# TRIANGULAR PLATE MODEL FOR NONLINEAR STRESS ANALYSIS

by

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# ABSTRACT

A triangular finite element model for analysis of nonlinear plane stress systems is presented herein. The general equations of the displacement method have been formulated from a differential point of view, and yield a set of simultaneous nonlinear first order ordinary differential equations. The fourth order Runge-Kutta integration scheme was used to solve these equations with proportional loads being applied to the system.

Nonlinear material properties were introduced into the element stiffnesses by Richard's equation for stress-strain, and Poisson's ratio was considered as being a function of the displacements. Although only material nonlinearities were considered in these examples, the method could be extended to include geometric nonlinearities.

Several examples were analyzed to demonstrate the feasibility of the model and the entire analysis was carried out on an IBM 7072 digital computer.

# CHAPTER 1

## INTRODUCTION

Methods for nonlinear stress analysis have received much attention in recent years, and are presently the subject of widespread research. Much of this interest has resulted from the requirement for faster aircraft and the demand of the space industry for greater payloads. In civil engineering structures the intensified interest in lightweight systems results from the demand that new methods be developed for analyzing structures that are basically nonlinear.

Many of the new materials which have recently been developed and are presently in use have material nonlinearities which must be considered in certain lightweight systems. Inherent properties such as brittleness of some of these newer materials also require that the structures be analyzed to a high degree of accuracy.

The classical theory of statically indeterminate structures is a linear theory which assumes that the structure to which this theory is being applied comprises only elements whose behavior can be described entirely by linear laws. However, a rigorous nonlinear analysis is far more complex than the linear analysis and, in general, very difficult to use. Therefore, much of the research in nonlinear analysis is presently based upon certain numerical and matrix analysis approaches.

Idealization of structures into discrete elements is a method which has a broad applicability to complex structures. The suitability of this method to automated digital computers has led to its general acceptance for application with either the force method or the displacement method as a satisfactory procedure for matrix structural analysis. Computer programs are presently available for use in predicting linear stresses and displacements throughout real, complex structures. The generally accepted matrix method for nonlinear analysis is essentially the same as for the linear analysis except that a different technique is utilized in formulating and solving the set of governing equations.

One approach to nonlinear analysis is the adoption of the incremental approach, i.e., the reduction of a nonlinear problem to a piecewise linear problem. Turner et.al. (1960) developed a procedure for the displacement method while Denke (1962a) and Warren (1962) provided similar procedures for the force method. Another well known technique used to account for the nonlinear behavior of the elements is the Prandtl-Reuss plastic flow theory used by Denke (1962b) for the plastic analysis of plane stress elements.

The purpose of this thesis is to present a new model which may be utilized in formulating and solving the set of governing equations characteristic to nonlinear structures, and to present some examples which have been solved by this model. Triangular finite elements, developed by Turner (1956), have been selected as the discrete elements for usage in the idealization of these examples. The stress-strain formula developed by Richard (1961) was employed to represent the nonlinear stress-strain relationship. Formulation and solution of the set of governing equations is based on a differential point of view and results in an initial value problem. Only material nonlinearities have been considered here; however, the method developed herein may be extended to include nonlinearities resulting from changes in structure geometry.

CHAPTER 2

# DEVELOPMENT OF THE MODEL

## General Description

Nonlinear methods presently employed in structural analysis are essentially a generalization of one of the two distinct and separate methods which are currently being used to analyze statically indeterminate structural systems. These two methods are the force method and the displacement method. In the force method, the redundant forces are treated as unknowns and the element forces are expressed in terms of the external forces and the redundants. Element displacements can then be expressed in terms of element forces by using the stress-strain relationship. By applying compatibility conditions a set of linear simultaneous equations is obtained which can be solved for the redundant forces. When this method is applied to nonlinear structures, a set of nonlinear algebraic equations results in the unknown forces which were chosen as the redundants. In general, an iterative technique must be used to solve these equations, and for large systems of equations there generally is no assurance of convergence to a solution. Another method has been developed (Richard and Goldberg, 1965) which treats the force method from a differential point of view and, when extended to apply to nonlinear systems, results in a system of simultaneous nonlinear ordinary differential equations. For systems of practical size and interest only numerical solutions of these equations are feasible.

In the displacement method, the displacements are chosen as the unknowns, and by enforcing compatibility the internal element displacements can be found in terms of the node displacements. Internal element stresses can then be obtained by using the stress-strain relationship, and the external forces necessary to cause these displacements can be determined by applying the conditions of equilibrium at the node points. In the case of linear systems the result is a set of linear simultaneous algebraic equations which can be solved for the displacements of the system. This method can be used for nonlinear structural systems as well as for linear systems. Goldberg and Richard (1963) extended the displacement method by using a differential point of view to obtain a set of simultaneous nonlinear ordinary differential equations which could be solved by numerical integration procedures.

Either of the above methods may be used to solve linear and nonlinear systems, but a severe drawback in the force method must be seen in the fact that considerable skill is needed in choosing suitable redundant forces. Forces causing only local effects which do not diffuse into the system are considered to be best for this method, and choosing these often becomes a difficult problem in real, complex systems. The displacement method does not have this problem because the displacements necessary to describe the displaced configuration are usually obvious, Therefore, the model developed herein will be used with the displacement method, since this method is more suitable for the analysis of the nonlinear systems treated in this thesis.

## General Equations

In developing the general equations for the model presented here the differential formulation of the displacement method was used. The following characteristic equations are derived from the algebraic equations which have been formulated for the linear displacement method.

For completeness and convenience both the linear and differential systems of equations are presented, and the equilibrium, stress-strain, and compatibility equations are expressed in matrix form.

In the following linear equations let x denote the column matrix of generalized displacements of the structural system, u denote the column matrix of the element boundary displacements of the structural elements, and B represent the rectangular matrix relating the element boundary displacements to the generalized displacements (compatibility). Let A be the rectangular matrix relating the element boundary forces to the applied forces associated with the generalized displacements (equilibrium), $\lambda$ be the factor relating the proportionality of the applied forces, $\lambda P$ be the column matrix of applied forces associated with the generalized displacements of the system, and F denote the column matrix of element boundary forces associated with the boundary displacements.

Then, the compatibility conditions are

$$u = Bx \tag{2-1}$$

and the equilibrium equations are

$$\lambda P = AF \tag{2-2}$$

Applying the principle of virtual work using virtual displacements the equation can be written as

$$\lambda P^t x_v = F^t u_v \tag{2-3}$$

where the superscript t indicates the transpose of the matrix and the subscript v indicates the virtual displacements. Transposing Equation (2-3) yields

$$x_v^t \lambda P = u_v^t F \tag{2-4}$$

Substituting Equations (2-2) and (2-1) into Equation (2-4) gives

$$x_v^t AF = x_v^t B^t F \tag{2-5}$$

or

$$x_v^t (A - B^t) F = 0 \tag{2-6}$$

since the matrix product is distributive and associative. Since F is any arbitrary set of element boundary forces and $x_v^t$ is any set of virtual external displacements, it follows that

$$(A - B^t) = 0 \tag{2-7}$$

or

$$A = B^t \tag{2-8}$$

This means that the equilibrium matrix is equal to the transpose of the compatibility matrix. Hence, Equation (2-2) can be rewritten as

$$\lambda P = B^t F \tag{2-9}$$

Now let k denote the square matrix relating the element boundary displacements to the element boundary forces and the stress-strain (force-displacement) relationship is written as

$$F = ku \tag{2-10}$$

Substituting Equation (2-1) into Equation (2-10) yields

$$F = kBx \tag{2-11}$$

Substituting Equation (2-11) into Equation (2-9) gives

$$\lambda P = B^t kBx \tag{2-12}$$

Equation (2-12) forms a system of simultaneous linear algebraic equations. Since the applied forces $\lambda P$ are known, these equations may be solved for the generalized displacements x, which in turn may be used in Equation

(2-11) to solve for the boundary forces of the individual elements.

In order to develop the differential system of equations characteristic of structures containing nonlinear elements, Equations (2-1),(2-9), and (2-10) are differentiated with respect to an applied load parameter, P. That is

$$\frac{du}{dP} = \frac{Bdx}{dP} \tag{2-13}$$

$$\frac{d(\lambda P)}{dP} = \frac{B^t dF}{dP} \tag{2-14}$$

and

$$\frac{dF}{dP} = \frac{\bar{k}du}{dP} \tag{2-15}$$

Since the elements of the stress-strain (force-displacement) matrix are not constant but are functions of the boundary displacements, $\bar{k}$ denotes the square matrix relating the differential boundary displacements to the differential boundary forces. Then as in the linear case

$$\frac{d(\lambda P)}{dP} = \frac{B^t \bar{k} Bdx}{dP} \tag{2-16}$$

in which

$\frac{d(\lambda P)}{dP} = \lambda$ = column matrix of constants in the case of proportional loading.

Equation (2-16) may be written as

$$\lambda = B^t \bar{k} B \dot{x} \qquad (2\text{-}17)$$

where the dot indicates the differentiation with respect to the applied load, P.

In the preceding formulation, the coefficients of the compatibility matrix, B, and therefore the coefficients of $B^t$, were treated as being independent of the applied loading. This is true if the concept of geometrical linearity inherent in small deflection theory is used, and in most structures it does not result in a significant loss of accuracy in the solutions. Equation (2-17) generates a set of simultaneous nonlinear ordinary differential equations which, when integrated, will yield the displacements of the structural system for any given proportional loading and initial conditions.

There are many numerical integration schemes available for solving systems of differential equations. The fourth order Runge-Kutta method of numerical integration was used here. This method is based on the Taylor series solution and is self starting, that is, does not require a past history of function values. The approximation to functions being integrated is obtained through several evaluations of the expression for the first derivatives. Although lower order Runge-Kutta schemes are well known, the fourth order process is probably the one best known and most widely used. It is presented in detail in several publications on numerical integration (Ince, 1926; McCracken, 1961) and therefore will not be developed here.

Stress-Strain Relationship

In nonlinear systems the k matrix relating the element boundary displacements to the element boundary forces is not a constant, and the nonlinear stress-strain relationship must be represented by a form which can be easily applied to computer programming. This form should be reasonably simple, yet one which will provide accurate analytical results. Richard's formula for stress-strain (Richard, 1961), a three parameter expression which represents the stress-strain relationship, is used here. It is written in the following form,

$$\sigma = \frac{E\varepsilon}{\left[1 + \left|\frac{E\varepsilon}{\sigma_o}\right|^n\right]^{1/n}} \tag{2-18}$$

where $\sigma$ denotes the stress, $\varepsilon$ is the strain, $E$ describes the initial linear relationship between $\sigma$ and $\varepsilon$, $\sigma_o$ is the maximum stress, and n is the parameter defining the general nonlinear relationship between $\sigma$ and $\varepsilon$. This function is well adapted for computer programming, and as it contains only three parameters is a relatively simple expression to use. Another advantage of this formula is the fact that it is single valued.

Figure 1 is a nondimensional representation of the function in terms of stress, $\sigma$, and a normalized strain, $\varepsilon$; and shows the different shapes of this function for several values of the parameter n. By choosing an appropriate value for the parameter n, the function can be used to represent a wide range of stress-strain curves, and hence the accuracy of this function will depend only on the selection of n.

For a uniaxial state of stress the stress-strain relationship is given by

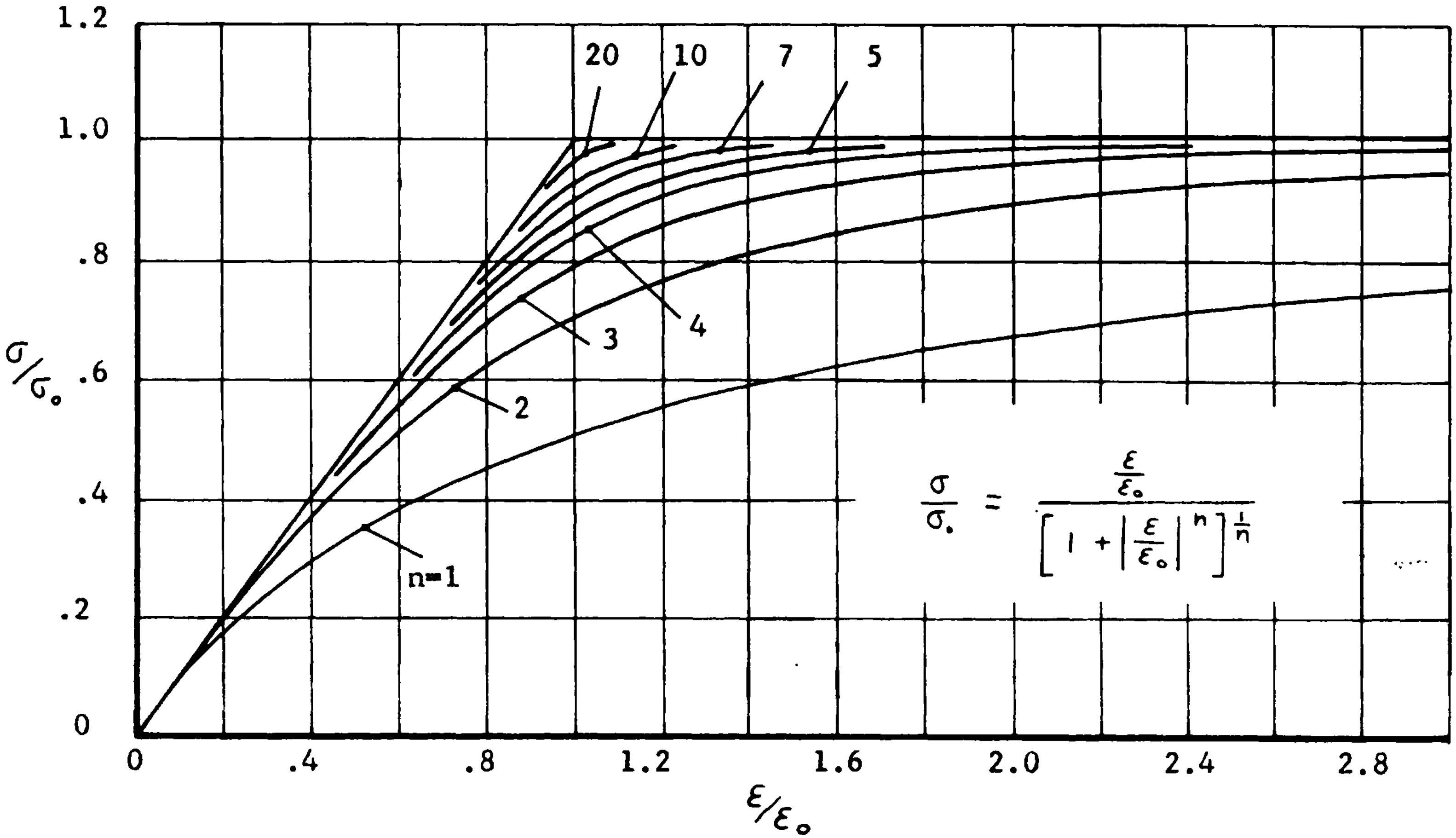



FIGURE 1 NONDIMENSIONAL STRESS - STRAIN RELATIONSHIPS

$$\sigma_e = E\,\varepsilon \tag{2-19}$$

where $\sigma_e$ is the elastic stress. However for the biaxial state of stress an expression similar to the von Mises' yield criterion

$$\sigma_e = \left[\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\,\tau_{xy}^2\right]^{1/2} \tag{2-20}$$

is used as the effective elastic stress. Therefore the stress-strain relationship for plane stress is

$$\sigma = \frac{E\,\varepsilon}{\left[1 + \left|\frac{\sigma_e}{\sigma_o}\right|^n\right]^{1/n}} \tag{2-21}$$

The tangent modulus can be expressed in the following form,

$$E_t = \frac{d\sigma}{d\varepsilon} = \frac{E}{\left[1 + \left|\frac{\sigma_e}{\sigma_o}\right|^n\right]^{\frac{n+1}{n}}} \tag{2-22}$$

where $E_t$ denotes the tangent modulus, $d\sigma$ is a differential value of stress, and $d\varepsilon$ is a differential value of strain. Figure 2 is a nondimensional representation of the tangent modulus in terms of a normalized strain, $\dot{\varepsilon}$ .

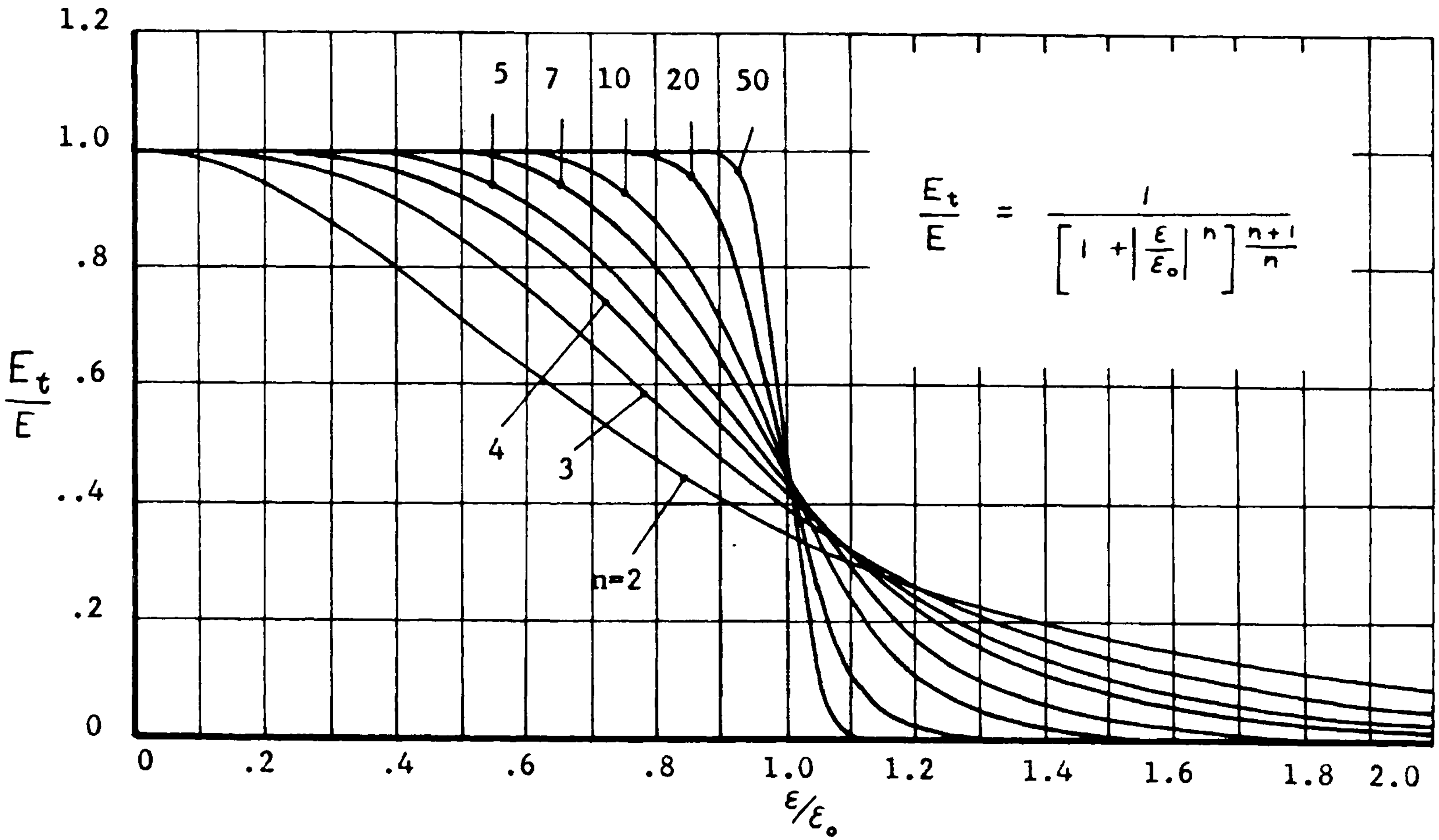



FIGURE 2 NONDIMENSIONAL TANGENT MODULUS

The expression for the secant modulus is

$$E_s = \frac{E}{\left[ 1 + \left| \frac{\sigma_e}{\sigma_0} \right|^n \right]^{1/n}} \tag{2-23}$$

where $E_s$ denotes the secant modulus. This function is also dependent on the value n and has the property of being single valued. A nondimensional representation of the secant modulus for several values of n is shown in Figure 3. This figure indicates that when the normalized strain, $\varepsilon$, reaches a value of approximately 2.0, the secant modulus is almost independent of the value of the parameter n and decreases at approximately the same rate for all n's. These functions are shown in Figure 4 for a given value of strain, $\varepsilon$, on a typical nonlinear stress-strain curve.

The parameter n which was used in the foregoing functions is determined from the properties of the stress-strain curve for any given material. In arriving at the value of n, an upper bound must be selected such that the stress-strain curve becomes asymptotic to this value for a given strain. By using Young's modulus a value of strain $\varepsilon_0$ associated with $\sigma_0$ can be found, and the stress $\sigma_1$ can then be determined from the nonlinear stress-strain curve of the given material. Now with the ratio of $\sigma_1/\sigma_0$

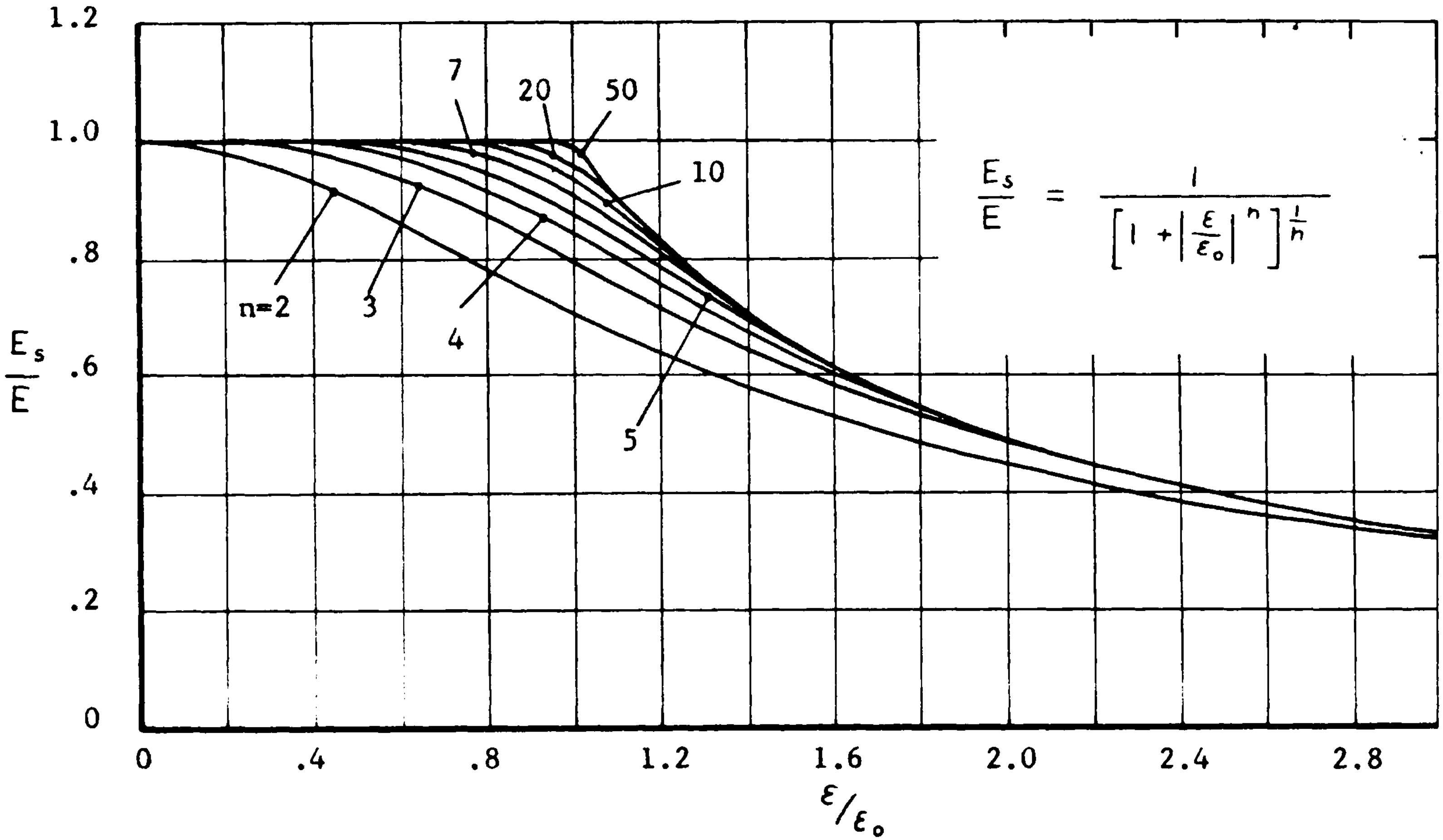



FIGURE 3 NONDIMENSIONAL SECANT MODULUS

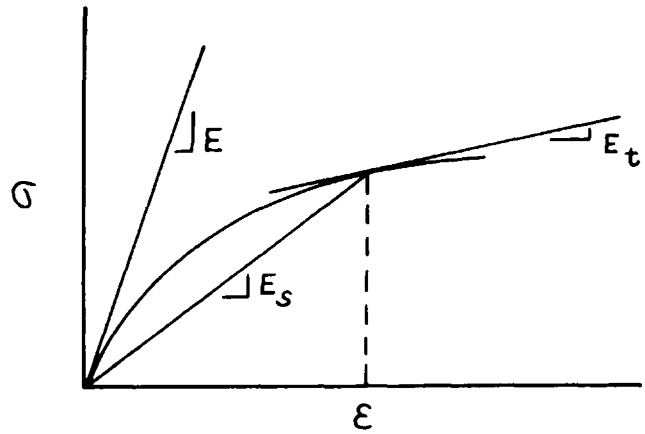



FIGURE 4 STRESS - STRAIN RELATIONSHIPS

the proper value of n can be determined from Figure 5. As this figure indicates, n approaches infinity as the stress-strain curve approaches the bilinear case.

<u>Element Stiffness</u>

In order for the finite element idealization to represent more nearly the continuum which it replaces, it is assumed that each element deforms similarly to the region it replaces. This objective may be accomplished by assuming a linear displacement field, that is, straight lines in the element remain straight in their displaced positions. With this assumption the boundary compatibility between elements is assured.

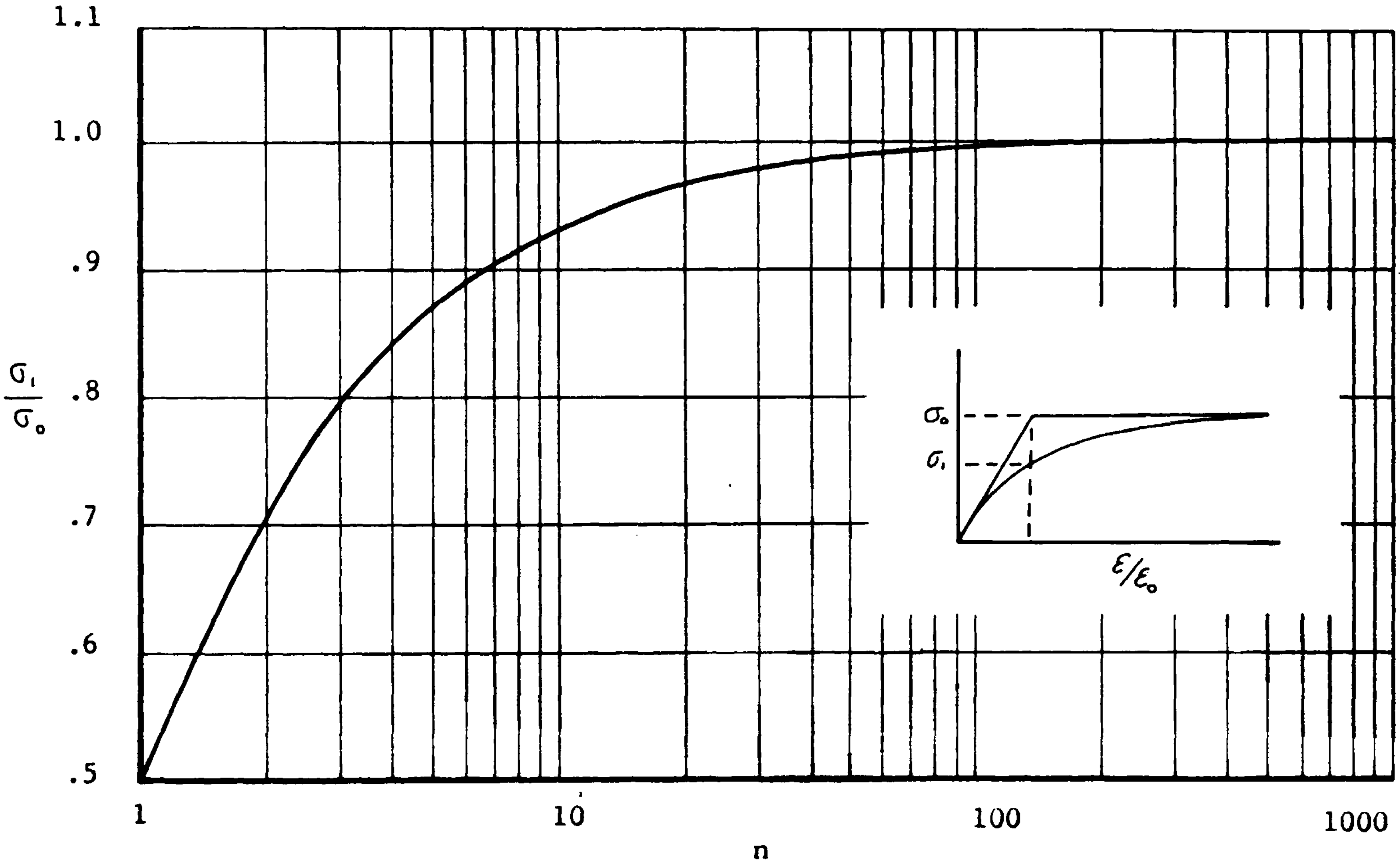



FIGURE 5 NONLINEAR PARAMETER n

The constant stress element procedure does not insure equilibrium of stresses along the element boundaries, which means that adjacent elements, in general, will not have the same stresses. Equilibrium will therefore be achieved only by applying artificial forces on the boundaries of the elements. These artificial loads are local and self equilibrating, and thus generally will have local effects on the behavior of the system.

Although the triangular element stiffness procedure is widely known (Turner, 1956; Wilson, 1963), it will be presented here for continuity. In its development a linear displacement field is assumed and the stresses which act on the edges of each element are therefore constant, and are replaced by stress resultants which act at the node points of the elements. Based on these assumptions it is possible to formulate the stiffness of a triangular element by use of nodal displacements and nodal forces.

Expressing the linear displacement field in terms of U(x, y) and V(x, y) it follows that

$$U(x, y) = U_i + C_1 x + C_2 y \tag{2-24}$$

$$V(x, y) = V_i + C_3 x + C_4 y \tag{2-25}$$

where i is some point in the structure and $C_1$, $C_2$, $C_3$, and $C_4$ are constants. Using the notation shown in Figure 6, the constants can be developed in terms of the nodal displacements and the geometry of the element

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} y_j-y_k & 0 & y_k-y_i & 0 & y_i-y_j & 0 \\ x_k-x_j & 0 & x_i-x_k & 0 & x_j-x_i & 0 \\ 0 & y_j-y_k & 0 & y_k-y_i & 0 & y_i-y_j \\ 0 & x_k-x_j & 0 & x_i-x_k & 0 & x_j-x_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} \tag{2-26}$$

where $D = (x_j-x_i)(y_k-y_i) - (x_k-x_i)(y_j-y_i)$.

Constant strains within the elements can be obtained from the assumed displacement field using the definitions of the strains

$$\varepsilon_x = \frac{\partial u}{\partial x} = C_1 \tag{2-27}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = C_4 \tag{2-28}$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_2 + C_3 \tag{2-29}$$

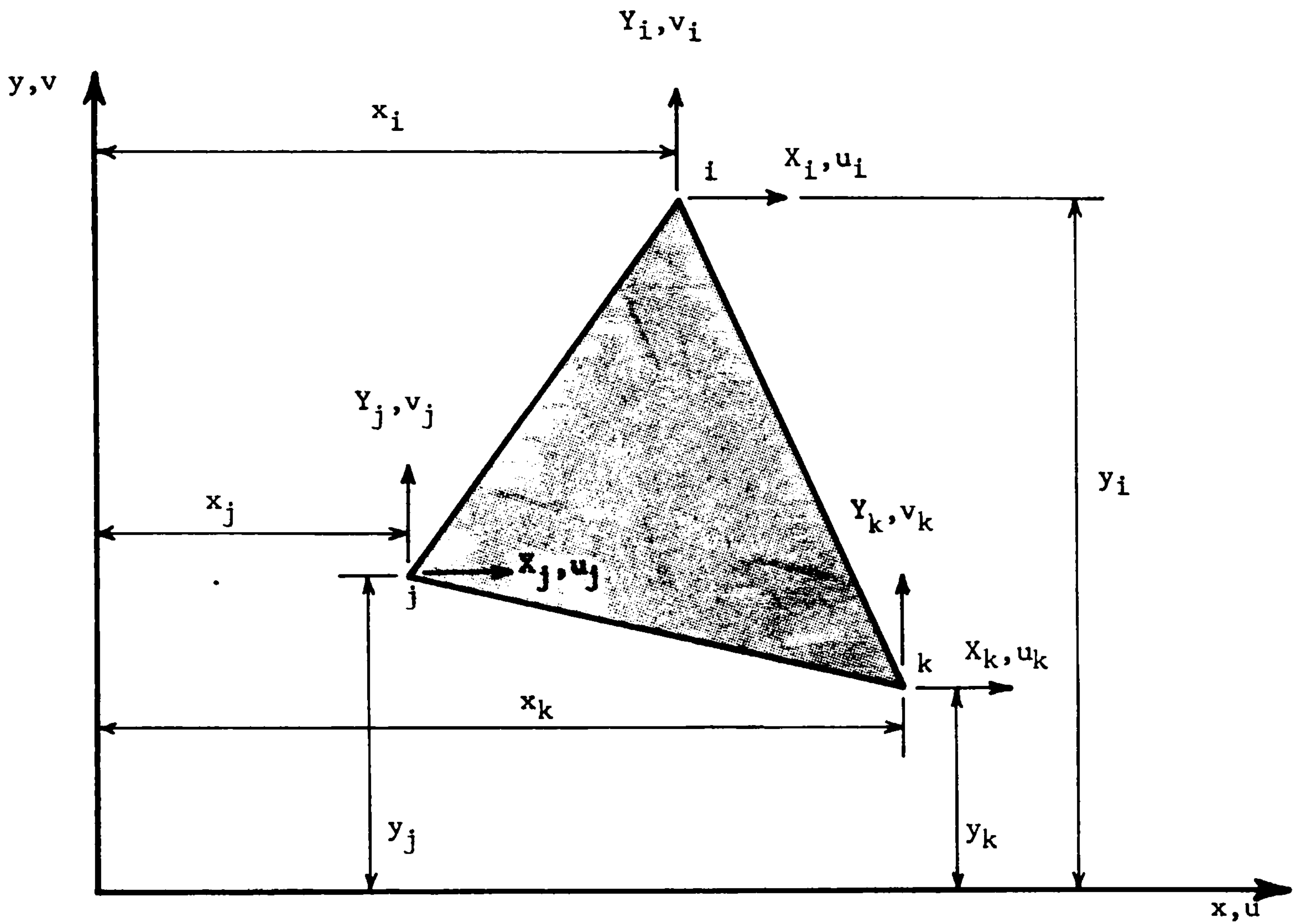



FIGURE 6 TRIANGULAR FINITE ELEMENT NOTATION

Thus the element strains, $\mathcal{E}$ , are expressed in terms of the nodal displacements, u, in the following matrix equation

(2-30)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} y_j - y_k & 0 & y_k - y_i & 0 & y_i - y_j & 0 \\ 0 & x_k - x_j & 0 & x_i - x_k & 0 & x_j - x_i \\ x_k - x_j & y_j - y_k & x_i - x_k & y_k - y_i & x_j - x_i & y_i - y_j \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix}$$

or in the matrix notation

$$\{\varepsilon\} = [b]\{u\} \qquad (2\text{-}31)$$

The stress-strain relationship for an isotropic linear elastic material in the state of plane stress is shown in matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (2\text{-}32)$$

where E is the initial linear relationship between stress and strain and $\mu$ is Poisson's ratio. For nonlinear materials the constant elastic modulus E, is replaced by the secant modulus $E_s$, as formulated in Equation (2-23).

It is known (Shanley, 1957) that for nonlinear materials, Poisson's ratio will vary from approximately .3 to a maximum of .5 as the material reaches the plastic range. In order to account for this, $\mu$ is expressed as

$$\mu_s = .5 - (.5-.3)\,\frac{E_s}{E} \qquad (2\text{-}33)$$

where $\mu_s$ represents Poisson's ratio as a function of the secant modulus, $E_s$. Replacing $E_s$ with the tangent modulus, $E_t$, Poisson's ratio is written as

$$\mu_t = .5 - (.5-.3)\,\frac{E_t}{E} \qquad (2\text{-}34)$$

where $\mu_t$ denotes Poisson's ratio as a function of the tangent modulus. Plots of the functions $\mu_s$ and $\mu_t$ are shown in Figures 7 and 8 respectfully for several values of the parameter n. These expressions for Poisson's ratio are, of course, only approximate, and their validity is left to further study.

Using the previous nonlinear expressions the stress-strain relationship can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E_s}{1-\mu_s^2} \begin{bmatrix} 1 & \mu_s & 0 \\ \mu_s & 1 & 0 \\ 0 & 0 & \frac{1-\mu_s}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (2\text{-}35)$$

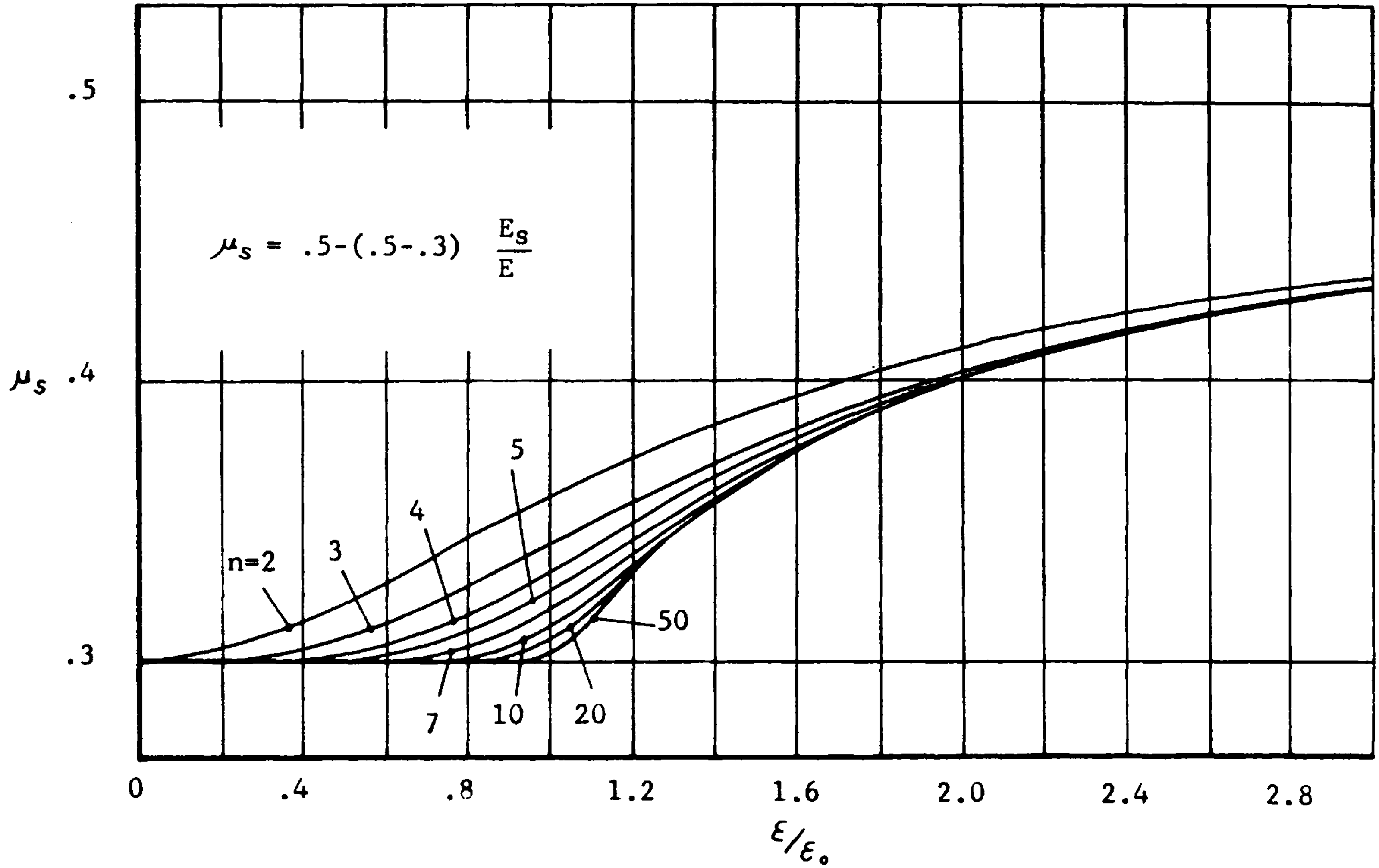



FIGURE 7 SECANT POISSON'S RATIO

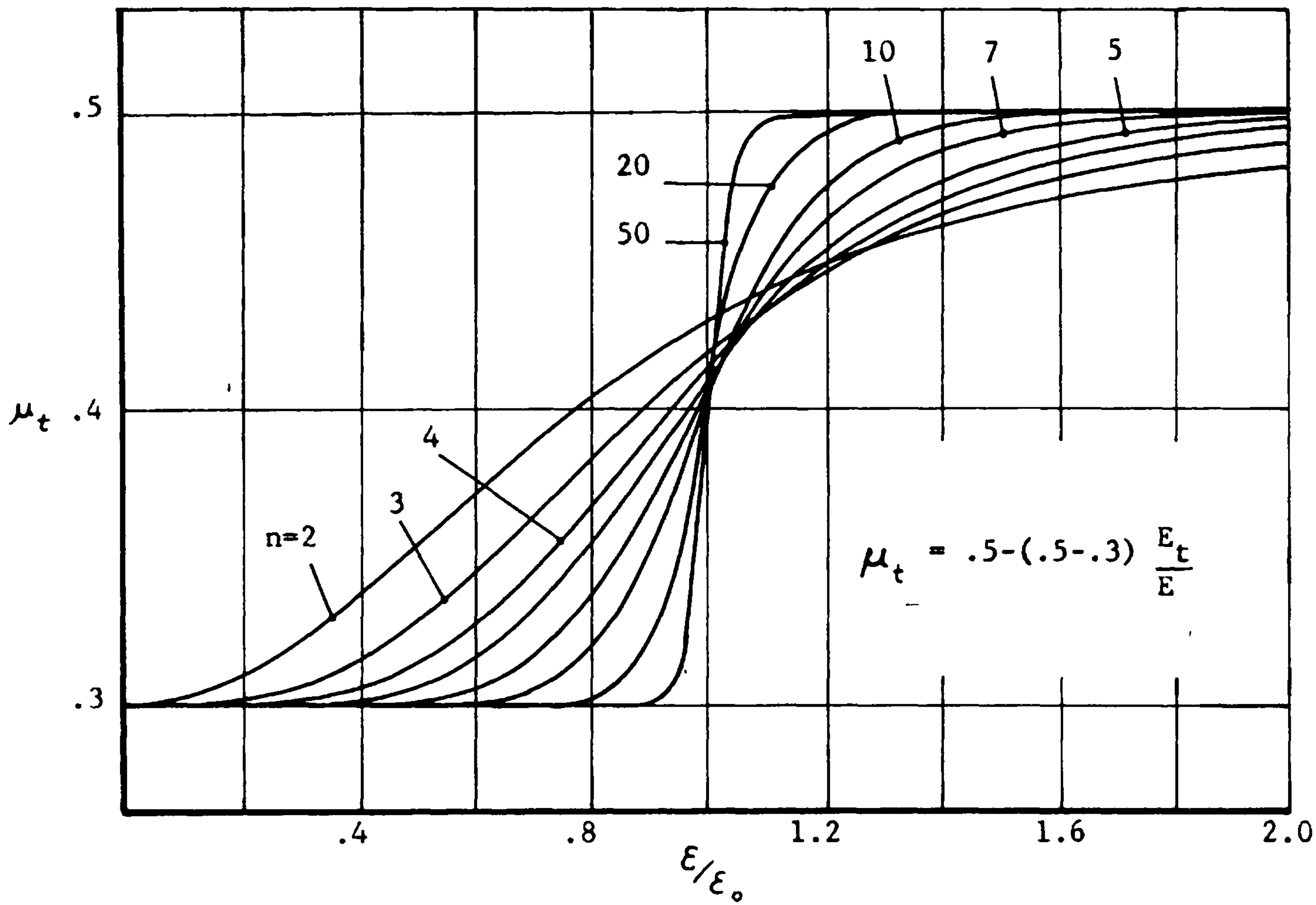



FIGURE 8 TANGENT POISSON'S RATIO

or in matrix notation

$$\{\sigma\} = [k_s]\{\varepsilon\} \qquad (2\text{-}36)$$

The uniform stresses along the elements are now replaced by the stress resultants acting at the node points. In considering an element of unit thickness, Figure 9 shows a set of statically equivalent nodal forces for each of the components of stress, as given by Wilson (1963). Formulating the nodal forces in terms of these stresses gives

$$\begin{Bmatrix} X_i \\ Y_i \\ X_j \\ Y_j \\ X_k \\ Y_k \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} y_j-y_k & 0 & x_k-x_j \\ 0 & x_k-x_j & y_j-y_k \\ y_k-y_i & 0 & x_i-x_k \\ 0 & x_i-x_k & y_k-y_i \\ y_i-y_j & 0 & x_j-x_i \\ 0 & x_j-x_i & y_i-y_j \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (2\text{-}37)$$

or in matrix notation

$$\{F\} = [c]\{\sigma\} \qquad (2\text{-}38)$$

The element stiffness can then be expressed in terms of the nodal displacements by substituting Equation (2-31) into Equation (2-36) thus

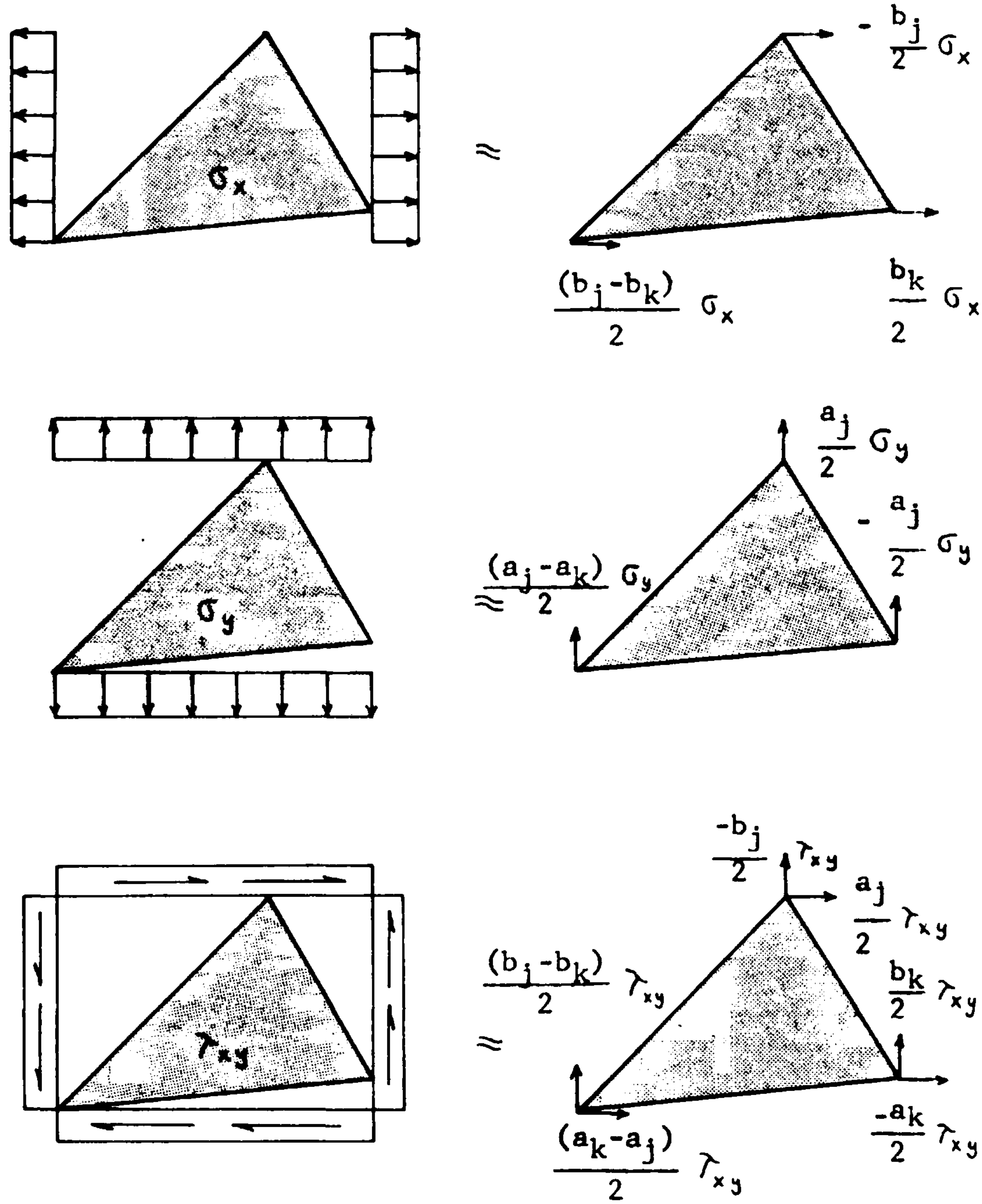



FIGURE 9 ELEMENT STRESS RESULTANTS

$$\{\sigma\} = [k_s][b]\{u\} \tag{2-39}$$

The substitution of Equation (2-39) into Equation (2-38) gives

$$\{F\} = [c][k_s][b]\{u\} \tag{2-40}$$

From matrices $[c]$ and $[b]$ it is seen that $[c]$ is the transpose of $[b]$ except for a scalar quantity. By letting G represent the product of the two scalar quantities, Equation (2-40) can be written as

$$\{F\} = G\,[b^t][k_s][b]\{u\} \tag{2-41}$$

or

$$\{F\} = [K]\{u\} \tag{2-42}$$

where $[K]$ is the 6x6 nonlinear stiffness matrix for the triangular element and is represented by

$$[K] = G\,[b^t][k_s][b] \tag{2-43}$$

Substituting the differential relationships $E_t$ and $\mu_t$ into Equation (2-35) for $E_s$ and $\mu_s$ yields the differential stress-strain relationship

$$\begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\tau_{xy} \end{Bmatrix} = \frac{E_t}{1-\mu_t^2} \begin{bmatrix} 1 & \mu_t & 0 \\ \mu_t & 1 & 0 \\ 0 & 0 & \frac{1-\mu_t}{2} \end{bmatrix} \begin{Bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\gamma_{xy} \end{Bmatrix} \tag{2-44}$$

or in matrix notation

$$\{d\sigma\} = [k_t]\{d\varepsilon\} \tag{2-45}$$

Now using the differential relationships of Equations (2-31) and (2-38), Equation (2-41) gives

$$\{dF\} = G[b^t][k_t][b]\{du\} \tag{2-46}$$

from which the differential nonlinear element stiffness, $[\bar{K}]$ is denoted by

$$[\bar{K}] = G[b^t][k_t][b] \tag{2-47}$$

It should be observed that the element stiffness matrix could have been derived by using energy considerations which lead to

$$[K] = \int [b^t][k_s][b]\, dV \tag{2-48}$$

where the integral is evaluated over the volume of the triangle. The purpose of using the stress resultant approach was to give the stiffness matrix a physical interpretation.

Computer Program

The computer program for the nonlinear finite element analysis studied here consisted of approximately 400 Fortran statements, and approximately 100 data cards were used for each example. It was written so that both the linear and nonlinear solutions could be obtained from the analysis, and a copy of the flow diagram may be found in Appendix A.

In order to reduce computer storage a reindexing procedure was used to minimize the storage requirement of the system stiffness matrix. This reindexing method was based on the fact that the system stiffness matrix is banded and that the band width can be minimized by carefully numbering the node points. For the examples presented here, this procedure reduced the normal storage requirement by approximately one half.

Essentially this computer program consists of a main program and six subprograms as shown below:

1) The subroutine which reindexes the node points

2) A matrix multiply subprogram

3) The program for calculating the element stiffness

4) The subroutine for solving the reindexed simultaneous equations

5) The program for calculating the element stresses from known displacements

6) The packing procedure which generates the system stiffness matrix from the element stiffnesses

The information required for each example was as follows:

1) The number of elements

2) The number of node points in the system

3) The number of loads

4) The location of the loads

5) The load at each node point

6) The number of supports

7) The points of support

8) The number of loading intervals

9) The band width of the system stiffness matrix

10) The location of the node points in reference to the coordinate system of the structure

11) The properties of the elements

Output from the program was displacements of the node points, strains for each finite element, and stresses within each element. All remaining parts of the program result from generalizations of programs for the well known displacement method and will not be discussed here.

# CHAPTER 3

# EXAMPLES

## Description

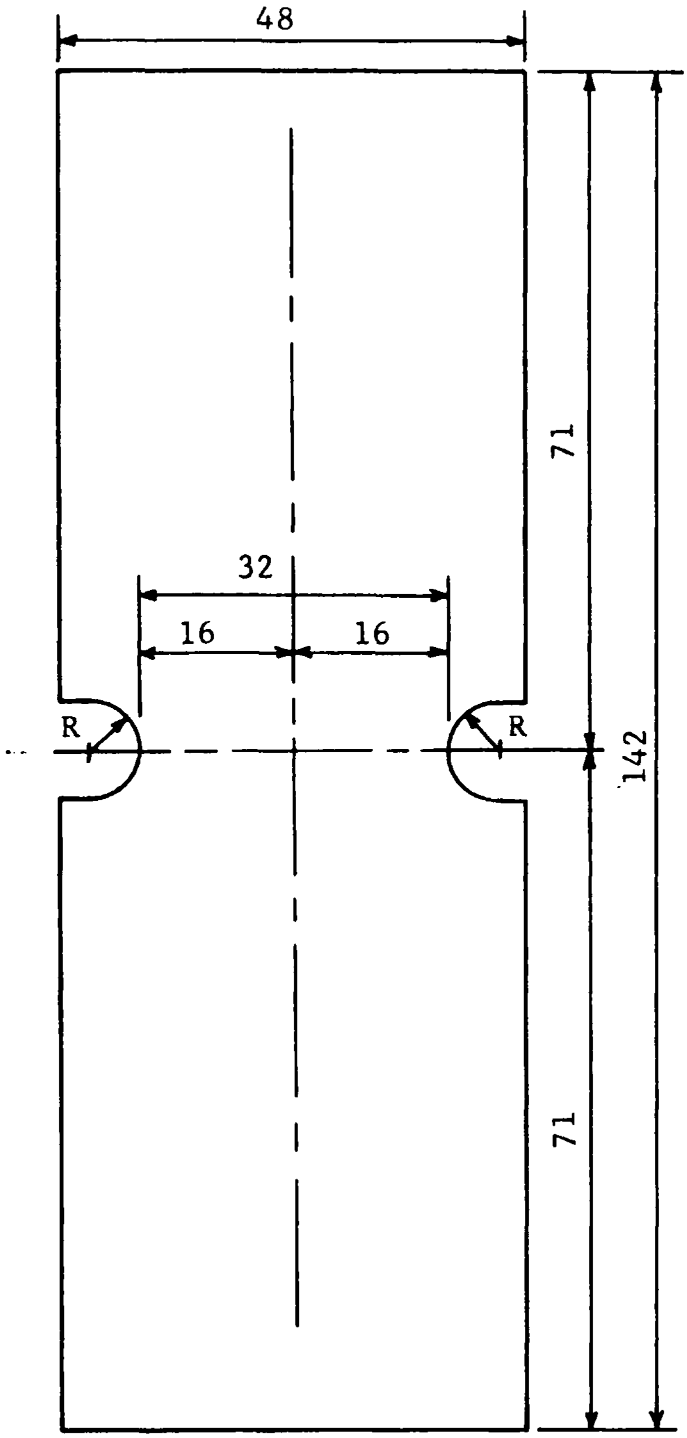
The examples selected to be analyzed here are stress concentrations caused by discontinuities in flat plates loaded in tension with in-plane axial forces in the plastic range, and were previously studied experimentally by Hardrath and Ohman (1951). In addition to providing experimental data in which to check the results of the analytical model, another reason for the selection of these examples was their adaptability to the limited computer storage available for this work. Many theoretical studies have been made of elastic stress concentrations (Timoshenko, 1934), but because of the analytical difficulties in the plastic range, theoretical work in this area has been limited.

In this paper the examples discussed consisted of three notched panels, herein designated by N, and three filleted panels, designated by F. These six panels were all 1/8 inch thick and made of 24S-T3 (2024) aluminum alloy. They were designated to have a nominal elastic stress concentration of 2, of 4, and of 6, which will herein be designated by N2, N4, N6, and F2, F4, and F6. The dimensions of each panel are shown in detail in Figures 10 and 11.

For calculating the nonlinear parameter n used in the stress-strain relationship, a yield stress of 56,000 psi was determined from the stress-strain curve of the aluminum alloy. This represents the stress level at which the stress-strain curve will become asymptotic, and using a value of 48,500 psi for $\sigma_1$, the nonlinear parameter n is found to equal 5, as shown by Figure 5.

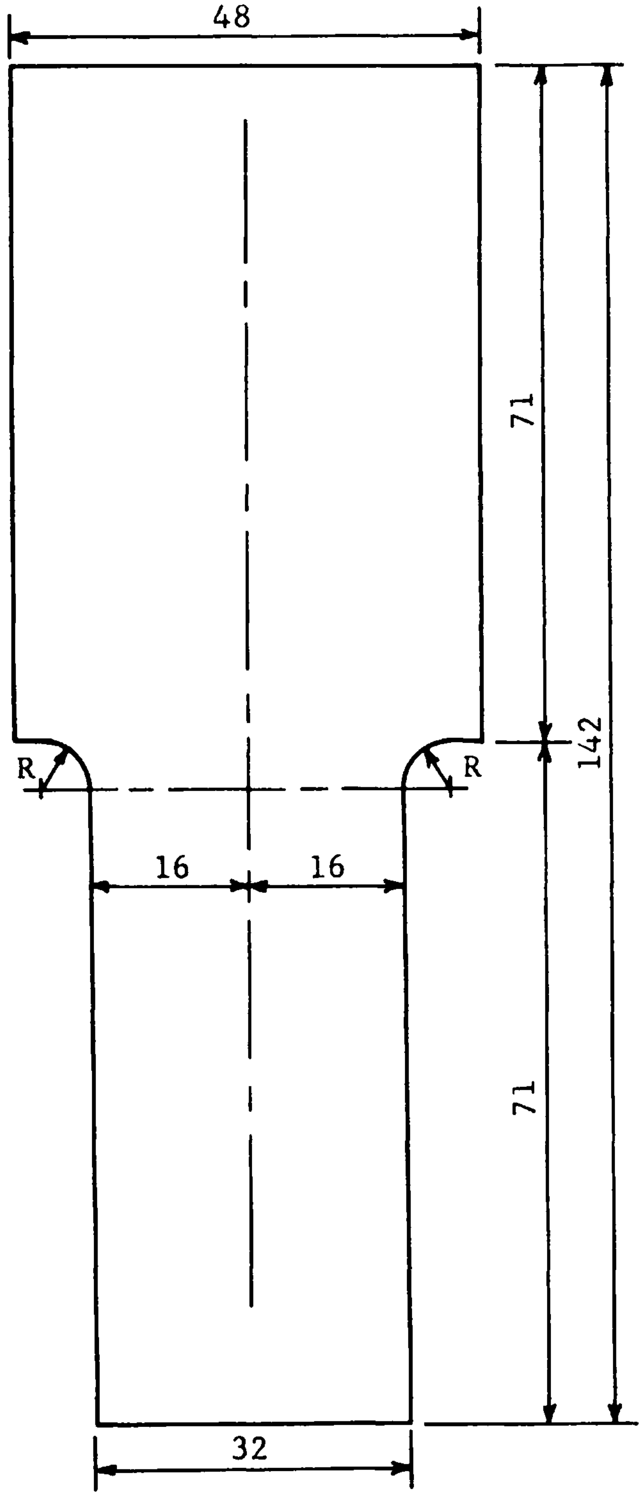
Development of Models

The development of models for the six examples studied herein was restricted due to limited storage capacity of the computer used. In order not to exceed the capacity of the computer the number of node points was limited to 40, with a total of two degrees of freedom at each node point. This allowed 80 degrees of freedom and a maximum of 60 finite elements. The knowledge gained from one example was used to improve the next model, with the




| Panel | R, in. |
|---|---|
| N2 | 6.750 |
| N4 | 1.187 |
| N6 | .484 |

FIGURE 10 DIMENSIONS OF NOTCHED PANELS




| Panel | R, in. |
|---|---|
| F2 | 3.703 |
| F4 | .656 |
| F6 | .266 |

FIGURE 11 DIMENSIONS OF FILLETED PANELS

only limitation being placed on the number of node points available.

In developing the stiffness matrix for the triangular elements it was assumed that a linear displacement field existed. The finite elements were then constant stress elements, and since the exact stress gradients were unknown, the best shape for the finite elements was assumed to be equilateral. It has been shown that the finite element method is a converging process depending on the size of the elements (Wilson, 1963). Therefore, in regions of steep stress gradients the discrete elements should be made as small as possible.

Because of the symmetry of the panels, the models developed for this computer program represented only a quarter of the notched panels and one half of the filleted panels. The models were loaded at node points and roller supports were placed at node points along the lines of symmetry of the panels. These rollers provided support normal to the edge of the panels and thus the straight lines of the boundary remained straight after nodal displacements.

The first example to be analyzed was the N2 panel, and the curved boundary of this panel was represented by a series of straight lines. In order to keep the boundary elements almost equilateral and not exceed the 40 node points, the elements in the regions of the steep stress gradients had to be made larger than desired. It is also noted that the transition from the small elements to larger elements caused several elements to be isosceles in shape.

Analyzed next was the N4 panel. It was found that since the curved boundary was much shorter than that of the N2 panel the boundary elements could be made smaller than before. This fact was very helpful in trying to make the elements in the regions of the steep stress gradients as small as possible. The problem of some of the elements being isosceles in shape was not alleviated however, since there were only a limited number of node points available.

A new problem was presented when the F2 plate was considered. Since this model represented one half of the actual panel the net section of the model extended 12 inches past the point where the net section was tangent to the fillet radius. The roller supports were previously placed along the boundary at the net section and the transition from the small elements to larger elements was not necessary at this point. With the extension of the net section for the filleted panels this transition became necessary, and required that more node points be used. This in turn caused all of the elements in the region of steep stress gradients to be larger than desired. The small radius used on the F4 plate alleviated somewhat the problem presented by the shape of the filleted panels, but the limited number of node points still affected the idealization of this filleted panel.

Last to be analyzed were the N6 and F6 panels. The discrete elements were made as small as possible as a result of the earlier examples and the fact that an extremely steep stress gradient would exist in the neighborhood of the maximum stress. In order to get some additional node points, elements were made larger on part of the boundary. Because the radius used on these panels was very small the elements on the curved boundary were smaller than before. The requirement that the elements be equilateral was relaxed a short distance from the highly stressed region to allow for the transition from small elements to larger ones. (See Appendix B for the dimensions and idealizations used in each of these examples.)

## Results

In general, the stresses from the finite element analysis were not in complete agreement with the actual stresses. One reason for this was that the finite elements were constant stress elements and each element had self equilibrating boundary stresses acting on it in order to maintain compatibility. Schemes have been developed for calculating the stresses internal to the system such as averaging the stress of all the elements which meet at one node point to obtain that node point stress. This procedure gives good results for interior points but less satisfactory results along the boundary. Another method has been developed for application to the boundary which is a "weighing procedure", (Wilson, 1963), and with its usage node stresses on the boundary can be found with good accuracy.

For the examples analyzed here it has been shown by photo-elastic procedures that the maximum stress in the linear range occurs at a point on the curved boundary about 10 degrees from the point where the radius is tangent to the net section. If the elements which meet at a node point in this region of the plate were to be averaged, the calculated maximum stress concentration of the system would be reduced somewhat. It therefore seems that a good assumption for the maximum stress in the system would be to assume that it is equal to the stress of the first element on the curved boundary near the net section. If this element is small in the region where the stress gradient is steep, it is apparent that the value obtained would contain only a small amount of error. By using the above concepts and defining the stress concentration factor as the maximum stress divided by the net section stress, the results shown in Figures 12 and 13 are obtained.

From values established for the N2 plate in the linear range, it is shown that ideally the elements should have been smaller near the point of maximum stress. As the structure became nonlinear and the maximum stress in the critical element approached the yield value the analytical stress concentration factor was in good agreement with the experimental results.

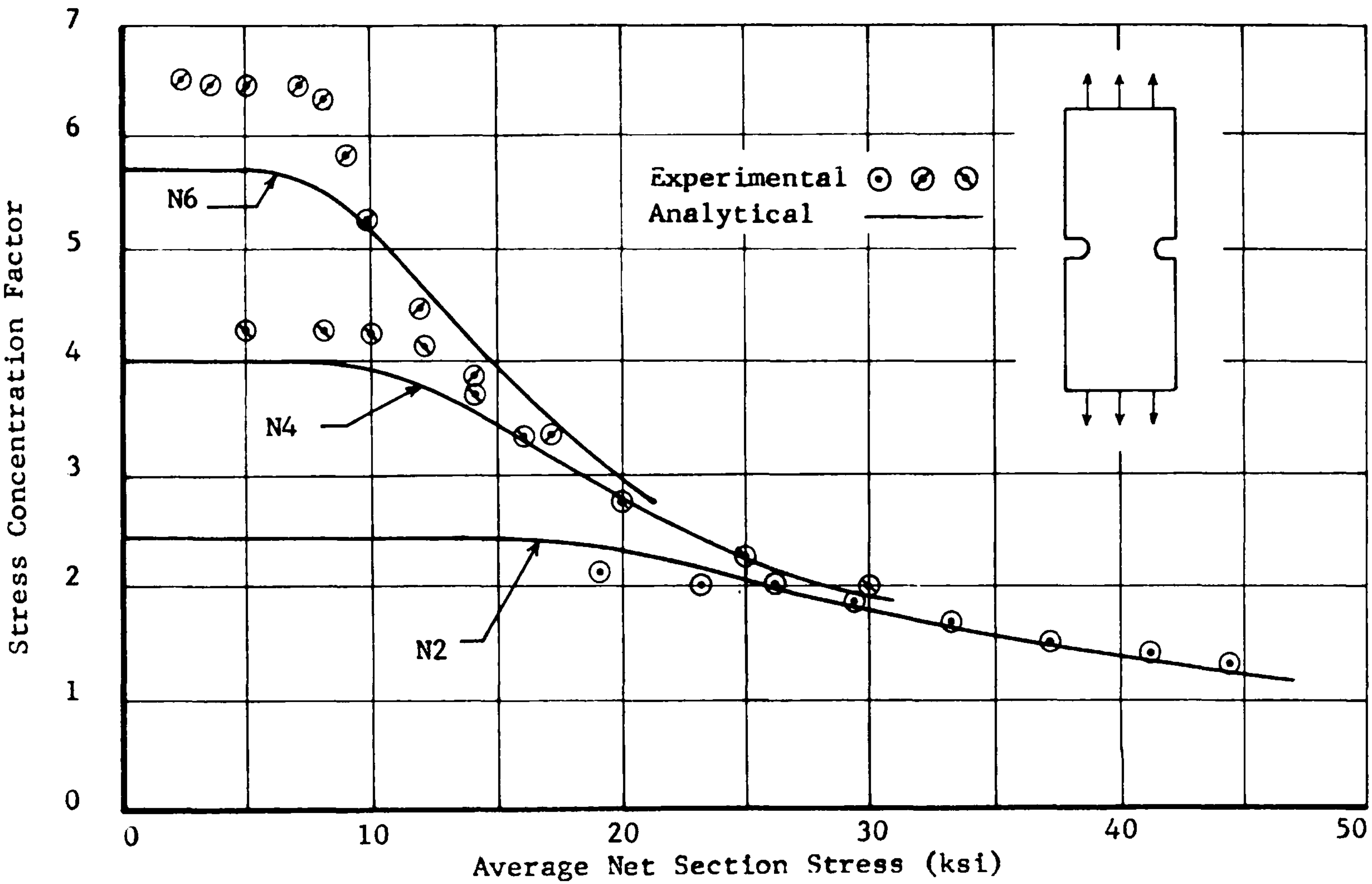



FIGURE 12 STRESS CONCENTRATION FACTORS (N PANELS)

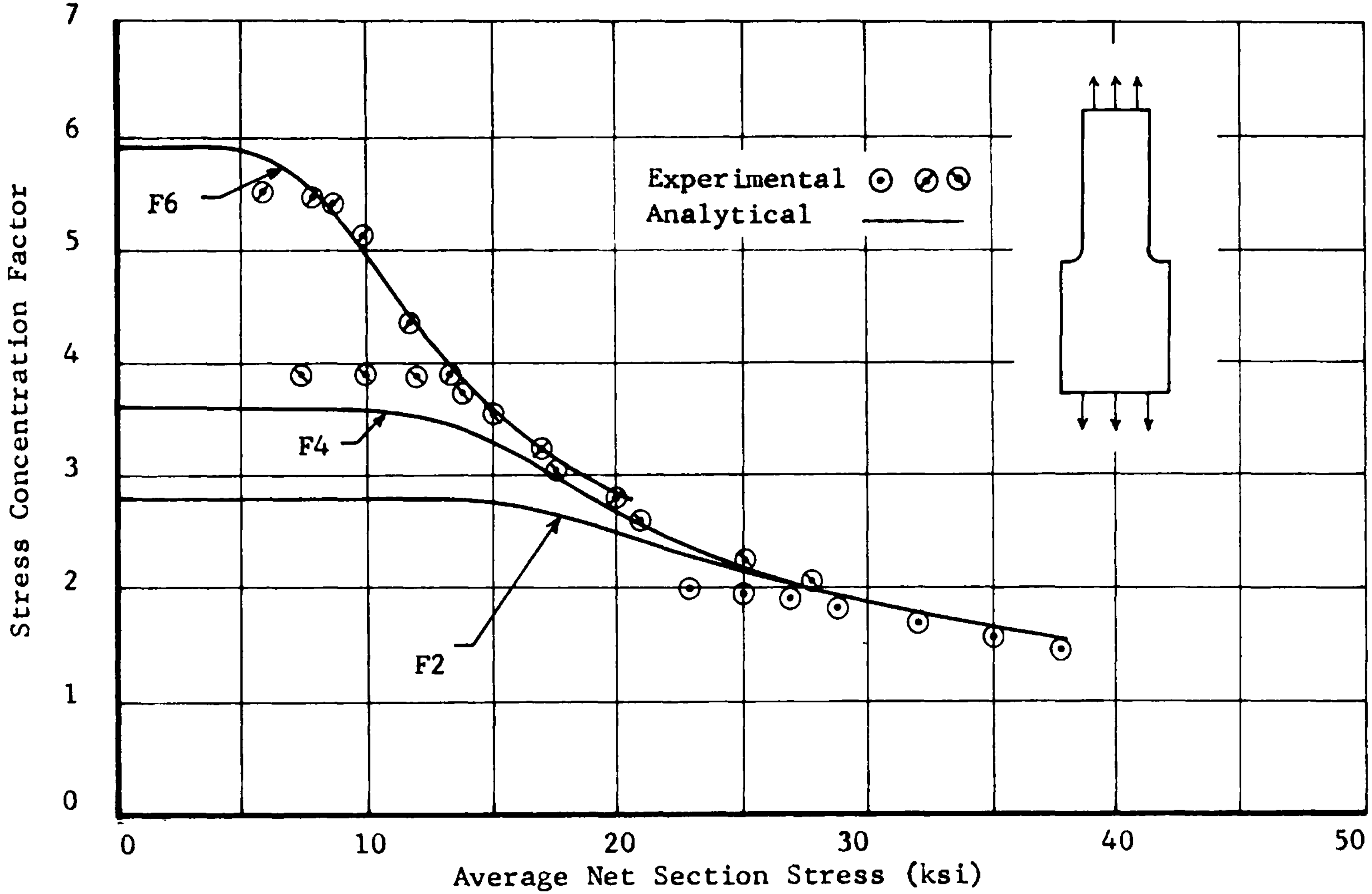



FIGURE 13 STRESS CONCENTRATION FACTORS (F PANELS)

Examination of the results of the N4 plate showed that there was good comparison between the experimental and analytical results over the complete range of loading. As mentioned before, if the linear results were required to be more precise, it would have been necessary for the finite elements in the region of the steep stress gradients to be made smaller until the exactness required was obtained.

The elements of the model for the N6 plate appeared at first to be small enough to give good results in the linear range. However, since the stress gradient was extremely steep at the boundary, it was apparent that the elements used were not small enough to obtain the maximum linear stress. As the structure became nonlinear the results from the analysis were in better agreement with the experimental results at the boundary.

Both the F2 and F4 plates gave similar results to those of the N2 and N4 plates when compared to the experimental results. Generally the same problems existed in the two filleted plates as were found in the two notched plates, and if more computer storage had been available the results could have been improved.

Last to be analyzed was the F6 plate, where the discrete elements were very small and almost equilateral on the curved boundary. The results obtained in the linear range of loading were found to be greater than the experimental results, however the magnitude of the error was small. As was shown in the previous examples, when the structure became nonlinear the results obtained from the finite element analysis were in good agreement with the experimental results.

Figure 14 shows the stress distribution across the net section as it was plotted for the N4 panel in three dimensions. The experimental results are denoted by the dotted line and the solid line represents the analytical results. Net section stresses were calculated by an extrapolation procedure as explained in Appendix C. Equal incremental loads were applied so that the net section stress increased in 5 ksi intervals up to a maximum of 30 ksi. As can be seen there was close agreement between the analytical and experimental results.

A three dimensional plot of the strain distribution across the net section of the N4 panel is illustrated in Figure 15. As before, the dotted line represents the

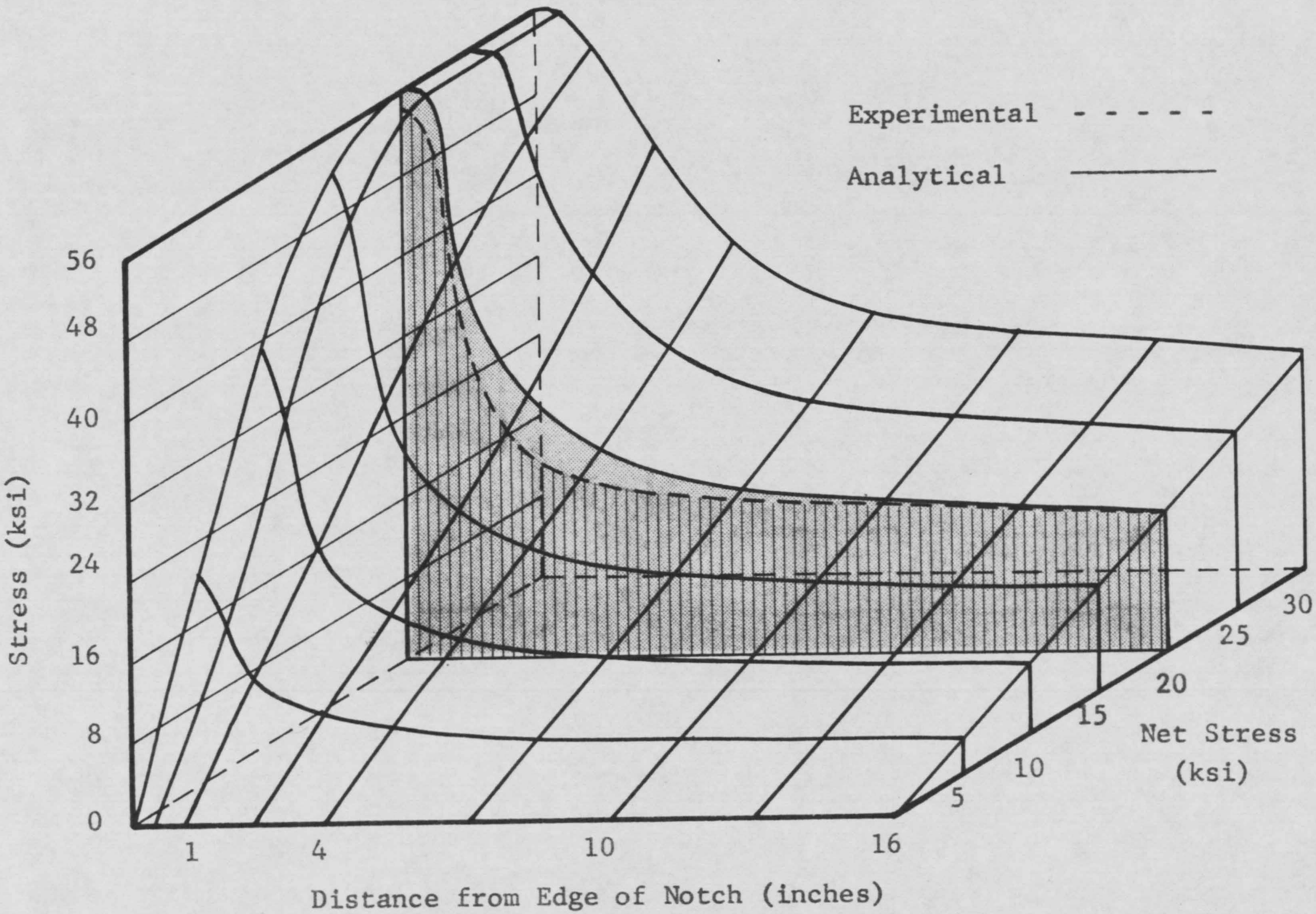



FIGURE 14 STRESS DISTRIBUTION FOR N4 PANEL

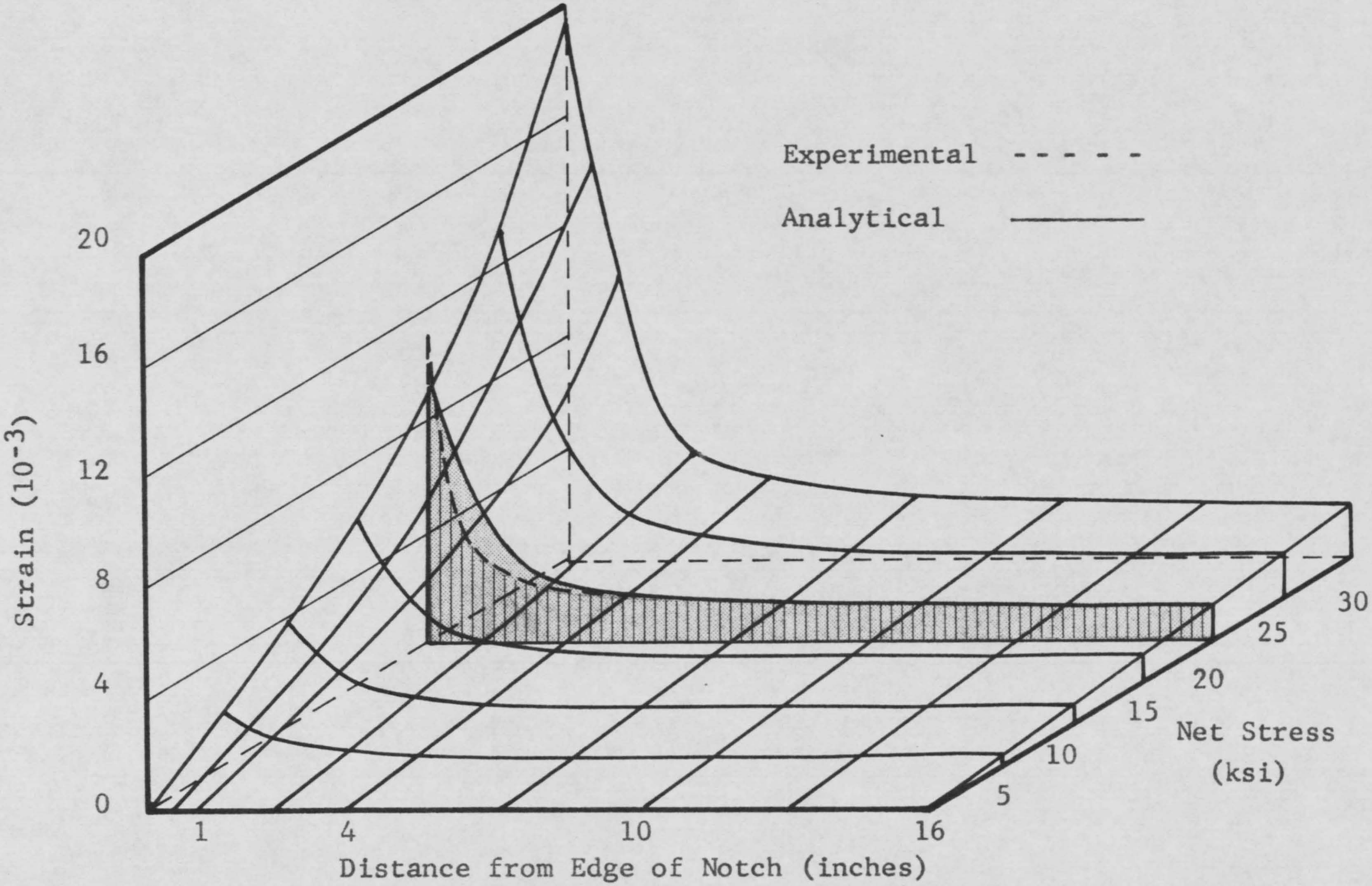



FIGURE 15 STRAIN DISTRIBUTION FOR N4 PANEL

experimental results while the solid line indicates the analytical results. Using the extrapolation procedure, the strains were plotted for the same increments of loading as were used in the stress distribution. Results obtained by this procedure were in accord with the experimental results except for the small error in the region of the steep stress gradient. Again this error appeared to be caused by the use of constant stress elements which were too large for the stress gradient which existed in their locations.

## SUMMARY AND CONCLUSION

A model for nonlinear stress analysis has been presented here using triangular plate elements for representation of the continuum. Since a numerical approach was used it was impossible to discuss this model in mathematically rigorous terms. Therefore, the accuracy of this model can only be determined by comparing its results with known experimental or exact results. The limitations of this discussion must be realized since it is based on a restricted amount of data.

Because the model presented was based on a differential point of view, the resulting set of equations was equivalent to a set of nonlinear first order ordinary differential equations. Material nonlinearities were introduced by the nonlinear stress-strain relationship of the individual structural elements. Although only material nonlinearities were considered here, this model could be extended to include nonlinearities resulting from changes in the structure geometry.

Solutions of the nonlinear differential equations were obtained by using the fourth order Runge-Kutta integration scheme. Other methods for solving nonlinear differential equations are available, and their use with this model should be employed and additional studies made.

It has been shown through the examples presented that the model yields results which are within an engineering tolerance. Also demonstrated was the fact that it is a relatively simple model, and is based on a conventional method presently being used for linear analysis. The computation time was not studied here and no comparison to other finite element models was made, however each example was analyzed in approximately 15 minutes on an IBM 7072-1401 digital computer.

Based on the results presented above, this nonlinear finite element model is feasible for the analysis of nonlinear systems and should be studied further. One area of study into which this model could be extended is that of stress reversal in nonlinear systems. Other finite element idealizations to be used with this model or a combination of several finite elements could also be studied. A method for analyzing anisotropic materials is also a possible area for additional study. Thus extensive research in nonlinear analysis is just beginning.

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## APPENDIX A

## FLOW DIAGRAM - NONLINEAR FINITE ELEMENT ANALYSIS

### Subroutines

RBIND - Reindex stiffness matrix elements into compact form to save storage.

MATMU - Matrix multiply.

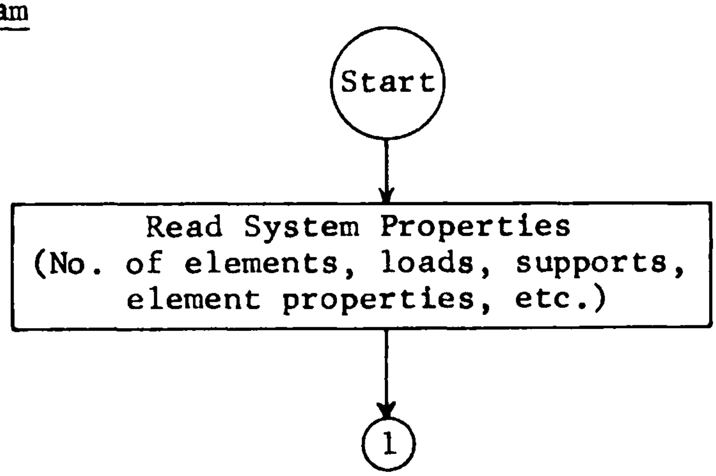
ELEMS - Form element stiffness matrix.

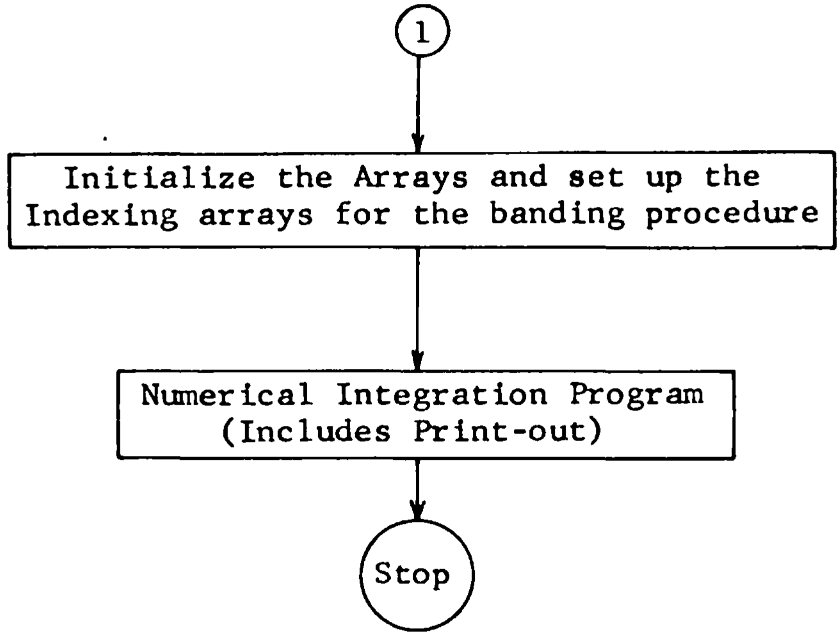
EQSOL - Solution of the reindexed band matrix by Gauss-Jordan lower triangulation followed by back substitution.

STRES - Using element corner displacements, solve for element stresses.

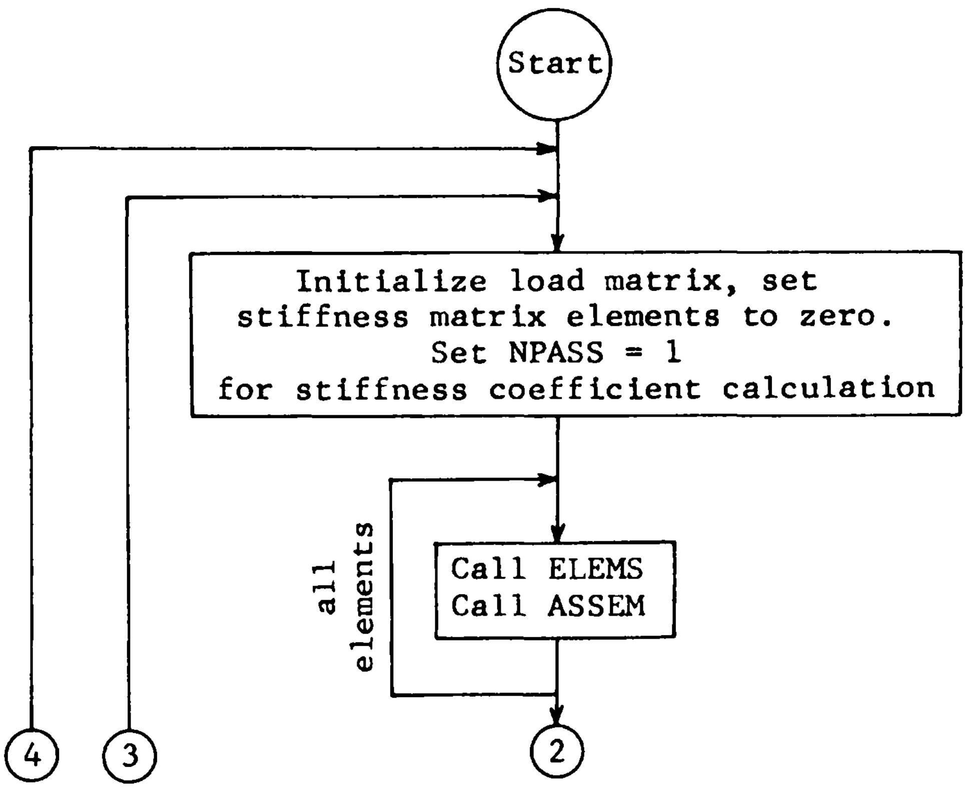
ASSEM - Assemble the system stiffness matrix into the reindexed band matrix.

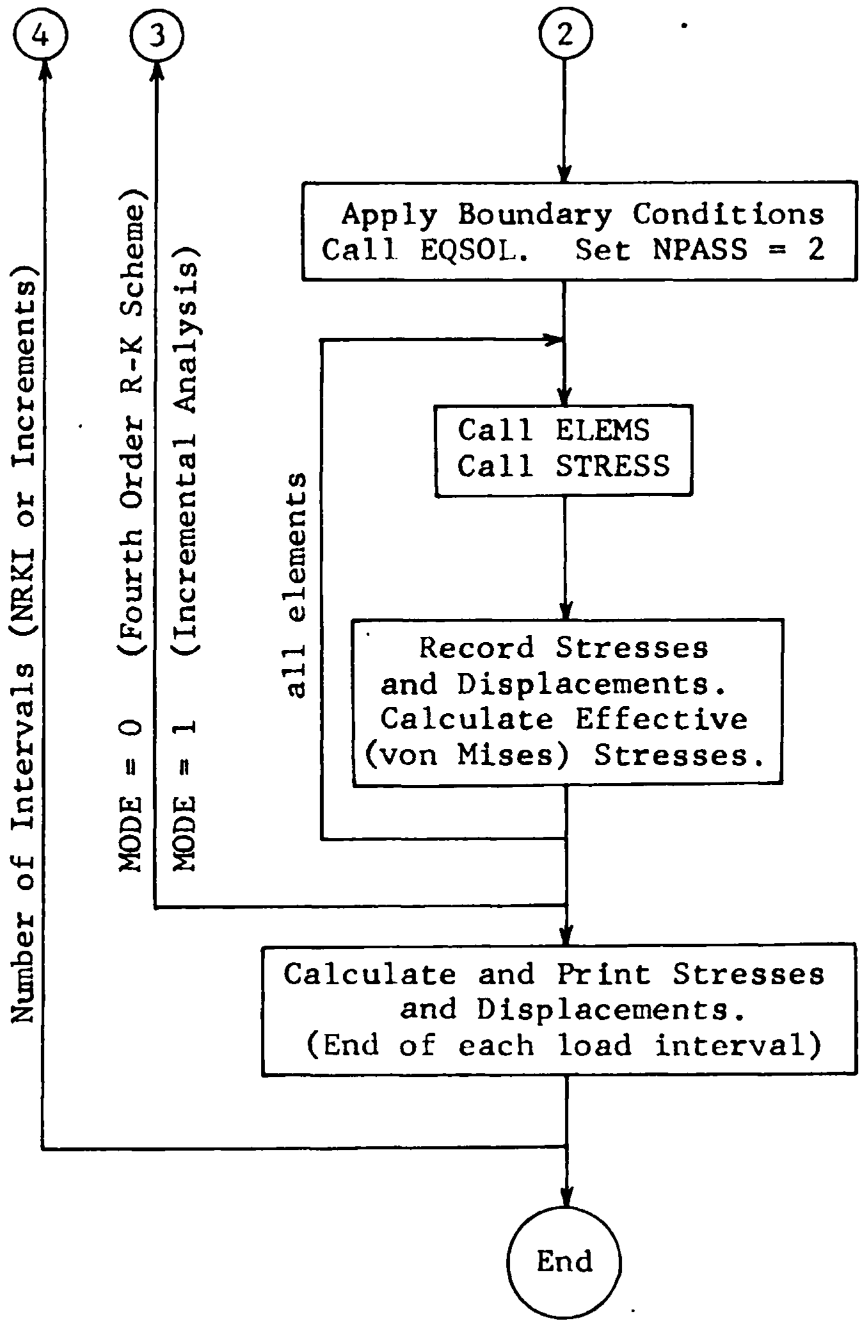
### Main Program






Numerical Integration Program




4
3
2
Apply Boundary Conditions
Call EQSOL. Set NPASS = 2
Call ELEMS
Call STRESS
Record Stresses
and Displacements.
Calculate Effective
(von Mises) Stresses.
all elements
MODE = 0 (Fourth Order R-K Scheme)
MODE = 1 (Incremental Analysis)
Number of Intervals (NRKI or Increments)
Calculate and Print Stresses
and Displacements.
(End of each load interval)
End

APPENDIX B

IDEALIZATIONS

Idealizations of the six examples studied are shown in the following Figures 16 through 21. These idealizations are not considered to be the ultimate representation for the different examples as they were developed without previous experience. However, their application in this study was satisfactory for the purposes desired.

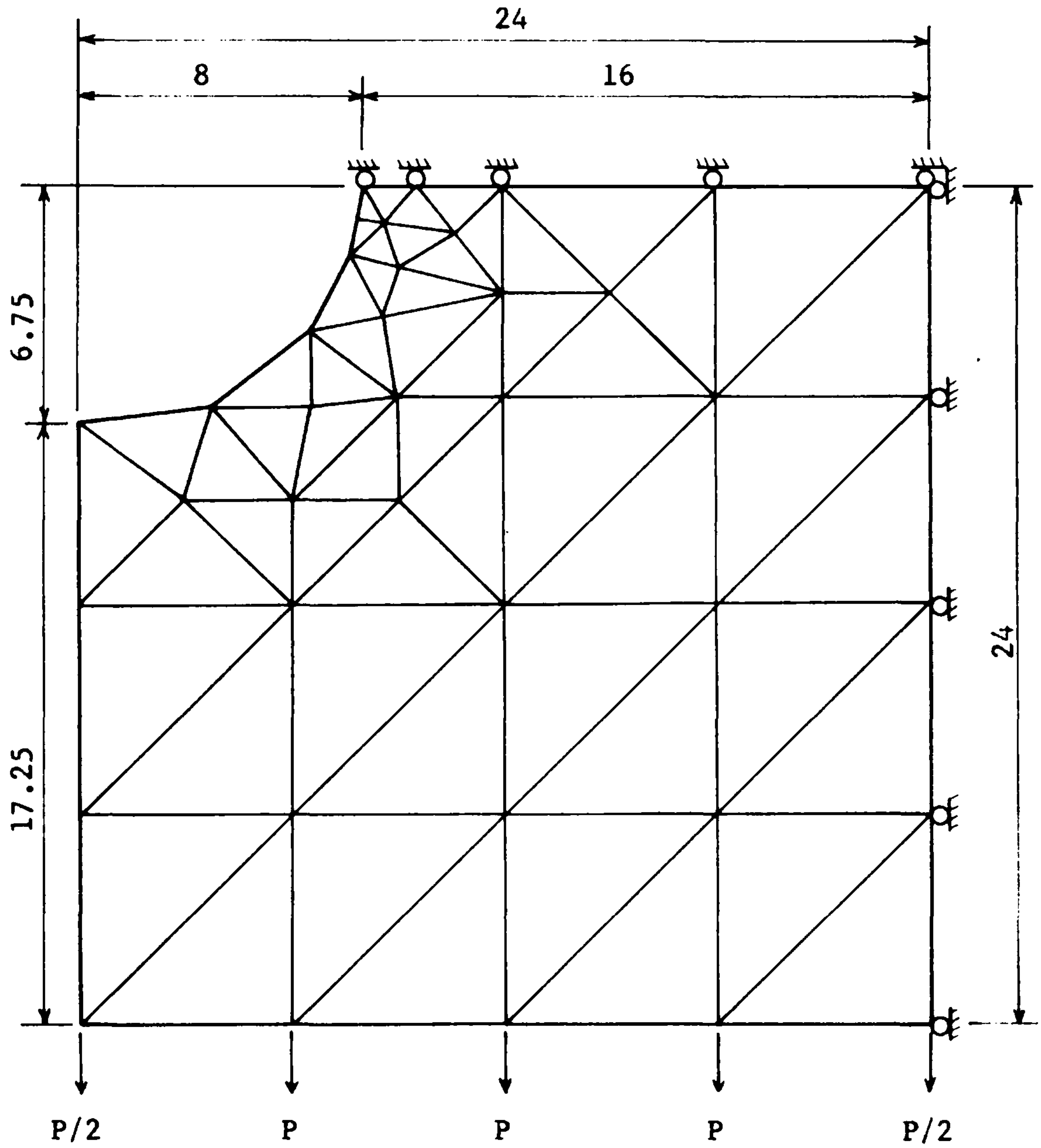



FIGURE 16 MODEL OF THE N2 PANEL

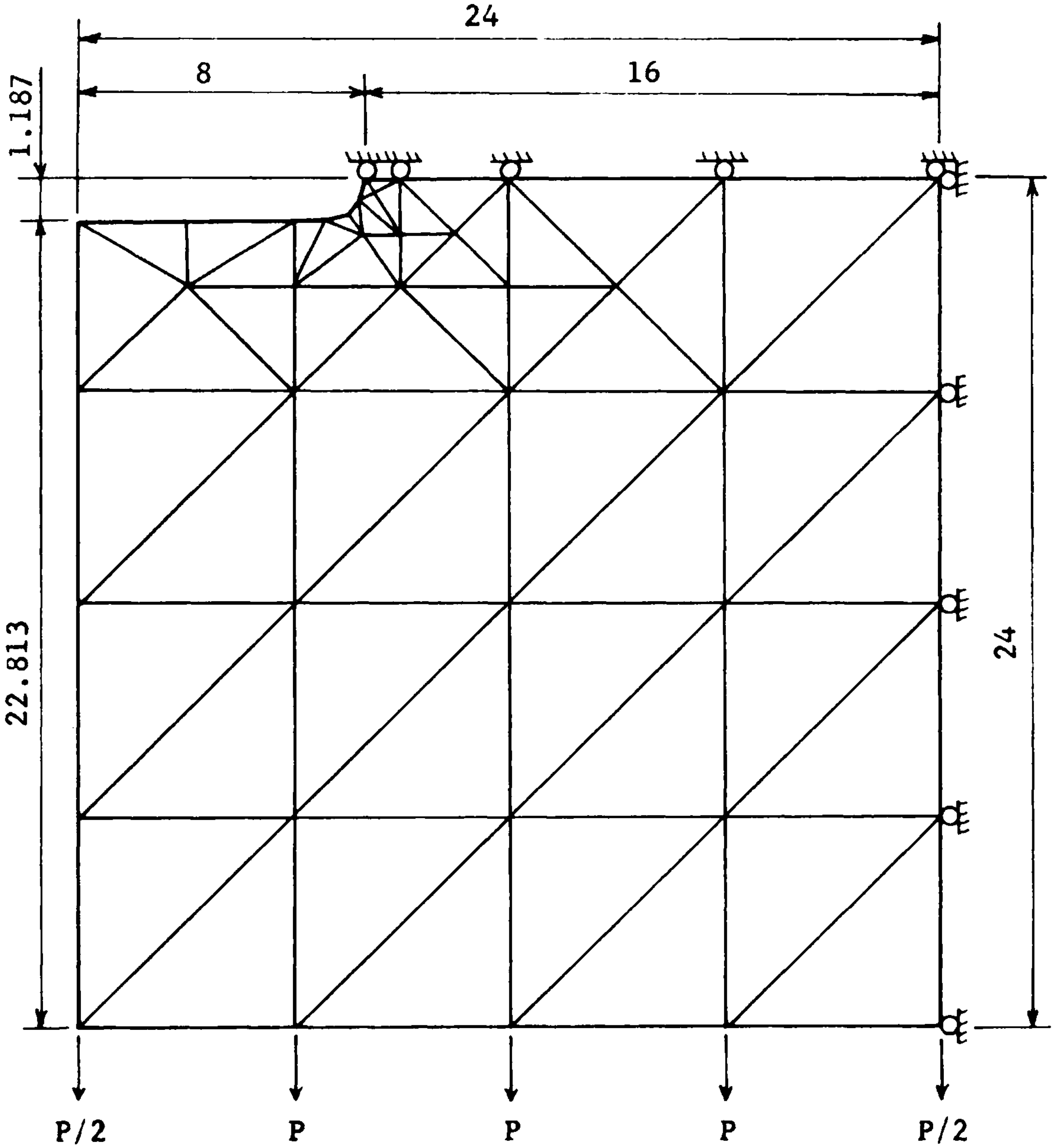



FIGURE 17 MODEL OF THE N4 PANEL

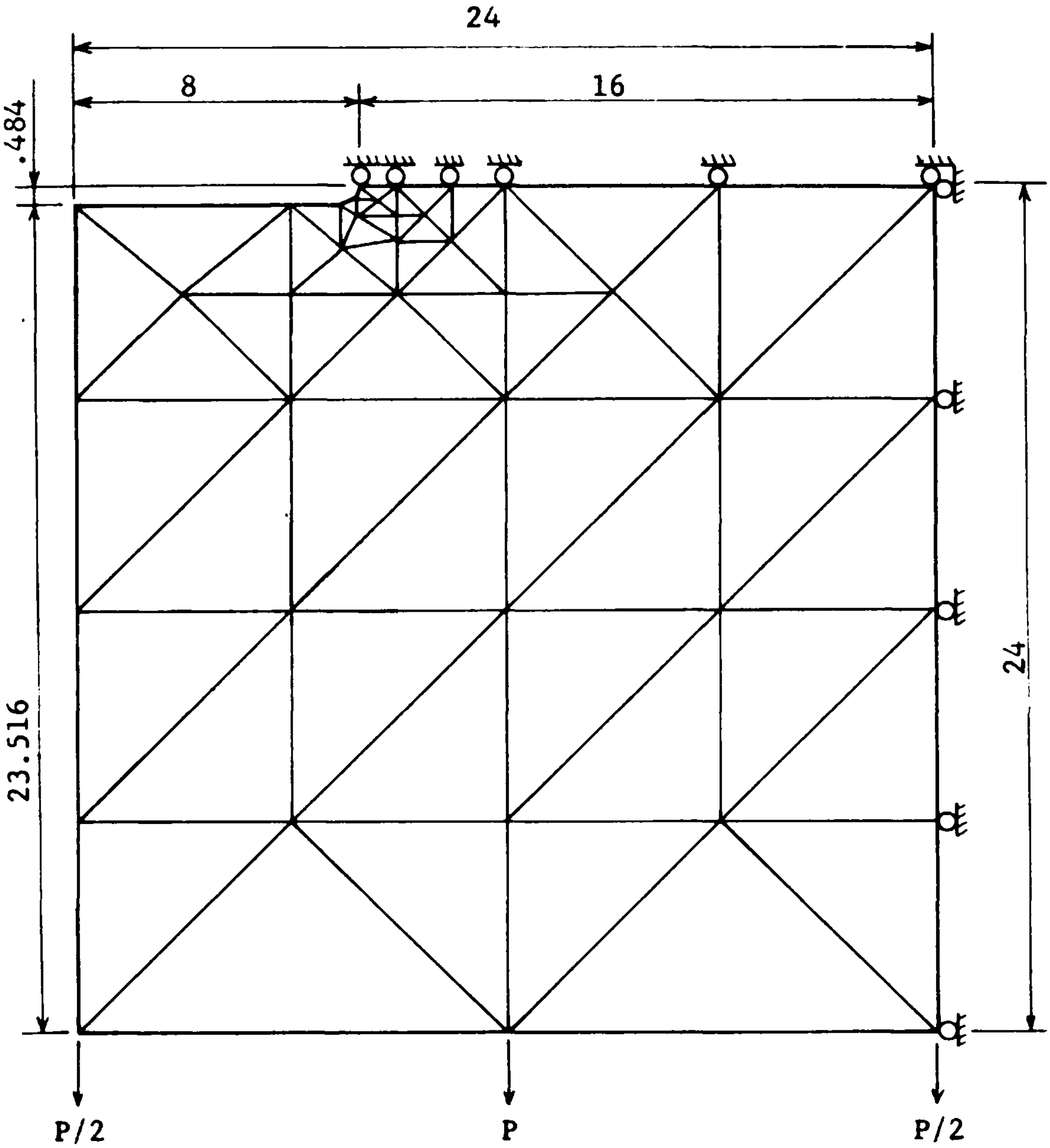



FIGURE 18 MODEL OF THE N6 PANEL

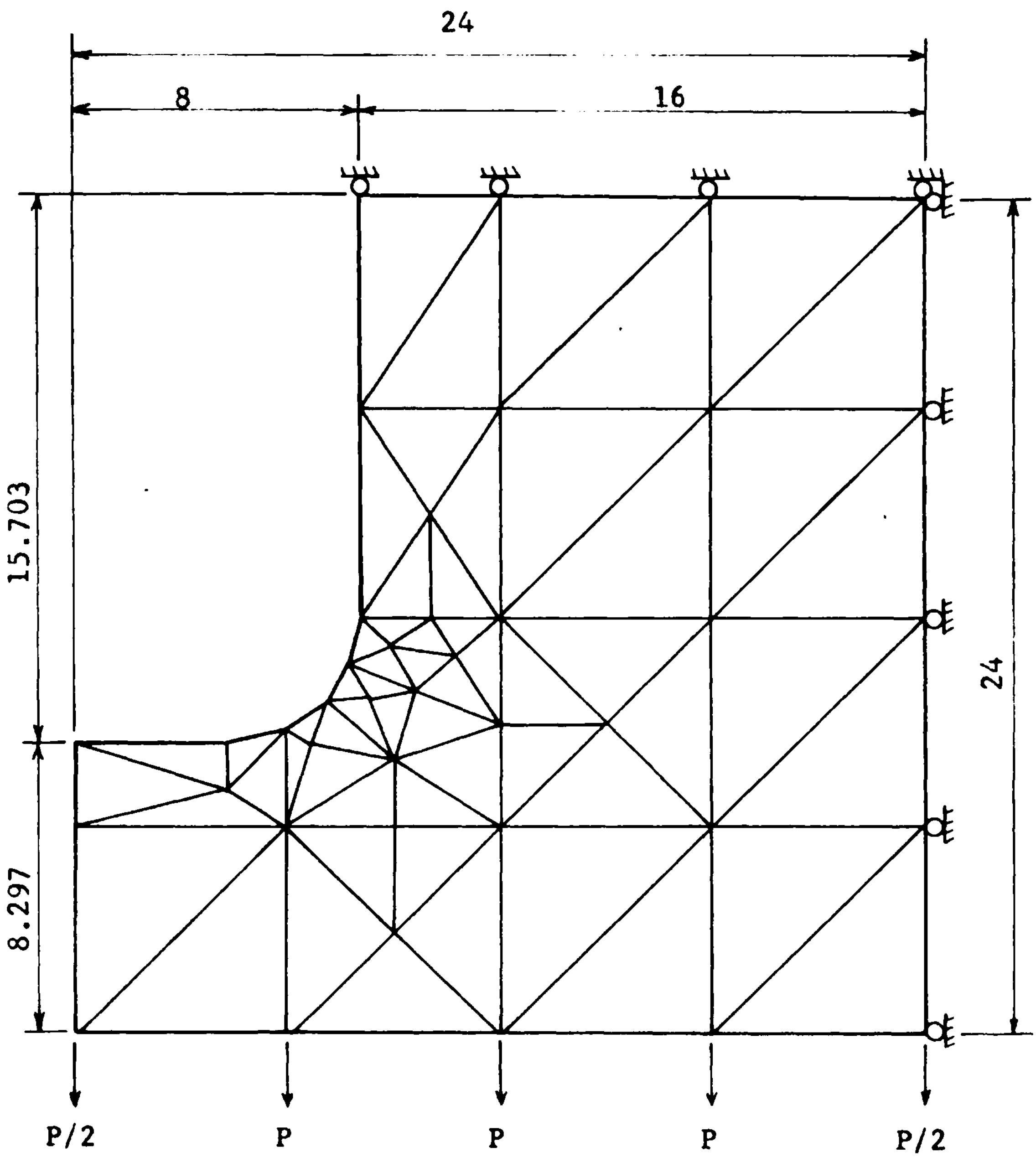



FIGURE 19 MODEL OF THE F2 PANEL

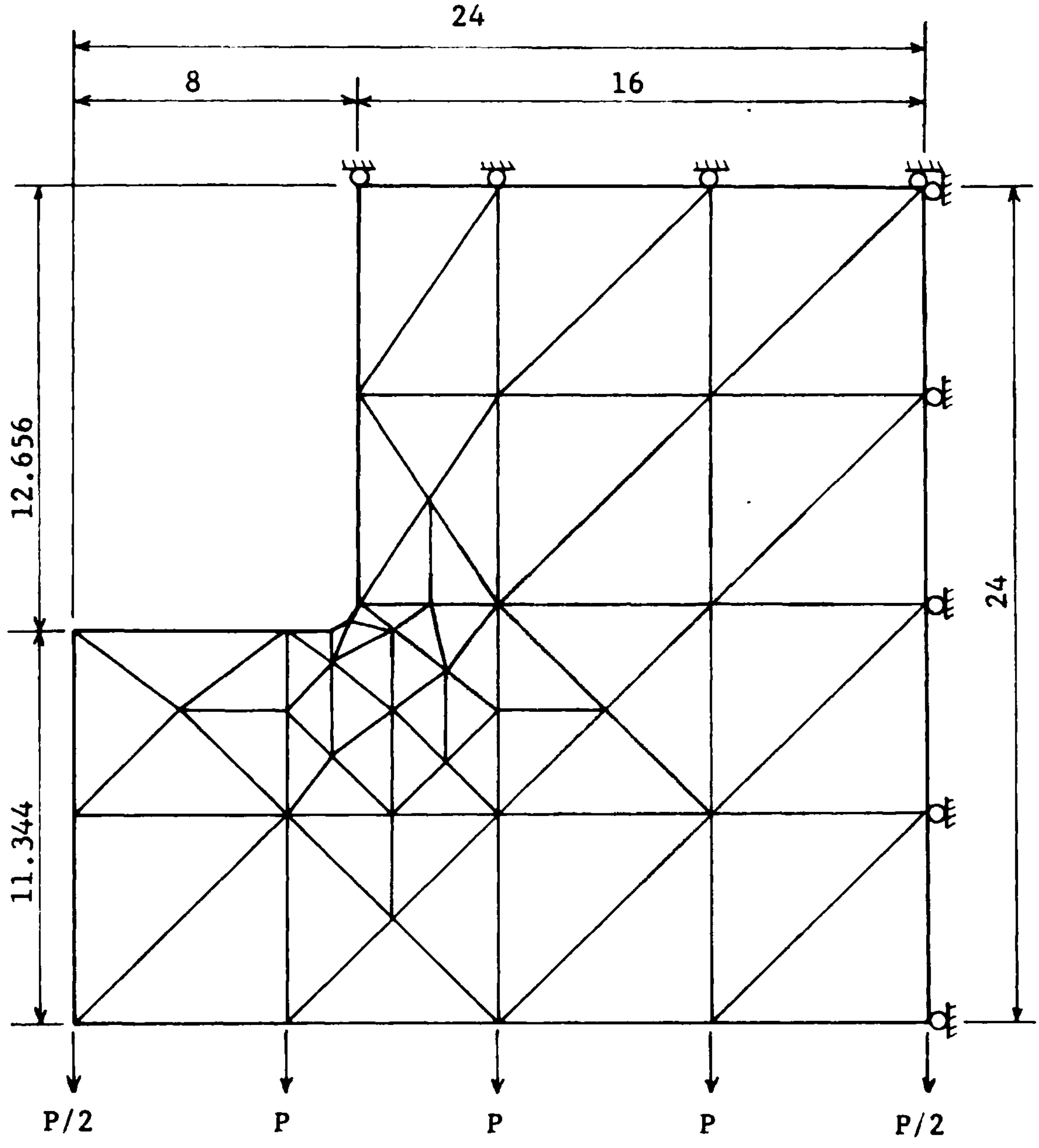



FIGURE 20 MODEL OF THE F4 PANEL

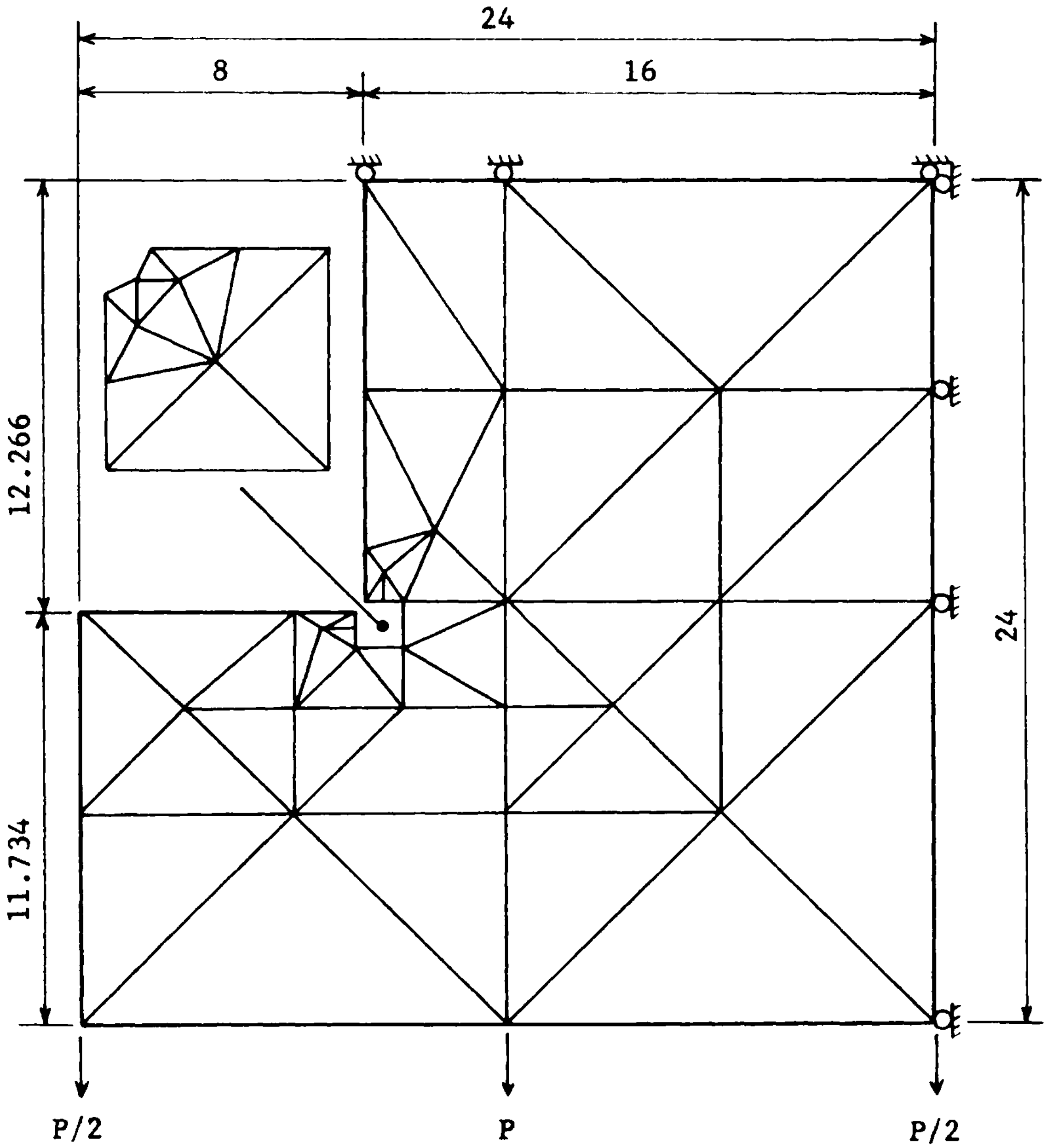



FIGURE 21 MODEL OF THE F6 PANEL

## APPENDIX C

## EXTRAPOLATION OF DATA

In determining the stresses across the net section the question arose as to what should be used for the node point stresses. Since the finite elements were constant stress elements the stresses in two neighboring elements often differed greatly. The procedure used here to calculate the net section stresses can not be rigorously proved, but it did yield acceptable results when compared to the experimental data. For this reason its use here seems justified.

Net section elements for the notched plates were defined as the elements along the net section boundary, and corresponding elements were assumed as net section elements for the filleted plates. The constant stress of the net section elements was plotted at the mid-point of the boundary edge for each of these elements, and the node point stress was assumed to be the average stress of all the net section elements which met at the node point. An exception to this generalization was made at the node point where the curved boundary was tangent to the net section. For this point the value of stress used was that of the finite element on the curved boundary which met at the node point. This exception was made in order to eliminate the averaging of the maximum stress in some unknown manner.

Using the above procedure a smooth curve was drawn through the points plotted, thus arriving at the net section stress distribution for the examples analyzed. These curves are shown in Figures 22 through 27 for different average net section stresses. Only the experimental data for the linear case was plotted and the solid lines are the analytical results.

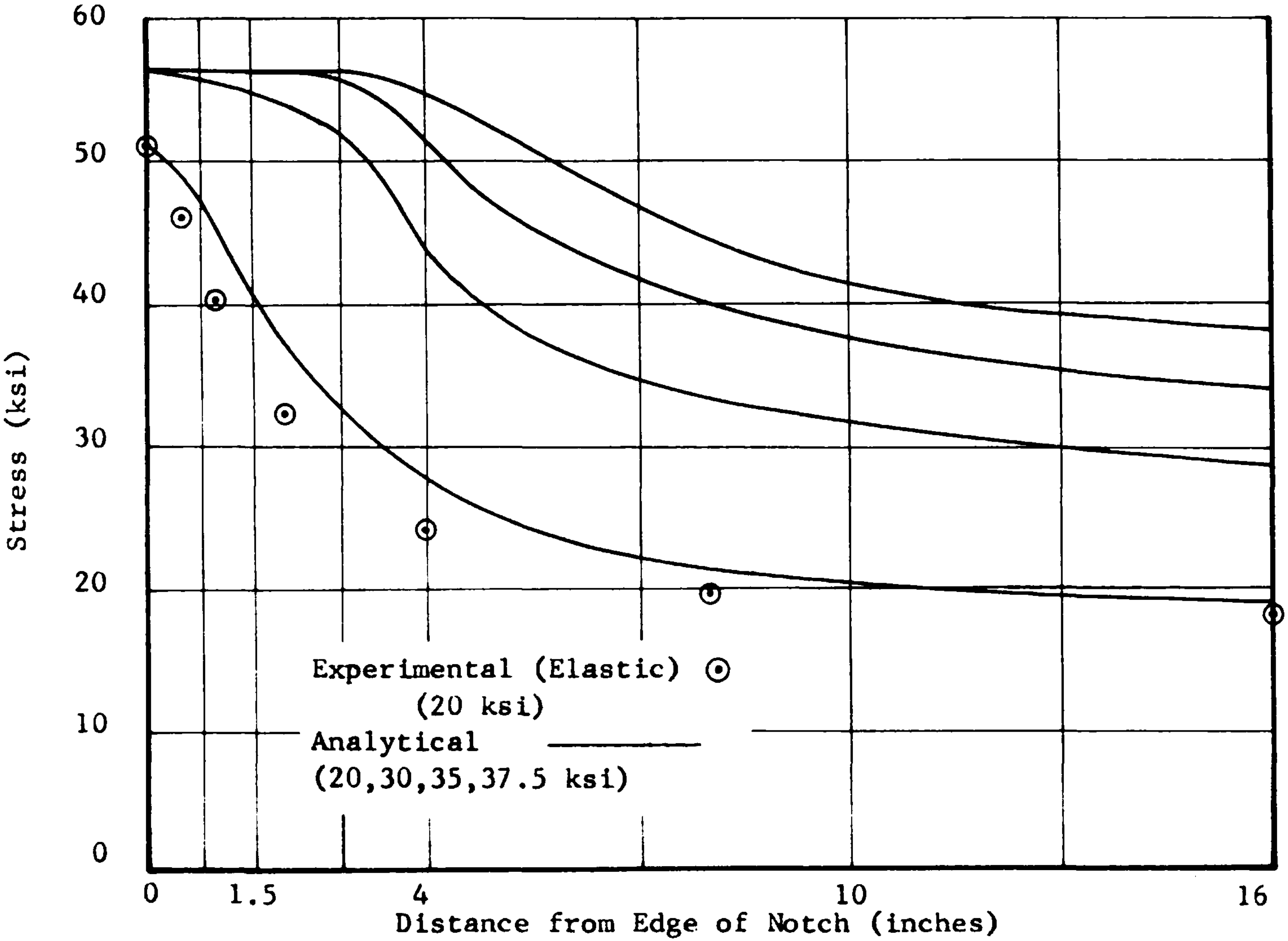



FIGURE 22 NET SECTION STRESS DISTRIBUTION (N2)

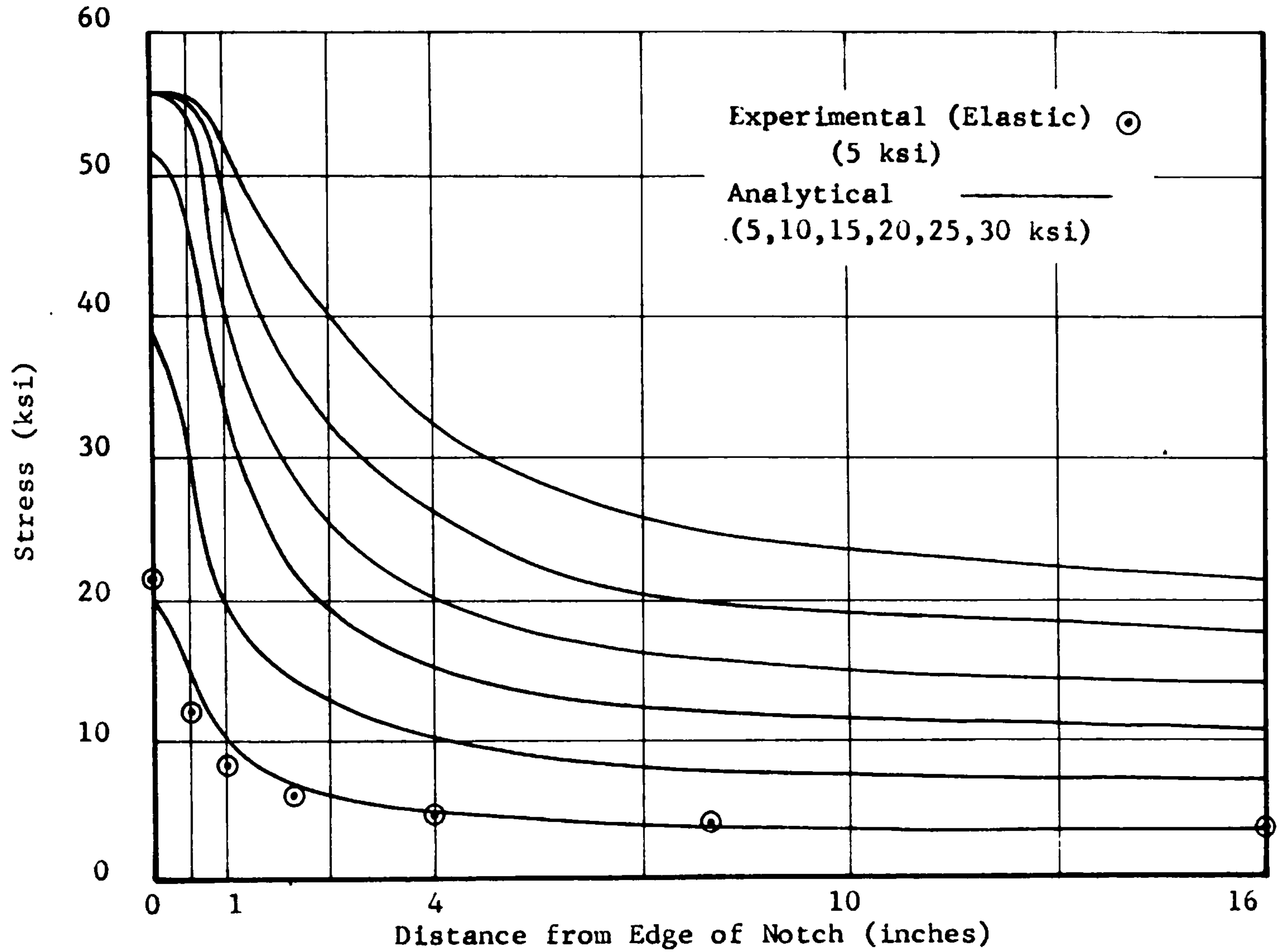



FIGURE 23 NET SECTION STRESS DISTRIBUTION (N4)

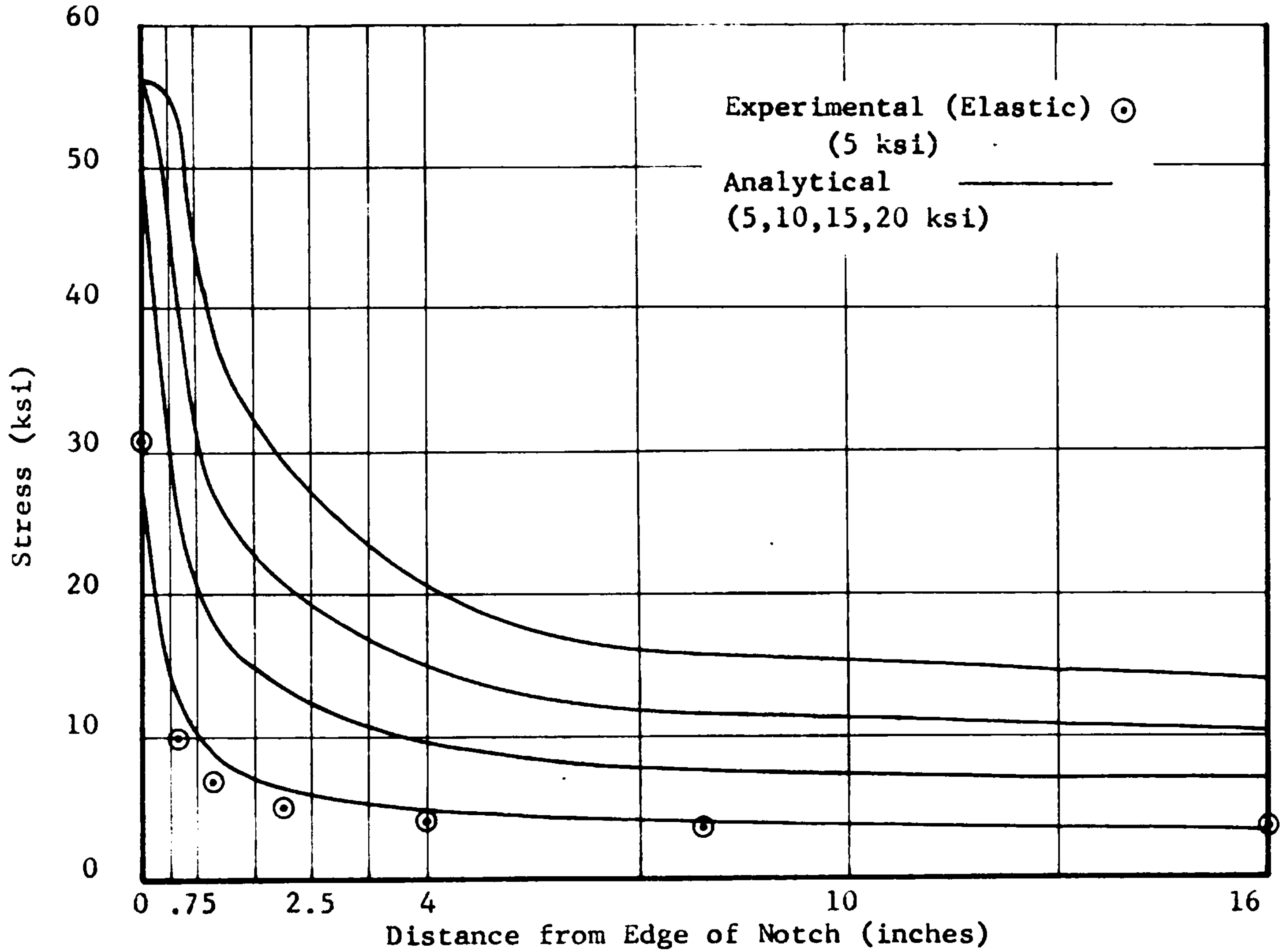



FIGURE 24 NET SECTION STRESS DISTRIBUTION (N6)

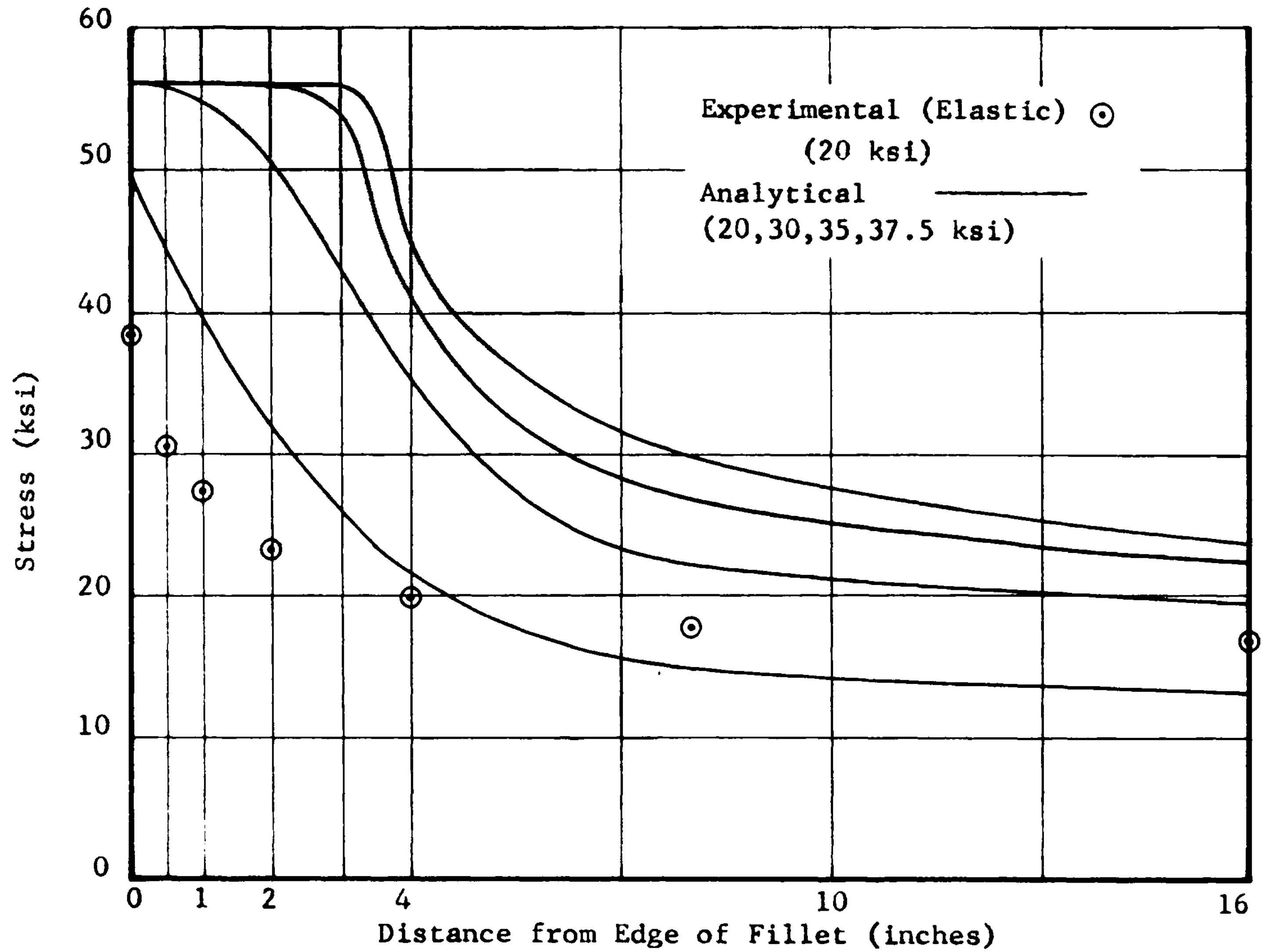



FIGURE 25 NET SECTION STRESS DISTRIBUTION (F2)

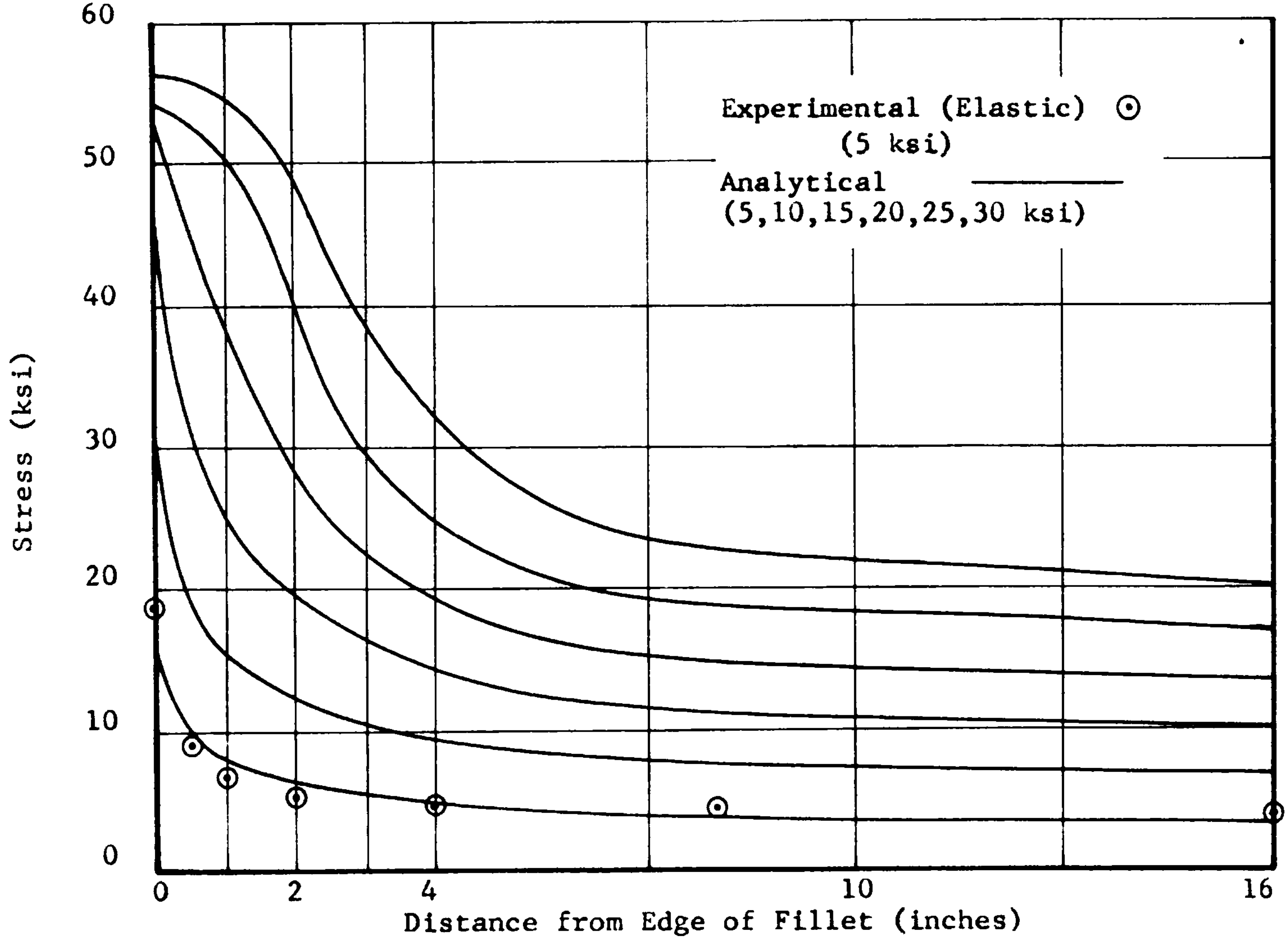



FIGURE 26 NET SECTION STRESS DISTRIBUTION (F4)

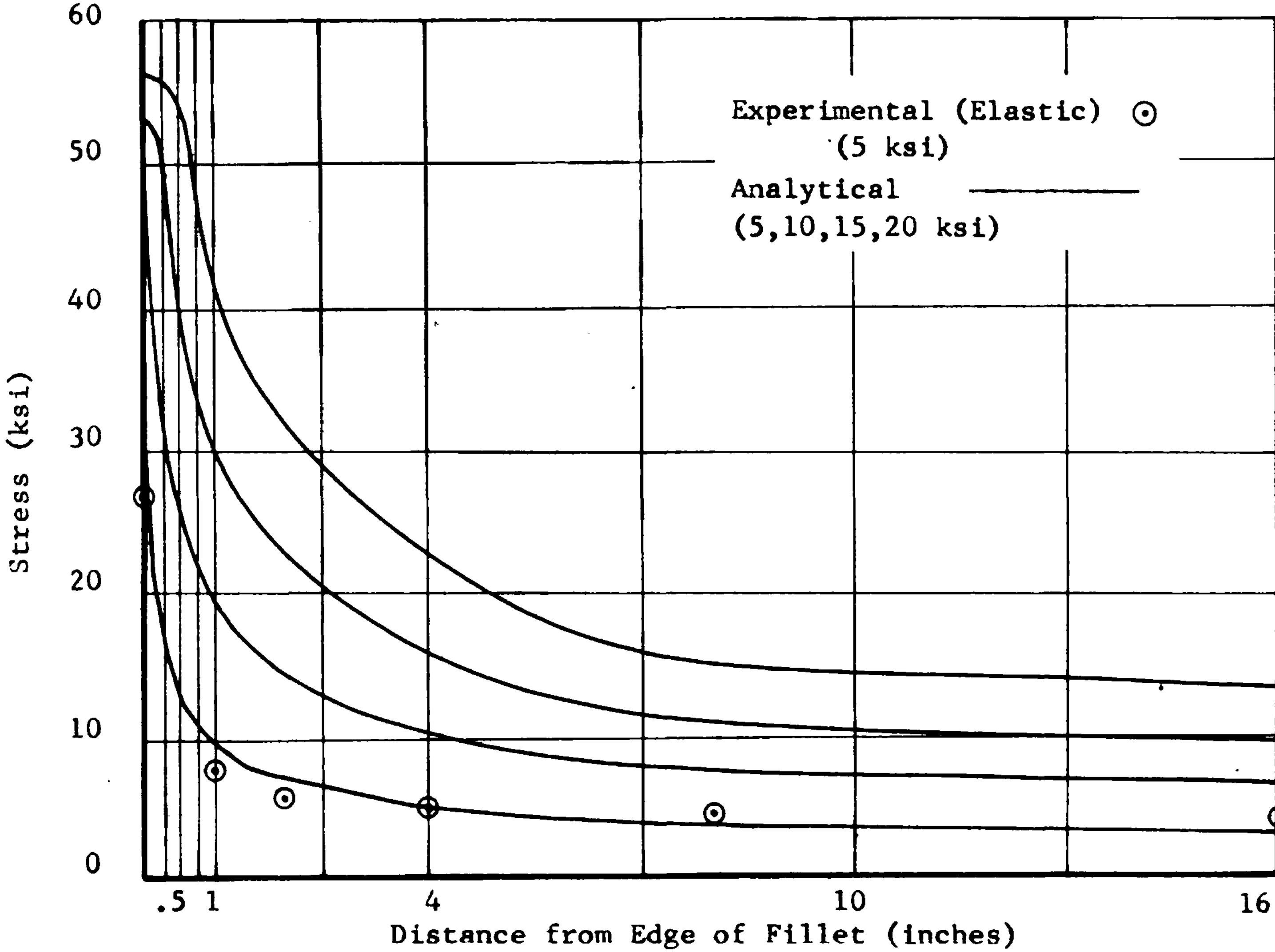



FIGURE 27 NET SECTION STRESS DISTRIBUTION (F6)