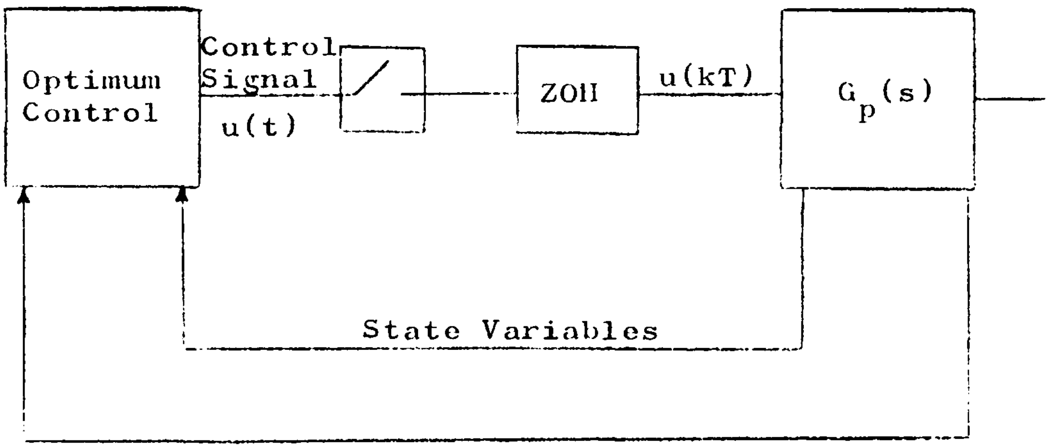


Fig. 2.1 Block Diagram Representation of a Sampled-Data Optimum Control System

etc. If one tries to simulate such plant by analog computers, this requires amplifiers, summing devices and n integrators. The outputs x_1, x_2, . . . x_n of the n integrators, at any instant of time, are the state of the plant at that instant (Kalman et al., 1958). It is to be noticed that the state variables can be chosen in different ways; the output is often one of them.

The plant is controlled by the control input u(kT). This control signal is assumed to be generated by the values of the state variables of the plant at the sample instants, if an optimum system is desired (see Chapters IV and V). It is assumed also that the plant is completely controllable and completely observable.

Specification of the Plant

For the present work it is sufficient to describe the plant by the linear differential equation with constant coefficients

$$
\begin{aligned}
\dot{\underline{x}}(t) &= A\,\underline{x}(t) + \underline{b}\,u(kT) \\
y(t) &= \underline{g}^t\,\underline{x}(t)
\end{aligned}
\qquad (2\text{-}1)
$$

where

$\underline{x}(t)$ is the state vector of the plant, assumed expressed in phase variables.

u(kT) is the control input.

y(t) is the scalar output.

A is the nxn state transition matrix for the continuous plant.

$\underline{b}$ is the distribution vector for the continuous plant.

$\underline{g}$ is the vector output of the plant.

Since a sampled-data feedback with zero order hold is assumed to be implemented at the control input, the description of the entire open loop plant as given in equation (2-1) may be written in sampled-data form by the two state equations

$$\underline{x}\,(k+1) = \emptyset\,(1)\,\underline{x}\,(k) + \underline{\beta}\,(1)\,u(k)$$
$$y\,(k+1) = \underline{g}^t\,\underline{x}\,(k+1) \qquad (2\text{-}2)$$

where

$\underline{x}(k)$ is the state vector of the plant at the time $t = k$ (the sampling period is assumed to be 1 for simplicity)

$\emptyset(1)$ is the nxn state transition matrix of the sampled-data system

$\underline{\beta}(1)$ is the nx1 distribution matrix of the system

$\underline{g}$ is the output vector

The state transition matrix $\emptyset(1)$ and $\underline{\beta}(1)$ may be found by a variety of manners. One of them is

Describe the differential equation of the continuous plant by its Laplace transformation in the following way: Transform equation (2-1) to the frequency domain and solve for $\underline{X}(s)$, as

$$\underline{X}(s) = \left[sI - A\right]^{-1} \underline{x}(0) + \left[sI - A\right]^{-1} \underline{b}\, u(s)$$

Inverting to time domain

$$\underline{x}(t) = \phi(t)\, \underline{x}(0) + \int_0^{t-} \phi\,(t - \tau)\, \underline{b}\, \underline{u}(\tau)\, d\tau \qquad (2\text{-}3)$$

where

$$\phi(t) = \mathcal{L}^{-1} \left[\left\{ (sI - A)^{-1} \right\} \right] \qquad (2\text{-}4)$$

At a particular time t = k

$$\underline{x}(k) = \phi(k)\, \underline{x}(0) + \int_0^{k-} \phi(k - \tau)\, \underline{b}\, u(\tau)\, d\tau$$

By taking into consideration the exponential properties of $\phi(t)$ and multiplying both members of Eq. (2-4) by $\phi(k+1-k)$

$$\phi(1)\, \underline{x}(k) = \phi(k+1)\, \underline{x}(0) + \int_0^{k-} \phi\,(k+1 - \tau)\, \underline{b}\, u(\tau) d\tau$$

or $$\phi(k+1)\underline{x}(0) = \phi(1)\underline{x}(k) - \int_0^{k-} \phi\,(k+1 - \tau)\, \underline{b}\, u(\tau) d\tau$$

Upon substitution in Eq. (2-3) where t = k+1

$$\underline{x}(k+1) = \phi(1)\, \underline{x}(k) + \int_k^{(k+1)-} \phi(k+1-\tau)\, \underline{b}\, u(\tau) d\tau$$

where

$$\phi(1) = \mathcal{L}^{-1}\left[\left\{ [sI - A]^{-1} \right\}\right] t = 1$$

$\beta(1)$ may be found from the second term of Eq. (2-3)

$$\int_{0}^{t-} \phi(t - \tau)\, \underline{b}\, u(\tau)\, d\tau$$

Since the system has a zero order hold, $u(\tau)$ is constant within the region of integration, or $u(\tau) = u(k)$. Also, since the system is specified in phase variables,

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

Therefore

$$\beta(1) = \int_{0}^{1} \phi\,(1 - \tau) \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} d\tau \qquad (2\text{-}6)$$

The following two examples are given by way of illustration, and to note a difficulty that arises in plants with zeroes.

<u>Example 2-1 (Plant with no zeroes)</u>

Find the state transition matrix of the open loop system of Fig. 2.2 (Assume T = 1 sec., as before).

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

In time domain

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$$

Let

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

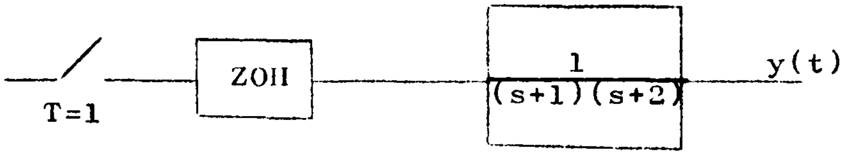
Then

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,u(t)$$

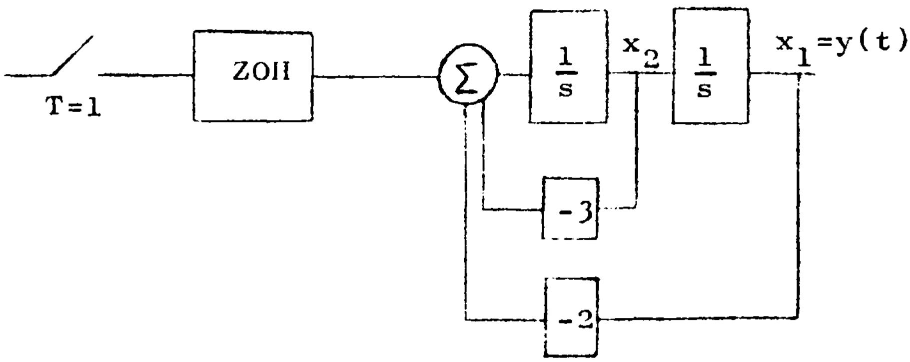
where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\left\{ \left[sI - A\right]^{-1} \right\}$$



(a) Block Diagram Representation of Open-Loop Plant



(b) Block Diagram Representation in Phase Variables

Fig. 2.2 Block Diagram Representation for Plant of Example 2-1

$$= \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \right\}$$

From Eqs. (2-5) and (2-6) one may find $\phi(1)$ and $\underline{\beta}(1)$

$$\phi(1) = \begin{bmatrix} 0.601 & 0.233 \\ -0.466 & -0.098 \end{bmatrix}$$

$$\underline{\beta}(1) = \begin{bmatrix} 0.200 \\ 0.233 \end{bmatrix}$$

In this example, the output $y(k+1)$ is only given by one state variable, $x_1(k+1)$, which facilitates the design for generalized performance index, as is explained in Chapter III.

<u>Example 2-2 (Plant with one zero)</u>

Given the open loop system shown in Fig. 2.3 derive the state differential equations and the state transition equations. Assume T = 1 sec.

$$G_p(s) = \frac{s+2}{s(s+1)}$$

Let x_1 and x_2 be the state variables as shown in same figure.

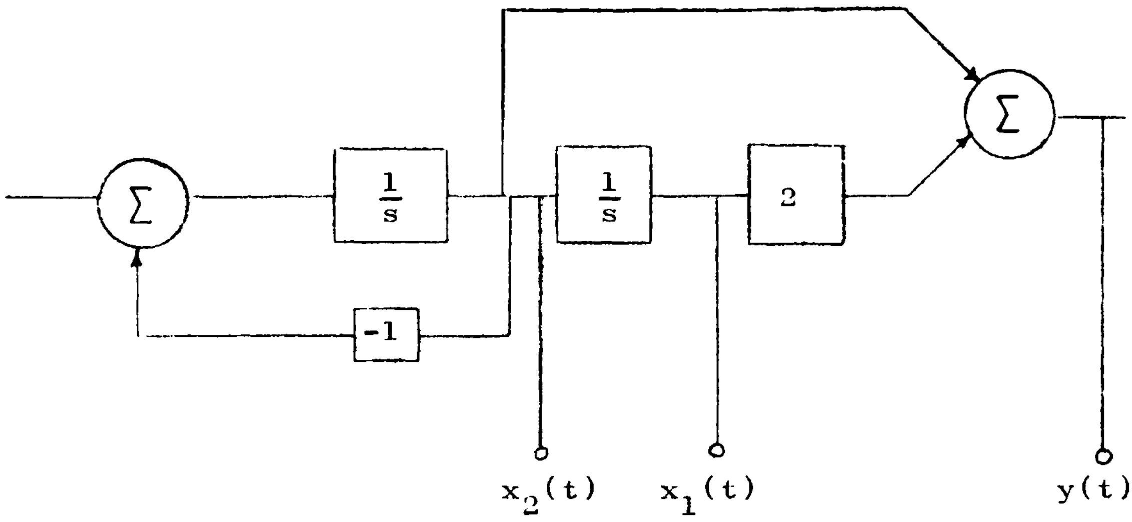


Fig. 2.3 Block Diagram Representation for Plant of Example 2-2

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + u$$

The output y(t) is given by

$$y(t) = 2x_1 + x_2$$

In matrix form

$$\dot{x}(t) = A\ \underline{x}(t) + \underline{b}\ u(t)$$

$$y(t) = \underline{g}^t \underline{x}(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

From Eq. (2-4)

$$\mathcal{L}\left|\phi(t)\right| = \left|sI - A\right|^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

and

$$\phi(t) = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\phi(1) = \begin{bmatrix} 1 & 0.632 \\ 0 & 0.368 \end{bmatrix}$$

$$\beta(1) = \int_0^1 \begin{bmatrix} 1 & 1 - e^{-(1-\tau)} \\ 0 & e^{-(1-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix}$$

The state equations are then given by

$$\underline{x}(k+1) = \begin{bmatrix} 1 & 0.632 \\ 0 & 0.368 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} u(k)$$

Note here that the output of the system, y(k+1), is given by a linear combination of the two state variables

$$y(k+1) = 2\,x_1(k+1) + x_2\,(k+1)$$

This relationship between the output and the state variables of the plant brings a slight problem in the performance criterion presented in Chapter III, as it will be explained in the same place.

Specification of the Control Input

It has been proved (Tou, 1964) that in order to obtain an optimum system, given a quadratic type performance index, it is necessary that the control input be generated by the present information of the state variables. To use the Second Method of Liapunov one has to assume that indeed the control input signal is given by a linear combination of all the states variables of the plant (see Chapter IV). For an nth order plant with state variables x_1, x_2, . . . x_n, the control input is then described by

$$u(k) = \underline{c}^t \; \underline{x}(k) \tag{2-7}$$

where $\underline{c}^t = \begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix}$ are the feedback coefficients of the system.

Specification of the Closed-Loop System

Once one has the plant and the control input specified (see Fig. 2.4), the closed-loop is described by

$$\begin{aligned} \underline{x}(k+1) &= \emptyset(1) \; \underline{x}(k) + \underline{\beta}(1) \; u(k) \\ y(k+1) &= \underline{g}^t \; \underline{x}(k+1) \\ u(k) &= \underline{c}^t \; \underline{x}(k) \end{aligned} \tag{2-8}$$

The entire closed-loop system (plant + controller) may be then defined as

$$\underline{x}(k+1) = \emptyset_s(1) \; \underline{x}(k) \tag{2-9}$$

where

$$\emptyset_s(1) = \emptyset(1) + \underline{\beta}(1) \; \underline{c}^t$$

Further discussions on state variables may be found in the references cited earlier.

The discussion on system design presented in the following chapters is based on the system description that has been proposed in the previous pages.

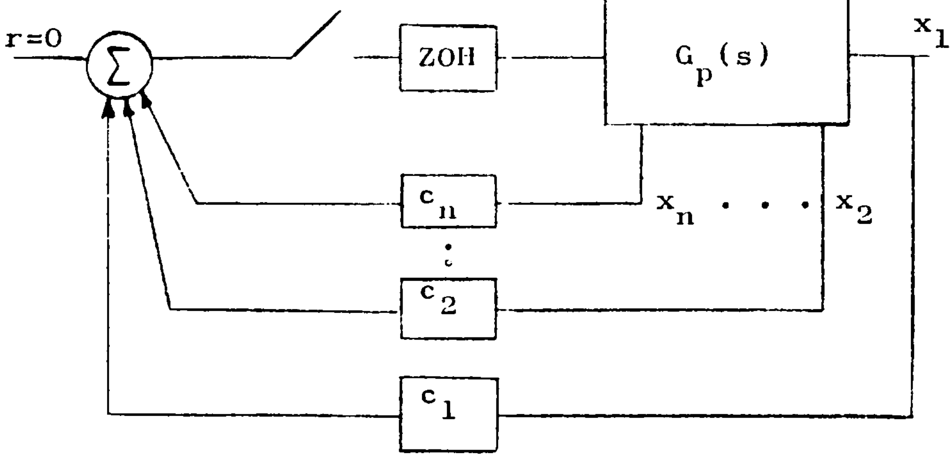


Fig. 2.4 Block Diagram Representation of an Autonomous Closed-Loop System

CHAPTER III

PERFORMANCE INDEX CRITERIA

3.1 Introduction

The analytical synthesis of control systems implies the existence of design criteria. It is the purpose of this chapter to discuss the design criterion used in this investigation. The design procedure is then primarily based on the optimization of a given performance criterion by using the Second Method of Liapunov.

Three generalized performance indices (the performance of the system is expressed in terms of the error and its derivatives), for sampled-data systems, are given by Tou (1963)

$$I\left[\underline{x}(t_o);\ \underline{u}\right] = E\sum_{k=1}^{N} F\ (\underline{x}(t_k)-\underline{r}(t_k),\underline{u}(t_k);t_k) \qquad (3\text{-}1)$$

$$I\left[\underline{x}(t_o);\ \underline{u}\right] = \sum_{k=1}^{N} F\left[\underline{x}(t_k),\underline{u}(t_k);t_k\right] \qquad (3\text{-}2)$$

$$I\left[\underline{x}(t_o);\ u\right] = F\left[\underline{x}(t_1),r(t_1),u(t_1);t_1\right] \qquad (3\text{-}3)$$

where N may be variable or fixed.

The performance index of interest in this work is of the form of equation (3-2), which may also be given by

$$I(N) = \sum_{k=1}^{N} \left[\underline{x}^t(k)\ Q_1\ \underline{x}(k) + \underline{u}^t(k-1)\ Q_2\ \underline{u}(k-1) \right] \qquad (3\text{-}4)$$

where $\underline{x}(k)$ is the state vector of the plant, Q_1 and Q_2 are positive definite matrices of order nxn and rxr respectively. n is the order of the system, r is the number of control inputs, and in general both are functions of the time k.

In Section 2 of this chapter, a discussion of the performance index (3-4) is given for the particular case that

$$N = \infty, \quad Q_1 = Q = \text{a constant matrix}, \quad Q_2 = 0$$

so that one may have

$$I = \sum_{k=1}^{\infty} \left[\underline{x}^t(k)\ Q\ \underline{x}(k) \right] \qquad (3\text{-}5)$$

The reason for doing this becomes clear when the modeling approach is explained.

In Section 3, a short discussion is presented for the case that both Q_1 and Q_2 influence I.

3.2 Modeling

One way of specifying overall system response is to specify some ideal trajectory in time. This is conveniently done by specifying that the trajectory be a solution to a particular differential equation with given initial conditions. Either the ideal trajectory or the differential equation is called the model. The object is to force the actual system to follow the model as closely as possible.

Aizerman (1963) has worked this problem for the case of continuous systems, with the simplest performance index of the form

$$I = \int_0^{\infty} \left[y^2 + \tau^2 \left(\frac{dy}{dt}\right)^2 \right] dt \tag{3-5}$$

where y(t) is the position output of the actual system.

Since the above performance index in general does not give enough flexibility to the designer, he suggests extending his criterion in the form

$$I = \int_0^{\infty} \left[y^2 + \tau_1^2 \left(\frac{dy}{dt}\right)^2 + \tau_2^4 \left(\frac{dy^2}{dt^2}\right)^2 + \cdots \right.$$

$$\left. + \tau_m^{2m} \left(\frac{dy^m}{dt^m}\right)^2 \right] dt \tag{3-6}$$

In trying to use the second order case, Higgins (1964) found that the ideal model must have a minimum damping factor $\xi \geqq 0.707$, which is a limitation on the choice one might have.

In view of the above limitation one could think of using a more applicable criterion such as

$$I = \int_0^{\infty} \left[y + \tau_1 \frac{dy}{dt} + \tau_2 \frac{d^2y}{dt^2} + \ldots + \tau_m \frac{dy^m}{dt^m} \right]^2 dt \qquad (3\text{-}7)$$

If one has a free system, i.e., a system with enough adjustable parameters, it is possible to make the actual system follow the model, obtaining in this way the absolute minimum value of the performance index. This absolute minimum value is zero, and is referred to as Iminmin. Since in general the adjustable parameters are limited, one must be satisfied in finding the parameter values for the "best" response, corresponding to the minimum value of I, or Imin. Obviously Imin > Iminmin. The procedure by which this can be done in the sampled-data case is explained later. Before considering the sampled-data case, however, it is interesting, to determine the significance of the model by examining the maximum possible deviation of the system response from the ideal response in the continuous case.

Consider in first place a second order continuous system. In this case assume a performance index defined by

$$I = \int_0^{\infty} \left[y(t) + \tau \frac{dy}{dt}(t) \right]^2 dt \qquad (3\text{-}8)$$

Let $y_o(t)$ be such that $y_o(t) + \tau \frac{dy_o}{dt} = 0$ with given initial conditions. $y_o(t)$ is then the response of the model to given initial conditions.

Now let y(t) be the actual response for the optimum system. The deviation $\delta y(t)$ of y(t) with respect to $y_o(t)$ is defined by

$$\delta y(t) \triangleq y(t) - y_o(t)$$

or y(t) is given by

$$y(t) = y_o(t) + \delta y(t) \tag{3-9}$$

Imin as a function of the deviation is then given by

$$\begin{aligned}
\text{Imin} &= \int_0^\infty \left[y_o(t) + \delta y(t) + \tau \dot{y}_o(t) + \tau \delta \dot{y}(t) \right]^2 dt \\
&= \int_0^\infty \left[\delta y(t) + \tau \delta \dot{y}(t) \right]^2 dt \\
&= \int_0^\infty \left[\delta y^2(t) + \tau^2 \delta \dot{y}^2(t) \right] dt + 2\int_0^\infty \tau \, \delta y(t) \delta \dot{y}(t) dt \\
&= \int_0^\infty \left[\delta y^2(t) + \tau^2 \delta \dot{y}^2(t) \right] dt + \delta y^2(\infty) - \delta y^2(0)
\end{aligned}$$

Assuming that the initial condition for the position of the model is the same as the one for the actual system, then one has

$$\delta y(0) = 0$$

If the system is asymptotically stable (for stability concepts see Chapter IV), one has

$$y(\infty) = 0$$

Therefore

$$\mathrm{Imin}(\delta y, t) = \int_0^\infty \left[\delta y^2(t) + \tau^2 \delta\dot{y}^2(t)\right] dt$$

$\delta y^2(t)$ can be expressed by

$$\delta y^2(t) = 2\int_0^t \delta y(t)\,\delta\dot{y}(t)\,dt$$

Using the Cauchy-Schwarz inequality

$$\int_0^t f_1(t) f_2(t)\,dt \leq \sqrt{\int_0^t f_1^2(t)\,dt \int_0^t f_2^2(t)\,dt}$$

By letting $\delta y(t) = f_1(t)$ and $\delta\dot{y}(t) = f_2(t)$ one has

$$\delta y^2(t) \leq 2\sqrt{\int_0^t \delta y^2(t)\,dt \int_0^t \delta\dot{y}^2(t)\,dt}$$

Since the integrands are positive for all t, one may write

$$\delta y^2(t) \leq 2\sqrt{\int_0^\infty \delta y^2(t)\,dt \int_0^\infty \delta\dot{y}^2(t)\,dt}$$

Multiplying and dividing by τ the right side of the inequality

$$\delta y^2(t) \leq \frac{2}{\tau}\sqrt{\int_0^\infty \delta y^2(t)\,dt \int_0^\infty \tau^2 \delta\dot{y}^2(t)\,dt}$$

Due to the inequality $\sqrt{2\,a\,b} \leqq a+b$, where in this particular case a and b are small numbers, one has

$$\delta y^2(t) \leqq \frac{2}{\tau} \int_0^\infty \left[\delta y^2(t) + \tau^2 \delta \dot{y}^2 \right] dt$$

and

$$\left|\delta y(t)\right| \leqq 2^{1/4} \sqrt{\frac{I_{min}}{\tau}} \qquad (3\text{-}10)$$

Notice the dependence of the deviation $\left|\delta y(t)\right|$ upon τ. If τ is small, $\left|\delta y(t)\right|$ is large. This is as might be expected from physical reasoning, since small τ requires fast rise time.

For a third or higher order system, however, an ideal model as described by equation (3-8) is unsatisfactory, since it does not give enough flexibility to describe the desired response. If one has a third order system, for example, more flexibility to describe the model is given by a second order differential equation. The performance index is then

$$I = \omega_n^4 \int_0^\infty \left[y(t) + \frac{2\zeta}{\omega_n} \dot{y}(t) + \frac{y(t)}{\omega_n^2} \right]^2 dt \qquad (3\text{-}11)$$

Again it is of interest to have an a priori idea about how close the maximum deviation of the actual response is with respect to the desired one.

Let $y_o(t)$ be such that $\ddot{y}_o(t) + 2\zeta\omega_n\dot{y}_o(t) + \omega_n^2 y_o(t) = 0$ so that $y(t) = y_o(t) + \delta y(t)$. Then

$$I_{min} = \omega_n^4 \int_0^\infty \left[\left[\delta y + \frac{2\zeta}{\omega_n}\delta\dot{y} + \frac{\delta\ddot{y}}{\omega_n^2}\right]^2\right] dt \qquad (3\text{-}12a)$$

$$I_{min} = \omega_n^4 \left\{ \int_0^\infty \left[\delta y^2 + \frac{4\zeta^2}{\omega_n}\delta\dot{y}^2\right] dt + \frac{4\zeta}{\omega_n}\int_0^\infty \delta y\,\delta\dot{y}\, dt + \frac{2}{\omega_n^2}\int_0^\infty \delta y\,\delta\ddot{y}\, dt + \frac{4\zeta}{\omega_n^3}\int_0^\infty \delta\dot{y}\,\delta\ddot{y}\, dt + \int_0^\infty \frac{\delta\ddot{y}^2}{\omega_n^2}\, dt \right\} \qquad (3\text{-}12b)$$

The above is obtained by letting $\delta y(0) = \delta\dot{y}(0) = 0$ and assuming asymptotic stability, so that $y(\infty) = \dot{y}(\infty) = 0$. By integrating the third term at the right side of Eq. (3-12b) by parts

$$\int_0^\infty \delta y\,\delta\ddot{y}\, dt = -\int_0^\infty \delta\dot{y}^2\, dt$$

So that Imin is now

$$I_{min}(\delta y, \delta\dot{y}, t) = \omega_n^4 \left[\int_0^\infty \left[\delta y^2 + \frac{2}{\omega_n^2}\left[\frac{4\zeta^2}{\omega_n^2} - \frac{2}{\omega_n^2}\right]\delta\dot{y}^2 + \frac{\delta\ddot{y}^2}{\omega_n^4}\right] dt \right]$$

$$=\omega_n^4\left[\int_0^{\infty}\left[\delta y^2+\frac{2}{\omega_n^2}(2\zeta^2-1)\delta\dot{y}^2+\frac{\delta\ddot{y}^2}{\omega_n^4}\right]dt\right]$$

Let

$$\frac{2\zeta^2-1}{\omega_n^2}=\tau^2\geqq 0$$

which holds for $\zeta\geqq 0.707$, then

$$Imin=\omega_n^4\left[\int_0^{\infty}\left[\delta y^2+\tau^2\delta\dot{y}^2\right]dt+\tau^2\int_0^{\infty}\left[\delta\dot{y}^2+\frac{\delta\ddot{y}^2}{\tau^2\omega_n^4}\right]dt\right]$$

(3-13)

But it has been proved already that

$$\int_0^{\infty}\left[\delta y^2+\tau^2\delta\dot{y}^2\right]dt\geqq\frac{\tau}{\sqrt{2}}\delta y^2$$

The second integral at the right of equation (3-12) has the same form as the first one, so

$$\int_0^{\infty}\left[\delta\dot{y}^2+\frac{\delta\ddot{y}^2}{\tau^2\omega_n^4}\right]dt\geqq\frac{\delta\dot{y}^2}{\tau\omega_n^2\sqrt{2}}$$

Hence

$$Imin\geqq\frac{\omega_n^4}{\sqrt{2}}\delta y^2+\frac{\tau\omega_n^2}{\sqrt{2}}\delta\dot{y}^2 \qquad (3\text{-}13)$$

and therefore

$$Imin\geqq\frac{\tau\omega_n^4}{\sqrt{2}}\delta y^2(t) \text{ and } Imin\geqq\frac{\tau\omega_n^2}{\sqrt{2}}\delta\dot{y}^2(t)$$

or

$$|\delta y| < \frac{2^{1/4}}{\omega_n^2} \sqrt{\frac{I_{min}}{T}} \text{ and } |\dot{\delta} y| < \frac{2^{1/4}}{\omega_n} \sqrt{\frac{I_{min}}{T}}$$

The restriction $\hat{\xi} \geqq 0.707$ does not mean in this case a limitation of the model, but that the deviation $|\delta y|$ can only be obtained for such values of $\hat{\xi}$.

For third or higher order models, one may show a relationship between Imin and the deviation, following the same arguments as for the first and the second order models. The relationship for an mth order model is found to be of the form

$$\text{Imin} \geqq k_1 \delta y^2 + k_2 \dot{\delta} y^2 + \ldots + k_{m-1} \delta^{(m-1)} y^2 \quad (3\text{-}14)$$

To extend the theory of modeling to include sampled-data systems, one must take into consideration that the state variables of the plant, $\underline{x}(t)$, although continuous in time, are actually known at the sampling times t = k k = 0, 1, . . . (see equation 2-8). Therefore a more significative performance index for the sampled-data case is given by

$$I = \sum_{k=1}^{\infty} \left[x_1(k) + a_2 x_2(k) + \ldots a_m x_m(k) \right]^2 \quad (3\text{-}15)$$

Where

$$x_1(k) = y(k)$$

$$x_2(k) = dy/dt\Big|_{t=k}$$

$$x_m(k) = d^m y/dt^m\Big|_{t=k}$$

and where y(k) is the output of the system at the sampling time t = k. m is the order of the model with $m \leq n-1$ for plants with no zeroes (n is the order of the plant). In the case of a plant with one zero, for example, the output y(k) depends on two state variables (see example 2-1). Therefore, when expressing I in terms of the state variables of the plant x_1, x_2, . . . x_n, in order to avoid cross products of a state variable with the control input, the order of the model must be $m \leq n-2$. This may be seen by examining the second order system of example 2-1. There

$$y(k) = 2\, x_1(k) + x_2(k)$$

Let the model be

$$y(t) + dy/dt = 0$$

In expressing the model in terms of the state variables of the plant one has

$$2\, x_1(t) + 3\, x_2(t) + \frac{d}{dt}\, x_2(t) = 0$$

But dx_2/dt is not a state variable. By solving dx_2/dt in terms of u(t), the performance index will include cross products of the state variables with the control input, which is undesired for the design procedure given later on.

In general for a plant with Z zeroes, the order of the model, m, must be $m \leq n-(Z+1)$.

By observing the expression for I as given in equation (3-15), one may notice that if one has a system with sufficient number of adjustable parameters, i.e., a free system, it is possible to approach the minimum-minimum I. Such Iminmin corresponds to the solution of the equation

$$x_1(k) + a_2 x_2(k) + \ldots + a_m x_m(k) = 0$$

In the actual system one is limited in the available number of adjustable parameters; so one must adjust these parameters for the "best" response, i.e., the response that most nearly corresponds to the response of the model in the sense dictated by the performance index. Again for sampled-data systems it would be of interest to find the deviation of the real system from the model. If one tries to use the same approach as in the continuous case, it is not possible to obtain a direct relationship between deviation and the optimum performance index. For example, by working in the simplest model one has

$$I = \sum_{k=1}^{\infty} \left[y(k) + \tau \left[dy/dt \right]_{t=k} \right]^2 \qquad (3\text{-}16)$$

Let $y_o(t)$ be the solution of the differential equation

$$y(k) + \tau\, dy(k)/dt = 0$$

At the sampling instant t = k, the actual response of the optimum system is

$$y(k) = y_o(k) + \delta y(k)$$

$$\text{Imin} = \sum_{k=1}^{\infty} \left[y_o(k) + \delta y(k) + \tau \dot{y}_o(k) + \tau \delta \dot{y}(k) \right]^2$$

$$= \sum_{k=1}^{\infty} \left[\delta y(k) + \tau \delta \dot{y}(k) \right]^2 = \sum_{k=1}^{\infty} \left[\delta y^2(k) + \tau^2 (\delta \dot{y}(k))^2 \right.$$

$$\left. + 2\tau \delta y(k) \delta \dot{y}(k)) \right]$$

But

$$\delta y^2(k) = 2\int_0^k \delta y(t) \delta \dot{y}(t)\, dt \leq 2 \sum_{i=1}^{k} \delta y(\xi_i) \delta y(\xi_i^{*})$$

where

$$\delta y(\xi_i) = \delta y_{max}(t) \qquad t_{i-1} \leq t \leq t_i$$

and

$$\delta \dot{y}(\xi_i^{*}) = \delta \dot{y}_{max}(t) \qquad t_{i-1} \leq t \leq t_i$$

Therefore, as with the continuous case,

$$y^2(k) \leq \frac{2}{\tau} \sqrt{\sum_{i=1}^{k} \delta y^2(\xi_i) \sum_{i=1}^{k} \tau^2 \delta y^2(\xi_i^{*})}$$

$$\leq \frac{2}{T} \sqrt{\sum_{i=1}^{\infty} \delta y^2(\xi_i) \sum_{i=1}^{\infty} T^2 \dot{\delta} y^2(\xi_i^*)}$$

$$\leq \frac{2}{T} \sum_{i=1}^{\infty} \left[\delta y^2(\xi_i) + T^2 \dot{\delta} y^2(\xi_i^*) \right] \qquad (3\text{-}17)$$

By observing the inequality (3-17), the bounds obtained for $\delta y^2(k)$ are unfortunately not directly related to Imin as it was the case for continuous systems. Nevertheless Eq. (3-10) is used as a reference in the example problems given in Chapter V.

3.3 Generalized Performance Index With Constraints in the Cost of the Control Input

In addition to considering errors as a measure of performance index, it is often desirable to give constraints to the control input, u(k). If only the errors are minimized, as is the case in Eq. (3-15), a design may result which needs larger values of u(k) than are physically realizable (Kalman et al., 1958). For this case, the performance index may be given as

$$I = \sum_{k=1}^{\infty} \left[\underline{x}^t(k)\ Q\ \underline{x}(k) + \alpha^2 u^2(k-1) \right] \qquad (3\text{-}18)$$

where Q is an nxn positive definite or semidefinite matrix and is referred to the model; α^2 is a constant which indicates a compromise between minimization of the errors and minimization of the cost of the control input. If the principal interest in the design is to minimize the error, one still could use Eq. (3-18) by having a constant factor λ^2 in Q such that $\lambda^2 > \alpha^2$. This is done in Chapter V where the computer program utilizes a performance index as given in Eq. (3-18) with $\alpha^2 = 1$.

3.4 Summary

A performance index is proposed in this chapter, which enables the designer to associate it to a prototype or model. A relationship between the maximum deviation of the optimum system with respect to the model is found for continuous systems. The sampled-data control system is worked out by analogy to the continuous case. Although in the latter the deviation is not related in a direct manner to the optimum performance index, the inequality obtained for the continuous case is used in the example problems given in Chapters IV and V.

CHAPTER IV

THE SECOND METHOD OF LIAPUNOV AND THE DESIGN OF SAMPLED-DATA CONTROL SYSTEMS

4.1 Introduction

The main object of Liapunov's direct method, commonly called the Second Method, is concerned with the stability of a system of equations. The Second Method provides means to answer questions concerning the stability of a system by using the given form of the differential or difference equations, but without requiring a knowledge of their solutions.

In order to be able to use the Second Method in system optimization, it is necessary to state the stability theorems in the sense of Liapunov. Once this is done, applications to system design and its limitations can be better developed. Among the references available on this theory are the papers by Kalman and Bertram (1960), Mantey (1964), and the book by Freeman (1965).

Section 2 of this chapter states the definitions and theorems for discrete-time systems. They were obtained from the references cited. In Section 3 two attempts to evaluate and minimize the performance index by using the Second Method of Liapunov are given for

1. Open-loop stable plants. In this case the performance index is not referred directly to the model.
2. Both stable and unstable plants. The performance index, I, is in this case associated to an ideal model. The system is then designed by finding the values of the feedback parameters which minimizes I. Difficulties arise in the analytical calculations for systems higher than the second order, due to the non-linear relationship between I and the adjustable parameters.

4.2 The Concept of Stability in the Sense of Liapunov

Consider the autonomous system defined by the vector difference equation

$$\underline{x}(k+1) = \emptyset_s(1)\ \underline{x}(k)$$

where k+1 and k indicate discrete values of time.

DEFINITION 1 The equilibrium state $\underline{x} = \underline{0}$ is said to be stable in the sense of Liapunov if for any t_o and any $\epsilon > 0$, there exists a number $\delta(\epsilon, t_o) > 0$ such that every initial state satisfying the inequality

$$x_1^2(0) + x_2^2(0) + \ . \ . \ . + x_n^2(0) < \delta(\epsilon, t_o)$$

implies that

$$x_1^2(t_k) + x_2^2(t_k) + \ldots + x_n^2(t_k) < \epsilon$$

for all $t_k > t_o$

DEFINITION 2 The equilibrium state $\underline{x} = \underline{0}$ is said to be asymptotically stable if it is stable in the sense of Liapunov and in addition there exists a $\eta(t_o) > 0$ such that

$$\lim_{k \to \infty} \left[x_1^2(t_k) + x_2^2(t_k) + \ldots + x_n^2(t_k) \right] = 0$$

for all $\|\underline{x}_o\| < \eta(t_o)$. If η is independent of t_o, the state is uniformly asymptotically stable.

The Second Method consists essentially of finding suitable functions of the state variables, with certain characteristics to ensure the desired kind of stability. These functions are called Liapunov or V functions, and the notation for them is $V(\underline{x}(k))$.

DEFINITION 3 A scalar function, $V(\underline{x}(k))$, is positive definite if within some region about the origin in the state space (a) $V(\underline{x}(k))$ is greater than zero away from the origin and (b) $V(\underline{x}(k))$ equals zero at the origin. $V(\underline{x}(k))$ is said to be negative definite if $-V(\underline{x}(k))$ is positive definite. A function is called semidefinite if it is allowed to be equal to zero at points other than the origin.

The theorem stated below is utilized in the next section where the performance indices are expressed in terms of $\Delta V(\underline{x}(k))$.

THEOREM 1 A discrete time, linear, autonomous, stationary, dynamical system

$$\underline{x}(k+1) = \phi_s(1)\ \underline{x}(k)$$

is asymptotically stable, if and only if, given any symmetric, positive definite matrix Q, there exists a symmetric, positive definite matrix P which is the unique solution of the linear equation

$$Q = P - \phi_s(1)^t\ P\ \phi_s(1)$$

and

$$V(\underline{x}(k)) = \underline{x}^t(k)P\underline{x}(k)$$

is a Liapunov function for the system, with

$$\Delta V(\underline{x}(k)) = -\underline{x}^t(k)\ Q\ \underline{x}(k)$$

The relationship between P and Q may be seen as follows. Let

$$V(\underline{x}(k)) = \underline{x}^t(k)\ P\ \underline{x}(k)$$

The first forward difference of $V(\underline{x}(k))$ is

$$\begin{aligned}\Delta V(\underline{x}(k)) &= V(\underline{x}(k+1)) - V(\underline{x}(k))\\ &= \underline{x}^t(k+1)\ P\ \underline{x}(k+1) - \underline{x}^t(k)\ P\ \underline{x}(k)\end{aligned}$$

Since

$$\underline{x}(k+1) = \phi_s(1)\underline{x}(k)$$

$$\Delta V(\underline{x}(k)) = \underline{x}^t(k)\phi_s(1)^t P\phi_s(1)\underline{x}(k) - \underline{x}^t(k)P\underline{x}(k)$$

By letting

$$\Delta V(\underline{x}(k)) = -\underline{x}^t(k)Q\underline{x}(k)$$

one has

$$-Q = \phi_s(1)^t P\phi_s(1) - P$$

4.3 System Optimization

It is assumed here that the overall closed loop system is given as an autonomous sampled-data control system with the control input generated by all the state variables of the plant. The problem then is to find the feedback parameters which make a given performance index, I, minimum.

Two approaches for the design by using the Second Method are given.

A. The plant of the system to be considered is described by the equation

$$\dot{\underline{x}} = A\,\underline{x} + \underline{b}\,u$$

or in sampled-data form

$$\underline{x}(k+1) = \phi(1)\underline{x}(k) + \underline{\beta}(1)u(k)$$

Here it is necessary to assume that Λ, and therefore $\emptyset$, are asymptotically stables. The problem is to find the feedback parameters of the entire system to minimize a performance index I(k)

$$I(k) = \underline{x}^t(k)\ P\ \underline{x}(k) \tag{4-1}$$

where P is a positive definite matrix. In order to relate I(k) to the model the following argument is presented.

Since the plant is asymptotically stable and by virtue of theorem stated in Section 4.2, given

$$\Delta V_1\ (\underline{x}(k)) = -\ \underline{x}^t(k)\ Q\ \underline{x}(k) \tag{4-2}$$

where subindex 1 in $\Delta V(\underline{x}(k))$ means that Eq. (4-2) is referred to the open loop plant. There exists a $V(\underline{x}) = x^t(k)\ P\ x(k)$ where P is the unique solution of the linear equation

$$-Q = \emptyset^t P \emptyset - P \tag{4-3}$$

$\emptyset(T)$ is the transition matrix of the plant with no control input applied.

Now assume that Q and not P is the given matrix. From Eq. (4-3) one may get the corresponding P. If one associates Q with the ideal model, the performance I(k) as given in Eq. (4-1) is indirectly related to the model, due to (4-3)

Now take into consideration the entire system (plant + controller): one may associate the same $V(\underline{x})$, previously obtained for the autonomous plant, with the overall closed loop system, provided that the controller does not increase the order of the system. The first forward difference of $V(\underline{x})$ is now

$$\Delta V_2\left|\underline{x}(k)\right| = V\left|\underline{x}(k+1)\right| - V\left|\underline{x}(k)\right|$$

$$= \underline{x}^t(k+1)P\underline{x}(k+1) - \underline{x}^t(k)P\underline{x}(k) \qquad (4\text{-}4)$$

where subindex 2 in $\Delta V(\underline{x}(k))$ means that Eq. (4-4) is referred to the closed loop system.

Since

$$\underline{x}(k+1) = \phi(1)\underline{x}(k) + \underline{\beta}^t(1)u(k)$$

one has

$$\Delta V_2\ (\underline{x}(k)) = \underline{x}^t(k)\left[\phi(1)^t\ P\ \phi(1) - P\right]\underline{x}(k)$$

$$+\ 2\underline{x}^t(k)\left[\underline{\beta}^t(1)\ P\ \phi(1)\right]u(k)$$

$$+\left[\underline{\beta}(1)^t\ P\ \underline{\beta}(1)\right]u^2(k)$$

If $u(k)$ is chosen to maintain $\Delta V_2(\underline{x}(k))$ as large as possible, then $V(\underline{x}(k))$ is made minimum, and therefore $I(k)$ is minimum.

Therefore by taking $\dfrac{\partial \Delta V_2}{\partial u} = 0$ (Kalman, 1960)

$$\frac{\partial}{\partial}\frac{\Delta V_2}{u} = 0 = 2\underline{\beta}^t P\phi\ \underline{x}(k) + 2\underline{\beta}^t P\underline{\beta}u(k)$$

$$u(k) = -(\underline{\beta}^t P \underline{\beta})^{-1} \underline{\beta}^t P \,\emptyset\, \underline{x}(k) = \underline{c}^t \underline{x}(k) \qquad (4\text{-}6)$$

An example is given as a way of illustration of the procedure.

Example 4-1

Given the sampled-data system shown in Fig. 4-1(a), find the feedback coefficients c_1 and c_2 to minimize the performance index

$$I(k) = \underline{x}^t(k)\ P\ \underline{x}(k) = V(\underline{x})$$

The first forward difference of $V(\underline{x})$ for the open-loop plant is

$$\Delta V_1 \left[\underline{x}(k)\right] = -x^t(k)\ Q\ \underline{x}(k)$$

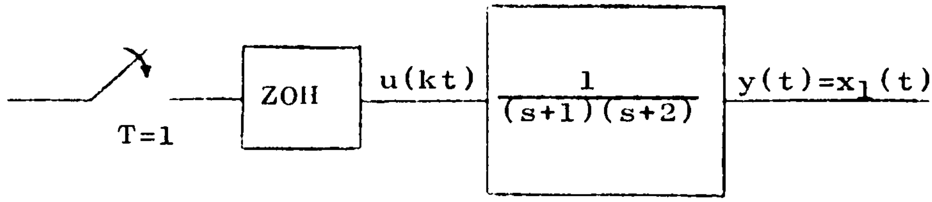
Let $\Delta V_1(k)$ be associated to the first order model $y(t) + \tau\dot{y}(t) = 0$ with given initial conditions. Then

$$Q = \begin{bmatrix} 1 & \tau \\ \tau & \tau^2 \end{bmatrix}$$

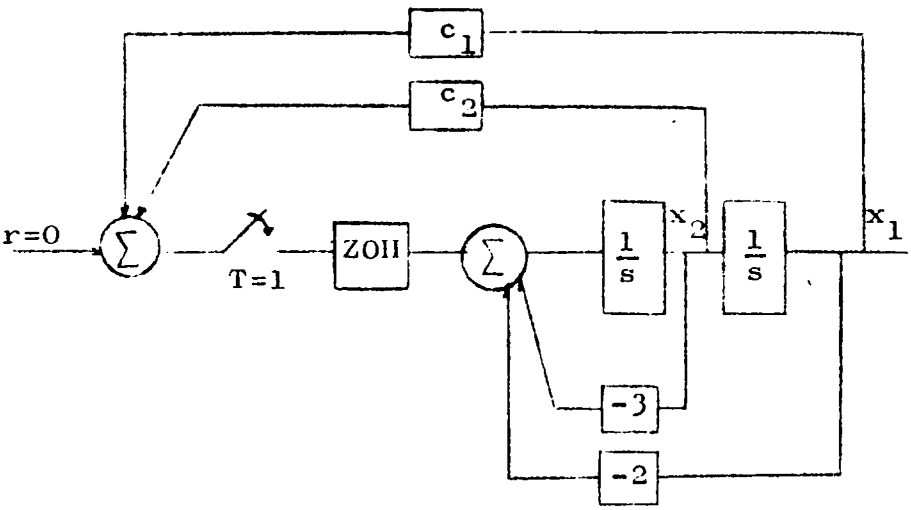
$\underline{x}(k)$ is the state of the plant, assumed in phase variables, and $\tau = 0$ and 0.5.

The initial condition at the output of the plant is $x_1(0) = 1$.

The state equations in phase variables were already obtained in Example 2-2. They are



(a) Block Diagram Representation of Open-Loop System



(b) Block Diagram Representation of the Closed-Loop Sampled-Data Control System

Fig. 4.1 Block Diagram Representation for System of Example 4-1

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.601 & 0.233 \\ -0.466 & -0.098 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.200 \\ 0.233 \end{bmatrix} u(k)$$

The open loop autonomous plant is defined by

$$\underline{x}(k+1) = \phi(1)\ \underline{x}(k)$$

where

$$\phi(1) = \begin{bmatrix} 0.601 & 0.233 \\ -0.466 & -0.098 \end{bmatrix}$$

Let a suitable Liapunov function for the open loop stable gain be $V\left[\underline{x}(k)\right] = \underline{x}^t(k)P\underline{x}(k)$ such that its first difference $\Delta V_1\ \underline{x}(k)$ is given by

$$\Delta V_1\ \underline{x}(k) = -\underline{x}^t(k)\ Q\ \underline{x}(k)$$

The elements of P are obtained from the linear equations

$$-Q = \phi^t(1)\ P\ \phi(1) - P$$

Since Q is a symmetric matrix, P is symmetric also. Solving for p_{11}, p_{12}, and p_{22} in terms of τ one gets

$$p_{11} = 0.276\tau^2 \quad -0.700\tau \quad +1.430$$

$$p_{12} = 0.070\tau^2 \quad +0.765\tau \quad +0.175$$

$$p_{22} = 1.010\tau^2 \quad -0.070\tau \quad +0.075$$

From Eq. (4-6), in order to have $-\Delta V_2\, \underline{x}(k)$ as large as possible, u(k) must be given by

$$u(k) = -(\underline{\beta}^t\, P\, \underline{\beta})^{-1}\, \underline{\beta}(1)^t\, P\, \phi(1)\underline{x}(k)$$

in this particular example:

$$\underline{\beta}^t\, P\, \underline{\beta} = 0.04\, p_{11} + 0.093\, p_{12} + 0.054\, p_{22}$$

$$\underline{\beta}^t\, P\, \phi = 0.12\, p_{11} + 0.048\, p_{12} - 0.108\, p_{22}\; 0.046\, p_{11}$$

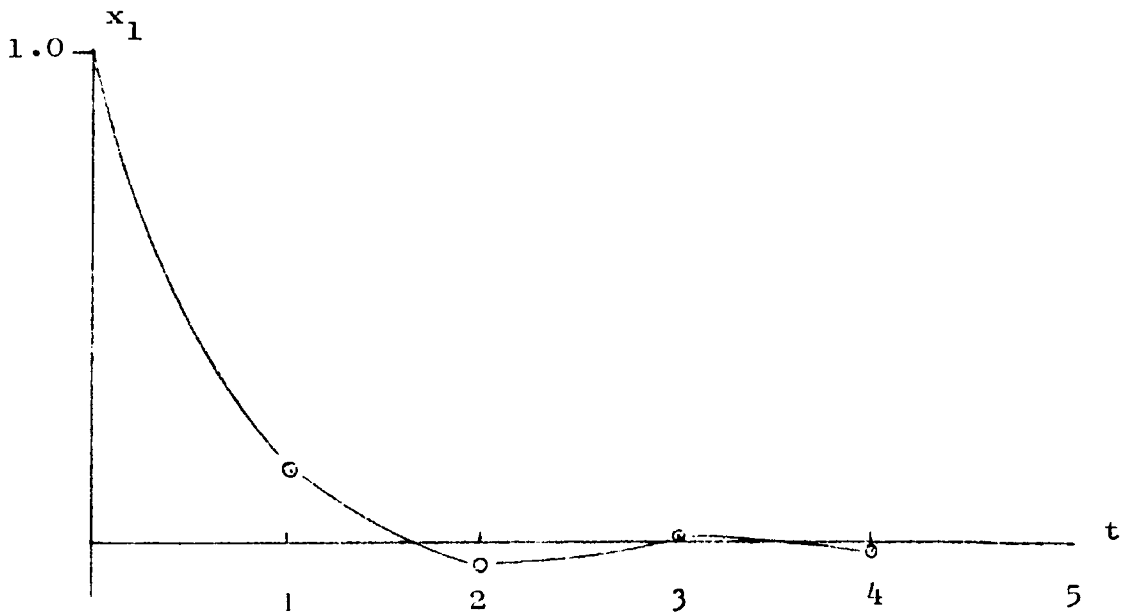
$$+ 0.034$$

Table 4-1 gives the values for c_1 and c_2 for the two values of $\tau = 0$ and $\tau = 0.5$. Fig. 4-1b shows the Block Diagram representation of closed-loop system in phase variables. Figures 4-2 (a) and (b) shows the transient responses for $\tau = 0$ and $\tau = .5$.

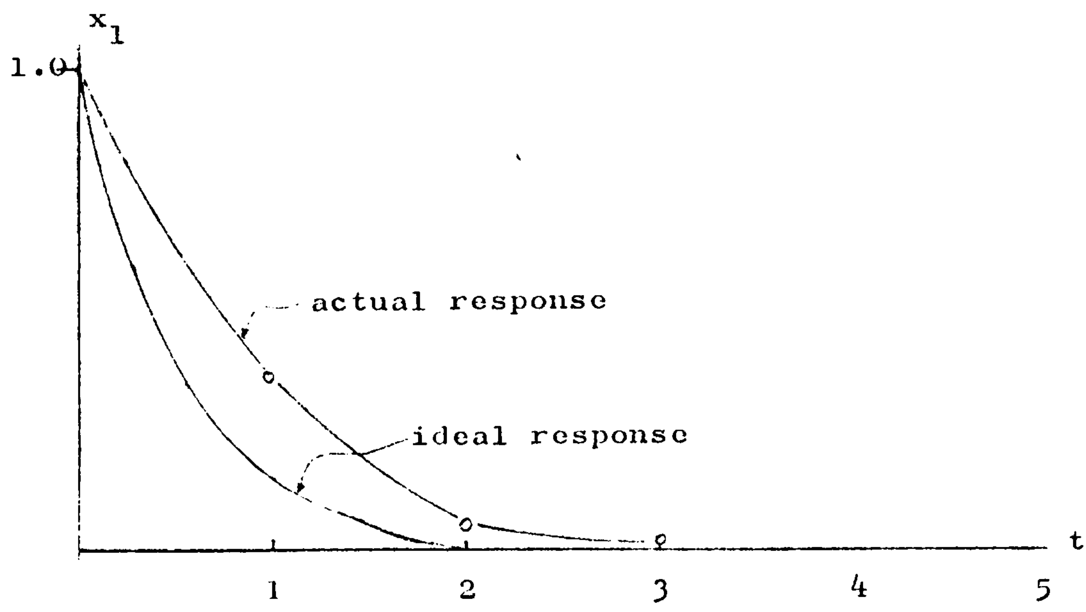
TABLE 4-1

RESULTS OF EXAMPLE 4-1 CALCULATIONS

	c_1	c_2
$\tau = 0$	- 2.280	- 0.920
$\tau = .5$	-1.160	- 0.575



(a) $\tau = 0$



(b) $\tau = 0.5$

Fig. 4.2 Transient Responses for System of Example 4-1

In the case of a pole at the origin, the plant is no longer open-loop stable. Nevertheless, it is possible to work the problem assuming the plant with poles in the left half plane and then finding the limit of the feedback coefficients when one pole goes to zero (Lindorff, 1965).

The problem in the above design is that although $I(k)$ is indirectly related to the model, minimization of $V(\underline{x}(k))$ does not imply a minimization of $\Delta V_1(\underline{x}(k))$ which is the function defined by the ideal model.

A design procedure is necessary for which the performance index of the system is referred to the model. This is done in part B of this section.

B. Consider again the plant and control input as defined by equation (2-8). This time domain description is repeated here for convenient reference

Plant

$$\underline{x}(k+1) = \emptyset(1)\underline{x}(k) + \underline{\beta}(1)u(k)$$

$$y(k+1) = \underline{g}^t \ \underline{x}(k+1)$$

Control input

$$u(k) = \underline{c}^t \ \underline{x}(k)$$

The entire system (plant + controller) is described by the state equation

$$\underline{x}(k+1) = \emptyset_s(1)\underline{x}(k)$$

where

$$\underline{\phi}_s(1) = \phi(1) + \underline{\beta}(1)\underline{c}^t$$

The adjustable parameters are all included in $\underline{c}$.

The performance index is given by

$$I = \sum_{k=1}^{\infty} \underline{x}^t(k)\ Q\ \underline{x}(k) + \alpha^2\ u^2(k-1)$$

where Q is a positive definite or semidefinite nxn matrix and α^2 is a constant which indicates a constraint on the control input.

Since

$$u(k-1) = \underline{c}^t\underline{x}(k-1)$$

and

$$\underline{x}\ (k) = \phi_s\underline{x}(k-1)$$

the performance index I, can also be given by

$$I = \sum_{k=1}^{\infty} \underline{x}^t(k-1)\ Q_T\ \underline{x}(k-1) \qquad (4\text{-}7)$$

where

$$Q_T = \phi_s^{\ t}\ (1)\ Q\ \phi_s(1) + \alpha^2\underline{c}\ \underline{c}^t \qquad (4\text{-}8)$$

Let $\Delta V\ (\underline{x},k)$ be given by

$$\Delta V(\underline{x}(k)) = -\ \underline{x}^t(k)Q\underline{x}(k)$$

so that

$$I = -\sum_{k=1}^{\infty} \Delta V(\underline{x}(k-1)) = -\sum_{k=1}^{\infty} \left[V(\underline{x}(k)) - V(\underline{x}(k-1))\right]$$

$$= -\left[V(\underline{x}(1))-V(\underline{x}(0))+V(\underline{x}(2))-V(\underline{x}(1))+ \ . \ . \ . \ +V(\underline{x}(k))\right.$$

$$\left. -V(\underline{x}(k-1))+\cdot\cdot\right] = + V(\underline{x}(0)) - V(\underline{x}(\infty))$$

If the closed-loop system is asymptotically stable

$$V(\underline{x}(\infty)) = V(\underline{0}) = 0$$

Hence

$$V(\underline{x}(0)) = \underline{x}^t(0) \ P \ \underline{x}(0) = \sum_{k=1}^{\infty}\left[\underline{x}^t(k) \ Q \ \underline{x}(k) \right.$$

$$\left. + \alpha^2 u^2(k-1)\right] \tag{4-9}$$

In the design problem it is desired to associate Q with the model. This is conveniently done by having a multiplier factor λ^2 in Q such that

$$\lambda^2 > \alpha^2$$

so that minimization of the performance index conducts the system to approach the model.

By having Q associated to the model described in Chapter III, it looks like Theorem 1 does no longer apply, since now Q is positive semidefinite. This restriction is specially troublesome when one is trying to design systems with unstable plants. The following theorem is required to include those cases. A similar theorem, applied for other problem is given by Mantey (1964).

THEOREM 2 Consider the dynamical plant described by Eq. (2-2) with a controller described by Eq. (2-7). A matrix S is defined such that $Q = S^t S$, which is possible for any symmetric nonnegative matrix.

If (S, ϕ) is completely observable, then $V(\underline{x}(0))$ given by Eq. (4-9) is positive for all $\underline{x}(0) \neq 0$, where $\underline{x}(0)$ is the state vector of the plant.

Assume $\underline{x}(0)$ variable: since Q and α^2 are non-negative, $V(\underline{x}(0))$ must be positive or zero. The proof of the theorem is obtained by showing that the only $\underline{x}(0)$ for which $V(\underline{x}(0)) = 0$ is $\underline{x}(0) = 0$.

Start with the first term in (4-9), given by k = 1: $S\underline{x}(1) = 0$ and $u(0) = 0$ otherwise $V(\underline{x}(0)) > 0$. Using the transition equation (2-2)

$$\underline{x}(1) = \phi(1)\, \underline{x}(0) + \underline{\beta}(1)^t u(0) = \phi(1)\underline{x}(0)$$

so that

$$S\underline{x}(1) = S\, \phi(1)\underline{x}(0) = 0$$

From the next term at k = 2, in order for $V(\underline{x}(0))$ to remain zero, it is necessary that

$$S\underline{x}(2) = S\left[\phi(1)\underline{x}(1) + \underline{\beta}^t u(1)\right] = 0$$

and

$$u(1) = 0$$

Thus

$$\underline{x}(2) = \phi(1)\underline{x}(1) = \phi^2(1)\ \underline{x}(0)$$

This process is continued until $k = n$, where

$$S\underline{x}(n) = S\ \phi^{n-1}\ \underline{x}(0) = 0$$

Thus, $\underline{x}(0)$ is orthogonal to all of the rows of the matrix

$$\begin{bmatrix} S\ \phi(1) \\ S\ \phi^2(1) \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ S\ \phi^n(1) \end{bmatrix}$$

This matrix may be used to determine the observability of (S,ϕ). The rows of this matrix span the whole state of the plant, since (S,ϕ) was assumed to be completely observable. Since $\underline{x}(0)$ is orthogonal to all of the rows of this matrix, it must be the zero vector. Thus, the only initial state of the plant for which $V(\underline{x}(0))$ is not positive is $\underline{x}(0) = 0$ for which $V(\underline{x}(0)) = 0$.

It is now possible to find a positive definite P from the n(n+1) linear equations

$$- \phi_s(1)^t \, P \, \phi_s(1) + P = Q_T$$

For the case of stable plants, one may have $\alpha^2 = 0$ in the performance index. This is done in example 4-2, in order to reduce the complexity of the problem. For this case P may be found by following the same procedure as before, but now one has

$$- \phi_s(1)^t \, P \, \phi_s(1) + P = Q \qquad (4\text{-}11)$$

Since ϕ_s includes all the adjustable feedback coefficients and Q is given, then P is also a function of the adjustable parameters, i.e.,

$$P = f(c_1, c_2, \ldots c_n)$$

and

$$I = F(c_1, c_2, \ldots c_n, x_1(0), x_2(0), \ldots x_n(0))$$

The minimum performance index may be obtained by taking the partial derivatives of I with respect to each parameter and setting them to zero. The optimum $\underline{c}^t$ is the one which gives 1 min.

Example 4-2

Given the sampled-data system shown in Fig. 4.3, find the feedback coefficient in x_2 which minimizes the performance index

$$I = \sum_{k=0}^{\infty} (x_1(k) + x_2(k))^2$$

The initial conditions of the plant are $x_1(0) = 1$, $x_2(0) = 0$

First the system is defined in sampled-data form as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.0 & 0.632 \\ 0.0 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} u(k)$$

The control input is given by

$$u(k) = - x_1(k) - c_2 \, x_2(k)$$

The entire system can be described by the state equation

$$\underline{x}(k+1) = \emptyset_s \, \underline{x}(k)$$

where

$$\emptyset_s(1) = \begin{bmatrix} 0.632 & 0.632-0.368c_2 \\ -0.632 & 0.368-0.632c_2 \end{bmatrix}$$

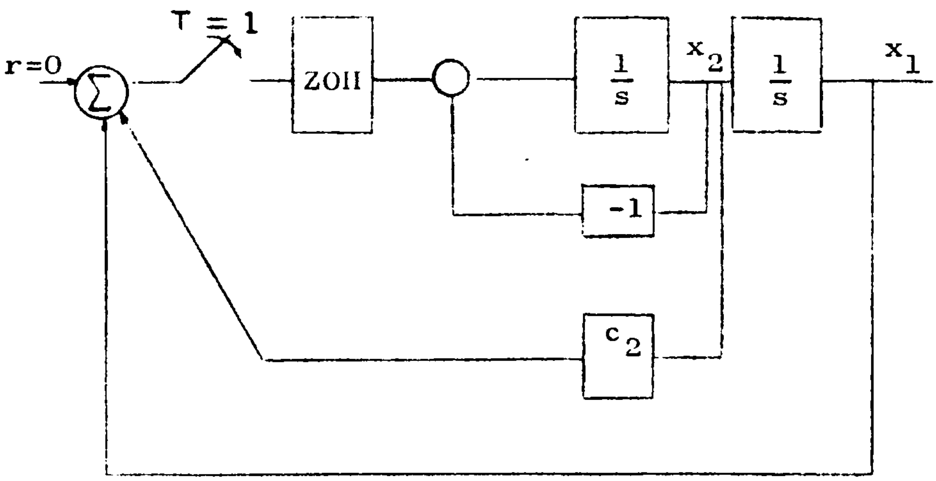


Fig. 4.3 Block Diagram Representation for System of Example 4-2

Let $V(\underline{x}(k)) = \underline{x}^t(k)\ P\ \underline{x}(k)$ be a suitable Liapunov function for the entire system with $\Delta V(\underline{x},k) = -\ x^t(k)Q\underline{x}(k)$. From the three linear equations

$$Q = P - \phi_s^{\ t}\ P\ \phi_s$$

one gets

$$-1.0 = -0.600\ p_{11}\ -0.796\ p_{12}\ -0.496\ p_{22}$$

$$-1.0 = (0.398-0.233c_2)p_{11}+(-1.1659-.166)p_{12}$$
$$+(-.232+.398)p_{22}$$

$$1.0 = (-.398+.465c_2-.135c_2^{\ 2})p_{11}+(-.465+1.07c_2$$
$$-.465c_2^{\ 2})p_{12}+(.865+.465c_2-.398c_2^{\ 2})p_{22}$$

The initial conditions are $x_1(0) = 1$, $x_2(0) = 0$; therefore

$$I = \underline{x}^t(0)P\underline{x}(0) = x_1(0)p_{11} = p_{11}$$

Solving for p_{11} in the above equations

$$p_{11} = \frac{0.31c_2^3-0.793c_2^2+1.0c_2-1.578}{0.006c_2^3+0.642c_2^2-0.945c_2-0.763}$$

By taking $dp_{11}/dc_2 = 0$, the following fourth order polynomial in c_2 is obtained

$$0.202\ c_2^4\ -\ 0.597\ c_2^3\ -\ 0.573\ c_2^2\ +\ 3.225\ c_2\ -\ 2.242\ = 0$$

For p_{11} and therefore I to be minimum, the value of c_2 is

$$c_2 = 0.985$$

In Fig. 4.4a I is shown as a function of c_2. For $c_2 \geqq 2.022$ matrix P is non positive so that $p_{11} \neq I$, as is indicated in the same Figure. Fig. 4.4b shows the actual response for $c_2 = 0.985$ and the ideal model which corresponds to the solution of the differential equation

$$y(t) + dy(t)/dt = 0$$

The maximum deviation of the actual response with respect to the ideal one is

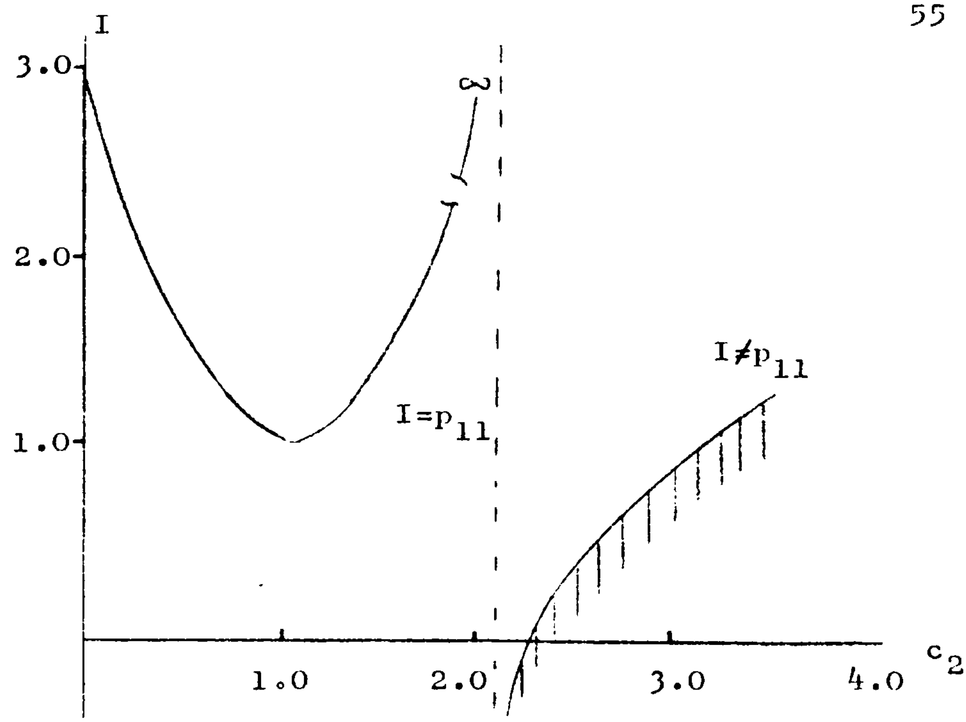
$$x = 0.265 < 2^{1/4}\sqrt{\frac{I}{T}} = 1.13$$

By considering now that both c_1 and c_2 are unknown, it might be possible to make the actual system approach closer to the model. The relationship between P and the two adjustable coefficients is now

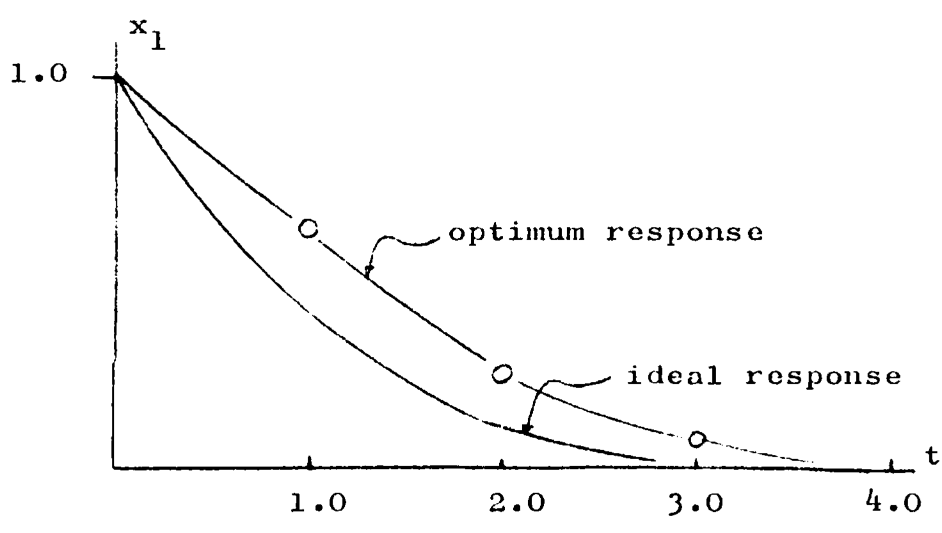
$$-1.0 = (1-(1-.368c_1)^2p_{11}+1.26c_1(1-.37c_1)p_{12}-.40p_{22}$$

$$-1.0 = -(1-.37c_1)(.63-.37c_2)p_{11}+(.63+.63c_1+.63c_2$$

$$-.47c_1c_2)p_{12}+.63c_1(.37-.63c_2)p_{22}$$



(a) Performance Index Versus Parameter c_2 in Example 4-2



(b) Transient Response for the Optimum Value of the Performance Index in Example 4-2

Fig. 4.4 Results for the Optimum System of Example 4-2

$$1.0 = -(.63-.37c_2)^2 p_{11} - 2(.63-.37c_2)(.37-.63c_2)p_{12}$$
$$+(1-(.37-.63c_2)^2)p_{22}$$

As it may be observed from this example, non-linear relations between the elements of P and the feedback coefficients arise in the evaluation of the equation obtained from the theorem stated in Section 2 of this chapter. The difficulties of procedure B can then be summarized as

1. In finding P, one has to solve the $n(n+1)/2$ linear equations given by equation (4-11).
2. Once P is expressed in terms of the adjustable parameters of the system, the relationship between P and $\underline{c}^t$ is non-linear.

The problem gets worse when one has to find the roots of $\frac{\partial I}{\partial c_i} = 0$ and then to check which one gives I min. It is necessary therefore, to look for another formulation of the system optimization by using the computer. Chapter V is concerned with this approach to the system design.

4.4 Summary

In this chapter stability definitions and theorems are stated in order to introduce design procedures using the Second Method of Liapunov.

Two approaches to the design have been proposed: in the first approach, the performance index is not referred to the model, and the system obtained is not optimum. In the second approach, although the performance index is referred to the model, difficulties arise mainly because the relationship between the Liapunov function and the feedback coefficients was non-linear. An alternative formulation by using the computer is suggested. This last approach is presented in Chapter V.

CHAPTER V

ALTERNATE APPROACH VIA DYNAMIC PROGRAMMING

5.1 Introduction

It was observed in Chapter IV that difficulties arise in the evaluation of the performance index through the use of the Second Method of Liapunov. This was clearly illustrated in Example 4-2, where only one adjustable parameter was given. The nonlinear relationship between the elements of the P matrix and the one adjustable parameter resulted in a fourth order polynomial when dp_{11}/dc_2 was taken. In systems with more than one free feedback coefficient, this difficulty is compounded in two ways. Not only is the order of the polynomial liable to increase, but the $\frac{\partial P}{\partial c_i}$ yields a function of all the c_i's. Thus, the roots of n high order polynomial must be found simultaneously. This presents an insurmountable obstacle as far as hand calculation is concerned. If one is forced to resort to machine calculations, then a number of alternate formulations exist. One of these is the use of dynamic programming. This chapter discusses the equivalence between the Second Method approach and that of dynamic programming. The chapter concludes with examples of system design to the performance index

$$I = \sum_{k=1}^{\infty} \left[\underline{x}^t(k)\ Q\ \underline{x}(k) + \alpha^2 u^2(k-1) \right]$$

from Chapter IV.

The computer program used to obtain these results is based on the dynamic programming methods.

5.2 System Optimization by Dynamic Programming

Consider again an nth order linear process defined by the vector-matrix differential equation

$$\dot{\underline{x}}(t) = A\ \underline{x}(t) + \underline{b}\ u(t)$$

In sampled-data form, the linear process may be characterized by the state-transition equation

$$\underline{x}(k+1) = \phi(1)\ \underline{x}(k) + \underline{\beta}(1)u(k)$$

The problem is to choose the control processes u(0), u(1), . . . u(N-1), so as to minimize the performance index I_N

$$I_N = \sum_{k=1}^{N} \left[\underline{x}^t(k)Q\underline{x}(k) + \alpha^2 u^2(k-1) \right] \tag{5-1}$$

Assuming N and $\underline{x}(0)$ to be variables, let the minimum value I_N be denoted by

$$f\left[\underline{x}(0),\ N\right] = \left\{ \min_{\substack{u(0)\\ u(1)\\ \cdots\\ u(N-1)}} \sum_{k=1}^{N} \underline{x}^t(k)Q\underline{x}(k) + \alpha^2u^2(k-1) \right\} \quad (5\text{-}2)$$

Invoking the Principle of Optimality (Bellman, 1957), which may be stated as an optimal sequence of N control signals u(0), u(1), . . . u(N-1) has the property that whatever the initial state $\underline{x}(0)$ and the initial choice of u(0) are, the remaining sequence u(1), . . . u(N-1) must constitute an optimal sequence with regard to the state $\underline{x}(1)$ resulting from the first choice u(0), then the minimum value of I_N may be given by

$$f\left[\underline{x}(0),\ N\right] = \min_{u(0)} \left[f\left[\underline{x}(0),\ 1\right] + f\left[\underline{x}(1),\ N-1\right] \right] \quad (5\text{-}3)$$

where

$$f\left[\underline{x}(0),\ 1\right] = \min_{u(0)} \left[\underline{x}^t(1)Q\underline{x}(1) + \alpha^2u^2(0) \right] \quad (5\text{-}4)$$

Assume that f is quadratic in $\underline{x}$, so that one has

$$f\left[\underline{x}(0),\ N\right] = \underline{x}^t(0)\ P(N)\ \underline{x}(0) \quad (5\text{-}5)$$

Then Eq. (5-1) becomes

$$\underline{x}^t(0)P(N)\underline{x}(0) = \min_{u(0)} \Big[\underline{x}^t(1)Q\underline{x}(1)+\alpha^2u^2(0) + \underline{x}^t(1)P(N-1)\underline{x}(1) \Big]$$

Upon substitution of $\underline{x}(1)$ by

$$\underline{x}(1) = \emptyset(1)\ \underline{x}(0) + \underline{\beta}(1)\ u(0)$$

one has

$$\underline{x}^t(0)P(N)\underline{x}(0) = \min_{u(0)} \Big[\underline{x}^t(0)\ \emptyset^t(1) \left[Q+P(N-1)\right] \emptyset(1)\ \underline{x}(0)$$

$$+ 2\underline{\beta}^t(1)\underline{x}^t(0) \left[Q+P(N-1)\right] \emptyset(1)u(0)$$

$$+ \left[\underline{\beta}^t(1)(Q+P(N-1))\underline{\beta}(1) + \alpha^2\right]u^2(0) \Big] \quad (5\text{-}6)$$

Let

$$S(N-1) = Q + P(N-1) \quad (A)$$

Then

$$\underline{x}^t(0)P(N)\underline{x}(0) = \min_{u(0)} \Big[\underline{x}^t(0)\emptyset^t(1)S(N-1)\emptyset(1)\underline{x}(0)$$

$$+ 2\underline{\beta}^t(1)\underline{x}^t(0)S(N-1)\emptyset(1)\ u(0)$$

$$+ \left[\underline{\beta}^t(1)S(N-1)\beta(1)+\alpha^2\right] u^2(0)\Big] \quad (5\text{-}7)$$

Since the N-stage decision process has been reduced to a sequence of N single-stage decision processes by the principle of Optimality, the minimization procedure may be readily carried out through ordinary differentiation. By taking the derivative of the functions inside the braces of Eq. (5-7), with respect to u(0), one has for u(0)

$$u^N(0) = -\left[\frac{\underline{\beta}^t S(N-1)\emptyset}{\underline{\beta}^t S(N-1)\underline{\beta}+\alpha^2} \right] \underline{x}(0)$$

Define

$$\underline{c}^t(N-1) = - \frac{\underline{\beta}^t(1)S(N-1)\phi(1)}{\underline{\beta}^t(1)S(N-1)\underline{\beta}(1)+\alpha^2} \qquad \text{(B)}$$

Then

$$u^N(0) = \underline{c}^t(N-1)\ \underline{x}(0) \qquad \text{(D)}$$

Notice that the optimum control law is a linear function of the state variables. Substituting back into Eq. (5-7), one has

$$P(N) = \left[\phi + \beta\underline{c}^t(N-1)\right]^t S(N-1)\left[\phi + \beta\underline{c}^t(N-1)\right]$$

$$+ \alpha^2\underline{c}(N-1)\ \underline{c}^t(N-1) \qquad \text{(C)}$$

The computer program used in this work (Melsa, 1965) primarily does the following

1. Start with $P(0) = 0$
2. Use expression (A) to find $S(i)$ (start with $i=0$)
3. Use expression (B) to find $\underline{c}^t(0)$
4. Use (C) to find $P(1)$
5. Go to (2)

5.3 Relationship to the Optimum Liapunov Function

With reference to the performance index defined in Eq. (5-1), let $N \to \infty$; then one has the performance index studied in Chapter IV, for the case of constraints in the control input.

$$I = \sum_{k=1}^{\infty} \underline{x}^t(k)\ Q\ \underline{x}(k) + \alpha^2 u^2(k-1)$$

where matrix Q is again associated to the model.

Equation (5-3) reduces to

$$f\left[\underline{x}(0)\right] = \min_{u(0)} \left[\underline{x}^t(1)Qx(1) + \alpha^2 u^2(0) + f\left[\underline{x}(1)\right]\right] \quad (5\text{-}8)$$

Assume that

$$f\left[\underline{x}(0)\right] = \underline{x}^t(0)P\underline{x}(0)$$

The Eq. (5-8) becomes

$$\underline{x}^t(0)P\underline{x}(0) = \min \left[\underline{x}^t(1)Q\underline{x}(1)+\alpha^2 u^2(0)+\underline{x}^t(1)P\underline{x}(1)\right] \quad (5\text{-}9)$$

Substitute in Eq. (5-9), $\underline{x}(1)$ by

$$\underline{x}(1) = \phi(1)\underline{x}(0) + \underline{\beta}(1)u(0)$$

so that

$$\begin{aligned}\underline{x}^t(0)P\underline{x}(0) = \min_{u(0)} \Big[& \underline{x}^t(0)\phi^t(1)\left[Q+P\right]\phi(1)\underline{x}(0) \\ & + 2\underline{\beta}^t\underline{x}^t(0)\left[Q+P\right]\phi(1)u(0)+\underline{\beta}^t(1)\left[Q+P\right]\underline{\beta}(1) \\ & + \alpha^2u^2(0)\Big]\end{aligned} \quad (5\text{-}10)$$

By differentiating the right side of Eq. (5-10) with respect to u(0) and setting the result to zero, one gets:

$$u(0) = -\frac{\underline{\beta}^t(1)\ \left[P+Q\right]\phi(1)}{\underline{\beta}^t(1)\ \left[P+Q\right]\underline{\beta}(1)+\alpha^2}\ \underline{x}(0) \quad (5\text{-}11)$$

Let

$$\underline{c}^t = - \frac{\underline{\beta}(1) \left[P+Q\right] \emptyset(1)}{\underline{\beta}^t(1) \left[P+Q\right] \underline{\beta}(1) + \alpha^2}$$

then

$$u(0) = \underline{c}^t \, \underline{x}(0)$$

where $\underline{c}$ is now a constant row matrix.

Substituting u(0) of Eq. (5-11) back into Eq. (5-10)

$$P = \left[\emptyset(1) + \underline{\beta}(1)\underline{c}^t\right]^t \left[P+Q\right] \left[\emptyset(1) + \underline{\beta}(1)\underline{c}^t\right] + \alpha^2 \underline{cc}^t$$

Solving for P one gets

$$P - \left[\emptyset(1) + \underline{\beta}(1)\underline{c}^t\right]^t \left[P\right] \left[\emptyset(1) + \underline{\beta}(1)\underline{c}^t\right]$$

$$= \left[\emptyset(1) + \underline{\beta}(1)\underline{c}^t\right]^t Q \left[\emptyset(1) + \beta(1)\underline{c}^t\right] + \alpha^2 \underline{cc}^t$$

But

$$\emptyset(1) + \underline{\beta}(1)\underline{c}^t = \emptyset_s(1)$$

Then

$$P - \emptyset_s^{\ t} \, P \, \emptyset_s = \emptyset_s^{\ t} \, Q \, \emptyset_s + \underline{cc}^t = Q_T \qquad (5\text{-}12)$$

which is the same expression for P obtained in Eq. (4-9) when constraints in the control input were taken into consideration. Therefore

$$I\text{min} = f\left[\underline{x}(0)\right] = \underline{x}^t(0) \, P \, \underline{x}(0) = V\left[\underline{x}(0)\right]$$

$V\left[\underline{x}(0)\right]$ is the Liapunov function for the optimum system.

The performance index used in the computer program by Melsa (1965) is given in Eq. (5-1). By letting N be as large as possible one may approximate $\underline{c}(N-1)$ to $\underline{c}$ of Eq. (5-11 b). This is conveniently done by defining a region of convergence for $\underline{c}(N-1)$. Also in this computer program $\alpha^2 = 1$ then one must have a weighting factor $\lambda^2 > \alpha^2$ in Q, if one wants the performance index to be referred to the model. In the example problems given later, $\lambda^2 = 5$.

The data required for the computer are summarized below

$\phi(1)$ = state-transition matrix of the plant with T = 1

$\underline{\beta}(1)$ = nx1 distribution matrix of the system

Q = nxn matrix which depends on the model

CONV = Region of convergence for $\underline{c}(N-1)$

5.4 Application to Specific Problems

Example Problem 5-1

Let the plant be given by

$$G(s) = \frac{1}{s(s+1)}$$

Find the feedback coefficients c_1 and c_2 (see Fig. 5.1), which minimizes the performance index

$$I = \sum_{k=1}^{\infty} \left[5\left[y(k) + \tau \dot{y}(k)\right]^2 + u^2(k-1) \right]$$

where τ = 0, 1, 2, 3, and 5.

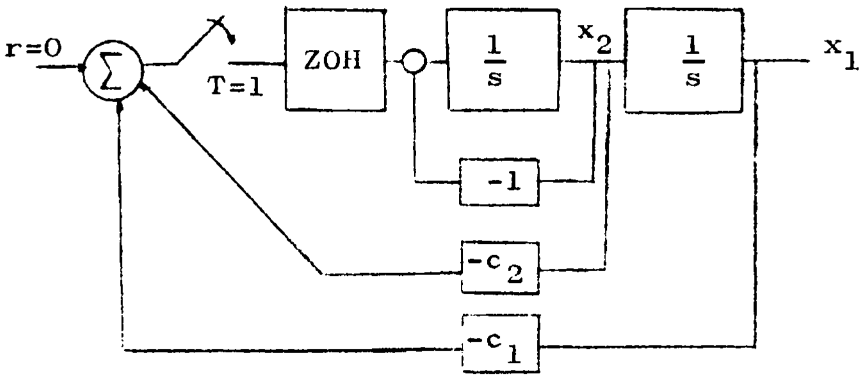


Fig. 5.1 Block Diagram Representation for System of Example 5-1

First one must find $\phi(1)$ and $\underline{\beta}(1)$ for the given plant:

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)}$$

$$y(t) + \dot{y}(t) = u(t)$$

Let

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

then one has

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{b}\, u(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad ; \qquad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\phi(t)$ is then given by

$$\phi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1} \right] = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\underline{\beta}(1) = \int_0^1 \phi(1-\tau)\ \underline{b}\ d\tau = \int_0^1 \begin{bmatrix} 1-e^{-(1-\tau)} \\ \\ e^{-(1-\tau)} \end{bmatrix} d\tau$$

The data required for the computer is then

$$\phi(1) = \begin{bmatrix} 1 & 0.632 \\ \\ 0 & 0.368 \end{bmatrix} \qquad \underline{\beta}(1) = \begin{bmatrix} 0.368 \\ \\ 0.632 \end{bmatrix}$$

$$Q = 5 \begin{bmatrix} 1 & \tau \\ \\ \tau & \tau^2 \end{bmatrix}$$

The region of convergence for $\underline{c}(N-1)$ is assumed to be 0.001. The results of the computer calculations are shown in Table 5.1.

The transient response of the optimum system may be obtained from

$$\underline{x}(k+1) = \phi_s(1)\ \underline{x}(k)$$

where

$$\phi_s(1) = \begin{bmatrix} 1-0.368c_1 & 0.632-0.368c_2 \\ \\ -0.632c_1 & 0.368-0.632c_2 \end{bmatrix}$$

Figures 5.2 and 5.3a through d show the transient responses of the actual system. In Table 5.1 the maximum deviation of the actual responses with respect to their respective models

TABLE 5.1

COMPUTER RESULTS AND CALCULATIONS FOR MAXIMUM DEVIATIONS OF ACTUAL SYSTEM REFERRED TO THE MODELS FOR EXAMPLE 5-2

	$\tau = 0$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 5$
c_1	-0.876	-0.784	-0.608	-0.458	-0.292
c_2	-0.813	-0.784	-0.734	-0.698	-0.665
$\lvert\delta\, ymax\rvert$	----	0.344	0.169	0.112	0.075
$2^{1/4}\sqrt{\frac{Imin}{\tau}}$	----	0.300	0.430	0.440	0.720

is given; also their respective bounds $2^{1/4}\sqrt{\frac{Imin}{\tau}}$ as given in inequality (3-10).

Example Problem 5-2

Let the plant be given by (see Fig. 5.4)

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

Find the feedback coefficients c_1, c_2, and c_3, which minimizes the performance indices

$$\text{a)}\quad I_1 = \sum_{k=1}^{\infty} 5\left[\, y(k) + \dot{y}(k)\right]^2 + u^2(k-1)$$

$$\text{b)}\quad I_2 = \sum_{k=1}^{\infty} 5\left[100y(k)+10\dot{y}(k)+y(k)\right]^2+u^2(k-1)$$

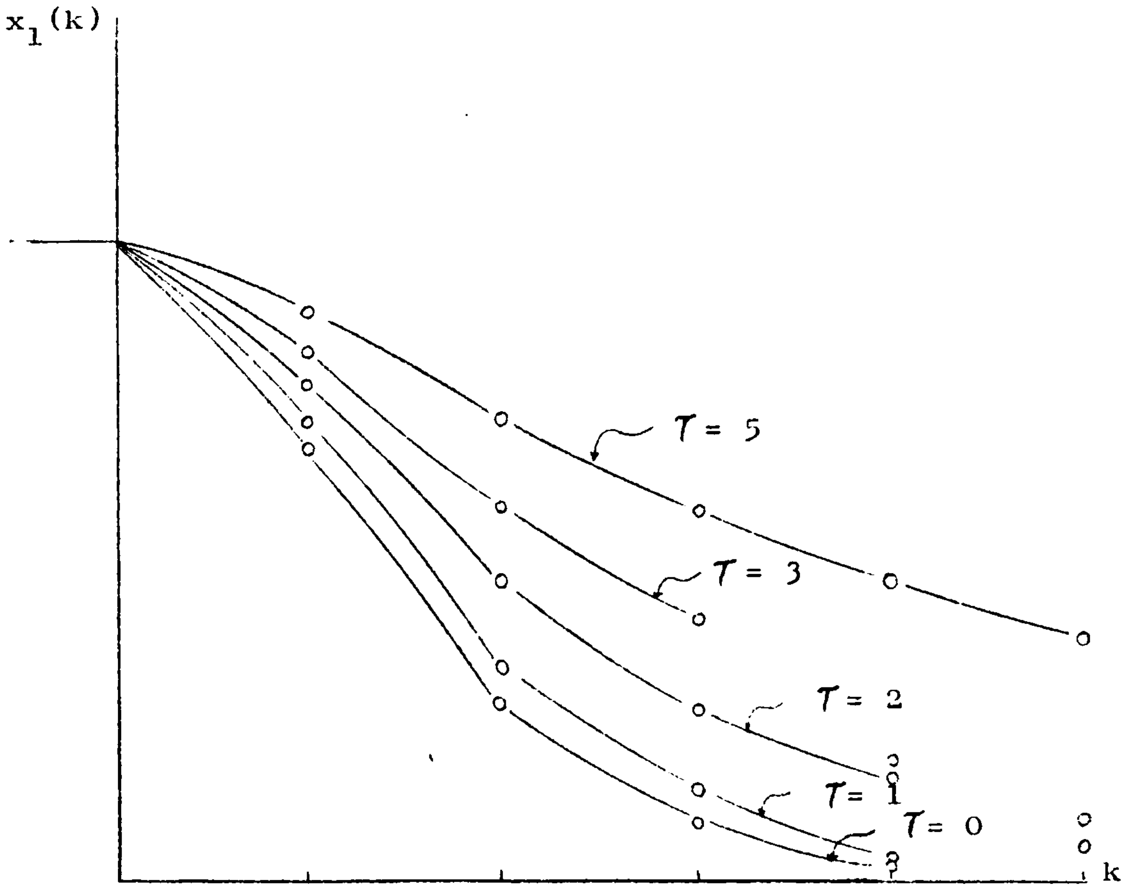


Fig. 5.2 Transient Responses for the Optimum System of Example 5-1

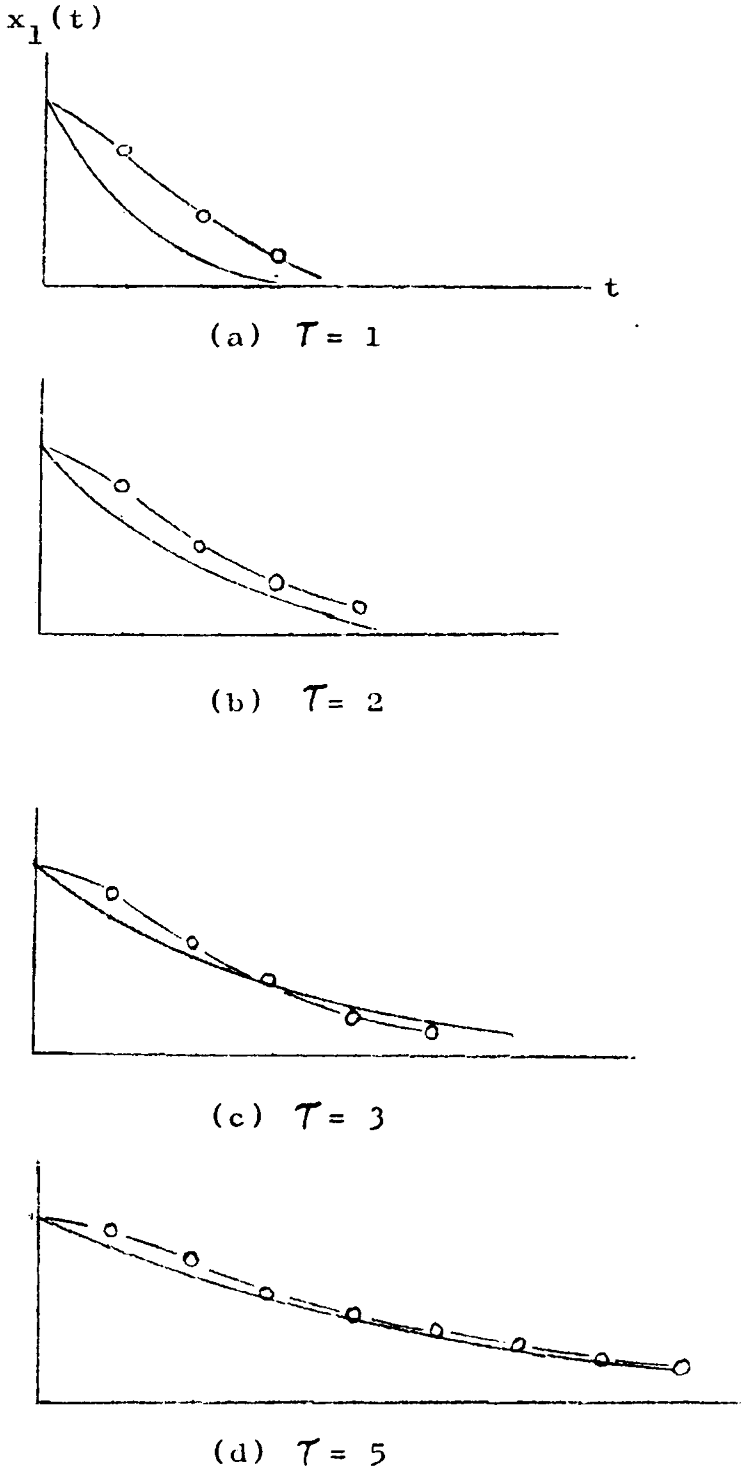


Fig. 5.3 Transient Responses for the Optimum System of Example 5-1

—o— Actual Response
——— Ideal Model With Time Instant

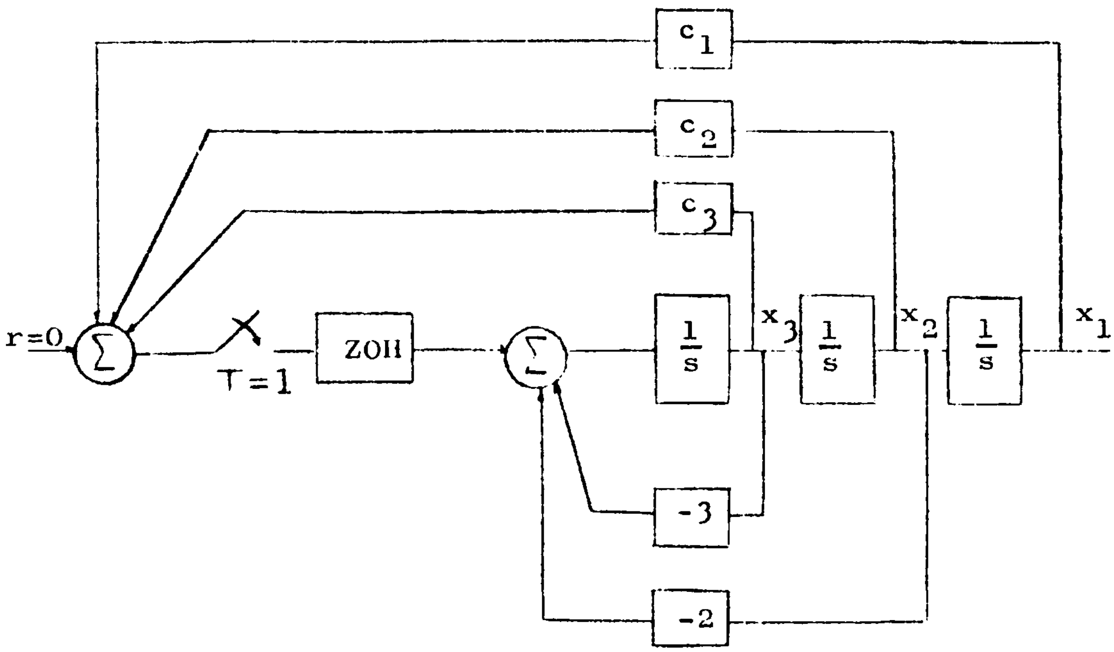


Fig. 5.4 Block Diagram Representation for Closed-Loop System of Example 5-2

The model of I_1 corresponds to the solution of the differential equation

$$y(t) + \dot{y}(t) = 0$$

In I_2 the model corresponds to the solution of the differential equation

$$\ddot{y}(t) + \frac{2\hat{\xi}}{\omega_n}\dot{y}(t) + \omega_n^2 y(t) = 0$$

where

$$\xi = 0.5, \omega_n = 10 \text{ radsec}^{-1}$$

First of all $\phi(1)$ and $\underline{\beta}(1)$ are determined; they are

$$\phi(1) = \begin{bmatrix} 1.00 & 0.831 & 0.200 \\ 0.00 & 0.601 & 0.233 \\ 0.00 & -0.466 & -0.098 \end{bmatrix}$$

$$\underline{\beta}(1) = \begin{bmatrix} 0.084 \\ 0.200 \\ 0.233 \end{bmatrix}$$

For I_1 the Q matrix is

$$Q = 5 \begin{bmatrix} 1.0 & 1.0 & 0.0 \\ 1.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

For I_2 the Q matrix is

$$Q = 5 \begin{bmatrix} 10^4 & 10^3 & 10^2 \\ 10^3 & 10^2 & 10^1 \\ 10^2 & 10^1 & 1 \end{bmatrix}$$

The convergence for $\underline{c}(N-1)$ is 0.001

The results of the computer calculations are

For I_1	For I_2
$c_1 = -1.777$	$c_1 = -6.128$
$c_2 = -2.660$	$c_2 = -6.068$
$c_3 = -0.866$	$c_3 = -1.637$

Figures 5.5a and 5.5b show the transient responses of the optimum systems for the performance indices I_1 and I_2 respectively.

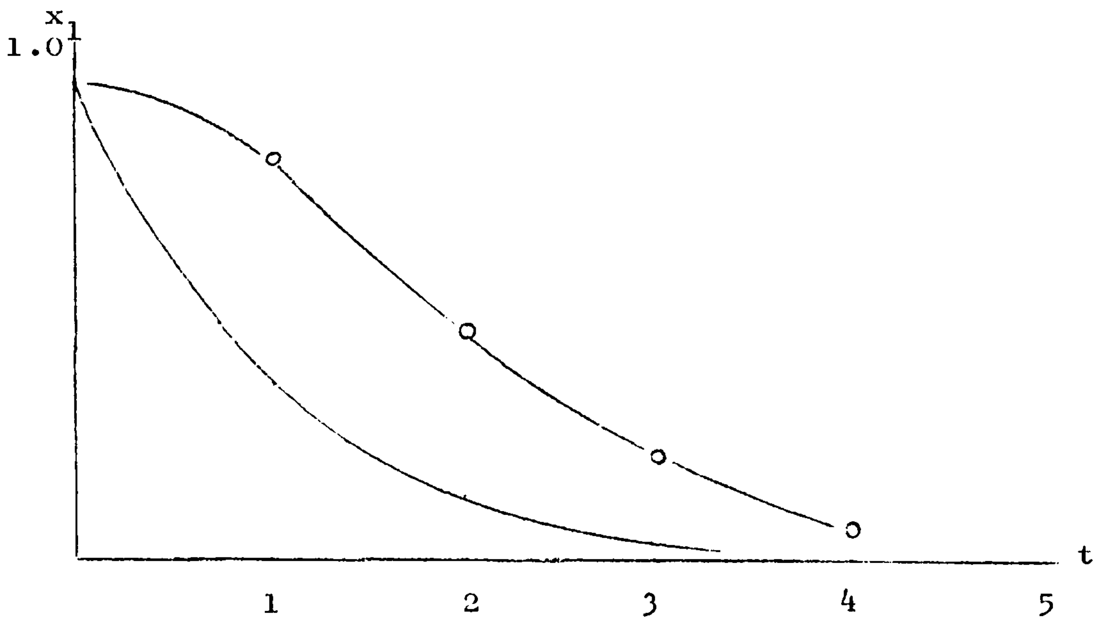
<u>Example 5-3</u>

Let the plant be given by

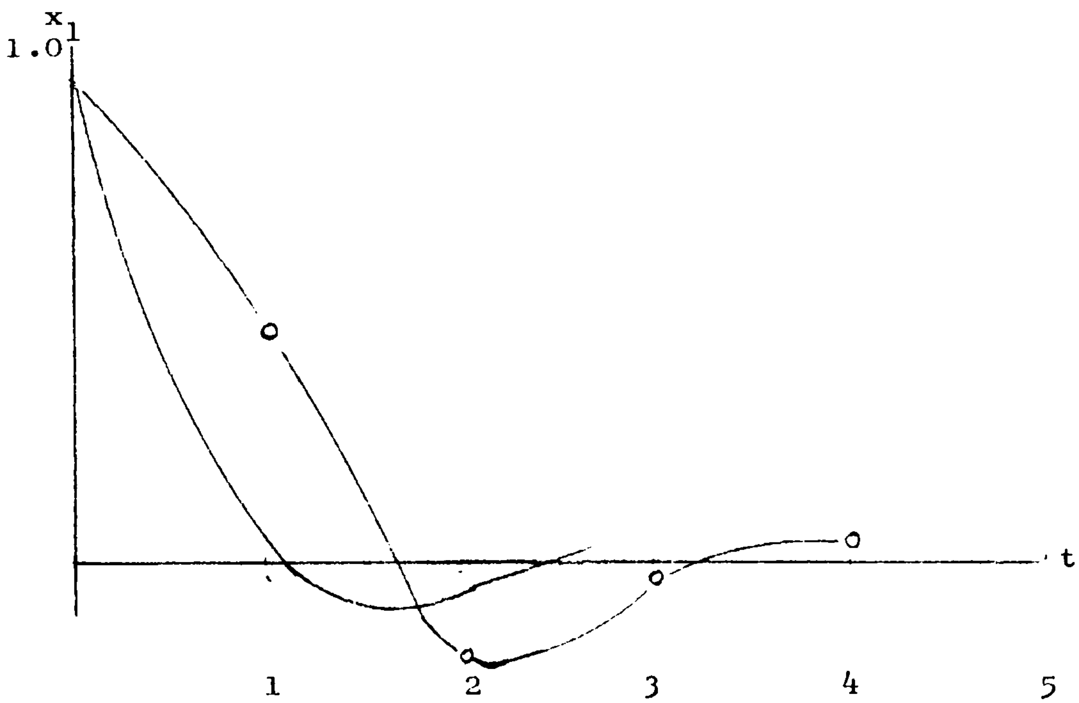
$$G(s) = \frac{s+0.5}{s(s+1)(s+2)}$$

Find the feedback coefficients c_1, c_2, and c_3, which minimizes the performance index

$$I = \sum_{k=1}^{\infty} 5\left[\, y(k) + y(k)\right]^2 + u^2(k-1)$$



(a) Optimum System for Performance Index I_1



(b) Optimum System for Performance Index I_2

Fig. 5.5 Transient Responses for the Optimum System of Example 5-2

By representing the block diagram of the plant as shown in Fig. 5-6, one has that $\phi(1)$ and $\underline{\beta}(1)$ are the same as in Example 5-2.

$$\phi(1) = \begin{bmatrix} 1.00 & 0.831 & 0.200 \\ 0.00 & 0.601 & 0.233 \\ 0.00 & -0.466 & -0.098 \end{bmatrix}$$

$$\underline{\beta}(1) = \begin{bmatrix} 0.084 \\ 0.200 \\ 0.233 \end{bmatrix}$$

The output y(k) is given by

$$y(k) = 0.5\, x_1(k) + x_2(k)$$

$$\dot{y}(k) = 0.5\, x_2(k) + x_3(k)$$

By expressing the model in terms of the state variables of the system, one has for the performance index

$$I = \sum_{k=1}^{\infty} 5\left[0.5x_1(k)+1.5x_2(k)+x_3(k)\right]^2 + u^2(k-1)$$

so that

$$Q = 5\begin{bmatrix} 0.25 & 0.75 & 0.50 \\ 0.75 & 2.25 & 1.50 \\ 0.50 & 1.50 & 1.00 \end{bmatrix}$$

The results of the computer calculations are

$$c_1 = -1.702$$

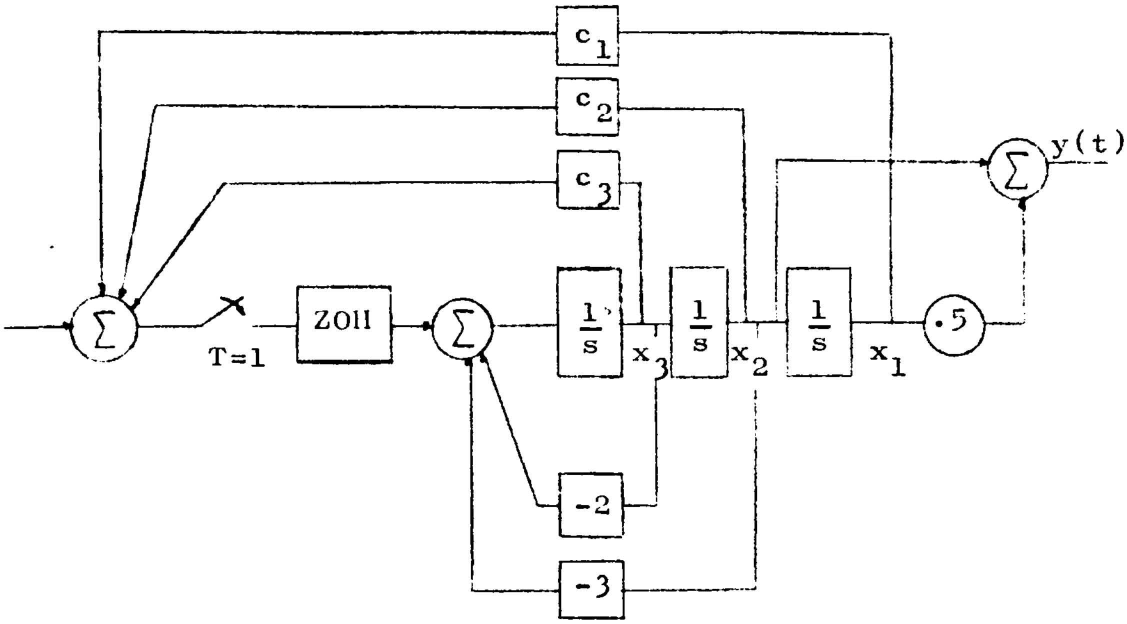


Fig. 5.6 Block Diagram Representation for Closed-Loop System of Example 5-3

$c_2 = -2.558$

$c_3 = -0.859$

Fig. 5.7 shows the transient responses of the optimum system and of its respective model.

5.5 Summary

As it is observed in the computer results, the actual systems approach to their respective models as expected. In the case of a first order model, for example, it is harder for the system to approach the response of those models with small time constant values. Also the relationship between the maximum deviation of the system trajectory from the model response as determined for continuous systems seems sufficiently conservative so that it is applicable to sampled-data systems as well. Evidence of this fact is given in Table 5.1.

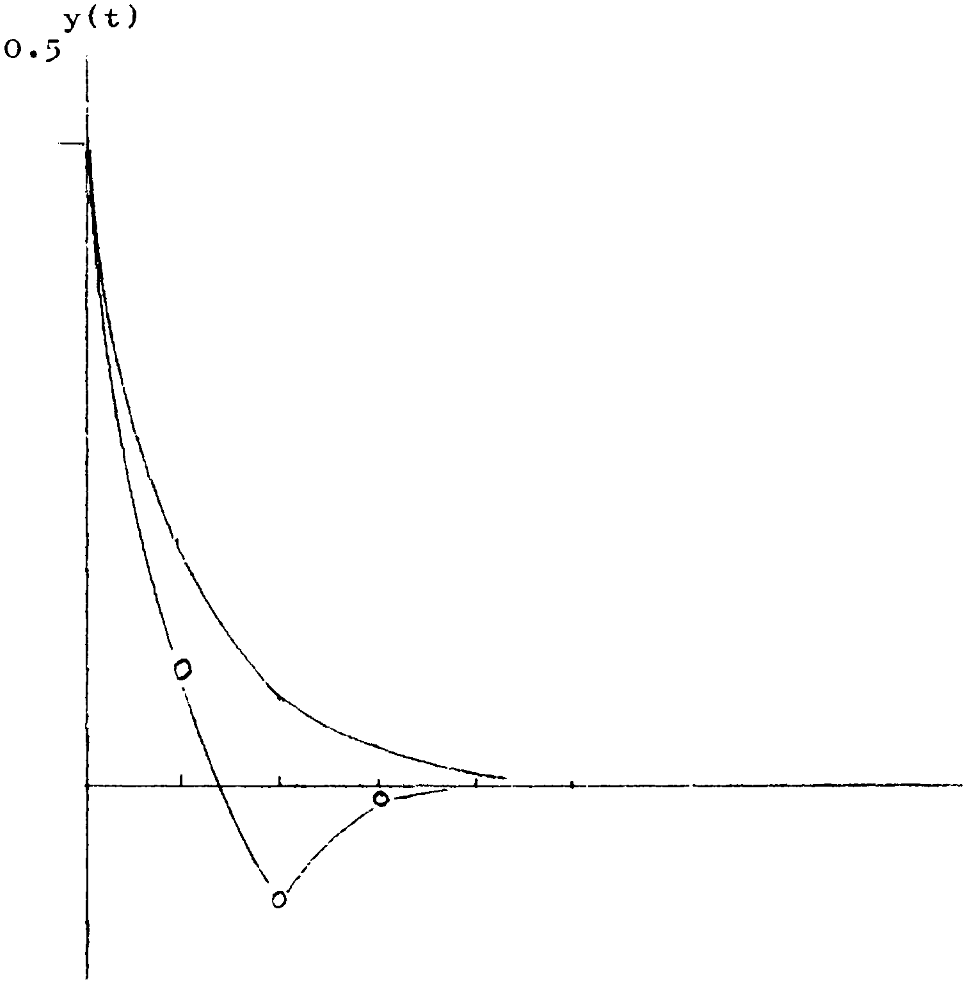


Fig. 5.7 Transient Response for Optimum System of Example 5-3. First order model with unity time constant

CHAPTER VI

CONCLUSIONS

6.1 Summary

In this thesis the general problem is the design of sampled-data control systems using quadratic type performance indices. The adjustable parameters are assumed to be the fixed feedback gains of the state variables of the plant.

The method of design presented is important since it enables the designer to associate the performance index to a model. This is not only useful for known plants but also for adaptive systems because it is possible to simulate derivatives of the output when the latter is known.

Deviation of the optimum system with respect to the ideal one is obtained for the integral type performance criterion. For a first order differential equation this deviation is found to be proportionally to the inverse of the time constant of the model.

The Second Method of Liapunov offers a general procedure for the evaluation and minimization of the performance index. By using the Second Method, the performance index, I, is expressed in terms of the initial conditions and the adjustable feedback parameters of the plant. Also, by expressing a suitable Liapunov function in a convergent series form, the association of the performance index to a model has been possible.

A disadvantage of the Second Method is that non-linear relations between the performance index and the adjustable parameters are obtained. Formulation of the problem based on dynamic programming is also presented.

6.2 Suggestions for Other Investigations

Three areas for future investigation have occurred to the author.

1. To extend the design of sampled-data control systems to time weighted performance indices.
2. To design sampled-data control systems by feeding back all of the state variables of the plant and at the same time to use digital compensators. The performance criterion should be other than quadratic type invariant with time. The increase in adjustable parameters should improve the transient responses of the optimum systems.
3. To extend the design procedure studied in this work to adaptive systems.

In any of the three cases just mentioned, it is advisable to study the stability considerations by using the Second Method, and then to minimize by using computer programming.

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