# THE DYNAMIC BEHAVIOR OF AN ARTIFICIAL SATELLITE STABILIZED BY GRAVITY-GRADIENT

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF MECHANICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

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# CHAPTER 1

## INTRODUCTION

The control of an artificial satellite's attitude may be accomplished by control devices installed in the satellite, proper utilization of natural forces or a combination of both. The natural means referred to are such forces as solar radiation pressure, earth's magnetic field, aerodynamics or gravity gradient. Depending upon the mission and geometry of the satellite, these forces may either be stabilizing or destabilizing.

Equations of motion for a gravity gradient stabilized satellite have been derived in reference (1). The satellite was of laterally isotropic shape and the motion was considered to be small. Euler angles were used to describe the satellite's attitude with respect to a reference coordinate system. Solutions to the equations of angular motion indicated that, although the Euler angles were assumed to be small, the displacement about the Earth-oriented axis may not be expected to be small. Since this result is contrary to the assumption that the Euler angles are all small, it is reasonable to consider next the satellite's angular motion when the displacement about the Earth-oriented axis is not assumed small.

The objective of the first portion of this thesis is to derive the equations for the small angular motion of an Earth-scanning satellite stabilized by gravity gradient moments, allowing the angular displacement around the Earth-oriented axis to be unrestricted.

Equations of motion derived for a satellite of arbitrary shape stabilized by gravity gradient are then used to investigate satellite stability. It may be said that a satellite can assume a number of various attitudes in which it will be in equilibrium. Therefore, an analysis is performed to determine what specific configurations will result in stable motion once the equilibrium is disturbed. It is assumed that only small motions are to be considered.

# CHAPTER 2

## EQUATIONS OF MOTION

Consider a satellite of arbitrary shape injected into an elliptical orbit about the earth. A coordinate system shall be fixed in the satellite with origin at the center of mass. The x,y,z axes of the coordinate system coincide with the principal axes of the body.

The angular momentum of the satellite may be expressed in matrix notation as:

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{2.1}$$

Where Hx, Hy, Hz represent the components of angular momentum on the body axes; Ix, Iy, Iz the moments of inertia about the body axes; and $\omega$x, $\omega$y, $\omega$z the components of satellite angular velocity on the body axes relative to a fixed frame of reference.

The time rate of change of angular momentum in vector notation may be defined as

$$\dot{\bar{H}} = \dot{\bar{H}}_r + \bar{\omega} \times \bar{H}$$

where $\dot{\bar{H}}_r$ is the relative rate of change of angular momentum as observed from the satellite. Therefore, an expression for the time rate of change of angular momentum of the satellite in terms of body axes is

$$\dot{\bar{H}} = \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ H_x & H_y & H_z \end{vmatrix} \tag{2.2}$$

or, substituting equations (2.1) into equations (2.2) and collecting terms

$$\begin{aligned} \dot{\bar{H}} = \; & \bar{i}\,[I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y)] + \\ & \bar{j}\,[I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z)] + \\ & \bar{k}\,[I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x)] \end{aligned} \tag{2.3}$$

Now $$\dot{\bar{H}} = \bar{Q}$$

$\dot{\bar{H}}$, the time rate of change of angular momentum of the satellite about the center of mass, is equal to $\bar{Q}$, the resultant moment of the external forces about the same point. Denoting the components of $\bar{Q}$ on the body axes as L, M and N, equations (2.3) may be written as

$$L = I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y) \tag{2.4a}$$

$$M = I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z) \tag{2.4b}$$

$$N = I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x) \tag{2.4c}$$

Consider now that the satellite is assigned the mission of orienting one of its axes toward the earth and maintaining that specific orientation. The reference coordinate system is defined as shown in figure (2.1).

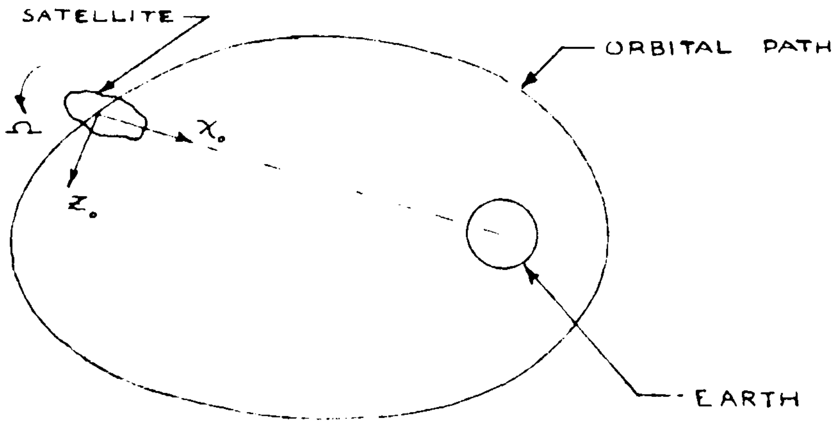



Figure (2.1). Reference Coordinate System.

The $x_o - z_o$ plane lies in the plane of the orbit and the $x_o$ axis is directed toward the earth's center. $\Omega$ is the angular velocity of rotation of the reference coordinate system about the $y_o$ axis. Appendix 1 defines the Euler angle transformation which relates the attitude of the satellite to the reference coordinate system. The body components of the angular velocity of the satellite can be related to the angular velocity of the reference coordinate system and the time rates of change of the Euler angles by the following equations in matrix notation.

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \Omega \\ 0 \end{Bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} \qquad (2.5)$$

Assume that the Euler angles $\theta$ , $\psi$ and the time derivatives of $\phi$ , $\theta$ and $\psi$ can be considered small but that the Euler angle $\phi$ is not necessarily small. Trigonometric terms involving $\phi$ then cannot be linearized. The second order approximation of equations (2.5) is

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 1 & \Psi & -\Theta \\ \Theta\sin\phi - \Psi\cos\phi & \Theta\Psi\sin\phi + \cos\phi & \sin\phi \\ \Theta\cos\phi + \Psi\sin\phi & \Theta\Psi\cos\phi - \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ \Omega \\ 0 \end{Bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & -\Theta \\ \Theta\sin\phi & \cos\phi & \sin\phi \\ \Theta\cos\phi & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} \tag{2.6}$$

Equations (2.6) can now be expressed as

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} \Psi \\ \Theta\Psi\sin\phi + \cos\phi \\ \Theta\Psi\cos\phi - \sin\phi \end{Bmatrix} \Omega + \begin{Bmatrix} -\Theta \\ \sin\phi \\ \cos\phi \end{Bmatrix} \dot{\Psi} + \begin{Bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{Bmatrix} \dot{\Theta} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \dot{\phi}$$

Therefore

$$\omega_x = \Psi\Omega - \Theta\dot{\Psi} + \dot{\phi} \tag{2.7a}$$

$$\omega_y = (\Theta\Psi\sin\phi + \cos\phi)\Omega + \dot{\Psi}\sin\phi + \dot{\Theta}\cos\phi \tag{2.7b}$$

$$\dot{\omega}_z = (\Theta\Psi\cos\phi - \sin\phi)\Omega + \dot{\Psi}\cos\phi - \dot{\Theta}\sin\phi \tag{2.7c}$$

and

$$\dot{\omega}_x = \Psi\dot{\Omega} + \Omega\dot{\Psi} - \Theta\ddot{\Psi} - \dot{\Psi}\dot{\Theta} + \ddot{\phi} \tag{2.8a}$$

$$\dot{\omega}_y = \theta\Psi\dot{\Omega}\sin\phi + \theta\Psi\Omega\dot{\phi}\cos\phi + \theta\dot{\Psi}\Omega\sin\phi + \dot{\theta}\Psi\Omega\sin\phi + \dot{\Omega}\cos\phi - \Omega\dot{\phi}\sin\phi + \ddot{\Psi}\sin\phi + \dot{\Psi}\dot{\phi}\cos\phi + \ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi \tag{2.8b}$$

$$\dot{\omega}_z = \theta\Psi\dot{\Omega}\cos\phi - \theta\Psi\Omega\dot{\phi}\sin\phi + \theta\dot{\Psi}\Omega\cos\phi + \dot{\theta}\Psi\Omega\cos\phi - \dot{\Omega}\sin\phi - \Omega\dot{\phi}\cos\phi + \ddot{\Psi}\cos\phi - \dot{\Psi}\dot{\phi}\sin\phi - \ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi \tag{2.8c}$$

Substituting equations (2.7) and (2.8) into equations (2.4), and eliminating third order and higher terms in $\Theta$ and $\Psi$ and the time derivatives of $\phi$, $\Theta$ and $\Psi$, the following moment equations result.

$$L = I_x(\dot{\Omega}\Psi + \Omega\dot{\Psi} - \theta\ddot{\Psi} - \dot{\theta}\dot{\Psi} + \ddot{\phi}) + (I_z - I_y)\left[(\dot{\theta}\dot{\Psi} + \dot{\Psi}\Omega + \theta\Psi\Omega^2)\cos 2\phi + \left(-\dot{\theta}\Omega + \frac{\dot{\Psi}^2}{2} - \frac{\dot{\theta}^2}{2} - \frac{\Omega^2}{2}\right)\sin 2\phi\right] \tag{2.9a}$$

$$M = I_y\left[(\theta\Psi\dot{\Omega} + \theta\dot{\Psi}\Omega + \dot{\theta}\Psi\Omega - \dot{\phi}\Omega + \ddot{\Psi} - \dot{\phi}\dot{\theta})\sin\phi + (\dot{\Omega} + \dot{\phi}\dot{\Psi} + \ddot{\theta})\cos\phi\right] + (I_x - I_z)\left[(\dot{\Psi}\dot{\phi} + \Psi\dot{\Psi}\Omega)\cos\phi + (\theta\dot{\Psi}\Omega - \dot{\phi}\Omega - \dot{\theta}\dot{\phi} - \Psi\Omega^2 + - \dot{\theta}\Psi\Omega)\sin\phi\right] \tag{2.9b}$$

$$N = I_z[(-\dot{\Omega} - \dot{\phi}\dot{\psi} - \ddot{\theta})\sin\phi + (\theta\psi\dot{\Omega} + \theta\dot{\psi}\Omega +$$
$$\dot{\theta}\psi\Omega - \dot{\phi}\Omega + \ddot{\psi} - \dot{\phi}\dot{\theta})\cos\phi] + (I_y - I_x)[(\dot{\phi}\Omega +$$
$$\dot{\phi}\dot{\theta} + \psi\Omega^2 - \theta\dot{\psi}\Omega + \dot{\theta}\psi\Omega)\cos\phi + (\dot{\phi}\dot{\psi} + \psi\dot{\psi}\Omega)\sin\phi] \qquad (2.9c)$$

Since no restriction has been placed on $\Omega$, it is reasonable to assume that $\Omega$ will be large in comparison with $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$. It has been stated that the $\theta$ and $\psi$ displacements and the time derivatives of $\phi$, $\theta$ and $\psi$ are assumed small. Utilization of the above two assumptions and elimination of products of these small quantities, equations (2.9) can be reduced to the following set of moment equations.

$$L = I_x(\psi\dot{\Omega} + \dot{\psi}\Omega + \ddot{\phi}) + (I_z - I_y)[\dot{\psi}\Omega\cos 2\phi + (-\dot{\theta}\Omega - \tfrac{\Omega^2}{2})\sin 2\phi] \qquad (2.10a)$$

$$M = I_y[(-\dot{\phi}\Omega + \ddot{\psi})\sin\phi + (\dot{\Omega} + \ddot{\theta})\cos\phi] + (I_x - I_z)(-\dot{\phi}\Omega - \psi\Omega^2)\sin\phi \qquad (2.10b)$$

$$N = I_z[(-\dot{\Omega} - \ddot{\theta})\sin\phi + (-\dot{\phi}\Omega + \ddot{\psi})\cos\phi] + (I_y - I_x)(\dot{\phi}\Omega + \psi\Omega^2)\cos\phi \qquad (2.10c)$$

Thus, equations (2.10) represent the angular motion of an artificial satellite in an elliptical orbit.

# CHAPTER 3

## GRAVITY GRADIENT MOMENTS

As a means of controlling the satellite's attitude, the principle of gravity gradient is useful. The gravity gradient torque can provide inherent static stability for a satellite with respect to the earth's gravitational field, providing external disturbances are small.

The gravitational potential of an element of mass located at ( $x_o$, $y_o$ , $z_o$ ) on the reference coordinate system as defined in Figure (2.1), is

$$dU = -g_o R^2 \frac{dm}{\sqrt{(r - x_o)^2 + y_o^2 + z_o^2}} \tag{3.1}$$

where $g_o$ is the acceleration of gravity at the earth's surface; R, the radius of the earth; and r, the distance from the center of mass of the earth to the origin of the reference coordinate system.

Rewriting equation (3.1)

$$dU = -\frac{g_o R^2}{r}\left[1 - \frac{2x_o}{r} + \left(\frac{x_o}{r}\right)^2 + \left(\frac{y_o}{r}\right)^2 + \left(\frac{z_o}{r}\right)^2\right]^{-\frac{1}{2}} dm$$

Expanding according to the binomial expansion,

$$dU = -\frac{g_o R^2}{r}\left\{1 - \frac{1}{2}\left[-\frac{2x_o}{r} + \left(\frac{x_o}{r}\right)^2 + \left(\frac{y_o}{r}\right)^2 + \left(\frac{z_o}{r}\right)^2\right] + \frac{3}{8}\left[-\frac{2x_o}{r} + \left(\frac{x_o}{r}\right)^2 + \left(\frac{y_o}{r}\right)^2 + \left(\frac{z_o}{r}\right)^2\right]^2 \cdots\right\} dm$$

and eliminating the 3rd order and higher terms,

$$dU = -\frac{g_o R^2}{r}\left[1 + \frac{x_o}{r} + \left(\frac{x_o}{r}\right)^2 - \frac{1}{2}\left(\frac{y_o}{r}\right)^2 - \frac{1}{2}\left(\frac{z_o}{r}\right)^2\right] dm \qquad (3.2)$$

The relationship of body and reference coordinates, shown in matrix notation below, may be obtained from the Euler angle transformations described in Appendix 1, assuming small external disturbances.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & \psi & -\theta \\ \theta\sin\phi - \psi\cos\phi & \theta\psi\sin\phi + \cos\phi & \sin\phi \\ \theta\cos\phi + \psi\sin\phi & \theta\psi\cos\phi - \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix}$$

The inverse relationship is given by

$$\begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} = \begin{bmatrix} 1 - \frac{\theta^2}{2} - \frac{\psi^2}{2} & -\psi\cos\phi + \theta\sin\phi & \psi\sin\phi + \theta\cos\phi \\ \psi & \left(1 - \frac{\psi^2}{2}\right)\cos\phi + \theta\psi\sin\phi & -\left(1 - \frac{\psi^2}{2}\right)\sin\phi + \theta\psi\cos\phi \\ -\theta & \left(1 - \frac{\theta^2}{2}\right)\sin\phi & \left(1 - \frac{\theta^2}{2}\right)\cos\phi \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad (3.3)$$

Then we can write

$$x_o^2 = \left[\left(1-\frac{\theta^2}{2}-\frac{\psi^2}{2}\right)x + (-\psi\cos\phi+\theta\sin\phi)y + (\psi\sin\phi+\theta\cos\phi)z\right]^2$$

$$y_o^2 = \left\{\psi x + \left[\left(1-\frac{\psi^2}{2}\right)\cos\phi+\theta\psi\sin\phi\right]y + \left[-\left(1-\frac{\psi^2}{2}\right)\sin\phi+\theta\psi\cos\phi\right]z\right\}^2$$

$$z_o^2 = \left\{-\theta x + \left[\left(1-\frac{\theta^2}{2}\right)\sin\phi\right]y + \left[\left(1-\frac{\theta^2}{2}\right)\cos\phi\right]z\right\}^2$$

It is advantageous, at this point, to define the static moments, products of inertia, and moments of inertia, respectively, prior to expanding the $x_o^2$ , $y_o^2$ , $z_o^2$ terms and integrating.

$$\int x\,dm = \int y\,dm = \int z\,dm = 0$$

$$\int xy\,dm = \int yz\,dm = \int xz\,dm = 0$$

$$\int (x^2+y^2)dm = I_z \quad , \quad \int (y^2+z^2)\,dm = I_x \quad , \quad \int (x^2+z^2)\,dm = I_y$$

Thus, integration of the $\frac{x_o}{r}$ term of equation (3.2) in terms of x, y and z and terms involving products of x, y, z are zero and therefore neglected. Expanding the squared terms and eliminating third order terms or higher in the small quantities $\theta$ or $\psi$ , the following expressions for $x_o^2$ , $y_o^2$ and $z_o^2$ result.

$$x_o^2 = (1-\theta^2-\psi^2)x^2 + (\psi^2\cos^2\phi - 2\theta\psi\sin\phi\cos\phi + \theta^2\sin^2\phi)y^2 + (\psi^2\sin^2\phi + 2\theta\psi\sin\phi\cos\phi + \theta^2\cos^2\phi)z^2 \tag{3.4a}$$

$$y_o^2 = \psi^2 x^2 + (\cos^2\phi - \psi^2\cos^2\phi + 2\theta\psi\sin\phi\cos\phi)y^2 + (\sin^2\phi - \psi^2\sin^2\phi - 2\theta\psi\sin\phi\cos\phi)z^2 \qquad (3.4b)$$

$$z_o^2 = \theta^2 x^2 + [(1-\theta^2)\sin^2\phi]y^2 + [(1-\theta^2)\cos^2\phi]z^2 \qquad (3.4c)$$

Substitution of equations (3.4) into equation (3.2) and integrating equation (3.2) utilizing the definitions of the moments of inertia yields

$$U = -\frac{g_o R^2}{r}\Big[m + \frac{1}{2r^2}(I_y + I_z - 2I_x) + \frac{3}{2r^2}\theta^2(I_x - I_z) + \frac{3}{2r^2}\psi^2(I_x - I_y) + \frac{3}{2r^2}(I_z - I_y)(\theta^2 - \psi^2)\sin^2\phi + - \frac{3}{r^2}(I_z - I_y)\theta\psi\sin\phi\cos\phi\Big] \qquad (3.5)$$

The partial derivatives of $U$ with respect to $\phi$ , $\theta$ and $\psi$ yield the components of the gravity gradient moments in the directions of increasing $\phi$, $\theta$, $\psi$ .

$$-\frac{\partial U}{\partial \phi} = \frac{3 g_o R^2}{r^3}(I_z - I_y)\Big[(\theta^2 - \psi^2)\sin\phi\cos\phi + - \theta\psi(\cos^2\phi - \sin^2\phi)\Big] \qquad (3.6a)$$

$$-\frac{\partial U}{\partial \theta} = \frac{3g_o R^2}{r^3}\Big\{(I_x - I_z)\theta +$$

$$(I_z - I_y)\left[\theta \sin^2\phi - \psi \sin\phi \cos\phi\right]\Big\} \tag{3.6b}$$

$$-\frac{\partial U}{\partial \psi} = \frac{3g_o R^2}{r^3}\Big\{(I_x - I_y)\psi +$$

$$-(I_z - I_y)\left[\psi \sin^2\phi + \theta \sin\phi \cos\phi\right]\Big\} \tag{3.6c}$$

The angular displacement $\phi$ is about the body x axis. However, $\theta$ is about an intermediate $y_1$ axis and $\psi$ about the reference $z_0$ axis. Therefore, the results of equations (3.6) must be transformed to the body axes. The Euler angle transformations of Appendix 1 can be utilized for this operation. Transforming the components of the gravity gradient moment to the body coordinate system, the following is obtained.

$$\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\frac{\partial U}{\partial \psi} \end{Bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ -\frac{\partial U}{\partial \theta} \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\frac{\partial U}{\partial \phi} \\ 0 \\ 0 \end{Bmatrix}$$

Combining matrices,

$$\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\theta \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} -\frac{\partial U}{\partial \phi} \\ -\frac{\partial U}{\partial \theta} \\ -\frac{\partial U}{\partial \psi} \end{Bmatrix} \tag{3.7}$$

Substitution of the equations (3.6) into equations (3.7) yields

$$L = \frac{3g_oR^2}{r^3}\Big\{(I_z - I_y)\big[(\theta^2 - \psi^2)\sin\phi\cos\phi + $$
$$- \theta\psi(\cos^2\phi - \sin^2\phi)\big] - (I_x - I_y)\theta\psi +$$
$$(I_z - I_y)\theta\left[\psi\sin^2\phi + \theta\sin\phi\cos\phi\right]\Big\} \tag{3.8a}$$

$$M = \frac{3g_oR^2}{r^3}\Big\{(I_x - I_z)\theta\cos\phi + (I_x - I_y)\psi\sin\phi +$$
$$(I_z - I_y)\cos\phi\left[\theta\sin^2\phi - \psi\sin\phi\cos\phi\right] +$$
$$-(I_z - I_y)\sin\phi\left[\psi\sin^2\phi + \theta\sin\phi\cos\phi\right]\Big\} \tag{3.8b}$$

$$N = \frac{3g_oR^2}{r^3}\Big\{-(I_x - I_z)\theta\sin\phi + (I_x - I_y)\psi\cos\phi +$$
$$-(I_z - I_y)\sin\phi\left[\theta\sin^2\phi - \psi\sin\phi\cos\phi\right] +$$
$$-(I_z - I_y)\cos\phi\left[\psi\sin^2\phi + \theta\sin\phi\cos\phi\right]\Big\} \tag{3.8c}$$

Substitution of the moment equations (3.8) into equations (2.10) results in the equations of motion for an earth orbiting satellite of unspecified shape stabilized by gravity gradient with unrestricted angular displacement about the earth-oriented axis.

$$I_x(\psi\dot{\Omega} + \dot{\psi}\Omega + \ddot{\phi}) + (I_z - I_y)\big[\dot{\psi}\Omega\cos 2\phi +$$
$$\left(-\dot{\theta}\Omega - \frac{\Omega^2}{2}\right)\sin 2\phi\big] + \frac{3g_oR^2}{r^3}\Big\{(I_y - I_z)\big[(\theta^2 +$$
$$-\psi^2)\sin\phi\cos\phi - \theta\psi(\cos^2\phi - \sin^2\phi)\big] +$$
$$(I_x - I_y)\theta\psi + (I_y - I_z)\theta\left[\psi\sin^2\phi + \theta\sin\phi\cos\phi\right]\Big\} = 0 \tag{3.9a}$$

$$I_y[(-\dot{\phi}\Omega+\ddot{\psi})\sin\phi + (\dot{\Omega}+\ddot{\theta})\cos\phi] +$$
$$(I_x - I_z)(-\dot{\phi}\Omega - \psi\Omega^2)\sin\phi +$$
$$\frac{3g_oR^2}{r^3}\{(I_z - I_x)\theta\cos\phi + (I_y - I_x)\psi\sin\phi +$$
$$(I_y - I_z)\cos\phi\,[\theta\sin^2\phi - \psi\sin\phi\cos\phi] +$$
$$(I_z - I_y)\sin\phi[\psi\sin^2\phi + \theta\sin\phi\cos\phi]\} = 0 \qquad (3.9b)$$

$$I_z[(-\dot{\Omega}-\ddot{\theta})\sin\phi + (-\dot{\phi}\Omega+\ddot{\psi})\cos\phi] +$$
$$(I_y - I_x)(\dot{\phi}\Omega + \psi\Omega^2)\cos\phi + \frac{3g_oR^2}{r^3}\{(I_x - I_z)\theta\sin\phi +$$
$$(I_y - I_x)\psi\cos\phi + (I_z - I_y)\sin\phi[\theta\sin^2\phi +$$
$$-\psi\sin\phi\cos\phi] + (I_z - I_y)\cos\phi[\psi\sin^2\phi +$$
$$\theta\sin\phi\cos\phi]\} = 0 \qquad (3.9c)$$

Eliminating products of small quantities $\theta$, $\psi$ reduce equations (3.9) to the following equations of motion.

$$I_x(\psi\dot{\Omega} + \dot{\psi}\Omega + \ddot{\phi}) + (I_y - I_z)[-\dot{\psi}\Omega\cos 2\phi + \qquad (3.10a)$$
$$(\dot{\theta}\Omega + \frac{\Omega^2}{2})\sin 2\phi] = 0$$

$$I_y[(-\dot{\phi}\Omega+\ddot{\psi})\sin\phi + (\dot{\Omega}+\ddot{\theta})\cos\phi] + (I_z - I_x)(\dot{\phi}\Omega +$$
$$\psi\Omega^2)\sin\phi] + \frac{3g_oR^2}{r^3}\{(I_z - I_x)\theta\cos\phi + (I_y - I_x)\psi\sin\phi +$$
$$(I_y - I_z)\cos\phi[\theta\sin^2\phi - \psi\sin\phi\cos\phi] +$$
$$-(I_y - I_z)\sin\phi[\psi\sin^2\phi + \theta\sin\phi\cos\phi]\} = 0 \qquad (3.10b)$$

$$I_z[(-\dot{\Omega} - \ddot{\theta})\sin\phi + (-\dot{\phi}\Omega + \ddot{\psi})\cos\phi] + (I_y - I_x)(\dot{\phi}\Omega + \psi\Omega^2)\cos\phi + \frac{3g_0R^2}{r^3}\{-(I_z + - I_x)\theta\sin\phi + (I_y - I_x)\psi\cos\phi + -(I_y - I_z)\sin\phi[\theta\sin^2\phi - \psi\sin\phi\cos\phi] + -(I_y - I_z)\cos\phi[\psi\sin^2\phi + \theta\sin\phi\cos\phi]\} = 0 \quad (3.10c)$$

For a satellite of laterally isotropic shape Iy = Iz = I. For a satellite injected into a circular orbit, the angular velocity of rotation of the reference coordinate system, $\Omega$, is constant. Therefore, $\dot{\Omega} = 0$ and $\Omega$ may be expressed as $\sqrt{\frac{g_0R^2}{r^3}}$ where r is now a constant. The following equations of motion, derived from equations (3.10), result for the special case of a laterally isotropic satellite in a circular orbit.

$$I_x(\dot{\psi}\Omega + \ddot{\phi}) = 0 \quad (3.11a)$$

$$I[(-\dot{\phi}\Omega + \ddot{\psi})\sin\phi + \ddot{\theta}\cos\phi] + (I - I_x)(\dot{\phi}\Omega + \psi\Omega^2)\sin\phi] + 3\Omega^2\{(I - I_x)\theta\cos\phi + (I - I_x)\psi\sin\phi\} = 0 \quad (3.11b)$$

$$I[-\ddot{\theta}\sin\phi + (-\dot{\phi}\Omega + \ddot{\psi})\cos\phi] + (I - I_x)(\dot{\phi}\Omega + \psi\Omega^2)\cos\phi + 3\Omega^2\{-(I - I_x)\theta\sin\phi + (I - I_x)\psi\cos\phi\} = 0 \quad (3.11c)$$

The terms in equations (3.11b) and (3.11c) represent moments about the y and z body axes respectively. Equation (3.11a) represents the equilibrium of moments about both the body x and reference $x_o$ axis since $\theta$ and $\psi$ angular displacements were assumed small. Equations representing the equilibrium of moments around the $y_o$ and $z_o$ axes may be derived from equations (3.11b) and (3.11c) as follows.

$$[\text{equation (3.11b)}]\cos\phi - [\text{equation (3.11c)}]\sin\phi \tag{3.12a}$$

$$[\text{equation (3.11c)}]\cos\phi + [\text{equation (3.11b)}]\sin\phi \tag{3.12b}$$

Performing the indicated operations of expressions (3.12) and rewriting equation (3.11a), the equations of motion become

$$\ddot{\phi} + \dot{\psi}\Omega = 0 \tag{3.13a}$$

$$\ddot{\theta} + 3\Omega^2\left(\frac{I - I_x}{I}\right)\theta = 0 \tag{3.13b}$$

$$\ddot{\psi} - \frac{I_x}{I}\Omega\dot{\phi} + 4\Omega^2\left(\frac{I - I_x}{I}\right)\psi = 0 \tag{3.13c}$$

Equations (3.13) are identical to those derived in reference (1) assuming small $\phi$ , $\theta$ and $\psi$ angular displacements. It can be concluded that the same satellite motion will result whether the angular displacement about the earth oriented axis of a laterally isotropic satellite is or is not restricted.

CHAPTER 4

STABILITY CONDITIONS FOR SATELLITE OF ARBITRARY SHAPE

For a satellite of arbitrary shape, there may be one or several attitudes it may assume in orbit for which the satellite is said to be in a condition of equilibrium. Once the equilibrium state has been established, the stability characteristics of the equilibrium must be determined. If a satellite in equilibrium encounters a disturbance, it will either tend to return to the equilibrium state or continue to move away from it. Motion of the former nature is considered stable and of the latter, unstable or divergent.

The following analysis is intended to determine what specific attitudes a satellite of arbitrary shape must assume for stable motion, once its equilibrium has been disturbed. Further qualification of the analysis requires that

1. the satellite is in a circular orbit, for which the angular velocity of rotation of the reference coordinate system, $\Omega$, is constant. Therefore, $\dot{\Omega} = 0$ and $\Omega$ may be expressed as $\sqrt{\dfrac{g_0 R^2}{r^3}}$.

2. the satellite angular motion is small. Thus, the $\Phi$ trigonometric terms can be linearized.

Application of the above criteria to equations (3.10) of Chapter 3 reduces them to the following linearized equations.

$$I_x \ddot{\phi} + \Omega^2 (I_y - I_z)\phi + \Omega(I_x - I_y + I_z)\dot{\psi} = 0 \qquad (4.1a)$$

$$I_y \ddot{\theta} + 3\Omega^2 (I_z - I_x)\theta = 0 \qquad (4.1b)$$

$$I_z \ddot{\psi} + 4\Omega^2 (I_y - I_x)\psi + \Omega(I_y - I_x - I_z)\dot{\phi} = 0 \qquad (4.1c)$$

Inspection of equations (4.1) indicates that the equation of motion in the plane of the orbit is uncoupled. Equation (4.1b) will therefore be investigated briefly before considering the two coupled equations. A more complete **analysis** of equation (4.1b) is performed in Chapter 5.

Rewriting equation (4.1b),

$$\ddot{\theta} + 3\Omega^2 \left(\frac{I_z - I_x}{I_y}\right)\theta = 0$$

Since $\Omega^2$ and $I_y$ are intrinsically positive, the $I_z$ quantity must be greater than the $I_x$ quantity for stable motion. A general solution to equation (4.1b) can then be written as

$$\theta = A \cos\sqrt{3\Omega^2\left(\frac{I_z - I_x}{I_y}\right)}\, t + B \sin\sqrt{3\Omega^2\left(\frac{I_z - I_x}{I_y}\right)}\, t$$

where A and B are arbitrary constants to be evaluated from the initial conditions.

The general solution is seen to be oscillatory from which it can be said that the motion in the orbital plane is stable.

Consider now the coupled equations (4.1a,c) which, when solved, represent the response $\phi(t)$ and $\psi(t)$.

$$\ddot{\phi} + \Omega^2\left(\frac{I_y - I_z}{I_x}\right)\phi + \Omega\left(\frac{I_x - I_y + I_z}{I_x}\right)\dot{\psi} = 0$$

$$\ddot{\psi} + 4\Omega^2\left(\frac{I_y - I_x}{I_z}\right)\psi + \Omega\left(\frac{I_y - I_x - I_z}{I_z}\right)\dot{\phi} = 0$$

Let $R = \dfrac{I_y - I_z}{I_x}$

$S = \dfrac{I_y - I_x}{I_z}$

The above two coupled equations can now be rewritten as

$$\ddot{\phi} + \Omega^2 R\phi + \Omega(1-R)\dot{\psi} = 0 \tag{4.2a}$$

$$\ddot{\psi} + 4\Omega^2 S\psi + \Omega(S-1)\dot{\phi} = 0 \tag{4.2b}$$

Assume solutions to equations (4.2) of the form

$$\phi = A_1 e^{pt}$$

$$\psi = A_2 e^{pt}$$

Upon substitution of the assumed solutions into equations (4.2) and dividing out the common factor $e^{pt}$, the following two equations can be obtained.

$$(p^2 + \Omega^2 R)A_1 + \Omega(1-R)pA_2 = 0$$

$$\Omega(S-1)pA_1 + (p^2 + 4\Omega^2 S)A_2 = 0$$

Set the determinant of the coefficients equal to zero for a non-trivial solution.

$$\begin{vmatrix} (p^2 + \Omega^2 R) & \Omega(1-R)p \\ \Omega(S-1)p & (p^2 + 4\Omega^2 S) \end{vmatrix} = 0$$

Expanding the determinant, the following characteristic equation results

$$p^4 + \Omega^2(3S + RS + 1)p^2 + 4\Omega^4 RS = 0$$

The solution to the characteristic equation is of the form

$$p^2 = -\frac{\Omega^2}{2}(3S + RS + 1) \pm \frac{\Omega^2}{2}\sqrt{(3S + RS + 1)^2 - 16RS} \qquad (4.3)$$

Inspection of equation (4.3) indicates the roots for $p^2$ may be of the nature:

1. Real $p^2 < 0$; in which event the roots are pure imaginary.
2. Real $p^2 > 0$ ; for which the roots are real, positive and negative.
3. Complex $p^2$ , with real part $> 0$ or $< 0$ ; for which the roots are complex, the real parts positive and negative.

The nature of the roots for Case 1 indicate stable motion can result. Cases 2 and 3, however, always have roots with positive real parts. The only acceptable condition for stable motion then is when the $p^2$ term is less than zero. Bearing this in mind, and focusing attention on the terms in equation (4.3), the ensuing deductions can be made.

The motion will always be unstable if the terms $3S + RS + 1$ and RS are less than zero. This then indicates that the terms $3S + RS + 1$ and RS must be greater than zero for stability. Should they be greater than zero, the motion will be stable only if $(3S + RS + 1)^2 > 16RS$.

# CHAPTER 5

## INVESTIGATION OF THE STABILITY CONDITIONS

The deduced conditions for stable motion from Chapter 4 require further analysis before drawing conclusions regarding satellite attitudes which can result in stable motion. An R-S diagram has been instituted to facilitate visualization of the investigated conditions. The R-S diagram resulting from the complete investigation of the stability conditions will show the stability regions as a function of R and S.

It is recalled from Chapter 4 that

$$R = \frac{Iy - Iz}{Ix} \quad \text{and } S = \frac{Iy - Ix}{Iz}$$

In addition, the conditions requiring further investigation are $RS > 0$, $3S + RS + 1 > 0$ and $(3S + RS + 1)^2 - 16\,RS > 0$ .

### Stability Analysis of $\Phi$, $\Psi$ Motions

Consider $RS > 0$ . Substitution of the expressions for R and S into the above inequality yields $\left(\frac{Iy - Iz}{Ix}\right)\left(\frac{Iy - Ix}{Iz}\right) > 0$

It becomes readily apparent that Iy must be either greater than Iz and Ix or less than both for the product of the two terms to be non-negative. The regions of R and S for which $RS > 0$ exist are as shown in Figure (5.1).

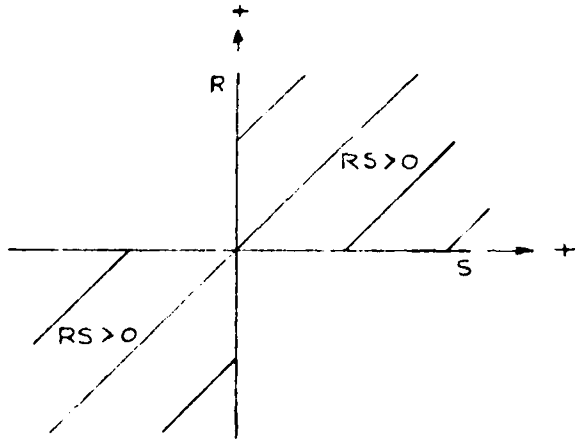



Figure (5.1). Regions of R,S for RS $> 0$

Consider next $3S + RS + 1 > 0$. Rewriting, $R > -\frac{1}{S} - 3$
For R,S $> 0$ and for R,S $< 0$ the regions where $3S + RS + 1 > 0$
exist are as shown in Figure (5.2).

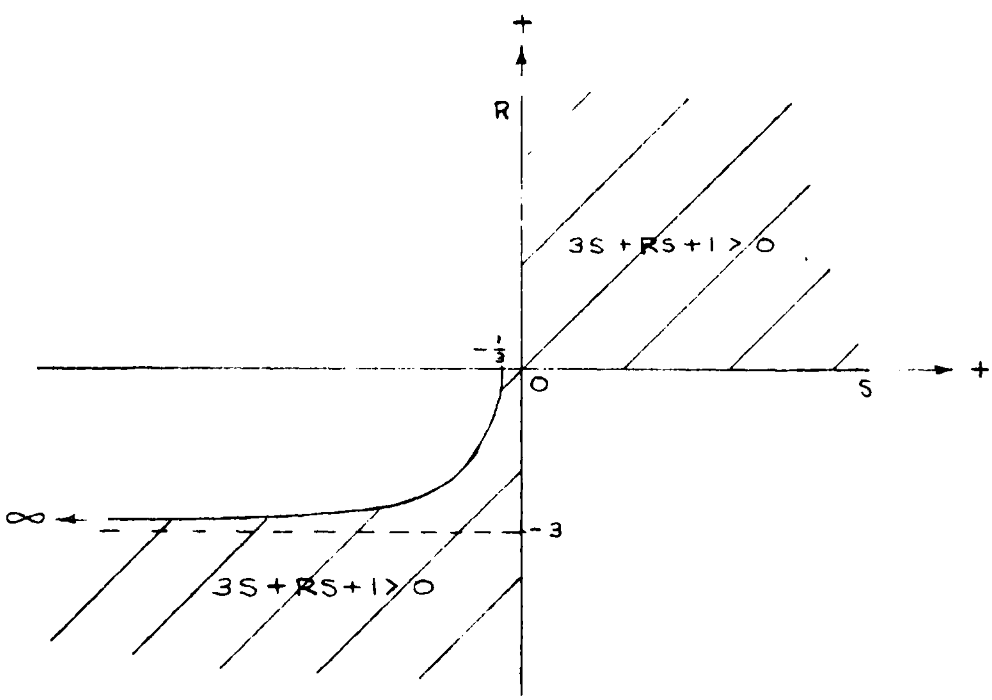



Figure (5.2). Regions of R,S for $3S + RS + 1 > 0$

Now consider $(3S + RS + 1)^2 - 16\,RS > 0$

Expanding the term under consideration, and equating to zero, we obtain $R^2 + (6 - \frac{14}{S})R + (9 + \frac{6}{S} + \frac{1}{S^2}) = 0$

for which the solution is

$$R = \frac{1}{S}\left[7 - 3S \pm 4\sqrt{3 - 3S}\right]$$

For $S > 0$ and $S < 0$ the regions for which $(3S + RS + 1)^2 - 16\,RS > 0$ exist are shown in figure (5.3)

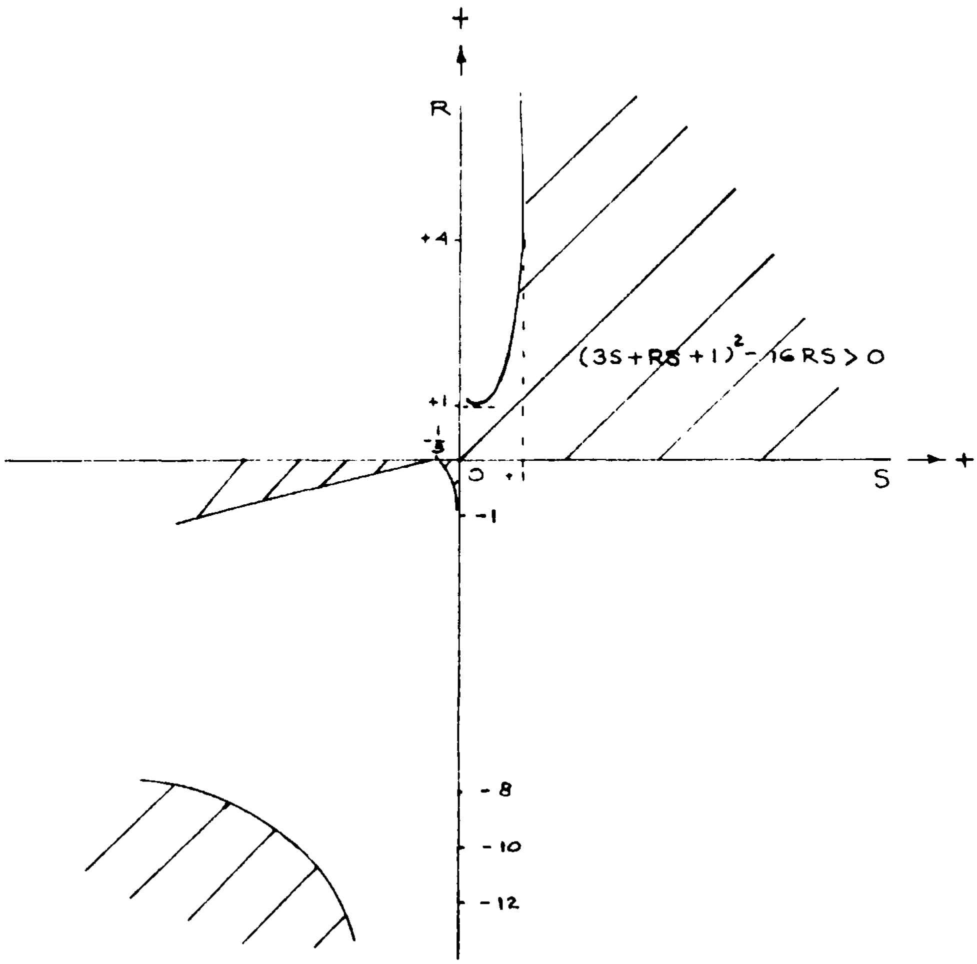



Figure (5.3) Regions of R,S for $(3S + RS + 1)^2 - 16\,RS > 0$

Prior to analyzing the $\theta$ motion, and before superimposing Figures (5.1 - 5.3), an analysis shall be performed to determine which regions of R and S yield negative moments of inertia.

From the expressions for R and S,

$$I_x R + I_z = I_y \text{ and } I_z S + I_x = I_y$$

then

$$\frac{I_z}{I_x} = \frac{(R-1)}{(S-1)}$$ and for non-negative moments of

inertia, $$\frac{I_y}{I_x} > 0 .$$

Thus,

$$\frac{I_y}{I_x} = R + \frac{I_z}{I_x} = \frac{RS-1}{S-1}$$

and now $$\frac{RS-1}{S-1} > 0 . \qquad (5.1)$$

Likewise, for non-negative moments of inertia $\frac{I_z}{I_x} > 0$

Then $$\frac{(R-1)}{(S-1)} > 0 \qquad (5.2)$$

Utilization of the inequalities (5.1) and (5.2) yield the regions shown in Figure (5.4) for which values of R and S result in positive moments of inertia.

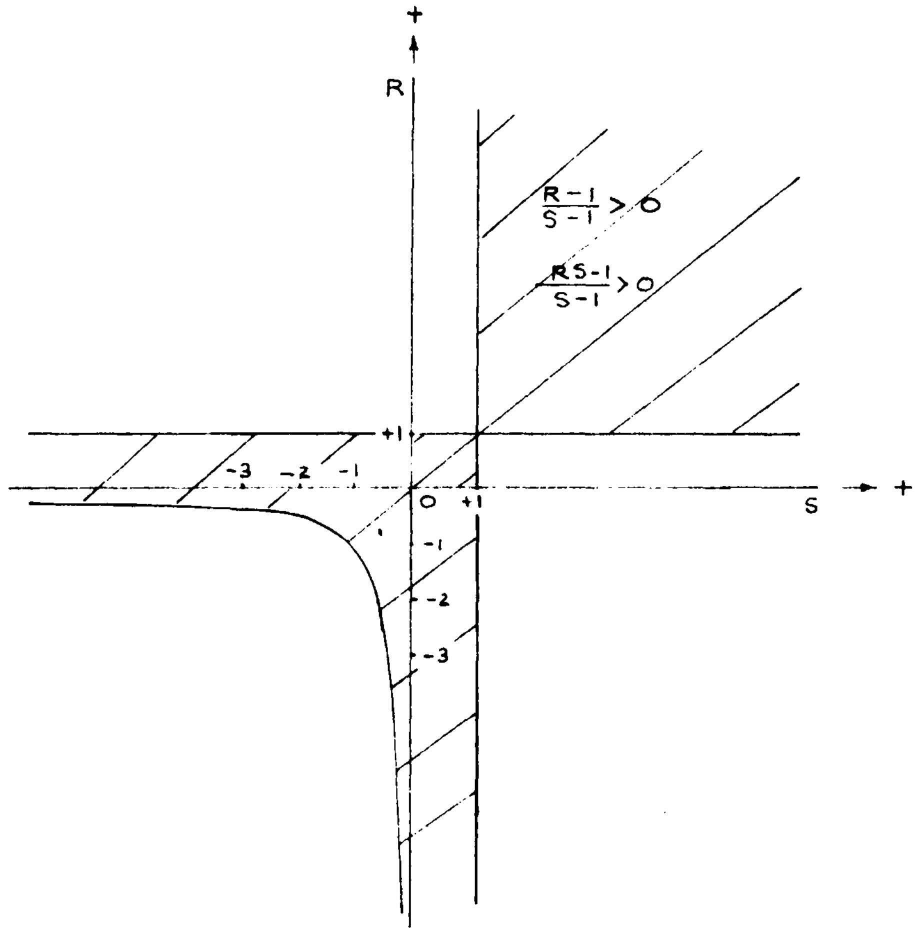



Figure (5.4). Regions of R,S for $\frac{RS-1}{S-1} > 0$ and $\frac{R-1}{S-1} > 0$

Superimposing Figures (5.1 - 5.4) shows the R,S regions in Figure (5.5) where stable motion can occur for the $\phi$ and $\psi$ motions.

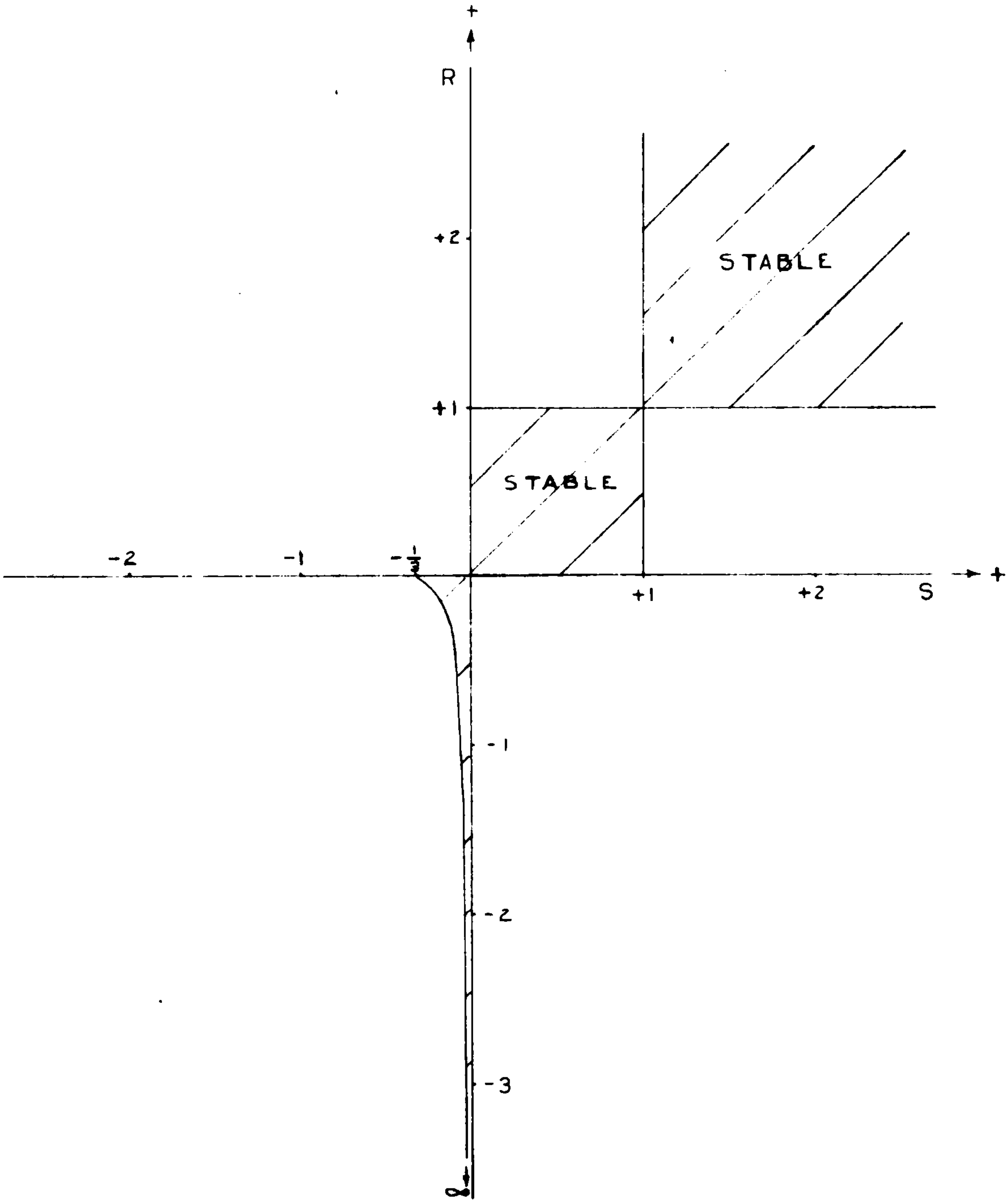



Figure (5.5) Stability Regions of R,S for $\Phi$ and $\Psi$ Motions.

## Stability Analysis of $\Theta$ Motion

As previously stated in Chapter 4, $\frac{Ix}{Iz} < 1$ is required for stability. Then $\frac{S-1}{R-1} < 1$ and the R,S regions where this applies are shown in Figure (5.6).

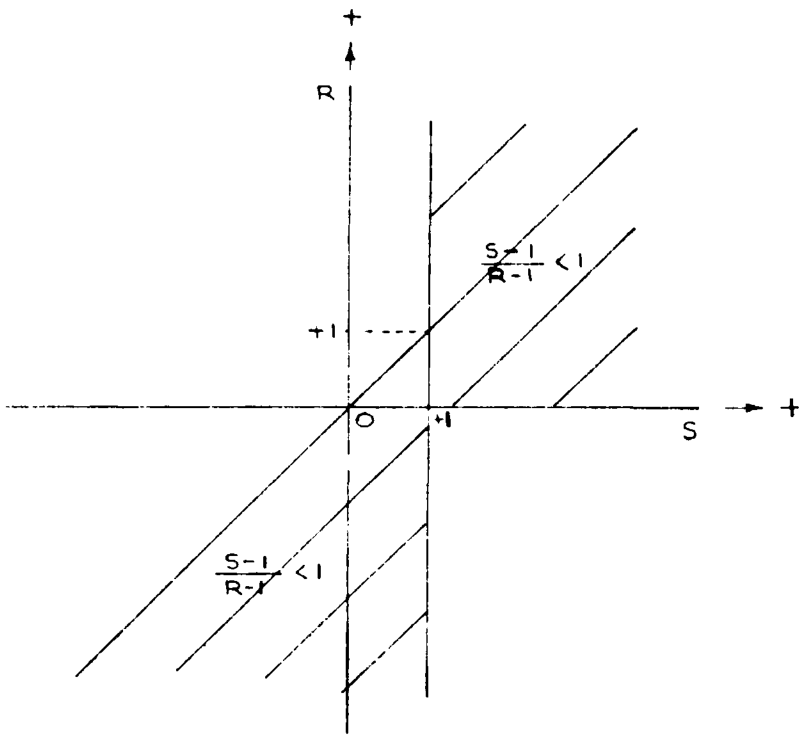



Figure (5.6) Regions of R,S for $\frac{S-1}{R-1} < 1$.

Superimposing Figures (5.4) and (5.6) show the R,S regions in Figure (5.7) where stable motion can occur for the $\Theta$ motion.

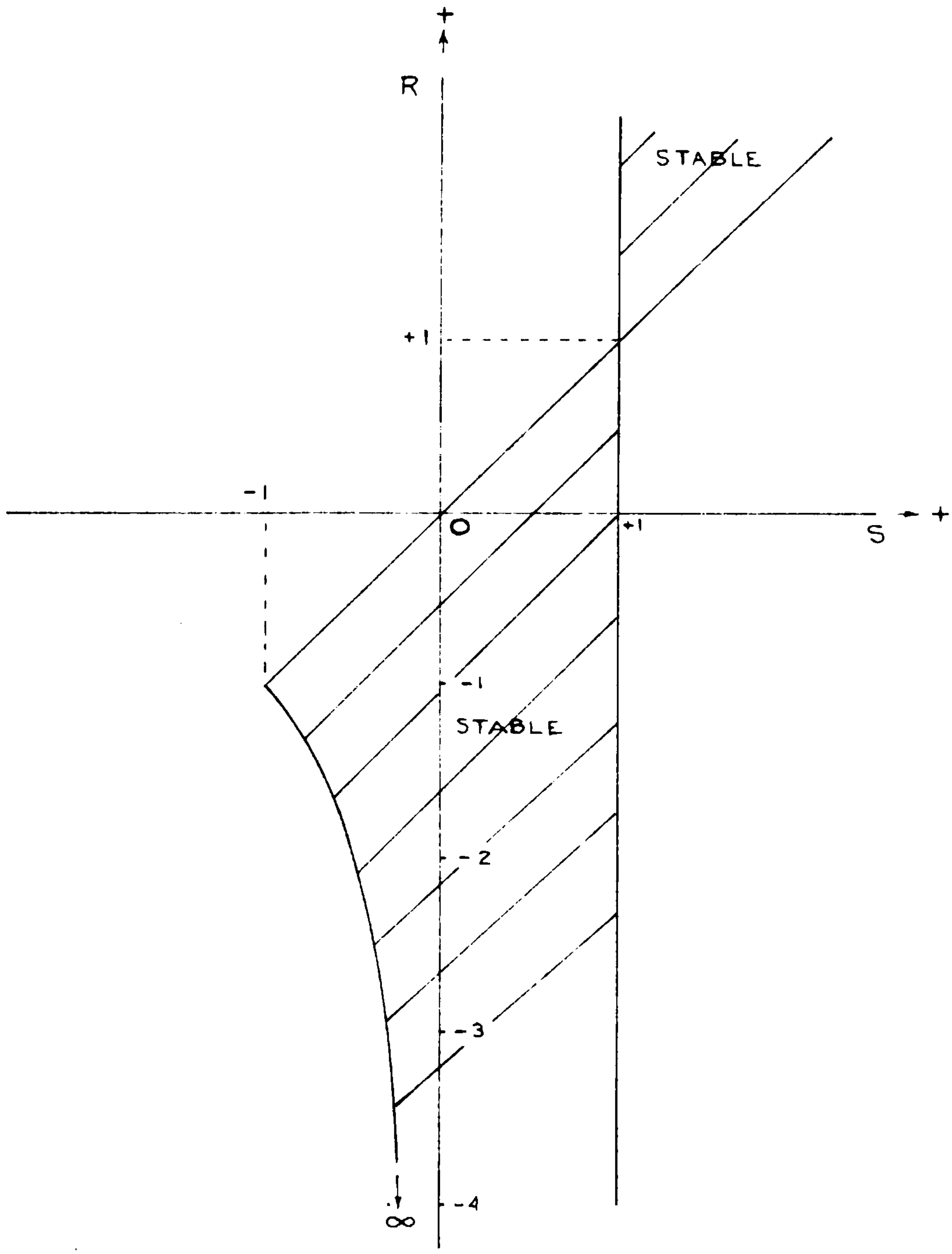



Figure (5.7) Stability Regions of R,S for Θ Motion.

Superimposing Figures (5.5) and (5.7) shows the complete picture of the regions for R and S which yield stable motion for the $\phi$, $\theta$, $\psi$ motions. These regions are shown in Figure (5.8).

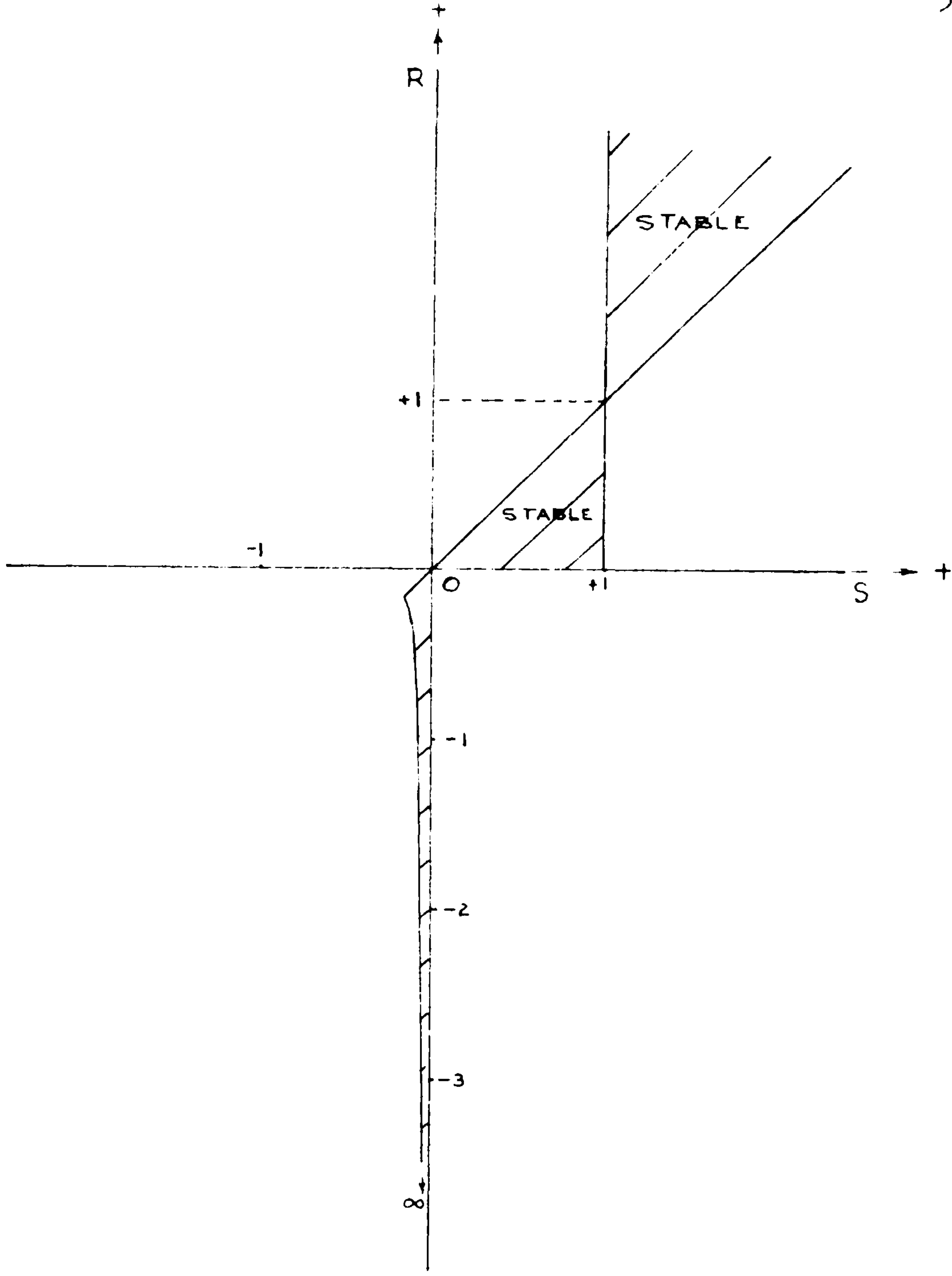



Figure (5.8). Stability Regions of R,S for Φ, Θ, Ψ Motions.

## CHAPTER 6

## APPLICATION OF RESULTS TO SPECIFICALLY SHAPED SATELLITES

Consider a homogeneous solid right circular cylinder of mass m with axis directed toward the earth's center, as shown in Figure (6.1). The x-z plane is in the orbital plane with y axis normal to it. A circular orbit and small satellite angular motions are assumed.

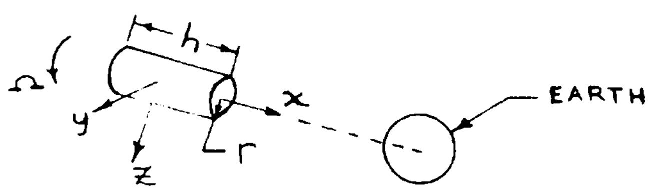



Figure (6.1). Characteristics of Circular Cylindrical Satellite

The principal moments of inertia are

$$I_x = \frac{mr^2}{2}$$

$$I_y = I_z = \frac{m}{12}(h^2 + 3r^2)$$

and

$$R = \frac{I_y - I_z}{I_x} = 0, \qquad S = \frac{I_y - I_x}{I_z} = \frac{h^2 - 3r^2}{h^2 + 3r^2}$$

Substitution of the values for R and S into equation (4.3) yields the following roots for the response $\phi$ (t) and $\psi$ (t).

$$P_{1,2} = 0$$

$$P_{3,4} = \pm i\,\Omega\sqrt{\frac{4h^2 - 6r^2}{h^2 + 3r^2}}$$

The $\theta$ motion can be obtained by substitution of the Iz and Ix values into equation (4.1b). The roots for $\theta(t)$ are $\pm\, i\, \Omega\sqrt{3\left(\frac{h^2-3r^2}{h^2+3r^2}\right)}$.

Consider first the stability of the motion in $\phi$ and $\psi$. The motion is evidently stable if $\frac{h}{r} > \sqrt{\frac{3}{2}}$ and unstable if $\frac{h}{r} < \sqrt{\frac{3}{2}}$. In terms of the parameter S, the motion is stable if $S > -\frac{1}{3}$ and unstable if $S < -\frac{1}{3}$. These conclusions are in agreement with the stability regions of Figure (5.5).

Consider now the stability of the $\theta$ motion. If $\frac{h}{r} > \sqrt{3}$ the motion is stable and unstable if $\frac{h}{r} < \sqrt{3}$. Or, in terms of S, the motion is stable if $S > 0$ and unstable if $S < 0$. These conclusions agree with the stability regions of Figure (5.7).

As our next application, consider a rectangular plate of infinitesimal width oriented in the orbital plane, as shown in Figure (6.2). Assume a circular orbit and small satellite angular motions.

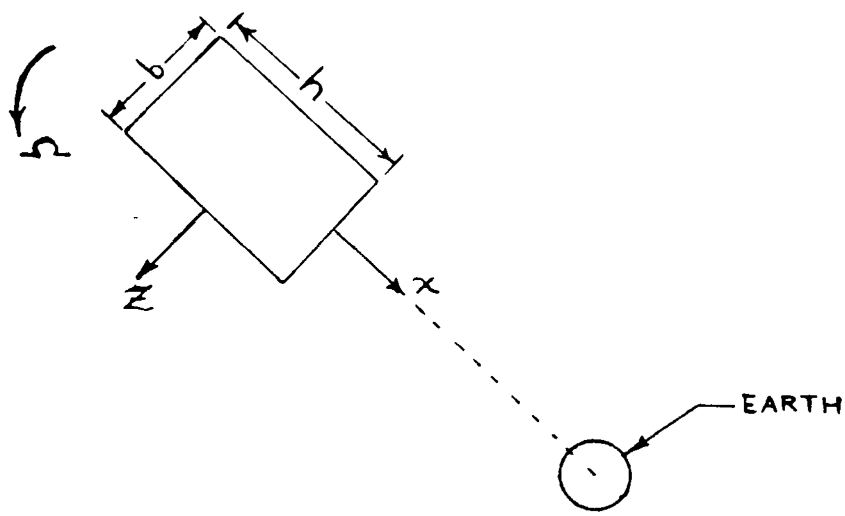



Figure (6.2). Characteristics of Rectangular Satellite

The principle moments of inertia are

$$I_x = \frac{hb^3}{12} \times \mu$$

$$I_y = \frac{bh}{12}\left(b^2 + h^2\right) \times \mu$$

$$I_z = \frac{bh^3}{12} \times \mu$$

where $\mu$ is the mass per unit area.

Now, $R = \dfrac{I_y - I_z}{I_x} = 1$ , $S = \dfrac{I_y - I_x}{I_z} = 1$

The roots for $\phi(t)$ and $\Psi(t)$ are

$$P_{1,2} = \pm i\Omega$$

$$P_{3,4} = \pm i2\Omega$$

The roots for the $\theta$ (t) response are $\pm i\Omega\sqrt{3\left(\frac{h^2-b^2}{h^2+b^2}\right)}$

Consider first the stability of the $\phi$ and $\psi$ motions. Apparently, b and h can assume any value and the motion will be stable. The values for R and S define a point on the boundary between stable and unstable motion of Figure (5.5) so that the results are inconclusive in this case. The roots, however, are definitive and the motion is stable.

Consider next the stability of the $\theta$ motion. The motion is evidently stable as long as $h > b$ and unstable if $h < b$. In terms of the parameters R and S, the values for R and S define a point on the boundary between stable and unstable motion of Figure (5.7). The results in this case are also inconclusive.

## CHAPTER 7

## CONCLUSIONS

The equations for the angular motion of a satellite of arbitrary shape stabilized by gravity gradient moments have been developed. The angular motion has been assumed small except that the angular displacement around the Earth-oriented axis is unrestricted. For the special case of a satellite which is a body of revolution around the Earth-oriented axis, the equations of motion are identical with those for a small motion.

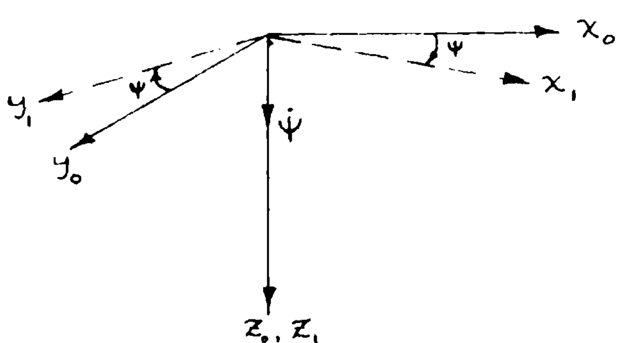
Making use of the equations of motion developed for a satellite of arbitrary shape, the dynamic behavior of such a satellite has been investigated. In order to linearize the equations of motion, a small motion is assumed.

The angular motion of the satellite in the orbital plane, represented by the Euler angle $\theta$ , is uncoupled from the other angular motion if the motion is small. The condition for stability is simply $I_z > I_x$. In terms of the parameters R,S, the condition for stability is shown in Figure (5.7).

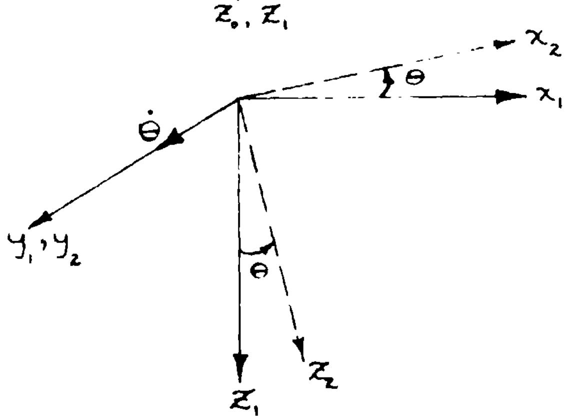
The angular motions of the satellite out of the orbital plane and around the Earth-oriented axis, represented by the Euler angles $\psi$ and $\phi$ , are coupled. Because of the complexity of the motion, the condition for stability cannot be stated simply in terms of the moments of inertia. In terms of the parameters R,S, the condition for stability is shown in Figure (5.5).

## APPENDIX 1

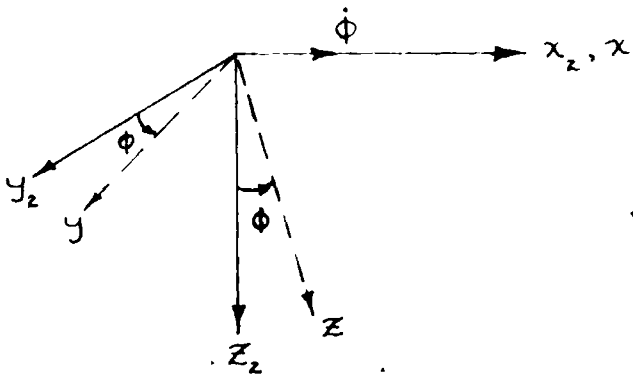
### Euler Angle Transformations




$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix}$$




$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$




$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}$$

## APPENDIX 2

## NOTATION

| Symbol | Quantity |
|---|---|
| $I$ | Moment of Inertia |
| $\omega$ | Angular Velocity |
| $\bar{H}$ | Angular Momentum |
| $\bar{i}, \bar{j}, \bar{k}$ | Unit Vectors |
| $\bar{Q}$ | Resultant Moment about the Center of Mass |
| $L, M, N$ | Components of $\bar{Q}$ on the Body Axes |
| $x, y, z$ | Body Axes (Appear Frequently as Subscripts) |
| $x_o, y_o, z_o$ | Reference Axes |
| $\Omega$ | Angular Velocity of Rotation of the Reference Coordinate System |
| $\phi, \theta, \psi$ | Euler Angles |
| $U$ | Gravity Potential |
| $R$ | Radius of the Earth |
| $g_o$ | Acceleration of Gravity on the Surface of the Earth |
| $m$ | Mass |
| $r$ | Distance from Center of Mass of the Earth to the Origin of the Reference Coordinate System |
| $A, B$ | Arbitrary Constants |
| $R$ | Parameter Equivalent to $\frac{I_y - I_z}{I_x}$ |
| $S$ | Parameter Equivalent to $\frac{I_y - I_x}{I_z}$ |

## APPENDIX 2
(Continued)

## NOTATION

| Symbol | Quantity |
|---|---|
| $e$ | Exponential |
| $\rho$ | Root of the Characteristic Equation |
| $t$ | Time |
| $>$ | Greater Than |
| $<$ | Less Than |
| $\mu$ | Mass Per Unit Area |

Dots over symbols indicate differentiation with respect to time.

LIST OF REFERENCES

(1) Rumsey, Jr., Frank A., The Dynamics of an Earth-Scanning Satellite Stabilized by Gravity-Gradient and Inertia Wheels, University of Arizona Thesis, E 9791, 1962, No. 26