DEFLECTIONS OF BELLEVILLE SPRINGS

by

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NOMENCLATURE

\( r, \theta \) Polar coordinates in planes perpendicular to the axis of symmetry

\( s \) Meridional coordinate (see Fig. 1b.)

\( z \) Thickness coordinate (see Fig. 1b.)

\( h \) Thickness of the shell

\( d \) Free cone height

\( a, b \) \( r \) coordinate of the inner and outer edges, respectively

\( \Phi \) Initial cone angle (see Fig. 1b.)

\( P \) Total axial edge load

\( u, w \) Components of displacement of the middle surface in the meridional direction and the direction normal to the middle surface, respectively

\( \theta \) Rotation of a meridian of the middle surface

\( e_r, e_\theta \) Extensional strains in \( r \) and \( \theta \) directions, respectively

\( E \) Modulus of elasticity in tension and compression

\( \nu \) Poisson's ratio

\( D \) Flexural rigidity of the shell

\( N_r, N_\theta \) Normal forces per unit length of sections of the shell perpendicular to the \( r \) and \( \theta \) directions, respectively.

\( M_r, M_\theta \) Bending moments per unit length of sections of the shell perpendicular to the \( r \) and \( \theta \) directions, respectively

\( Q \) Shear force per unit length of a section of the shell perpendicular to the \( r \) direction

\( A \) Ratio of \( a \) to \( b \)

Auxiliary notation is explained in the text.
CHAPTER I
INTRODUCTION

The Belleville spring, or disk spring,\(^1\) is a truncated shallow conical shell of uniform thickness. It may be simpler to think of it as an annular plate that has been dished slightly into the shape of a cone. Such a spring is shown in Figure 1a.

The Belleville spring is usually loaded only at its edges by circumferentially uniform loads, axial in direction and with a sense such that they tend to reduce the cone angle. In most cases the edges are completely free to move. However, certain applications require that either the outer edge be restrained from radial expansion, as would be the case if the spring were inserted in a cylinder, or that the inner edge be restrained from radial contraction, as would be the case if a shaft were inserted through the spring. Some applications may require that both of these restrictions be imposed.

HISTORY OF BELLEVILLE SPRINGS

Belleville springs were first used in this country about 1890 as counter-recoil springs in certain large guns. Production difficulties were such that no private company would undertake to produce the springs, and the U. S. Army, after considerable experimentation, produced those necessary. Shortly thereafter, however, these early Belleville springs were replaced by helical springs, which were found to be cheaper and easier to fabricate.

During World War I the U. S. Army obtained designs for gun carriages from the French government in which Belleville springs were used to place initial compression on packings which had to be capable of holding a pressure of more than 100 atmospheres. Inasmuch as Belleville springs were ideally suited for this purpose, it became necessary to produce them in great quantity. These springs were also used at that time as rebound springs on caissons. After some modification of production procedure, these springs proved satisfactory, and after the war the Army undertook to design more powerful guns involving larger recoil mechanisms. Consequently, Belleville springs of different dimensions and characteristics were needed. Since there was no satisfactory design

procedure existing at that time, the Army Ordnance Department carried on extensive research between 1919 and 1929. Although this research shed considerable light on the subject of Belleville springs, it did not yield a satisfactory design procedure.

Not until 1936 was this need for a satisfactory design procedure filled at which time J. O. Almen and A. Laszlo published a paper which contained an approximate solution\(^3\) to the Belleville spring problem and outlined a method of design employing this solution. This solution yielded results in good agreement with experimental data and was later made the basis for a design manual\(^4\) for Belleville springs published by the Society of Automotive Engineers. The Almen-Laszlo solution has been used almost exclusively for the last 24 years as a basis for the design of Belleville springs, and it has given adequate results for most design purposes.

However, G. A. Wempner has recently presented\(^5\) a numerical solution to the governing differential equations for the Belleville spring which indicates that there is considerable


\(^4\) Publication SP-63, available from S.A.E. Special Publications Department, 29 West 39th Street, New York 18, N.Y.

room for improvement in the Almen-Laszlo solution. Also G. A. Wempner\(^6\) and R. Schmidt and G. A. Wempner\(^7\) have recently advanced approximate solutions to this problem which appear to be better than the Almen-Laszlo solution. It is believed, however, that these recent approximate solutions can be improved upon, and it is the endeavor of this thesis to do so.

**CHARACTERISTICS OF BELLEVILLE SPRINGS**

The simplest type of spring, a prismatic tension specimen, exhibits a characteristic which is typical of structural members in the elastic range: a linear load-deflection curve. Accordingly, the helical spring, the volute spring, the ring spring, and most common springs have a straight line relationship between load and deflection. On the other hand, Belleville springs usually have a nonlinear relationship between load and deflection, and it is possible to design Belleville springs with many differently shaped load-deflection curves. By varying the fundamental parameters, it is possible to obtain positive, zero, and even negative spring rates in given portions of the load-deflection curve.

Most authorities\(^8\) report that the most important

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\(^6\)"Axially Symmetrical Deformations..." *op. cit.*


\(^8\)J. O. Almen and A. Laszlo, J. J. Ryan and A. M. Wahl use this criterion in their papers listed in the Bibliography.
parameter in varying the shape of the load-deflection curve is the ratio of the free cone height, \( d \), to the shell thickness, \( h \). For values of \( d/h \) near 2.0, a curve such as that labeled "A" in Fig. 2 will occur. This curve demonstrates the "snap-through" characteristic which is a result of instability of the cone as the cone angle is reduced to the vicinity of zero. Consequently, large additional deflections occur in this region accompanied by a decrease in load. Of course, once the spring has reached a stable position after "snap-through", increase in deflection is accompanied by increase in load. For values of \( d/h \) near 1.5, this "snap-through" action is reduced to the point that after a certain load, deflection is increased considerably with no appreciable change in load, as is shown by curve "B" in Fig. 2. This type of Belleville spring is the so-called "constant-load" spring. For values of \( d/h \) less than 1.5, increasing load is accompanied by increasing deflection, as is shown by curves "C" and "D" in Fig.2. It is interesting to note that Ryan\(^9\) has found that for values of \( d/h \) between 0.11 and 0.17 the Belleville spring exhibits a linear load-deflection curve over a considerable range.

Another characteristic of interest is the fact that compared to helical, volute, elliptic leaf springs, and other

common springs of comparable size, the Belleville spring is a high load, small deflection spring.

Also of interest is the fact that the flexibility of disk springs is a function of the ratio, A, of the inside to the outside diameter of the spring. For instance, Almen and Laszlo\textsuperscript{10} report that for an initially flat Belleville spring, that is \(d/h\) is zero, the maximum flexibility occurs for a value of A near 0.5.

**ADVANTAGES AND VERSATILITY OF BELLEVILLE SPRINGS**

To state that the Belleville spring is superior in any given respect to another type of spring is somewhat misleading. Certainly there are spring applications in which other types of springs are far superior. However, in a great many instances, the Belleville spring has definite advantages.

By far the greatest advantage of disk springs and the predominating reason why they are used in various applications is the wide range of load-deflection curve shapes that can be obtained. In applications calling for a nonlinear relationship between load and deflection, the Belleville spring is very useful.

In cases where heavy loads and only moderate deflections are desired, the Belleville spring is again useful due to its

\textsuperscript{10}{"The Uniform Section Disk Spring," pp. 307-308.}
compactness in the direction of loading. A helical spring, for instance, with these characteristics would require a very heavy coil, probably with a diameter of about the same dimension as the free cone height of an equivalent conical spring. Since usually at least three loops are required to make a helical spring, the equivalent Belleville spring would be several times more compact in the direction of loading.

In cases where heavy loads and relatively large deflections are required, several Belleville springs stacked in series\textsuperscript{11} may be used. As might be expected, this procedure does not decrease the load-carrying capacity but increases the allowable deflection by the number of springs used.\textsuperscript{12} Although series stacking decreases the compactness of a spring mechanism consisting of Belleville springs, it is probable that this mechanism is more compact than an equivalent spring of another type. Another method of arranging disk springs, parallel stacking,\textsuperscript{13} enables one to increase the load-carrying capacity of a spring mechanism over that of an individual disk. In addition, such multiple disk mechanisms have the advantage over other types of springs in that failure

\textsuperscript{11}See Fig. 3a.

\textsuperscript{12}Almen and Laszlo, "The Uniform Section Disk Spring," pp. 307-308.

\textsuperscript{13}See Fig. 3b.
of one of the disks will not cause complete loss of flexibility, and in the case of series stacking such a failure will not increase the load on the remaining disks. Also, it is a simple matter to replace a broken disk.

Another advantage of multiple disk mechanisms is the fact that it is possible to vary the friction damping effect. In the case of series stacking, virtually no interspring friction occurs, and only slight friction occurs between the outermost springs and the supporting rings, as is apparent from the slight hysteresis loop which can be obtained on a load-deflection diagram for such a mechanism. In the case of parallel stacking, however, considerable interspring friction causes a large hysteresis loop. Hence, by employing either or both methods of stacking disks in a spring mechanism, it is possible to vary friction damping.

Brecht and Wahl report that uniform heat treatment in the case of disk springs is considerably easier than it is in the case of heavy helical springs. Also, Belleville springs will tolerate lateral as well as axial loading.

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16 Ibid.
17 Ibid.
19 Ibid.
Almen and Laszlo\textsuperscript{20} also point out that the characteristics of a given Belleville spring may be varied considerably without altering the spring. This variation is brought about by applying either or both of the circumferentially uniform loads on the face of the disk between the inner and outer edges as is shown in Fig. 4.

Several authorities\textsuperscript{21} have pointed out the need to improve the efficiency -- that is, the uniformity of stress distribution throughout a disk under load -- and the flexibility of the Belleville spring. According to Brecht and Wahl,\textsuperscript{22} the flexibility of the disk spring can be improved by radially tapering its thickness in a manner shown in Fig. 5. It should be noted that a disk spring which has any variation in thickness is no longer called a Belleville spring.

Although none of the more complicated loading arrangements and disk alterations mentioned above are considered here in detail and only the radially tapered disk spring is considered at length in any of the references given, these variations do point out the versatility of this type of spring. Certainly further study of these more complicated cases is warranted.

\textsuperscript{20}"Disk Spring Facilitates Compactness," p. 42.

\textsuperscript{21}J. O. Almen and A. Laszlo, W. A. Brecht and A. M. Wahl, and Joseph Kaye Wood (see discussion of paper by D. A. Gurney, p. 17 of that reference).

\textsuperscript{22}"The Radially Tapered Disk Spring," p.45.
APPLICATIONS OF BELLEVILLE SPRINGS

Only a few examples from the wide range of applications of disk springs will be cited.

The most common application of Belleville springs is of the "constant-load" type. In applications where it is desirable not to exceed a certain value of load and it is not possible to control deflection too closely, the "constant-load" spring is very useful. One application of this type has already been cited: the device used on large guns to hold a specified pressure on certain packings. Here it would clearly be difficult to control the deflection of the spring; yet, if the deflection of each spring can be held to a range of 0.8h to 2.25h, the pressure on the packings can be controlled to within 5 per cent.23 In another application of this type, "constant-load" springs are arranged to take the load from the bearings of the live tail-stock center of a lathe. Thus it is guaranteed that these bearings will not be overloaded due to expansion of the material being machined.24 A similar application of disk springs has been made to support commutator bearings in electric motors. In pressing, stamping, and punching machines, Belleville springs are very useful because of their high load capabilities. Here also the "constant-load" characteristic is of importance since loads

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are applied with considerable impact, and if the deflection can be held to the value prescribed above, the load applied by the pressing, stamping, or punching device can be maintained at a constant value. In applications of this type, Ryan\textsuperscript{25} points out that Belleville springs have been used on high speed machines in constant operation for many months without excessive failures. It may therefore be concluded that Belleville springs have good fatigue resistance.

In applications where springs with heavy load capabilities are needed and only limited space is available, Belleville springs are of great utility. It is for just such a reason that Belleville springs were chosen to serve as counter-recoil springs on large guns. Ryan\textsuperscript{26} has made a unique and interesting application of the compactness and heavy load capabilities of Belleville springs. For values of d/h between 0.11 and 0.17 he found that disk springs exhibit a linear relationship between load and deflection. Using springs with this specification, he was able to build a portable tensiometer weighing 6 pounds, having an over-all length of 9 inches, and capable of measuring suddenly applied loads of the order of magnitude of those in towlines attached to ships or barges. So successful was this application, that Mr. Ryan feels that the greatest use of Belleville springs is in instrument applications.

\textsuperscript{25} "Characteristics of Dished-Plate (Belleville) Springs as Measured by Portable Recording Tensiometers," p. 438.

\textsuperscript{26} \textit{ibid.}, pp. 431-438.
A solution of the disk spring problem requires consideration of larger deflections than are usually taken into account in "the small deflection theory". In the following analysis the so-called "large deflection theory" will be used.

**EQUILIBRIUM EQUATIONS**

Reference is made to Fig. 6a in which a typical element is shown which has been cut out from a conical shell of uniform thickness by two meridional and two circumferential sections normal to the middle surface. By taking the sum of the projections of forces acting on this element in the


meridional direction, neglecting magnitudes of higher order, and limiting consideration only to shallow\textsuperscript{29} shells, we obtain

\[ N_r + r \frac{dN_r}{dr} - N_t = 0 \] (1)

For the same element, by taking moments of all forces with respect to an axis perpendicular to one of the bounding meridional sections and by using the foregoing criteria, we obtain the second equilibrium equation:

\[ M_t - M_r - r \frac{dM_r}{dr} = -rQ \] (2)

In Fig. 6b the middle portion of the shell has been cut out by a circumferential section that is normal to the middle surface. The third equilibrium equation may be obtained by summing the projections of all forces in the axial direction which act on this portion of the shell. Thus,

\[ \frac{P}{2\pi} + r N_r (\beta + \Phi) = -rQ \] (3)

for small values of \( \beta \) and \( \Phi \).

**EXPRESSIONS FOR STRAINS**

Reference is made to Fig. 7a which shows a differential

\textsuperscript{29}The initial cone angle \( \Phi \) should be less than 0.10 radians.
element of a meridional line on the middle surface before and after deformation of the shell. By use of the Pythagorean theorem, we find

$$\frac{d\xi^*}{ds} = \left[\left(\frac{dw}{ds}\right)^2 + \left(\frac{du}{ds}\right)^2 + 2\frac{du}{ds} + 1\right]^{1/2}$$  \hspace{1cm} (4a)

By expanding the right side of Eq. (4a), neglecting magnitudes of higher order, restricting consideration to shallow shells, and using the following definition of strain:

$$e = \frac{d\xi^* - ds}{ds}$$  \hspace{1cm} (4b)

we obtain

$$e_r = \frac{du}{dr} + \left(\frac{a}{2} + \phi\right)b$$  \hspace{1cm} (4c)

since $ds \approx dr$.

By means of Eq. (4b) and Fig. 7b we find

$$e_t = \frac{u}{r}$$  \hspace{1cm} (5)

**OTHER IMPORTANT RELATIONSHIPS**

Multiplying both sides of Eq. (5) by $r$ and differentiating with respect to $r$, we obtain an expression for $\frac{du}{dr}$. Substituting this expression in Eq. (4c), we obtain the following compatibility equation:

$$e_r = e_t + r\frac{de_t}{dr} + \beta\left(\frac{a}{2} + \phi\right)$$  \hspace{1cm} (6)
From Hooke's law we find

\[ e_t = \frac{1}{Eh}(N_t - \nu N_r) \quad \text{and} \quad e_r = \frac{1}{Eh}(N_r - \nu N_t) \]  

(7a)

\[ N_t = \frac{Eh}{1 - \nu^2}(e_t + \nu e_r) \quad \text{and} \quad N_r = \frac{Eh}{1 - \nu^2}(e_r + \nu e_t) \]  

(7b)

Eq. (6) may now be written in terms of \( N_r, N_t \) and \( \beta \) if Eqs. (7a) are used to eliminate \( e_r \) and \( e_t \). Eliminating \( N_t \) from the resulting expression by means of Eq. (1), we obtain the first governing equation,

\[ r^2 \frac{d^2 N_r}{dr^2} + 3r \frac{dN_r}{dr} = -Eh\beta \left( \frac{\beta}{r^2} + \Phi \right) \]  

(8)

In the work that follows it is convenient to put Eq. (8) in the following form:

\[ \frac{1}{r} \frac{d}{dr}(r^3 \frac{dN_r}{dr}) = -Eh\beta \left( \frac{\beta}{r^2} + \Phi \right) \]  

(8a)

The expressions for bending moments in large deflection theory are identical to those in small deflection theory and are given\(^{30}\) by

\[ M_r = -D \left( \frac{\partial \beta}{r} + \nu \frac{\partial}{r} \right), \quad M_t = -D \left( \frac{\beta}{r} + \nu \frac{\partial}{dr} \right) \]  

(9)

\(^{30}\)Timoshenko and Woinowsky-Krieger, p.52.
By eliminating $Q$ between Eqs. (2) and (3), we obtain an expression in terms of $M_t$, $M_r$, $N_r$, and $\beta$. $M_t$ and $M_r$ may be eliminated from this expression by means of Eqs. (9). Thus the second governing equation is

$$\frac{d^2 \beta}{dr^2} + \frac{1}{r} \frac{d \beta}{dr} - \frac{\beta}{r} = \frac{P}{2\pi Dr} + \frac{N_r}{D}(\beta + \Phi)$$

(10)

This equation may be put in the following equivalent form:

$$\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r \beta) \right] = \frac{P}{2\pi Dr} + \frac{N_r}{D}(\beta + \Phi)$$

(10a)

It may be desirable to proceed in a given problem in terms of displacements only. If this is the case, the first governing equation is obtained by eliminating $N_r$ and $N_t$ from Eq. (1) by means of Eqs. (7b). Values for $e_t$ and $e_r$ given by Eqs. (4c) and (5) are substituted in the resulting expression, and we obtain

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r} = -\frac{(1-\nu)}{r} \left( \frac{\partial}{\partial} + \Phi \right) \beta - \left( \beta + \Phi \right) \frac{d^2 \beta}{dr}$$

(11)

Rewriting this equation as in the case of Eq. (10), we obtain

$$\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru) \right] = -\frac{(1-\nu)}{r} \left( \frac{\partial}{\partial} + \Phi \right) \beta - \left( \beta + \Phi \right) \frac{d^2 \beta}{dr}$$

(11a)
The second governing equation in terms of displacements is obtained from Eq. (10a) with the aid of Eqs. (4c), (5), and the second of Eqs. (7b). Thus,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi)\right] = \frac{P}{2\pi Dr} + \frac{12}{R^2}\left[\frac{d\lambda}{dr} + \left(\frac{\beta + \phi}{2}\right)\beta + \nu\frac{\lambda}{r}\right](\beta + \phi)$$  (12)
A solution of either Eqs. (8a) and (10a) or Eqs. (11a) and (12) which satisfies the appropriate boundary conditions will be an exact solution to the disk spring problem. Although the general form of such a solution is known, it would be extremely difficult to find a specific solution which satisfies the appropriate boundary conditions. Therefore, one must resort to either numerical or approximate methods of solution. One method of obtaining such an approximate solution is the Ritz method.

If a system is in a state of stable equilibrium, its total energy is a minimum. In the case of large deflections of a shallow shell, the total energy, \( I \), consists of three terms: the strain energy due to bending, \( V_1 \), the strain energy due to stretching of the middle surface, \( V_2 \), and the potential energy of the load acting on the shell, \( V_3 \). These expressions in their respective order are given as follows:

\[
V_1 = \pi D \int_a^b \left[ \left( \frac{d\theta}{dr} + \frac{1}{r}\beta \right)^2 - \frac{2(1-\nu)}{r} \beta \frac{d\theta}{dr} \right] r dr
\]  

(13) \(^{32}\)

\[
V_2 = 2\pi \int_a^b \left[ \frac{N_r e_r}{2} + \frac{N_\theta e_\theta}{2} \right] r dr
\]  

(14) \(^{33}\)

\(^{31}\)Timoshenko and Woinowsky-Krieger, pp. 343-346.

\(^{32}\)Ibid., p. 345.

\(^{33}\)Ibid., p. 400.
\[ V_3 = -2\pi \int_{a}^{b} (wq) r dr \]  

(15)

and

\[ I = V_1 + V_2 + V_3 \]  

(16)

We may assume that the rotation at any point of the shell can be represented in the form of a series

\[ \varphi = K_1 F_1(r) + K_2 F_2(r) + K_3 F_3(r) + \ldots K_n F_n(r) \]  

(17)

in which \( F_1, F_2, F_3, \ldots F_n \) are functions chosen so as to suitably represent the deflection surface and satisfy the boundary conditions. Substitution of Eq. (17) in Eq. (16) results in an expression for \( I \) in terms of the coefficients \( K_1, K_2, K_3, \ldots K_n \). In order that \( I \) be a minimum, these coefficients must be chosen such that

\[ \frac{\partial I}{\partial K_1} = 0, \quad \frac{\partial I}{\partial K_2} = 0, \quad \frac{\partial I}{\partial K_3} = 0, \ldots \quad \frac{\partial I}{\partial K_n} = 0 \]  

(18)

These conditions yield a system of \( n \) algebraic equations in \( K_1, K_2, K_3, \ldots K_n \), and each of these coefficients can then

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34 Timoshenko and Woinowsky-Krieger, p. 345.
be determined. By a wise choice of the functions $F_1, F_2, F_3, \ldots, F_n$ we may obtain an approximate solution which is very close to the exact solution.
**DIMENSIONLESS FORMS OF EQUATIONS**

In the work that follows, it was found convenient to introduce dimensionless variables which are defined as follows:

\[ \alpha = \frac{c}{b}, \quad \eta = \frac{u}{b} \quad (19a) \]

\[ N = \frac{1 - \nu^2}{Eh} N_r, \quad \overline{N} = \frac{1 - \nu^2}{Eh} N_t \quad (19b) \]

\[ M = -\frac{b}{D} M_r, \quad \overline{M} = -\frac{b}{D} M_t \quad (19c) \]

In terms of these new variables, Eq.(1) is

\[ N + \alpha \frac{dN}{d\alpha} - \overline{N} = 0 \quad (1a) \]

By using dimensionless variables and eliminating Q between Eq.(2) and (3) we obtain

\[ \frac{d(\alpha M)}{d\alpha} - \overline{M} = \frac{12b^2}{h} \alpha N (\beta + \Phi) + \frac{Pb}{2\pi D} \quad (2a) \]

In terms of dimensionless variables, equations (4c), (5), (6), (7a), (7b), (8a), (9), (10a), (11a), and (12), respectively, may be written as follows:
\[ e_r = \frac{dm}{d\alpha} + (\frac{\theta}{2} + \phi) \beta \] (4d)

\[ e_t = \frac{\eta}{\alpha} \] (5a)

\[ e_r = e_t + \alpha \frac{de_t}{d\alpha} + (\frac{\theta}{2} + \phi) \beta \] (6a)

\[ e_t = \frac{N - \nu N}{1 - \nu^2} \quad , \quad e_r = \frac{N - \nu N}{1 - \nu^2} \] (7c)

\[ N = e_r + \nu e_t \quad , \quad \bar{N} = e_t + \nu e_r \] (7d)

\[ \frac{1}{\alpha} \frac{d}{d\alpha} (\frac{\alpha^3 dN}{d\alpha}) = -(1 - \nu^2) \beta (\frac{\theta}{2} + \phi) \] (8b)

\[ M = \frac{d\beta}{d\alpha} + \nu \frac{\beta}{\alpha} \quad , \quad \bar{M} = \nu \frac{d\beta}{d\alpha} + \beta \] (9a)

\[ \frac{d}{d\alpha} \left[ \frac{1}{\alpha} \frac{d}{d\alpha} (\alpha \beta) \right] = \frac{12 \beta N}{\hbar^2} (\beta + \Phi) + \frac{P_{\beta}}{2\pi D\alpha} \] (10b)

\[ \frac{d}{d\alpha} \left[ \frac{1}{\alpha} \frac{d}{d\alpha} (\alpha \eta) \right] = -\frac{\beta}{\alpha} (1 - \nu) (\frac{\theta}{2} + \phi) - (\beta + \Phi) \frac{d\beta}{d\alpha} \] (11b)
In the specific case of the Belleville spring, if dimensionless variables are used, Eqs. (13), (14), and (15) may be written as follows:

\[
\frac{d}{d\alpha} \left[ \frac{1}{\alpha} \frac{d}{d\alpha} (\alpha \beta) \right] = \frac{P \beta}{2\pi D \alpha} + \frac{12B^2}{h^2} (\beta + \Phi) \left[ \frac{d\eta}{d\alpha} + \frac{\beta}{2} (\beta + \Phi) \beta + \nu \frac{m}{\alpha} \right]
\]  

(12a)

\[V = \pi D \int_0^1 \left[ \frac{d}{d\alpha} \left( \frac{d\beta}{d\alpha} \right)^2 + \frac{\beta}{\alpha} \right] \alpha d\alpha
\]  

(13a)

\[
V_2 = \frac{24DB^2 \pi}{(1+\nu)h^2} \int_0^1 \left[ N^2 + \alpha N \frac{dN}{d\alpha} + \frac{\alpha^2}{2(1-\nu)} \left( \frac{dN}{d\alpha} \right)^2 \right] \alpha d\alpha
\]  

(14a)

\[V_3 = -P w_a \text{ (where } w_a \text{ is the deflection at } r = a)\]  

(15a)

If it is desirable to work in terms of displacements only, substituting Eqs. (4d), (5a), and (7b) in Eq. (14) we find

\[
V_2 = \frac{\pi EhB^2}{1-\nu^2} \int_0^1 \left[ \left( \frac{d\eta}{d\alpha} \right)^2 + 2 \left( \frac{d\eta}{d\alpha} \right) \left( \frac{\beta}{\alpha} + \Phi \right) \beta + \frac{\beta}{2} \left( \beta + \Phi \right) \beta^2 \right.
\]

\[
+ \left. \left( \frac{\eta}{\alpha} \right)^2 + 2\nu \left( \frac{\eta}{\alpha} \right) \frac{d\eta}{d\alpha} + 2\nu \left( \frac{\eta}{\alpha} \right) \left( \frac{\beta}{2} + \Phi \right) \beta \right] \alpha d\alpha
\]  

(14b)
CHAPTER III

PRESENTATION OF SOLUTIONS

CASE I

The most practical and commonly used applications of Belleville springs are those in which the edges are completely free to move. The boundary conditions in this case are

\[ N = 0 \text{ at } \alpha = 1; \quad N = 0 \text{ at } \alpha = A \quad (20a) \]
\[ M = 0 \text{ at } \alpha = 1; \quad M = 0 \text{ at } \alpha = A \quad (20b) \]

As stated in the foregoing chapter, an exact solution to this problem would be very difficult. Hence, an approximate solution will be presented using the Ritz method.

As a first step in the application of the Ritz method, we must make an assumption as to the shape of the deflection surface. It has been found by F. Dubois\textsuperscript{35} that the stress distribution in a shallow truncated conical shell has the same character as that in a circular plate with a hole at the center. This discovery suggests that the deflections in the

\textsuperscript{35}ibid., p. 564.
case of a Belleville spring may be similar to those in an annular plate. Guided by this reasoning, we take

$$\beta = K\beta_s = K\left[\alpha \ln \alpha + B\alpha + \frac{\Gamma}{\alpha}\right]$$  \hspace{1cm} (21)

in which $\beta_s$ is the rotation of a meridian on the middle surface of an annular plate as given by the small deflection theory.\textsuperscript{36} Eq.(21) is a simplification of Eq.(17) in which only the first term in the series is taken. The constants, $B$ and $\Gamma$, are given as follows:

$$B = \frac{A^2 \ln A}{1 - A^2} - \frac{1}{1 + \nu} \hspace{1cm} (21a)$$

$$\Gamma = \frac{1 + \nu}{1 - \nu} \left(\frac{A^2 \ln A}{1 - A^2}\right) \hspace{1cm} (21b)$$

Substituting the assumed expression for $\beta$, Eq.(21), in the first governing equation, Eq.(8b), we obtain, after one integration,

$$\frac{dN}{d\alpha} = -(1-\nu^2)\left\{K^2\left[\frac{L}{\phi}(\ln \alpha)^2 + \Delta \alpha \ln \alpha + \frac{\Gamma}{\alpha}\right] + \frac{\lambda}{\alpha} + \frac{1}{2} \frac{\ln \alpha}{\alpha^2} + \frac{\omega}{\alpha^3}\right\} + K\Phi\left[\frac{1}{3} \ln \alpha + \frac{\Gamma}{\alpha^2} + \frac{T}{\alpha^3}\right]$$ \hspace{1cm} (22)

\textsuperscript{36}Ibid., p. 59.
in which the additional constants, $\Delta, \xi, \gamma, \lambda, \psi$ and $T$, are defined as follows:

$$\Delta = \frac{1}{16}\left[\frac{4A^2 \ln A}{1-A^2} - \frac{5+\nu}{1+\nu}\right]$$  \hspace{1cm} (22a)$$

$$\xi = \frac{1}{A}\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{A^2 \ln A}{1-A^2}\right)\left[\frac{2A^2 \ln A}{1-A^2} - \frac{3+\nu}{1+\nu}\right]$$  \hspace{1cm} (22b)$$

$$\gamma = \frac{1}{9}\left[\frac{3A^2 \ln A}{1-A^2} - \frac{4+\nu}{1+\nu}\right]$$  \hspace{1cm} (22c)$$

$$\lambda = \frac{1}{16}\left\{\frac{2A^2 \ln A}{1-A^2}\left[\frac{A^2 \ln A}{1-A^2} - \frac{1}{2} - \frac{3+\nu}{1+\nu}\right] + \frac{3+\nu}{(1+\nu)^2} + \frac{1}{4}\right\}$$  \hspace{1cm} (22d)$$

$$\psi = \frac{1}{16(1-A^2)}\left[2A^2(\ln A)^2(A^2 + 4\Gamma) - \ln A(2A^4 - \Delta A^4 +$$

$$-32A^2 + 8\Gamma^2)\right] - \frac{1}{16}\left[A^2 - 8\Delta A^2 + 16\Lambda A^2 - 4\Gamma^2\right]$$  \hspace{1cm} (22e)$$

$$T = \frac{1}{1-A^2}\left[\frac{2}{3}A^3 \ln A + \frac{2}{3}A^2(1-A) - 2\Gamma A^2(1-A) - 2\Delta A(1-A)\right]$$  \hspace{1cm} (22f)$$
We now write Eq. (14a) as follows:

\[
V_2 = \frac{\pi Eh b^2}{(1-\nu^2)^2} \left\{ \int_\alpha^1 2(1-\nu)(\alpha N_0) d(\alpha N_0) + \int_\alpha^1 \alpha^3 \left( \frac{dN}{d\alpha} \right)^2 d\alpha \right\}
\]  

(14b)

Since the boundary conditions require that \( N=0 \) at \( \alpha = 1 \) and \( \alpha = A \), we find that the first integral in Eq. (14b) vanishes. Hence, we have

\[
V_2 = \frac{\pi Eh b^2}{(1-\nu^2)^2} \int_\alpha^1 \alpha^3 \left( \frac{dN}{d\alpha} \right)^2 d\alpha
\]  

(14c)

Substituting Eq. (22) in Eq. (14c), we obtain after integration

\[
V_2 = C_1 K^4 + C_2 \phi K^3 + C_3 \phi^2 K^2
\]  

(23)

The expressions for the constants, \( C_1, C_2 \) and \( C_3 \), are extremely long and are presented in the appendix to this thesis.

Substituting the expression for \( \phi \), Eq. (21), in Eq. (13a) and integrating, we find

\[
V_1 = C_4 K^2
\]  

(24)
where

\[
C_4 = \pi D \left\{ \frac{1 - A^2}{2} + 2\Gamma (1 - v) \left[ \frac{(1 - A^2)\Gamma}{2A^2} + \ln A \right] + 
- (1 + v) \left[ A^2 (\ln A)^2 + 2BA^2 \ln A - (1 - A^2)B^2 \right] \right\}
\]  

Writing Eq. (21) in the form

\[
\omega_\alpha = -\int_b^1 \beta^3 d\alpha = -\int_b^1 \beta \left[ \alpha \ln \alpha + B \alpha + \frac{\Gamma}{\alpha} \right] d\alpha
\]

we obtain after integration

\[
V_3 = -P\omega_\alpha = C_5 K,
\]

where

\[
C_5 = -\frac{Pb}{4} \left[ 2\ln A(A^2 + 2\Gamma) + (1 - A^2)(1 - 2B) \right]
\]

Substituting Eqs. (23), (24) and (25) in Eq. (16), we obtain the following expression for the total energy of the system:

\[
I = C_1 K^4 + C_2 \Phi K^3 + C_3 \Phi^2 K^2 + C_4 K^2 + C_5 K
\]

Since the total energy is a minimum for a system in the state of stable equilibrium,

\[
\frac{\partial I}{\partial K} = 4C_1 K^3 + 3C_2 \Phi K^2 + 2C_3 \Phi^2 K + 2C_4 K + C_5 = 0
\]
If we let $A = 0.5$ and $\nu = 0.3$, divide all terms by $\frac{2\pi EhbCs}{(1-\nu^2)^P}$ and introduce the dimensionless parameters

$$\gamma = \frac{h}{b}, \quad P = \frac{(1-\nu^2)P}{2\pi Ehb},$$

(27)

Eq. (26a) becomes

$$0.046919K^3 - 0.089875\Phi K^2 + 0.019777\Phi^2 K + 0.166667\gamma^2 K = P$$

(26b)

Using Eq. (25), we may determine the value of $K$ for any desired deflection at the inner edge of the shell, $w_a$. Hence, Eq. (26b) gives us the relationship between load and deflection. This relationship is represented graphically in Figs. 8 and 9 in which $P$ is plotted against $w_a$. For comparison, these figures also include load-deflection curves for a numerical solution of the governing equation as determined by G. A. Wempner. 37

A special case of interest is the annular plate of uniform thickness. For this case $\Phi$ is zero, and Eq. (26b) becomes

$$0.046919K^3 + 0.166667\gamma^2 K = P$$

(26c)

Assuming that $P = 0.000041,325$, $\gamma = 0.05$ and solving Eq. (26c) for $K$, we obtain

$$K = 0.066326$$

(26d)

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Hence, for the particular value of load given above, using the value of \( K \) together with Eqs. (1a), (9a), (21), (22) and the integrated form of Eq. (22), we may obtain numerical values for \( \beta, M, \bar{M}, N \), and \( \bar{N} \) at any point of the plate. Numerical values for these quantities are given at various points of the plate in Table 1. For comparison, numerical values for these same quantities, as given by a series solution determined by Wempner and Schmidt,\(^3^8\) are also included.

CASE II

Although not as common as Case I, applications of disk springs in which both edges are restrained from lateral displacements, but are still free to rotate, are also of interest. In this case, the boundary conditions are

\[ \eta = 0 \text{ at } \alpha = 1; \quad \eta = 0 \text{ at } \alpha = A \quad (28a) \]
\[ M = 0 \text{ at } \alpha = 1; \quad M = 0 \text{ at } \alpha = A \quad (28b) \]

Due to the excessive length of the previous solution, we shall no longer use Eq.(21), but instead will make the following assumption:

\[ M = \frac{d\alpha}{d\alpha} + \gamma \beta = 0 \quad (29) \]

It is not immediately apparent that there is any merit in such an assumption; however, examination of Table 1 indicates that \( M \) is very small compared to \( \bar{M} \) and hence may be considered negligible. Furthermore Schmidt and Wempner\(^{39}\) found that such an assumption gave excellent results in the case of free edges, and there is reason to believe that it will give satisfactory results for other boundary conditions also.

\(^{39}\)"The Nonlinear Conical Spring."
Solving Eq. (29) for $\beta$, we obtain

$$\beta = -C \alpha^{-\nu}$$  \hspace{1cm} (30)

Substituting Eq. (30) in Eq. (11b) and integrating the resulting equation, with due regard to the boundary conditions, we obtain

$$\eta = \frac{(1-3\nu)C^2}{8\nu(1-\nu)} \left[ \alpha^{1-2\nu} - F_1 \alpha + \frac{F_2}{\alpha} \right] + \frac{(1-2\nu)C\Phi}{\nu(2-\nu)} \left[ \alpha^{1-\nu} - F_3 \alpha + \frac{F_4}{\alpha} \right]$$ \hspace{1cm} (31)

where

$$F_1 = \frac{1-A^{2-2\nu}}{1-A^{2\nu}} \quad , \quad F_3 = \frac{1-A^{2-\nu}}{1-A^{2\nu}}$$

$$F_2 = \frac{A^{2(1-A^{-2\nu})}}{1-A^{2\nu}} \quad , \quad F_4 = \frac{A^{2(1-A^{-\nu})}}{1-A^{2\nu}}$$ \hspace{1cm} (31a)

Substituting Eqs. (30) and (31) in Eq. (14b) and integrating, we find

$$V_z = K_1 C^4 + K_2 C^3 + K_3 C^2$$ \hspace{1cm} (32)

where

$$K_1 = \frac{1}{64\nu^2(1-\nu)^2} \left\{ \frac{1}{1-2\nu} \left( \frac{1-A^{2-2\nu}}{1-A^{2\nu}} \right) \left[ 1 - 4\nu + 11\nu^2 - 18\nu^3 + 14\nu^4 \right] + 
\left[ 1 - 2\nu + 2\nu^2 - 7\nu^3 \right] F_1 \right\}$$

$$K_2 = -\frac{1}{4\nu^2(1-\nu)(2-\nu)} \left\{ \frac{1}{2-3\nu} \left( \frac{1-A^{2-3\nu}}{1-A^{2\nu}} \right) \left[ 2 - 7\nu + 16\nu^2 - 22\nu^3 + 13\nu^4 \right] + 
\left[ 1 - 2\nu + 2\nu^2 - 7\nu^3 \right] F_1 F_3 \right\}$$ \hspace{1cm} (32a)

$$K_3 = -\frac{1}{4\nu^2(1-\nu)(2-\nu)} \left\{ \frac{1}{2-3\nu} \left( \frac{1-A^{2-3\nu}}{1-A^{2\nu}} \right) \left[ 2 - 7\nu + 16\nu^2 - 22\nu^3 + 13\nu^4 \right] + 
\left[ 1 - 2\nu + 2\nu^2 - 7\nu^3 \right] F_1 F_3 \right\}$$ \hspace{1cm} (32b)
Substituting the expression for Eq. (30), in the energy integral, Eq. (13a), we obtain

\[ V_1 = K_4 C^2 \]  

where

\[ K_4 = \frac{\pi E h^3 (A^{-2\nu} - 1)}{24\nu} \]  

Integrating Eq. (30), as in the case of Eq. (25), we obtain

\[ V_3 = K_5 C = -Pw_a \]  

where

\[ K_5 = \frac{Pb}{1-\nu} (A^{1-\nu} - 1) \]  

The total energy of the system is, therefore, the sum of Eqs. (32), (33), and (34):

\[ I = K_1 C^4 + K_2 \Phi C^3 + K_3 \Phi^2 C^2 + K_4 C^2 + K_5 C \]  

and

\[ \frac{\partial I}{\partial C} = 4K_1 C^3 + 3K_2 \Phi C^2 + 2K_3 \Phi^2 C + 2K_4 C + K_5 = 0 \]
If we let $A = 0.5$ and $\nu = 1/3$ and divide all terms by $\frac{\pi \mu h^2 K}{(1-\nu^2)P}$, Eq. (35a) becomes

$$0.506,25C^3 - 1.641,75 \Phi C^2 + 0.821,73 \Phi^2 C + 0.094,493 \gamma C^2 = P$$

In the case of an annular plate, Eq. (35b) reduces to

$$0.506,25C^3 + 0.094,493 \gamma^2 C = P$$

Setting $C = 0.1$ and $\gamma = 0.05$, we find

$$P = 0.000,184,960 Eb^2$$

Since it was not possible to find another solution to the disk spring problem which satisfies the boundary conditions, Eqs. (28), we must resort to another method of checking the results given by Eqs. (35b) and (35c). Although the assumption, Eq. (30), is an approximate solution to the problem at hand (that is, the disk spring loaded by uniform axial edge loads), it will be the exact solution of the disk spring problem with something other than uniform axial edge loads. Hence, by computing some function of the load, such as the shear, $Q$, by means of the assumption, Eq. (30), and comparing this assumed shear with the actual shear, we may obtain a check.

We shall consider first the case of an annular plate for which $A = 0.5$ and $\nu = 1/3$. By substituting Eqs. (30) and (31) in Eqs. (4d) and (5a) and substituting the resulting expressions for $e_r$ and $e_t$ in the first of Eqs. (7d), we obtain
Then by substituting Eqs. (30) and (36) in Eq. (10b) and multiplying the resulting expression by \( \frac{D}{b^2} \), we obtain the following expression for the shear:

\[
Q' = \frac{8DC}{9b^2\alpha^3} + \frac{9EhC^3}{16\alpha}
\]  

(37)

By comparing the shear \( Q' \) as given by Eq. (37) with the actual shear \( Q \) given by

\[
Q = \frac{P}{2\pi b\alpha}
\]  

(38)

we may obtain a check of Eq. (34c). Such a comparison is represented graphically in Fig. 10.

A similar comparison was made in the case of Eq. (35b) yielding doubtful results. For a value of \( \Phi \) of 0.08 radians, it was found that when the load was applied in the usual sense, the assumed shear deviated from the true shear by as much as several hundred per cent. However, when the load was taken in the direction opposite to that shown in Fig. 4a, the assumed shear differed from the actual shear by no more than the deviation between these values as found in the case of the annular plate. Nevertheless, since the direction of the load is as shown in Fig. 4a in most practical applications, it was decided not to include these results here.
CHAPTER IV

DISCUSSION

CASE I

Since we have limited our consideration to small values of $\Phi$, it seems reasonable to assume that the conical disk spring will act in a manner similar to the annular plate, and it has been verified by F. Dubois\textsuperscript{40} that the stress distribution is similar in these two cases. Therefore, it seems quite logical to assume as in Eq. (21) that, in the range of small deflections, the rotation at any point in the conical disk will be of the same character as that given by the exact linear solution for an annular plate. We might expect, therefore, that Eq. (21) would give excellent results in the range of small deflections. Since Figs. 8 and 9 indicate that there is little deviation between the numerical and approximate solutions in this range, we may conclude that such a prediction is quite accurate.

As we might expect, this assumption is not as accurate for excessively large deflections, and Figs. 8 and 9 show deviation between the approximate and numerical solutions in

\textsuperscript{40}Timoshenko and Woinowsky-Krieger, p. 564.
This deviation is not too important, however, since in almost all applications of disk springs, deflection cannot continue after the spring is flattened. In other words, for most practical applications, $|\beta|$ is less than $|\phi|$. In this range, Figs. 8 and 9 indicate that the assumption stated in Eq. (21) gives good results.

For the boundary conditions stated in Eqs. (20a) and (20b), other approximate solutions have been offered that have shortcomings not present here. Almen and Laszlo\textsuperscript{41} have offered an approximate solution in which they assume that $\beta$ is constant and that the meridional strain is negligible. These assumptions do not satisfy the boundary conditions, Eqs. (20), and there is reason to believe that they will not give good results for small values of $A$. In order to substantiate the last conclusion, consider the case in which $A=0$ and $\phi=0$. We have thus reduced the problem to that of a solid plate with a concentrated load at the center. For this case, $\beta$ is zero at the center and increases to a maximum value at the outer edge. It seems likely, therefore, that $\beta$ will also vary considerably in cases where $\phi$ is not zero and $A$ is small. G. A. Wempner\textsuperscript{42} has found another approximate solution in which he assumes only that $\beta$ is

\textsuperscript{41}"The Uniform Section Disk Spring," pp. 305-314.

\textsuperscript{42}"Axially Symmetrical Deformations of a Shallow Conical Shell."
constant. Here again the boundary conditions are not completely satisfied and it is believed that results will be poor for small values of $A$. A third approximate solution was discovered by R. Schmidt and G. A. Wempner\textsuperscript{43} in which it was assumed that $M=0$, as in Case II above. Although this assumption satisfies the boundary conditions, Eqs. (20a) and (20b), there is reason to believe that it will not be valid for small values of $A$. If we consider the case where $A=0$, we reduce the problem to that of a solid cone with a concentrated load at the apex. For such a problem, we find that the assumption that $M=0$ gives an infinite value for $\beta$ under the load. This statement follows from Eq. (30) in which Poisson's ratio $\nu$ is a positive constant and $C$ is not zero when $A$ is zero. It appears reasonable to assume, therefore, that Eq. (30) will result in very large values of $\beta$ for values of $A$ near zero, and we may conclude that the assumption, $M=0$, is not valid for small holes.

On the other hand, the function assumed for $\beta$ given by Eq. (21) is continuous when $A=0$ and $\alpha = 0$. Also it has been shown\textsuperscript{44} that, for small deflections, Eq. (30) reduces to the exact solution for a plate with a concentrated load at the center when $A=0$. We may therefore conclude that Eq. (21) is valid for all values of the parameter $A$. Furthermore, Figs. 8

\textsuperscript{43} "The Nonlinear Conical Spring," pp. 681-682.

\textsuperscript{44} Timoshenko and Woinowsky-Krieger, p. 60.
and 9 and Table 1 indicate that, for comparable values of parameters, Eq.(21) gives results at least as accurate as any of the approximate solutions mentioned above.

 Apparently, the major shortcoming of the solution presented in Case I is the length of the final expression, Eq.(26a). However, substitution of numerical values for $A$ and $\nu$ in this expression reduces it to one which is short and contains all of the other parameters as variables. Therefore, although it would be a laborious process, it is possible to present the constants $C_1, C_2, C_3, C_4$ and $C_5$ graphically as functions of $A$. We would then be able to evaluate Eq.(26a) for any value of $A$, and it would be a simple matter to obtain values for $w, \beta, M, \bar{M}, N$ and $\bar{N}$ for any combination of parameters. Also, the solution given in Case I could be programmed for a digital computer, and the variables, $w, \beta, M, \bar{M}, N$, and $\bar{N}$, could be evaluated very rapidly for a disk spring of any dimensions.

Since the assumption, Eq.(21), was obtained from the exact linear solution for an annular plate, we would expect the results obtained in Case I to be excellent in the case of a plate. Table 1 substantiates such a belief. With the exception of $M$, none of the values given by the approximate solution differs from the exact values by more than 2%, and many of the approximate values agree with the exact values to three significant figures. Even in the case of the meridional bending moment, $M$, the absolute difference between values
given by the approximate and exact solutions is of about the same magnitude as the absolute difference between the approximate and exact values of \( \bar{M} \). Furthermore, compared to \( \bar{M} \), the magnitude of \( M \) is very small and would therefore never be critical in the analysis of disk springs. We may therefore conclude that the large percentage errors in values of \( M \) are not as serious as they seem to be.

CASE II

Since the small deflection theory does not differentiate between the boundary conditions stated in Eqs. (20a) and (20b) and those stated in Eqs. (28a) and (28b), Eq. (21) could have been employed in Case II also. However, it was considered desirable to find a somewhat shorter solution than that obtained in Case I, and the assumption that the bending moment \( M \) be zero was used instead. Such an assumption is based on the fact that \( M \) is very small compared to \( \bar{M} \) in a great many cases, and also it was found to give excellent results in certain cases for disk springs with free edges.

In the case of the annular plate, Fig. 10 seems to indicate that the assumption, \( M=0 \), gives good results. Also, since deflections are relatively insensitive to small variations in the loading function, it may well be that the deflections given by Eq. (35a) are even closer to the exact deflections than the assumed shear in Fig. 10 is to the exact shear. We should keep in mind, however, that the same arguments used in the discussion of Eq. (30) in the first section
of Chapter IV hold here also, so that the assumption, Eq.(30), should not be expected to give good results for small values of $A$.

Since such promising results were obtained in the case of the annular plate, we may conclude that the assumption, Eq.(30), will give good results in the case of a cone also, provided that $\Phi$ is very small. However, it was discovered that for values of $\Phi$ even as large as 0.05, Eq.(35a) gives very large discrepancies when shears are compared as in Fig. 10. We may therefore conclude that, for relatively large values of $\Phi$, the boundary conditions, Eqs.(28a) and (28b), are so restrictive that considerable bending of the meridian lines is necessary before appreciable deflection may take place. Therefore, $M$ may not be neglected in such cases.

It is interesting to note that if the direction of loading is taken opposite to the usual direction of loading, results comparable to those shown in Fig. 10 may be obtained. Such results should probably be expected, since when the load is taken in the direction opposite to that shown in Fig. 4a, there is a tendency for the meridian lines to be extended rather than compressed. Since it is compressive forces which cause the bending of the meridian lines mentioned in the previous paragraph, it seems reasonable to assume that extending the meridian lines will reduce their tendency to bend.
Therefore, it seems likely that $M$ will be at least as small in the case of a conical disk with the load reversed as it is in the case of an annular plate. Although such reasoning leads us to believe that Eq. (30) may give good results in the case of a conical disk with the load reversed, such a case is of little significance in practical applications of Belleville springs.
FIG. 1a DISK SPRING APPROXIMATELY TO SCALE

FIG. 1b SCHEMATIC DRAWING OF DISK SPRING SHOWING THE SENSE OF VARIOUS QUANTITIES
FIG 2 (SEE RYAN, P. 432)
FIG. 3a SPRING MECHANISM OF BELLEVILLE SPRINGS STACKED IN SERIES

FIG. 3b SPRING MECHANISM OF BELLEVILLE SPRINGS STACKED IN PARALLEL
FIG. 4 VARIATIONS IN METHODS OF LOADING BELLEVILLE SPRINGS

FIG. 5 THE RADially TAPERED DISK SPRING
FIG. 6a  FREE BODY DIAGRAM OF A SEMI-INFINITESIMAL ELEMENT OF THE SHELL. ALL QUANTITIES ARE SHOWN IN THEIR POSITIVE SENSE.

FIG. 6b  FREE BODY DIAGRAM OF THE MIDDLE PORTION OF THE SHELL. ALL QUANTITIES ARE SHOWN IN THEIR POSITIVE SENSE.
FIG. 7a  DIFFERENTIAL ELEMENT OF A MERIDIAN LINE ON THE MIDDLE SURFACE BEFORE AND AFTER DEFORMATION OF THE SHELL. ALL QUANTITIES ARE SHOWN IN THEIR POSITIVE SENSE.

FIG. 7b  DIFFERENTIAL ELEMENT OF A CIRCUMFERENTIAL LINE ON THE MIDDLE SURFACE BEFORE AND AFTER DEFORMATION
A = 0.5 \quad \phi = 0.1 \quad \gamma = 0.05

FIG. 8 LOAD vs. DEFLECTION
A = 0.5 \ \phi = 0.1 \ \tau = 0.03

--- APPROXIMATE SOLUTION
--- NUMERICAL SOLUTION

FIG 9 LOAD vs. DEFLECTION
FIG. 10 A COMPARISON OF THE ACTUAL SHEAR WITH THAT CALCULATED FROM THE ASSUMPTION

\[ A = 0.5 \quad \phi = 0 \quad \gamma = 0.05 \]

- True shear, \( Q \)
- Assumed shear, \( Q' \)
Denotes Series Solution
- Denotes Approximate Solution

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TABLE 1. COMPARISON OF CERTAIN VALUES AS GIVEN BY THE FIRST APPROXIMATE SOLUTION AND AN EXACT SERIES SOLUTION BY G. A. WEMPNER AND R. SCHMIDT. VALUES PERTAIN TO AN ANNULAR PLATE FOR WHICH A = 0.5, $\nu = 0.3$, $\gamma = 0.05$, $p = 0.000,041,325$.

BIBLIOGRAPHY


The expressions for the constants, C₁, C₂ and C₃, in Eq. (23) are given as follows:

\[
C₁ = -\pi \text{E}_{\text{b}} \left\{ \frac{A^6}{384} \left[ (ln\,A)^4 - \frac{2}{3}(ln\,A)^3 + \frac{1}{3}(ln\,A)^2 - \frac{1}{9}(ln\,A) + \frac{1}{54}(\frac{1}{A^3}) \right] + \right. \\
\left. \frac{\Delta A^6}{24} \left[ (ln\,A)^3 - \frac{1}{2}(ln\,A)^2 + \frac{1}{6}(ln\,A) - \frac{1}{36} + \frac{1}{36}(\frac{1}{A^6}) \right] + \frac{\Lambda A^6}{24} \left[ (ln\,A)^2 - \frac{1}{3}(ln\,A) + \frac{1}{18}(\frac{1}{A^6}) \right] + \frac{\Gamma A^6}{32} \left[ (ln\,A)^3 - \frac{3}{4}(ln\,A)^2 + \frac{3}{8}(ln\,A) - \frac{3}{32} + \frac{3}{32}(\frac{1}{A^6}) \right] + \frac{\Sigma A^6}{16} \left[ (ln\,A)^2 - \frac{1}{2}(ln\,A) + \frac{1}{8} - \frac{1}{8}(\frac{1}{A^6}) \right] + \frac{\Lambda A^6}{6} \left[ (ln\,A)^2 - \frac{1}{3}(ln\,A) + \frac{1}{18} \right] + \frac{\Lambda A^6}{3} \left[ ln\,A - \frac{1}{6} + \frac{1}{6}(\frac{1}{A^6}) \right] + \frac{\Delta A^6}{4} \left[ (ln\,A)^2 - \frac{1}{2}(ln\,A) + \frac{1}{8} - \frac{1}{8}(\frac{1}{A^6}) \right] + \right. \\
\left. \frac{\Delta A^6}{2} \left[ ln\,A - \frac{1}{4} + \frac{1}{4}(\frac{1}{A^6}) \right] + \frac{\Gamma^2 A^2}{2} \left[ (ln\,A)^2 - (ln\,A) + \frac{1}{2} - \frac{1}{2}(\frac{1}{A^6}) \right] + \right. \\
\left. \Lambda^2 A^2 \left[ ln\,A - \frac{1}{2} + \frac{1}{2}(\frac{1}{A^6}) \right] + \frac{\Delta A^4}{6} \left[ A^2 - 1 \right] + \frac{\Gamma A^4}{4} \left[ ln\,A - \frac{1}{2} + \frac{1}{2}(\frac{1}{A^6}) \right] + \Lambda^2 A^2 \right\} 
\]
\[
C_e = -\pi \varepsilon h b^3 \left\{ \frac{A^4}{60} \left[ (\ln A)^3 - \frac{3}{8} (\ln A)^2 + \frac{3}{25} (\ln A) - \frac{1}{25} + \frac{6}{125} \left( \frac{1}{A^3} \right) \right] + \frac{\tau A^5}{20} \left[ (\ln A)^2 - \frac{7}{3} (\ln A) + \frac{2}{25} - \frac{7}{25} \left( \frac{1}{A^4} \right) \right] + \frac{\tau A^5}{12} \left[ (\ln A)^2 - \frac{2}{3} (\ln A) + \frac{7}{9} - \frac{2}{3} \left( \frac{1}{A^3} \right) \right] + \frac{TA^2}{6} \left[ (\ln A)^2 - \ln A + \frac{1}{2} - \frac{1}{2} \left( \frac{1}{A^2} \right) \right] + \frac{2\Delta A^3}{15} \left[ (\ln A)^2 - \frac{7}{3} (\ln A) + \frac{2}{25} - \frac{2}{25} \left( \frac{1}{A^4} \right) \right] + \frac{2\Delta A^4}{5} \left[ \ln A - \frac{1}{5} + \frac{1}{5} \left( \frac{1}{A^5} \right) \right] + \frac{2\Delta A^5}{3} \left[ \ln A - \frac{1}{3} + \frac{1}{3} \left( \frac{1}{A^6} \right) \right] + \frac{2\Delta A^5}{5} \left[ A^5 - 1 \right] + \Delta T A^5 \left[ \ln A - \frac{1}{2} + \frac{1}{2} \left( \frac{1}{A^2} \right) \right] + \frac{2\Delta A^5}{15} \left[ \ln A - \frac{1}{5} + \frac{1}{5} \left( \frac{1}{A^5} \right) \right] + \frac{2\Delta A^5}{3} \left[ A^5 - 1 \right] + \Delta T T \left[ A^2 - 1 \right] + \frac{\Delta T}{9} \left[ (\ln A)^2 - \frac{7}{3} (\ln A) + \frac{2}{9} - \frac{2}{9} \left( \frac{1}{A^3} \right) \right] + \frac{\Delta T A^3}{3} \left[ \ln A - \frac{1}{3} + \frac{1}{3} \left( \frac{1}{A^3} \right) \right] + \Delta T A^3 \left[ \ln A - \frac{1}{2} - \frac{1}{2} \left( \frac{1}{A^2} \right) \right] + \frac{\Delta T A^4}{2} \left[ (\ln A)^2 - 2 (\ln A) + 2 - 2 \left( \frac{1}{A} \right) \right] + \frac{\Delta T A^5}{3} \left[ A^5 - 1 \right] + \Delta T A^5 \left[ \ln A - \frac{1}{2} + \frac{1}{2} \left( \frac{1}{A^2} \right) \right] + \frac{\Delta T A^6}{3} \left[ A^6 - 1 \right] + \frac{\Delta T A^7}{2} \left[ \ln A + \frac{1}{2} - \frac{1}{2} A^2 \right] + \frac{\Delta T A^8}{2} \left[ A^2 - 1 \right] + \frac{\Delta T A^9}{2} \left[ A^2 - 1 \right] \right\} - A^4 \left[ \frac{1}{A^4} - \frac{1}{A^5} \right] - \frac{1}{A^2} \left[ A^2 - 1 \right] - \frac{1}{A^3} \left[ A^3 - 1 \right] - \frac{1}{A^4} \left[ A^4 - 1 \right] - \frac{1}{A^5} \left[ A^5 - 1 \right] - \frac{1}{A^6} \left[ A^6 - 1 \right] - \frac{1}{A^7} \left[ A^7 - 1 \right] - \frac{1}{A^8} \left[ A^8 - 1 \right]
\]

\[
C_3 = -\pi \varepsilon h b^3 \left\{ \frac{A^4}{36} \left[ (\ln A)^2 - \frac{1}{2} (\ln A) + \frac{1}{8} - \frac{1}{8} \left( \frac{1}{A^3} \right) \right] + \frac{TA^4}{6} \left[ \ln A - \frac{1}{4} + \frac{1}{4} \left( \frac{1}{A^4} \right) \right] + \frac{TA^4}{3} \left[ \ln A - \frac{1}{2} + \frac{1}{2} \left( \frac{1}{A^2} \right) \right] + \frac{2TA^5}{3} \left[ \ln A + \frac{1}{2} - \frac{1}{2} A^2 \right] + \frac{TA^6}{4} \left[ A^6 - 1 \right] + \Delta T \left[ A^2 - 1 \right] + 2\Delta T \left[ A - 1 \right] + \frac{1}{A} \left[ (\ln A)^2 - 2 (\ln A) + 2 - 2 \left( \frac{1}{A} \right) \right] + \frac{2\Delta T}{3} \left[ A^3 - 1 \right] + \frac{2\Delta T A^4}{2} \left[ A^4 - 1 \right] + \frac{\Delta T A^5}{2} \left[ A^5 - 1 \right] + \frac{T^2}{2A^2} \left[ A^2 - 1 \right] \right\} - A^4 \left[ \frac{1}{A^4} - \frac{1}{A^5} \right] - \frac{1}{A^2} \left[ A^2 - 1 \right] - \frac{1}{A^3} \left[ A^3 - 1 \right] - \frac{1}{A^4} \left[ A^4 - 1 \right] - \frac{1}{A^5} \left[ A^5 - 1 \right] - \frac{1}{A^6} \left[ A^6 - 1 \right] - \frac{1}{A^7} \left[ A^7 - 1 \right] - \frac{1}{A^8} \left[ A^8 - 1 \right]
\]