MICROWAVE MEASUREMENT OF THE ELECTRON DENSITY PROFILE OF A CYLINDRICAL PLASMA COLUMN

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF ELECTRICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

1968

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ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. R. N. Carlile for his encouragement during the course of this work, and to the National Science Foundation who supported this research under Grant GK-203.

TABLE OF CONTENTS

TABLE OF CONTENTS--Continued

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS--Continued

ABSTRACT

The electron number density in the positive column of a mercury vapor discharge is a function of the radius from the center of the column. A model for the positive column was developed and the dispersion relation for the plasma surface wave modes was obtained using this model. Solutions for the dispersion relation were obtained numerically. Experimental data was collected on the symmetric and dipolar surface wave modes. From this data the radial variation of electron density in the positive column was determined.

The effect of a transverse ambipolar current on the propagation of a plasma surface wave was examined. This transverse current flow is shown to result in attenuation of the surface wave. However, this attenuation is found to be negligible compared to that due to collisions in the positive column of a mercury vapor discharge.

I. INTRODUCTION

THE PLASMA WAVEGUDE

This paper describes the use of plasma surface waves for determining the radial variation in electron density in the positive column of a mercury vapor discharge. A surface wave propagates along the interface between the surrounding dielectric and the plasma column, and is characterized by large energy densities near the interface and no propagation in the radial direction. A cross-section of the plasma waveguide system is shown in figure 1.

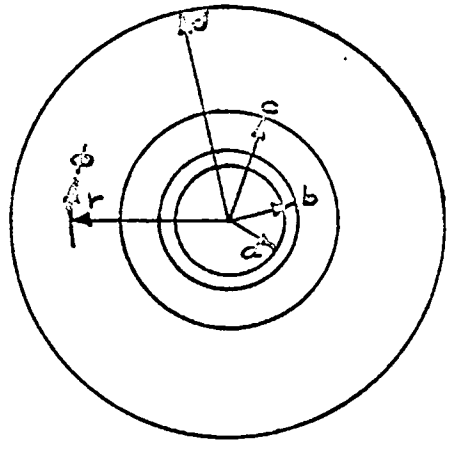

Figure 1. Cross-section of plasma waveguide

The isotropic plasma is located at the center of the waveguide ($0 < r < a$). It is maintained in this position by a quartz cylinder ($b < r < c$) which has a thin coating of titanium dioxide ($a < r < b$) on its inner surface. The quartz tube is surrounded by air ($c < r < d$) and the entire system is bounded by a copper waveguide at $r = d$. In this paper the dielectric regions and the copper waveguide are assumed to be lossless. We shall call this plasma waveguide configuration the four region system.

PLASMA SURFACE WAVES

Trivelpiece[1] investigated wave propagation in a waveguide partially filled by a plasma both in the presence and absence of an axial d.c. magnetic field. For the case of no axial d.c. magnetic field Trivelpiece found a set of modes which could propagate in the waveguide at frequencies below the electron plasma frequency, ω_p, which is the natural frequency of oscillation of the plasma electrons. These modes are the surface wave modes. The field components of the surface wave modes have an azimuthal variation of $\exp(in\phi)$ and an axial variation of $\exp(i\beta z)$ where n is an integer, β is the propagation constant of the wave, ϕ is the azimuthal angle in cylindrical coordinates, and the z-axis is along the axis of the plasma column. Because of the azimuthal variation exhibited by these modes,

the n = 0 mode is frequently referred to as the symmetric mode and the n = 1 mode is termed the dipolar mode. Trivelpiece showed theoretically and experimentally that the symmetric mode was a low pass mode. Carlile[2] found that the dipolar mode was a backward wave, band pass mode. As early as 1959, Schumann[3] in Germany had observed a backward wave in a similar system but had not identified it as the dipolar mode. In Japan Akao and Ida[4] have also found the dipolar mode to be a backward wave.

The investigations of Trivelpiece and Carlile, as well as Akao and Ida, were confined to a three region system consisting of the plasma, the quartz cylinder, and the air region. Madison[5] extended the theoretical work of Trivelpiece to the four region system. Madison obtained results for the four region system with and without an axial, d.c. magnetic field present. Johnson[6] constructed the four region system described by Madison and experimentally verified the effect of the thin titanium dioxide layer on the propagation of the two lowest order surface wave modes. Johnson's experiments were performed with an isotropic plasma, that is, with no axial magnetic field present.

The energy of the plasma surface wave is bound to the interface between the plasma and the adjacent dielectric. For small values of β (long wavelengths) this binding is slight, and the energy is distributed throughout the plasma column and adjoining dielectric

regions in the waveguide. When β becomes very large the energy concentrates at the edge of the plasma column. A graph of ω versus β is shown in figure 2. In this figure the angular frequency, ω, of the wave has been normalized to the plasma frequency, ω_p, and β has been normalized to the radius of the plasma column, "a". As βa becomes very large both the symmetric and dipolar modes approach an asymptotic limit of $\omega/\omega_p = (1+\epsilon_r)^{-1/2}$, where ϵ_r is the relative dielectric constant of the dielectric region immediately adjacent to the plasma.

The unique properties of the surface wave make it an interesting tool for plasma diagnostics, that is, determining properties of the plasma such as electron number density and density variations. For small values of βa the energy of the symmetric mode is distributed throughout the plasma. Carlile, Swinford, and Spiegel[7] have shown that this portion of the symmetric mode can be used to determine values of average electron densities with an error of less than 5%. As βa becomes large the energy of the surface modes becomes confined to the edge of the plasma column and the frequency of the wave approaches an asymptotic limit, $\omega = \omega_p(1+\epsilon_r)^{-1/2}$ which is determined by the electron density of the plasma through which the wave propagates since the plasma frequency is proportional to the square root of electron density. If the electron density is a function of the radius

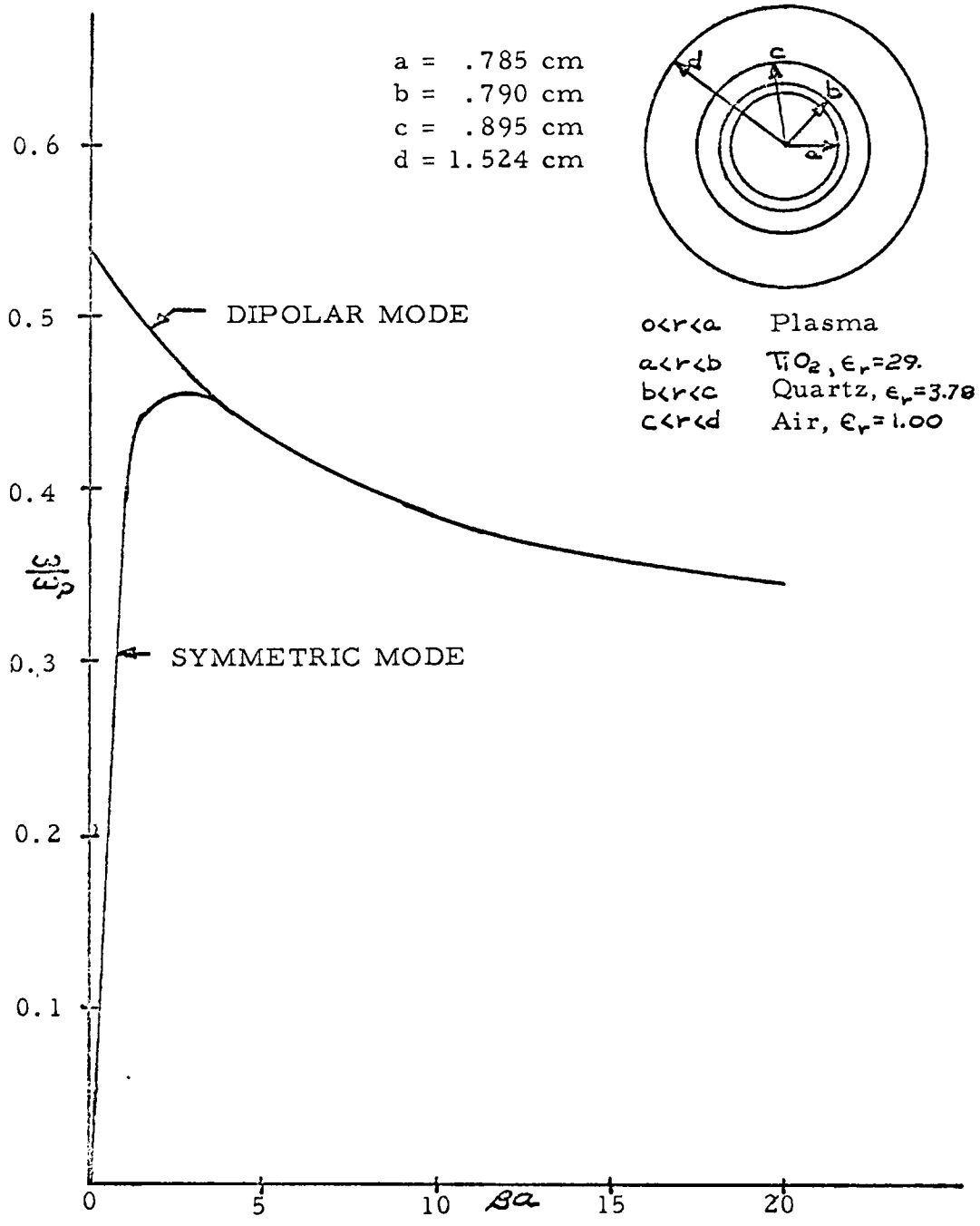

Figure 2. ω-β diagram of the symmetric and dipolar modes for the four region system.

from the center of the column the asymptotic limit of the frequency will vary with βa . This suggests that the large βa region of the surface wave modes might be used to measure radial variations in electron density. It is necessary to use the dipolar mode for measuring the electron density variations since the symmetric mode is a double valued function of frequency in the range of large βa and attempts to excite the mode in this frequency range invariably lead to a wave with the smaller value of βa . This paper is primarily concerned with the use of the dipolar mode to investigate radial electron density variations. In Chapter II a model for the inhomogeneous positive column is presented and the dispersion relation for this model is obtained. In Chapter III the experimental method is described briefly and the experimental results are presented and compared with the theoretical results of Chapter II. Chapter IV discusses the effect of transverse current flow on the propagation of the plasma surface wave, and Chapter V is the conclusion.

II. THE INHOMOGENEOUS PLASMA COLUMN

THE POSITIVE COLUMN

The mercury vapor discharge used in many laboratory discharge tubes can be separated into three distinct regions; the cathode region, the positive column, and the anode region. It is the positive column which is of interest in this paper. Two regions can be distinguished in the positive column itself, the "plasma" region at the center of the discharge tube, and the "sheath" region which is adjacent to the dielectric structure which contains the column. The structure of the positive column is such as to enhance the motion of the ions toward the surrounding wall and to retard the flow of the high velocity electrons to the wall in order that the ion current and electron current actually arriving at the wall will be of equal magnitude. These two currents must be of equal magnitude when the column is in the steady state so that the net current arriving at the wall will be zero as it must be since the charge on the wall remains constant.

In the plasma region of the positive column the number of ions per unit volumne, n_i, is only slightly greater than the number of electrons per unit volume, n_e. This difference is sufficient enough,

however, to establish a potential drop which accelerates the ions toward the wall.

In the sheath region near the wall the ion density greatly exceeds the electron density and a potential drop of the order of 5 electron volts develops across the sheath.[8] The sheath serves to further accelerate the ions and to reflect all but the fastest electrons.

Parker[9] has made a theoretical determination of the electron density as a function of radius in the positive column. For continuity a brief summary of Parker's analysis will be presented.

It is assumed that the electron density has a simple Boltzmann dependence

$$n_e(r) = n_{eo} \exp[eV(r)/kT] \quad , \qquad \text{II.1}$$

and that the ion generation is proportional to the electron density

$$G(r) = \nu n_e(r) \quad , \qquad \text{II.2}$$

where n_{eo} is the electron density on the axis and $V(r)$ is the electric potential in the column which is taken to be zero on the axis and decreases monotonically with radius. ν is a constant of proportionality related to the ionization cross section. The ion current density is then written as the integral of the source function

$$J_i = e\int_0^r G(\rho)\rho/r\, d\rho \quad . \qquad \text{II. 3}$$

The ion current can also be written as $J_i = n_i e v_i$ where v_i is the velocity of the ions. Equating these two expressions for ion current density enables one to solve for the ion number density

$$n_i(r) = \int_0^r \frac{G(\rho)\,\rho/r\, d\rho}{\left\{\left(\frac{2e}{m_i}\right)\left[V(\rho)-V(r)\right]\right\}^{1/2}} \quad . \qquad \text{II. 4}$$

Poisson's equation is

$$\nabla^2 V(r) = e/\epsilon_o\,(n_e - n_i)$$

Combining equations II. 1 and II. 4 with Poisson's equation and changing to the dimensionless variables

$$s = r\nu(m_i/2kT)^{1/2} \qquad \text{and} \qquad \eta = -eV(r)/kT$$

gives the equation which determines the potential within the column

$$\tfrac{1}{2}\xi^2\left(s\frac{d^2\eta}{ds^2} + \frac{d\eta}{ds}\right) = \int_0^s \frac{\exp[-\eta(\sigma)]\,\sigma\, d\sigma}{\left[\eta(s)-\eta(\sigma)\right]^{1/2}} - s\cdot\exp\left[-\eta(s)\right] \quad , \qquad \text{II. 5}$$

where $\xi^2 = (2/\nu^2)(n_{eo}\, e^2/m_i\epsilon_o)$.

From numerical solutions to equation II. 5 one obtains the potential $\eta(s)$ and consequently can determine the normalized electron density $n_{e(r)}/n_{eo}$ for various values of the parameter ξ^2 . The graphs of the normalized electron density versus the normalized radius r/r_w (r_w is the radius of the wall) for a mercury plasma are reproduced in figure 3. The parameter in this figure is ξ^2 .

PLASMA MODEL

A laboratory plasma is in general quite complex. However, various idealized models of a plasma have been devised to allow the study of the interaction of electromagnetic waves with a plasma in a mathematically tractable way. The most elementary of these models is the cold plasma model. This model assumes that the electrons and ions are stationary in the absence of disturbing electric fields. It is also assumed that the electrons respond to high frequency electric fields but that the ions do not respond because they are orders of magnitude heavier. They merely serve as a neutralizing background for the electrons. It is convenient to first determine an equivalent conductivity for the plasma by considering the response of the electrons to a harmonic time-varying electric field and then, second, to cast this result into a form in which the effect of the plasma is accounted for by a frequency dependent dielectric constant. The relative

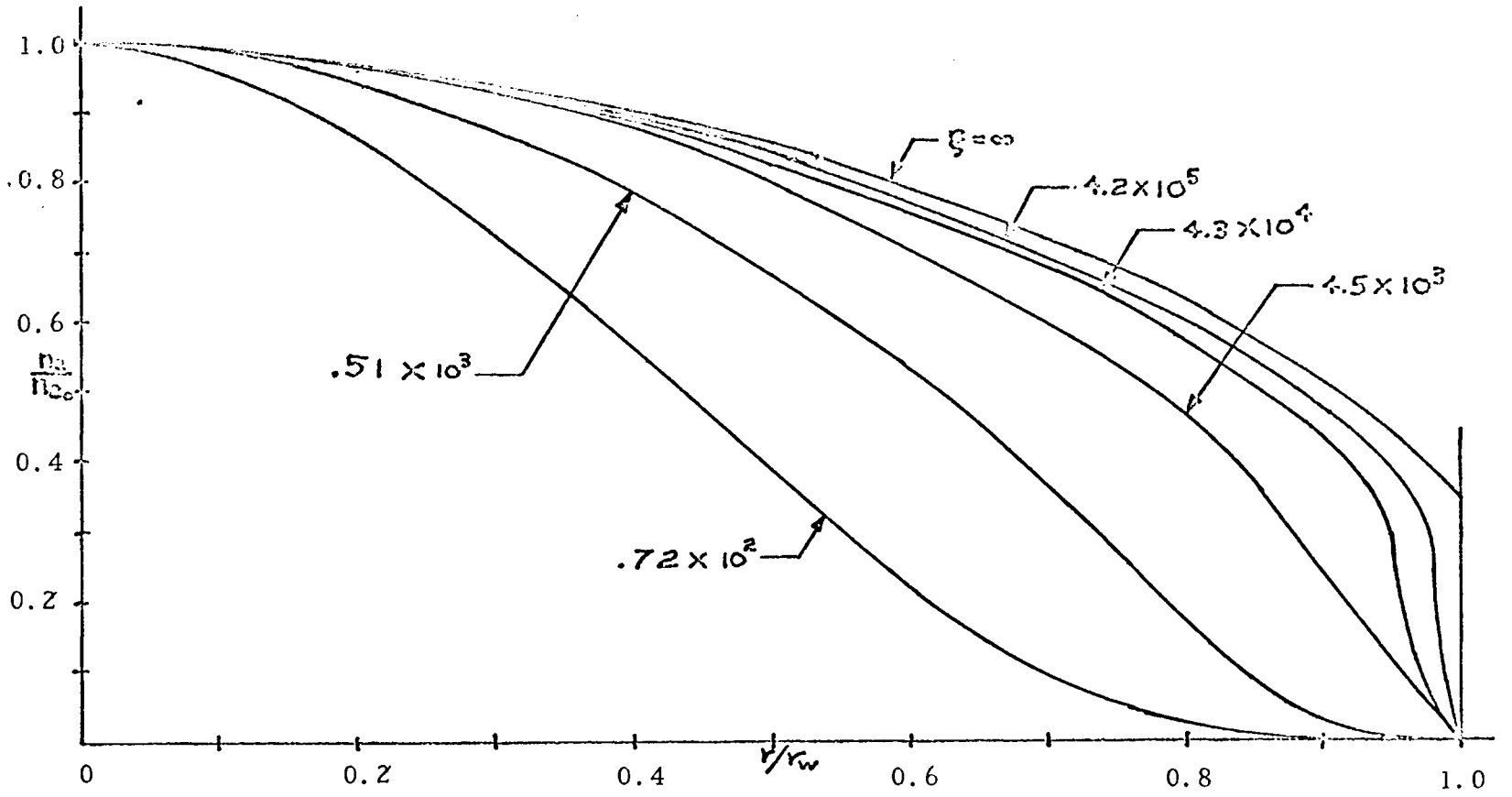

Figure 3. Ne/Neo versus r/r_w for a mercury vapor plasma for various values of ξ^2.

dielectric constant for a plasma in the absence of an external magnetic field is

$$\epsilon_p = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad , \qquad \text{II.6}$$

where ω is the frequency of the electromagnetic wave in radians per second and ω_p is again the plasma frequency of the electrons which is

$$\omega_p = \left(\frac{n_e e^2}{m_e \epsilon_o}\right)^{1/2} \quad . \qquad \text{II.7}$$

n_e , e , and m_e are as previously defined and ϵ_o is the permittivity of a vacuum. This equivalent dielectric constant is then employed in Maxwell's equations in the usual manner.

One more assumption regarding the plasma model used in this paper remains to be discussed. This is the quasi-static approximation. If field variations of the form $\exp(i\omega t - i\beta z)$ are assumed, Maxwell's equations can be written

$$E_r = -\left(\frac{1}{k^2 - \beta^2}\right)\left[i\beta \frac{\partial E_z}{\partial r} + \frac{i\omega\mu_o}{r} \frac{\partial H_z}{\partial \phi}\right] \qquad \text{II.8}$$

$$E_\phi = \left(\frac{1}{k^2 - \beta^2}\right)\left[\frac{-i\beta}{r} \frac{\partial E_z}{\partial \phi} + i\omega\mu_o \frac{\partial H_z}{\partial r}\right] \qquad \text{II.9}$$

$$H_r = \left(\frac{1}{k^2 - \beta^2}\right)\left[\frac{i\omega\epsilon_r\epsilon_o}{r}\frac{\partial E_z}{\partial \phi} - i\beta\frac{\partial H_z}{\partial r}\right] \qquad \text{II. 10}$$

$$H_\phi = \left(\frac{-1}{k^2 - \beta^2}\right)\left[i\omega\epsilon_r\epsilon_o\frac{\partial E_z}{\partial r} + \frac{i\beta}{r}\frac{\partial H_z}{\partial \phi}\right] \qquad \text{II. 11}$$

where k is the propagation constant

$$k^2 = \omega^2\mu_o\epsilon_o\epsilon_p$$

Our results are assumed to be valid only when $\beta^2 \gg k^2$. Consider the total electric field vector, $\overline{E}$, to be composed of its TM component $\overline{E}'$ and its TE component $\overline{E}''$, then

$$\overline{E} = \overline{E}' + \overline{E}'' \; . \qquad \text{II. 12}$$

The TM solutions to equations II. 8 and II. 9 can be found by setting $H_z = 0$. We find

$$\overline{E}' = -\left[\frac{\partial}{\partial r}\left(\frac{E_z}{i\beta}\right)\overline{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{E_z}{i\beta}\right)\overline{a}_\phi + \frac{\partial}{\partial z}\left(\frac{E_z}{i\beta}\right)\overline{a}_z\right] . \qquad \text{II. 13}$$

By inspection we can then write

$$\overline{E}' = -\nabla\psi \qquad \text{II. 14}$$

where ψ is of course a scalar. By similar reasonsing one could obtain the slow TE magnetic field solutions as the negative gradient of a scalar magnetic potential. It can be shown, however, that pure slow TE waves cannot satisfy the boundary conditions in closed waveguide like that used in these experiments.[3] We therefore seek only TM solutions whose field compoenets satisfy the appropriate boundary conditions. Equation II. 14 expresses the quasi-static approximation, that is $\overline{E}' \approx \overline{E} \approx -\nabla\psi$.

PARABOLIC APPROXIMATION TO THE POSITIVE COLUMN

Inspection of figure 3 shows that the positive column is inhomogeneous in the radial direction since the electron density decreases with radius. From figure 3 it can be seen that for the three larger values of ξ^2 this decrease is a slow function of the radius in the plasma regions of the column, but that the electron density drops abruptly toward zero in the sheath region of the column near the wall. A mathematically tractable model for this behavior can be obtained. The electron density variation in the plasma region can be approximated quite closely by a parabola, or

$$n_e(r/a) = n_{eo}\left[1 - \alpha(r/a)^2\right] \quad . \qquad \text{II. 15}$$

n_{eo} is again the electron density on the axis of the column, a is the radius of the plasma region, and α is the parameter which determines the degree of parabolaticity. This parabolic approximation will match the actual density variation quite well outside the sheath region. Within the sheath region the electron density and consequently the plasma frequency is much lower than it is in the plasma region. In fact, in the sheath region the plasma frequency is so low that $\omega_p^2/\omega^2 \ll 1.0$. Therefore, $\epsilon_p \approx 1.0$ which is the same as saying that the electron density is negligible in the sheath. As part of our model of the positive column we will take the electron density to be zero in the sheath. In figure 4 Parker's result for $\xi^2 = 4.3 \times 10^4$ is shown as the dotted curve and the electron density approximation as given by the model is shown as the solid line for a value of α of 0.7.

Since the electron density is a function of radius in the positive column, the effective dielectric constant is also then a function of the radius. For the parabolic electron density approximation the effective dielectric constant for the positive column becomes

$$\epsilon_p(r/b) = \begin{cases} 1 - \frac{e^2 n_{eo}}{\omega^2 m \epsilon_o}\left[1 - \alpha\left(\frac{r}{b}\right)^2\right] & r/b < T \\ 1 & T < r/b < 1 \end{cases} \quad . \qquad \text{II. 16}$$

This is the model of the positive column that will be used in this paper. It should be noted that our model is characterized by the two parameters, α and T.

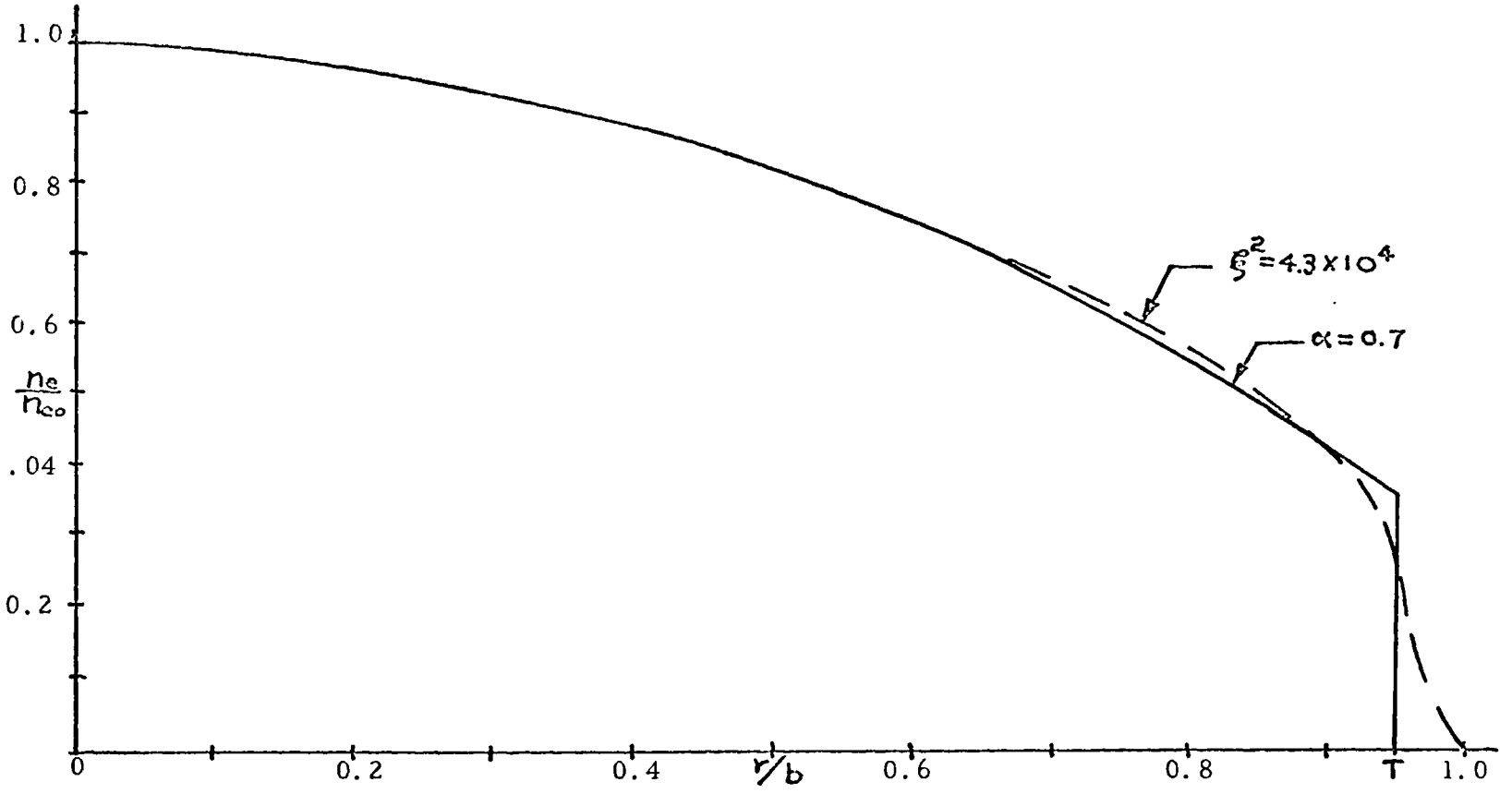

Figure 4. n_e/n_{eo} versus r/b as determined by Parker and as given by the parabolic approximation.

DISPERSION RELATION FOR THE PARABOLIC ELECTRON DENSITY APPROXIMATION

One of Maxwell's equations for the plasma region can be written

$$\nabla \times \overline{H} = i\omega \epsilon_o \epsilon_p \overline{E} \quad . \qquad \text{II. 17}$$

Taking the divergence of both sides of equation II. 17 gives

$$\nabla \cdot \nabla \times \overline{H} = i\omega \epsilon_o \nabla \cdot \epsilon_p \overline{E} = 0 \qquad \text{II. 18}$$

since the divergence of the curl of any vector is zero. By equation II. 14, the electric field can be represented as the negative gradient of a scalar potential. Substituting this into equation II. 18 yields

$$\nabla \cdot \epsilon_p \nabla \psi = 0 \qquad \text{II. 19}$$

as the differential equation which must be solved to determine the fields within the plasma due to the time varying electric field. This can be expanded in cylindrical coordinates as

$$\epsilon_p \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{\partial r}\frac{\partial \epsilon_p}{\partial r} + \frac{\epsilon_p}{r}\frac{\partial \psi}{\partial r} + \frac{\epsilon_p}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \epsilon_p \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad \text{II. 20}$$

Trivelpiece[1] has also derived this equation and obtained solutions for the symmetric mode. Assuming a product solution of the form

$$\psi(r, \phi, z) = R(r)\Phi(\phi)Z(z) \qquad \text{II. 21}$$

gives three ordinary differential equations

$$\frac{d^2 Z}{dz^2} + \beta^2 Z = 0 \quad , \qquad \text{II.22}$$

$$\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0 \quad , \qquad \text{II.23}$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{\epsilon_p}\frac{d\epsilon_p}{dr}\frac{dR}{dr} + \frac{1}{r}\frac{dR}{dr} - R\left(\beta^2 + \frac{n^2}{r^2}\right) = 0 \quad . \qquad \text{II.24}$$

β is a positive real number and n is a positive or negative integer. By making a change of variables, equation II.24 can be written

$$\frac{d^2 R(\xi)}{d\xi^2} + \left[\frac{1}{\xi} + \frac{2\xi}{\xi^2 - G}\right]\frac{dR(\xi)}{d\xi} - \left(\frac{n^2}{\xi^2} + 1\right)R(\xi) = 0 \qquad \text{II.25}$$

where $\xi = \beta r$, $G = \frac{(1-f^2)(\beta a)^2}{\alpha}$, and $f = \frac{\omega}{\omega_{p_0}}$. ω_{p_0} is the plasma frequency on the axis of the column. A solution may be obtained using the method of Frobenius. A power series solution of the form

$$R_n(\xi) = \sum_{i=0}^{\infty} C_i \xi^{i+n} \qquad \text{II.26}$$

can be assumed for equation II.25. Substituting this series into the differential equation and equating the coefficients of like powers of

ξ equal to zero determines the recursion formula for the coefficients

$$C_{i+4} = \frac{1}{(i+4)(2n+i+4)} \left\{ \frac{[G+2n(i+3)+(i+4)(i+2)]C_{i+2} - C_i}{G} \right\} . \qquad \text{II. 27}$$

We normalized the function $R_n(\xi)$ by setting $C_o = 1$. $R_n(\beta r)$ describes the radial variation of the plasma surface wave field components within the plasma.

In the dielectric regions exterior to the plasma the permittivity is not a function of position so it may be taken outside the divergence operator and we are left with Laplace's equation,

$$\nabla \cdot \nabla \psi = 0 . \qquad \text{II. 28}$$

This partial differential equation can again be separated into three ordinary differential equations by assuming a product solution. The equations for the azimuthal and longitudinal variations are identical with equations II. 23 and II. 22. The equation for the radial variations, however, is Bessel's equation whose solutions are the modified Bessel functions of the first and second kind.

The dispersion relation for the plasma surface wave is the relation between the frequency and propagation constant of the wave. This relation is readily obtained for the plasma surface wave by the standard technique of matching boundary conditions. We require that the tangential electric field and the normal displacement be continuous

across the plasma-dielectric and dielectric-dielectric boundaries within the waveguide. We also require that the tangential electric field be zero at the waveguide wall. Imposing these conditions on the field components leads to the following dispersion relation for the inhomogeneous plasma surrounded by four lossless dielectric regions enclosed in a metal waveguide

$$\left[1-\frac{\omega_{p_0}^2}{\omega^2}(1-\alpha)\right]\frac{R_n'(\beta a)}{R_n(\beta a)}=$$

$$\epsilon_2\frac{\epsilon_2\left[I_n'(\beta a)K_n'(\beta b)-I_n'(\beta b)K_n'(\beta a)\right]+\frac{Q(\beta b)}{\epsilon_0}\left[I_n(\beta b)K_n'(\beta a)-I_n'(\beta a)K_n(\beta b)\right]}{\epsilon_2\left[I_n(\beta a)K_n'(\beta b)-I_n'(\beta b)K_n(\beta a)\right]+\frac{Q(\beta b)}{\epsilon_0}\left[I_n(\beta b)K_n(\beta a)-I_n(\beta a)K_n(\beta b)\right]} \qquad \text{II.29}$$

where

$$\frac{Q(\beta b)}{\epsilon_0}=\epsilon_3\frac{\epsilon_3\left[I_n'(\beta b)K_n(\beta c)-I_n(\beta c)K_n'(\beta b)\right]+\frac{Q(\beta c)}{\epsilon_0}\left[I_n(\beta c)K_n'(\beta b)-I_n'(\beta b)K_n(\beta c)\right]}{\epsilon_3\left[I_n(\beta b)K_n'(\beta c)-I_n'(\beta c)K_n(\beta b)\right]+\frac{Q(\beta c)}{\epsilon_0}\left[I_n(\beta c)K_n(\beta b)-I_n(\beta b)K_n(\beta c)\right]}$$

$$\frac{Q(\beta c)}{\epsilon_0}=\epsilon_4\frac{\epsilon_4\left[I_n'(\beta c)K_n'(\beta d)-I_n'(\beta d)K_n'(\beta c)\right]+\frac{Q(\beta d)}{\epsilon_0}\left[I_n(\beta d)K_n'(\beta c)-I_n'(\beta c)K_n(\beta d)\right]}{\epsilon_4\left[I_n(\beta c)K_n'(\beta d)-I_n'(\beta d)K_n(\beta c)\right]+\frac{Q(\beta d)}{\epsilon_0}\left[I_n(\beta d)K_n(\beta c)-I_n(\beta c)K_n(\beta d)\right]}$$

$$\frac{Q(\beta d)}{\epsilon_0}=\frac{\epsilon_5\left[I_n(\beta e)K_n'(\beta d)-I_n'(\beta d)K_n(\beta e)\right]}{\left[I_n(\beta c)K_n(\beta d)-I_n(\beta d)K_n(\beta e)\right]}\,.$$

ω_β is the plasma frequency on the axis of the discharge. This may be related to the average plasma frequency, $\langle\omega_p\rangle$, by the expression

$$\langle\omega_p\rangle=\omega_{p_0}(1-\alpha/2)^{1/2}\,.$$

We call this the average plasma frequency because it is computed using an electron density, $\langle n_e \rangle$, which is obtained by averaging the true electron density, $n_e(r)$, across the cross-section of the column, that is

$$\langle n_e \rangle = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a n_e(r) r \, dr \, d\phi \quad .$$

a, b, c, d, e and ϵ_2 , ϵ_3 , ϵ_4 , ϵ_5 are the radii and relative dielectric constants of the regions in the waveguide as shown in the insert of figure 5.

Equation II. 29 is solved numerically to give $\omega/\langle \omega_p \rangle$ versus βa. Figure 5 shows the effect of the radial electron density variation on the ω-β curves for the two lowest order surface wave modes. The ω-β curve for the uniform plasma column is of course the curve labeled $\alpha = 0$. In figure 5 the sheath thickness has been taken as zero so that the effect of the sheath would not obscure the effect of the radial density variation. It may be seen from this figure that the effect of the radial variation is to reduce the large βa , frequency asymptote.

Figure 6 shows the effect of the sheath thickness, T , on the ω-β curves. In this figure αT^2 has a constant value of 0.7 and the sheath thickness is the variable parameter. It can be seen that the effect of increasing the sheath thickness is to raise the frequency asymptote for βa between 2.0 and 6.0 .

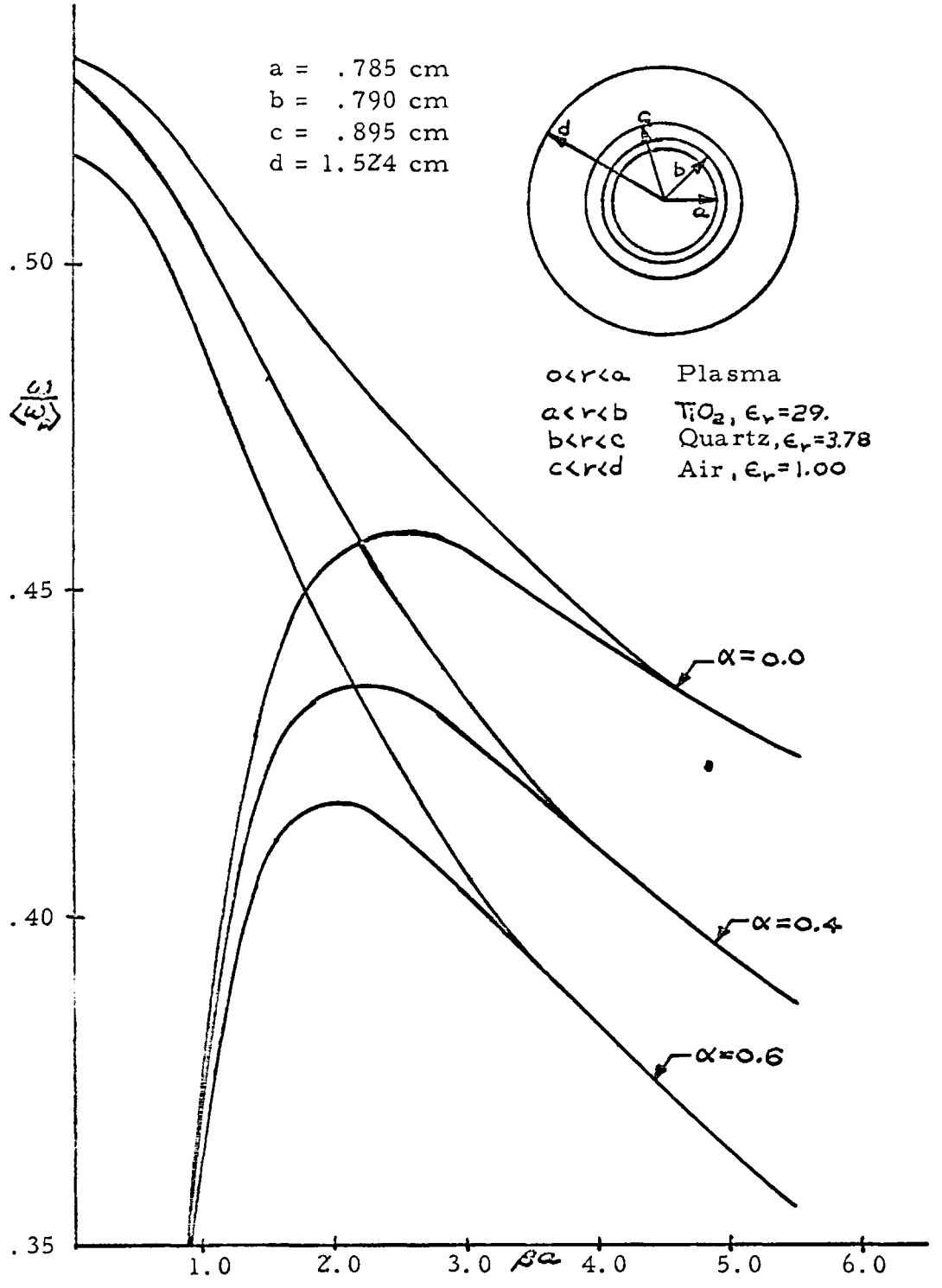

Figure 5. ω-β diagram of the symmetric and dipolar modes for varying α and constant sheath thickness.

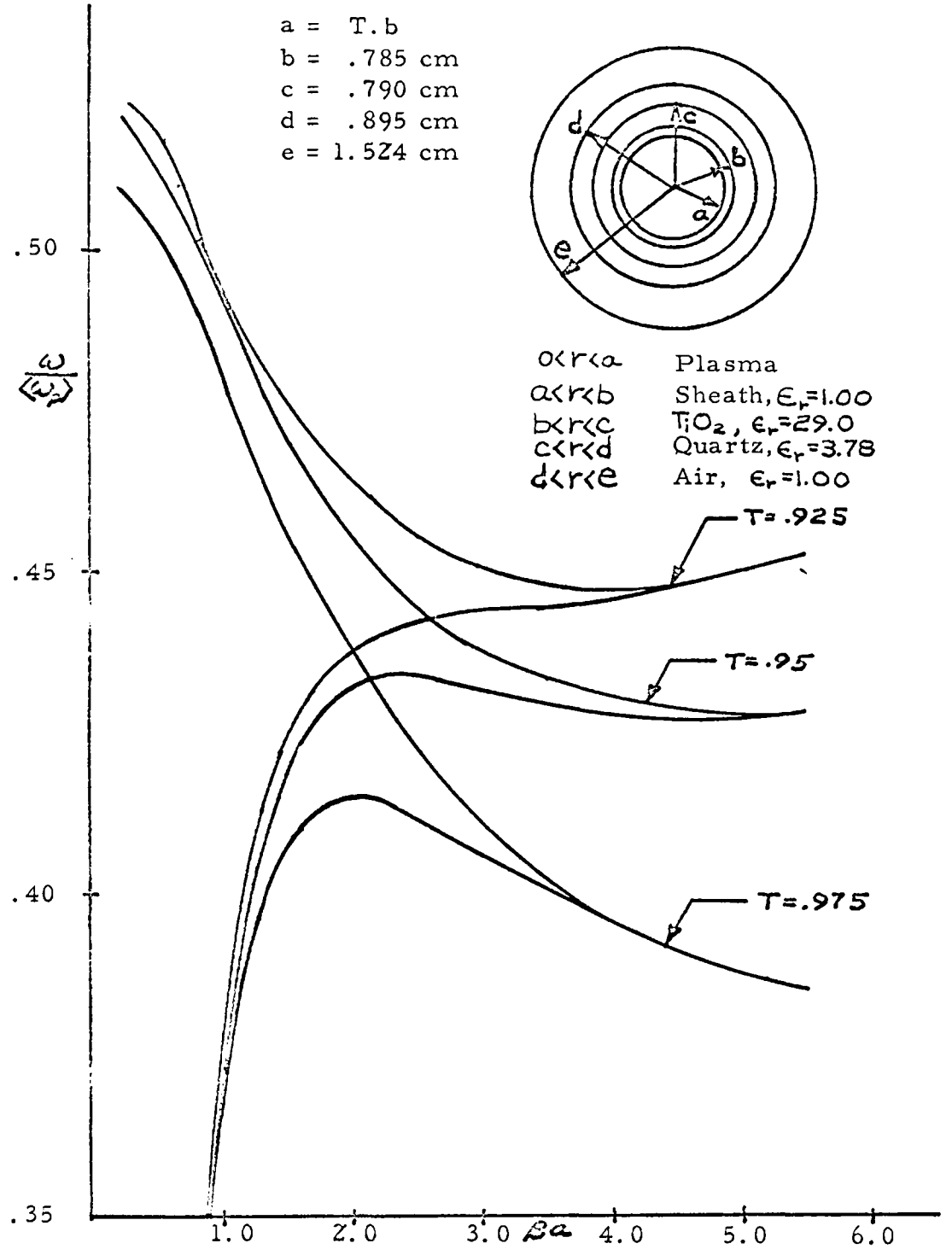

Figure 6. ω-β diagram of the symmetric and dipolar modes for varying sheath thickness and αT^2 = constant = 0.7 .

Figures 5 and 6 show that the dipolar mode is sensitive to the effects of the radial electron density variation and the sheath thickness for even the small values of βa whereas the symmetric mode is not. For this reason the symmetric mode can be used to measure average electron density, while the dipolar mode is more suited to investigating electron density variations.

III. EXPERIMENTAL INVESTIGATION

THE EXPERIMENTAL SYSTEM

The experimental system used in this work consisted of a cylindrical waveguide mounted coaxially about the positive column of a hot cathode mercury vapor discharge tube. The discharge tube was about 63 centimeters long and 1.79 centimeters in outer diameter. The waveguide is slotted to admit a small radial probe which samples the E_r component of the surface wave. The slot is filled by a sliding carriage which carries the probe. The wave is launched onto the tube by a surface wave coupler[2,6] which is mounted at the cathode end of the column. A tapered aqua-dag attenuator was painted on the tube at the anode end of the system to reduce reflections.

The wavelength of the surface wave was measured using a microwave bridge circuit. The signal from the probe is added to a reference signal from the oscillator whose amplitude and phase are adjustable by means of a variable attenuator and a line stretcher. The magnitude of the sum of these two signals is measured by a standing wave meter after demodulation by a crystal detector. The reference signal can be adjusted to be equal in magnitude to the signal coming from the probe, but opposite in phase. This creates a null on the

standing wave meter. Similar nulls are found at one wavelength intervals as the probe is moved down the tube. In the actual system the probe carriage is driven down the tube by an electric motor, and a d.c. voltage proportional to the probe position is used as the x-axis input to an oscilloscope. A signal proportional to the standing wave meter reading is used as the y-axis input to the oscilloscope. A photograph of the face of the oscilloscope is taken as the probe moves down the tube. The distance between the minima on the photograph is made equal to a wavelength of the surface wave.

Gould[10] has suggested that the orientation of small magnetic fields (such as that of the earth) with respect to a plasma discharge can affect the propagation of plasma waves. To eliminate the possibility of day to day variations in the orientation of the earth's magnetic field in the vicinity of the plasma waveguide system, two sets of Helmholtz coils were mounted adjacent to the waveguide. One set was mounted to produce a field along the tube axis, and the second set was mounted to produce a field perpendicular to the tube axis. These two sets of coils were used to keep the field in the vicinity of the waveguide completely transverse to the axis of the waveguide.

THE EXPERIMENTAL METHOD

The experimental approach used in this paper consisted of measuring frequency versus wavelength for the system. The

propagation constant, β , is then computed from the wavelength ($\beta = 2\pi/\lambda$) and normalized by multiplying it by the tube radius. The frequency is normalized to the average plasma frequency. This average plasma frequency is determined by using the technique developed by Carlile[7], et al. This method uses the symmetric surface wave mode at frequencies such that $\beta a < 1.0$, and has been shown to have an error of less than 5% . This normalized experimental data is then compared to thoretical curves which were obtained for various values of the two parameters α and T which completely characterize our model of the positive column as discussed in the previous chapter. The actual value of α and T in the discharge tube is that combination of α and T whose theoretical curve best fits the experimental data.

THE EXPERIMENTAL RESULTS

In figure 7 the experimental data points are shown plotted against the theoretical curve for $\alpha = .775$ and $T = 0.95$. I_D is the tube discharge current and B_T is the magnitude of the residual magnetic field in the vicinity of the tube.

It is possible to change both α and T by small amounts and still obtain a reasonable fit between experimental and theoretical data. The experimental data shown in figure 7 will also give a reasonable fit to theoretical curves for $\alpha = 0.70$, $T = 0.9625$ or

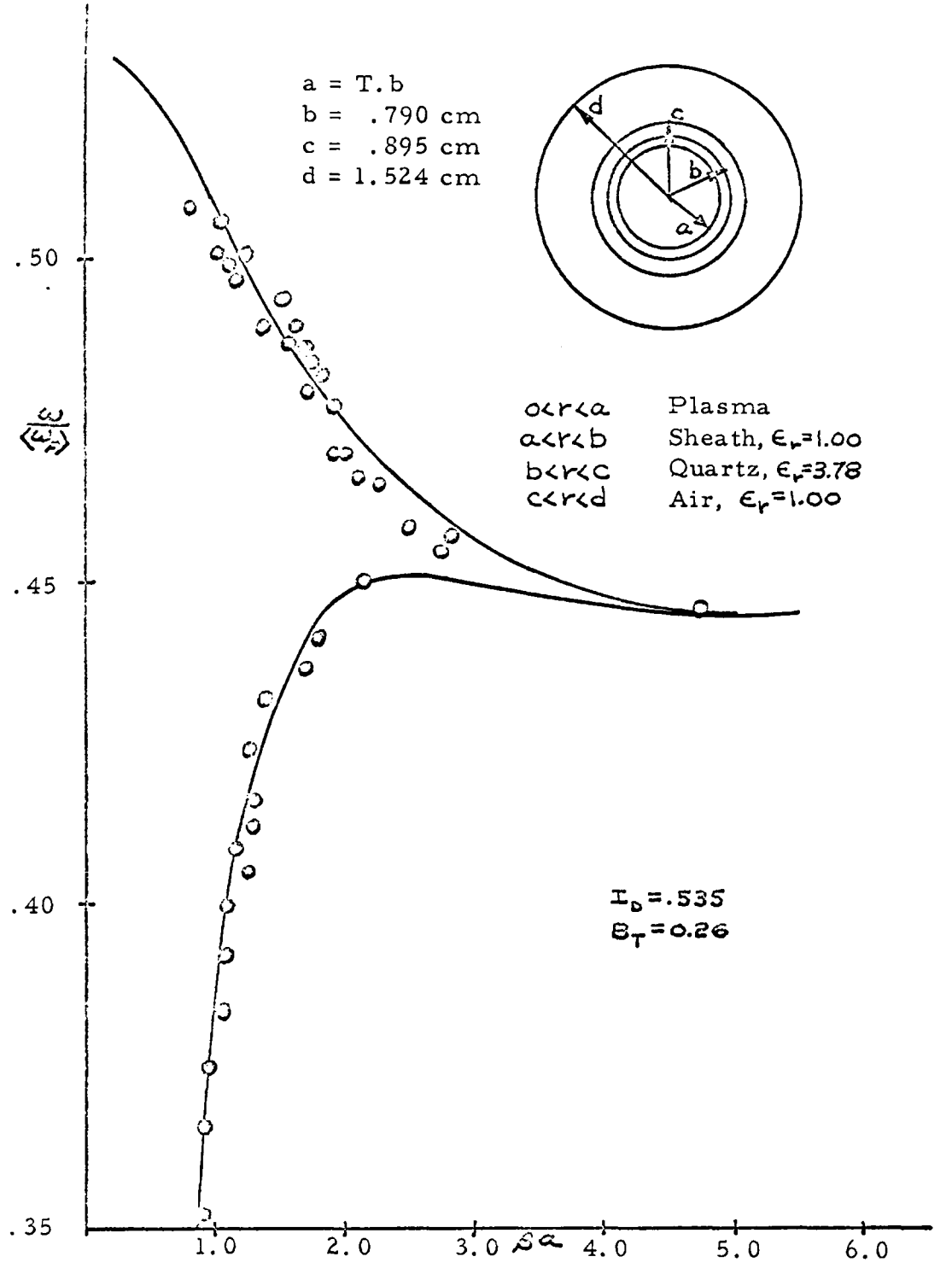

Figure 7. Experimental data points plotted against theoretical curves for α = .775 and T = 0.95 .

$\alpha = .876$, $T = 0.925$. This is shown in figure 8. These values of α and T lie on either side of the nominal values of $\alpha = .775$ and $T = 0.95$. They give an estimate of the possible range in which the true α and T must be. The best conclusion that can be obtained from the experimental data, therefore, is that $0.70 \leq \alpha \leq 0.876$ and $0.925 \leq T \leq 0.9625$ with the nominal values being $\alpha = .775$ and $T = 0.95$. It should be pointed out that this accuracy in T is the result of obtaining experimental data points at values of βa which are greater than 3.0. Without large βa points it is difficult to determine which of the curves for varying T is best fitted by the experimental data due to the scattering of the dipolar mode experimental data as can be seen in figure 8.

It is of interest to compare these experimental results with the theoretical prediction of electron density variation due to Parker[9]. To do this one must determine the value of Parker's parameter, ξ^2, for the experimental conditions. At the discharge currents used for these experiments the average plasma frequency was about 14.5×10^9 radians/sec. If one assumes an electron temperature of $20,000^{\circ}K$, ξ^2 is 4.3×10^4. Parker's theoretical curve for this value of ξ^2 was shown in figure 4 along with the approximation to this curve as given by our model. The values of α and T for this approximation in figure 4 are $\alpha = 0.70$ and $T = 0.95$. Our experimentally

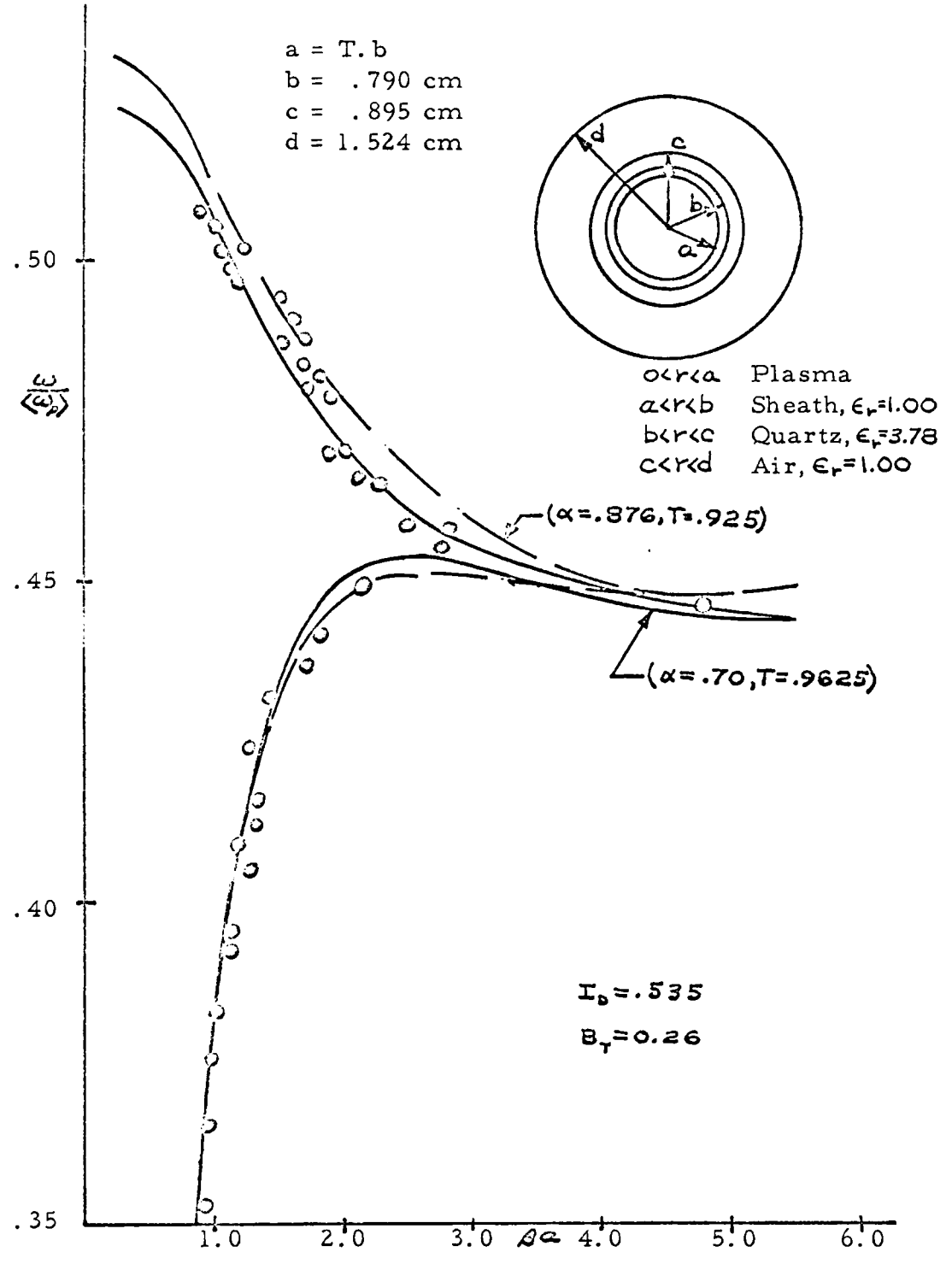

Figure 8. Experimental data points plotted against theoretical curves for $\alpha = .70$, $T = .9625$ and $\alpha = .876$, $T = .925$.

determined radial variation in the electron density is therefore in agreement with the theoretically predicted variation.

IV. PROPAGATION OF A PLASMA SURFACE WAVE ACROSS A TRANSVERSE AMBIPOLAR CURRENT

In the qualitative discussion of the positive column in Chapter II it was mentioned that there is a continuous flow of ions and electrons to the wall enclosing the discharge. This current flow is transverse to the direction of propagation of the plasma surface wave. Since this current consists of both positive and negative charge carriers we shall henceforth refer to it as an ambipolor current. We now wish to determine the effect of this transverse ambipolar current on the plasma surface wave. To do this we shall consider an elementary model of this current flow in rectangular geometry.

SYMBOLS

The following symbols will be used throughout this chapter.

n_{eo}	steady state electron density
$\tilde{n}_e$	r.f. portion of electron density
n_e	total electron density
n_{io}	steady state ion density
$\tilde{n}_i$	r.f. portion of ion density
n_i	total ion density

u_e	electron drift velocity
$\tilde{v}_e$	r.f. portion of electron velocity
$\bar{v}_e$	total electron velocity
u_i	ion drift velocity
$\tilde{v}_i$	r.f. portion of ion velocity
$\bar{v}_i$	total ion velocity
m_e	electron mass
m_i	ion mass
ω	frequency in radians/sec.
$\bar{k}$	wave number in the plasma
$\bar{k}_1$	wave number in the dielectric
$\bar{a}_x$, $\bar{a}_y$, $\bar{a}_z$	unit vectors in cartesian coordinates
K_1	relative permittivity of the dielectric
$\bar{j}_i$	r.f. ion current density
$\bar{j}_e$	r.f. electron current density
Π_e	electron plasma frequency
Π_i	ion plasma frequency
ϵ_o	permittivity of a vacuum
μ_o	permeability of a vacuum
c	speed of light in a vacuum
e	charge of an electron
$\bar{E}$	r.f. electric field vector

$\overline{H}$ r.f. magnetic field vector

DERIVATION OF THE EQUIVALENT DIELECTRIC CONSTANT FOR THE PLASMA

Consider the geometry shown in figure 9. The half-space $x>0$ is a semi-infinite plasma consisting of singly ionized ions and electrons. We shall assume that the ions have a steady drift speed, u_i, toward the wall and that the electrons have a steady drift speed, u_e, toward the wall. We shall also assume that the plasma is homogeneous. The half-space $x<0$ is a semi-infinite ideal dielectric. We shall consider

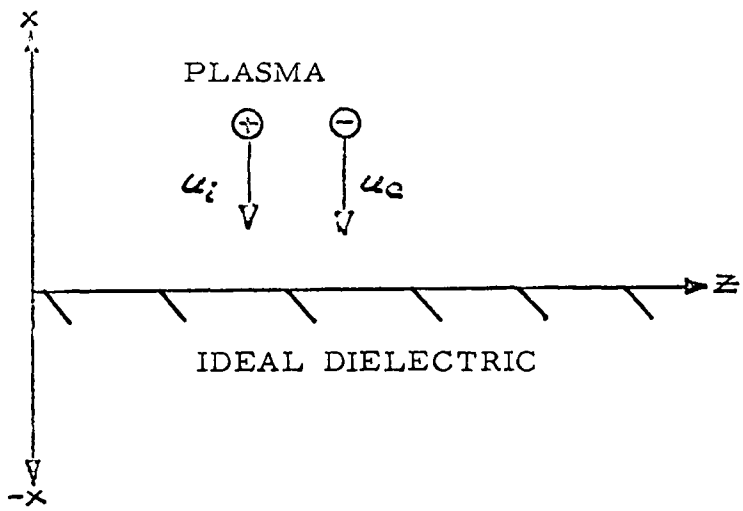

Figure 9. Geometry of the problem.

a plasma surface wave propagating in the positive z direction along the plasma-dielectric interface. We shall assume a variation of $\exp(i\omega t-\bar{k}\cdot\bar{r})$ in the plasma and $\exp(i\omega t-\bar{k}_1\cdot\bar{r})$ in the dielectric, where $\bar{k}=-\alpha\bar{a}_x+i\beta\bar{a}_z$, $\bar{k}_1=-\alpha_1\bar{a}_x+i\beta_1\bar{a}_z$ and $\bar{r}=x\bar{a}_x+z\bar{a}_z$. We also assume that the total ion and electron number densities and velocities consist of a non-varying or d.c. part plus an r.f. portion which is due to the propagating wave. It should be pointed out that all equations will be written in linearized form, that is, products of r.f. quantities will be neglected.

We write the total electron and the total ion number densities as

$$n_e=n_{eo}+\tilde{n}_e \qquad \text{IV.1}$$

$$n_i=n_{io}+\tilde{n}_i \qquad \text{IV.2}$$

and the total electron and the total ion velocities as

$$\bar{v}_e=-u_e\,\bar{a}_x+\tilde{v}_e \qquad \text{II.3}$$

$$\bar{v}_i=-u_i\,\bar{a}_x+\tilde{v}_i \qquad \text{II.4}$$

The equation of motion for the electrons is

$$m_e\frac{d\bar{v}_e}{dt}=m_e\left[\frac{\partial\bar{v}_e}{\partial t}+(\bar{v}_e\cdot\nabla)\bar{v}_e\right]=-e\bar{E} \qquad \text{II.5}$$

where we have neglected the Lorentz force on the electrons due to the time varying magnetic field. This is possible if the electrons are non-relativistic. Equation IV.5 may be expanded to

$$m_e\left[\frac{\partial \tilde{v}_e}{\partial t} - u_e \frac{\partial}{\partial x}(-u_e \bar{a}_x + \tilde{v}_e) + (\tilde{v}_e \cdot \nabla)(-u_e \bar{a}_x + \tilde{v}_e)\right] \quad . \qquad \text{II.6}$$

The third term in the brackets is a second order term which we neglect to get

$$m_e\left[i\omega \tilde{v}_e - u_e \alpha \tilde{v}_e\right] = -e\bar{E}$$

and therefore

$$\tilde{v}_e = \frac{-e\bar{E}}{m_e(i\omega - \alpha u_e)} \quad . \qquad \text{II.7}$$

By similar reasoning we can obtain an expression for the r.f. ion velocity as

$$\tilde{v}_i = \frac{e\bar{E}}{m_i(i\omega - \alpha u_i)} \quad . \qquad \text{II.8}$$

Now the linearized, r.f. electron and ion current densities are

$$\bar{j}_e = -n_{e_o} e \tilde{v}_e + \tilde{n}_e e u_e \bar{a}_x$$

and

$$\bar{j}_i = n_{i_o} e \tilde{v}_i - \tilde{n}_i e u_i \bar{a}_x \quad .$$

Substituting equations IV.7 and IV.8 the current densities become

$$\bar{j}_e = \frac{\epsilon_o \Pi_e^2 \bar{E}}{(i\omega - \alpha u_e)} + e\tilde{n}_e u_e \bar{a}_x \qquad \text{II.9}$$

and

$$\bar{j}_i = \frac{\epsilon_0 \Pi_i^2 \bar{E}}{(i\omega - \alpha u_i)} - e\tilde{n}_i u_i \bar{a}_x \quad . \qquad \text{II. 10}$$

We may obtain an expression for the perturbed electron density, $\tilde{n}_e$, from the continuity equation for the electron current,

$$\nabla \cdot \bar{j}_e + i\omega(-e\tilde{n}_e) = 0 \quad .$$

Substituting equation IV. 9 and expanding we obtain

$$\frac{\epsilon_0 \Pi_e^2 \nabla \cdot \bar{E}}{(i\omega - \alpha u_e)} - e\tilde{n}_e(i\omega - \alpha u_e) = 0$$

and therefore

$$e\tilde{n}_e = \frac{\epsilon_0 \Pi_e^2}{(i\omega - \alpha u_e)^2} \nabla \cdot \bar{E} \qquad \text{II. 11}$$

Similarly for ions we find

$$-e\tilde{n}_i = \frac{\epsilon_0 \Pi_i^2}{(i\omega - \alpha u_i)^2} \nabla \cdot \bar{E} \qquad \text{II. 12}$$

Now the divergence of the r.f. electric field is

$$\nabla \cdot \bar{E} = e/\epsilon_0(\tilde{n}_i - \tilde{n}_e) \quad . \qquad \text{IV. 13}$$

Upon substitution of equations IV. 11 and IV. 12 into equation IV. 13 we obtain

$$\left\{1 + \frac{\Pi_e^2}{(i\omega - \alpha u_e)^2} + \frac{\Pi_i^2}{(i\omega - \alpha u_i)^2}\right\} \nabla \cdot \bar{E} = 0 \quad . \qquad \text{IV. 14}$$

Now either the expression in brackets is zero or the divergence of the r.f. electric field is zero. Setting the expression in brackets equal to zero leads to the double stream amplifier solution[11, 12] when u_e and u_i are not zero. Should u_e and u_i be zero we would obtain the plasma oscillations of Tonks and Langmuir.[13] In our analysis we have assumed discrete velocities for the ions and electrons. Had we assumed an infinite number of discrete velocities or a distribution in velocity for the electrons we would have obtained the dispersion relation of Bohm and Gross[14] by setting the corresponding bracketed expression equal to zero. Setting $\nabla \cdot \bar{E} = 0$ leads to the surface wave solution. We note then that $\tilde{n}_e = \tilde{n}_i$.

The total r.f. current density is the sum of the electron and ion current densities which is

$$\bar{J} = \left[\frac{\Pi_e^2}{(i\omega - \alpha u_e)} + \frac{\Pi_i^2}{(i\omega - \alpha u_i)}\right] \epsilon_0 \bar{E} + e\left[\tilde{n}_e u_e - \tilde{n}_i u_i\right] \bar{a}_x \ .$$

Applying the continuity equation to this total current density leads to the expression

$$\tilde{n}_e u_e - \tilde{n}_i u_i = 0 \qquad \text{IV. 15}$$

Since $\tilde{n}_e = \tilde{n}_i$ and in general u_e and u_i are not equal the solution to equation IV. 15 must be

$$\tilde{n}_e = \tilde{n}_i = 0 \qquad \text{IV. 16}$$

Under this condition the total r.f. current density becomes

$$\bar{J} = \left[\frac{\Pi_e^2}{(i\omega - \alpha u_e)} + \frac{\Pi_i^2}{(i\omega - \alpha u_i)} \right] \epsilon_o \bar{E} \quad . \qquad \text{IV.17}$$

Equation IV. 17 relates the r.f. current density to the r.f. electric field. We now use this relationship to determine the equivalent dielectric constant for this plasma model.

One of Maxwell's equations is

$$\nabla \times \bar{H} = \bar{J} + i\omega \epsilon_o \bar{E} \quad .$$

Substituting for $\bar{J}$ we find

$$\nabla \times \bar{H} = i\omega \epsilon_o \left\{ 1 + \frac{\Pi_e^2}{i\omega(i\omega - \alpha u_e)} + \frac{\Pi_i^2}{i\omega(i\omega - \alpha u_i)} \right\} \bar{E} = i\omega \epsilon_o K_p \bar{E} \qquad \text{IV.18}$$

where

$$K_p = 1 + \frac{\Pi_e}{i\omega(i\omega - \alpha u_e)} + \frac{\Pi_i^2}{i\omega(i\omega - \alpha u_i)} \qquad \text{IV.19}$$

is the relative dielectric constant of the plasma for our plasma model.

THE DISPERSION RELATION

Maxwell's wave equation in the plasma half-space is

$$\nabla^2 \bar{E} + \frac{\omega^2}{c^2} K_p \bar{E} = 0$$

which with our assumed spacial and temporal variation becomes

(Note that $\partial/\partial y = 0$)

$$(\alpha^2 - \beta^2 + \frac{\omega^2}{c^2} K_p) \bar{E} = 0 \quad . \qquad \text{IV.20}$$

The non-trivial solution to equation IV. 20 is

$$\alpha^2 - \beta^2 + \frac{\omega^2}{c^2} K_p = 0 \quad . \qquad \text{IV.21}$$

Similarly in the dielectric half-space we find

$$\alpha_1^2 - \beta_1^2 + \frac{\omega^2}{c^2} K_1 = 0 \quad . \qquad \text{IV.22}$$

Since the tangential component of the electric field, E_z , must be continuous across the boundary at every point on the boundary

$$\beta_1 = \beta$$

and therefore

$$\alpha^2 + \frac{\omega^2}{c^2} K_p = \alpha_1^2 + \frac{\omega^2}{c^2} K_1 \quad . \qquad \text{IV.23}$$

The relationship between the two components of the electric field vector can be determined from the divergence equation

$$\nabla \cdot \bar{E} = \alpha E_x - i\beta E_z = 0 \quad . \qquad \text{IV.24}$$

Using equation IV. 24 and Maxwell's second equation,

$$\nabla \times \bar{E} = -i\omega\mu \bar{H} \quad , \qquad \text{IV.25}$$

we can obtain the magnetic field vector in the plasma,

$$\bar{H} = \frac{E_z}{i\omega\mu} \left[\frac{\alpha^2 - \beta^2}{\alpha} \right] e^{(i\omega t - \bar{k} \cdot \bar{r})} \bar{a}_y \quad , \qquad \text{IV.26}$$

and in the dielectric

$$\overline{H_1} = \frac{E_z}{i\omega\mu}\left[\frac{\alpha_1^2-\beta^2}{\alpha_1}\right] e^{(i\omega t-\bar{k}_1\cdot\bar{r})}\bar{a}_y \quad . \qquad \text{IV.27}$$

Requiring that the tangential magnetic field be continuous across the boundary leads to

$$\frac{\alpha_1^2-\beta^2}{\alpha_1} = \frac{\alpha^2-\beta^2}{\alpha} \qquad \text{IV.28}$$

Equations IV.23 and IV.28 can be combined to yield the dispersion relation,

$$\alpha^2 + \frac{\omega^2}{c^2}\frac{K_p^2}{K_p+K_1} = 0 \quad . \qquad \text{IV.29}$$

This is the simplest form of the dispersion relation. In reality we must recall that K_p is a function of α . When this substitution is made one obtains a sixth order polynomial in α which has been solved by numerical means to obtain values of α and consequently of β through equation IV.21.

The values of β obtained are complex

$$\beta = \beta' + i\beta'' \qquad \text{IV.30}$$

for all non-zero values of u_e and u_i . β' is the portion of β which results in phase shift as the wave progresses in the z-direction

and β'' is the portion of β which results in attenuation of the wave as it travels.

An ω-β' diagram is shown in figure 10 for various values of u_e. The dashed curve is for the case of $u_e = 0$. It is immediately apparent that the transverse ambipolar current has a definite effect on the propagation. For non-zero values of u_e, β' no longer goes to infinity as ω/π_e approaches $(1+K_1)^{-1/2}$ but instead reaches a finite value and then begins to decrease as ω gets larger. Carlile[2] found a similar ω-β' behavior in his investigation of the effects of collisions on the attenuation of a plasma surface wave. Indeed, this behavior may be characteristic of loss phenomena in general.

In figure 11 the attenuation in db. per wavelength is plotted versus frequency for various values of u_e. The dashed curve shows the attenuation which would result from collisions. We have assumed a collision frequency of $\nu/\pi_e = .001$.

Let us now extend our analysis to determine how a transverse ambipolar current might affect the propagation of a plasma surface wave along the positive column of a low pressure mercury vapor discharge. In the sheath region which is adjacent to the dielectric boundary in the positive column the ion density, n_{io}, exceeds the electron density, n_{eo}. Since the net d.c. current arriving at the wall must be zero, the electron drift velocity, u_e, must exceed the ion drift velocity,

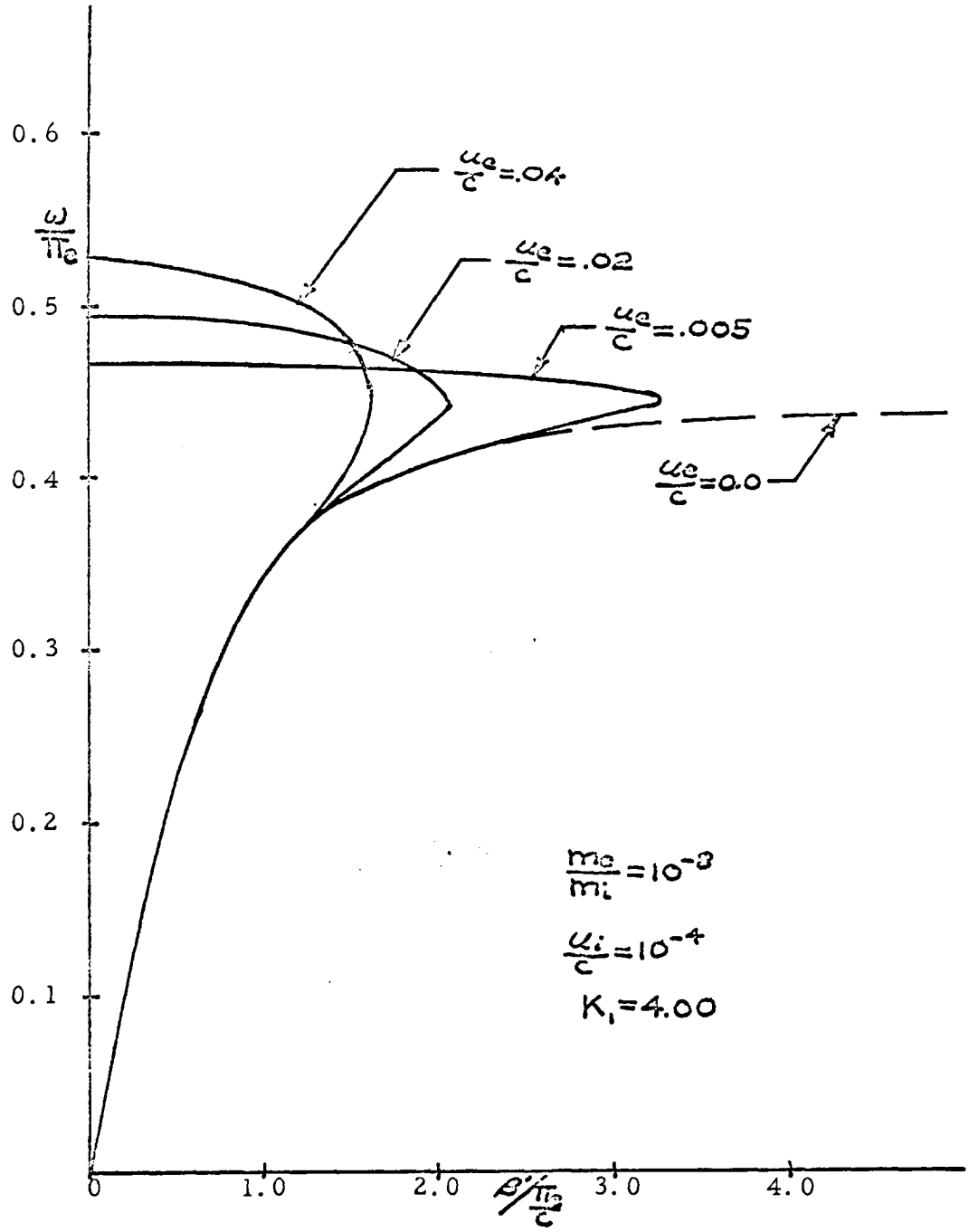

Figure 10. ω-β' curves for varying values of u_e.

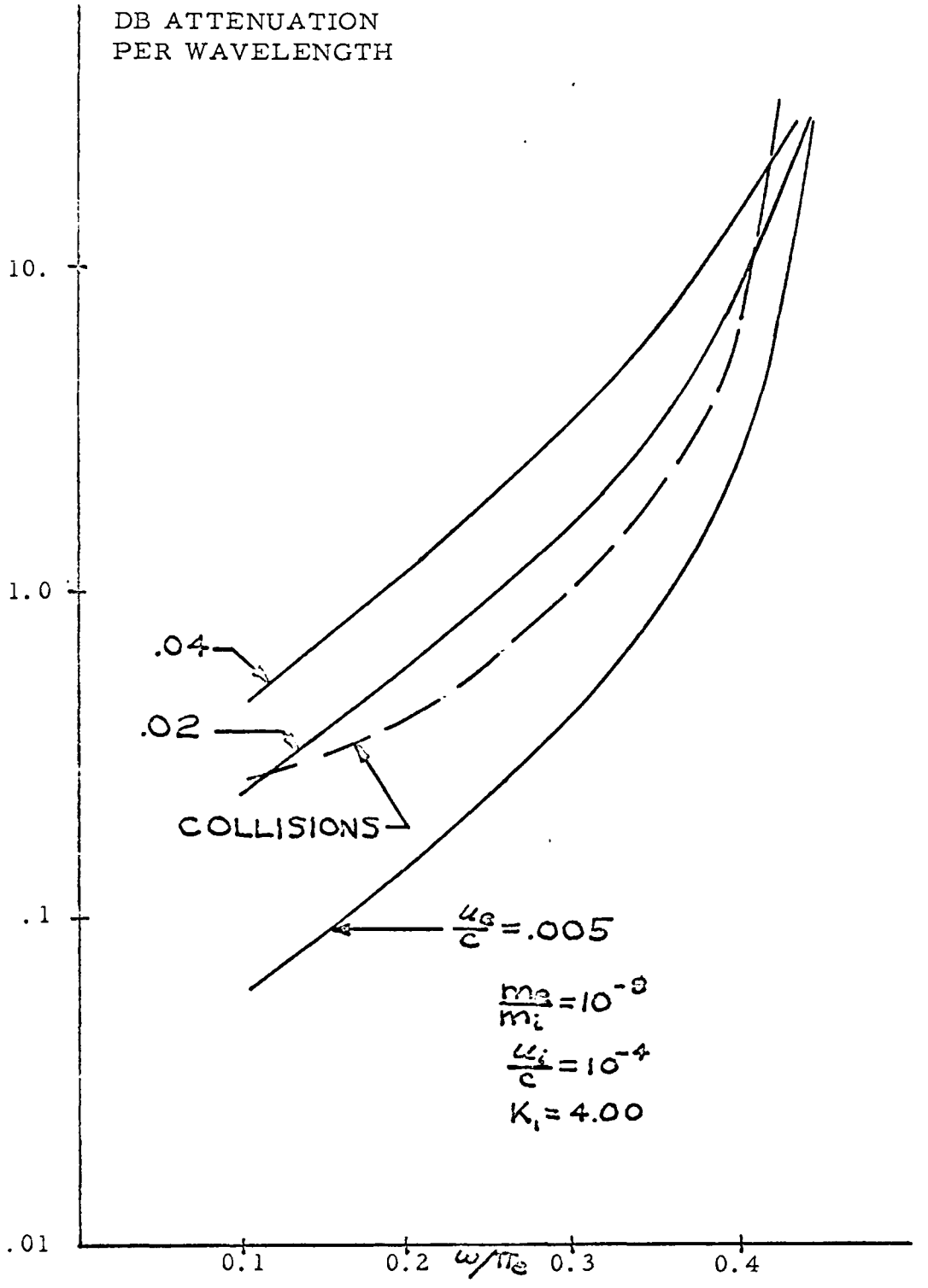

Figure 11. Attenuation in db per wavelength versus frequency for varying u_e .

u_i, by the same factor. In Crawford's[6] discussion of the mercury vapor discharge he gives a radial ion velocity of about 2×10^5 cm/sec for ions near the wall. If we assume that the ion density is ten times the electron density then the electron velocity must be 2×10^6 cm/sec. A ω-β'' diagram for this condition is shown in figure 12 and the attenuation versus frequency in figure 13. By comparing figures 11 and 13 we see that as ω/π_e varies from 0.1 to 0.4 the attenuation in db. per wavelength due to the ambipolar current ranges from 1.2×10^{-3} to about 5.0×10^{-2} whereas the attenuation due to collisions ranges from 2.7×10^{-1} to about 6.0 db. per wavelength. The attenuation due to collisions is two orders of magnitude higher than that due to the transverse ambipolar current. We therefore conclude that the loss due to the transverse ambipolar current is insignificant compared with that due to collisions.

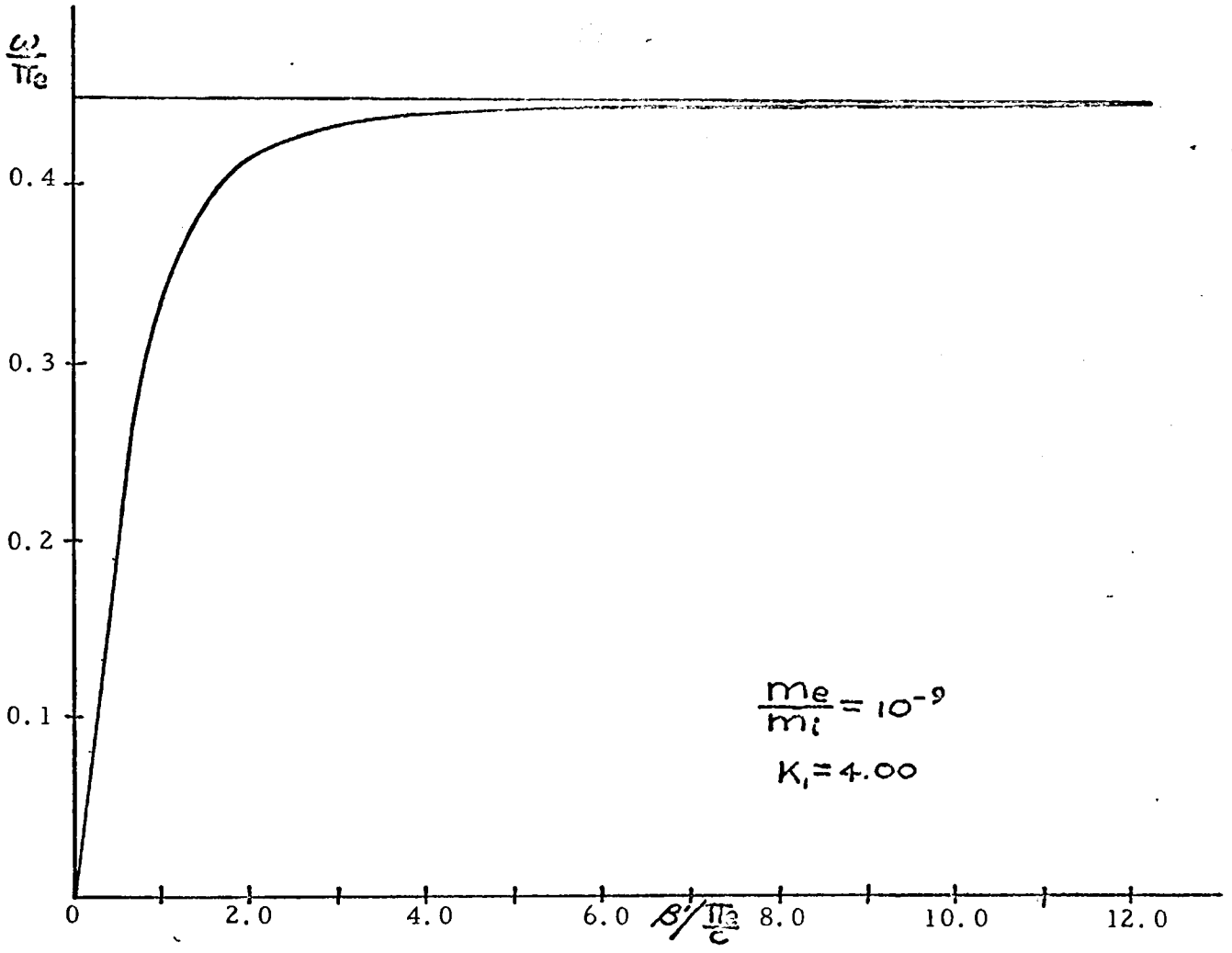

Figure 12. ω-β' diagram for $u_e = 2 \times 10^6$ cm/sec and $u_i = 2 \times 10^5$ cm/sec.

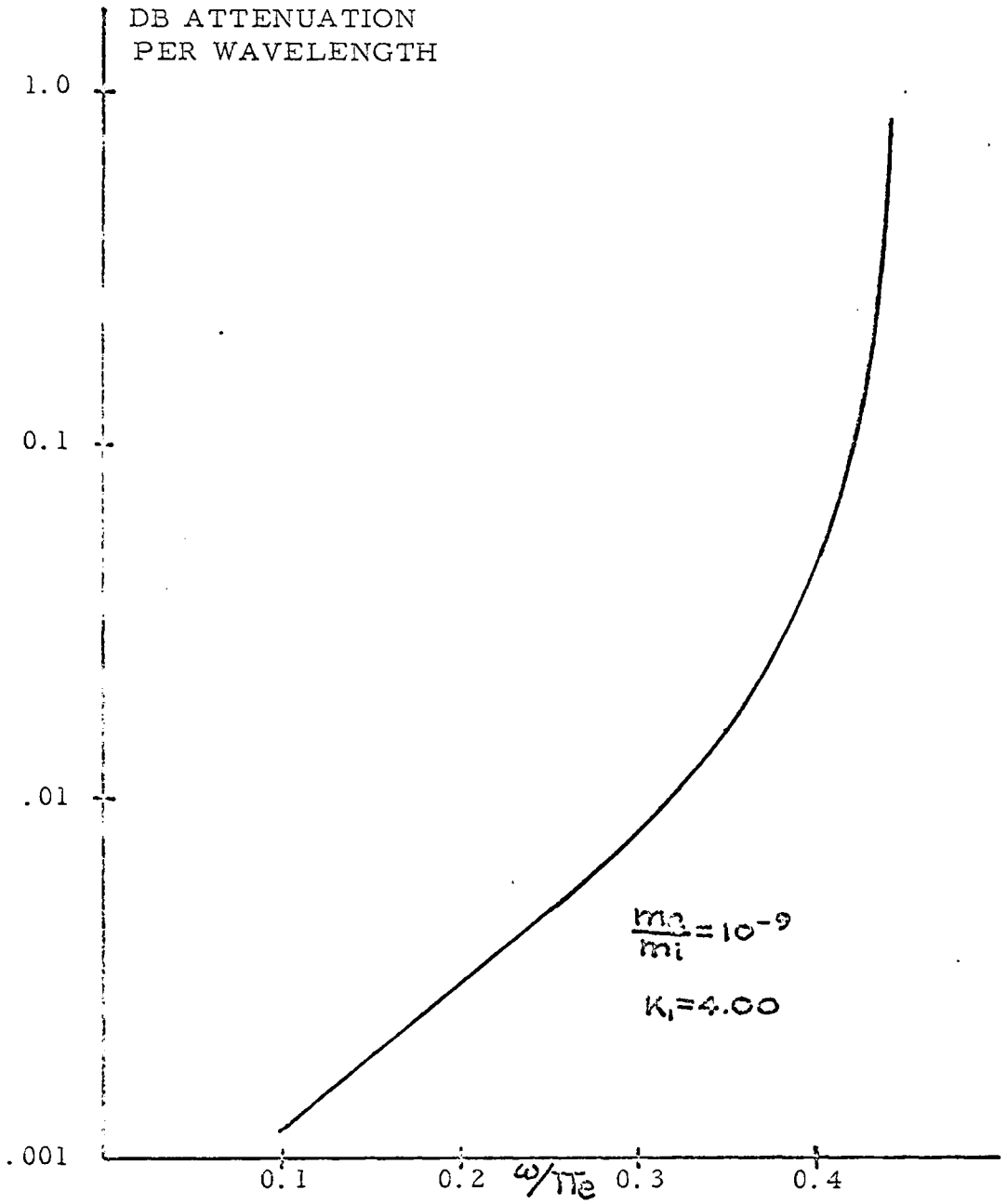

Figure 13. Attenuation in db per wavelength versus frequency for $u_e = 2 \times 10^6$ cm/sec and $u_i = 2 \times 10^5$ cm sec.

V. CONCLUSIONS

In this paper a model for the positive column has been presented. The disperson relation for this model placed coaxially in a cylindrical waveguide has been obtained and solved numerically. Although the efforts toward the application of surface waves to plasma diagnostics are preliminary in nature, it has been demonstrated that surface waves can be used to determine the radial variations in electron density in a mercury vapor plasma discharge.

The dispersion relationship for the propagation of a surface wave across a transverse ambipolar current has been derived and numerical solutions have been obtained. It has been demonstrated that this transverse current flow represents a loss mechanism to the surface wave. However, the attenuation due to this transverse current flow is probably small compared to the attenuation which results from collisions.

SUGGESTIONS FOR FURTHER STUDY

The experimental investigations described in this paper were performed exclusively using a mercury vapor discharge plasma. The value of Parker's parameter, ξ^2 , was 4.3×10^4 for this discharge.

It is extremely difficult to vary this parameter by more than an order of magnitude in a mercury vapor discharge. The investigation of radial electron density variations with plasma surface waves should be extended to discharges in gases in which the parameter, ξ^2 , can be varied over a range of several orders of magnitude. For lower values of ξ^2 such as 10^3, a new model of the discharge column would be required. The model presented in this paper assumes that the transition from the plasma region to the sheath region is rather sharp. At lower values of ξ^2 this transition is a gradual one, therefore necessitating the use of a new model

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