LARGE DEFLECTION OF A CIRCULAR PLATE SUBJECTED TO A CONCENTRATED LOAD AT THE CENTER

by

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APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

Robert Schmidt — May 11, 1960

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Date

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TABLE OF CONTENTS

NOMENCLATURE

r, θ	Polar coordinates
h	Thickness of the plate
P	Concentrated load at the center of the plate
σ_r	Radial stress in polar coordinates
σ_t	Tangential stress in polar coordinates
u, w	Components of displacements
ϵ_r	Radial strain im polar coordinates
ϵ_t	Tangential strain in polar coordinates
E	Modulus of elasticity in tension and compression
ν	Poisson's ratio
D	Flexural rigidity of plate
M_r, M_t	Radial and tangential moments in polar coordinates
Q_r	Radial shearing force per unit length
N_r, N_t	Normal forces per unit length, in radial and tangential directions, respectively
b	Radius of the plate

Auxiliary notation is explained in the text

CHAPTER I

INTRODUCTION

Nonlinear boundary value problems of practical importance occur in many branches of engineering and physical science. Unfortunately, very few exact solutions to problems of this kind have been found. Thus, it becomes necessary to resort to the approximate methods of analysis, such as successive approximations, step-by-step integration, iteration, and the energy methods.

This thesis contains a solution of von Karman's equations[1] for a rigidly clamped circular plate subjected to a concentrated load at the center. The problem is governed by a system of two nonlinear ordinary differential equations the exact solution of which is unknown. Therefore, a successive approximation method, commonly known as "perturbation method" is employed. This method was used recently in the analysis of a simply supported circular plate under a uniformly distributed load by M. Stippes and A. H. Hausrath[2] who also suggested the application of the method to the analysis of a circular plate subjected to a

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[1] S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, 2d ed. rev., McGraw-Hill Book Co., Inc., 1959, p. 418.

concentrated load:

> "The method also has the advantage that it can be used directly to compute deflections in the sense of von Karman for concentrated loads which, to the author's knowledge, have not been treated theoretically in the literature."[2]

Since the appearance of Stippes and Hausrath's paper the problem of a circular plate with a simply supported moveable edge was solved numerically by Naghdi[3]. Also, in the second edition of "Theory of Plates and Shells"[4] approximate solutions, obtained by the Ritz-Galerkin method, are presented for various boundary conditions. However, all of these results are very approximate in nature, and the problem still awaits a rigorous treatment. The advantage of the perturbation method used in this thesis is that it gives practically exact results for small values of the load parameter, thus the range of validity of the linear solution of the problem can be established.

The main text of the thesis will require the reader to have some knowledge of the large deflection theory of plates. For this reason the appendix contains a resume of this theory.

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[2] M. Stippes and A. H. Hausrath, "Large Deflections of Circular Plates," J. Appl. Mech., Trans. ASME, Vol. 19, No. 3, September, 1952, pp. 287-292

[3] P. M. Naghdi, "Bending of Elastoplastic Circular Plates With Large Deflections," J. Appl. Mech., Trans. ASME, Vol. 19, No. 3, September, 1952, pp. 293-300

[4] Timoshenko and Woinowsky-Krieger, op.cit., pp.400-427

CHAPTER 2

GOVERNING EQUATIONS

Two fundamental assumptions are usually made in the theory of plates: small deflections and linear stress-strain relations (Hooke's law). Under these assumptions, the stress distribution in an elastic body is governed by linear differential equations. If we relinquish one or both of the assumptions, the governing equations become nonlinear.

The problem considered in this thesis retains the assumption that the stress components are linear functions of the strain components, but the deflections are not small in comparison with the thickness of the plate, although they are still small as compared with other dimensions. In this case the nonlinear deflection terms in the expressions for strain components become significant in magnitude and cannot be neglected, as they usually are in the small deflection theory. Hence, the strain in the middle plane of the plate must be included in the analysis.

The problem under consideration is a circular plate with a clamped and immovable edge subjected to a transverse concentrated load at the center. The problem possesses axial symmetry, hence, the governing equations are ordinary differential equations.

The following expressions, necessary for the proposed analysis were taken from "Theory of Plates and Shells." [5] A list of the nomenclature to be used is found on page ix.

Using a polar coordinate system with the origin at the center of the plate we can express the radial and tangential moments as

$$M_r = - D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$
$$M_t = - D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \qquad (2.1)$$

where w is the displacement component, of a point in the middle plane of the plate, perpendicular to the plane of the plate. Denoting the radial displacement component of a point in the middle plane of the plate by u, the strain components of the middle plane in the radial and tangential directions are, respectively,

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad (2.2)$$

$$\epsilon_t = \frac{u}{r} \qquad (2.3)$$

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[5] Timoshenko and Woinowsky-Krieger, <u>op.cit.</u>, pp. 396-405.

Denoting the corresponding tensile forces per unit length by N_r and N_t and using Hooke's law, we obtain

$$N_r = \frac{Eh}{1-\nu^2}(\epsilon_r + \nu\epsilon_t) \tag{2.4}$$

$$N_t = \frac{Eh}{1-\nu^2}(\epsilon_t + \nu\epsilon_r) \tag{2.5}$$

The three equilibrium equations may be presented in the following form:

$$N_r - N_t + r\frac{dN_r}{dr} = 0 \tag{2.6}$$

$$Q_r = -\frac{P}{2\pi r} - N_r\frac{dw}{dr} \tag{2.7}$$

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = -\frac{Q_r}{D} \tag{2.8}$$

Eliminating Q_r in Eq. (2.7) by means of Eq. (2.8) and using Eqs. (2.2) to (2.5) we can express the equilibrium equations (2.6) and (2.7) in terms of displacement components in the following form:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -\frac{1-\nu}{2r}\left(\frac{dw}{dr}\right)^2 - \frac{dw}{dr}\frac{d^2w}{dr^2} \tag{2.9}$$

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{12}{h^2}\frac{dw}{dr}\left[\frac{du}{dr} + \nu\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right] + \frac{P}{2\pi Dr} \tag{2.10}$$

These are the governing differential equations in terms of displacements. However, for our purpose, it is more convenient to write the governing equations in a different form.

Eq. (2.9) is equivalent to

$$N_r - N_t + r \frac{dN_r}{dr} = 0 \tag{2.11}$$

and Eq. (2.10) may be written as

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{1}{D} N_r \frac{dw}{dr} + \frac{P}{2\pi D}\frac{1}{r} \tag{2.12}$$

Eqs. (2.11) and (2.12) provide us with two equations involving three unknown functions w, N_r and N_t. A third equation can be derived in the following manner. From Eqs. (2.2) and (2.3) we derive a compatibility equation:

$$\epsilon_r = \epsilon_t + r \frac{d\epsilon_t}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{2.13}$$

Eq. (2.4) and (2.5) can be solved for ϵ_t and ϵ_r :

$$\epsilon_t = \frac{1}{hE}(N_t - \nu N_r) \tag{2.14}$$

$$\epsilon_r = \frac{1}{hE}(N_r - \nu N_t) \tag{2.15}$$

Substituting (2.14) and (2.15) in (2.13) and making use of (2.11), we obtain

$$\frac{d^2N_r}{dr^2} + \frac{3}{r}\frac{dN_r}{dr} = -\frac{Eh}{2}\frac{1}{r^2}\left(\frac{dw}{dr}\right)^2 \tag{2.16}$$

Letting $\frac{dw}{dr} = B$, Eqs. (2.16) and (2.12) become

$$\frac{d^2N_r}{dr^2} + \frac{3}{r}\frac{dN_r}{dr} = -\frac{Eh}{2}\frac{B^2}{r^2} \qquad (2.17)$$

$$\frac{d^2B}{dr^2} + \frac{1}{r}\frac{dB}{dr} - \frac{B}{r^2} = \frac{1}{D}N_rB + \frac{P}{2\pi D}\frac{1}{r} \qquad (2.18)$$

These two equations, containing the two unknown functions N_r and B are used to solve the current problem.

CHAPTER 3

SIMPLIFICATION OF THE GOVERNING EQUATIONS

By making suitable substitutions, Eqs. (2.17) and (2.18) can be put into a dimensionless form which renders them more convenient for our use. If we let $a = \frac{r}{b}$, where b is the external radius of the plate, Eqs. (1.17) and (1.18) become

$$\frac{d^2N_r}{da^2} + \frac{3}{a}\frac{dN_r}{da} = -\frac{Eh}{2}\frac{B^2}{a^2} \tag{3.1}$$

$$\frac{d^2B}{da^2} + \frac{1}{a}\frac{dB}{da} - \frac{B}{a^2} = \frac{b^2}{D} N_r B - \frac{Pb}{2\pi D}\frac{1}{a} \tag{3.2}$$

Letting

$$N = \epsilon_r + \nu \epsilon_t = \frac{1-\nu^2}{Eh} N_r \tag{3.3}$$

$$\bar{N} = \epsilon_t + \nu \epsilon_r = \frac{1-\nu^2}{Eh} N_t \tag{3.4}$$

Eqs. (3.1) and (3.2) become

$$\frac{d^2N}{da^2} + \frac{3}{a}\frac{dN}{da} = -\frac{1-\nu^2}{2}\frac{B^2}{a^2} \tag{3.5}$$

$$\frac{d^2B}{da^2} + \frac{1}{a}\frac{dB}{da} - \frac{1}{a^2} B = \frac{b^2Eh}{D(1-\nu^2)} NB + \frac{Pb}{2\pi D}\frac{1}{a} \tag{3.6}$$

Since the flexural rigidity of the plate, D, is defined as

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Eq. (3.6) may be rewritten as

$$\frac{d^2B}{da^2} + \frac{1}{a}\frac{dB}{da} - \frac{1}{a^2}B = \frac{12b^2}{h^2}NB + \frac{Pb}{2\pi D}\frac{1}{a} \qquad (3.7)$$

If we introduce the notation $x = a^2$, Eqs. (3.5) and (3.7) become

$$\frac{d^2N}{dx^2} + \frac{2}{x}\frac{dN}{dx} = -\frac{1-\nu^2}{8}\frac{B^2}{x^2} \qquad (3.8)$$

$$\frac{d^2B}{dx^2} + \frac{1}{x}\frac{dB}{dx} - \frac{B}{4x^2} = \frac{3b^2}{h^2}\frac{NB}{x} + \frac{Pb}{8\pi D}\frac{1}{x\sqrt{x}} \qquad (3.9)$$

Eqs. (3.8) and (3.9) will assume their final form by introducing the following notation:

$$I = \frac{1-\nu^2}{8}$$

$$J = \frac{3b^2}{h^2}$$

$$P = \frac{Pb\sqrt{IJ}}{8\pi D}$$

$$y = J\,a^2 N = J\,x\,N$$

$$z = \sqrt{IJ}\,a\,B = \sqrt{IJ}\,\sqrt{x}\,B$$

$$N = \frac{y}{Jx}$$

$$B = \frac{1}{\sqrt{IJ}}\,x^{-\frac{1}{2}}\,z$$

With this notation Eqs. (3.8) and (3.9) become, respectively,

$$y'' = -\frac{z^2}{x^2} \qquad (3.10)$$

$$z'' = \frac{yz}{x^2} + \frac{p}{x} \tag{3.11}$$

and the first of Eqs. (2.1) may be written as

$$M_r = -\frac{D}{b\sqrt{IJ}}\left[2\ z' - (1-\nu)\frac{z}{x}\right] \tag{3.12}$$

where the primes denote derivatives with respect to x.

The transformation of the governing differential equations to a more convenient form necessitates a corresponding transformation of the boundary conditions. In the original formulation of the problem the boundary conditions required that the radial displacement u and the rotation $\frac{dw}{dr} = B$ of the middle plane vanish at the edge and center, i.e.,

$$B\big|_{r=b} = 0 \tag{3.13}$$

$$u\big|_{r=b} = 0 \tag{3.14}$$

$$B\big|_{r=0} = 0 \tag{3.15}$$

$$u\big|_{r=0} = 0 \tag{3.16}$$

An additional condition is that the deflection w vanish at the edge, i.e,

$$w\big|_{r=b} = 0 \tag{3.17}$$

Referring to the notation on page we see that at $r = b$ (i.e., $x = 1$) the condition corresponding to (3.13) would be

$$z = 0 \qquad (3.18)$$

since

$$z = \sqrt{IJ}\; x^{\frac{1}{2}}\; B$$

The condition corresponding to (3.14) is obtained in the following manner. The radial displacement u, from Eq. 2.3 can be expressed as

$$u = r\,\epsilon_t \qquad (3.19)$$

N and $\overline{N}$ were previously defined as

$$N = \epsilon_r + \nu\epsilon_t \doteq \frac{1-\nu^2}{Eh} N_r$$

$$\overline{N} = \epsilon_t + \nu\epsilon_r = \frac{1-\nu^2}{Eh} N_t$$

Solving these two equations for we obtain

$$\epsilon_t = \frac{\overline{N} - \nu N}{1-\nu^2} \qquad (3.20)$$

With the above notation Eq. (2.6) becomes

$$\overline{N} = N + r\,\frac{dN}{dr} \qquad (3.21)$$

Substituting this into Eq. (3.20), we obtain

$$\epsilon_t = \frac{r\,\frac{dN}{dr} + (1-\nu)\,N}{1-\nu^2}$$

and finally with the aid of Eq. (3.19),

$$u = \frac{r}{1-\nu^2}\left[r\,\frac{dN}{dr} - (1-\nu)N\right] \qquad (3.22)$$

In terms of the variable x (remembering that $x = \frac{r^2}{b^2}$) this equation becomes

$$u = \frac{b\sqrt{x}}{1-\nu^2}\left[2x\,\frac{dN}{dx} - (1-\nu)N\right] \tag{3.23}$$

and since $N = \frac{y}{xJ}$

$$u = \frac{b\sqrt{x}}{J(1-\nu^2)}\left[2\,\frac{dy}{dx} - (1+\nu)\,\frac{y}{x}\right]$$

Hence, the condition u = 0 at r = b (x = 1) becomes

$$\frac{dy}{dx} - \frac{1+\nu}{2}\,y = 0 \tag{3.24}$$

For u to vanish at the center, r = x = o, requires that N be finite. Since y = JxN the boundary condition corresponding to (3.16) becomes

$$y\Big|_{x=0} = 0 \tag{3.25}$$

The boundary conditions to be used may be summerized as follows:

$$z\Big|_{x=0} = 0$$

$$y\Big|_{x=0} = 0$$

$$z\Big|_{x=1} = 0$$

$$\left[\frac{dy}{dx} - \left(\frac{1+\nu}{2}\right)y\right]_{x=1} = 0$$

CHAPTER 4

PERTURBATION METHOD

A complete statement of the boundary value problem under consideration would be:

$$y'' = -\frac{z^2}{x^2} \tag{4.1}$$

$$z'' = \frac{yz}{x^2} + \frac{p}{x} \tag{4.2}$$

$$z\big|_{x=0} = 0 , \quad y\big|_{x=0} = 0$$

$$z\big|_{x=1} = 0 , \quad \left[\frac{dy}{dx} - \left(\frac{1+\nu}{2}\right)y\right]_{x=1} = 0$$

where p is a constant parameter proportional to the load.

The method employed in solving this boundary value problem is the so-called "perturbation method." For this purpose, we assume that y and z depend analytically on the load parameter p, so that these two dependent variables may be represented by the following power series in p,

$$y = y_2\ (x)\ p^2 + y_4\ (x)\ p^4 + y_6\ (x)\ p^6 + ---- ; \tag{4.3}$$

$$z = z_1(x)\ p + z_3(x)p^3 + z_5(x)\ p^5 + z_7(x)p^7 + ----- ; \tag{4.4}$$

Substituting these series in the governing equations (4.1) and (4.2), and equating the coefficients of equal powers of p, we obtain an infinite system of linear differential equations

$$y_2'' = -\frac{z_1^2}{x^2}$$

$$y_4'' = -\frac{2z_1 z_3}{x^2}$$

$$y_6'' = -\left(\frac{z_3^2 + 2z_1 z_5}{x^2}\right)$$

$$\cdots\cdots\cdots$$

$$z_1'' = \frac{1}{x}$$

$$z_3'' = \frac{y_2 z_1}{x^2}$$

$$z_5'' = \frac{y_2 z_3 + z_1 y_4}{x^2}$$

$$z_7'' = \frac{y_2 z_5 + y_4 z_3 + y_6 z_1}{x^2}$$

$$\cdots\cdots\cdots$$

the successive solutions of which enable us to determine the unknown functions y_2, y_4, y_6, ····· and z_1, z_3, z_5,····· in the series (4.3) and (4.4). Of course, it is physically impossible to solve an infinite number of equations. Hence, we limit ourselves to the determination of the seven functions y_2, y_4, y_6 and z_1, z_3, z_5, z_7. It may be noted at this point that the expression for $(z_1 p)$ will be the linear solution of the problem under consideration.

The actual expressions obtained for these functions will not be presented herein because of their length. Only numerical and graphical results will be stated.

CHAPTER 5

RESULTS

We shall now present the numerical results for the deflection under the load, w ; the forces N_{r_0} at the center and N_{r_1}, at the edge; and the moment at the edge, M_{r_1}. The expressions for these quantities are

$$w_0 = -\frac{b}{2\sqrt{IJ}}\int_0^1 \frac{z}{x}\,dx$$

$$N_{r_0} = \frac{Eh}{J(1-\nu^2)}\left.\frac{y}{x}\right|_{x=0}$$

$$N_{r_1} = \frac{Eh}{J(1-\nu^2)}\left.\frac{y}{x}\right|_{x=1}$$

$$M_{r_1} = -\frac{2D}{b\sqrt{IJ}}\left.\frac{dz}{dx}\right|_{x=1}$$

The corresponding quantities in the tangential direction are

$$N_{t_0} = N_{r_0}$$

$$N_{t_1} = \nu N_{r_1}$$

$$M_{t_1} = \nu M_{r_1}$$

For convenience these quantities can be put into dimensionless form. Using the notation previously stated and Eqs. (4.3) and (4.4) we can write

$$\frac{w_0}{h} = -\frac{b}{2\sqrt{IJ}\,h}\int_0^1 \frac{p}{x}\,(z_1 + z_3p^2 + z_5p^4 + z_7p^6 + \cdots)\,dx \quad (5.1)$$

$$\frac{b^2 N_r}{Eh^3} = \frac{4D}{Eh^3}\frac{1}{x}\left(Y_2 p^2 + Y_4 p^4 + y_6 p^6 + \cdots\cdots\right) \tag{5.2}$$

$$\frac{b^2 M_r}{Eh^4} = -\frac{2bD}{Eh^4\sqrt{IJ}}\; p\left(z_1' + z_3' p^2 + z_5' p^4 + z_7' p^6 + \cdots\right)\Big|_{x=1} \tag{5.3}$$

where $$p = \frac{27(1-\nu^2)^3}{32\pi^2}\left(\frac{Pb^2}{Eh^4}\right)^2 \tag{5.4}$$

Substituting the results of the perturbation process into Eqs. (5.1), (5.2) and (5.3) and taking $\nu = 0.3$, we obtain

$$-\frac{w_o}{h} = 0.217{,}246{,}5\,\frac{Pb^2}{Eh^4} - 0.004{,}561{,}83\left(\frac{Pb^2}{Eh^4}\right)^3$$
$$+ 0.000{,}299{,}065{,}01\left(\frac{Pb^2}{Eh^4}\right)^5 - 0.000{,}026{,}201{,}231\left(\frac{Pb^2}{Eh^4}\right)^7$$

$$\frac{b^2 N_{r_o}}{Eh^3} = 0.058{,}152{,}264\left(\frac{Pb^2}{Eh^4}\right)^2 - 0.002{,}329{,}816{,}6\left(\frac{Pb^2}{Eh^4}\right)^4$$
$$+ 0.000{,}176{,}187{,}84$$

$$\frac{b^2 N_{r_1}}{Eh^3} = 0.016{,}855{,}7\left(\frac{Pb^2}{Eh^4}\right)^2 - 0.000{,}740{,}490\left(\frac{Pb^2}{Eh^4}\right)^4$$
$$+ 0.000{,}056{,}871{,}5\left(\frac{Pb^2}{Eh^4}\right)^6$$

$$\frac{b^2 M_{r_1}}{Eh^4} = -0.079{,}577{,}5\left(\frac{Pb^2}{Eh^4}\right) + 0.001{,}402{,}01\left(\frac{Pb^2}{Eh^4}\right)^3$$
$$- 0.000{,}092{,}015{,}4\left(\frac{Pb^2}{Eh^4}\right)^5 + 0.000{,}008{,}068{,}56\left(\frac{Pb^2}{Eh^4}\right)^7$$

Graphical representation of these quantities is given in Fig. 5.1.

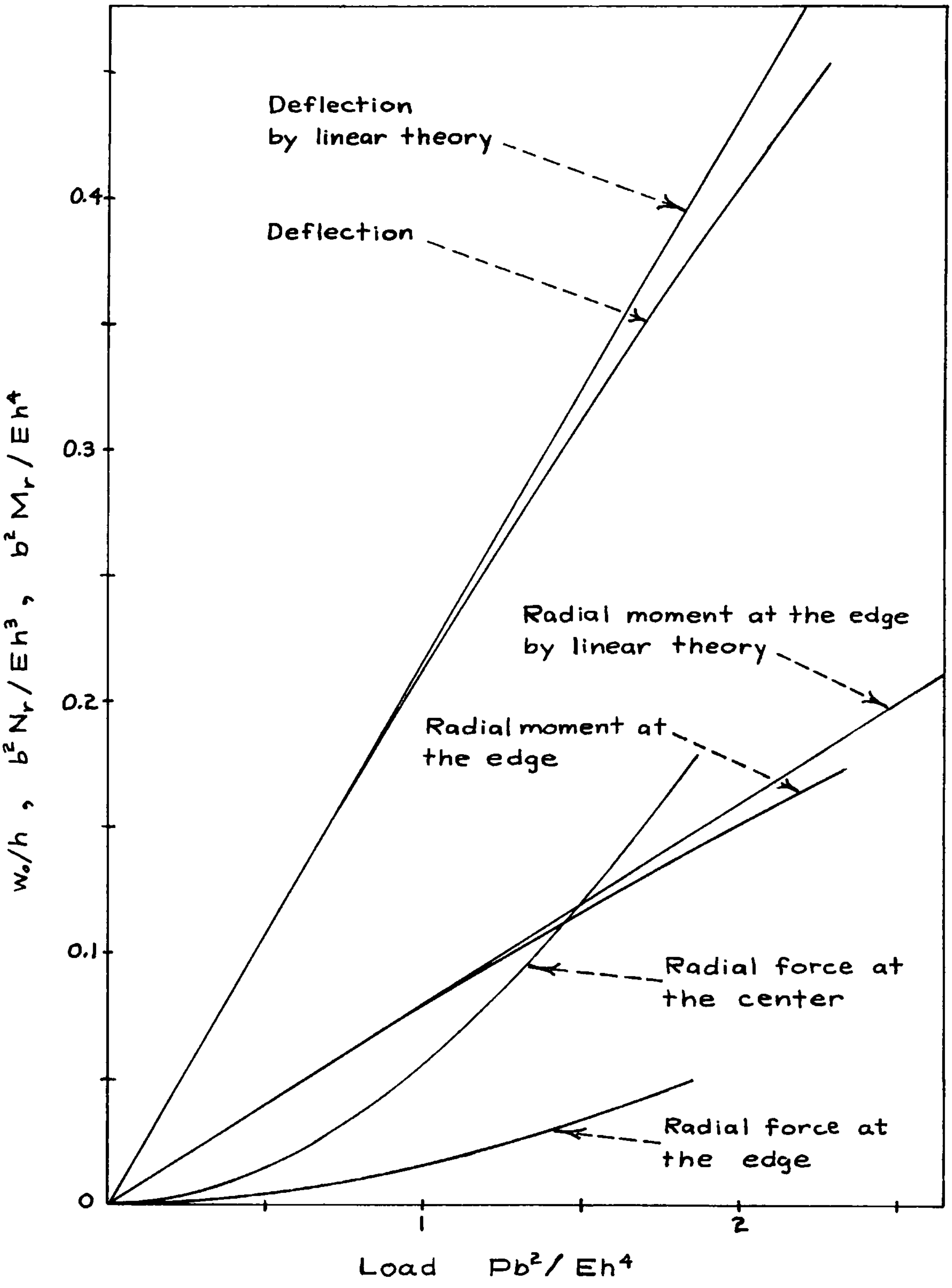
Deflection
by linear theory
Deflection
Radial moment at the edge
by linear theory
Radial moment at
the edge
Radial force at
the center
Radial force at
the edge
0.4
0.3
0.2
0.1
0
1
2
w_o/h , $b^2 N_r/Eh^3$, $b^2 M_r/Eh^4$
Load Pb^2/Eh^4

Fig. 5.1

CHAPTER 6

DISCUSSION OF RESULTS

Fig. 5.1 shows graphically the results obtained for the problem under consideration. The convergence of the series (4.3) and (4.4) is slow for large values of the load parameter. For this reason the curves cannot be extended beyond the positions shown, as the labor in calculating the additional coefficients of p^8, p^9, p^{10}..... becomes prohibitive. On the other hand the perturbation method does show the direct dependence of significant quantities on the load parameter and it also establishes a range of validity for the linear theory. When the maximum deflection w is of the order of 0.5h, the linear theory predicts a maximum deflection about 10 per cent higher. Therefore we may conclude that the linear solution is fairly accurate until the maximum deflection approaches one half the thickness of the plate.

The solution presented in this thesis agrees fairly well with an approximate solution presented in "Theory of Plates and Shells."[6]

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[6] Timoshenko and Woinowsky-Krieger, op.cit., p.415.

APPENDIX

The following is a resume of large deflection theory of plates. Consider a circular plate (Fig. A.1) with the coordinate system shown. Let AB be an infinitesimal line element dr in the middle plane of the unstrained plate. After strain, AB is displaced to A'B' (Fig. A.2), and the displacement components of A are equal to u and w. Let the length of A'B' be dr^*. Engineering strain is defined as

$$\epsilon_r = \frac{dr^* - dr}{dr} \tag{A.1}$$

The length of A'B' may be written as

$$dr^* = \sqrt{dw^2 + (dr + du)^2} \tag{A.2}$$

Dividing this equation by dr, we obtain

$$\frac{dr^*}{dr} = \sqrt{1 + \left[2\frac{du}{dr} + \left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2\right]} \tag{A.3}$$

Expanding the right hand member of Eq. (A.3) in series, we have

$$\frac{dr^*}{dr} = 1 + \frac{1}{2}\left[2\frac{du}{dr} + \left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2\right] - \cdots \cdots \text{(A.4)}$$

(2.1)

Substituting the first two terms of the series (A.4) in Eq. (A.1), we obtain

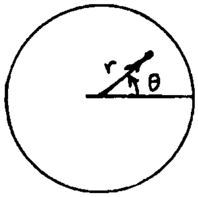

Fig. A.1

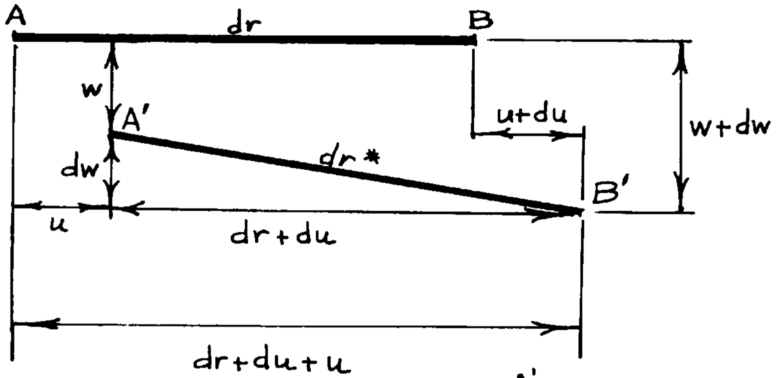

Fig. A.2

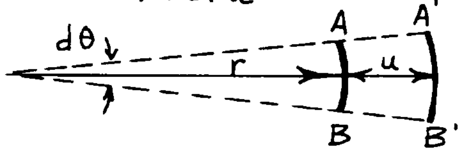

Fig. A.3

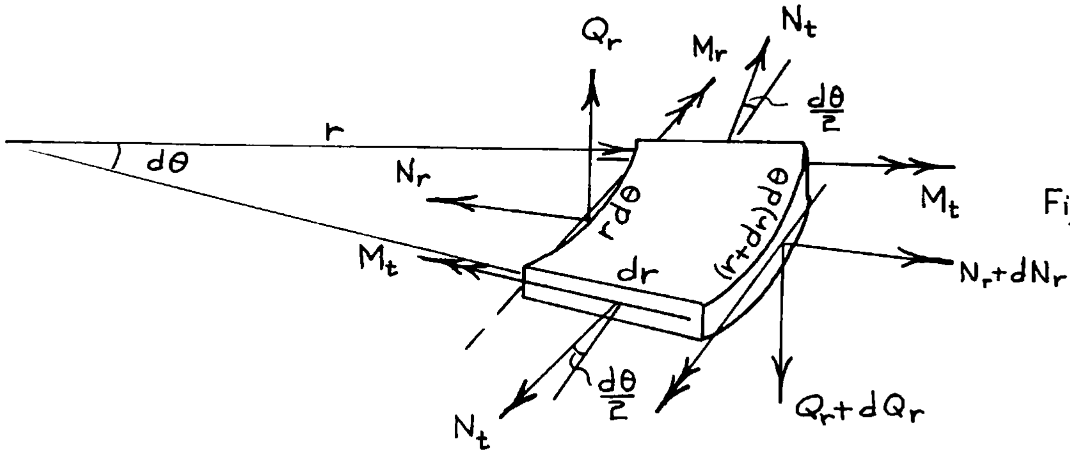

Fig. A.4

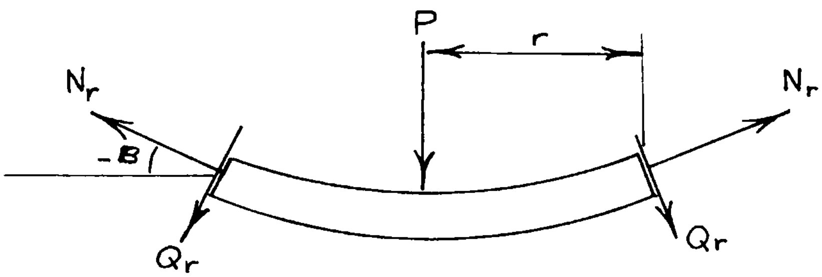

Fig. A.5

$$B = \frac{dw}{dr}$$

$$\epsilon_r = \frac{dr^*}{dr} - 1 \cong \frac{du}{dr} + \frac{1}{2}\left(\frac{du}{dr}\right)^2 + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (A.5)$$

du must be small compared to dr since it is assumed that strains will be small, and we have therefore

$$\frac{du}{dr} \gg \left(\frac{du}{dr}\right)^2$$

The quantity $\left(\frac{dw}{dr}\right)^2$ is not a negligible quantity since the rotations in large deflection theory are large. Eq. (A.5) reduces then to

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (A.6)$$

or, since $\frac{dw}{dr} = B$,

$$\epsilon_r = \frac{du}{dr} - \frac{1}{2} B^2 \quad (A.7)$$

The strain in the tangential direction can be obtained by considering the infinitesimal line element AB (Fig.A.3) in the undeformed body which is displaced to A'B' in the deformed body. The expression for the tangential strain is derived in the following sequence of steps.

$$AB = rd\theta$$

$$A'B' = (r + u)\, d\theta$$

$$\epsilon_t = \frac{A'B' - AB}{AB}, \text{ by definition}$$

$$\epsilon_t = \frac{(r + u)d\theta - rd\theta}{rd\theta}$$

$$\epsilon_t = \frac{u}{r} \quad (A.8)$$

Radial and tangential stress can be expressed in terms of strain, by applying Hooke's law:

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_t) \qquad (A.9)$$

$$\sigma_t = \frac{E}{1-\nu^2}(\epsilon_t + \nu\epsilon_r) \qquad (A.10)$$

Combining Eqs. (A.6, 8, 9, 10), expressions for the tensile forces per unit length may be stated as

$$N_r = \frac{Eh}{1-\nu^2}(\epsilon_r + \nu\epsilon_t) = \frac{Eh}{1-\nu^2}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] \qquad (A.11)$$

$$N_t = \frac{Eh}{1-\nu^2}(\epsilon_t - \nu\epsilon_r) = \frac{Eh}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right], \qquad (A.12)$$

Since $N_r = h\sigma_r$ and $N_t = h\sigma_t$ two equilibrium equations can be derived by considering the forces which act on an infinitesimal element. (Fig. A.4)

Summing forces in the radial direction, we obtain

$$N_r + r\frac{dN_r}{dr} - N_t = 0 \qquad (A.13)$$

Summing moments in the radial direction, we obtain

$$M_r + r\frac{dM_r}{dr} - M_t - Q_r = 0 \qquad (A.14)$$

Substituting[7] $M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$

and $M_t = -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right)$

in Eq. (A.14), we obtain

$$Q_r = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \qquad (A.15)$$

An expression for the shearing force Q_r can be obtained by considering the equilibrium of an inner circular portion of the plate (Fig. A.5). Summing forces in the vertical direction gives

$$Q_r = -\frac{P}{2\pi r} - N_r\frac{dw}{dr} \qquad (A.16)$$

Eqs. (A.13), (A.15) and (A.16) are Eqs. (2.6), (2.8) and (2.7) respectively, in Chapter 1.

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[7] The expressions for M_r and M_t were derived using the small deflection theory of plates but they are also valid in the large deflection theory (S. Timoshenko and S. Woinowsky-Krieger, op.cit., p. 52 and p. 398).

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