NOTES ON THE GENERALIZATION OF THE GEOMETRIC OPTICS OF ISOTROPIC MEDIA

by

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CHAPTER I

INTRODUCTION

Geometric optics is the phase of optics dealing with electromagnetic phenomena which can be adequately described by three laws: the laws of reflection, refraction, and rectilinear propagation. These laws, which follow directly from empirical evidence, may also be deduced from theoretical considerations.

Employing Maxwell's field equations, one can formulate the wave equations which provide the laws of reflection and refraction at the boundary of two dielectrics. The assumption that the electric field possesses a simple periodic dependence on time leads to the scalar wave equation.[1] If it is further assumed that the electric field varies in a simple periodic manner with position, one arrives at an equation which, upon deletion[2] of several terms, takes the form of Hamilton's equation,* $\nabla S \cdot \nabla S = \mu^2$. The optical path can be deduced[4] from Hamilton's equation.

Another approach to geometric optics lies in the application and analysis of the construction of Huygens.[5] Hamilton's equation and the laws of refraction and reflection can be derived[6] from an

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*Bruns' equation of the eiconal corresponds to Hamilton's equation, but Bruns derived the equation about sixty years after Hamilton. We will accept Synge's claims[3] that Hamilton's name and not Bruns' should be associated with the equation.

analysis of Huygen's construction.

The principle of Fermat affords another means[7] of obtaining the equations of geometric optics. Using Fermat's principle and the calculus of variations, one obtains[8] an elegant description of the rays in heterogenous anisotropic media.

Of great importance is the fact that in using the methods of geometric optics, one is accepting the equation of Hamilton as a valid description of the wave phonomena considered. Freehafer has shown[9] that there are two conditions to be satisfied if this equation is to yield an adequate approximation to the scalar wave equation. In effect, the first condition states that the fractional change in index of refraction within a distance of one wavelength must be very much smaller than unity. The second condition requires that the fractional change in cross section of a ray bundle within a path distance of one wavelength must also be very much smaller than unity. A further requirement is that the cross section of a ray bundle does not vanish. It is therefore necessary to carefully examine each problem to determine the applicability of geometric optics.

Synge utilizes[10] Fermat's principle as a basis for some very enlightening mathematical discussions of Hamilton's method in geometric optics. It is the purpose of this thesis to extend parts of Synge's discussions to curvilinear coordinates, and to more fully discuss the analogies between optics and mechanics. The solutions of several optical problems have been included to facilitate application of the equations.

CHAPTER II

OUTLINED DERIVATION OF THE GENERALIZED RAY EQUATIONS FOR A SYSTEM OF HETEROGENEOUS ISOTROPIC MEDIA

Let $\mu(x^r)$ represent the index of refraction of the system and let the infinitesimal distance along the ray be ds. Latin indices will be employed for the range 1,2,3; summation is implied when indices are repeated. The optical path length from S_1 to S_2 is defined as $\int_{S_1}^{S_2} \mu \, ds$ and will be denoted by $[S_1 S_2]$. For an arbitrary curvilinear coordinate system

$$ds = (g_{mn}\, dx^m dx^n)^{\frac{1}{2}}, \tag{1}$$

where g_{mn} is the metric tensor. Equation (1) may also be written

$$\begin{aligned} ds &= (g_{mn}\, \dot{x}^m \dot{x}^n)^{\frac{1}{2}} ds \\ &= (g_{mn}\, \dot{x}^m \dot{x}^n)\, ds , \end{aligned} \tag{2}$$

where $\dot{x}^i = \frac{dx^i}{ds}$, and the bracketed quantity is always equal to unity. A general expression for the optical path length is then

$$\begin{aligned} [S_1 S_2] &= \int_{S_1}^{S_2} \mu\, (g_{mn}\, \dot{x}^m \dot{x}^n)^{\frac{1}{2}} ds \\ &= \int_{S_1}^{S_2} (\bar{g}_{mn}\, \dot{x}^m \dot{x}^n)^{\frac{1}{2}} ds , \end{aligned} \tag{3}$$

where $\bar{g}_{mn} = \mu^2 g_{mn}$. Note that the element of optical path length is just the line element in a coordinate system with metric tensor $\bar{g}_{mn}$.

Fermat's principle requires the optical path length to be an extremum. To extremize the path length, the methods of the calculus of variations are applied[11] to (3) in a manner which allows no variation at the points S_1 and S_2. For the sake of generality it is assumed that $\mu(\chi^r)$ is piecewise continuous. In regions where μ is continuous the problem is that of finding a function

$$\psi = (\bar{g}_{mn}\,\dot{\chi}^m \dot{\chi}^n)^{\frac{1}{2}} \tag{4}$$

which will satisfy the Euler-Lagrange differential equations

$$\frac{\partial \psi}{\partial \chi^r} - \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) = 0\,. \tag{5}$$

In the language of differential geometry Fermat's principle states that the optical path is a geodesic in a coordinate system with metric $\bar{g}_{mn} = \mu^2 g_{mn}$. The assumption of piecewise continuity of $\mu(\chi^r)$ yields the equation

$$\left(\Delta \frac{\partial \psi}{\partial \dot{\chi}^r}\right)\delta\chi^r = 0 \tag{6}$$

which combines the laws of reflection and refraction at each surface of discontinuity of μ. The term in parentheses in equation (6) is the increment of $\frac{\partial \psi}{\partial \dot{\chi}^r}$ across the surface of discontinuity, and $\delta\chi^r$ denotes an arbitrary variation of coordinates; the variation taken on the surface of discontinuity.

If a surface is defined by the equation $P(\chi^r) = 0$, then the unit gradient of P is the unit normal to the surface. The scalar product of the unit normal at a point with a vector tangent to the surface at that point must vanish, so that the statement

$$g_{nr}\, n^n \delta\chi^r = (\partial P/\partial \chi^t)\left[g^{ij}(\partial P/\partial \chi^i)(\partial P/\partial \chi^j)\right]^{-\frac{1}{2}} \delta\chi^t \tag{7}$$

must follow. Let P be the surface of discontinuity and let $\delta\chi^r$ denote a vector tangent to P. The vector will represent infinitesimal coordinate variations at the point of tangency. With the use of (4) and (7), equations (5) and (6) can be put into the forms

$$\frac{\partial \mu}{\partial x^p} = \mu \dot{x}^m \dot{x}^n [mn, p] + g_{jp} \frac{d}{ds}(\mu \dot{x}^j) \qquad (8)$$

$$\Delta(\mu \dot{x}^j) = \theta g^{ji} \partial P/\partial x^i \left[g^{mn} (\partial P/\partial x^m)(\partial P/\partial x^n) \right]^{-\frac{1}{2}}, \qquad (9)$$

where $[mn, p]$ is the Christoffel symbol of the first kind, and θ is an undetermined scalar multiplier. Derivations of (8) and (9) are found in appendix A, and their expressions in several coordinate systems are given in appendix B. In appendix C it is shown that the laws of reflection, refraction and rectilinear propagation follow from the four equations, (8) and (9).

The solutions of (8) and (9) afford a complete description of the ray through a system of heterogeneous isotropic media; however, the problem of solving (8) is not generally a simple matter. For the purpose of obtaining a solution for (8) the methods of Hamilton and Jacobi are employed, for these methods reduce the problem to one of solving first order ordinary differential equations.

As a first step in the development of the Hamilton-Jacobi equation, notice that equation (3) may be written

$$[S_1 S_2] = \int_{x^r_{(1)}}^{x^r_{(2)}} \mu g_{mn} \dot{x}^m \, dx^n. \qquad (10)$$

Assume that there is at least one ray which will pass from S_1 to S_2. Then S_1 and S_2 determine a ray which generates the optical path length $[S_1 S_2]$, and the optical path length is thus a characteristic function of the system. The value of the characteristic function depends only on the choice of end points S_1 and S_2. Let $V(S_1 S_2)$ denote the characteristic function defined as

$$V(s,s_2) = \int_{x^r_{(1)}}^{x^r_{(2)}} \left(\partial \bar{V}/\partial x^i \right) dx^i \equiv [s_1 s_2]. \tag{11}$$

The limits of the integrals in (10) and (11) are arbitrary and in appendix D it is shown that (10) and (11) require

$$\frac{\partial \bar{V}}{\partial x^i} = \mu g_{mi} \dot{x}^m, \tag{12}$$

and that (12) provides

$$g^{nj} \frac{\partial V}{\partial x^n} \frac{\partial \bar{V}}{\partial x^j} = \mu^2. \tag{13}$$

This is the equation of Hamilton which was investigated by Jacobi; the solution of the equation when substituted into (12) corresponds to reducing (8) to first order ordinary differential equations. To solve (13) assume

$$\bar{V} = V_1(x^1) + V_2(x^2) + \bar{V}_3(x^3) \tag{14}$$

and that μ is of a form which makes (13) separable. If these are valid assumptions, (13) may be written

$$F\left(\partial \bar{V}/\partial x^1, x^1, \alpha^1\right) = G\left(\partial \bar{V}/\partial x^2, \partial \bar{V}/\partial x^3, x^2, x^3, \alpha^1\right) = \alpha^2, \tag{15}$$

where α^r are constants; and G may then be separated into

$$H\left(\partial \bar{V}/\partial x^2, x^2, \alpha^1, \alpha^2\right) = I\left(\partial \bar{V}/\partial x^3, x^3, \alpha^1, \alpha^2\right) = \alpha^3. \tag{16}$$

Eliminate V from F, H, and I by using (12). The ray problem is then reduced to solving a system of first order ordinary differential equations. More complete treatments of the solution of the Hamilton-Jacobi equation are found in references.[12]

CHAPTER III

DISCUSSION OF OPTICAL-MECHANICAL ANALOGIES

In a generalized[13] coordinate system consider the motion of a mass point[13] with potential energy $\varphi(q^r)$ and kinetic energy.

$$T = \tfrac{1}{2} m g_{rs} \dot{q}^r \dot{q}^s \tag{16}$$

where now $\dot{q}^r = dq^r/dt$. Note that φ is independent of $\dot{q}^r$ so that the Lagrangian, defined to be $L \equiv T - \varphi$ is homogeneous quadratic in $\dot{q}^r$. The generalized momenta[13] p_r are then given by

$$\frac{\partial L}{\partial \dot{q}^r} = p_r = \frac{\partial T}{\partial \dot{q}^r}, \tag{17}$$

and by Euler's theorem[14]

$$p_r \dot{q}^r = 2T. \tag{18}$$

The Hamiltonian is defined by

$$H \equiv p_r \dot{q}^r - L = T + \varphi, \tag{19}$$

and is shown[15] to be a constant if $\partial L/\partial t = 0$. For a conservative system, this constant is the total energy.

Hamilton's principle requires that $\int_{t_1}^{t_2} L\, dt$ be an extremum; so that (19) provides

$$\delta \int_{t_1}^{t_2} (p_r \dot{q}^r - H)\, dt = 0. \tag{20}$$

If $\partial L/\partial t = 0$, then (20) becomes

$$\delta \int_{t_1(q)}^{t_2(q)} p_r\, dq^r = 0. \tag{21}$$

It is now desired to determine a function which characterizes[13] the motion of a mass point. The function, like V, is to depend only on the end points of the trajectory. The function satisfying the above requirements is shown [16] to be the action

$$A \equiv \int_{t_1}^{t_2} 2T\, dt = \int_{t_1(q)}^{t_2(q)} p_r\, dq^r,$$

and by a proof similar to that in appendix D

$$\partial A / \partial q^s = m\, g_{rs}\, \dot{q}^r$$

$$= p_s ,$$

and

$$(\partial A/\partial q^s)(\partial A/\partial q^i)\, g^{si} = 2m\,(E - \Phi), \tag{22}$$

where E is the total energy for a conservative system. Equation (22) represents the Hamilton-Jacobi equation for a mechanical system, and the development of the equation follows closely the development of Hamilton's equation for optical systems. A comparison of (13) and (22) leads to the conclusion that a light ray in Euclidean space corresponds to the trajectory of a mass point in configuration space; provided

$$\mu = [2m\,(E - \Phi)]^{\frac{1}{2}}$$

A thorough analysis of the optical-mechanical analogies may be found in references.[17]

CHAPTER IV

APPLICATION OF THE RAY EQUATIONS

A. Slab with Linearly Varying Refractive Index

Consider a heterogeneous isotropic medium in the form of a slab. Let the slab be bounded by a homogeneous isotropic medium. It is desired to know the deviation ϵ of an arbitrary ray that has passed through the slab as indicated in Fig. 1.

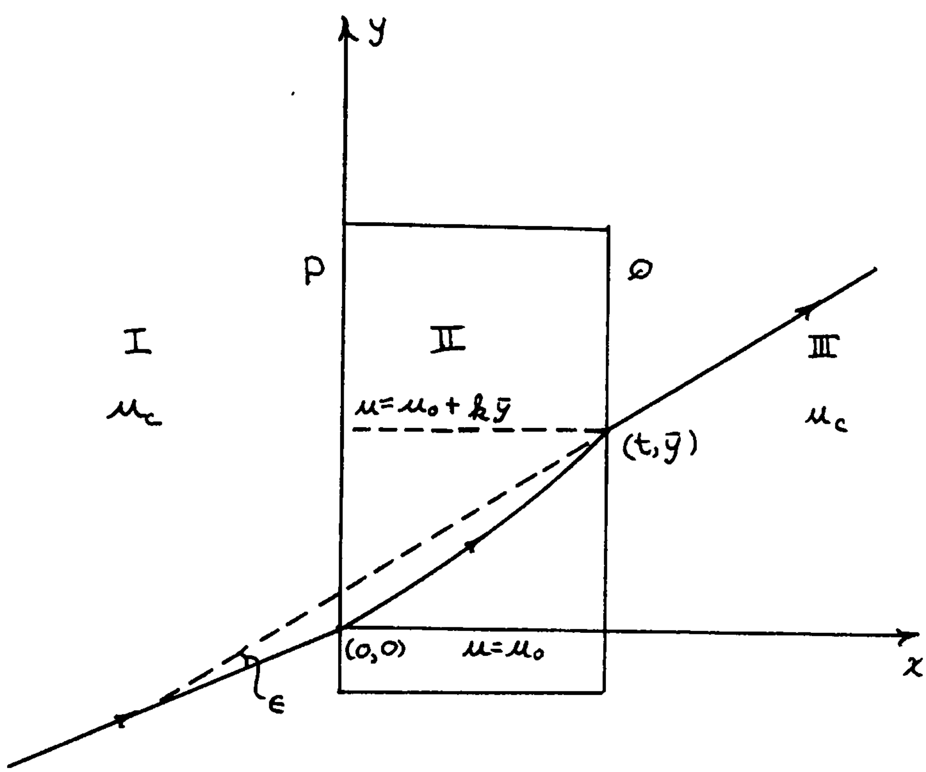

Fig. 1

The slab is oriented with edges perpendicular to the xy plane and the incident ray is restricted to the xy plane; consequently the problem is reduced to two dimensions.

The following notation will be employed:

t = the thickness of the slab

λ_0 = wavelength (in vacuum) of electromagnetic radiation considered

$\beta = 2\pi/\lambda_0$ = a constant

μ_c = a constant = the index of the bounding medium

$\mu = \mu(y)$

$= \mu_0 + k y''$, where μ_0 and k are known constants characteristic of the slab

P and Q are the lines of intersection of the surfaces of discontinuity and the xy plane

y' and x'; y'' and x''; y''' and x'''; denote the ray coordinates in regions I, II, and III respectively (see Fig. 1)

A dot over a coordinate, e.g. $\dot{x}'$ signifies $\frac{dx'}{ds}$, which is a direction cosine of the tangent to the ray

$\dot{x}'_{(P)}$ means $\dot{x}'$ evaluated on the line P

In this problem the parameters will be selected such that the criteria[2] for the application of geometric optics are satisfied. In dealing with single rays the only criterion is

$$|\nabla\mu|/(\beta\mu^2) \ll 1; \tag{23}$$

however $\mu = \lambda_0/\lambda$, and (23) becomes

$$|\nabla\lambda|/(2\pi) \ll 1. \tag{24}$$

The inequality is strengthened if 2π is replaced by unity, and since

$\lambda = \lambda(y)$, $$|\partial\lambda/\partial y| \ll 1. \tag{25}$$

Thus (23) requires that magnitude of the fractional increment of λ per displacement of λ be much smaller than unity. For the proposed problem

$$\mu = \mu_0 + k y = \lambda_0/\lambda$$

and

$$\frac{d\lambda}{dy} = \frac{k\lambda_0}{\mu^2} \ll 1. \tag{26}$$

Note that $\mu \geq 1$ for any medium; therefore (26) may be strengthened if μ is replaced by unity. The condition which determines the selection of parameters for the problem being considered is then

$$k\lambda_0 \ll 1 \tag{27}$$

. . . . .

The equation of P is $x=0$; consequently the unit normal has only one component $n_x = 1$. Equations (B-2) provide

$$\mu_c \dot{y}'_{(P)} - \mu_0 \dot{y}''_{(P)} = 0 \tag{28}$$

and it is seen from (2) that

$$\begin{aligned} \dot{x}''_{(P)} &= \left[1 - (\dot{y}''_{(P)})^2\right]^{\frac{1}{2}} \\ &= \left[1 - \left(\frac{\mu_c}{\mu_0}\dot{y}'_{(P)}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{29}$$

Within the slab equations (B-1) state that

$$\mu_0 \dot{x}''_{(P)} - (\mu_0 + k\bar{y})\dot{x}''_{(Q)} = 0,$$

where $\bar{y}$ is the y coordinate of the point of emergence of the ray. Now with the use of (2)

$$\begin{aligned} \dot{y}''_{(Q)} &= \left[1 - (\dot{x}''_{(Q)})^2\right]^{\frac{1}{2}} \\ &= \left[1 - \left(\frac{\mu_0}{\mu_0 + k\bar{y}}\dot{x}'_{(P)}\right)^2\right]^{\frac{1}{2}}; \end{aligned} \tag{30}$$

however $\dot{x}''_{(P)}$ can be eliminated from (30) with the help of (29), and

$$\dot{y}''_{(P)} = \left\{ 1 - \frac{\mu_0^2 [1-(\mu_c/\mu_0)^2(\dot{y}'_{(P)})^2]}{(\mu_0 + k\bar{y})^2} \right\}^{\frac{1}{2}}. \tag{31}$$

Application of (B-2) at Q yields

$$\mu_c \dot{y}'''_{(Q)} - (\mu_0 + k\bar{y})\dot{y}''_{(Q)} \quad ,$$

and from (31)

$$\dot{y}'''_{(Q)} = \frac{\mu_0 + k\bar{y}}{\mu_c} \left\{ \frac{(\mu_0 + k\bar{y})^2 - \mu_0^2 [1-(\mu_c/\mu_0)^2(\dot{y}'_{(P)})^2]}{(\mu_0 + k\bar{y})^2} \right\}^{\frac{1}{2}}$$

$$= (1/\mu_c) \left\{ (\mu_0 + k\bar{y})^2 - \mu_0^2 [1-(\mu_c/\mu_0)^2(\dot{y}'_{(P)})^2] \right\}^{\frac{1}{2}}. \tag{32}$$

Equations (32) and (2) will afford the final direction cosines of the ray once $\bar{y}$ is determined. To find $\bar{y}$, utilize (B-1) and (29);

$$\mu \dot{x}'' = \text{a constant}$$

$$= \mu_0 \dot{x}''_{(P)}$$

$$= [1 - (\mu_c/\mu_0)^2 (\dot{y}'_{(P)})^2]^{\frac{1}{2}}. \tag{33}$$

Call the right side of equation (33) A. Equation (2) provides

$$\dot{y}'' = \left[1-(\dot{x}'')^2\right]^{\frac{1}{2}}, \tag{34}$$

but (33) is a means of eliminating $(\dot{x}'')^2$ from (34). Thus,

$$\dot{y}'' = (1/\mu)(\mu^2 - A^2)^{\frac{1}{2}}, \tag{35}$$

and division of (35) by (33) gives

$$dy''/dx'' = (1/A)(\mu^2 - A^2)^{\frac{1}{2}}. \tag{36}$$

Equation (36) may be written

$$A\int_0^{\bar{y}} \frac{dy''}{(\mu^2 - A^2)^{\frac{1}{2}}} = \int_0^t dx'',$$

and integration yields

$$\bar{y} = \frac{\varphi^2 + A^2 - \mu_0^2}{2k(\mu_0 + \varphi)}, \tag{37}$$

where

$$\varphi = \left\{\left[(\mu_0^2 - A^2)^{\frac{1}{2}} + \mu_0\right] \exp(kt/A) - \mu_0\right\}.$$

Thus, $\bar{y}$ may be determined from (37), and the final direction cosines follow from (32) and (2). The deviation ϵ of the ray is

$$\epsilon = \cos^{-1}\alpha''' - \cos^{-1}\alpha',$$

where $\quad \dot{x}'_{(0)} = \cos\alpha',$

and $\dot{x}'''_{(q)} = \cos\alpha'''$.

If, in the above problem, μ were to be a function of x only, then the deviation of the ray would vanish. To prove this, note that at the surfaces which intersect the xy plane on P and Q respectively, equations (B-2) provide

$$\Delta(\mu\dot{y}'') = 0$$

$$\Delta(\mu\dot{z}'') = 0,$$

and within the slab (B-1) provides

$$d(\mu\dot{y}'') = 0$$

$$d(\mu\dot{z}'') = 0.$$

These equations express the conservation of $\mu\dot{y}'''$ and $\mu\dot{z}'''$ along the ray; thus for a ray entering and passing completely through the slab

$$\mu_c\dot{y}' = \mu_c\dot{y}'''$$

$$\dot{y}' = \dot{y}''',$$

Similarly,

$$\dot{z}' = \dot{z}''',$$

and the deviation thus vanishes.

B. Deviation of Rays by a Conical Shock Zone*

Let a shock zone be oriented as shown in Fig. 2.

*In the design of airborne electromagnetic radiation collectors and transmitters it is often necessary to know the effect of the aerodynamic shock zone on the ray.

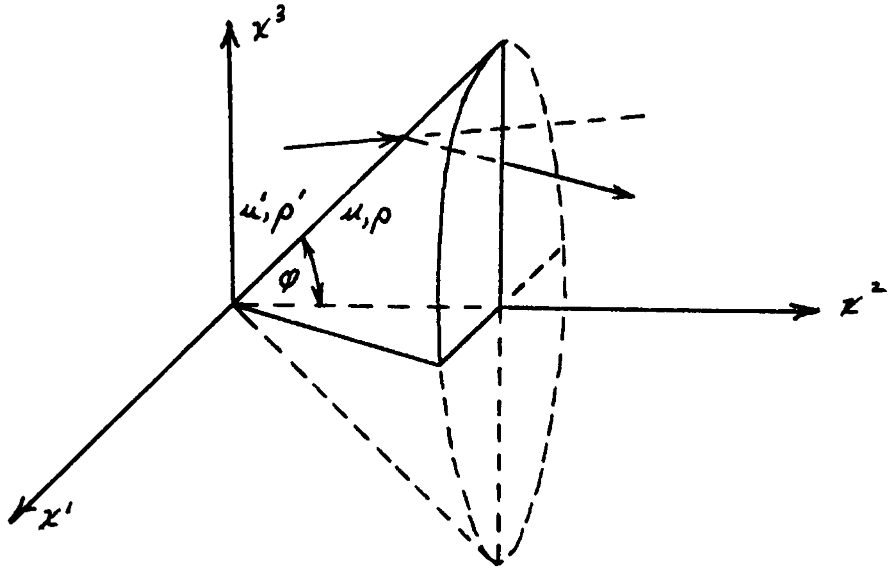

Fig. 2

For simplifying calculations assume $\varphi = \pi/4$, then the equation of the surface of discontinuity is

$$P(\chi^r) = 0$$

$$= (\chi')^2 - (\chi^2)^2 + (\chi^3)^2. \tag{38}$$

The use of (B-2) for deflection calculations requires the magnitude of the surface gradient; from (38) it is found that

$$\left[\left(\partial P/\partial \chi'\right)^2 + \left(\partial P/\partial \chi^2\right)^2 + \left(\partial P/\partial \chi^3\right)^2\right]^{\frac{1}{2}} = 2\left[(\chi')^2 + (\chi^2)^2 + (\chi^3)^2\right]^{\frac{1}{2}}. \tag{39}$$

Now χ' and χ^2 may be eliminated from (39) with the use of (38), and

$$\left[\left(\partial P/\partial \chi'\right)^2 + \left(\partial P/\partial \chi^2\right)^2 + \left(\partial P/\partial \chi^3\right)^2\right]^{\frac{1}{2}} = \sqrt{8}\,(\chi^2). \tag{40}$$

Let the index of refraction on the right and left sides of the shock zone be μ and μ' respectively. The direction cosines of the incident ray and the refracted ray will be represented by $\dot{\chi}_r$ and $\dot{\chi}_r'$. Equations (B-2) can be used to calculate the deviation of a ray; they provide

$$\mu \dot{\chi}_1 - \mu' \dot{\chi}_1' = \theta\,(2\chi_1)/(\sqrt{8}\,\chi_2)$$

$$\mu \dot{\chi}_2 - \mu' \dot{\chi}_2' = \theta\,(-2\chi_2)/(\sqrt{8}\,\chi_2) \tag{41}$$

$$\mu \dot{\chi}_3 - \mu' \dot{\chi}_3' = \theta\,(2\chi_3)/(\sqrt{8}\,\chi_2)\,.$$

From (2)

$$(\dot{\chi}_1')^2 + (\dot{\chi}_2')^2 + (\dot{\chi}_3')^2 = 1\,. \tag{42}$$

Given the direction cosines of an incident ray, and the coordinates of the point of incidence, equations (41) and (42) can be solved for the direction cosines of the deflected ray.

Let the shock zone be bounded on the left and right by air of density ρ' and ρ respectively. It is of interest to find the deflection of a ray parallel to χ_2 and in the $\chi_2\chi_3$ plane. The index of refraction of air has been shown[18] to be related to its density by

$$(\mu-1)10^6 = 276.6\,\rho/\rho_0\,, \tag{43}$$

where ρ_0 is the density at a pressure of one atmosphere and a temperature of 288° Kelvin. For wavelengths in the interval 0.29 to 0.86 microns, (43) provides values of μ which vary less than 3% from the values obtained for a wavelength of 0.658 microns.

Assume that $\rho/\rho' \leqslant 6$, then from (43) $\mu \leqslant 6\mu' - 5$. Using the NACA[19] standard atmosphere tables, it is found that at an altitude of 65,000 feet $\rho/\rho_0 = 0.07$. From (43)

$$\mu' = 1.000019362,$$

and since $\mu \leqslant 6\mu' - 5$,

$$\mu \leqslant 1.000116172$$

for a wavelength of 0.6583 microns. From this information the deviation of a ray is found from (C-1) not to exceed 0.1 milliradian.

APPENDIX

APPENDIX A

DERIVATION OF THE GENERALIZED RAY EQUATIONS FOR A SYSTEM OF HETEROGENEOUS ISOTROPIC MEDIA

The optical path length from point S_1 to point S_2 is defined to be

$$[S_1 S_2] \equiv \int_{S_1}^{S_2} \mu \, ds, \tag{A-1}$$

where μ denotes the index of refraction of the medium, and ds represents the element of distance along the ray. For an arbitrary curvilinear coordinate system

$$ds^2 = g_{mn} \, dx^m dx^n. \tag{A-2}$$

Latin indices signify a range (1,2,3) and summation is implied when indices are repeated. Divide (A-2) by ds^2 and write $\dot{x}^i$ for dx^i/ds

$$1 = (g_{mn} \dot{x}^m \dot{x}^n)^{\frac{1}{2}}. \tag{A-3}$$

In view of (A-3), (A-1) may be written

$$[S_1 S_2] = \int_{S_1}^{S_2} \mu \, (g_{mn} \dot{x}^m \dot{x}^n)^{\frac{1}{2}} ds, \tag{A-4}$$

for (A-4) is the result of multiplying (A-1) by unity.

Fermat's principle requires that the variation of the optical path length vanish;

$$\therefore \quad \delta [S_1 S_2] = 0$$

$$= \delta \int_{S_1}^{S_2} \mu \, (g_{mn} \dot{x}^m \dot{x}^n)^{\frac{1}{2}} ds. \tag{A-5}$$

Assume μ is piecewise continuous along the path and let

$$\psi = \mu \, (g_{mn} \dot{x}^m \dot{x}^n)^{\frac{1}{2}} \tag{A-6}$$

It is shown[20] that integration and differentiation with respect to the parameter s commute with variation; therefore (A-5) and (A-6) combine to

$$0 = \int_{s_1}^{s_2} \left[\frac{\partial \psi}{\partial x^r} \delta x^r + \frac{\partial \psi}{\partial \dot{x}^r} \delta \dot{x}^r \right] ds$$

$$= \int_{s_1}^{s_2} \left[\frac{\partial \psi}{\partial x^r} \delta x^r + \frac{\partial \psi}{\partial \dot{x}^r} \frac{d}{ds} (\delta x^r) \right] ds$$

$$= \int_{s_1}^{s_2} \frac{\partial \psi}{\partial x^r} \delta x^r ds + \int_{s_1}^{s_2} \frac{\partial \psi}{\partial \dot{x}^r} d(\delta x^r). \tag{A-7}$$

Let $\quad W_r = \partial \psi / \partial \dot{x}^r \; ; \; dz^r = d(\delta x^r);$

then $\quad dW_r = \frac{d}{ds} \left(\frac{\partial \psi}{\partial \dot{x}^r} \right) ds$

$$z^r = \delta x^r ,$$

and the second integral in (A-7) can be integrated by parts. Notice, however, that W_r will be piecewise continuous along the path so that it is necessary to integrate separately in each region of continuity and sum the results. For example, let discontinuities occur at points a and b along the path. Partial integration affords (for an extremely small $\epsilon > 0$)

$$\int_{s_1}^{s_2} \frac{\partial \psi}{\partial \dot{\chi}^r} d(\delta \chi^r) = z^r W_r \Big|_{s_1}^{a-\epsilon} + z^r W_r \Big|_{a+\epsilon}^{b-\epsilon} + z^r W_r \Big|_{b+\epsilon}^{s_2}$$

$$- \int_{s_1}^{a-\epsilon} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds - \int_{a+\epsilon}^{b-\epsilon} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds - \int_{b+\epsilon}^{s_2} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds$$

(A-8)

but z^r is continuous so that

$$z^r(b-\epsilon) = z^r(b+\epsilon) = z^r(b)$$

and

$$z^r(a-\epsilon) = z^r(a+\epsilon) = z^r(a)$$

Equation (A-1) may now be written

$$\int_{s_1}^{s_2} \frac{\partial \psi}{\partial \dot{\chi}^r} d(\delta \chi^r) = W_r \Big|_{a+\epsilon}^{a-\epsilon} z^r(a) + W_r \Big|_{b+\epsilon}^{b-\epsilon} z^r(b) + W_r z^r \Big|_{s_1}^{s_2}$$

$$- \int_{s_1}^{a-\epsilon} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds - \int_{a+\epsilon}^{b-\epsilon} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds$$

$$- \int_{b-\epsilon}^{s_2} \frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{\chi}^r}\right) \delta \chi^r ds.$$

(A-9)

Now let $\epsilon \rightarrow 0$, and (A-9) becomes

$$\int_{s_1}^{s_2} \frac{\partial \psi}{\partial \dot{\chi}^r} d(\delta \chi^r) ds = \Delta W_r(a) z^r(a) + \Delta W_r(b) z^r(b) + z^r W_r \Big|_{s_1}^{s_2}$$

$$+ \lim_{\epsilon \rightarrow 0} \left[- \int_{s_1}^{a-\epsilon} - \int_{a+\epsilon}^{b-\epsilon} - \int_{b+\epsilon}^{s_2} \right],$$

(A-10)

where $\Delta W_r(a)$ and $\Delta W_r(b)$ represent the jumps or increments of W_r across points a and b respectively. The limit of the sum of the integrals enclosed in the brackets may be written as the sum of the integrals each of which is taken within a region of continuous u. Equation (A-10) may be generalized to any integral number of surfaces of discontinuity, and (A-7) may now be written

$$0 = \sum \int \left[\partial\psi/\partial\chi^r - \frac{d}{ds}\left(\partial\psi/\partial\dot{\chi}^r\right)\right]\delta\chi^r\,ds + \sum \Delta\left(\partial\psi/\partial\dot{\chi}^r\right)\delta\chi^r$$

$$+ \frac{\partial\psi}{\partial\dot{\chi}^r}\,\delta\chi^r\Big|_{s_1}^{s_2}. \qquad \text{(A-11)}$$

For a system possessing no discontinuities the second summation vanishes. Since the variational method selected requires that the variation at the end points vanish, the third term must vanish. The symbol $\delta\chi^r$ represents an arbitrary variation so that the bracketed quantity within the integrand must also vanish; thus

$$\partial\psi/\partial\chi^r - \frac{d}{ds}\left(\partial\psi/\partial\dot{\chi}^r\right) = 0. \qquad \text{(A-12)}$$

To obtain the deflection of a ray at a surface of discontinuity, evaluate (A-11) within an interval which includes the point of intercept of the ray and the surface, and let the length of this interval approach zero, i.e. let

$$[s_1 s_2] \longrightarrow 0$$

The result is

$$\left(\Delta\, \partial\psi/\partial\dot{\chi}^r\right)\delta\chi^r = 0. \qquad \text{(A-13)}$$

To put (A-12) and (A-13) into the form of (8) and (9) take the derivatives of ψ as indicated in (A-12) and (A-13) and utilize (A-3):

$$\frac{\partial \psi}{\partial x^r} = \frac{1}{2}\,\frac{\partial g_{mn}}{\partial x^r}\,\dot{x}^m(\mu \dot{x}^n) + (g_{mn}\dot{x}^m\dot{x}^n)^{\frac{1}{2}}\,\frac{\partial \mu}{\partial x^r}$$

$$= \frac{1}{2}\,\frac{\partial g_{mn}}{\partial x^r}\,\dot{x}^m(\mu \dot{x}^n) + \frac{\partial \mu}{\partial x^r}\,; \tag{A-14}$$

$$\frac{\partial \psi}{\partial \dot{x}^r} = \frac{\mu}{2}\,g_{mn}\,\frac{\partial \dot{x}^m}{\partial \dot{x}^r}\,\dot{x}^n + \frac{\mu}{2}\,g_{mn}\,\frac{\partial \dot{x}^n}{\partial \dot{x}^r}\,\dot{x}^m$$

$$= \frac{\mu}{2}\,g_{mn}\,\delta_r^m\,\dot{x}^n + \frac{\mu}{2}\,g_{mn}\,\delta_r^n\,\dot{x}^m$$

$$= \frac{\mu}{2}\,g_{rn}\,\dot{x}^n + \frac{\mu}{2}\,g_{mr}\,\dot{x}^m\,; \ldots$$

if the dummy index m is changed to n and the equation $g_{ij} = g_{ji}$ is used,

$$\frac{\partial \psi}{\partial \dot{x}^r} = \mu\, g_{rn}\,\dot{x}^n, \tag{A-15}$$

and

$$\frac{d}{ds}\left(\frac{\partial \psi}{\partial \dot{x}^r}\right) = g_{rn}\,\frac{d}{ds}(\mu \dot{x}^n) + \mu\,\frac{d g_{rn}}{ds}\,\dot{x}^n$$

$$= g_{rn}\,\frac{d}{ds}(\mu \dot{x}^n) + \mu\,\frac{\partial g_{rn}}{\partial x^m}\,\dot{x}^m\dot{x}^n\,. \tag{A-16}$$

The second term on the right side of (A-16) may be written

$$\mu\,\frac{\partial g_{rn}}{\partial x^m}\,\dot{x}^m\dot{x}^n = \dot{x}^m\dot{x}^n\,\frac{\mu}{2}\,\frac{\partial g_{rn}}{\partial x^m} + \dot{x}^m\dot{x}^n\,\frac{\mu}{2}\,\frac{\partial g_{rn}}{\partial x^m}\,.$$

In the second term on the right side of the equation let m = n and

$n = m$ and substitute the result into (A-16). The new form of (A-16) and equations (A-14) are substituted into (A-11). The result is

$$0 = \frac{1}{2} \frac{\partial g_{mn}}{\partial x^r} \dot{x}^m (\mu \dot{x}^n) + \frac{\partial \mu}{\partial x^r} - g_{rn} \frac{d}{ds}(\mu \dot{x}^n)$$

$$- \frac{\mu}{2} \dot{x}^m \dot{x}^n \frac{\partial g_{rm}}{\partial x^n} - \frac{\mu}{2} \dot{x}^m \dot{x}^n \frac{\partial g_{rn}}{\partial x^m}$$

$$0 = \mu \dot{x}^m \dot{x}^n [mn, r] + g_{rn} \frac{d}{ds}(\mu \dot{x}^n) - \frac{\partial \mu}{\partial x^r} , \qquad \text{(A-16)}$$

where $[mn,r]$ is the Christoffel symbol of the first kind. Substitution of (A-15) into (A-13) gives

$$\Delta (\mu g_{rn} \dot{x}^n) \delta x^r = 0. \qquad \text{(A-17)}$$

APPENDIX B

EXPRESSION OF THE RAY EQUATIONS IN SEVERAL COORDINATE SYSTEMS

Let the primes be used to denote values of quantities in one medium, and let unprimed quantities refer to another medium.

For a <u>rectangular Cartesian system</u>

$$(x^1, x^2, x^3) = (x, y, z)$$

$$g_{mn} = \delta_{mn} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[mn, p] = 0,$$

and equations (8) become

$$\begin{aligned} \frac{\partial \mu}{\partial x} &= \frac{d}{ds}(\mu \dot{x}) \\ \frac{\partial \mu}{\partial y} &= \frac{d}{ds}(\mu \dot{y}) \\ \frac{\partial \mu}{\partial z} &= \frac{d}{ds}(\mu \dot{z}) ; \end{aligned} \tag{B-1}$$

whereas equations (9) become

$$\begin{aligned} \mu \dot{x} - \mu' \dot{x}' &= (\theta\, \partial P/\partial x) A \\ \mu \dot{y} - \mu' \dot{y}' &= (\theta\, \partial P/\partial y) A \\ \mu \dot{z} - \mu' \dot{z}' &= (\theta\, \partial P/\partial z) A, \end{aligned} \tag{B-2}$$

where $$A = [(\partial P/\partial x)^2 + (\partial P/\partial y)^2 + (\partial P/\partial z)^2]^{-\frac{1}{2}}.$$

For <u>cylindrical polar coordinates</u>

$$(x^1, x^2, x^3) = (r, \theta, z);$$

$$g_{mn} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and nonvanishing first order Christoffel symbols are

$$[22,1] = -r$$

$$[21,2] = [12,2] = r.$$

Equations (8) become

$$\begin{aligned} \frac{\partial \mu}{\partial r} &= -\mu r \dot{\theta}^2 + \frac{d}{ds}(\mu \dot{r}) \\ \frac{\partial \mu}{\partial \theta} &= 2\mu r \dot{r} \dot{\theta} + r^2 \frac{d}{ds}(\mu \dot{\theta}) \qquad \text{(B-3)} \\ \frac{\partial \mu}{\partial z} &= \frac{d}{ds}(\mu \dot{z}), \end{aligned}$$

and equations (9) are written

$$\begin{aligned} \mu \dot{r} - \mu' \dot{r}' &= (\beta \, \partial P/\partial r) B \\ \mu \dot{\theta} - \mu' \dot{\theta}' &= \frac{1}{r^2} (\beta \, \partial P/\partial \theta) B \qquad \text{(B-4)} \\ \mu \dot{z} - \mu' \dot{z}' &= (\beta \, \partial P/\partial z) B, \end{aligned}$$

where $$B = [(\partial P/\partial r)^2 + \frac{1}{r^2}(\partial P/\partial \theta)^2 + (\partial P/\partial z)^2]^{-\frac{1}{2}},$$

and β an undetermined multiplier. For plane polar coordinates the first two equations of (B-3) and of (B-4) apply.

For <u>spherical polar coordinates</u>

$$(x^1, x^2, x^3) = (r, \theta, \phi)$$

$$g_{mn}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix},$$

and the nonvanishing first order Christoffel symbols are:

$$[22,1] = -r$$

$$[33,1] = -r \sin^2\theta$$

$$[12,2] = [21,2]$$

$$= r$$

$$[13,3] = [31,3]$$

$$= r \sin^2\theta$$

$$[23,3] = [32,3]$$

$$= r^2 \sin^2\theta \cos^2\theta$$

$$[33,2] = -r^2 \sin^2\theta \cos^2\theta.$$

Equations (8) are then

$$\frac{\partial \mu}{\partial r} = -\mu r \dot{\theta}^2 - \mu r \dot{\varphi}^2 \sin^2\theta + \frac{d}{ds}(\mu \dot{r})$$

$$\frac{\partial \mu}{\partial \theta} = 2\mu r \dot{r} \dot{\theta} - \mu r^2 \dot{\varphi}^2 \sin\theta \cos\theta + r^2 \frac{d}{ds}(\mu \dot{\theta}) \qquad \text{(B-5)}$$

$$\frac{\partial \mu}{\partial \varphi} = 2\mu r \dot{r} \dot{\varphi} \sin^2\theta + 2\mu r^2 \dot{\theta} \dot{\varphi} \sin\theta \cos\theta + r^2 \sin^2\theta \frac{d}{ds}(\mu \dot{\varphi})$$

and equations (9) become

$$\mu \dot{r} - \mu' \dot{r}' = (\beta\, \partial P/\partial r) D$$

$$\mu \dot{\theta} - \mu' \dot{\theta}' = \frac{1}{r^2}(\beta\, \partial P/\partial \theta) D \qquad \text{(B-6)}$$

$$\mu \dot{\varphi} - \mu' \dot{\varphi}' = \frac{1}{r^2 \sin^2\theta}(\beta\, \partial P/\partial \varphi) D,$$

where β is an undetermined multiplier and

$$D = \left[(\partial P/\partial r)^2 + \frac{1}{r^2}(\partial P/\partial \theta)^2 + \frac{1}{r^2 \sin^2\theta}(\partial P/\partial \varphi)^2\right]^{-\frac{1}{2}}.$$

APPENDIX C

DERIVATION OF THE BASIC LAWS OF GEOMETRIC OPTICS FROM THE RAY EQUATIONS AND THE JUMP EQUATIONS

Let the point of intercept of the ray with the surface be the origin of a rectangular Cartesian coordinate system. Let the system be oriented so that x_2 is along the surface normal and the $x_2\,x_3$ plane contains the ray. Equations (9) provide

$$\mu \dot{x}_3 - \mu' \dot{x}_3' = 0 \tag{C-1}$$

where $\dot{x}_3$ and $\dot{x}_3'$ are cosines of angles which are complements of the angles between the ray and the normal; thus

$$\mu \cos(\pi/2 - \varphi) = \mu' \cos(\pi/2 - \varphi')$$

or

$$\mu \sin\varphi = \mu' \sin\varphi', \tag{C-2}$$

where φ and φ' are the angles between the surface normal and the incident and refracted ray respectively—equation (C-2) is Snell's law.

A reflected ray may be defined as a ray which does not pass through the boundary; therefore (C-1) yields

$$\dot{x}_3 = \dot{x}_3'$$

but

$$\dot{x}_2' = \pm[1 - (\dot{x}_3)^2]^{\frac{1}{2}} \tag{C-3}$$

$$= -\dot{x}_2,$$

where the negative root is selected to satisfy our definition of a reflected ray. Equation (C-3) states that $\cos\theta' = -\cos\theta$ which is the law of reflection.

If μ is constant within a medium, then equations (8) in in rectangular Cartesian coordinates become

$$\frac{d}{ds}(\dot{x}_1) = 0$$

$$\frac{d}{ds}(\dot{x}_2) = 0.$$

Therefore $\dot{x}_1$ and $\dot{x}_2$ are constants along the ray. This is the law of rectilinear propagation, for if the direction cosines of the ray tangent are constant, then the ray is a straight line.

APPENDIX D

DERIVATION OF THE HAMILTON-JACOBI EQUATION

In the derivation of the Hamilton-Jacobi equation it is necessary to deduce

$$\frac{\partial V}{\partial x^i} = \mu g_{mi} \dot{x}^m \tag{D-1}$$

from the equation

$$\int_A^B \frac{\partial V}{\partial x^i} dx^i = \int_A^B \mu g_{mn} \dot{x}^m dx^n . \tag{D-2}$$

The limits of the integrals in (D-2) are arbitrary so that

$$\frac{\partial V}{\partial x^i} dx^i = \mu g_{mn} \dot{x}^m dx^n \tag{D-3}$$

must be true for any dx^i along the path. Vectorially this statement says that

$$\bar{A} \cdot \bar{B} = \bar{C} \cdot \bar{B}, \tag{D-4}$$

where $\bar{B}$ is an arbitrary non zero vector. Equation (D-4) requires

$$\bar{A} = \bar{C}$$

and by similar reasoning (D-3) requires

$$\frac{\partial V}{\partial x^j} = \mu g_{nj} \dot{x}^n \tag{D-5}$$

which establishes (12). Inner multiplying the left and right sides of (D-1) by the left and right sides of (D-5) results in

$$\frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} = \mu^2 g_{mi} g_{nj} \dot{x}^m \dot{x}^n .$$

Now inner multiplication by g^{ij} and use of $g_{mn} \dot{x}^m \dot{x}^n = 1$ gives

$$g^{ij}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j} = \mu^2 g_{mi}\,\delta^i_n\,\dot{x}^m\dot{x}^n$$

$$= \mu^2 g_{mn}\dot{x}^m\dot{x}^n$$

$$= \mu^2$$

which establishes the Hamilton-Jacobi equation (13).

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