

A COMPARISON OF METHODS FOR
MULTIVARIABLE CONTROL SYNTHESIS

by

Robert Lee White

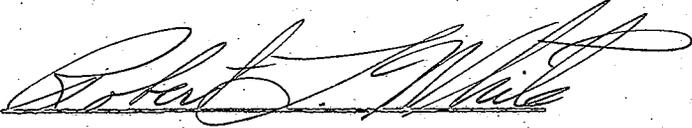
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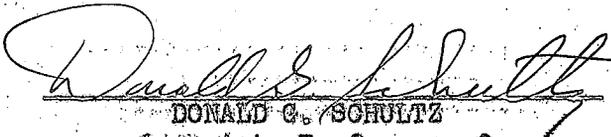
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TABLE OF CONTENTS

Chapter	Page
LIST OF ILLUSTRATIONS.....	v
ABSTRACT.....	vii
1 INTRODUCTION.....	1
2 PRELIMINARY DESIGN MECHANICS.....	3
3 MULTIVARIABLE SYNTHESIS THEORY.....	10
4 NONINTERACTION.....	24
5 PLANT PARAMETER VARIATION.....	32
6 SUMMARY AND SUGGESTIONS FOR FURTHER STUDY.....	41
APPENDIX I DETERMINATION OF MATRIX INVERSE BY SIGNAL FLOW DIAGRAMING.....	43
APPENDIX II MULTIVARIABLE SIGNAL FLOW DIAGRAMING.....	49
LIST OF REFERENCES.....	52

LIST OF ILLUSTRATIONS

Figure		Page
1	Multivariable Plant Block Diagram.....	6
2	Overall Multivariable System Excluding Compensators.....	6
3	Multivariable Plant with Cascade Compensation.....	6
4	General Two-Degree-of-Freedom Configuration.....	9
5	Other Two-Degree-of-Freedom Configurations.....	9
6	Multivariable Electro-Mechanical Plant.....	14
7	Block Diagram of Multivariable Electro-Mechanical Plant..	14
8	Block Diagram of Electro-Mechanical Plant Inverse.....	16
9	Desired Transfer Function Block Diagram.....	16
10a	Actual Structure of Compensated System with Supplemented Plant Matrix.....	26
10b	Artificial Structure of Compensated System with Supplemented Plant Matrix.....	26
11	Physical System, Electro-Mechanical Plant with Unity Feedback.....	29
12	Block Diagram, Electro-Mechanical Plant with Unity Feedback.....	29
13	Loop Flow Diagram.....	33
14	Plant Signal Flow Diagram.....	45
15	Grouped Plant Signal Flow Diagram.....	45
16	Reduced Plant Signal Flow Diagram.....	45
17	Inversion Path.....	47
18	Inverse Plant Signal Flow Diagram.....	47
19	Reduced Inverse Plant Signal Flow Diagram.....	47

LIST OF ILLUSTRATIONS--Continued

Figure		Page
20	Simple Signal Flow Diagram.....	50
21	Overall System Signal Flow Diagram.....	50

ABSTRACT

A linear time-invariant multivariable plant may be compensated by forward and feedback elements to yield the desired overall transfer functions between inputs and outputs. A general two-degree-of-freedom configuration is adopted and various design methods are investigated with regard to design simplicity, plant parameter variations, and practicality.

CHAPTER 1

INTRODUCTION

A multivariable control system may be defined as a physical device whose set of m variables termed outputs are dependently related to a set of n variables termed inputs with $m \geq 1$ and $n > 1$.

The dependence of outputs upon inputs is governed by physical laws which can be expressed mathematically by inter-differential equations. The scope of this thesis is limited to control systems whose functional dependencies can be mathematically represented by linear differential equations with constant coefficients.

Several applications of multivariable control systems are familiar. A turbo-jet engine constitutes a multivariable control system with input quantities of fuel flow rate, exhaust nozzle area, altitude and with output quantities of engine speed and engine temperature. Process control in the chemical industry requires various input quantities such as concentrations of materials, time duration, and temperature of processes be constantly monitored and controlled to insure quality and uniformity in the output product. The single-variable control system may be considered as a special case of the multivariable theory.

In the past two decades control theory has experienced growth at a rapidly expanding rate. As may be noted in all fields of endeavor where the present needs surpass the boundaries of well-developed theory, in the past much of the multivariable control design was carried out on a specific problem basis with design techniques founded upon the intuition

of the designer and his ability to adapt experience gained in related problems to the synthesis of multivariable controllers. This approach has definite limitations however, and as more severe specifications and constraints were imposed upon the systems and more complicated plants handled, the need for a general multivariable control theory became mandantory.

One of the first attempts at a systematic design for the control of a multivariable plant was pioneered in the late 1940's by Boksenbom and Hood, who developed a method for the synthesis of gas-turbine engine controllers.^{1,2} Though this work is the root of most present day multivariable synthesis techniques, its applications are specialized and a general theory is not explicitly developed. It was not until 1956 that Kavanaugh developed a more generalized approach.³ Since that time a number of publications have appeared supplementing and augmenting the original theory to aid in solution of the various aspects of the multivariable synthesis problem.

As is to be expected, the problems involved have been approached in a different manner by each author. It is the purpose of this thesis to investigate several aspects of the design and present to the reader the methods which have the most merit.

A great deal of the value in a multivariable synthesis technique is dependent upon the versatility of the approach used and its ability to constrain the complexity of the mathematics. Therefore, a large portion of the design burden lies in the initial stage of development. One must first choose the proper controller configuration, design approach, and mathematical representation. Having made these selections, the theory must then be extended to handle the desired specifications.

CHAPTER 2

PRELIMINARY DESIGN MECHANICS

The control synthesis problem may be stated as one of adapting a control system to a given plant such that certain variables are controlled in a specified manner. A general form for the solution of this problem is given by the following steps:⁴

1. A set of desired performance specifications are established.
2. A mathematical representation of the given plant is determined.
3. The plant is analyzed to determine if its performance meets the desired specifications.
4. If specifications are not met, a method of compensation is used until further analysis yields the desired overall results.

These general steps are applicable to multivariable control synthesis as well as single-variable control synthesis. With synthesis techniques for single-variable theory well developed, the natural trend has been for multivariable synthesis to parallel these techniques. The type of performance specifications established by the designer is largely dependant upon the synthesis technique to be used. An optimal or statistical approach is concerned with minimization of certain system characteristics, as in driving a system from one state to another in the shortest amount of time or the least amount of energy. A frequency domain approach where design is determined by the open- and closed-loop transfer functions must meet specifications such as steady-state error, overshoot, sensitivity to plant parameter variations, etc. It is with

this latter synthesis criteria that this thesis is concerned.

In attempting to formulate a mathematical representation of the multivariable system and to choose a controller configuration upon which to base the synthesis, two important guides should be kept in mind. Any correlation between multivariable techniques and the more familiar single-variable problem should be considered. The compensator configuration should express enough freedom to be adaptable to a variety of problems.

In the development of a multivariable synthesis procedure, the following notation will be utilized:

- p_{12} - Lower-case letters with duo-number subscripts found in the equations and matrices represent transfer functions of the frequency variable, s .
- c_1 - Lower-case letters with a singular-number subscript represent frequency dependant parameters of the system.
- P - Upper-case letters represent matrices of the lower-case letters defined above. The lower-case letters with duo-number subscripts are found in square and rectangular matrices, whereas the lower-case letters with singular-number subscripts are found in column matrices.
- P^T - The superscript T denotes the transpose of the matrix.
- P^{-1} - The superscript (-1) denotes the inverse of the matrix.
- adj. P - A matrix notation prefixed by the letters, adj., denotes the adjoint of the matrix.
- Δ_P - Delta with a matrix notation subscript denotes the determinant of the matrix.

Consider the multivariable plant P shown in figure 1 with n inputs and m outputs where $n \geq m$. The case where $m > n$ is feasible but impractical. The output variables are related to the system inputs by the following equations:

$$\begin{aligned} c_1 &= p_{11}x_1 + p_{12}x_2 + \dots + p_{1n}x_n \\ c_2 &= p_{21}x_1 + p_{22}x_2 + \dots + p_{2n}x_n \\ &\vdots \\ c_m &= p_{m1}x_1 + p_{m2}x_2 + \dots + p_{mn}x_n. \end{aligned} \quad (2.1)$$

Since the plant P is represented by a set of simultaneous linear equations, the adoption of a matrix notation is a logical step. This not only offers the designer an abbreviated form of notation but also the possibility for application of theorems from matrix algebra. Equation (2.1) may be expressed in matrix form by

$$C = PX, \quad (2.2)$$

where C is an $(m \times 1)$ column matrix; X, an $(n \times 1)$ column matrix; and P, an $(m \times n)$ rectangular matrix. In expanded form, equation (2.2) takes the shape of

$$\begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{pmatrix} = \begin{pmatrix} p_{11} & \cdot & \cdot & p_{1n} \\ \cdot \\ \cdot \\ \cdot \\ p_{m1} & \cdot & \cdot & p_{mn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad (2.3)$$

After adoption of a matrix notation, the next step is to determine a controller configuration which will adequately compensate the given plant. Consider the following multivariable control problem. Given the plant matrix P whose elements, p_{ij} , are functions of the

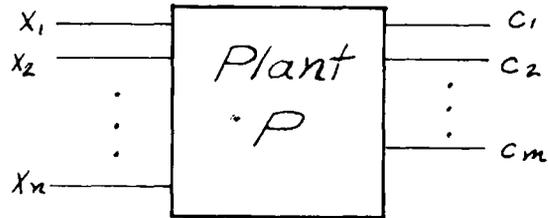


Fig. 1.--Multivariable Plant Block Diagram

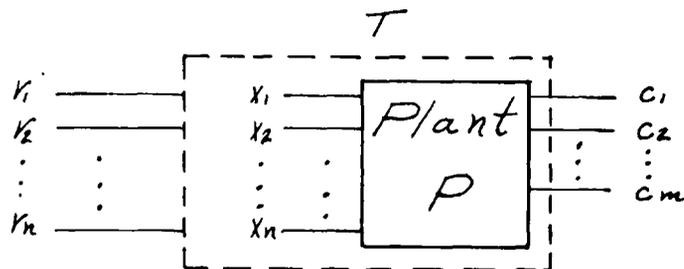


Fig. 2.--Overall Multivariable System Excluding Compensators

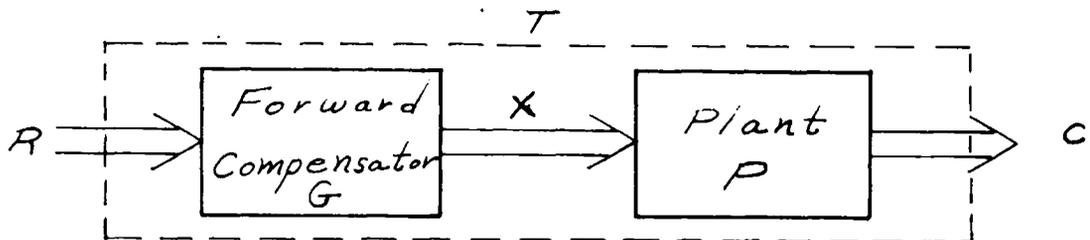


Fig. 3.--Multivariable Plant
with Cascade Compensation

complex frequency variable, s , relating the plant output, c_i , to the plant input, x_j , compensate the plant so that the overall system will exhibit a transfer matrix, T . The individual elements of T , t_{ij} , are equal to $c_i/r_j \Big|_{r_i = 0}$, where $i \neq j$. The overall system inputs may be represented by an $(n \times 1)$ column matrix, R . T is given by an $(m \times n)$ rectangular matrix. The configuration as thusfar described is shown in figure 2.

The simplest solution would seem to be a forward compensator, G , cascaded with the plant P such that

$$C = PGR, \quad (2.4)$$

as illustrated in figure 3. Since

$$C = TR, \quad (2.5)$$

the forward compensator can be represented by an $(n \times n)$ matrix given by

$$G = P^{-1}T. \quad (2.6)$$

If P is non-singular, the compensator matrix can then be determined from equation (2.6). The requirement of non-singularity must be placed upon P since a singular matrix has an undefined inverse. This open-loop compensation has several disadvantages. The parameters of P must be specified within the tolerances of the T parameters. P parameters must be known not to vary more than the T tolerance limits.

From single-variable theory, it is known that these ills are often cured by feedback. Assuming that the plant may not be varied,

that there are no internal nodes within the plant available for feedback, and there is no signal leakage between plant output and input, the output signals, C , are determined by X alone. As illustrated in figure 2, the only signals of which X can be composed are C and R . Therefore, a completely general feedback compensator structure (figure 4) is composed of a forward controller G between the plant inputs, X , and the system inputs, R , and a feedback controller, H , between the plant outputs, C , and the plant inputs, X . G represents an $(n \times n)$ square matrix and H , an $(n \times m)$ rectangular matrix. This structure has two degrees of freedom since there are n^2 overall transfer functional elements, t_{ij} 's, to realize and $(mn + n^2)$ compensator functional elements, h_{ij} 's and g_{ij} 's, to specify. Other possible two-degree-of-freedom configurations are shown in figure 5. It has been shown that, with regard to meeting most design objectives, no other two-degree-of-freedom configuration possesses any more potentiality than the structure of figure 4.⁵

A format for a means of representing the multivariable system and a general configuration of compensators to control the multivariable plant have now been established. A method for determining the values of the compensator matrices which will force the system to meet a given set of specifications must now be developed.

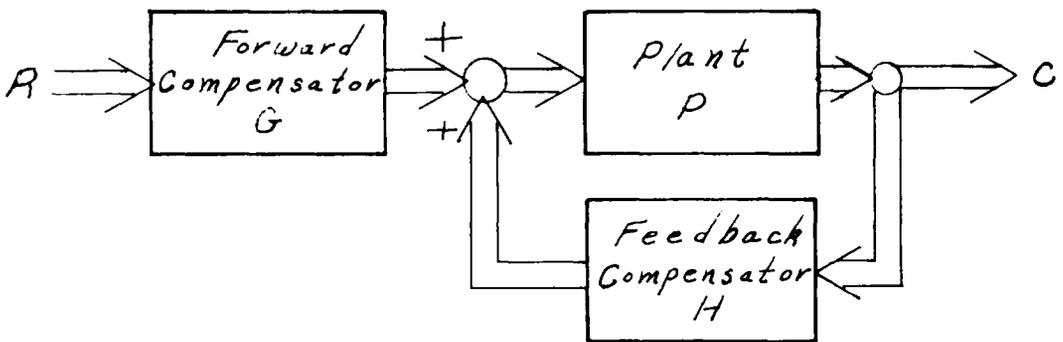


Fig. 4--General Two-Degree-of-Freedom Configuration

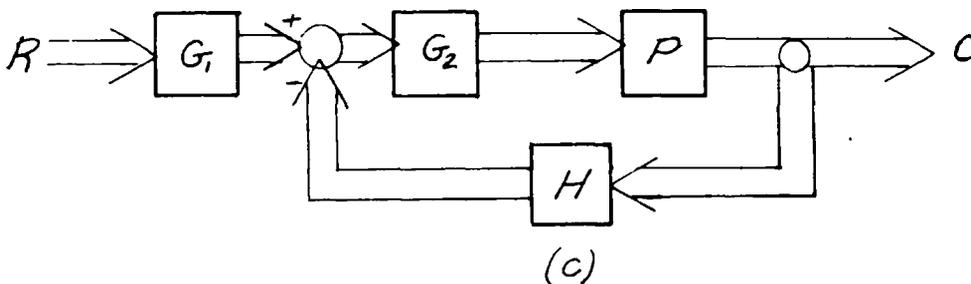
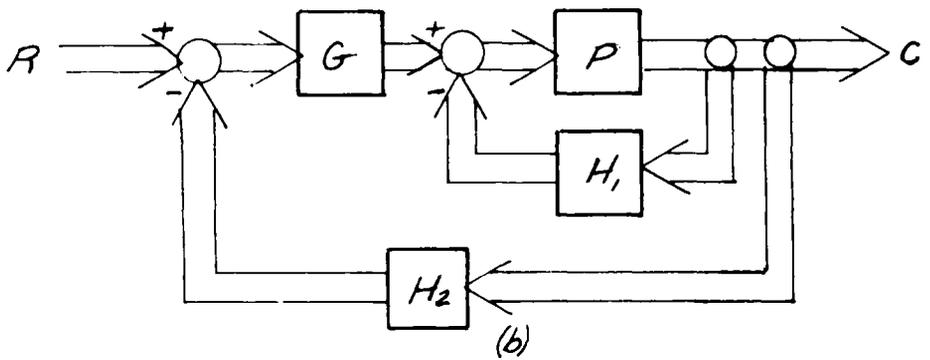
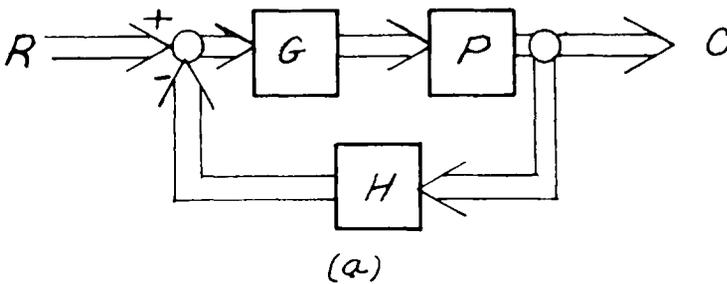


Fig. 5.--Other Two-Degree-of-Freedom Configurations

CHAPTER 3

MULTIVARIABLE SYNTHESIS THEORY

The problem to be approached in this chapter may be stated in the following manner. Given in the problem is the transfer relationships between inputs and outputs of the plant. These transfer relationships can be represented by the matrix P as described in the last chapter. The forward and feedback compensators must then be determined which will yield the desired overall transfer functions. The t_{ij} 's must be chosen such that the specifications on the desired overall transfer function are met. Limiting the poles of the t_{ij} 's to the left-half plane insures stability of the final system.

One approach to the problem is illustrated by figure 3 in chapter 2. A forward controller, given by equation (2.6), is cascaded with the plant to obtain the desired transfer characteristics. However, as stated in the previous chapter, the absence of feedback places severe tolerance limitations upon the plant and allows no opportunity to compensate for plant parameter variations.

In the synthesis of single-variable control systems, a great many problems can be solved from an analysis point of view, supplemented with a cut and try procedure. A method of multivariable synthesis has been suggested by Fuch and Povejsil⁶ which is carried out in this manner. This approach utilizes feedback compensators only in attempting to shape the characteristic function of the overall transfer function to equal a desired characteristic function determined

from the specifications. The method is limited in scope by the assumption that the desired transfer function may be determined from the characteristic equation only. It is also severely limited in application due to the complexity arising from the large number of variables involved.

A more direct synthesis approach proves to be more rewarding than the techniques mentioned above. Two similar methods have been developed by Kavanaugh³ and Freeman⁷ which utilize the matrix notation and general controller configuration described in chapter 2. The following describes the approach developed by Kavanaugh. Freeman followed the same approach with the addition of partitioning in the matrices.

It will initially be assumed that all specifications imposed upon the design are incorporated in the desired overall transfer matrix, T. Using the matrix notation and general two-degree-of-freedom configuration developed in chapter 2, it is now necessary to obtain an expression which will allow determination of values for the controller matrices, G and H, such that the plant transfer matrix, P, is compensated to give a desired overall system transfer function of T.

For the general configuration of figure 4,

$$\begin{aligned}
 x_1 &= h_{11}c_1 + \dots + h_{1m}c_m + g_{11}r_1 + \dots + g_{1n}r_n \\
 x_2 &= h_{21}c_1 + \dots + h_{2m}c_m + g_{21}r_1 + \dots + g_{2n}r_n \\
 &\vdots \\
 x_n &= h_{n1}c_1 + \dots + h_{nm}c_m + g_{n1}r_1 + \dots + g_{nn}r_n
 \end{aligned} \tag{3.1}$$

Expressed in matrix form equation (3.1) is given by

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} h_{11} & \dots & h_{1m} \\ \vdots & & \\ h_{n1} & & h_{nm} \end{bmatrix} \times \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} + \begin{bmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \\ g_{n1} & & g_{nn} \end{bmatrix} \times \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \quad (3.2)$$

or

$$X = HC + GR.$$

With the aid of equations developed in chapter 2 and equation (3.2), the matrix T may be determined.

$$C = PX$$

$$C = P(HC + GR)$$

$$C = PHC + PGR$$

$$(I - PH)C = PGR$$

$$C = (I - PH)^{-1}PGR \quad (3.3)$$

From equation (2.5), $C = TR$. Therefore, the overall transfer function may be expressed by

$$T = (I - PH)^{-1}PG, \quad (3.4)$$

where (I) represents an identity matrix. Equation (3.4) is an analysis equation. Given P, H, and G, the overall transfer matrix, T, may be determined. However, in its present form equation (3.4) is not amenable to synthesis. Given the values of P and T, the values of G and H which will satisfy equation (3.4) are not at all evident. Multiplying both sides of equation (3.4) by (I - PH) and rearranging gives

$$T = PG + PHT. \quad (3.5)$$

Equation (3.5) is more adaptable to synthesis than was equation (3.4). Equation (3.5) represents a set of $(m \times n)$ simultaneous equations with $(m \times n) + (n^2)$ unknown variables. One is therefore allowed a choice of n^2 variables of G and H . The $(m \times n)$ simultaneous equations may then be solved for $(m \times n)$ unknowns.

When the arbitrary elements may be chosen such that one complete controller is specified, the solution of simultaneous equations may be simplified by means of matrix algebra. For the present, it is assumed that P and T are square. Solving equation (3.5) for G and H gives

$$G = P^{-1}T - HT \quad (3.6)$$

$$H = P^{-1} - GT^{-1} \quad (3.7)$$

The use of these equations may best be illustrated by an example.

Figure 6 shows an electro-mechanical plant with inputs of voltages e_1 and e_2 applied to coils 1 and 2, and outputs of coil core positions x_1 and x_2 . The plant may be mathematically represented by the following differential equations:

$$\begin{aligned} e_1 &= i_1 r_1 + L_1 \dot{i}_1 + K_0 \dot{x}_1 \\ e_2 &= i_2 r_2 + L_2 \dot{i}_2 + K_0 \dot{x}_2 \\ F_{c1} &= K_L i_1 \\ F_{c2} &= K_L i_2 \\ F_{c1} &= M_1 \ddot{x}_1 + (B_1 + B_2) \dot{x}_1 + (K_1 + K_2) x_1 - B_2 \dot{x}_2 - K_2 x_2 \\ F_{c2} &= -B_2 \dot{x}_1 - K_2 x_1 + M_2 \ddot{x}_2 + B_2 \dot{x}_2 + K_2 x_2 \end{aligned} \quad (3.8)$$

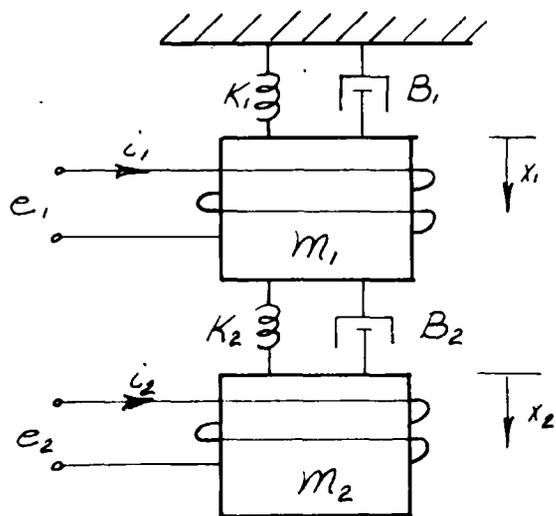


Fig. 6.--Multivariable Electro-Mechanical Plant

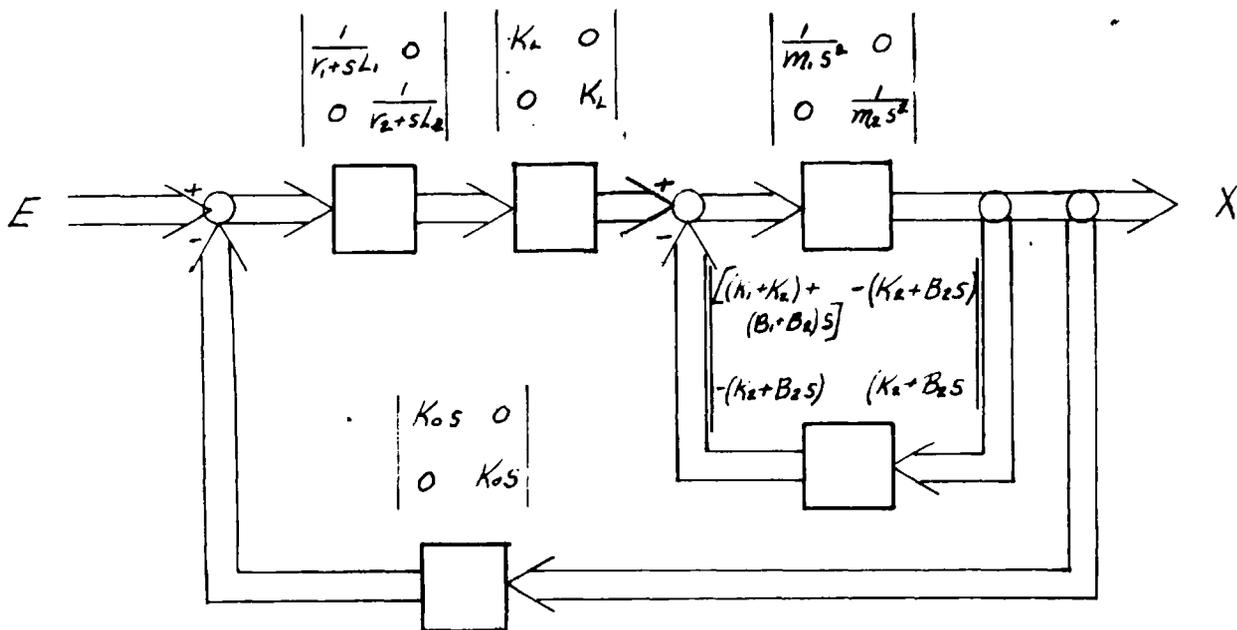


Fig. 7.--Block Diagram of Multivariable Electro-Mechanical Plant

where

K_1, K_2 = spring constants

B_1, B_2 = damping factors

M_1, M_2 = coil core masses

L_1, L_2 = inductance of coils

r_1, r_2 = resistance of coils

F_{c1}, F_{c2} = forces applied by coils to cores.

This set of differential equations is represented in block diagram form in figure 7. The overall transfer function from input E to output X is equal to the plant transfer function, P.

It may be seen from equations (3.6) and (3.7) that it is not necessary to determine the value of P for the solution of G and H. Only the inverse need be determined. Without the use of a computer program the inverse of a large matrix is often difficult to determine. A method for determining the inverse of a matrix from its block diagram without solving for the matrix itself is outlined in Appendix I. This method is utilized here to determine P^{-1} from the block diagram of figure 7. For purposes of illustration, assume the following parameter values:

$$M_1 = M_2 = 1,$$

$$K_1 = 4, K_2 = 2,$$

$$B_1 = B_2 = 2,$$

$$L_1 = L_2 = 1,$$

$$r_1 = r_2 = 3,$$

$$K_o = 1, K_L = 5.$$

By the procedure outlined in Appendix I, the block diagram of P^{-1} , as shown in figure 8, may be determined.

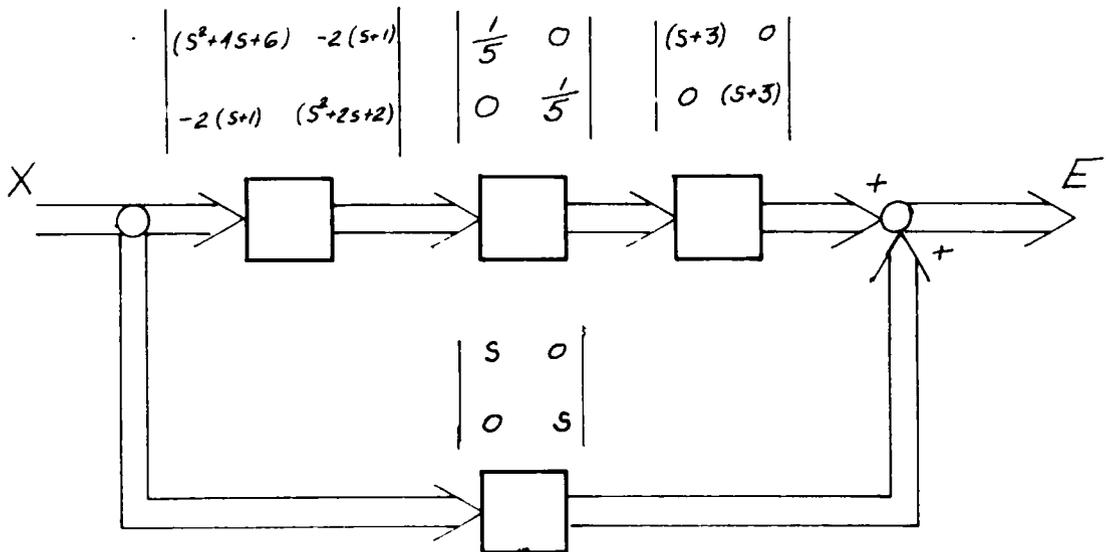


Fig. 8.--Block Diagram of Electro-Mechanical Plant Inverse

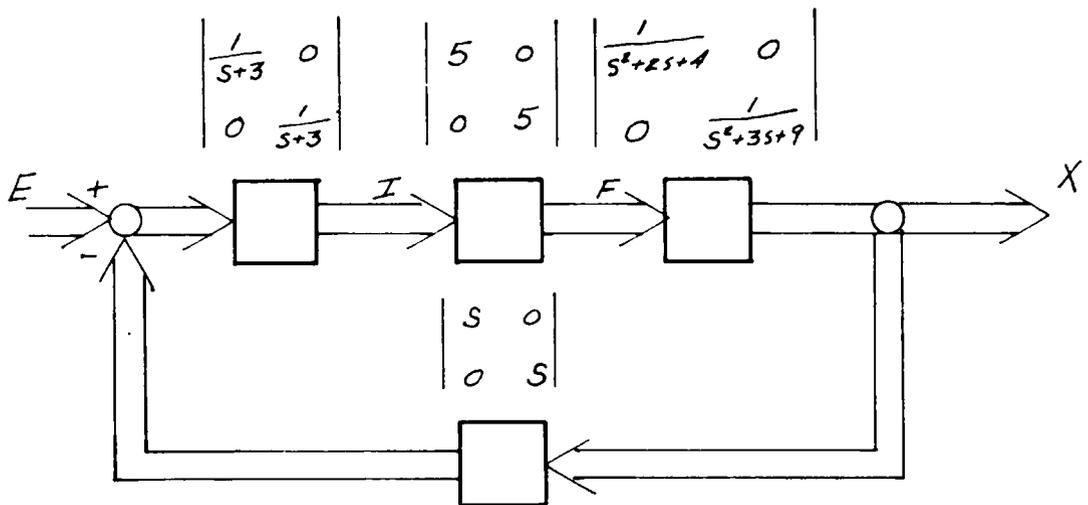


Fig. 9.--Desired Transfer Function Block Diagram

$$P^{-1} = 1/5 \begin{vmatrix} (s^3 + 7s^2 + 23s + 8) & -2(s^2 + 4s + 3) \\ -2(s^2 + 4s + 3) & (s^3 + 5s^2 + 13s + 6) \end{vmatrix} \quad (3.9)$$

Assume it is desired the response between the displacement, x_1 , and the force exerted upon the mass, M_1 , by the first coil be characterized by a quadratic with a damping factor of 0.5 and a natural frequency of 2. Similarly, the transfer between x_2 and F_{c2} is characterized by a quadratic with damping factor of 0.5 and natural frequency of 3. It is also desired that M_2 not be effected by forces acting upon M_1 and M_1 not effected by forces acting upon M_2 . A block diagram of the overall desired transfer function is shown in figure 9. The desired overall transfer function is given in matrix form by

$$T = \begin{vmatrix} 5/(s^3 + 5s^2 + 15s + 12) & 0 \\ 0 & 5/(s^3 + 6s^2 + 23s + 27) \end{vmatrix} \quad (3.10)$$

In order to make use of equations (3.6) and (3.7), the element values of either the forward or feedback controller must be specified. This constitutes a choice of n^2 element values. For a first case let the forward controller G equal an identity matrix;

$$G = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \quad (3.11)$$

The value of H may now be determined from equation (3.7).

$$H = P^{-1} - T^{-1} \quad (3.12)$$

$$H = \begin{vmatrix} 2(s^2 + 4s + 3)/5 & -2(s^2 + 4s + 3)/5 \\ -2(s^2 + 4s + 3)/5 & -(s^2 + 10s + 2)/5 \end{vmatrix} \quad (3.13)$$

Physically, each term of the H matrix represents a combination of position, rate, and acceleration feedback from outputs to inputs.

As a second case only unity position feedback will be used and the values of a voltage transfer network in cascade with the plant will be determined. The network, represented in the block diagram by the forward controller, G, will be determined such that the desired overall transfer function, T, will remain the same as the previous case. For the value of

$$H = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad (3.14)$$

equation (3.6) yields

$$G = (P^{-1} - H)T \quad (3.15)$$

$$G = \begin{vmatrix} \frac{s^3 + 7s^2 + 23s + 12}{s^3 + 5s^2 + 15s + 12} & \frac{-2(s^2 + 4s + 3)}{s^3 + 6s^2 + 23s + 27} \\ \frac{-2(s^2 + 4s + 3)}{s^3 + 5s^2 + 15s + 12} & \frac{s^3 + 5s^2 + 13s + 1}{s^3 + 6s^2 + 23s + 27} \end{vmatrix} \quad (3.16)$$

In the physical model inputs to G are voltages applied to the overall system and outputs are the voltages applied to the coils.

In the previous problem the choice of n^2 element values was arbitrary. This is not always the case. The choice of element values may be prescribed by the specifications, constrained by the nature of the system, or may be completely arbitrary. The following example

illustrates this fact. Given a multivariable plant represented by the matrix

$$P = \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix} \quad (3.17)$$

with two outputs, two desired inputs, and one disturbance input. Input 1 will be termed the disturbance input. Equation (3.17) could represent the electro-mechanical system of figure 6 where outputs are displacement values x_1 and x_2 , inputs 2 and 3 are the voltages applied to the coils, and input 1 is an undesirable force acting upon one of the masses. In this example it is desired to have the overall transfer function, t_{ij} , equal the plant transfer functions for the desired input-output transfer functions and attenuate the overall disturbance transfer function by a factor of k . Therefore the overall transfer matrix, T , is given by

$$T = \begin{vmatrix} p_{12}/k & p_{12} & p_{13} \\ p_{21}/k & p_{22} & p_{23} \end{vmatrix} \quad (3.18)$$

Equation (3.4) may now be expressed as

$$\begin{vmatrix} p_{11}/k & p_{12} & p_{13} \\ p_{11}/k & p_{12} & p_{13} \end{vmatrix} = \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{11} & p_{12} & p_{12} \end{vmatrix} \times \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \quad (3.19)$$

$$+ \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix} \times \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{vmatrix} \times \begin{vmatrix} p_{11}/k & p_{12} & p_{13} \\ p_{21}/k & p_{22} & p_{23} \end{vmatrix}$$

Equation (3.19) represents six simultaneous equations with fifteen unknown variables. Therefore, nine variables must be specified to obtain a unique solution. Because input 1 is a disturbance input, it may not be operated upon and therefore imposes the following natural constraints upon the design.

$$\begin{aligned}
 \varepsilon_{11} &= 1 & \varepsilon_{12} &= 0 \\
 \varepsilon_{21} &= 0 & \varepsilon_{13} &= 0 \\
 \varepsilon_{31} &= 0 & h_{11} &= 0 \\
 h_{12} &= 0
 \end{aligned} \tag{3.20}$$

This leaves two element values to be arbitrarily determined. Letting ε_{23} and ε_{32} equal 0 diagonalizes the G matrix. Equation (3.19) now reduces to

$$\begin{aligned}
 \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{vmatrix} &= \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix} x \begin{vmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{23} \end{vmatrix} \\
 + \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix} x \begin{vmatrix} 0 & 0 \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{vmatrix} x \begin{vmatrix} p_{11}/k & p_{12} & p_{13} \\ p_{21}/k & p_{22} & p_{23} \end{vmatrix} & \tag{3.21}
 \end{aligned}$$

Equations (3.21) represent six simultaneous equations with six unknowns. Solution of these equations, one of which is represented below, yields the necessary values of the compensator elements.

$$t_{11} = p_{11} + (p_{12}h_{21} + p_{13}h_{31})p_{11}/k + (p_{12}h_{22} + p_{13}h_{32})p_{21}/k \tag{3.22}$$

In appendix II, it is shown that equation (3.22) may also be determined

by the methods of multivariable signal flow diagraming.

As was noted in the first example of this chapter, when the arbitrary elements may be chosen such that one complete controller is specified, the solution of simultaneous equations may be simplified by means of matrix algebra. As an example, assume that in the last example all three inputs are actual inputs and no disturbance signals enter the system. All other aspects of the problem being the same, the synthesis equation is represented by equation (3.19) with none of the fifteen unknown variables being specified. Let us assume for an arbitrary choice of variables that $G = I$. Equation (3.19) with this substitution would then represent six simultaneous equations with six unknowns. The solution would follow the form of the previous example. If the P and T matrices are square, H may be determined directly from equation (3.7). To solve explicitly for the value of H, it is necessary that the inverses of P and T exist. The fact that the plant P has more inputs than outputs, causing it to exhibit a non-square matrix, proves to be a serious deterrent to the solution of the multivariable synthesis problem.

A rectangular P matrix may be supplemented to obtain a square matrix in the following manner. By specifying $(n - m)$ virtual outputs for the plant, $(n - m)$ rows are added to the plant matrix P. This addition squares the matrix. The values of the added plant elements are specified such that a 1 transfer function exists between the extra inputs and the virtual outputs and a 0 transfer function exists between the essential inputs and the virtual outputs. If P is squared, T must also be squared. The added elements, $t_{(m+1)(m+1)} \dots t_{nn}$, represent the

desired overall transfer function between the overall system inputs, $r_{m+1} \dots r_n$, and the plant inputs, $x_{m+1} \dots x_n$. Adding a row to square the matrices of equations (3.17) and (3.18) gives

$$P' = \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{vmatrix}, \quad (3.23)$$

$$T' = \begin{vmatrix} p_{11}/k & p_{12} & p_{13} \\ p_{21}/k & p_{22} & p_{23} \\ 0 & 0 & 1 \end{vmatrix}. \quad (3.24)$$

The primes indicate the matrices have been supplemented to make them square. The value of $t'_{33} = 1$ chosen in equation (3.24) indicates it is desired that the value of the third plant input be equal to the value of the third system input. From equation (3.7) and the choice of $G = I$, H is given by

$$H = (P'^{-1} - T'^{-1}). \quad (3.25)$$

T' of equation (3.24) can also be represented by

$$T' = \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1/k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (3.26)$$

The inverse of the T' matrix may now be determined as a function of the inverse P' matrix.

$$T^{-1} = \begin{vmatrix} 1/k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}^{-1} P^{-1} \quad (3.27)$$

From equation (3.25) and (3.27), the values for the elements of H are given by

$$H = \frac{1}{P_{11}P_{22} - P_{12}P_{21}} \begin{vmatrix} p_{22}(1-k) & -p_{12}(1-k) & (p_{12}p_{23} - p_{13}p_{22})(1-k) \\ -p_{21} & p_{11} & (p_{21}p_{13} - p_{11}p_{23}) \\ 0 & 0 & (p_{11}p_{22} - p_{21}p_{22}) \end{vmatrix} \quad (3.28)$$

This method of supplementing a rectangular matrix will prove to be of further value when the constraint of noninteraction is imposed upon the synthesis.

CHAPTER 4

NONINTERACTION

A constraint imposed in multivariable control synthesis which proves to have both practical importance and a simplifying effect upon the design is noninteraction.⁸ Noninteraction specifies that each system input controls only one system output and that output is controlled by the one input only. The cross-coupling terms of the plant are canceled out by compensation. This allows any output quantity to be adjusted independently of the other output quantities. It will be shown that constraint of noninteraction upon the problem reduces the multivariable synthesis to a number of single-variable synthesis problems.⁹

The problem in chapter 3 involving the two coils (figure 6) was actually a noninteraction problem. It was specified that in the compensated system forces acting upon each mass were not to effect the position of the other mass. Since the applied voltages, inputs to the system, are directly proportional to the forces applied to the masses, the overall transfer from inputs, e , to outputs, x , is characterized by noninteraction. The following simplified example will further illustrate the use of noninteraction.

A plant P is given whose transfer matrix is represented by

$$P = \begin{vmatrix} 1/(s + 1) & 2/(s + 5) & 1/(s + 2) \\ 3/(s + 1) & 8/(s + 5) & 1/(s + 2) \end{vmatrix}. \quad (4.1)$$

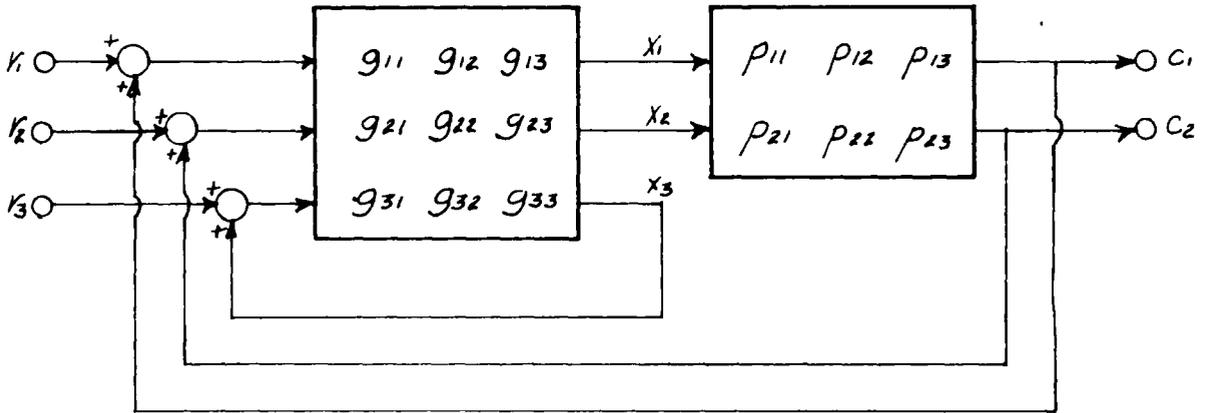
The plant represented by equation (4.1) has three inputs and two outputs. For noninteraction to exist, the T matrix must be diagonal. This, in turn, requires that the matrix be square. If $n > m$, the P and T matrices may be supplemented as previously described in chapter 3 to yield square ($n \times n$) matrices. The $(n - m)$ plant inputs are then controlled independently by $(n - m)$ system inputs. The values of these additional inputs might be chosen to maintain a desirable operating region. In figure 10, it may be observed that treating the added row to the plant matrix as describing an actual plant output is an artifice to satisfy the matrix algebra. In the figure, H has been set equal to the negative of the forward controller matrix G, giving negative feedback around the complete system. In the actual structure, the third plant input, x_3 , will be controlled by the third system input, r_3 , if noninteraction is specified.

The plant represented by equation (4.1) has three inputs and two outputs. Squaring the P matrix yields

$$P' = \begin{vmatrix} 1/(s+1) & 2/(s+5) & 1/(s+2) \\ 3/(s+1) & 8/(s+5) & 1/(s+2) \\ 0 & 0 & 1 \end{vmatrix} \quad (4.2)$$

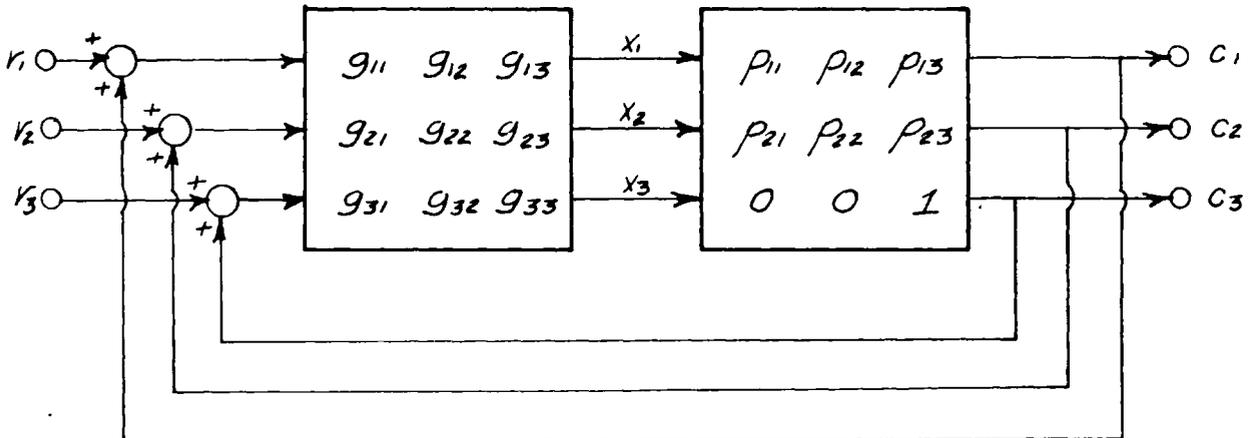
The desired overall system transfer function, representing a noninteraction system, is given as

$$T = \begin{vmatrix} 1/(s+1) & 0 & 0 \\ 0 & 1/(s+5) & 0 \\ 0 & 0 & 1/(s+1) \end{vmatrix} \quad (4.3)$$



(a)

Fig. 10a.--Actual Structure of Compensated System with Supplemented Plant Matrix



(b)

Fig. 10b.--Artificial Structure of Compensated System with Supplemented Plant Matrix

Note that t_{33} states the desired relationship between the plant input, x_3 , and the system input r_3 . From equation (3.5), restated here, it may be seen that a choice of variables exists. As an arbitrary choice, let $H = -G$.

$$T = PG + PHT \quad (3.5)$$

Solving equation (3.5) for G with the chosen value of H gives

$$G = (P^{-1}T)(I - T)^{-1} \quad (4.4)$$

The plant inverse is found to be

$$P^{-1} = \begin{vmatrix} 4(s+1) & -(s+1) & -3(s+1)/(s+2) \\ -3(s+5)/2 & (s+5)/2 & (s+5)/(s+2) \\ 0 & 0 & 1 \end{vmatrix} \quad (4.5)$$

Substitution of equations (4.3) and (4.5) into equation (4.4) gives

$$G = \begin{vmatrix} 4(s+1)/s & -(s+1)/(s+4) & -3(s+1)/s(s+2) \\ -3(s+5)/2s & (s+5)/2(s+4) & (s+5)/s(s+2) \\ 0 & 0 & 1/s \end{vmatrix} \quad (4.6)$$

With the proper choice of variables, the specification of noninteraction upon the control problem can provide opportunities to relate familiar single-variable design techniques to the multivariable design problem. The general structure of figure 4 with the choice of variables $H = -G$ reduces to a plant with forward controller inside the loop and negative unity feedback from system output to system input as shown by figure 10. The system transfer function for this

$$T = (I + PG)^{-1}PG \quad (4.7)$$

Letting the open loop transfer function equal D, equation (4.7) becomes

$$T = (I + D)^{-1}D, \quad (4.8)$$

where

$$D = PG. \quad (4.9)$$

In order for the T matrix to be diagonal, D must also be diagonal.

The diagonal elements of the T matrix are then

$$t_{ii} = d_{ii}/(1 + d_{ii}). \quad (4.10)$$

The multivariable problem is now reduced to a number of individual single-variable problems. The diagonal elements of the open-loop transfer matrix, D, may now be determined such that the overall transfer functions will meet required specifications. Solving for the elements of the compensator from equation (4.8) gives

$$G = P^{-1}D. \quad (4.11)$$

The electro-mechanical system of chapter 3 may again be used to illustrate this method. Allowing the feedback matrix of the general configuration (figure 4) to equal (-G) places a negative unity feedback loop around the complete system. A physical interpretation of this configuration is given by figure 11 and a block diagram, by figure 12. Generally, the desired open-loop transfer function is known and the problem is to compensate the plant to give this desired open-loop transfer function. D may also be determined from T by equation (4.12).

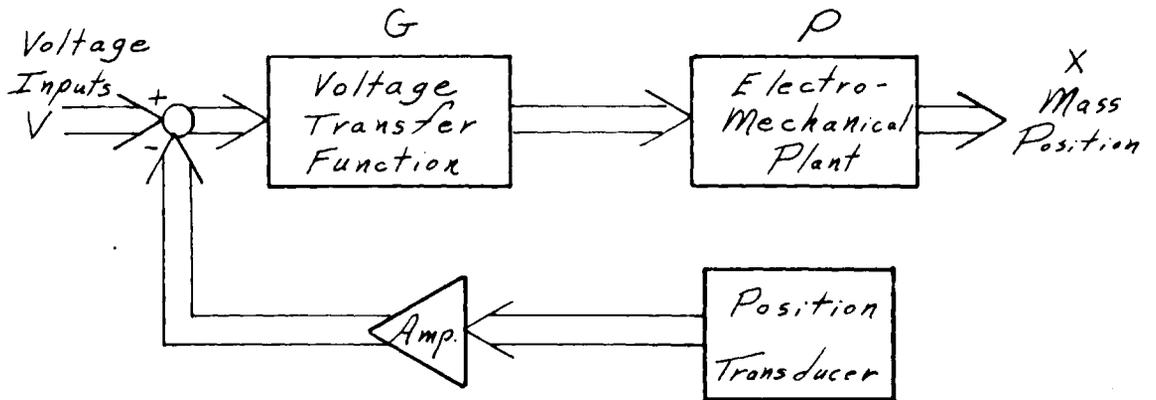


Fig. 11.--Physical System, Electro-Mechanical Plant with Unity Feedback

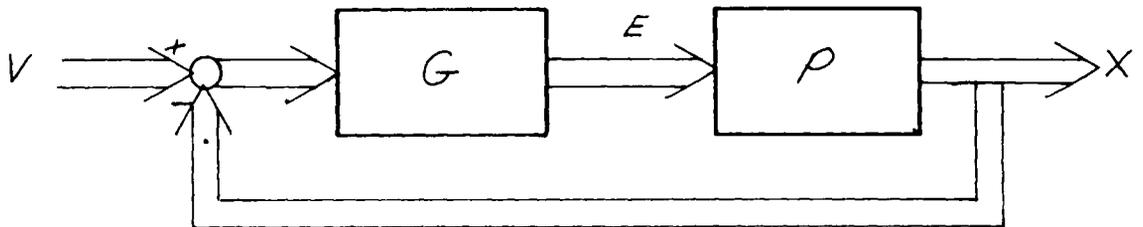


Fig. 12.--Block Diagram, Electro-Mechanical Plant with Unity Feedback

$$D = (I - T)^{-1}T \quad (4.12)$$

When T is diagonal,

$$d_{ii} = t_{ii}/(1 - t_{ii}). \quad (4.13)$$

For this example D is found from equations (3.10) and (4.13) to equal

$$D = \begin{vmatrix} 1/(s^3 + 5s^2 + 15s + 7) & 0 \\ 0 & 1/(s^3 + 6s^2 + 23s + 22) \end{vmatrix} \quad (4.14)$$

Substituting equations (3.9) and (4.14) into equation (4.11) yields

$$G = \begin{vmatrix} \frac{5(s^3 + 7s^2 + 23s + 8)}{(s^3 + 5s^2 + 15s + 7)} & \frac{-10(s^2 + 4s + 3)}{(s^3 + 6s^2 + 23s + 22)} \\ \frac{-10(s^2 + 4s + 3)}{(s^3 + 5s^2 + 15s + 7)} & \frac{5(s^3 + 5s^2 + 13s + 6)}{(s^3 + 6s^2 + 23s + 22)} \end{vmatrix}. \quad (4.15)$$

The example represented by equations (4.1), (4.2), and (4.3) could have also been solved in this manner. From equations (3.10) and (4.13), D is given by

$$D = \begin{vmatrix} 1/s & 0 & 0 \\ 0 & 1/(s + 4) & 0 \\ 0 & 0 & 1/s \end{vmatrix} \quad (4.16)$$

G may then be determined from equations (4.5), (4.11), and (4.16).

This value of G is the same as that given by equation (4.6).

As previously mentioned, the constraint of noninteraction has simplified the multivariable problem to a number of single-variable

problems. This is particularly advantageous to the last synthesis method discussed since design of each input-output transfer pair may be determined from its open-loop transfer function, d_{ii} . The stability and transfer characteristics of each transfer pair may be specified by the proper choice of d_{ii} . This selection may be determined by any of the basic methods presented in single-variable control theory. Another advantage of individual-loop design is that the synthesis of each overall transfer function is not obscured by the mathematics. This is particularly beneficial when several loops of an existing system must be modified.

CHAPTER 5

PLANT PARAMETER VARIATION

In chapter 3 it was pointed out that without any form of feedback, variations in the overall transfer function, T , are proportional to variations in the plant parameters, P . Feedback around the plant has the effect of reducing the change in the values of the overall transmission elements due to a variation in the values of the plant parameters. To determine a synthesis procedure which will take into effect plant parameter variation, one must first develop a mathematical relationship between the loop gain and plant parameter variation effects.¹⁰

In a general form the overall transmission between system input and output may be expressed by

$$T = (I + L)^{-1}PG, \quad (5.1)$$

where P is the plant matrix; G , the forward controller matrix, and L , the negative of the loop transmission matrix. This is a more general equation than equation (3.4) in which case L equals $(-PH)$ for the configuration of figure 4. L represents the negative loop gain matrix for any two-degree-of-freedom configuration. A physical significance may be gained for L by the following procedure. In figure 13 the feedback loops of the (2×2) plant are opened just before the output. Each element of the matrices is represented by flow paths. With both feedback paths opened, a (-1) signal is injected at c_1' . The signals

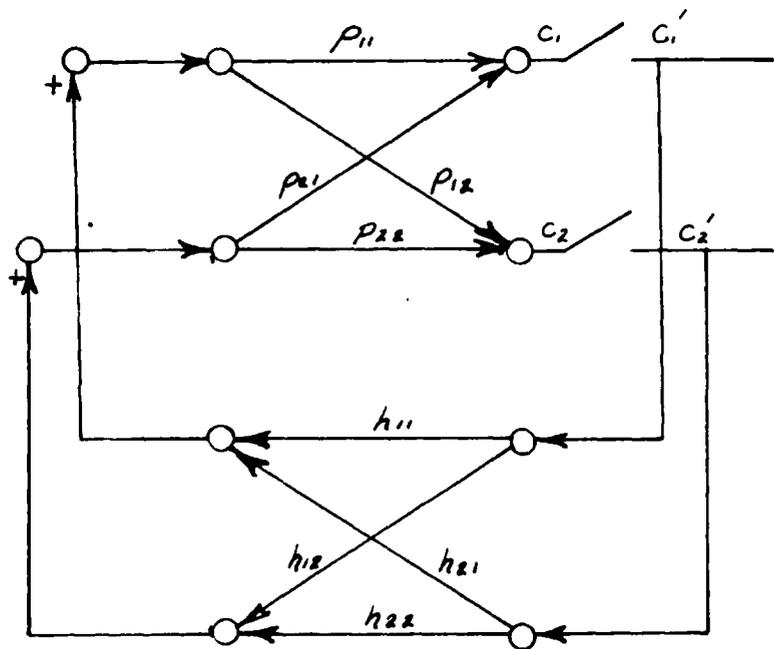


Fig. 13.--Loop Flow Diagram

then appearing at c_1 and c_2 represent l_{11} and l_{21} . Similarly the values of l_{11} and l_{21} appear at c_1 and c_2 when a (-1) signal is injected at c_2' . In this case the loop transmission matrix is represented by

$$L = \begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} . \quad (5.2)$$

The matrix may also be represented by

$$L = PM, \quad (5.3)$$

where M is the negative of the feedback transfer matrix between plant output, C , and plant input, X .

In the following equations, the primed quantities represent the value of the matrices after the plant parameters have varied from their nominal values; thus

$$P' = P + \Delta P \quad T' = T + \Delta T \quad L' = P'M \quad (5.4)$$

To control the effects of plant parameter variation, it is necessary to develop an expression which will allow the designer to choose values of loop gain which will maintain the values of T within their specified tolerances. From equation (5.1), it is determined that

$$T^{-1} = G^{-1}P^{-1}(I + L) \quad (5.5)$$

$$T' = (I + L')^{-1}P'G \quad (5.6)$$

$$T'^{-1} = G^{-1}P'^{-1}(I + L') \quad (5.7)$$

From equations (5.7), (5.5), and (5.3), it may be shown that

$$L' = P'M \quad (5.8)$$

$$\begin{aligned} T'^{-1} - T^{-1} &= G^{-1}P'^{-1}(I + P'M) - G^{-1}P^{-1}(I + PM), \\ T'^{-1} - T^{-1} &= G^{-1}(P'^{-1} - P^{-1}) \quad . \end{aligned} \quad (5.9)$$

Premultiplying (5.9) by T and post-multiplying the resulting product by $-T^{-1}$ yields

$$\mathcal{J}T = T' - T = TG^{-1}(P^{-1} - P'^{-1}) \quad . \quad (5.10)$$

Rearranging (5.1) shows that

$$TG^{-1} = (I + L)^{-1}P \quad . \quad (5.11)$$

Substituting (5.11) into (5.10) gives

$$\mathcal{J}T = (I + L)^{-1}(I - PP'^{-1})T' \quad . \quad (5.12)$$

Equation (5.12) relates the overall system variation to the loop gains, plant parameter variations, and the resulting overall transmission function when the plant has varied from its nominal value. Note that the value of the forward control matrix, G , has no effect upon $\mathcal{J}T$.

Since T' is unknown, it would seem advantageous to substitute $T' = T + \mathcal{J}T$, group terms and solve for $\mathcal{J}T$ in terms of L , P , P' , and T . The product of this procedure is given by equation (5.13).

$$\mathcal{J}T = (L + PP'^{-1})^{-1}(I - PP'^{-1})T \quad (5.13)$$

The variation of the overall transfer function is now expressed as a function of T . Solving equation (5.13) explicitly for L gives

$$L = T\mathcal{J}T^{-1} - PP'^{-1}\{T(\mathcal{J}T)^{-1} + I\} \quad . \quad (5.14)$$

In order to obtain the necessary values for the loop gains, the elements of the P' matrix must be varied independently within their limits to maximize the right hand side of each equation represented by equation (5.14). This approach to the problem encounters difficulty at this point due to the large number of variables involved in the maximization. For anything but the simplest of cases, the maximum values of the loop gains are obscured by the mathematics.

As was the case previously, the use of a diagonal matrix will aid in simplifying the complexity of the problem considerably. First, specify the loop transfer matrix, L , to be diagonal. This does not require the overall system transfer to be noninteracting since no restraint is placed upon the value of G . A further simplification of the problem may be gained by reconsidering equation (5.12). If only small variations in T are allowed, T' in equation (5.12) may be replaced by T .

$$\mathcal{J}T = (I + L)^{-1}(I - PP'^{-1})T . \quad (5.15)$$

Note that in equation (5.15) the varying (PP'^{-1}) terms are not added to the unknown L terms, as is the case in equation (5.13) due to the inversion of $(L + PP'^{-1})$. Equation (5.15) may also be derived from equation (5.13) by means of the following reasoning. If variations are small, $P \cong P'$ and

$$PP'^{-1} \cong I. \quad (5.16)$$

Substitution of equation (5.16) into the first term of the right-hand side of equation (5.13) yields equation (5.15).

Values for the loop gains necessary to hold the effects of parameter variation within given limits may now be determined from equation (5.15). This is illustrated in the following example.

Given the plant

$$P = \begin{vmatrix} k_{11}/(s+1) & k_{12}/(s+1) \\ k_{21}/(s+1) & k_{22}/(s+1) \end{vmatrix} \quad (5.17)$$

where the plant parameter k_{11} varies from 2 to 5; k_{12} , from 0.5 to 1; k_{21} , from 0.25 to 0.5; and k_{22} , from 1 to 3; determine the values of loop gain which will set the desired overall transfer elements equal to 1, allowing no greater than a 10% variation. The nominal values of k will be considered the normal state of the plant.

$$P = \begin{vmatrix} 2/(s+1) & 0.5/(s+1) \\ 0.25/(s+1) & 1/(s+1) \end{vmatrix} \quad (5.18)$$

$$P' = \begin{vmatrix} k_{11}'/(s+1) & k_{12}'/(s+1) \\ k_{21}'/(s+1) & k_{22}'/(s+1) \end{vmatrix} \quad (5.19)$$

Values for the loop gains may now be determined from equation (5.15).

$$P'^{-1} = \frac{(s+1) \begin{vmatrix} k_{22}' & -k_{12}' \\ -k_{21}' & k_{11}' \end{vmatrix}}{k_{11}'k_{22}' - k_{12}'k_{21}'} \quad (5.20)$$

$$\Delta t_{11} = \Delta t_{12} = \left[1 + \frac{0.5k_{21}' + 2k_{12}' - 0.5k_{11}' - 2k_{22}'}{k_{11}'k_{22}' - k_{12}'k_{21}'} \right] / 1 + l_{11} \quad (5.21)$$

$$\mathcal{S}t_{21} = \mathcal{S}t_{22} = \left[1 + \frac{k_{21}' + 0.25k_{12}' - 0.25k_{22}' - k_{11}'}{k_{11}'k_{22}' - k_{12}'k_{21}'} \right] / (1 + l_{22}) \quad (5.22)$$

By cut and try, the loop gains may be determined for the values of k which maximize the right-hand side of equation (5.21) and (5.22). It is found that the overall transmission tolerance limits are met if the value of l_{11} is greater than 4 and l_{22} , greater than 5.4.

The elements of P in equation (5.17) were chosen with a common denominator for the sake of simplifying the problem. When more realistic problems are investigated, a great deal of cut and try must be used to determine the values of k which will give the maximum deviations.

When the diagonal matrix is used, elements in the x th row of $(I - PP'^{-1})T$ are divided by $(1 + l_{xx})$. In the example, this was no problem since the same amount of loop gain was desired for each element. Suppose, however, that the specified values for the first column of the

T matrix must be less than 0.1 and the elements of the second column must be less than 0.2. Since l_{11} controls the elements of the first row and l_{22} , the elements of the second row, their values must be set to satisfy the most restrictive condition which, in this case, is the 10% tolerance specification. The question might be asked of what advantage a non-diagonal matrix for L would be in this case. Considering a general (2×2) case as in the example, it is found that the non-diagonal case does not necessarily lead to lower loop gains.

In equation (5.15), let

$$V \triangleq (I - PP^{-1})T. \quad (5.23)$$

For the (2 x 2) case, equation (5.15) then becomes

$$\mathcal{S}T = \frac{\begin{vmatrix} v_{11}(1 + l_{22}) - v_{21}l_{12} & v_{12}(1 + l_{22}) - v_{22}l_{12} \\ v_{21}(1 + l_{11}) - v_{11}l_{21} & v_{22}(1 + l_{11}) - v_{12}l_{21} \end{vmatrix}}{(1 + l_{11})(1 + l_{22}) - l_{12}l_{21}} \quad (5.24)$$

Considering a specific element of the $\mathcal{S}T$ matrix of equation (5.23),

$$\mathcal{S}t_{11} = \frac{v_{11}(1 + l_{22}) - v_{21}l_{12}}{(1 + l_{11})(1 + l_{22}) - l_{12}l_{21}} \quad (5.25)$$

it may be seen that the value of the added elements of the L matrix lies in their ability to maintain at least a partial cancellation of terms in the numerator, minimizing the overall transfer variation. However, the elements of the V matrix are independently varying since they are functions of the elements of the P matrix which are variable quantities. Partial cancellation may only be accomplished when the variances of V are small and known within specified limits.

In the previous chapters, the stability of the overall transfer matrix has been insured by properly specifying the value of T. However, variations in plant parameters could cause instability in the overall transfer function. A means of insuring stability in the synthesis procedure is not apparent if plant parameter variation is to be taken into effect. However, a check of the stability for a system may be determined by solving equation (5.12) for T'. Substituting $(T' - T) = \mathcal{S}T$

into equation (5.12) and grouping terms yields

$$T' = (L + PP'^{-1})^{-1}(I + L)T . \quad (5.26)$$

Any right-half plane poles of equation (5.26) must appear as zeros of the determinant of the matrix $(L + PP'^{-1})$.

This treatment of the plant parameter variation problem can not be compared with any other since this reference¹⁰ was the only one found by the author which treats the subject. The approach seems cumbersome in that the value of the loop gain must first be determined and then, using this value of L , the plant must be further compensated to yield the desired overall transfer function, T . This could be accomplished by means of two loops if some over-design is acceptable.

In the previous example illustrated in figures 6 and 7, assume that the masses are varying quantities. If the spring constants and damping factors may be set at such a value that effects of plant parameter variation may be lessened to the desired value, forward and feedback controllers may then be used as previously discussed to achieve the wanted value of T .

CHAPTER 6

SUMMARY AND SUGGESTIONS FOR FURTHER STUDY

Much of multivariable design technology has developed along lines parallel to the well-developed single-variable design theory. It is found that, due to the large number of variables involved, multivariable design is best handled by a synthesis approach rather than a cut and try analysis technique. The use of matrix notation proves beneficial both in abbreviating the notation and in offering the designer computational aids by use of matrix theorems. Though matrix notation is disadvantageous in that the synthesis of each individual loop is obscured by the mathematics, it seems to be the only systematic way to handle the large number of parameters involved.

In multivariable synthesis, a choice of values for compensator elements is often left to the discretion of the designer. These choices are, at times, diminished by natural constraints but are often left available to the designer. The selection of these element values by applying a constraint of noninteraction can lessen the complexity of the problem by a considerable amount.

A need exists for greater knowledge of the effects of interaction upon the overall signal transmission. What effect would considering only the diagonal elements of a non-diagonal plant matrix have upon the overall transmission? This would be particularly advantageous for the open-loop design procedure discussed in chapter 4.

The method presented in chapter 5 to consider the effects of plant parameter variations is mathematically cumbersome. The more pliable synthesis equation involves an approximation that the overall transmission matrix remains approximately the same before and after variations in the plant parameters. The determination of values of compensation to diminish the effects of plant parameter variations to a desired degree and of values to give the desired overall transmission function are two indirectly related steps.

From a review of the literature, it may be observed that developments in the field of multivariable control have been cyclic. After Boksenbour and Hood's work in the late 1940's, little was done until 1956. For several years a number of articles appeared on the subject. Another lull appeared which was again broken by recent publications in the past two years. This new interest was brought on by new "fuel for the fire" supplied by recent advances in single-variable control technology. It is in these areas that the most opportunity for further study lies. Publications relating to optimization of multivariable systems are presently appearing at upward from five a month in the journals. Sample data multivariable techniques have also appeared in recent publications.

APPENDIX I

DETERMINATION OF MATRIX INVERSE BY SIGNAL FLOW DIAGRAMING

A majority of the techniques for multivariable synthesis require the inversion of the plant matrix, P . If the matrix P is given, a prescribed set of steps laid down by the laws of matrix algebra may be followed to determine the P^{-1} matrix. However, in a practical problem this approach may not be feasible. In initially determining the plant transfer function, the mathematical relationships between the various plant parameters must be found. In the general case, transfer functions for portions of the plant are first determined and then incorporated into the overall plant transfer function. However, it may be advantageous to work with the plant diagram in a form which retains the identity of its internal parameters when some specification is dependant upon the value of these parameters. It will be shown that the inverse of the plant may be determined by signal-flow diagraming techniques while retaining the identity of the plant's internal nodes.¹¹

A presentation of signal flow diagraming techniques may be found in both references (12) and (13). A knowledge of these techniques is assumed in the following discussion.

The adaptation of single-variable flow graph techniques to multivariable systems is a simple one. Transfer branch functions are replaced by transfer matrix branch functions and nodal values are replaced by a column matrix of values. As is the case for single-variable flow graphs, the value of a node is determined only by the branches

entering the node, and not by those leaving the node.

Determination of the plant inverse by means of signal flow diagramming may best be illustrated by an example. The flow graph shown in figure 14 represents a plant with two outputs, y_1 and y_2 ; two inputs, x_1 and x_2 ; and internal parameters, r and s . The letters, a through k , represent linear functions of the complex frequency variable. In figure 15, the flow branches are divided into groups, each of which may be represented by a (2×2) matrix. This grouping could have been done in a number of ways. A practical choice for grouping might be influenced by a condition where certain portions of the plant are subjected to different environmental conditions than others. In this manner parameter variations due to temperature might be handled more easily. These groupings are represented by the following matrices:

$$\begin{aligned}
 A &= \begin{vmatrix} a & c \\ 0 & b \end{vmatrix} & B &= \begin{vmatrix} d & f \\ g & e \end{vmatrix} \\
 C &= \begin{vmatrix} h & 0 \\ 0 & i \end{vmatrix} & D &= \begin{vmatrix} 0 & 0 \\ 0 & j \end{vmatrix} \\
 E &= \begin{vmatrix} k & 0 \\ 0 & 0 \end{vmatrix}
 \end{aligned} \tag{I.1}$$

The A, B, C matrices are of rank 2, whereas the D and E matrices are of rank 1. No inverse exists for the D and E matrices. Nodes are represented by

$$W = \begin{vmatrix} r \\ s \end{vmatrix} \quad X = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \quad Y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \tag{I.2}$$

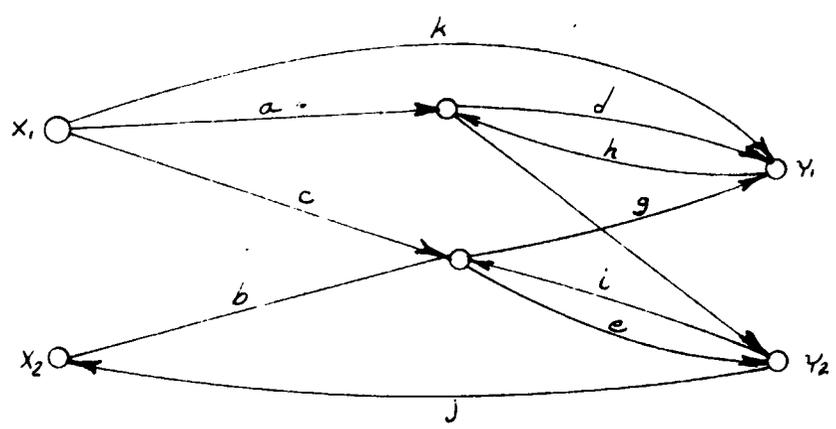


Fig. 14.--Plant Signal Flow Diagram

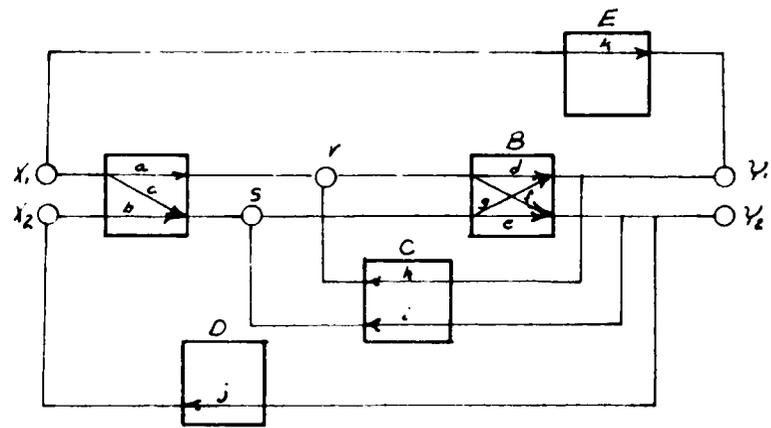


Fig. 15.--Grouped Plant Signal Flow Diagram

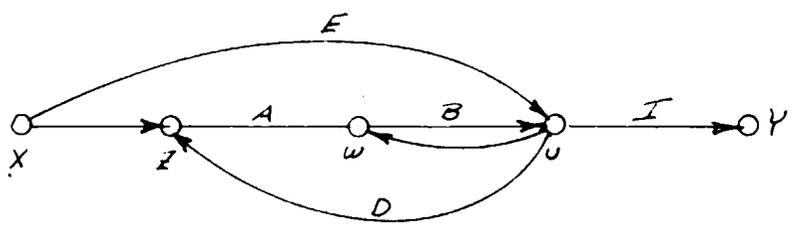


Fig. 16.--Reduced Plant Signal Flow Diagram

The creation of the U and Z nodes is due to a rule of signal flow diagramming which states that no branch may enter an input node or leave an output node.

The primary structuring technique for the generation of P inverse from a P signal flow graph will be the branch-preserving inversion. Inverting the plant in the example will make the x 's a function of the y 's. The rules for branch-preserving inversion will now be stated. Choose a path from input to output. Along this path, termed the inversion path, reverse the direction of flow so that flow is from output to input. Now invert each transfer function along this path. Divide each node along the inversion path into a source node and a sink node connected by an identity matrix. The original node is preserved as the source node. This will be shown in the example to be necessary for proper signal routing. Each branch that enters the inversion path at any node along the inversion path, including the input and output, has its sign changed but remains uninverted. The transfer function from Y to X represented by the flow diagram in this form is equal to P^{-1} .

In the flow diagram of figure 16, it would seem that two possible inversion paths are possible; one through A and B, and the other through E. E is ruled out, however, since it is a singular matrix and its inverse does not exist. The first step of reversing the flow and inverting the functions is shown by figure 17. The z , u , and w along the inversion path are separated into source and sink nodes, and the signs are reversed on branch E, C, and D entering the inversion path. The primed nodes are the sink nodes in figure 18. It the values of the

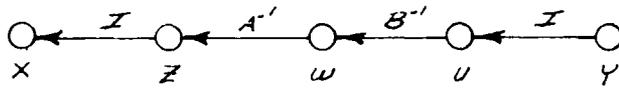


Fig. 17.--Inversion Path

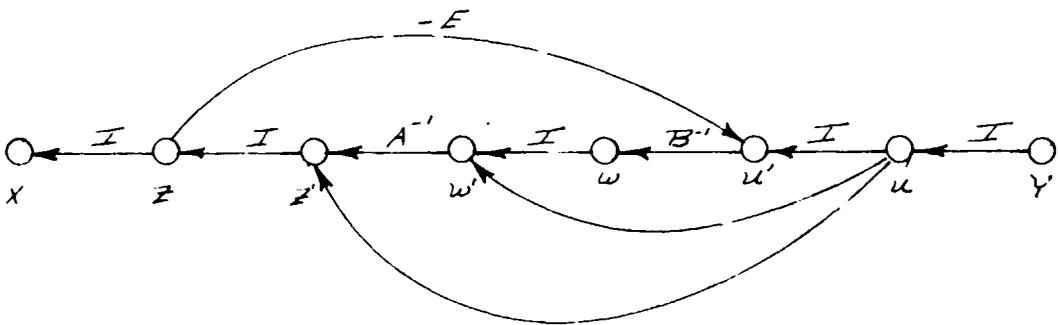


Fig. 18.--Inverse Plant Signal Flow Diagram

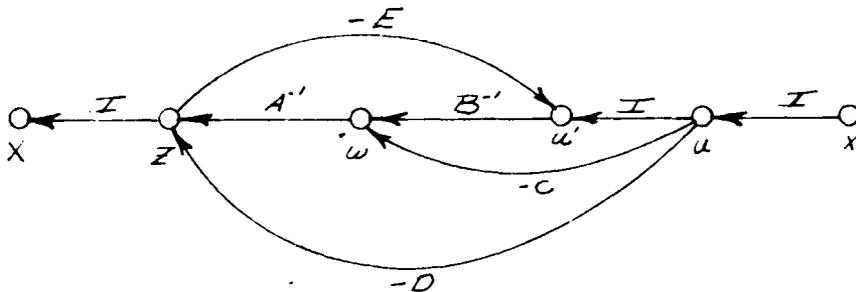


Fig. 19.--Reduced Inverse Plant Signal Flow Diagram

source and sink nodes are now equal, they may be reunited into one node. This would mean that all the signal appearing at one also appears at the other. In figure 18 it may be seen that the node pairs z, z' and w, w' are equal. The node u, u' are not equal and can not be combined.

Figure 19 represents the inverse flow graph. From this the plant inverse may be determined by the following steps:

$$X = -DY + A^{-1}W$$

$$W = -CY + B^{-1}U'$$

$$U' = Y - EZ$$

$$X = Z = -DU + A^{-1}(-CY + B^{-1}Y - B^{-1}EZ)$$

$$X = (I + A^{-1}B^{-1}E)^{-1}(A^{-1}B^{-1} - A^{-1}C - D)Y \quad (I.3)$$

The value found by writing the overall transfer function for the flow graph of figure 16 and inverting is the same as that of equation (I.3)

APPENDIX II

MULTIVARIABLE SIGNAL FLOW DIAGRAMING

Theory of signal flow diagraming states that the overall transfer function of the flow graph shown in figure 20 equals the open loop transfer function divided by one minus the sum of the loop transfer functions.

$$A = B / (1 - \sum l_i) = B / (1 - W) \quad (\text{II.1})$$

A represents the overall transfer function; B, the open loop transfer function; l_i , the loop transfer functions; and W, the sum of the loop transfer functions.

For the example in chapter 3, the t_{11} element of the desired overall transfer function T is given by equation (3.22) as

$$t_{11} = p_{11}/k = p_{11} + (p_{12}h_{21} + p_{13}h_{31})p_{11}/k + (p_{12}h_{22} + p_{13}h_{32})p_{21}/k \quad (3.22)$$

By means of a flow graph and a simple substitution into equation (3.22), it will be seen how the mathematics of chapter 3 has produced a set of feedback elements which compensate the original plant to give the desired overall transfer function. It may be noted that the p_{11}/k term of equation (3.22) and the $B/(1 - W)$ term of equation (II.1) are similar in form. Substituting $(1 - W)$ for k into equation (3.22) and solving for W yields

$$W = h_{11}p_{12} + h_{31}p_{13} + p_{21}h_{22}(p_{12}/p_{11}) + p_{21}h_{32}(p_{13}/p_{11}) . \quad (\text{II.2})$$

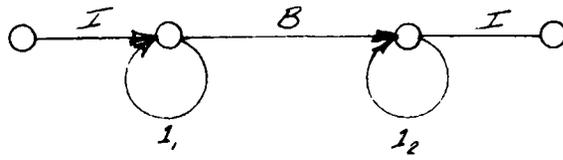


Fig. 20.--Simple Signal Flow Diagram

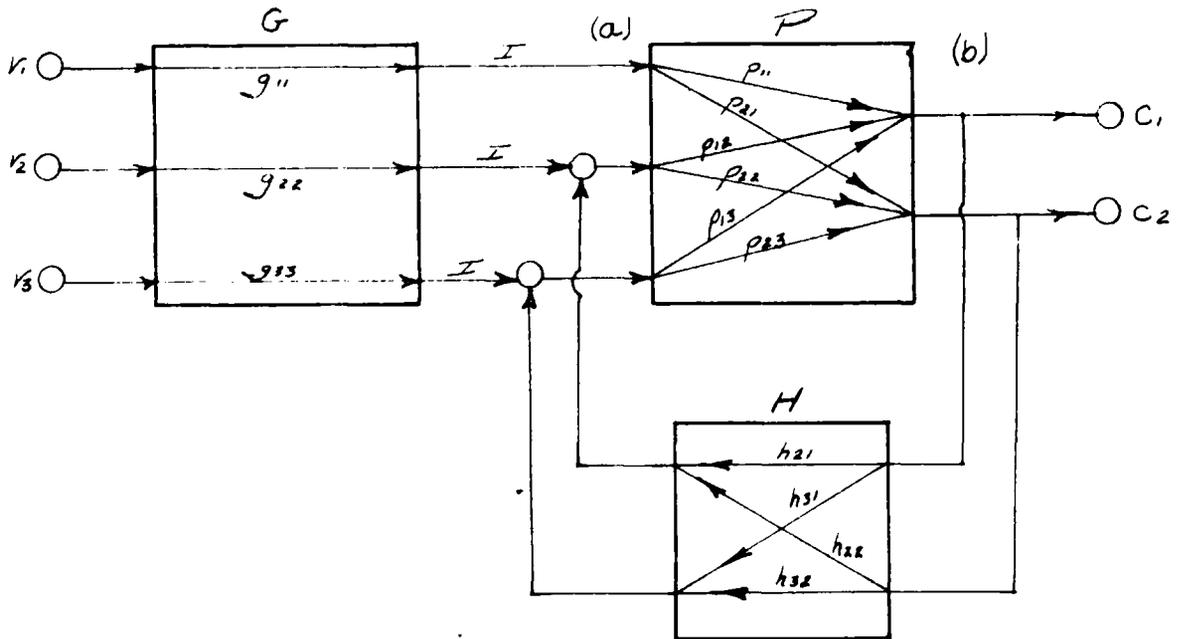


Fig. 21.--Overall System Signal Flow Diagram

According to signal flow diagramming theory, each term of equation (II.2) must represent a loop closing on the path given by p_{11} . The signal flow diagram for equation (3.21) is shown in figure (21). It may be seen that the overall transfer function between output, c_1 , and input, r_1 , is equal to the open loop transfer function, p_{11} , with a series of four loops closing on it in a similar manner to figure 20. Loops $h_{21}p_{12}$ and $h_{31}p_{13}$ close at point (b) and loops $p_{21}h_{22}(p_{12}/p_{11})$ and $p_{21}h_{32}(p_{13}/p_{11})$ close at point (a). These values are the same as those derived from equation (3.22) and given in equation (II.2).

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