ANALYSIS OF THE DEVELOPMENT OF LAMINAR FLOW IN A CIRCULAR CYLINDER FROM A QUIESCENT STATE

by

Subin Banharnsupavat

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SIGNED: Subin Banhamaupavat.

APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

M R Bottaccini
MANFRED R. BOTTACCINI
Professor of Mechanical Engineering

14 January 1962
Date

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SYMBOLS AND NOMENCLATURES

a	radius of the circular cylinder, ft.
g	gravitational acceleration, ft/sec^2
$K(z)$	constant value of parameter, $\frac{1}{\rho}\frac{\partial p}{\partial z}$, ft/sec^2
$\overline{K}(z)$	Hankel Transforms of $K(z)$
p	pressure, lb/ft^2
r	radial distance from the center line of the cylinder, ft.
t	time in seconds
T	time in seconds
u	velocity in the angular direction, ft/sec
v	velocity in the radial direction, ft/sec
$w=w(r,t)$	velocity in the axial direction, ft/sec
$w^*(r,t)$	mean velocity in the axial direction, ft/sec
$\overline{w}_a(r,t)$	Hankel Transforms of $w(r,t)$
z	direction along the center line
$J_n(\xi_i r)$	Bessel function of order n
ξ_i	eigenvalue of equation $J_0(\xi_i r) = 0$
ψ	dimensionless parameter, $\frac{\nu t}{a^2}$
θ	angular direction
ρ	density of fluid, slug/ft^3
γ	specific weight in lb/ft^3
μ	dynamic viscosity in lb-sec/ft^2
ν	kinematic viscosity, ft^2/sec
τ	shear stress, lb/ft^2
Ω	gravitational potential, ft^5/sec^2

CHAPTER 1

INTRODUCTION

The stability of laminar shear flow has received and continues to receive much attention because it is of fundamental importance. A great deal of theoretical and experimental work has been done on the stability of laminar boundary layers and on the determination of criteria for transition.

In order to understand and establish such criteria, there are certain characteristic physical features of the flow that need to be studied. It is known in general that real fluid flow can be influenced by many factors, such as, pressure gradient, surface roughness, Reynolds number, etc. If we introduce any disturbance into the flow regime, for example, by discharging fluid through a valve, the perturbation in the fluid velocity will depend on the history of the motion as well as on future actions. The formulation of the complete problem leads to so complex a system that solution becomes nearly impossible. In order to obtain a manageable formulation we shall restrict ourselves to the study of a limited number of variables, namely, velocity, shear stress, viscosity, piezometric head, time and radius of the cylinder. The problem will be treated in cylindrical co-ordinates since this leads to a steady state solution dependent on only two space variables.

The paper presents a theoretical analysis and calculation for the determination of the velocity profile, the mean velocity, and the shear stress.

CHAPTER 2

ANALYSIS

2.1 Determination of the Velocity

The flow co-ordinates and velocities are shown in the following figure:

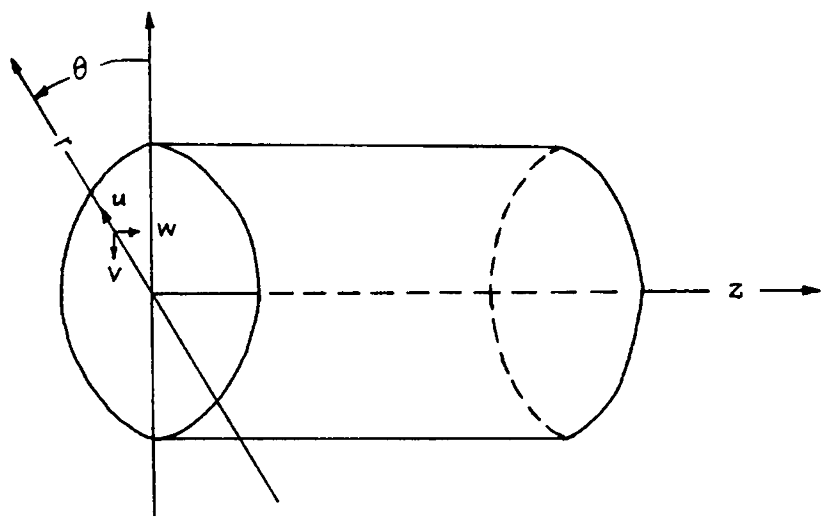

Figure 2.1 Flow Co-ordinates and Velocities

We assume that there exists a fluid at rest in a circular cylinder, and we disturb the fluid by changing the piezometric head at $t = 0^+$. Stated analytically the problem can be represented by the Navier-Stokes equation [1]*:

---

*Number in brackets designate References.

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = \frac{\partial}{\partial r}(p+\rho\Omega)$$

$$+\mu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) \qquad (2.1)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = -\frac{1}{r}\frac{\partial}{\partial \theta}(p+\rho\Omega)$$

$$+\mu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right) \qquad (2.2)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial z}(p+\rho\Omega) + \mu\nabla^2 w \qquad (2.3)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

ρ = density of fluid

u = velocity in the angular direction

v = velocity in the radial direction

w = velocity in the axial direction

θ = angular direction

p = pressure

t = time

Ω = gravitational potential

The continuity equation [1] may be written as

$$\frac{1}{r}\frac{\partial (ur)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \qquad (2.4)$$

It is reasonable to assume symmetry in θ, and we shall also limit ourselves to consider the case of uniform parallel flow in the axial direction. Therefore, equation (2.4) is reduced to

$$\frac{\partial w}{\partial z} = 0 \qquad (2.5)$$

and hence, w is independent of z. So w is only a function of r and t.

The variation in gravitational potential may be neglect due to the flow configuration. Equations (2.1) and (2.2) may be then reduced to

$$(2.1) \qquad \frac{\partial}{\partial r}(p + \rho\Omega) = \frac{\partial}{\partial r}(p) = 0 \qquad (2.6)$$

$$(2.2) \qquad \frac{1}{r}\frac{\partial}{\partial \theta}(p) = 0 \qquad (2.7)$$

Thus, we deduce from equations (2.6) and (2.7) that p is not a function of r and θ.

Then,

$$p = p(z,t) \qquad (2.8)$$

From (2.3),

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}(p) + \mu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right)$$

$$\frac{\partial w}{\partial t} = -K(z,t) + \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right) \qquad (2.9)$$

where,

$$K(z,t) = \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\nu = \text{kinematic viscosity} = \mu/\rho$$

The problem is completely defined by the boundary and initial conditions:

(1) $w(r,t) = 0$, for $r = a$

(2) $w(r,0) = 0$, for $t = 0$

We now multiply equation (2.9) by $rJ_o(\xi_i r)dr$ and integrate from 0 to a, noting that [2] ,

$$-\xi_i^2 \overline{w}_a(r,t) = \int_0^a r \left[\frac{\partial^2 w(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,t)}{\partial r}\right] J_o(\xi_i r)\, dr \qquad (2.10)$$

where

$J_o(\xi_i r)$ = Bessel function of zero order

ξ_i = eigenvalue of equation $J_n(\xi_i r) = 0$

$\overline{w}_a(r,t)$ = Hankel Transforms of $w(r,t)$

Define

$$\overline{K}(z,t) = \int_0^a K(z,t)\, r\, J_o(\xi_i r)\, dr \qquad (2.11)$$

and

$$\overline{w}_a(r,t) = \int_0^a w(r,t)\, r\, J_o(\xi_i r)\, dr \qquad (2.12)$$

then,

$$\frac{\partial \overline{w}_a(r,t)}{\partial t} = \overline{K}(z,t) + \nu\left[-\xi_i^2 \overline{w}_a(r,t)\right] \qquad (2.13)$$

$$\frac{\partial \overline{w}_a(r,t)}{\partial t} + \nu \xi_i^2 \overline{w}_a(r,t) = \overline{K}(z,t)$$

$$\frac{\partial}{\partial t}\left[\overline{w}_a(r,t)\, e^{\nu \xi_i^2 t}\right] = \overline{K}(z,t)\, e^{\nu \xi_i^2 t}$$

$$\overline{w}_a(r,t)\, e^{\nu \xi_i^2 t} = \int_0^t \overline{K}(z,t)\, e^{\nu \xi_i^2 T}\, dT \qquad (2.14)$$

We shall assume that the initial disturbance is introduced instantaneously or impulsively. Thus, we can write

$$\overline{w}_a(r,t)\, e^{\nu \xi_i^2 t} = \int_0^t \overline{K}(z)\, e^{\nu \xi_i^2 T}\, dT$$

and

$$\overline{w}_a(r,t) = \overline{K}(z) \int_0^t e^{-\nu \xi_i^2 (t-T)}\, dT$$

$$= \frac{\overline{K}(z)}{\nu \xi_i^2} [\, 1 - e^{-\nu \xi_i^2 t}\,] \tag{2.15}$$

from (2.1), we obtain [2]

$$\overline{K}(z) = \int_0^a K(z)\, r\, J_0(\xi_i r)\, dr$$

$$= \frac{K(z)}{\xi_i} [\, a\, J_1(\xi_i a)\,] \tag{2.16}$$

If (2.15) is substituted into (2.14), then

$$\overline{w}_a(r,t) = \frac{K(z)}{\nu \xi_i^3} J_1(\xi_i a) [\, 1 - e^{-\nu \xi_i^2 t}\,] \tag{2.17}$$

By inversion [2] of $\overline{w}_a(r,t)$, we find that

$$w(r,t) = \frac{2}{a^2}\sum_i \frac{J_0(\xi_i r)}{[J_1(\xi_i a)]^2} \left[\frac{aK(z)}{\nu \xi_i^3}\right] J_1(\xi_i a) \left[1 - e^{-\nu \xi_i^2 t}\right]$$

$$= \frac{2}{a}\sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)} \left[\frac{K(z)}{\nu \xi_i^3}\right]\left[1 - e^{-\nu \xi_i^2 t}\right]$$

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3}\left[1 - e^{-(\xi_i a)^2 \frac{\nu t}{a^2}}\right]$$

$$= \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3}\left[1 - e^{-(\xi_i a)^2 \psi}\right] \qquad (2.18)$$

where $\psi = \frac{\nu t}{a^2}$

2.2 Determination of the Mean Velocity, $w^*(a,t)$

$$w(r,t) = \frac{2}{a}\sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)}\left[\frac{K(z)}{\nu \xi_i^3}\right]\left[1 - e^{-(\xi_i a)^2 \psi}\right]$$

$$w^*(a,t) = \frac{2}{a^2}\int_0^a w(r,t)\, r\, dr$$

$$= \frac{2K(z)}{a^2}\sum_i \frac{1}{\nu \xi_i^4}\left[1 - e^{-(\xi_i a)^2 \psi}\right] \qquad (2.18a)$$

2.3 Determination of the Shear Stress, $\tau(r,t)$

From equation (2.18), we have

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3}\left[1 - e^{-(\xi_i a)^2 \psi}\right]$$

but,

$$\tau(r,t) = \mu \frac{\partial w(r,t)}{\partial r} \qquad (2.19)$$

therefore,

$$\tau(r,t) = -\frac{2\mu}{a^2\nu} K(z) \sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\xi_i^2} [1 - e^{-(\xi_i a)^2 \psi}]$$

or,

$$\frac{\tau(r,t)}{\frac{2\rho}{a} K(z)} = -\sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\xi_i^2} [1 - e^{-(\xi_i a)^2 \psi}] \qquad (2.20)$$

CHAPTER 3

RESULT AND DISCUSSION

3.1 Theoretical Results

The equationsobtained in the previous chapter are restated here:

(a) Velocity Profile

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3} \left[1 - e^{-(\xi_i a)^2 \psi} \right] \qquad (3.1)$$

(b) Shear Stress

$$\frac{\tau(r,t)}{\frac{2\rho}{a}K(z)} = -\sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\xi_i^2} \left[1 - e^{-(\xi_i a)^2 \psi} \right] \qquad (3.2)$$

(c) Mean Velocity

$$w^*(a,t) = \frac{2K(z)}{a^2} \sum_i \frac{1}{\nu \xi_i^4} \left[1 - e^{-(\xi_i a)^2 \psi} \right] \qquad (3.3)$$

3.2 Discussion of Theoretical Results

The velocity of the fluid in a circular cylinder following the application of a given piezometric head at the entrance is given by the equation (3.1). For the case $\psi \longrightarrow \infty$

$$w(r,t) = \frac{2K(z)}{a\nu} \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3} \tag{3.4}$$

Considering the steady state of equation (2.13) and assuming that K(z) is established impulsively, we obtain

$$\nu[\xi_i^2 \bar{w}_a(r)] = \bar{K}(z)$$

then,

$$\bar{w}_a(r) = \frac{\bar{K}(z)}{\nu \xi_i^2} \tag{3.5}$$

but,

$$\bar{K}(z) = \frac{aK(z)}{\xi_i} J_1(\xi_i a) \quad , \qquad \text{see equation (2.16)}$$

thus,

$$\bar{w}_a(r) = \frac{aK(z)}{\nu \xi_i^3} J_1(\xi_i a) \tag{3.6}$$

whence, by inversion [2] ,

$$w(r) = \frac{2}{a^2} \sum_i \frac{\bar{w}_a(r) J_0(\xi_i r)}{[J_1(\xi_i a)]^2} \tag{3.7}$$

and, therefore,

$$w(r) = \frac{2K(z)}{a\nu} \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)} \tag{3.8}$$

We now consider equation (2.12)

$$\frac{\partial w(r,t)}{\partial t} = -K(z,t) + \nu\left[\frac{\partial^2 w(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,t)}{\partial r}\right]$$

for a steady state condition

$$\frac{\partial w(r,t)}{\partial t} = 0$$

and thus,

$$-K(z) + \nu \left[\frac{\partial^2 w(r)}{\partial r^2} + \frac{1}{r} \frac{\partial w(r)}{\partial r} \right] = 0$$

in which

$$w(r) = -\frac{K(z)}{4\nu} (a^2 - r^2) \tag{3.9}$$

Equation (3.9) is the well-known equation of a parabolic flow, but equations (3.4), (3.8) and (3.9) all represent the identical flow pattern. Thus, it may be concluded that for a steady state condition,

$$w(r) = \frac{2K(z)}{a\nu} \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3} \left[1 - e^{-(\xi_i a)^2 \psi} \right]$$

has a parabolic shape.

3.3 Computation Procedure

The relationship of the velocity profile and shear stress are presented in equation (3.1) and (3.2). The complete calculations are in the appendix. The computations are tabulated on the ordinary computing machine, and, therefore, will provide us only a fair amount of accuracy. Briefly, the method proceeds as follows:

(a) Determine the value of $J_0(\xi_i r)$, $J_1(\xi_i r)$ and $\xi_i a$ from Reference [3]

(b) Calculate the value of $e^{-(\xi_i a)^2}$

(c) Assume a value of the dimensionless group, $\psi = \frac{\nu t}{a^2}$

(d) For each point on the radial direction, compute the value of the velocity function, $\frac{w(r,t)\nu}{\frac{2}{a}K(z)}$, and the shear function, $\frac{\tau}{2a\rho K(z)}$, by using the result obtained from (a) to (c) and substituting equations (3.1) and (3.2).

3.4 Sample Calculation

(a) Velocity Function

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i r)\xi_i^3} - \sum_i \frac{J_0(\xi_i r)}{J_1(\xi_i a)\xi_i^3} e^{-(\xi_i a)^2\psi}$$

Assume that $\psi = 0.05$, the value of $\xi_i a$, $J_0(\xi_i a)$, $J_1(\xi_i a)$ and $e^{-(\xi_i a)^2}$ for r/a = 0 may be obtained as follows:

i	$\frac{r}{a}$	$\xi_i a$	$J_0(\xi_i r)$	$J_1(\xi_i a)$	$J_1(\xi_i r)$	$e^{-(\xi_i a)^2\psi}$
1	0	2.4048	1	0.5191	0	0.749012
2	0	5.5201	1	-0.3403	0	0.217840
3	0	8.6537	1	0.2715	0	0.023750
4	0	11.7915	1	-0.2325	0	0.000959
5	0	14.9390	1	0.2065	0	0.0000146

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \left[\frac{1}{(0.5191)\frac{14}{a^3}} - \frac{1}{(0.3403)\frac{170}{a^3}} + \frac{1}{(0.2715)\frac{650}{a^3}} - \frac{1}{(0.2325)\frac{1650}{a^3}} + \ldots\right]$$

$$- \left[\frac{0.749012}{(0.5191)\frac{14}{a^3}} - \frac{0.21784}{(0.3404)\frac{170}{a^3}} + \frac{0.02375}{(0.2715)\frac{650}{a^3}} - \frac{0.000959}{(0.2325)\frac{1650}{a^3}} + \ldots\right]$$

$$= a^3\,[\,0.1376 - 0.0172857 - 0.00567 - 0.00261 - \ldots\ldots\,]$$

$$- a^3\,[\,0.1031 - 0.003765517 - 0.0001346625 - 0.000002503 - \ldots\ldots\,]$$

$$= 0.248076a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.248076$$

(b) Shear Function

$$\frac{\tau}{\frac{2}{a}\rho K(z)} = - \sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\,\xi_i^2} + \sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\,\xi_i^2} e^{-(\xi_i a)^2}$$

Assume that $\psi = 0.05$, the value of $\xi_i a$, $J_1(\xi_i r)$, $J_1(\xi_i a)$ and $e^{-(\xi_i a)^2}$ for $r/a = 0$ may be obtained in the same manner as in part (a).

In this case, it is found that

$$\frac{\tau}{2a\,\rho K(z)} = 0$$

3.5 Discussion of the Numerical Result

It can be seen that the series of the velocity function and the shear function converge from the computation obtained in the section 3.4 and, therefore, assure that the solution exists. It is also to be noted that as the value of the dimensionless group, $(\xi_i a)^2\psi$, increases (see the appendix), the value of $e^{-(\xi_i a)^2\psi}$ diminishes. Thus, it is sufficient for us to compute by using the first few terms of the Bessel Series.

From Figure 3.1, the velocity vanishes at $r = a$ for any value of t. The velocity is also zero at $t = 0$. Thus, the solution satisfies the boundary and initial conditions as stated.

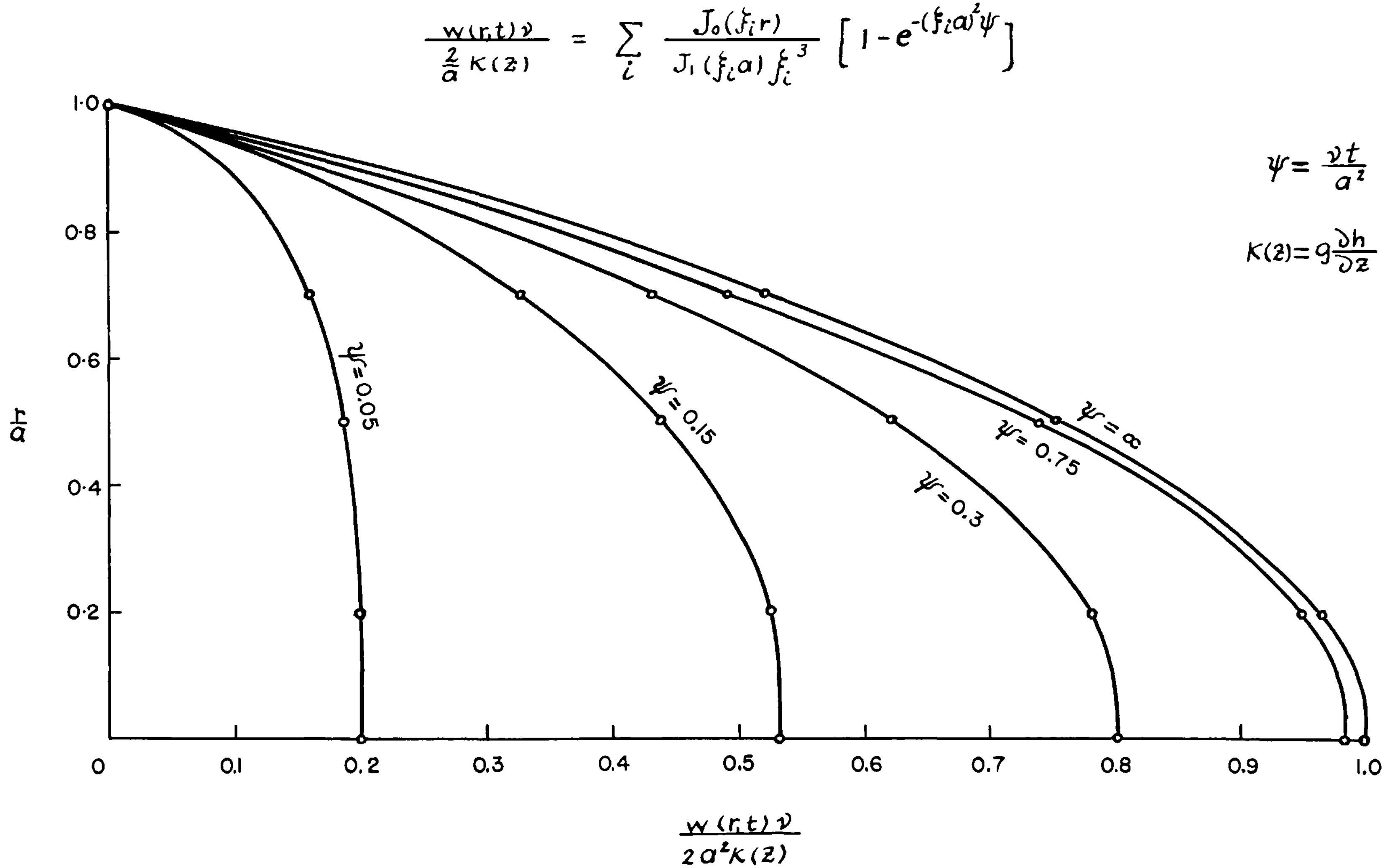

FIGURE 3.1 VELOCITY DISTRIBUTION

$$\frac{\tau}{\left(\frac{2\rho}{a}\right)K(z)} = \sum_{i} \frac{J_1(\xi_i r)}{J_1(\xi_i a)\,\xi_i^2}\left[\, e^{-(\xi_i a)^2\psi} - 1 \,\right]$$

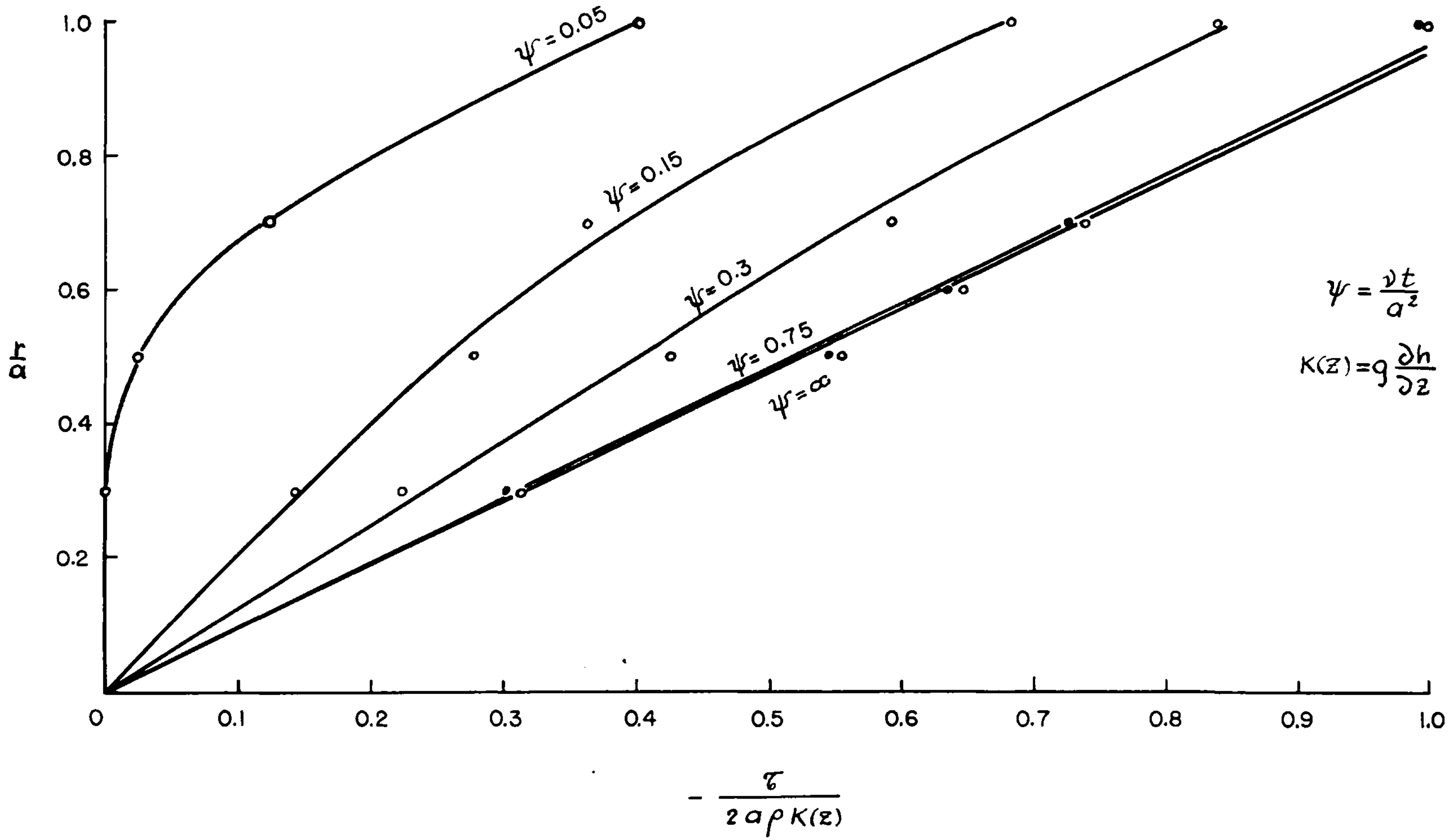

FIGURE 3.2 SHEAR DISTRIBUTION

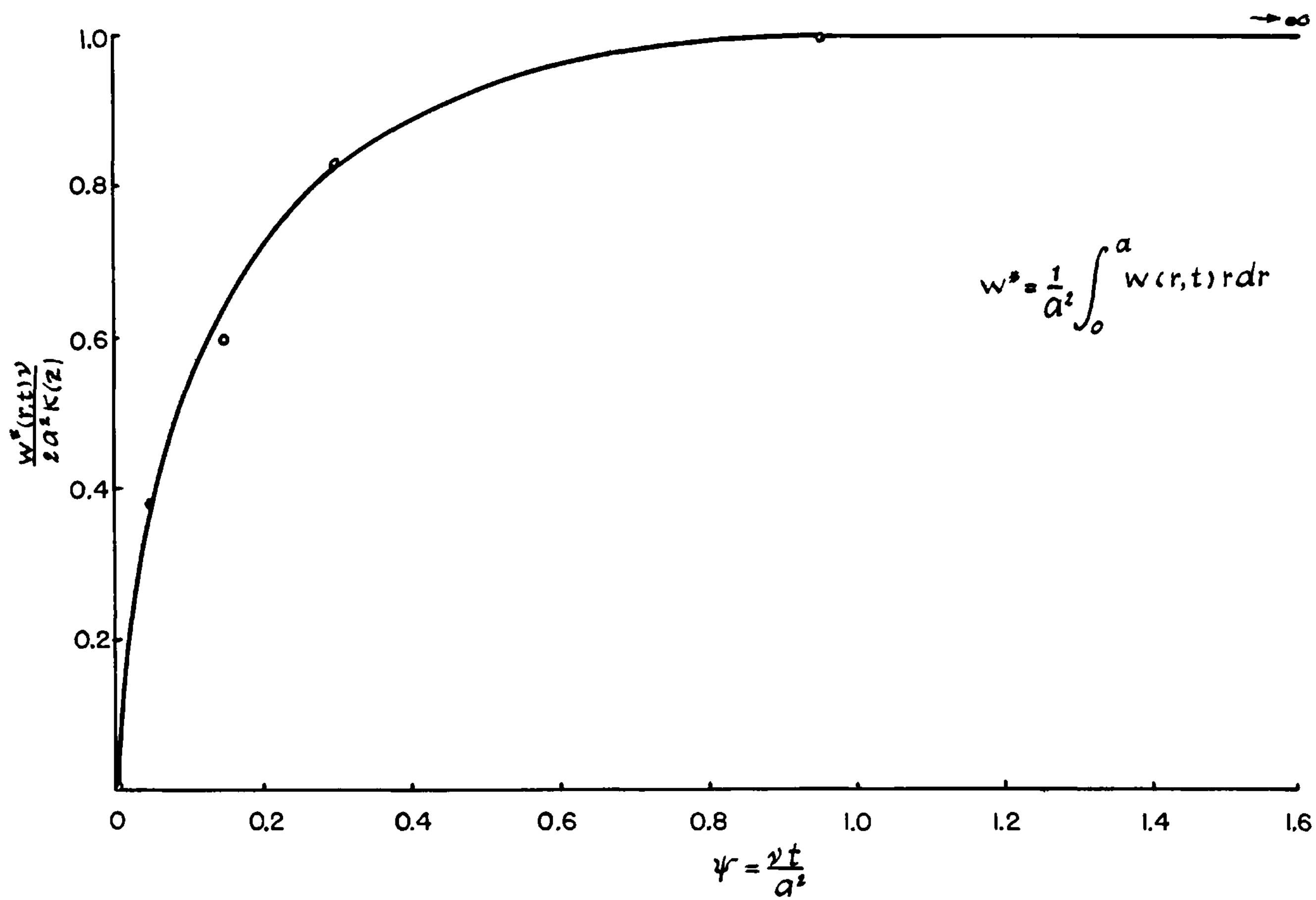

FIGURE 3.3 THE RELATIONSHIP BETWEEN THE MEAN VELOCITY AND ψ

CHAPTER 4

CONCLUSION AND RECOMMENDATION

4.1 Conclusion

The shapes of the velocity profile obtained through the use of Hankel Transforms are in agreement with the results of Szymanski [4]. In the steady state case the velocity profile has a parabolic shape. The mean velocity approaches a limiting value as the steady state condition is reached.

The shear stress increases its intensity and reaches its maximum value at the boundary as expected. It is also increased with time.

4.2 Recommendation For Future Work

A. The problem of the flow with variable piezometric head is of interest. For example, let us consider that K(z) or $\frac{1}{\rho}\frac{\partial p}{\partial z}$ is also a function of time. From equation (2.12) we have

$$\overline{w}_a(r,t) = \int_0^t \overline{K}(z,T)\, e^{-\nu \xi_i^2 (t-T)}\, dT$$

where

$$\overline{K}(z,T) = K(z,T)\frac{a}{\xi_i} J_1(\xi_i a)$$

We could assume that K(z,T) has a certain function of time, for example, $K(z,T) = c + dT + eT^2$ (or other appropriate function).

We then can solve the equation (2.12) for the value of velocity with a certain type of variable piezometric head.

B. The measurement of the shear stress is also very important in the flow analysis. Valuable in formation may be obtained through the understanding of the shear stress.

APPENDICES

APPENDIX A
Computation of $J_0(\beta_i r)$

i	$\beta_i a$	$\frac{r}{a}$	$(\beta_i a)\frac{r}{a}$	$J_0(\beta_i r) = J_0\left((\beta_i a)\frac{r}{a}\right)$
1	2.4048	0	0	1
2	5.5201	0	0	1
3	8.6537	0	0	1
4	11.7915	0	0	1
5	14.9309	0	0	1
1	2.4048	0.2	0.48096	+0.9432
2	5.5201	0.2	1.10402	+0.7196
3	8.6537	0.2	1.73074	+0.3806
4	11.7915	0.2	2.35830	+0.0235
5	14.9309	0.2	2.98618	−0.2566
1	2.4048	0.5	1.20240	+0.6711
2	5.5201	0.5	2.76005	−0.1684
3	8.6537	0.5	4.32685	−0.3557
4	11.7915	0.5	5.89575	+0.1220
5	14.9309	0.5	7.46545	+0.2703
1	2.4048	0.7	1.68336	+0.4095
2	5.5201	0.7	3.86407	−0.4026
3	8.6537	0.7	6.05759	+0.1669
4	11.7915	0.7	8.25405	+0.1092
5	14.9309	0.7	10.45163	−0.2403

Computation of $J_0(\xi_i r)$ (cont'd)

i	$\xi_i a$	$\frac{r}{a}$	$(\xi_i a)\frac{r}{a}$	$J_0(\xi_i r) = J_0\ (\xi_i a)\frac{r}{a}$
1	2.4048	1.0	2.4048	+0.0025
2	5.5201	1.0	5.5201	0.0000
3	8.6537	1.0	8.6537	-0.0003
4	11.7915	1.0	11.7915	+0.0020
5	14.9309	1.0	14.9309	+0.0043

Computation of $J_1(\xi_i r)$

				$J_1(\xi_i r) = J_1\ (\xi_i a)\frac{r}{a}$
1	2.4048	0	0	0
2	5.5201	0	0	0
3	8.6537	0	0	0
4	11.7915	0	0	0
5	14.9309	0	0	0
1	2.4048	0.3	0.72144	+0.3372
2	5.5201	0.3	1.65603	+0.5751
3	8.6537	0.3	2.59611	+0.4708
4	11.7915	0.3	3.53745	+0.1206
5	14.9309	0.3	4.47927	-0.2256

Computation of $J_1(\xi_i r)$ (cont'd)

i	$\xi_i a$	$\frac{r}{a}$	$(\xi_i a)\frac{r}{a}$	$J_1(\xi_i r) = J_1\left((\xi_i a)\frac{r}{a}\right)$
1	2.4048	0.5	1.20240	+0.4983
2	5.5201	0.5	2.76005	+0.4228
3	8.6537	0.5	4.32685	-0.1814
4	11.7915	0.5	5.89575	-0.2951
5	14.9309	0.5	7.46545	+0.1277
1	2.4048	0.7	1.68336	+0.5765
2	5.5201	0.7	3.86407	-0.0114
3	8.6537	0.7	6.05759	-0.2645
4	11.7915	0.7	8.25405	+0.2622
5	14.9309	0.7	10.45163	-0.0673
1	2.4048	1	2.4048	+0.5202
2	5.5201	1	5.5201	-0.3403
3	8.6537	1	8.6537	+0.2716
4	11.7915	1	11.7915	-0.2325
5	14.9309	1	14.9309	+0.2066

Computation of Eigenvalues

i	$\xi_i a$	$J_1(\xi_i a)$	ξ_i	ξ_i^3	ξ_i^2
1	2.4048	+0.5191	$\frac{2.4048}{a}$	$\frac{14}{a^3}$	$\frac{5.78}{a^2}$
2	5.5201	-0.3403	$\frac{5.5201}{a}$	$\frac{170}{a^3}$	$\frac{30.47}{a^2}$
3	8.6537	+0.2715	$\frac{8.6537}{a}$	$\frac{650}{a^3}$	$\frac{74.89}{a^2}$
4	11.7915	-0.2325	$\frac{11.7915}{a}$	$\frac{1650}{a^3}$	$\frac{139.04}{a^2}$
			$\frac{14.9309}{a}$	$\frac{3340}{a^3}$	$\frac{222.93}{a^2}$

Computation of $e^{-(\xi_i a)^2}$

$\psi = 0.05$

i	$\xi_i a$	$(\xi_i a)^2$	$(\xi_i a)^2$	$e^{-(\xi_i a)^2}$
1	2.4048	5.7831	0.2892	0.749012
2	5.5201	30.472	1.5236	0.21784
3	8.6537	74.8865	3.7443	0.02375
4	11.7915	139.0395	6.952	0.000959
5	14.9309	222.9318	11.1466	0.0000146
6	18.0711	326.5647	16.3282	-

$\psi = 0.15$

1	0.8675	0.4198
2	4.5708	0.01035
3	11.2298	0.0000134
4	20.8640	-

$\psi = 0.3$

1	1.7349	0.1764
2	9.1416	0.000108
3	22.4660	-

Computation of $e^{-(\xi_i a)^2}$

$\underline{i}$	$\xi_i a$	$(\xi_i a)^2$	$(\xi_i a)^2$	$e^{-(\xi_i a)^2}$
$\psi = 0.75$				
1			4.3373	0.013
2			22.854	-
$\psi = \infty$				
1				0

APPENDIX B

Computation of the Velocity Function

$$\frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \sum_i \frac{J_0(\beta_i r)}{J_1(\beta_i a)\,\beta_i^3}\left[1 - e^{-(\beta_i a)^2}\right]$$

$$= \sum_i \frac{J_0(\beta_i r)}{J_1(\beta_i a)\,\beta_i^3} - \sum_i \frac{J_0(\beta_i r)}{J_1(\beta_i a)\,\beta_i^3}\, e^{-(\beta_i a)^2}$$

- - - - - - - - - - - - - -

$\psi = 0.05$

$$\frac{r}{a} = 0;\ \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \left[\frac{1}{(0.5191)\frac{14}{a^3}} - \frac{1}{(0.3403)} + \frac{1}{(0.2715)} - \frac{1}{(0.2325)}\ ..\right]$$

$$- \left[\frac{0.749012}{(0.5191)\frac{14}{a^3}} - \frac{0.21784}{(0.3404)\frac{170}{a^3}} + \frac{0.02375}{(0.2715)\frac{650}{a^3}} - \frac{0.000959}{(0.2325)\frac{1650}{a^3}} + ..\right]$$

$$= a^3[0.1376 - 0.017257 + 0.00567 - 0.00261 + ...]$$

$$-a^3[0.1031 - 0.003765517 + 0.0001346625 - 0.000002503 + ...]$$

$$= a^3[0.1233743] - a^3[0.0994646]$$

$$= 0.248076a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.248076$$

$$\frac{r}{a} = 0.2;\ \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \left[\frac{0.9432}{(0.5191)\frac{14}{a^3}} - \frac{0.7196}{(0.3403)\frac{170}{a^3}} + \frac{0.3806}{(0.2715)\frac{650}{a^3}}\right.$$

$$\left. - \frac{0.0235}{(0.2325)\frac{1650}{a^3}} + ...\right]$$

$$-\left[\frac{0.9432}{(0.5191)\frac{14}{a^3}}(0.749012) - \frac{0.7196}{(0.3403)\frac{170}{a^3}}(0.21784) + \right.$$

$$\left.\frac{0.3806}{(0.2715)\frac{650}{a^3}}(0.02375) - \frac{(0.0235)(0.000959)}{(0.2325)\frac{1650}{a^3}} + \ldots\right]$$

$$= a^3[0.1298 - 0.01245 + 0.002158 - 0.00006134 + \ldots]$$

$$- a^3[0.097222 - 0.002712 + 0.0000512525 - 0.00000005883 + \ldots]$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.2430$$

$\underline{\psi = 0.05}$

$$\frac{r}{a} = 0.5;\ \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = \left[\frac{0.6711}{(0.5191)\frac{14}{a^3}} + \frac{0.1684}{(0.3403)\frac{170}{a^3}} - \frac{0.3557}{(0.2715)\frac{650}{a^3}}\right.$$

$$\left.- \frac{0.1220}{(0.2325)\frac{1650}{a^3}} + \ldots\right]$$

$$-\left[\frac{0.6711}{(0.5191)\frac{14}{a^3}}(0.749012) + \frac{0.1684}{(0.3403)\frac{170}{a^3}}(0.21784)\right.$$

$$\left.- \frac{0.3557}{(0.2715)\frac{650}{a^3}}(0.02375) - \frac{(0.12200)(0.000959)}{(0.2325)\frac{1650}{a^3}} + \ldots\right]$$

$$= a^3[0.09234 + 0.002913 - 0.002017 - 0.0003184 - \ldots]$$

$$- a^3[0.0422068262 + 0.0015172556 + 0.00002247462 - 0.000000273315 + \ldots]$$

$$= 0.02023\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.02023$$

$\underline{\psi = 0.15}$

$$\frac{r}{a} = 0;\ \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.1376 - 0.0172857 + 0.00561 - 0.00264 + \ldots.]$$

$$- a^3[0.1376(0.4198) - 0.017285(0.01035) + 0.00561(0.0000134) + \ldots]$$

$$= a^3[0.1233743] - a^3[0.05758566]$$

$$= 0.0657886\ a^3$$

$$\frac{w(r,t)\nu}{2a^2\ K(z)} = 0.0657886$$

$$\frac{r}{a} = 0.2;\ \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = [\ \frac{0.9432}{(0.5181)\frac{14}{a^3}} - \frac{0.7196}{(0.3403)\frac{170}{a^3}} + \frac{0.3806}{(0.2715)\frac{650}{a^3}}$$

$$- \frac{0.0235}{(0.2325)\frac{1650}{a^3}} + \ldots.]$$

$$- [\frac{0.9432}{(0.5191)\frac{14}{a^3}}(0.4198) - \frac{0.7196}{(0.3403)\frac{170}{a^3}}(0.01035)$$

$$+ \frac{0.3806}{(0.2715)\frac{650}{a^3}}(0.0000134) + \ldots.]$$

$$= a^3[0.1298 - 0.01245 + 0.002158 - 0.000061335 + \ldots.]$$

$$- a^3[0.05449 - 0.0001288575 + 0.0000000289172 + \ldots.]$$

$$= [0.11944666]\ a^3 - [0.0543611704]\ a^3$$

$$= 0.06509\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.06509$$

$$\frac{r}{a} = 0.5; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.09234 + 0.002913 - 0.002017 - 0.0003184 + \ldots]$$

$$- a^3[(0.09234)(.4198) + 0.002913)(0.01035) - (0.002017)(0.0000134)$$

$$+ \ldots.]$$

$$= [0.0929176] a^3 - [0.038764332 + 0.00003014955$$

$$- 0.0000000270278 + \ldots.] \ a^3$$

$$= [0.0929176 - 0.03879446] a^3$$

$$= 0.05412 \ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.05412$$

$$\frac{r}{a} = 0.7; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.05635 + 0.506965 + 0.0009463 - 0.0002855 + \ldots]$$

$$- a^3[(0.05635)(0.4198) + (0.006965)(0.01035)$$

$$+ (0.0009463)(0.0000134) + \ldots.]$$

$$= a^3[0.0639763] - a^3[0.02365573 + 0.00007208775$$

$$+ 0.00000001268 + \ldots.]$$

$$= a^3[0.0639763] - a^3[0.023727820043]$$

$$= 0.04025 \ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.04025$$

$\underline{\psi = 0.3}$

$$\frac{r}{a} = 0; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.1376 - 0.0172857 + 0.00561 - 0.00261 + \ldots]$$
$$- a^3[(0.1376)(0.1764) - (0.0172857)(0.000108) + \ldots]$$
$$= a^3[0.1233743] - a^3[0.0242708]$$
$$= 0.0991035\ a^3$$
$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.0991035$$

$$\frac{r}{a} = 0.2; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.1298 - 0.01245 + 0.002158 - 0.00006134 + \ldots]$$
$$- a^3[(0.1298)(0.1764) - (0.01245)(0.000108) + \ldots]$$
$$= a^3[0.11944666] - a^3[0.02289672 - 0.0000013446]$$
$$= a^3[0.11944666] - a^3[0.022895376]$$
$$= 0.096551\ a^3$$
$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.096551$$

$$\frac{r}{a} = 0.5; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.09234 + 0.002913 - 0.002017 - 0.0003184 + \ldots]$$
$$- a^3[(0.09234)(0.1764) + (0.002913)(0.000108) + \ldots]$$
$$= a^3[0.0929176] - a^3[0.016288776 + 0.0000003146 + \ldots]$$
$$= 0.07663\ a^3$$
$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.07663$$

$\underline{\psi = 0.3}$

$$\frac{r}{a} = 0.7; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.05635 + 0.006965 + 0.0009463 - 0.000285 + \ldots]$$

$$- a^3[(0.05635)(0.1764) + (0.006965)(0.000108) + \ldots]$$

$$= a^3[0.0639763] - a^3[0.00994014 + 0.00000075222]$$

$$= a^3[0.0639763] - a^3[0.009940899]$$

$$= 0.05404\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.05404$$

$\underline{\psi = 0.75}$

$$\frac{r}{a} = 0; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.1376 - 0.0172857 + 0.00561 - 0.00261 + \ldots]$$

$$- a^3[(0.1376)(0.013) + \ldots]$$

$$= a^3[0.1233743] - a^3[0.0017888]$$

$$= 0.1215855\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.1215855$$

$$\frac{r}{a} = 0.2; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.1298 - 0.01245 + 0.002158 - 0.00006134 + \ldots]$$

$$- a^3[(0.1298)(0.013) + \ldots]$$

$$= a^3[0.11944666] - a^3[0.0016874]$$

$$= 0.11776\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.11776$$

$\underline{\psi = 0.75}$

$$\frac{r}{a} = 0.7; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = a^3[0.05635 + 0.006965 + 0.0009463 - 0.000285 + \ldots]$$

$$- a^3[(0.05635)(0.013) + \ldots]$$

$$= a^3[0.0639763] - a^3[0.0073255]$$

$$= 0.05665\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.05665$$

$\underline{\psi = \infty}$

$$\frac{r}{a} = 0; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = 0.1233743\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.1233743$$

$$\frac{r}{a} = 0.2; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = 0.11945\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.11945$$

$$\frac{r}{a} = 0.5; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = 0.09292\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.09292$$

$$\frac{r}{a} = 0.7; \quad \frac{w(r,t)\nu}{\frac{2}{a}K(z)} = 0.06398\ a^3$$

$$\frac{w(r,t)\nu}{2a^2K(z)} = 0.06398$$

APPENDIX C

Computation of the Mean Velocity

$$\frac{\nu w^*}{\frac{2}{a^2} K(z)} = \sum_i \frac{K(z)}{\xi_i^4} [1 - e^{-(\xi_i a)^2}]$$

- - - - - - - - - - - - - - - -

$\psi = 0.05$

$$\frac{\nu w^*}{\frac{2}{a^2} K(z)} = a^4 [\frac{0.351}{33.41} + \frac{0.882}{928.42} + \frac{0.9760}{5638.51} + \frac{0.99904}{19332.02} + \frac{0.999985}{45697.28} + \ldots]$$

$$= a^4 [0.0106 + 0.00095 + 0.000173 + 0.0000518 + 0.0000218 + \ldots]$$

$$\frac{\nu w^*}{2a^2 K(z)} = 0.0118$$

- - - - - - - - - - - - - - - -

$\psi = 0.15$

$$\frac{\nu w^*}{2a^2 K(z)} = \frac{0.5802}{33.41} + \frac{0.9896}{928.42} + \frac{0.999986}{5638.51} + \frac{1}{19332.02} + \ldots.$$

$$= 0.0187$$

- - - - - - - - - - - - - - - -

$\underline{\psi = 0.30}$

$$\frac{\nu w^*}{2a^2\ K(z)} = \frac{0.8236}{33.41} + \frac{0.9999}{928.42} + \frac{1}{5638.51} + \cdots$$

$$= 0.0259$$

$\underline{\psi = 0.75}$

$$\frac{\nu w^*}{2a^2\ K(z)} = \frac{0.987}{33.41} + \frac{1}{928.42} + \cdots$$

$$= 0.031^-$$

$\underline{\psi = \infty}$

$$\frac{\nu w^*}{2a^2\ K(z)} = \frac{1}{33.41} + \cdots$$

$$= 0.031^+$$

APPENDIX D

Computation of the Shear Stress Function

$$\frac{\tau}{\frac{2}{a}\rho K(z)} = -\sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\,\xi_i^2} + \sum_i \frac{J_1(\xi_i r)}{J_1(\xi_i a)\,\xi_i^2}\, e^{-(\xi_i a)^2}$$

$\psi = \infty$

$$\frac{r}{a} = 0; \quad \frac{\tau}{2a\rho K(z)} = 0$$

$$\frac{r}{a} = 0.3; \quad \frac{\tau}{2a\rho K(z)} = -\Big[\frac{0.3372}{(.5191)(5.78)} - \frac{0.5751}{(0.3403)(30.47)} + \frac{0.4708}{(0.2715)(74.89)}$$

$$- \frac{0.1206}{(0.2325)(139.04)} - \frac{0.2256}{(0.2065)(222.93)} + \dots\Big]$$

$$= -\Big[\frac{0.3372}{3.0} - \frac{0.5751}{10.3689} + \frac{0.4708}{20.337} - \frac{0.1206}{32.3268} - \frac{0.2256}{46.035} + \dots\Big]$$

$$= -\big[0.1124 - 0.055463935 + 0.023154478$$

$$- 0.00373065 - 0.004898703 + \dots\big]$$

$$= -0.07136$$

$$\frac{r}{a} = 0.5; \quad \frac{\tau}{2a\rho K(z)} = -\Big[\frac{0.4983}{3.0} - \frac{0.4228}{10.3689} - \frac{0.18114}{20.333} + \frac{0.2951}{32.3268} + \frac{0.1277}{46.035} + \dots\Big]$$

$$= -\big[0.1661 - 0.0407753882 - 0.00890867 + 0.0091286486$$

$$+ 0.002772892 + \dots\big]$$

$$= -0.12829$$

$\underline{\psi = \infty}$

$$\frac{r}{a} = 0.7; \quad \frac{\tau}{2a\rho K(z)} = -\left[\frac{0.5765}{3.0} + \frac{0.0114}{10.3689} - \frac{0.2645}{20.333} - \frac{0.2622}{32.3268} - \frac{0.0673}{46.035} + \cdot\cdot\right]$$

$$= -[0.19216666 + 0.0010994416 - 0.01300841$$

$$- 0.008110917 - 0.0014619311 + \cdots]$$

$$= -0.170685$$

$$\frac{r}{a} = 1.0; \quad \frac{\tau}{2a\rho K(z)} = -\left[\frac{0.5202}{3.0} + \frac{0.3404}{10.3689} + \frac{0.2716}{20.333} + \frac{0.2325}{32.3268} + \frac{0.2066}{46.035} + \cdot\cdot\right]$$

$$= -[0.1734 + 0.03282894 + 0.013357596 + 0.0071921748$$

$$+ 0.0044878896 + \cdots]$$

$$= -0.2312$$

$\underline{\psi = 0.05}$

$$\frac{r}{a} = 0; \quad \frac{\tau}{2a\rho K(z)} = 0$$

$$\frac{r}{a} = 0.3; \quad \frac{\tau}{2a\rho K(z)} = -[0.1124 - 0.055463935 + 0.023154478 - 0.003730651$$

$$- 0.004898703] + [(0.1124)(0.749012) - (0.055463955)$$

$$(0.21284) + (0.023154478)(0.02375) - (0.00373065)$$

$$(0.000959) - (0.004898703)(0.0000146) + \cdots]$$

$$= - \; 0.07136 \; + [0.08418895 - 0.012082267 + 0.000549919$$

$$- 0.00000035776 - 0.0000000714967 + \cdots]$$

$$= -0.07136 + 0.071912$$

$$= 0.000552$$

$\psi = 0.05$

$$\frac{r}{a} = 0.5; \quad \frac{\tau}{2a\rho K(z)} = -0.12829 + [(0.1661)(0.749012) - (0.040775388)(0.21784) - (0.00890867)(0.02375) + (0.0091286486)(0.000959) + (0.00277289)(0.000146) + \ldots]$$

$$= -0.12829 + [0.12441089 - 0.000888251 - 0.000211581 + 0.00000875437 + 0.00000040484 + \ldots]$$

$$= -0.12829 + 0.123321$$

$$= -0.004969$$

$$\frac{r}{a} = 0.7; \quad \frac{\tau}{2a\rho K(z)} = -0.170685 + [(0.1921666)(0.749012) + (0.00109944)(0.21784) - (0.0130084)(0.02375) - (0.008110917)(0.000959) - (0.0014619311)(0.0000146) + \ldots]$$

$$= -0.170685 + [0.143935089 + 0.000239502 - 0.000308949 - 0.000007778369 - 0.0000000021344 + \ldots]$$

$$= -0.170685 + 0.14385784$$

$$= -0.02682716$$

$$\frac{r}{a} = 1; \quad \frac{\tau}{2a\rho K(z)} = -0.2312 + [0.1734\,(0.749012) + (0.0328294)(0.21784) + (0.013357596)(0.02375) + (0.0071921749)(0.000959) + (0.0044878896)(0.0000146) + \ldots]$$

$$= -0.2312 + [0.12987868 + 0.00715556649 + 0.000317243 + 0.00000689729 + 0.0000000655232 + \ldots]$$

$$= -0.2312 + 0.13735256$$

$$= -0.093895$$

$\underline{\psi = 0.15}$

$$\frac{r}{a} = 0; \quad \frac{\tau}{2a\rho K(z)} = 0$$

$$\frac{r}{a} = 0.3; \quad \frac{\tau}{2a\rho K(z)} = -0.7136 + [(0.1124)(0.4198) - (0.55463935)(0.01035)$$
$$+ (0.023154478)(0.0000134) + \ldots]$$
$$= -0.07136 + [0.04718552 - 0.005740517$$
$$+ 0.000000309613 + \ldots]$$
$$= -0.07136 + 0.041445$$
$$= -0.029915$$

$$\frac{r}{a} = 0.5; \quad \frac{\tau}{2a\rho K(z)} = -0.12829 + [(0.1661)(0.4198) - (0.0407754)(0.01035)$$
$$- (0.00890867)(0.0000134) + \ldots]$$
$$= -0.12829 + [0.06972878 - 0.0004220254 -$$
$$0.0000001193762 + \ldots]$$
$$= -0.12829 + 0.0693066$$
$$= -0.058983$$

$$\frac{r}{a} = 0.7; \quad \frac{\tau}{2a\rho K(z)} = -0.170685 + [(0.19216666)(0.4198)$$
$$+ (0.00109944)(0.01035) - (0.01300841)(0.0000134) + \ldots]$$
$$= -0.170685 + [0.0865699 + 0.0000113792$$
$$- 0.000000174312 + \ldots]$$
$$= -0.170685 + 0.0865811$$
$$= -0.084104$$

$\underline{\psi = 0.15}$

$$\frac{r}{a} = 1.0; \quad \frac{\tau}{2a\rho K(z)} = -0.2312 + [(0.1734)(0.4198) + (0.03282894)(0.01035)$$

$$+ (0.013357596)(0.0000134) + \ldots]$$

$$= -0.2312 + [0.07279332 + 0.0003404529 + 0.000000178991$$

$$+ \ldots]$$

$$= -0.2312 + 0.07313395$$

$$= -0.158066$$

$\underline{\psi = 0.3}$

$$\frac{r}{a} = 0; \quad \frac{\tau}{2a\rho K(z)} = 0$$

$$\frac{r}{a} = 0.3; \quad \frac{\tau}{2a\rho K(z)} = -0.07136 + [(0.1124)(0.1764) - (0.05546394)$$

$$(0.000108) + \ldots]$$

$$= -0.07136 + [0.01982736 - 0.000005990105 + \ldots]$$

$$= -0.07136 + 0.0198213$$

$$= -0.05154$$

$$\frac{r}{a} = 0.5; \quad \frac{\tau}{2a\rho K(z)} = 0.12829 + [(0.1661)(0.1764) - (0.0407754)$$

$$(0.000108) + \ldots]$$

$$= -0.12829 + [0.0293 - 0.0000044404 + \ldots]$$

$$= -0.12829 + 0.029296$$

$$= -0.098994$$

$\underline{\psi = 0.3}$

$\frac{r}{a} = 0.7; \frac{\tau}{2a\rho K(z)} = 0.170685 + [(0.19216666)(0.1764)$

$+ (0.00010994415)(0.000108) + \dots]$

$= -0.170685 + [0.0338982 + 0.000000011874 + \dots]$

$= -0.170685 + 0.0338982$

$= -0.1368$

$\frac{r}{a} = 1; \frac{\tau}{2a\rho K(z)} = -0.2132 + [(0.1734)(0.1764) + (0.0328294)$

$(0.000108) + \dots]$

$= -0.2132 + [0.03058776 + 0.00000354553 + \dots]$

$= -0.2132 + 0.0305913$

$= -0.1826$

$\underline{\psi = 0.75}$

$\frac{r}{a} = 0; \frac{\tau}{2a\rho K(z)} = 0$

$\frac{r}{a} = 0.3; \frac{\tau}{2a\rho K(z)} = -0.07136 + [(0.1124)(0.013) + \dots]$

$= -0.07136 + 0.0014612$

$= -0.069899$

$\frac{r}{a} = 0.5; \frac{\tau}{2a\rho K(z)} = -0.12829 + [(0.1661)(0.013) + \dots]$

$= -0.12829 + 0.0021593$

$= -0.12613$

$\frac{r}{a} = 0.7; \frac{\tau}{2a\rho K(z)} = -0.170685 + [(0.1921666)(0.013) + \dots]$

$= -0.170685 + 0.002498166$

$= -0.16811$

$\underline{\psi = 0.75}$

$$\frac{r}{a} = 1.0; \quad \frac{\tau}{2a\rho K(z)} = -0.2312 + [(0.1734)(0.013) + \ldots]$$

$$= -0.2312 + 0.0022542$$

$$= -0.228946$$

TABLES

TABLE 1

THE RELATIONSHIP BETWEEN THE VELOCITY, RADIUS AND ψ

ψ	$\frac{r}{a}$	Computed Value of $\frac{w(r,t)\nu}{2a^2K(z)}$	Normalize Value of $\frac{w(r,t)\nu}{2a^2K(z)}$
∞	0	0.12337	1
	0.2	0.11945	0.9682
	0.5	0.09292	0.7532
	0.7	0.06398	0.5186
0.05	0	0.2481	0.2011
	0.2	0.2430	0.1970
	0.5	0.02327	0.1886
	0.7	0.02023	0.1640
0.15	0	0.06579	0.5333
	0.2	0.06509	0.5276
	0.5	0.05412	0.4387
	0.7	0.04025	0.3263
0.3	0	0.0991	0.8033
	0.2	0.096551	0.7826
	0.5	0.07663	0.6211
	0.7	0.05404	0.4380
0.75	0	0.12159	0.9856
	0.2	0.11776	0.9545
	0.5	0.09172	0.7435
	0.7	0.05665	0.4592

TABLE 2

THE RELATIONSHIP BETWEEN THE MEAN VELOCITY AND ψ

ψ	Computed Value of $\frac{w^*(r,t)\nu}{2a^2K(z)}$	Normalize Value of $\frac{w^*(r,t)\nu}{2a^2K(z)}$
0	0	0
0.05	0.0118	0.38
0.15	0.0187	0.60
0.30	0.0259	0.83
0.75	0.031^-	1.00^-
	0.031^+	1.00^+

TABLE 3

THE RELATIONSHIP BETWEEN THE SHEAR

STRESS, RADIUS AND ψ

ψ	$\frac{r}{a}$	Computed Value of $-\frac{\tau}{2a\rho K(z)}$	Normalize Value of $-\frac{\tau}{2a\rho K(z)}$
	0	0	0
	0.3	0.07136	0.3086
	0.5	0.12829	0.5549
	0.7	0.170685	0.7383
	1.0	0.2312	1
0.05	0	0	0
	0.3	-0.000552	-0.0024
	0.5	0.004969	0.0215
	0.7	0.02683	0.1258
	1.0	0.093895	0.4026
0.15	0	0	0
	0.3	0.029915	0.1403
	0.5	0.058983	0.2764
	0.7	0.084104	0.3638
	1.0	0.158066	0.6837
0.3	0	0	0
	0.3	0.05154	0.2229
	0.5	0.098994	0.4265
	0.7	0.1368	0.5917
	1.0	0.1826	0.7898
0.75	0	0	0
	0.3	0.069899	0.3023
	0.5	0.12613	0.5455
	0.7	0.16811	0.7268
	1.0	0.228946	0.99025

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