LIBRATIONS OF A SATELLITE RESULTING FROM A GRAVITATIONAL FORCE GRADIENT

by

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ABSTRACT

The equation of motion for the angular motions of a satellite are derived. For the special case of a circular orbit the equation is solved and motion of the satellite subsequent to injection into orbit is examined. The more general case of the angular motions of a satellite in an elliptical orbit is then subjected to a brief investigation.

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LIST OF SYMBOLS

a	-	semimajor axis of orbit
e	-	eccentricity of orbit
h	-	angular momentum per unit mass
Ix	-	moment of inertia about the x axis
Iy	-	moment of inertia about the y axis
Iz	-	moment of inertia about the z axis
m	-	mass
n	-	mean angular motion
P	-	orbital period
p	-	orbital parameter
r	-	distance from geocenter to center of mass of satellite
T	-	kinetic energy
	-	period of the satellite libration
t	-	time
U	-	potential energy
v	-	true anomaly
μ	-	gravitational constant
Ω	-	angular velocity of non librating satellite
θ	-	angular deviation of satellite from position of stable equilibrium

CHAPTER I

INTRODUCTION

For over 150 years man has known that the Earth's natural satellite, the Moon, always maintains essentially the same face towards the Earth. Strictly speaking, the Moon experiences oscillations about its center of mass relative to the line of sight from the Earth which cause roughly 18% of its surface to be alternately visible and not visible from the Earth. These oscillations are termed lunar librations.

As the Earth's natural satellite librates, so can the artificial satellite of any body in space librate. The gravitational force on a mass particle in a satellite varies with its distance from the body about which it is revolving. As a result, the center of gravity of the satellite generally does not coincide with its center of mass. (1)* If, in addition, the line of action of the gravitational force is non-coincident with the line between the

*Numbers in parentheses refer to References at end of paper.

center of gravity and center of mass of the satellite, a torque is established about the vehicle's mass center. The angular motions resulting from this gravitational torque, and superimposed on the orbital motion, are the satellite librations.

The motion of a satellite about its mass center is a major concern in many satellite missions, principally those which require directional stabilization of the vehicle. Clearly, the capability to maintain control of a revolving vehicle's orientation greatly enhances the utility of artificial satellites. The first problem in the study of attitude control is the determination of the satellite's response to torques from various sources. Although much is still to be learned about the magnitudes and effects of many sources, such as meteorites or electrical and magnetic fields, it is estimated that, beyond the atmosphere of the primary, the effects of other sources are small compared to that of gravity.[2] This paper is intended as a brief examination of the librations, in the plane of the orbit, resulting from a gravitational force on a satellite of arbitrary shape in both circular and elliptical orbits.

In the development of the equations of motion the following assumptions are made:

1. The earth is a perfect homogeneous sphere. This is reasonable so long as the length characterizing oblateness

of the body is small compared to the distance from the geocenter to the satellite.

2. The mass of the primary is many times greater than that of the vehicle.

3. The orbit is sufficiently far from the body so that atmospheric drag is negligible.

4. The gravitational attractions of all other celestial bodies are negligible.

5. The orbital motion of the mass center is independent of the librational motions about the mass center. Strictly speaking, this is not the case, but for real motions it can be assumed true with a high degree of accuracy.[3]

6. The effects of other perturbing factors are ignored.

Although the analysis conducted in subsequent chapters is perfectly general, terms used to describe position and characteristics of the orbit will be those customarily employed when the Earth is the primary (e.g., geocenter, perigee). This in no way restricts the analysis to geocentric orbits.

CHAPTER II

GRAVITATION POTENTIAL

The two classical methods utilized in mechanics for developing the equations of motion of a dynamical system of particles are the force method of Newton and the method of Lagrange. This paper will employ the latter of the two, the first step of which is to determine the Lagrangian, an expression which is simply the difference between the kinetic and potential energies of the system. First, let us determine the potential energy of a satellite, restricting our investigation to motions in the orbital plane.

Consider a satellite of arbitrary shape (Figure 1) at a distance r from the geocenter. Fixed in the vehicle, with its origin at the center of mass of the vehicle, is a coordinate set of principal axes of inertia, x, y, and z. Another set of coordinate axes, x_o, y_o, and z_o, is established such that its origin is also at the center of mass of the satellite and its x axis is always directed toward the geocenter. In addition, the x-o-z and the x_o- o-z_o planes are both in the plane of the orbit. Hence, it is seen that θ represents the angular deviation of the x axis from the x_o axis in the plane of the orbit.

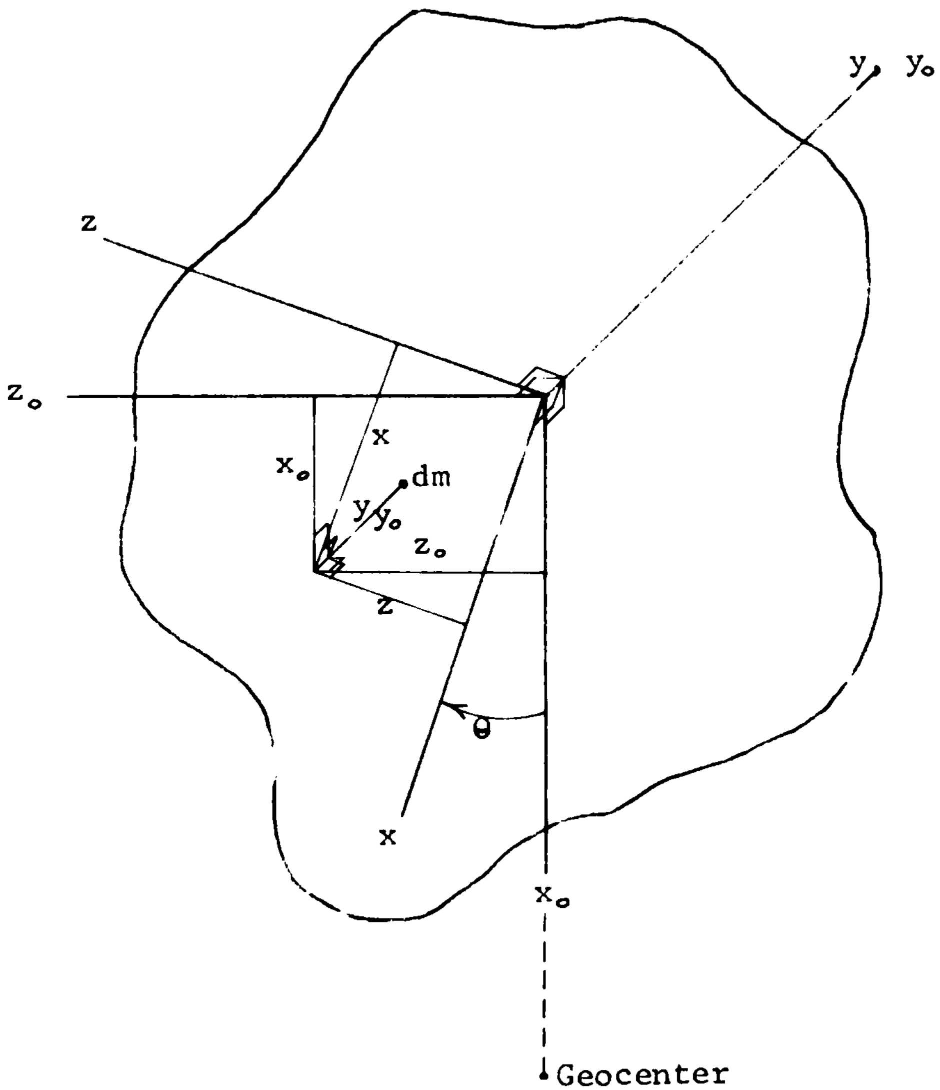

FIGURE 1

Satellite of Arbitrary Shape

The potential energy dU of a small element of mass, dm, of the vehicle is given by

$$dU = -\frac{\mu dm}{\sqrt{(r-x_o)^2+y_o^2+z_o^2}}$$

$$= -\frac{\mu dm}{r} \frac{1}{\sqrt{1-2\frac{x_o}{r}+\left(\frac{x_o}{r}\right)^2+\left(\frac{y_o}{r}\right)^2+\left(\frac{z_o}{r}\right)^2}}$$

where $\mu = Gm_1$ in which G = gravitational constant

and m_1 = mass of the primary

Expanding the radical and neglecting terms in $\frac{x_o}{r}$, $\frac{y_o}{r}$, and $\frac{z_o}{r}$ of power greater than two (since r is much greater than x_o, y_o, and z_o) yields

$$dU = -\frac{\mu dm}{r}\left[1+\frac{x_o}{r}+\left(\frac{x_o}{r}\right)^2-\tfrac{1}{2}\left(\frac{y_o}{r}\right)^2-\tfrac{1}{2}\left(\frac{z_o}{r}\right)^2\right]$$

From (Figure 1) it is seen that

$$x_o = x \cos \theta - z \sin \theta$$

$$y_o = y$$

$$z_o = x \sin \theta + z \cos \theta$$

Therefore, the expression for the potential energy of dm in terms of x, y, and z becomes

$$dU = -\frac{\mu dm}{r}\left[1+\left(\frac{x}{r} \cos \theta-\frac{z}{r} \sin \theta\right)+\left(\frac{x}{r} \cos \theta-\frac{z}{r} \sin \theta\right)^2 - \tfrac{1}{2}\left(\frac{y}{r}\right)^2-\tfrac{1}{2}\left(\frac{x}{r} \sin \theta + \frac{z}{r} \cos \theta\right)^2\right]$$

Integrating this equation over the entire satellite gives the potential energy of the satellite.

$$U = -\frac{\mu}{r}\left[\int dm + \frac{\cos\theta}{r}\int xdm - \frac{\sin\theta}{r}\int zdm + \frac{\cos^2\theta}{r^2}\int x^2dm\right.$$

$$-\frac{2\cos\theta\sin\theta}{r^2}\int xzdm + \frac{\sin^2\theta}{r^2}\int z^2dm - \frac{1}{2r^2}\int y^2dm$$

$$\left.-\frac{\sin^2\theta}{2r^2}\int x^2dm - \frac{\sin\theta\cos\theta}{r^2}\int xzdm - \frac{\cos^2\theta}{2r^2}\int z^2dm\right]$$

Since x, y, and z are principal axes, the product of inertia term $\int xzdm$ is zero. In addition, the terms involving $\int xdm$ and $\int zdm$ are zero because the origin of the coordinate system is at the center of mass of the satellite. Using these facts and the following definitions of the moments of inertia of the satellite with respect to the x, y, z axes,

$$Ix = \int (y^2 + z^2)\ dm$$

$$Iy = \int (x^2 + z^2)\ dm$$

$$Iz = \int (x^2 + y^2)\ dm,$$

the preceding equation reduces to

$$U = -\frac{\mu}{r}\left[m + \frac{1}{2r^2}(Ix + Iy + Iz - 3Ix) + \frac{3\sin^2\theta}{2r^2}(Ix - Iz)\right], \qquad (2.1)$$

the expression for the potential energy of the satellite which will be employed in the subsequent analysis.

CHAPTER III

CIRCULAR ORBIT

Let us now consider the satellite, described in Chapter I, in a circular orbit (Figure 2) where Ω represents the constant angular velocity of the nonlibrating vehicle. It might also be considered that Ω is the angular velocity of the x_o, y_o, z_o, coordinate system.

The kinetic energy of the satellite is readily seen to be

$$T = \tfrac{1}{2}\, Iy\, (\dot{\theta} - \Omega)^2 + \tfrac{1}{2}\, mr^2 \Omega^2$$

From this and Equation (2.1), utilizing the method of Lagrange, the equation of motion is found to be

$$\ddot{\theta} + \frac{3\mu \sin\theta \cos\theta}{r^3} \; \frac{Iz - Ix}{Iy} = 0$$

Remembering that for a circular orbit $\frac{\mu}{r^3} = \Omega^2$ and employing the relationship $\sin 2\theta = 2 \sin\theta \cos\theta$, the equation of motion can be reduced to

$$\ddot{\theta} + \frac{3}{2}\Omega^2 \frac{Iz-Ix}{Iy} \sin 2\theta = 0$$

This equation indicates that for dynamic stability of the satellite about the axis $\theta = 0$, the coefficient must be

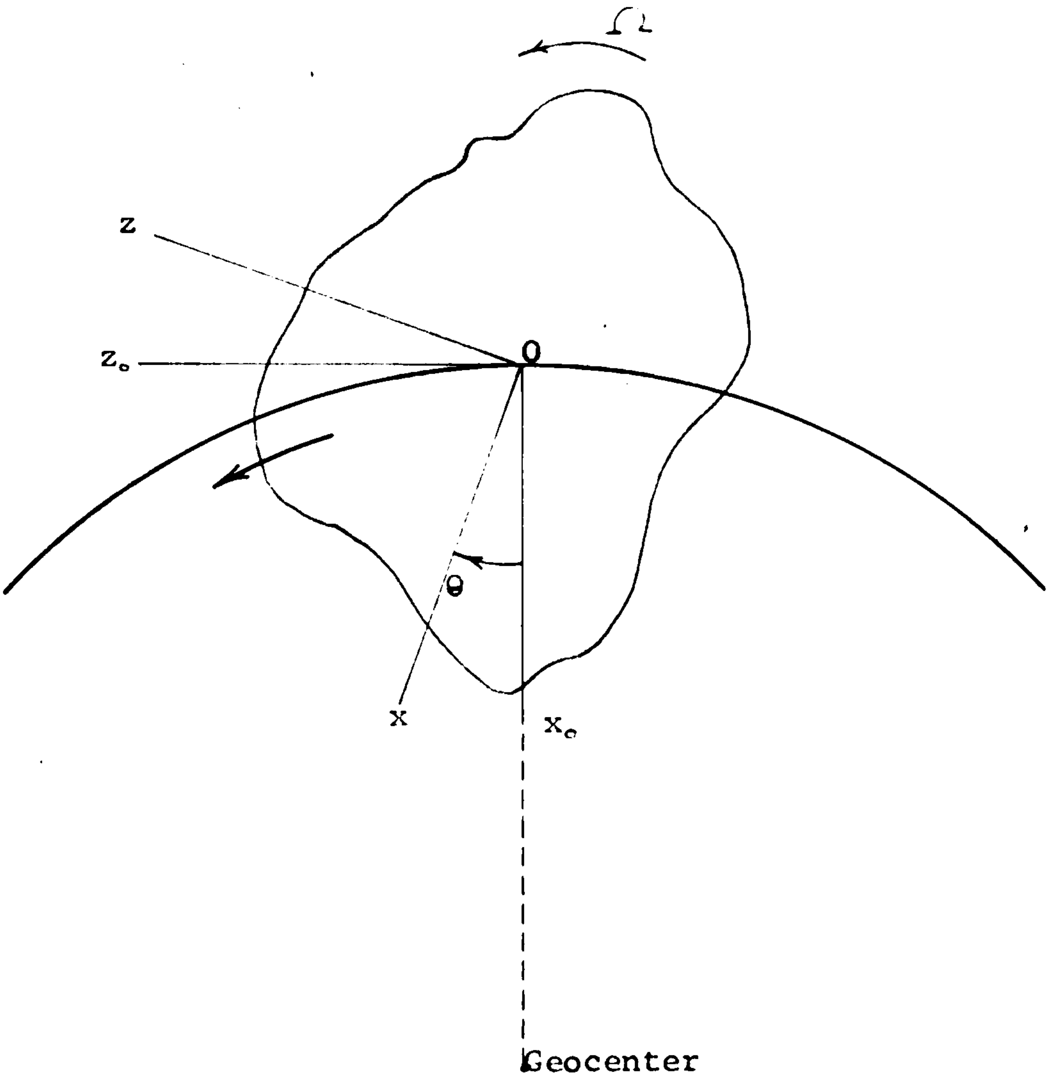

FIGURE 2

Satellite in Circular Orbit

positive. It will then represent a restoring force for small Θ. If $Iz > Ix$, the attitude of the satellite when $\Theta = 0$ represents a position of stable equilibrium; if $Iz = Ix$, it represents a position of neutral equilibrium (that is, for $\dot{\Theta}_o = 0$ and $\Theta_o \neq 0$ the satellite will orbit with no change in this angular deviation; and for $\dot{\Theta}_o \neq 0$ and $\Theta_o = 0$ the satellite will rotate indefinitely at $\dot{\Theta}_o$); and if $Iz < Ix$, a position of unstable equilibrium. Therefore, throughout this analysis, the configuration of the satellite will be restricted to the extent that $Iz > Ix$. Hence it is seen that, under this restriction, the satellite will be in a position of stable equilibrium when the x and x_o axes coincide and in a position of unstable equilibrium when the x and z_o axes coincide.

The equation can be readily integrated if it is first multiplied by $\dot{\Theta}$ and adjusted to

$$\dot{\Theta}\,\ddot{\Theta}\,dt = -\frac{3}{2}\,\Omega^2\,\frac{Iz-Ix}{Iy}\,\sin 2\,\Theta\,d\,\Theta$$

Integrating once yields

$$\tfrac{1}{2}\,\dot{\Theta}^2 - \tfrac{1}{2}\,\dot{\Theta}_o^2 = \frac{3}{4}\Omega^2\,\frac{Iz-Ix}{Iy}\,(\cos 2\Theta - \cos 2\Theta_o)$$

where the subscript, zero, represents values at time $t = 0$. This can be put in the form

$$\dot{\Theta} = \sqrt{\dot{\Theta}_o^2 + \frac{3}{2}\,\Omega^2\,\frac{Iz-Ix}{Iy}\,(\cos 2\Theta - \cos 2\Theta_o)} \qquad (3.1)$$

This equation is convenient for an investigation of maximum values of Θ under different initial conditions. Θ_{max} will occur when $\dot{\Theta}$ equals zero.

$$0 = \dot{\Theta}_o^2 + \frac{3}{2}\Omega^2 \frac{Iz-Ix}{Iy} (\cos 2\Theta_{max} - \cos 2\Theta_o)$$

or

$$\cos 2\Theta_{max} = \cos 2\Theta_o - \frac{\dot{\Theta}_o^2}{\Omega^2} (\frac{2}{3} \frac{Iy}{Iz-Ix}) \tag{3.2}$$

1. If, at time t = 0 (time of injection into orbit), $\Theta_o = 0$ and $\dot{\Theta}_o \neq 0$, then

$$\cos 2\Theta_{max} = 1 - \frac{2}{3} \frac{\dot{\Theta}_o^2}{\Omega^2} (\frac{Iy}{Iz - Ix})$$

Values of Θ_{max} are plotted against values of $\dot{\Theta}_o$ on Figure 3. It can be seen from the figure that when $\dot{\Theta}_o > \Omega\sqrt{3\frac{Iz-Ix}{Iy}}$ the value of $\cos 2\Theta_{max} < -1$. This is, of course, impossible and indicates that Θ_{max} does not exist, a condition that can occur only when the satellite is tumbling.

2. If at time t = 0, $\dot{\Theta}_o = 0$ and $\Theta_o \neq 0$, the relationship reduces to

$$\Theta_{max} = \Theta_o ,$$

a trivial conclusion which can be readily anticipated from physical reasoning.

Equation (3.1) can be written in the form

$$t = \int_{\Theta_o}^{\Theta} \frac{d\Theta}{\sqrt{\dot{\Theta}_o^2 + \frac{3}{2}\Omega^2 \frac{Iz-Ix}{Iy}(\cos 2\Theta - \cos 2\Theta_o)}} \tag{3.3}$$

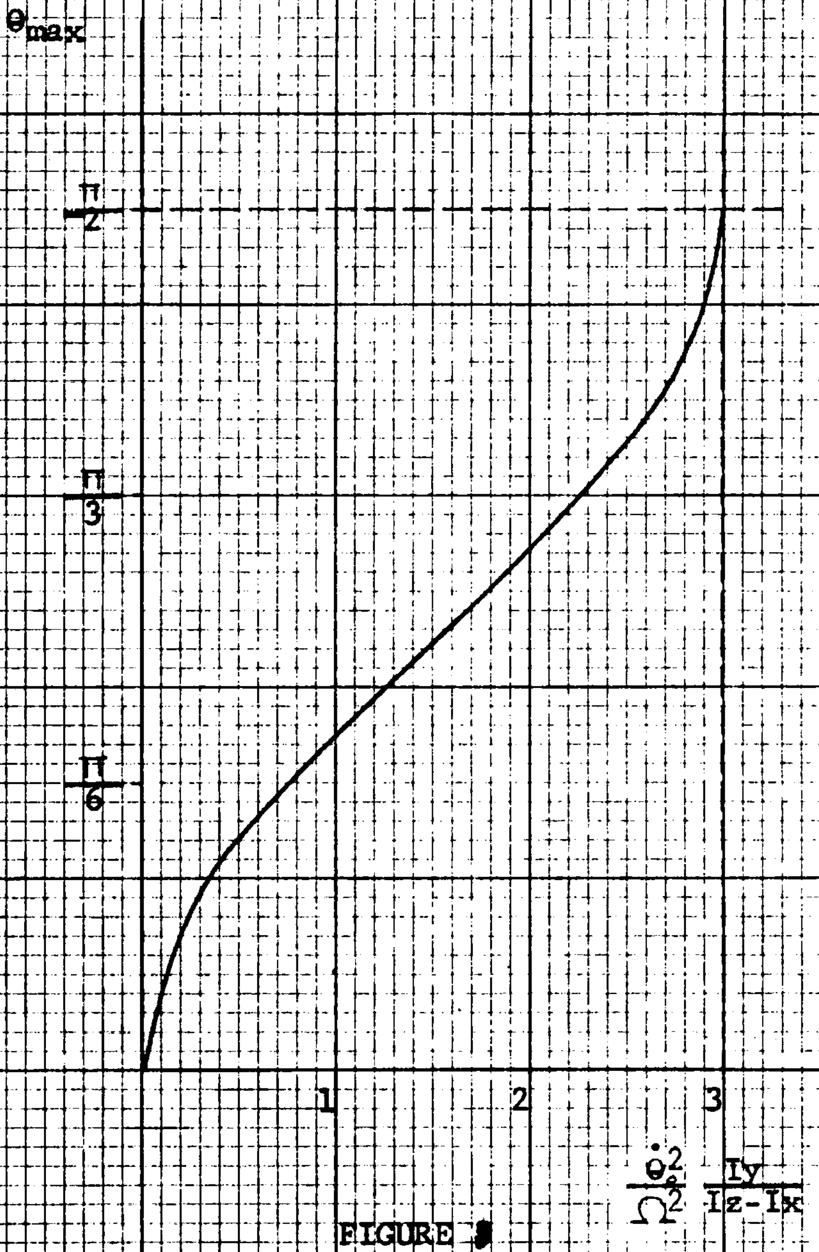

FIGURE [illegible]

Maximum Values of θ for Satellite Injected Into Circular Orbit at Initial Conditions

$\theta_0 = 0, \dot{\theta}_0 \neq 0$

Let us consider this equation with various initial conditions imposed. First, assume that at time t = 0, $\dot{\theta}_o \neq 0$ and $\theta_o = 0$. Then Equation (3.3) becomes

$$t = \int_0^\theta \frac{d\theta}{\sqrt{\dot{\theta}_o^2 + \frac{3}{2}\Omega^2 \frac{Iz-Ix}{Iy}(\cos 2\theta - 1)}}$$

which can be reduced to

$$t = \frac{1}{\dot{\theta}_o} \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{3\Omega^2}{\dot{\theta}_o^2} \frac{Iz-Ix}{Iy} \sin^2\theta}} \qquad (3.4)$$

This integral behaves differently depending upon whether

$$\frac{3\Omega^2}{\dot{\theta}_o^2} \frac{Iz-Ix}{Iy} \lesseqgtr 1$$

1. Assume

$$\frac{3\Omega^2}{\dot{\theta}_o^2} \frac{Iz-Ix}{Iy} < 1$$

The expression under the radical is always positive, and the angle increases without limit. That is, if the initial angular velocity is such that $\dot{\theta}_o > \Omega \sqrt{3 \frac{Iz-Ix}{Iy}}$, the satellite rotates in one direction indefinitely.

2. Assume

$$\frac{3\Omega^2}{\dot{\theta}_o^2} \frac{Iz-Ix}{Iy} = 1$$

As $\theta \longrightarrow \pi/2$, the value of the integral $\longrightarrow \infty$. This means that θ approaches $\pi/2$ as a limit but does not reach it in finite time.

3. Assume

$$\frac{3\Omega^2}{\dot{\theta}_o^2}\frac{Iz-Ix}{Iy} > 1$$

The expression under the radical is real only if θ is restricted between the limits $\pm\alpha$, where

$$\sin^2\alpha = \frac{\dot{\theta}_o^2}{3\Omega^2}\frac{Iy}{Iz-Ix}$$

For θ so restricted an oscillatory motion of amplitude α is obtained.

All three cases are shown qualitatively in Figure 4.

Confining ourselves to the case where oscillatory motion results from the initial $\dot{\theta}$, Equation (3.4) can be written

$$t = \frac{1}{\dot{\theta}_o}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{\sin^2\theta}{\sin^2\alpha}}}$$

This integral from 0 to α gives the time for one quarter of the period of the oscillation. It is more convenient to work with limits of 0 and $\pi/2$.

Let

$$\frac{\sin\theta}{\sin\alpha} = \sin\phi$$

Then

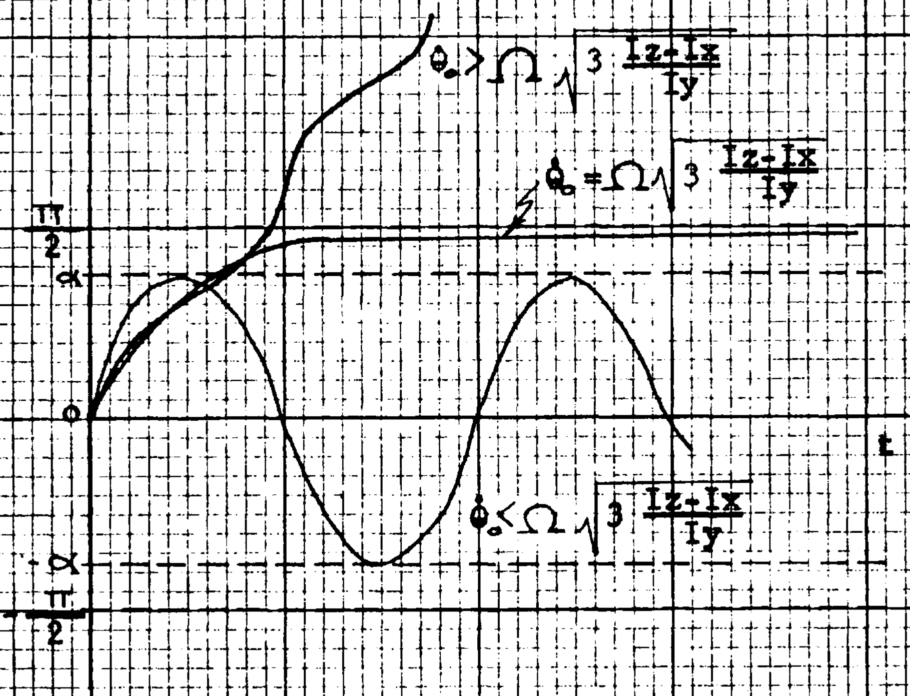

FIGURE 4

Motion of Satellite Injected Into
Circular Orbit at Initial Conditions
$\theta_o = 0, \dot{\theta}_o \neq 0$

$$d\Theta = \frac{\sin\alpha \;\; \cos\phi}{\cos\Theta}\, d\phi$$

And

$$t = \frac{1}{\dot{\Theta}_o} \int_0^{\phi} \frac{\frac{\sin\alpha \;\; \cos\phi}{\cos\Theta}\, d\phi}{\sqrt{1 - \sin^2\phi}}$$

which, by proper manipulation reduces to

$$t = \frac{1}{3\,\Omega} \sqrt{\frac{3\,Iy}{Iz-Ix}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2\alpha \;\; \sin^2\phi}} \qquad (3.5)$$

This is an elliptic integral of the first kind with modulus $\sin\alpha$ and amplitude ϕ. A plot of Θ versus $\dfrac{t}{\frac{1}{3\,\Omega}\sqrt{\frac{3Iy}{Iz-Ix}}}$ for values of α is shown in Figure 5.

At the point at which $\Theta = \alpha$ (the maximum value of Θ as prescribed by the restrictions on Θ), $\phi = \pi/2$, and t is one quarter of the period of oscillation. Equation (3.5) can then be written

$$T = \frac{4}{3\,\Omega}\sqrt{\frac{3Iy}{Iz-Ix}}\; F\,(\alpha\,,\,\phi) \quad (\phi = \pi/2)$$

where T = the period of one oscillation

and $F\,(\alpha,\,\phi) = \displaystyle\int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2\alpha\,\sin^2\phi}}$

$$(\phi = \frac{\pi}{2})$$

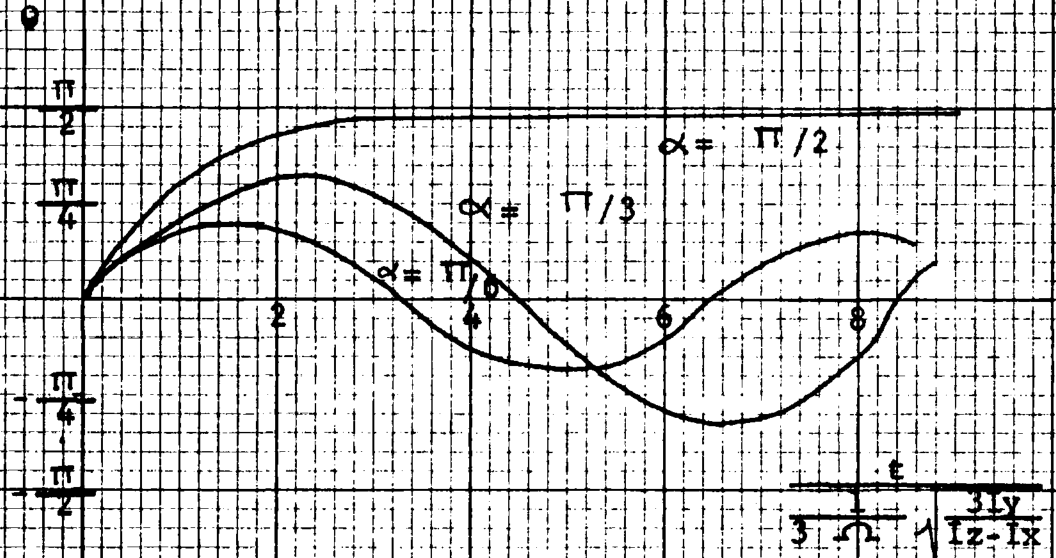

FIGURE 5

Motion, for Values of α
of Satellite Injected into Circular Orbit
at Initial Conditions $\theta_0 = 0, \dot{\theta}_0 \neq 0$

It is interesting at this point to show the relationship between the period of the oscillation of the satellite and θ_{max}. For purposes of illustration let us consider the special case of a satellite in the shape of an elongated body of revolution, (Iy = Iz, Ix negligible). This shape is chosen since it exhibits a high degree of stability; and, as can be seen from the preceding equation, it has a short period of oscillation and, therefore, a high frequency.

The period of a circular orbit, P, is related to the constant angular velocity of the satellite by the expression

$$P = 2\pi \frac{1}{\Omega}$$

The period of oscillation of the illustrative satellite is given by

$$T = \frac{4\sqrt{3}}{3\Omega} F(\alpha, \phi) \quad (\phi = \pi/2)$$

From this and the previous equation it is seen that

$$\frac{T}{P} = \frac{2\sqrt{3}}{3\pi} F(\alpha, \phi) \quad (\phi = \pi/2)$$

$F(\alpha, \phi)$ is tabulated in standard mathematical tables for values of α and ϕ. In the present example $\alpha = \theta_{max}$ and $\phi = \pi/2$. Therefore, we can show graphically the relationship between θ_{max} and T/P, actually between θ_{max} and T since the period of any particular orbit will be known. This is done in Figure 6. If is of note that, as $\theta_{max} \to 0$ the ratio of the two periods is greater than ½ which means that the satellite is unable to complete even two cycles in one orbital period. It is also of interest that θ_{max} must be in the neighborhood of 75° before the period of the oscillation and that of the orbit will be equal.

Now we shall briefly examine Equation (3.3) with other initial conditions imposed. If, at time t = 0, $\dot{\theta}_o = 0$, Equation (3.3) becomes

$$t = \int_{\theta_o}^{\theta} \frac{d\theta}{\sqrt{\frac{3}{2}\,\Omega^2\,\frac{Iz-Ix}{Iy}(\cos 2\theta - \cos 2\theta_o)}}$$

The solution to this integral is unnecessary for our purpose since the behavior of the satellite under the condition $\dot{\theta}_o = 0$ can be determined by physical reasoning. If $\theta_o = 0$, the satellite has been injected into orbit in a position of stable equilibrium and will remain there. If $0 < \theta_o < \pi/2$, the satellite has been injected at its maximum angular deviation from the axis and will oscillate about the stable equilibrium position. If $\theta_o = \pi/2$, it has been injected in a position of unstable equilibrium and will not oscillate. Finally, if $\pi/2 < \theta_o < \pi$, the satellite will again oscillate about the position of stable equilibrium. This is shown in Figure 7. It is interesting to note that the curves for the cases $\dot{\theta}_o = 0$, $0 < \theta_o < \pi/2$ and $\dot{\theta}_o < \Omega\sqrt{3\,\frac{Iz-Ix}{Iy}}$, $\theta_o = 0$ (see Figure 4) will have exactly the same shapes, except for a phase difference of one quarter period, if θ_o in the

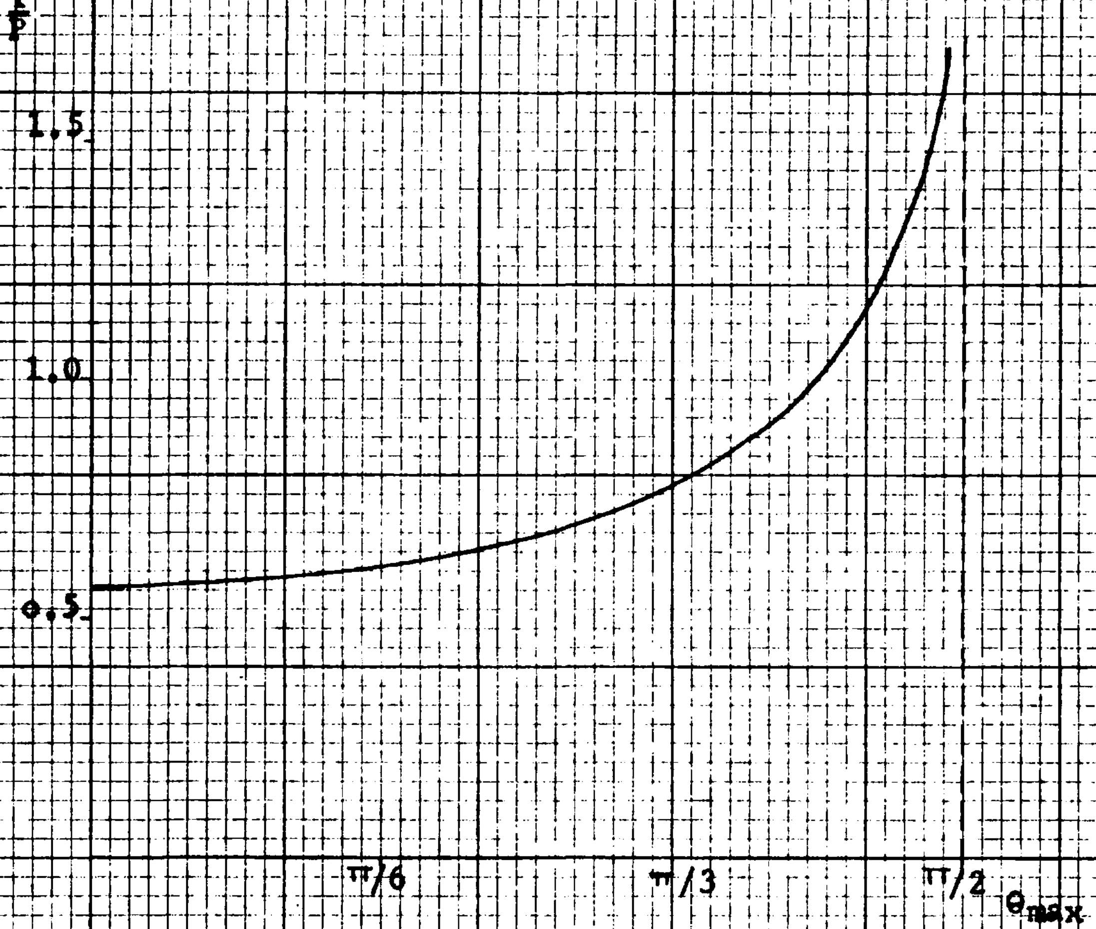

FIGURE 6

Period of Oscillation versus θ_{max} for an Elongated Body of Revolution

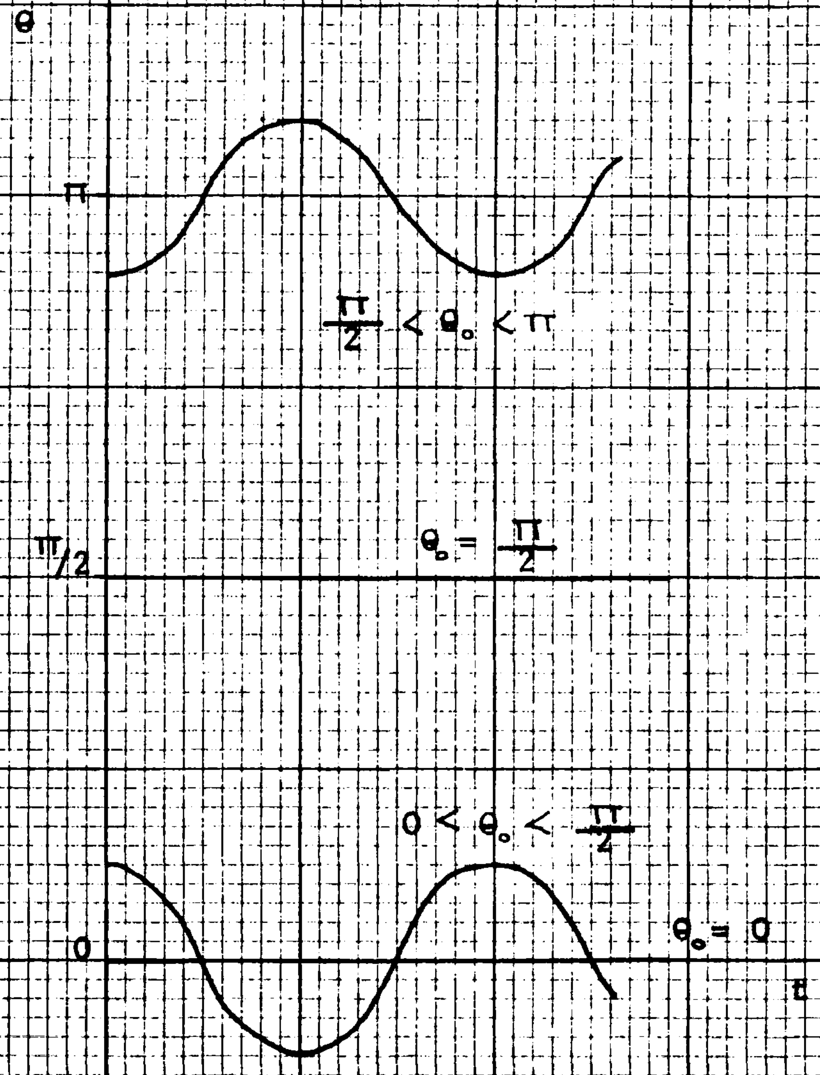

FIGURE 7

Motion of Satellite Injected into Circular Orbit at Initial Conditions $\theta_0 = \theta_0, \dot{\theta}_0 = 0$

first instance is made to equal α (i.e., θ_{max}) in the second. This means, of course, that a particular effect can be attained by either injecting the satellite at $\theta_o = 0$, $\dot{\theta}_o \neq 0$ or at $\theta_o \neq 0, \dot{\theta}_o = 0$. It is evident that this effect applies to periodicity as well as to amplitude, indicating that Figure 6 is applicable to an elongated body of revolution regardless of the initial conditions imposed.

It is more probable, however, that at the moment of injection the satellite will have both an angular deviation and an angular velocity. In this eventuality Equation (3.3) becomes

$$t = \int_{\theta_o}^{\theta} \frac{d\theta}{\sqrt{\dot{\theta}_o^2 + \frac{3}{2}\Omega^2 \frac{Iz-Ix}{Iy}(\cos 2\theta - \cos 2\theta_o)}}$$

or

$$t = \frac{1}{\dot{\theta}_o} \int_{\theta_o}^{\theta} \frac{d\theta}{\sqrt{1 - \frac{3\Omega^2}{\dot{\theta}_o^2} \frac{Iz-Ix}{Iy} \sin(\theta + \theta_o)\sin(\theta - \theta_o)}}$$

Again, a solution is not essential for this analysis. Representative curves depend upon the magnitude of θ_{max}, which can be determined from Equation (3.2). For given θ_o, particular values of $\dot{\theta}_o$ will cause θ_{max} to be the amplitude of an oscillatory curve, to approach $\pi/2$ as a limit, or to increase without limit. It is evident, however, that whichever the case, the slope of the curve at $t = 0$ will be equal to $\dot{\theta}_o$.

CHAPTER IV

ELLIPTICAL ORBIT

The expressions for the kinetic and potential energies of a satellite in an elliptical orbit (Figure 8) are

$$T = \tfrac{1}{2}\, Iy\, (\dot{\theta} - \Omega)^2 + \tfrac{1}{2}\, m\, (\dot{r}^2 + r^2 \Omega^2)$$

$$U = -\frac{\mu}{r}\left[m + \frac{1}{2r^2}(Ix + Iy + Iz - 3Ix) + \frac{3\sin^2\theta}{2r^2}(Ix - Iz) \right]$$

Three equations of motion will result since there are three independent variables, θ, Ω and r. Of the three only the equation in θ is here important. Employing the method of Lagrange, this equation of motion is seen to be

$$\ddot{\theta} + \frac{3\mu}{r^3}\left(\frac{Iz - Ix}{Iy}\right) \sin\theta \cos\theta = \dot{\Omega} \qquad (4.1)$$

Since θ and Ω are more readily related to position than to time, it is convenient to transform the equation into one more useful. This involves a change in the independent variable to the true anomaly, v.

$$\dot{\theta} = \frac{d\theta}{dv}\frac{dv}{dt} = \frac{h}{r^2}\frac{d\theta}{dv},$$

where h = angular momentum per unit mass.

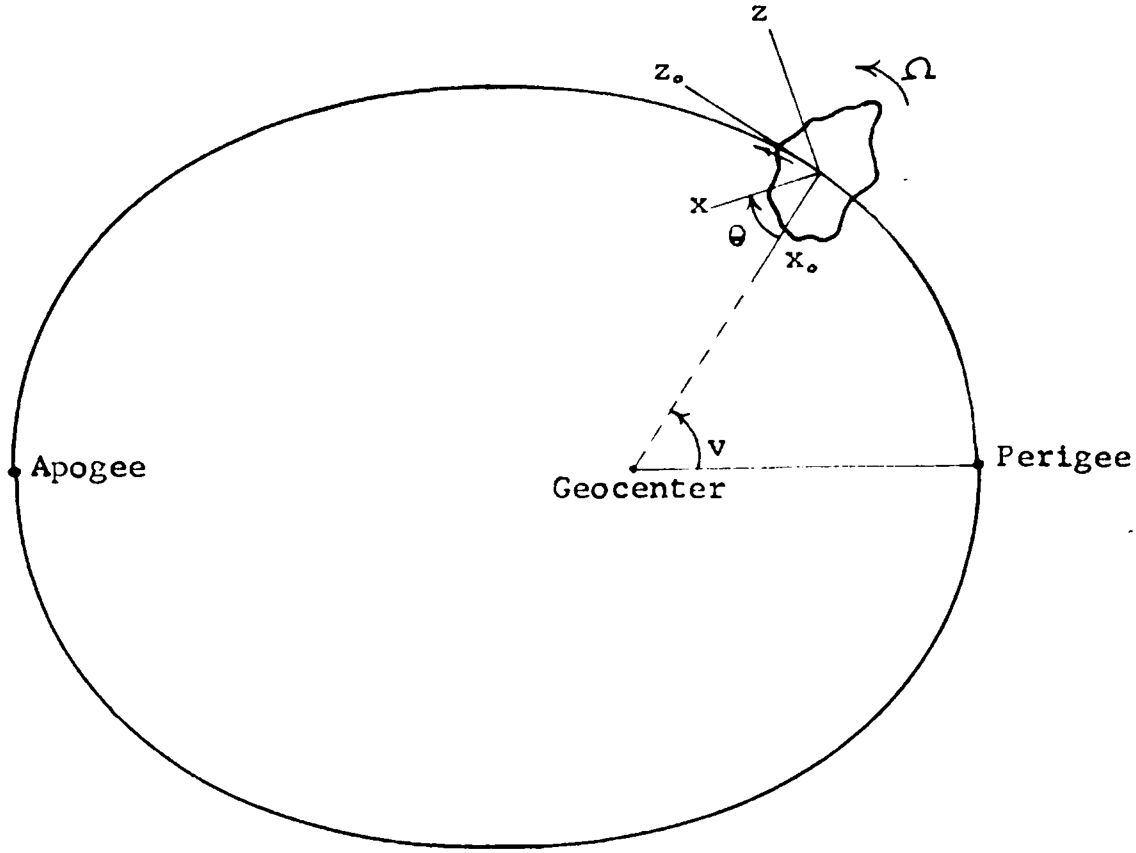

FIGURE 8

Satellite in Elliptical Orbit

$$\ddot{\theta} = \frac{h^2}{r^4}\frac{d^2\theta}{dv^2} - 2\,\frac{h\dot{r}}{r^3}\frac{d\theta}{dv}$$

$$\Omega = \dot{v} = \frac{h}{r^2}$$

$$\dot{\Omega} = \ddot{v} = -\frac{2h\dot{r}}{r^3}$$

Substituting these expressions into Equation (4.1) yields

$$\frac{d^2\theta}{dv^2} - 2\,\frac{r\dot{r}}{h}\frac{d\theta}{dv} + \frac{3\mu r}{h^2}\left(\frac{Iz-Ix}{Iy}\right)\sin\theta\cos\theta = -\frac{2r\dot{r}}{h}$$

which can be further simplified to

$$\frac{d^2\theta}{dv^2} - 2\,\frac{e\sin v}{1+e\cos v}\frac{d\theta}{dv} + \frac{3}{1+e\cos v}\left(\frac{Iz-Ix}{Iy}\right)\sin\theta\cos\theta$$

$$= -2\,\frac{e\sin v}{1+e\cos v}\,, \qquad (4.2)$$

where e = the eccentricity of the orbit,

by noting that

$$\dot{r} = \frac{-p(-e\sin v)\dot{v}}{(1+e\cos v)^2} = \frac{p\,e\sin v\,\frac{h}{r^2}}{(1+e\cos v)^2} = \frac{h\,e\sin v}{r(1+e\cos v)}$$

where p (the orbital parameter) = r(1+e cos v).

Equation (4.2) is a nonlinear equation and therefore impossible to solve directly. The coefficients are now, however, functions of the independent variable.

We can attempt to put Equation (4.2) in a form amenable to solution by requiring e and/or θ to be very small.

1. Assume e << 1.

Let
$$\Theta = \Theta_c + \Delta\Theta$$

where Θ_c satisfies Equation (4.2) for e = 0,

$$\frac{d^2\Theta_c}{dv^2} + 3\left(\frac{Iz-Ix}{Iy}\right)\sin\Theta_c\cos\Theta_c = 0$$

and $\Delta\Theta$ is small.

Then Equation (4.2) can be written

$$\frac{d^2\Theta_c}{dv^2} + \frac{d^2\Delta\Theta}{dv^2} - 2\frac{e\sin v}{1+e\cos v}\left(\frac{d\Theta_c}{dv} + \frac{d\Delta\Theta}{dv}\right)$$

$$+ \frac{3}{2(1+e\cos v)}\left(\frac{Iz-Ix}{Iy}\right)\sin 2(\Theta_c + \Delta\Theta) = -2\frac{e\sin v}{1+e\cos v}$$

which, after manipulation, becomes

$$\frac{d^2\Delta\Theta}{dv^2} + 3\frac{Iz-Ix}{Iy}\cos 2\Theta_c\Delta\Theta = 2e\sin v\left(\frac{d\Theta_c}{dv} - 1\right)$$

This is still not in a form that lends itself to solution.

2. Assume Θ very small. Then Equation (4.2) is readily seen to be

$$\frac{d^2\Theta}{dv^2} - 2\frac{e\sin v}{1+e\cos v}\frac{d\Theta}{dv} + \frac{3}{1+e\cos v}\left(\frac{Iz-Ix}{Iy}\right)\Theta = -2\frac{e\sin v}{1+e\cos v}, \quad (4.3)$$

a linear, nonhomogeneous equation with non constant coefficients and again in a form not readily solved by normal methods.

3. Finally, assume that both e and Θ are small. Although we have attempted to make this analysis perfectly general, the assumption that e and Θ are small is actually

quite reasonable since many natural satellites are in nearly circular orbits, and it is reasonable to assume that in artificial satellites an attempt is made to restrict θ to very small angles. Small e has no immediate effect on the form of the equation. Equation (4.3) which resulted from assuming θ small can be written

$$(1+e \cos v)\theta'' - 2 e \sin v \, \theta' + 3 \left(\frac{Iz-Ix}{Iy}\right)\theta = - 2 e \sin v$$

where

$$\theta'' = \frac{d^2\theta}{dv^2}$$

Now let

$$\theta = \theta_c + \Delta\theta$$

where θ_c is the angular motion for a circular orbit,

$$\theta_c'' + 3 \left(\frac{Iz-Ix}{Iy}\right) \theta_c = 0 .$$

Then

$$\theta_c = c_1 \cos k v + c_2 \sin k v$$

where

$$k = \sqrt{3 \frac{Iz-Ix}{Iy}} .$$

Applying the initial conditions that at perigee (v = 0), $\theta_c = \theta_o$ and $\theta_c' = \theta_o'$. Under these conditions

$$\theta_c = \theta_o \cos k v + \frac{\theta_o'}{k} \sin k v$$

Substituting these expressions, Equation (4.3) takes the form

$$(1+e\cos v)(\Theta_c''+\Delta\Theta'')-2e\sin(\Theta_c'+\Delta\Theta')+k^2(\Theta_c+\Delta\Theta)=-2e\sin v$$

which can be further simplified to

$$\Delta\Theta''+k^2\Delta\Theta = e\Big[-2\sin v\,(1-k\Theta_o\sin k\,v+\Theta_o'\cos k\,v)$$
$$+\cos v\,(k^2\Theta_o\cos k\,v+k\Theta_o'\sin k\,v)\Big]$$

and then to

$$\Delta\Theta''+k^2\Delta\Theta = -2e\sin v$$

since the products $e\Theta_o$ and $e\Theta_o'$ will be negligible. With the conditions imposed that at $v = 0$, $\Delta\Theta = \Delta\Theta' = 0$, the solution for $\Delta\Theta$ is

$$\Delta\Theta = \frac{-2e}{k^2-1}\left(\sin v - \frac{1}{k}\sin k\,v\right)$$

and the solution of Equation (4.3) is therefore seen to be

$$\Theta = \Theta_c + \Delta\Theta = \frac{-2e}{k^2-1}\sin v + \frac{\Theta_o' - \frac{2e}{k^2-1}}{k}\sin k\,v + \Theta_o\cos k\,v \qquad (4.4)$$

In utilizing this solution let us again employ the special case of a satellite in the shape of an elongated body of revolution. Equation (4.4) becomes

$$\Theta = -e\sin v + \frac{\Theta_o' - e}{\sqrt{3}}\sin\sqrt{3}\,v + \Theta_o\cos\sqrt{3}\,v \qquad (4.5)$$

Θ' is a rather nebulous quantity; however, an equivalent, but more meaningful, expression can be determined as shown below.

$$\theta' = \frac{\dot{\theta}_o}{\dot{v}_o}$$

$$h = \sqrt{\mu p} = a^2 (1 - e)^2 \dot{v}_o$$

where a = the semi-major axis of the orbit.

$$\dot{v}_o = \frac{\sqrt{\mu a(1-e^2)}}{a^2(1-e)^2} = n \frac{\sqrt{\frac{1+e}{1-e}}}{1-e}$$

in which n = mean angular motion

$$\theta_o' = \frac{\dot{\theta}_o}{n} (1-e) \sqrt{\frac{1-e}{1+e}} = \frac{\dot{\theta}_o}{n} (1 - 2e)$$

Using this relationship Equation (4.5) can be written

$$\theta = - e \sin v + \frac{1-2e}{\sqrt{3}} \left[\frac{\dot{\theta}_o}{n} - \frac{e}{1-2e} \right] \sin \sqrt{3}\, v + \theta_o \cos \sqrt{3}\, v$$

It is instructive now to consider the maximum values of θ when $\dot{\theta}_o$ and θ_o are alternately set equal to zero.

1. When $\dot{\theta}_o = 0$ and $\theta_o \neq 0$ Equation (4.6) takes the form

$$\theta = - e \sin v - \frac{e}{\sqrt{3}} \sin \sqrt{3}\, v + \theta_o \cos \sqrt{3}\, v$$

and θ_{max} is given by the expression

$$\theta_{max} = e + \sqrt{\theta_o^2 + \frac{e^2}{3}}$$

Figure 9 shows that, for a given Θ_o, Θ_{max} increases with increasing eccentricity.

2. When $\Theta_o = 0$ and $\dot{\Theta}_o \neq 0$ Equation (4.6) becomes

$$\Theta = -e \sin v + \frac{1-2e}{\sqrt{3}} \left(\frac{\dot{\Theta}_o}{n} - \frac{e}{1-2e}\right) \sin \sqrt{3}\, v$$

and Θ_{max} is given by

$$\Theta_{max} = e + \frac{1-2e}{\sqrt{3}} \left| \frac{\dot{\Theta}}{n} - \frac{e}{1-2e} \right|$$

Figure 10 shows graphically the relationship between Θ_{max} and $\frac{\dot{\Theta}_o}{n}$ for values of e. It is interesting to note that, for a constant e, Θ_{max} initially decreases with increasing $\dot{\Theta}_o$ until reaching a minimum. Beyond that point Θ_{max} increases with increasing $\dot{\Theta}_o$. This is interpreted to indicate that, in order to achieve minimum angular deviation from a position of stable equilibrium, it is desirable to impart some angular motion to the satellite at the point of injection into the orbit.

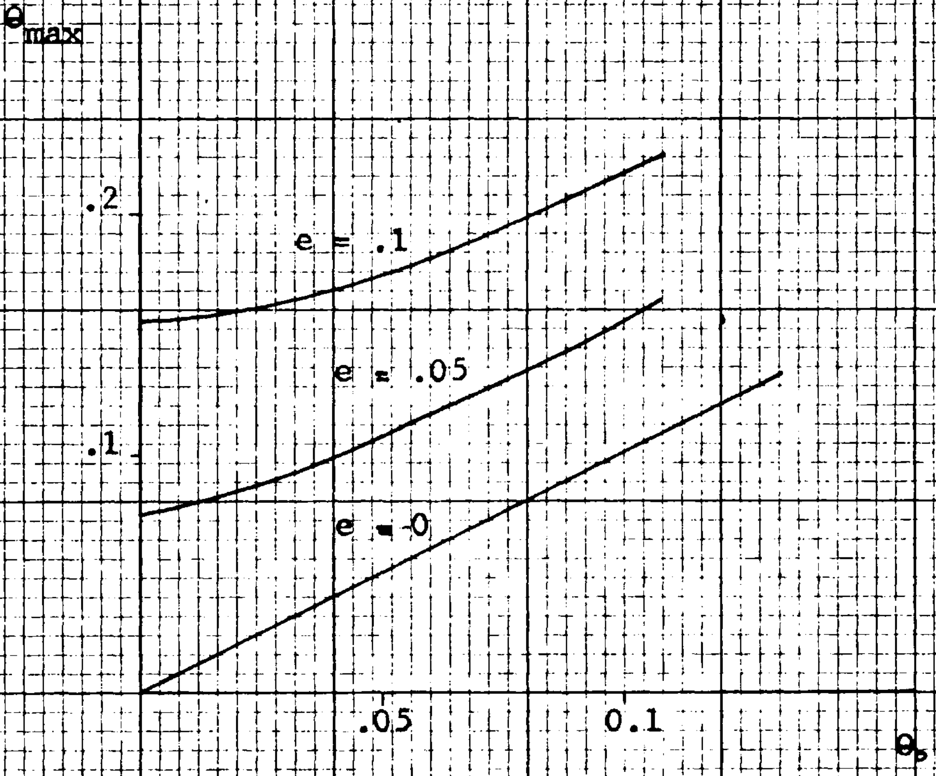

FIGURE 9

Maximum values of θ, for values of e, for an Elongated Body of Revolution Injected into Elliptical Orbit at Initial Conditions $\dot{\theta}_o = 0, \theta_o \neq 0$ (θ and e small)

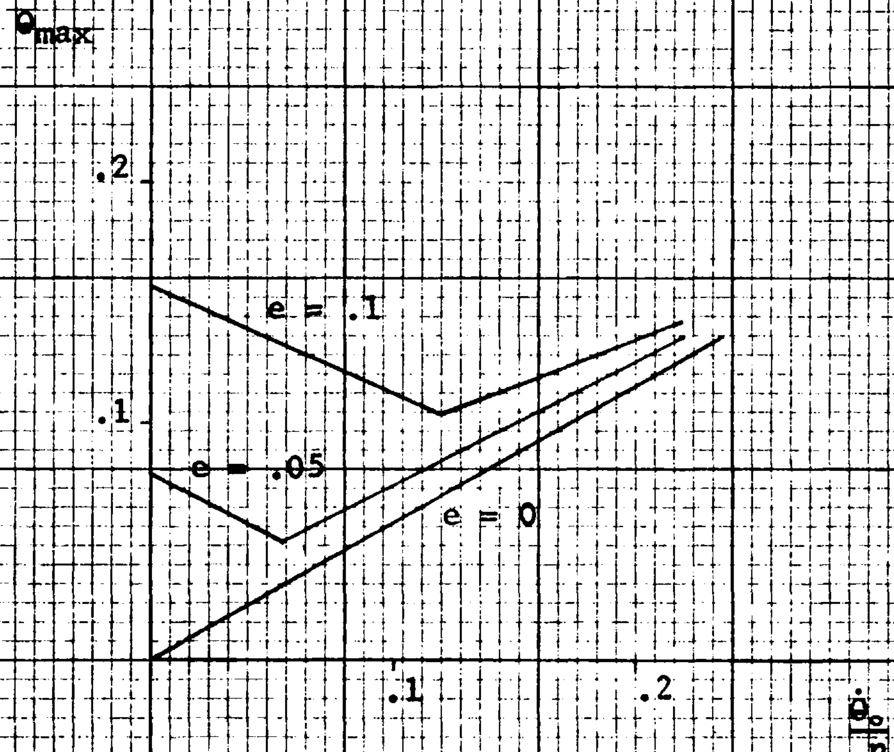

FIGURE 10

Maximum values of θ, for values of e,
for an Elongated Body of Revolution
Injected into Elliptical Orbit at
Initial Conditions $\theta_o = 0, \dot{\theta}_o \neq 0$
(θ and e small)

CHAPTER V

SUMMARY AND AREAS FOR FURTHER STUDY

The preceding analysis indicates that a satellite tends to stabilize itself in a position of stable equilibrium, the motion being quite similar to that of a pendulum. The dominant factor in the angular motion of an revolving vehicle is the angular velocity imparted to it at injection into orbit. If, in a circular orbit, the initial deviation from a position of stable equilibrium is zero, the angular velocity causes the satellite to oscillate (librate) about that equilibrium position, to approach a position of unstable equilibrium, or to tumble, depending upon the magnitude of the angular velocity. The maximum angular deviation varies accordingly and, of course, becomes infinite if the satellite tumbles. If, however, the initial angular velocity is zero, the satellite oscillates about the position of stable equilibrium unless it is injected at either stable or unstable equilibrium, in which case no angular motion occurs. The maximum angular deviation is equal to the initial angular deviation. The period of the oscillation is seen to be relatively short for the illustrative case of a satellite in the shape of an elongated body of revolution. It is, in fact, equal to

$1/\sqrt{3}$ times the orbital period when the maximum angular deviation is held to a minimum value. It is of longer duration for vehicles of lesser elongation, relative to girth.

The equation of motion for a satellite in elliptical orbit can be solved in a closed form only if the conditions are imposed that eccentricity and angular deviation are both small. If, in addition, the angular velocity is initially zero, and the satellite is injected into orbit at perigee, it is seen that the maximum angular deviation increases with increasing eccentricity for a given initial angular displacement. When the displacement at perigee injection is initially zero, for a given eccentricity the maximum angular deviation will decrease with increasing angular velocity until a minimum is reached and then will increase. This indicates that to achieve a minimum angular deviation it is necessary for the initial angular deviation to be zero and for the initial angular velocity to have a particular optimum value.

The treatment in this study could very profitably be expanded for the case of the elliptical orbit. This has to a large degree been neglected in the literature. Solutions for the equation of motion, allowing large angular displacement or large eccentricity or both, by computer or numerical methods would be a fruitful, though challenging, endeavor.

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