CALCULUS OF VARIATIONS SOLUTIONS
TO PROBLEMS OF VERTICAL FLIGHT

by
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STATEMENT BY AUTHOR

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[Signature]    MAY 7, 1963

Edwin K. Parks     Date
Professor of Mechanical Engineering
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CHAPTER I

INTRODUCTION

The modern development of the calculus of variations began with the formulation of the brachistochrone problem by John Bernoulli in 1696. Bernoulli proposed a model consisting of a particle moving under the influence of a constant gravitational force and asked the question, "What path must the particle follow in order to arrive at a particular point in a minimum time?" It soon became evident that this problem was not solvable by the methods of elementary calculus. Investigation of the brachistochrone excited a great interest and eventually led to the development of the calculus of variations by Euler, Lagrange, Hamilton, Jacobi, and others.

Whereas the theory of maxima and minima of elementary differential calculus may be used to find extremal values of a function of a finite number of real variables, the calculus of variations may be used to determine extremal values of a so-called functional which has functions as its independent variables.

In recent years the calculus of variations has become an important tool for optimizing trajectories of flight vehicles. Notable contributions have been made
with the aid of this tool by numerous individuals. Minimum
time problems have received much attention with outstanding
work being done by Miele,\textsuperscript{10} Cicala,\textsuperscript{2} Schindler,\textsuperscript{11} and
Theodorsen\textsuperscript{12} to name but a few. Leitman's book, "Optimi-
zation Techniques with Applications to Aerospace Systems,"\textsuperscript{7}
provides a collection of works by several well-known authors.
The calculus of variations is the subject of innumerable
texts including "An Introduction to the Calculus of Vari­
tions" by C. Fox\textsuperscript{5} and "Calculus of Variations" by A. R.
Forsyth.\textsuperscript{4}

The purpose of this paper is to investigate some
optimum trajectories of a flight vehicle which is confined
to motion in a vertical plane. This investigation is dir-
eted toward finding the relative effects of weight, thrust,
and drag on the optimum trajectory as well as the lift
program required to fly the trajectory. In addition to the
derivation of the differential equations which govern the
particular flight path, techniques used to solve these
equations with the aid of an analog computer will also be
given.

Specifically, two problems are dealt with here.
First, the minimum time path between two fixed points is
discussed. Then, a solution is found to the problem of
maximizing range to a final predetermined altitude and
velocity. In each case the integral to be extremized is
written in parametric form with arc length as the independent
variable. Next, the equations of constraint to which all solutions must comply are decided upon and introduced by using Lagrange multipliers. The well-known Euler-Lagrange equations can then be written and simplified by eliminating the multipliers and evaluating constants of integration with the aid of the pertinent transversality, or end conditions. The resulting differential equations are, in general, highly non-linear and may not be solved analytically with ease. However, trajectories obtained with the aid of an analog computer are presented and discussed for some characteristic examples.

As in ordinary differential calculus where the second derivative determines whether a maximum, a minimum, or an inflection point has been obtained, the calculus of variations uses the second variation to determine whether a maximum or a minimum function has been achieved. No attempt has been made in this paper to study second variations. In some cases interpretation of the results indicates whether a maximum or minimum value has been obtained while in other cases the solutions are compared with previously established optimum solutions.

In the two problems discussed here no restriction is placed on the initial flight path angle. Each choice of an initial angle corresponds to a particular trajectory and a particular end point on that curve. When the problem
requires a computer solution, the optimum initial angle for a particular path to a designated point must be found by a trial-and-error process. In some cases an undetermined constant of integration remains in the equation and must also be found in an iterative fashion. An attempt has been made to outline some definite iterative schemes for the two problems dealt with here. These schemes could be applied directly or in an altered form to other similar problems.
CHAPTER II

FORMULATION OF THE PROBLEM

In each of the problems dealt with here an integral of the form,

\[ I = \int F(x, y, v, x', y', v')\, ds \quad 2.0 \]

must be optimized subject to the constraining equations of the form

\[ \psi(x, y, v, x', y', v') = 0 \quad 2.0a \]
\[ \varphi(x, y, v, x', y', v') = 0 \quad 2.0b \]

The pertinent Euler-Lagrange equations which describe the optimum paths are, as derived by Fox,\(^5\)

\[ \frac{\partial G}{\partial x} - \frac{d}{ds} (\frac{\partial G}{\partial x'}) = 0 \quad 2.1a \]
\[ \frac{\partial G}{\partial y} - \frac{d}{ds} (\frac{\partial G}{\partial y'}) = 0 \quad 2.1b \]
\[ \frac{\partial G}{\partial v} - \frac{d}{ds} (\frac{\partial G}{\partial v'}) = 0 \quad 2.1c \]

Using Lagrange multipliers, \(\lambda_1\) and \(\lambda_2\), to introduce the constraints, the function \(G\) is given by the sum,

\[ G = F + \lambda_1 \psi + \lambda_2 \varphi \]

In addition to equations 2.1 another, alternate, optimizing equation may be used. As reduced for the
particular problems of minimum time and maximum range, the alternate equation is,

\[ G - \frac{\partial G}{\partial x'} x' - \frac{\partial G}{\partial y'} y' - \frac{\partial G}{\partial v'} v' = B \quad 2.2 \]

This equation, however, is not independent of equations 2.1 and may not be treated as such. T. Vincent\textsuperscript{13} shows that equation 2.2 is equivalent to equations 2.1. The value of the alternate equation is that it is sometimes simpler than one of the Euler-Lagrange equations or may be used in conjunction with one of the Euler-Lagrange equations to eliminate the derivative of one of the Lagrange multipliers but not the multiplier itself, of course.

Equations 2.1, 2.2, 2.0a, and 2.0b constitute five independent equations, and a total of six equations one of which is not independent. These equations contain five dependent variables \( x, y, v, \lambda_1 \) and \( \lambda_2 \). The solution of this set of equations requires the use of the natural boundary conditions or transversality conditions,

\[ [(G - \frac{\partial G}{\partial x'} x' - \frac{\partial G}{\partial y'} y' - \frac{\partial G}{\partial v'} v')ds + \frac{\partial G}{\partial x'} dx + \frac{\partial G}{\partial y'} dy + \frac{\partial G}{\partial v'} dv]^2 = 0 \quad 2.3 \]

The application of these conditions follows closely the procedure used by Miele.\textsuperscript{7}

Assuming a flat earth with constant gravitational field and a constant mass vehicle Newton's second law may be written tangent to the flight path,

\[ m\ddot{y} = -gmsin\gamma + F_t \]
where g is a gravitational constant, \( \gamma \) is the angle of inclination of the flight path above the horizontal, and \( F_t \) includes thrust and drag and is, in general, a function of altitude and velocity only.

Figure 2.1 shows that \( \sin \gamma = y' \). Also, the acceleration, \( \ddot{v} \), can be written as \( \frac{d^2y}{ds^2} \) or \( vv' \). Since all motion of the vehicle must obey Newton's law as written above, the equation

\[
\ddot{v} = m\ddot{v} + mg\gamma' - F_t = 0
\]

is an equation of constraint.

If all motion is confined to the vertical plane, changes in altitude, range, and arc length are related by

\[
dx^2 + dy^2 = ds^2 \quad \text{or} \quad x'^2 + y'^2 = 1
\]

A second equation of constraint is the geometric relationship

\[
\phi = x'^2 + y'^2 - 1 = 0
\]

These two equations of constraint apply to each of the two problems discussed in chapters three and four.
Graphic relation between incremental arc length and incremental altitude and range

Figure 2.1

Force diagram for model vehicle

Figure 2.2
CHAPTER III

THE MINIMUM TIME PATH BETWEEN TWO POINTS

The first problem to be discussed is the minimum time path between two points which are restricted to the vertical plane. The time consumed in traversing any path from point 1 to point 2 is the line integral

\[ \int = \int_{1}^{2} dt \quad 3.0 \]

Since it is more convenient to use arc length, \( s \), as the independent variable, time is converted as follows.

\[ dt = \frac{dt}{ds} ds \]

\[ \frac{dt}{ds} = (\frac{ds}{dt})^{-1} = v^{-1} = \frac{1}{v} \]

\[ dt = \frac{ds}{v} \]

and equation 3.0 may be written as

\[ \int = \int_{1}^{2} \frac{ds}{v} \quad 3.0a \]

The Euler-Lagrange equations, 2.1 and 2.2, may now be applied with \( G = \frac{1}{v^2} + \lambda_1 \Psi + \lambda_2 \phi \). Remembering that \( F_t \) is a function of \( y \) and \( v \) only, the following partial derivatives can be evaluated.

\[ \frac{\partial G}{\partial s} = 0 \]
\[
\frac{\partial G}{\partial x} = 0
\]
\[
\frac{\partial G}{\partial y} = -\lambda \frac{\partial F_t}{\partial y}
\]
\[
\frac{\partial G}{\partial v} = -\frac{1}{v^2} + \lambda_1 mv' - \lambda \frac{\partial F_t}{\partial v}
\]
\[
\frac{\partial G}{\partial x^1} = 2\lambda_2 x'
\]
\[
\frac{\partial G}{\partial y'} = \lambda_1 mg + 2\lambda_2 y'
\]
\[
\frac{\partial G}{\partial v'} = \lambda_1 mv
\]

Substituting the values for the partial derivatives, equations 2.1 reduce to,

\[
- \frac{d}{ds}(2\lambda_2 x') = 0 \quad 3.1a
\]
\[
- \lambda \frac{\partial F_t}{\partial y} - \frac{d}{ds}(\lambda_1 mg + 2\lambda_2 y') = 0 \quad 3.1b
\]
\[
- \frac{1}{v^2} + \lambda_1 mv' - \lambda \frac{\partial F_t}{\partial v} - \frac{d}{ds}(\lambda_1 mv) = 0. \quad 3.1c
\]

In addition, the alternate equation, 2.2, becomes,

\[
\left( \frac{1}{v} + \lambda_1 mv' + \lambda_1 mg y' - \lambda_1 F_t + \lambda_2 x'^2 + \lambda_2 y'^2 - \lambda_2
\right.
\]
\[
- 2\lambda_2 x'^2 - 2\lambda_2 y'^2 - \lambda_1 mg y' - \lambda_1 mv y' \right) = B
\]

which reduces to

\[
\left( \frac{1}{v} - \lambda_1 F_t - \lambda_2 x'^2 - \lambda_1 y'^2 \right) = B.
\]
Consider the last three terms.

\[-\lambda_2 x'^2 - \lambda_2 y'^2 - \lambda_2 = -\lambda_2 (x'^2 + y'^2 - 1) - 2\lambda_2 = -\lambda_2 \theta - 2\lambda_2 = -2\lambda_2 \quad \theta = 0.\]

Finally

\[
\frac{1}{v} = \lambda_1 F_t - 2\lambda_2 = B \quad 3.2
\]

Equation 3.1a may be integrated directly giving

\[2\lambda_2 x' = A \quad \text{or} \]

\[2\lambda_2 = \frac{A}{x'} \quad 3.3\]

By applying the transversality conditions, 2.3, the constants, A and B, may be evaluated. Since the final velocity and arc length are free to vary, the coefficients of ds and dv must be zero at the final point. The coefficients of dx and dy are not required to meet this condition because x-final and y-final are specified and dx = dy = 0 at the final point. The coefficients of ds and dv are respectively,

\[G - \frac{\partial G}{\partial x} x' - \frac{\partial G}{\partial y} y' - \frac{\partial G}{\partial v'} v' \bigg|_2 = 0 \quad 3.4a\]

\[\frac{\partial G}{\partial v'} \bigg|_2 = 0 \quad 3.4b\]

The second of these conditions implies that

\[[\lambda_1^{mv}]_2 = 0]
which reduces 3.4a to

\[
\left[ G - \frac{\partial G}{\partial x'} - \frac{\partial G}{\partial y'} \right]_2 = 0
\]

\[
\left[ \frac{1}{v} + \lambda_1 \Psi + \lambda_2 \Phi - (2\lambda_2 x') x' - (\lambda_1 mg + 2\lambda_2 y') y' \right]_2 = 0
\]

\[
\left[ \frac{1}{v} - 2\lambda_2 x'^2 - 2\lambda_2 y'^2 - \lambda_1 mg y' \right]_2 = 0
\]

Then, since \(- \lambda_1 mg y' = \lambda_1 mv v' - \lambda_1 F_t\),

\[- 2\lambda_2 (x'^2 + y'^2) = -2\lambda_2 \quad \text{and} \quad [\lambda_1 mv]_2 = 0,
\]

\[
\left[ \frac{1}{v} - 2\lambda_2 - \lambda_1 F_t \right]_2 = 0
\]

Using this relationship to evaluate the constant of equation 3.2, it is seen that \(B = 0\) and

\[
\frac{1}{v} - \lambda_1 F_t - 2\lambda_2 = 0
\]

3.5

Rearranging shows

\[2\lambda_2 = \frac{1}{v} - \lambda_1 F_t\]

Also from 3.3

\[2\lambda_2 = \frac{A}{x'}\]

So \[\frac{A}{x'} = \frac{1}{v} - \lambda_1 F_t\] or \[\frac{mvA}{x'} = m - \lambda_1 mv F_t\]

But from 3.4b \([\lambda_1 mv]_2 = 0\) so

\[\left[ \frac{mvA}{x'} \right]_2 = m\]

\[A = \left[ \frac{x'}{v} \right]_2\]

Again referring to Figure 2.1 \(x' = \cos \gamma\)

So \[A = \left[ \frac{\cos \gamma}{v} \right]_2\] 3.6
Now proceeding with the solution of the optimizing equations, the substitution of \( \frac{d}{ds}(\lambda_1^{\text{mv}}) = \lambda_1^{\text{mv}} + \lambda_1^{\text{mv'}} \) into 3.1c reduces that equation to

\[
\frac{1}{v^2} + \lambda \frac{\delta F_t}{\delta y} + \lambda_1^{\text{mv}} = 0
\]

From equation 3.3

\[
2\lambda_2 y' = A \frac{v'}{x'} = A \frac{\sin \gamma}{\cos \gamma} = \text{Atan} \gamma
\]

since \( y' = \sin \gamma \) and \( x' = \cos \gamma \). Substituting for \( 2\lambda_2 y' \) in equation 3.1b

\[
- \lambda \frac{\delta F_t}{\delta y} - \frac{d}{ds}(\lambda_1^{\text{mg}} + \text{Atan} \gamma) = 0
\]

or

\[
- \lambda \frac{\delta F_t}{\delta y} - \lambda_1^{\text{mg}} - \text{Asec}^2 \gamma y' = 0
\]

From equation 3.5

\[
\lambda_1 = \frac{1}{F_t} \left( \frac{1}{v} - 2\lambda_2 \right) = \frac{1}{F_t} \left( \frac{1}{v} - \frac{A}{\cos \gamma} \right)
\]

and from equation 3.7

\[
\lambda_1' = \frac{1}{mv^3} = \frac{\lambda_1}{mv} \frac{\delta F_t}{\delta y} \quad \text{or}
\]

\[
\lambda_1' = - \frac{1}{mv^3} - \left( \frac{1}{mv^2} - \frac{A}{mv \cos \gamma} \right) \frac{1}{F_t} \frac{\delta F_t}{\delta y}
\]

Substitution of the above expression for \( \lambda_1 \) and \( \lambda_1' \) in equation 3.8 results in

\[
- \frac{1}{F_t} \left( \frac{1}{v} - \frac{A}{\cos \gamma} \right) \frac{\delta F_t}{\delta y} + \frac{g}{v^3} + \left( \frac{1}{v^2} - \frac{A}{v \cos \gamma} \right) \frac{g}{F_t} \frac{\delta F_t}{\delta y}
\]

\[- \text{Asec}^2 \gamma y' = 0
\]

After some simplification and rearranging

\[
y' = - \frac{1}{F_t} \frac{\delta F_t}{\delta y} \left( \frac{\cos^2 \gamma}{Av} - \cos \gamma \right) + \frac{g \cos^2 \gamma}{Av^2}
\]

\[
+ \left( \frac{\cos^2 \gamma}{Av^2} - \frac{\cos \gamma}{v} \right) \frac{g}{F_t} \frac{\delta F_t}{\delta y}
\]
Noting that \( v\gamma' = \frac{ds}{dt} \frac{dv}{ds} = \frac{dv}{dt} = \dot{v} \)

\[
\dot{v} = (v\cos\gamma - \frac{\cos^2\gamma}{A}) \frac{1}{F_t} \frac{\partial F_t}{\partial y} + \frac{g\cos^2\gamma}{Av^2}
\]

\[
+ (\frac{\cos^2\gamma}{Av} - \cos\gamma) \frac{F_t}{F_t} \frac{\partial F_t}{\partial v}
\]

This equation along with the equations for \( x, y, \) and \( v, \)

\[
\dot{x} = v\cos\gamma
\]

\[
\dot{y} = v\sin\gamma
\]

constitute four first order, non-linear, differential equations with four dependent variables \( x, y, v, \) and \( \gamma. \)

The solution of these equations is possible for several choices of \( F_t(y, v). \)

**Simple Brachistochrone**

For the first example, the classic brachistochrone problem, a solution may be obtained without the aid of a computer. This solution will be used as a guide and a basis for comparing later solutions.

For the problem of the minimum time path between two points in a vertical plane it is possible to obtain the classic brachistochrone solution by setting the tangential forces equal to zero. This is equivalent to the situation of a ball sliding down a frictionless wire under the influence of a constant gravitational field. Equations 3.9a and 3.9b are seen to reduce to
\[ \dot{y} = g \frac{\cos \gamma}{v^2} \quad 3.10a \]

\[ \dot{v} = -g \sin \gamma \quad 3.10b \]

Also equation 3.5 becomes \( 2\lambda_2 = \frac{1}{v} \) but, from 3.3, \( 2\lambda_2 = \frac{A}{\cos \gamma} \)
\[ \text{so } A = \frac{\cos \gamma}{v} = \text{constant}. \]

Then \( \dot{\gamma} = g \frac{\cos \gamma}{v} = \text{constant}. \)

If the initial velocity is zero, the velocity at any depth, \(-y\), is \( \sqrt{-2gy} \). Then \( \frac{\cos \gamma}{\sqrt{-2gy}} = \text{constant} = m \) or \[ y = \cos^2 \gamma \text{ where } a = (-2gm)^{-1}. \]
So \( \text{dy} = -2a \sin \gamma \cos \gamma \text{ d}y \) and since \( \frac{dv}{dx} = \tan \gamma \)
\[ \tan \gamma = -2a \sin \gamma \cos \gamma \frac{dv}{dx} \quad \text{or} \]
\[ dx = -2a \cos^2 \gamma \text{ d}y. \]

By direct integration \( x = b - \frac{1}{2}a(2\gamma + \sin 2\gamma) \) and \( y = a \cos^2 \gamma \)
which are the equations of a cycloid in parametric form.

This result agrees with the standard solution obtained by more direct means and Fox\(^5\) shows that this solution is indeed the minimum time solution.

**Brachistochrone with Drag**

Another interesting problem is that of a glider attempting to get to a point in minimum time. This is similar to the simple brachistochrone except that drag has been added. First, assuming drag is a function of the form \(-kv^2\), equations 3.9a and 3.9b simplify to
\[ \dot{y} = -2g \frac{\cos \gamma}{v} + 3g \frac{\cos^2 \gamma}{Av^2} \quad 3.11a \]

\[ \dot{v} = -g \sin \gamma - \frac{k}{m} v^2 \quad 3.11b \]
The solution to equations 3.11 may be obtained with the aid of an analog computer but a definite iterative procedure must be employed in order to terminate the trajectory at a chosen point. The final point of any particular minimum time curve which is a solution to equations 3.11 depends on the choice of the constant, \( A \), as well as the initial flight path angle, \( \gamma \).

The close relation between the drag problem and the simple brachistochrone leads one to the idea of using a no-drag model as an initial approximation to the drag problem. This may be done after equations 3.11 have been programmed by setting the drag equal to zero and picking the constant, \( A \), from the initial conditions. Now the choice of the initial angle determines the constant, \( A \), and it is a simple matter to find a curve through a desired end point. Figure 3.1 shows a simple no-drag curve through a final point, \( P \).

Since the drag trajectory is dissipative, it might be expected that the minimum time drag path to point \( P \) will lie above the no-drag path. So, the initial angle for the first attempt at a drag curve through point \( P \) is chosen closer to the horizontal. Several curves are obtained without regard to the end condition 3.6. The constant, \( A \), is adjusted until a trajectory which passes through point \( P \) is found. Then the end point of this optimum curve is determined from condition 3.6. If this end point is beyond
Brachistochronic trajectory with no drag or thrust - 24 seconds consumed from start to point P

Figure 3.1

Brachistochronic trajectory with drag only - final point beyond point P

Figure 3.2

Brachistochronic trajectory with drag only - final point short of point P

Figure 3.3

Comparison of brachistochronic trajectories with and without drag - initial velocity is 500 fps in both cases - drag path consumes 29 seconds

Figure 3.4
point \( P \), as in Figure 3.2, the initial angle should be closer to the initial angle of the no drag curve. If the end point is short of the desired target point \( P \), as in Figure 3.3, the initial angle should be closer to the horizontal. After several tries the end point of the drag trajectory can be made to coincide with the desired end point \( P \) as in Figure 3.4.

**Thrust Problem**

Now that some of the effects of drag have been investigated one might ask, "What does the addition of thrust do to the simple brachistochrone trajectory?"

Considering a constant thrust force the tangential forces are \( F_t = T_o = \text{constant} \) and equations 3.9a and 3.9b reduce to

\[
\dot{v} = -g \sin \gamma + \frac{T_o}{m} \tag{3.12b}
\]

\[
\dot{\gamma} = \frac{g \cos^2 \gamma}{Av^2} \tag{3.12a}
\]

Once again the choice of the constant \( A \) and the initial flight path angle, \( \gamma \), determines an optimum curve to a particular point in the plane. To find an optimum trajectory to a pre-chosen point an iterative procedure must again be employed. The system outlined for the drag problem may be used if the final point is within the scope of brachistochronic curves. If, however, a desired final point is at too great an altitude to be reached by a
brachistochrone, a new iterative scheme must be used. This new scheme requires the establishment of a net of end points as in Figure 3.5. To draw this net a table, such as Table 3-1, is compiled by picking an \( A \), finding curves for several initial flight path angles, recording the coordinates of each end point, and repeating this procedure for a variety of \( A \)'s.

Direct graphic interpolation of the net gives very good results as shown in Figure 3.6. Of course, if the net is completed for many values of \( A \) and the initial flight path angle, the interpolation is much easier and normally an end point can be reached with the first try.

The lift to weight ratios required to fly the optimum trajectory to give the minimum time between two points are given in Figure 3.8 for the model without drag and thrust, the model with drag only, and the model with thrust only. None of these trajectories require unreasonable lift to weight ratios.
The coordinates of final points of trajectories along with the initial angle and $\frac{1}{A}$ for each path. The initial point is $x_0$ and $y_0$ and the initial velocity is 500 fps.

Table 3-1

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Locii of end points for the maximum range problem with lines of constant $\gamma_f$ and $\frac{1}{A}$.

Figure 3.5
A first attempt to obtain an optimum path to point P by using the net in Figure 3.5. The initial angle is $-45^\circ$, the initial velocity 500 fps, and $\frac{1}{A} = 2500$.

Figure 3.6

Minimum time problem with thrust but no drag - initial velocity 500 fps - initial angle $-60.2^\circ$ - $\frac{1}{A} = 1420$ - total time consumed 20 seconds

Figure 3.7

Lift to weight curves for minimum time problems. Curve a is for no drag or thrust, curve b for drag only, and curve c for thrust only. The trajectories for these lift programs are given in figures 3.7 and 3.4.

Figure 3.8
CHAPTER IV
THE MAXIMUM RANGE PATH

The second problem to be discussed in the maximum range path to a specified altitude. The range is the line integral

\[ \int = \int_1^2 dx \]  \hspace{2cm} 4.1

Again it is more convenient to use arc length, \( s \), as the independent variable so range is converted as follows:

\[ dx = \frac{dx}{ds} ds = x'ds \]

and equation 3.0 may be written as

\[ \int = \int_1^2 x'ds \]  \hspace{2cm} 4.1a

When applying the Euler-Lagrange equations, 2.1 and 2.2, to the range problem the function \( G \) is

\[ G = x' + \lambda_1 y' + \lambda_2 \varphi \]

Again evaluating the necessary partial derivatives, holding \( g \) constant and \( F_t \) a function of altitude and velocity only,

\[ \frac{\partial G}{\partial s} = 0 \]

\[ \frac{\partial G}{\partial x} = 0 \]

\[ \frac{\partial G}{\partial x'} = 1 + 2\lambda_2 x' \]

\[ \frac{\partial G}{\partial y'} = \lambda_1 mg + 2\lambda_2 y' \]
Using these partial derivatives, equations 2.1 reduce to

\[ - \frac{\partial F_t}{\partial y} = \lambda_1 \frac{\partial^2 G}{\partial y^2} \]
\[ \frac{\partial G}{\partial y} = \lambda_1 \frac{\partial^2 G}{\partial y^2} \]
\[ \frac{\partial G}{\partial v} = \lambda_1 \frac{\partial^2 G}{\partial v^2} \]

The alternate equation, 2.2, becomes

\[ (x' + \lambda_1 mvv' + \lambda_1 mg'y' - \lambda_1 F_t + \lambda_2 x'^2 + \lambda_2 y'^2 - \lambda_2 - x' - 2\lambda_2 x'^2 - 2\lambda_2 y'^2 - \lambda_1 mg'y' - \lambda_1 mvv') = B \]

which reduces to

\[ - \lambda_1 F_t - \lambda_2 \left( x'^2 + y'^2 - 1 \right) - 2\lambda_2 = B \]

or

\[ (\lambda_1 F_t + 2\lambda_2) = B \]

Equation 4.2a may also be integrated directly giving

\[ 1 + 2\lambda_2 x' = A \quad \text{or} \quad 2\lambda_2 = \frac{A - 1}{x'} \]

Again applying the transversality conditions, 2.3, the constants may be evaluated. Since the final range and arc length are unrestricted, the coefficients of dx and ds must be zero. These coefficients are

\[ G - \frac{\partial G}{\partial x} x' - \frac{\partial G}{\partial y} y' - \frac{\partial G}{\partial v} v' \bigg|_2 = 0 \]
\[
\left. \frac{\partial G}{\partial x'} \right|_2 = 0 \tag{4.5b}
\]

\[
\frac{\partial G}{\partial x'} = 1 + 2\lambda_2 x' \quad \text{so} \quad 1 + 2\lambda_2 x' \bigg|_2 = 0
\]

This condition requires that the constant \(A\) be zero.

Condition 4.5a requires that the constant \(B\) be zero. Then

\[
\lambda_1 F_t + 2\lambda_2 = 0 \tag{4.6}
\]

and

\[
2\lambda_2 = -\frac{1}{x'} \tag{4.7}
\]

Equation 4.2c may be simplified to

\[
\lambda_1 x' - \lambda_1 \frac{\partial F_t}{\partial v} - \lambda_1 v' \lambda_1 = -\lambda_1 \frac{\partial F_t}{\partial v} - \lambda_1 v' = 0 \tag{4.8}
\]

so

\[
\lambda_1 = -\frac{\lambda_1}{x'} \frac{\partial F_t}{\partial v} \tag{4.9}
\]

Expanding 4.2b as before,

\[
\lambda_1 \frac{\partial F_t}{\partial y} - \lambda_1^{mg} + \sec^2 \gamma' = 0
\]

Now using 4.9, 4.7 and 4.6 the multipliers may be eliminated, leaving,

\[
\dot{y} = \frac{v \cos \gamma}{F_t} \frac{\partial F_t}{\partial y} - \frac{g \cos \gamma}{F_t} \frac{\partial F_t}{\partial v} \tag{4.10}
\]

This equation along with 3.9b, c, d constitute a system of four equations in the four dependent variables \(x, y, v\) and \(\gamma\). The solution of these equations is the desired optimum trajectory.

A simple problem in elementary calculus is the finding of the optimum firing angle for a projectile in
constant gravitational field with zero drag. The acceleration in the horizontal direction is zero and in the vertical direction is \(-g\). Then \(\ddot{x} = 0\) and \(\ddot{y} = -g\). Integrating twice with \(x_1 = y_1 = 0\), \(x = \dot{x}_1 t\) and \(y = -\frac{1}{2}gt^2 + \dot{y}_1 t\).

The horizontal component of the initial velocity is \(v_1 \cos \gamma_1\) and the vertical component is \(v_1 \sin \gamma_1\), so \(x = v_1 \cos \gamma_1 t\) and \(y = -\frac{1}{2}gt^2 + v_1 \sin \gamma_1 t\). Since \(x = x_2\) when \(y = y_2 = 0\) and \(t = t_2\),

\[
0 = -\frac{1}{2}gt^2 + v_1 \sin \gamma_1 t.
\]

This equation has two roots; \(t_2 = 0\) and \(t_2 = \frac{2v_1 \sin \gamma_1}{g}\). The latter is obviously the correct value for \(t_2\). Then \(x_2 = v_1 \cos \gamma_1 t_2\) or \(x_2 = \frac{2v_1^2}{g} \sin \gamma_1 \cos \gamma_1\). So \(x_2\) is a function of \(\gamma_1\) and \(v_1\) and the maximum value for \(x_2\) may be found by setting the first derivative with respect to \(\gamma_1\) equal to zero and solving for \(\gamma_1\)

\[
\frac{2v_1^2}{g} (\cos^2 \gamma_1 - \sin^2 \gamma_1) = 0
\]

Then \(\cos^2 \gamma_1 = \sin^2 \gamma_1\) and, if it is required that \(\gamma_1\) be a positive angle, the optimum is \(\gamma_1 = 45^\circ\).

A plot of this curve for \(v_1 = 500\) feet per second is given in Figure 4.1. Since there is no energy loss, the final velocity is equal to the initial velocity and the curve is entirely symmetrical about a vertical line through the apex.
Maximum range trajectory for a constant gravitational field but no drag or thrust and no aerodynamic lift. The initial velocity, 500 fps and the initial angle, 45°.

A comparison of the NACA standard atmosphere density with an exponential density.

Figure 4.1

Figure 4.2
If drag is added to the above problem, the trajectory which maximizes the range cannot be found through the use of elementary calculus. However, the calculus of variations provides the necessary tool to solve this problem. This solution might apply, roughly, to a tactical rocket which has a very short burning time and is remotely controlled by some variable lift device which keeps the rocket on the desired optimum path. It could also apply to a manned glider which has just been released from a towing vehicle.

The next step is to choose an analytic representation of the drag. Starting with \( D = \frac{1}{2} \rho v^2 C_d \), a good approximation is \( D = k v^2 \), where the constant, \( k \), includes density and the drag coefficient. This is a good expression for drag if variations in altitude are kept small and speeds are kept in the compressible range. With \( F_t = -k v^2 \) the optimizing equations, 3.19 and 3.9b, reduce to

\[
\ddot{y} = -\frac{2g \cos \gamma}{v} \quad 4.11a
\]

\[
\dot{v} = -g \sin \gamma - \frac{k v^2}{m} \quad 4.11b
\]

A more realistic drag is given by \( D = k v^2 e^{-\beta y} \) where the exponential takes into account the variation of density with altitude. By choosing \( \beta \) equal to \( 3.25 \times 10^{-4} \) l/feet, the atmospheric density curve in Figure 4.2 agrees very well with the NACA standard atmosphere (Dwinell) from sea level to fifty thousand feet. Now the tangential forces
are \( F_t = -kv^2 e^{-\beta y} \) where \( k \) includes the coefficient of drag and the sea level density.

For the choice of \( F_t = -kv^2 e^{-\beta y} \) the equations for maximum range reduce to

\[
\begin{align*}
\dot{y} &= -\beta v \cos \gamma - 2g \frac{\cos \gamma}{v} \quad &4.12a \\
\dot{\theta} &= -g \sin \gamma - \frac{k}{m} v^2 e^{-\beta y} \quad &4.12b
\end{align*}
\]

It is seen that the second term in the acceleration equation decreases with altitude. Therefore, slightly greater ranges can be achieved with the density variation included in the drag term than with a constant density drag. That is, the optimum solution takes advantage of the less dense higher atmosphere in order to minimize energy dissipation along the flight path.

Figure 4.3 shows several solutions to equations 4.12. In each case the final altitude is zero but the final velocity varies with each solution. Each of these curves give the maximum range trajectories for each of their respective end conditions. It is seen that there is an upper limit on final velocity. For the examples in Figure 4.4 this upper limit occurs when the initial flight path angle is 55° and gives a final velocity of 443 feet per second. In each case the range increases with final velocity. An energy consideration shows that the greater the final velocity at a specified final altitude, the smaller is the net energy loss along the
Maximum range for varying initial angles and same initial velocity of 300 fps. An initial angle of 60° appears to give more final kinetic energy and range, if the initial velocity is 300 fps.

Figure 4.3

Maximum range with initial velocity of 400 fps. For this initial velocity it appears that an initial angle of about 55° will give maximum final kinetic energy and range.

Figure 4.4
flight path. Naturally maximum range is attained if the net energy loss along the trajectory is held to a minimum. Figure 4.5 indicates that this analysis is correct, since the final potential energies are the same in each case but the kinetic energies vary with the square of the velocities.

In order to obtain a pre-chosen final velocity at a specified final altitude no complex iterative procedure is required. It is merely required that, by varying the initial flight path angle, several curves be generated, stopping each at the desired final altitude. The final velocity of each curve is recorded as in Table 4-1.

<table>
<thead>
<tr>
<th>initial angle</th>
<th>final range</th>
<th>final velocity</th>
</tr>
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<tr>
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<td></td>
</tr>
<tr>
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<td>281</td>
</tr>
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<td>279</td>
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<tr>
<td>50</td>
<td>1620</td>
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<tr>
<td>initial velocity - 400 fps</td>
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<td></td>
</tr>
<tr>
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<td>2850</td>
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<td>368</td>
</tr>
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<td>initial velocity - 500 fps</td>
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<td></td>
</tr>
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<td>443</td>
</tr>
<tr>
<td>50</td>
<td>4220</td>
<td>441</td>
</tr>
</tbody>
</table>

TABLE 4-1
Maximum range with initial velocity of 500 fps. For this initial velocity it appears that an initial angle of about 55° will again give maximum final energy and range.

Figure 4.5
Simple graphic interpolation will result in the pre-chosen final velocity at that particular altitude. So obtainable end conditions are limited both as to maximum altitude and maximum velocity.

**Thrust and Drag**

If a thrust which varies with density is added to the maximum range problem, the tangential forces become

\[ F_t = (T_0 - kv^2)e^{-\beta y} \]

and the optimizing equations reduce to

\[ \dot{\gamma} = -\beta v \cos{\gamma} + \frac{2kg}{T_0 - kv^2} \cos{\gamma} \]  
\[ \dot{v} = -g \sin{\gamma} + \frac{T_0 - kv^2}{m}e^{-\beta y} \]

It might seem that these equations pose no new difficulties over the equations for the range problem without thrust. That this is not the case may be seen by a close inspection of equation 4.13a. Since the second term may be positive for small velocities, there exists a possibility of oscillation. This oscillation is demonstrated by the solution in Figure 4.6. The optimum maximum range trajectory with thrust and drag forces passes through certain altitudes more than once raising the question, "If the end conditions are met more than once by the solution to the optimizing equations, which meeting of the end conditions determines the final point?"

Figure 4.7 shows that no velocity is repeated at any
MAXIMUM RANGE
\[ R = (10 - 2.5 \times 10^{-4}v^2) e^{-0.06y} \]
\[ \theta_1 = 60^\circ \]
\[ v_1 = 400 \text{ fps} \]

**Figure 4.6**

Altitude-Velocity plot shows velocity doesn't repeat at any altitude.

**Figure 4.7**
altitude so the end conditions are not met more than once and there is no difficulty in locating the final point. It is interesting to note that, although the thrust trajectories oscillate, they do tend to damp down to a non-oscillating curve. This curve appears to be a quasi-steady climb as indicated by Figure 4.8 which shows that the acceleration and the rate of change of the flight path angle are both damping to zero. It is possible to pick initial conditions which result in a non-oscillatory trajectory (Figure 4.9).

Some of the sharp dips in the trajectories require fairly high lift to weight ratios but none require unreasonably high ratios so it is possible to fly these optimum trajectories.
Velocity and rate of change of the flight path angle versus range. The initial angle, $30^\circ$ and the initial velocity, 500 fps. $F_t = (10 - 2.5 \times 10^{-5}v^2)e^{-\beta y}$

Figure 4.8

Non-oscillatory trajectory. The initial angle, $10^\circ$ and the initial velocity, 478 fps. $F_t$ same as in Figure 4.8.

Figure 4.9
Lift to weight ratio versus range for $\gamma_l = 60^\circ$ and $v_l = 400$ fps.

Lift to weight ratio versus range for $\gamma_l = 30^\circ$ and $v_l = 400$ fps.

Figure 4.10
CHAPTER V

CONCLUSIONS

Although analog computer solutions to problems of optimized trajectories are not extremely accurate, they are valuable as a basic for qualitative analysis and as a first approximation to actual optimum trajectories.

Much insight can be gained in a particular problem by studying the differential equations governing the optimum flight path. However, a distinct advantage is gained by actually seeing a graphic representation of that flight path even if this representation is only approximate. For example, in the case of finding a minimum time path between two points, solutions containing drag or thrust effects cannot be obtained with ease analytically. An investigation of equation 3.11a or 3.12a will show the general shape of the desired curve. However, Figures 3.4 and 3.7 indicate the relative effects of drag and thrust by comparing each solution to the brachistochronic curve through the same end point. In addition to finding the optimum trajectory it is possible to determine the actual times consumed in each case with enough accuracy to make order of magnitude comparisons for the effects of drag and thrust on flight time.
With the aid of an analog computer it is also possible to obtain lift programs as in Figure 3.8 for each solution.

These curves are certainly accurate enough to determine the practicability of each solution from a maximum lift point of view.

In the solutions presented here some accuracy was lost due to equipment limitations. For instance, initial condition resistance pots were of the one turn type which lead to inaccuracies in the setting of initial conditions. Large errors in initial conditions usually result in very large changes in the trajectories. Replacement of these pots by ten turn type pots is one minor change which would increase the over-all accuracy a considerable amount. Also the transcendental functions, \( \sin \gamma \) and \( \cos \gamma \), were generated by means of an inner loop which tends to be unstable if the trajectories are lengthy as in Figure 4.6. This instability could be reduced if a servo-driven resolver were used in place of the inner loop.

As soon as analog computer accuracy can be increased problems with several undetermined constants, such as the constant \( A \) in the minimum time problem, can be solved by producing nets like the one in Figure 3.5.
REFERENCES


NOMENCLATURE

A, B constants of integration
D drag

\( F_t \) components of forces tangent to flight path excluding gravitational force

\( g \) acceleration of gravity, 32.2 feet per second

I integral to be optimized

L lift

m mass of the vehicle

P a prepicked point in the vertical plane

s arc length

T thrust

v velocity

x range

y altitude

1 initial point of trajectory

2 final point of trajectory

\( \beta \) \( 3.25 \times 10^{-4} \) l/feet

\( \gamma \) angle of inclination of the flight path with respect to the horizontal

\( \Phi \) constraining equations

\( \lambda_1, \lambda_2 \) Lagrange multipliers

\( \dot{x}, \dot{y}, \dot{v}, \dot{\gamma} \) derivatives with respect to time

\( x', y', v', \gamma' \) derivatives with respect to arc length, \( s \)