SOME CLASSES OF ARTIFICIAL SATELLITE ORBITS

by

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ABSTRACT

Because of the influence of small perturbing forces, the general orbit of an artificial earth satellite is not the Kepler ellipse. However, two special classes of closed orbits are found to exist: (1) circular orbits of zero inclination, and (2) polar elliptical orbits with their centers at the center of mass of the earth.

The class of circular orbits is examined in a non-rotating spherical polar coordinate system. Under the assumption of an axially symmetric gravitational potential, these orbits are shown to lie in a plane south of the equatorial plane and are also shown to be stable.

The class of elliptical polar orbits is studied in an elliptic-hyperbolic coordinate system, and the Lagrangian formulation is employed in setting up the equations of motion. The earth's potential is approximated by that of a uniform ellipsoid of revolution. A necessary condition for the existence of the closed elliptical orbit is found, and an expression relating time to position of the satellite is derived. Also, expressions are derived giving the period of the satellite and the velocity and geocentric distance in terms of latitude.

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CHAPTER I

INTRODUCTION

Although celestial mechanics is one of the oldest areas of science to be treated successfully by the application of the laws of physics, the recent successful launching of artificial earth satellites has caused a revival of interest in this subject and has created many new problems. Aside from purely academic interest, much of the work which has been done in this field has obvious scientific and technological applications. As a result, many papers have been written in recent years dealing with the problems concerned with the motion of orbiting artificial satellites.

If the earth were perfectly spherical and the only force on a satellite was a central gravitational force, the orbit of the satellite would be the Kepler ellipse; and the usual statements of Kepler's laws would apply to the motion of the satellite. The ellipse would lie in a plane passing through the earth's center with the center of the earth at one focus of the ellipse, and the plane of the orbit would remain fixed in space.

However, in the actual case of an orbit about the earth, there are several small additional forces acting on the satellite. These perturbing forces result in non-central force motion and cause the orbit of the satellite to deviate from the Kepler ellipse; and, hence, the satellite does not return to the same point in space on each successive revolution about the earth. The largest of these additional forces is that produced by the earth's equatorial bulge. For orbits lying close to the earth, atmospheric drag forces are also important. Other smaller forces on the satellite are caused by the asymmetrical mass distribution about the equatorial plane (resulting in the so-called "pear shape" of the earth), longitudinal variations in the earth's gravitational field, gravitational attractions of other celestial bodies, electromagnetic forces, and solar radiation pressure.

However, subject to certain assumptions, two special classes of closed orbits do exist, and these will be investigated in detail in this thesis. These two classes consist of circular orbits lying in a plane parallel to that of the equator and closed elliptical orbits lying in a plane containing the earth's polar axis. The only forces taken into account here will be those caused by the earth's gravitational field.

CHAPTER II

DISCUSSION OF THE GENERAL ORBIT ABOUT THE EARTH

Before considering the special classes of satellite orbits in detail, a brief discussion of satellite motion in general is in order.

The general solution of Laplace's equation for the gravitational potential will yield an expression involving an infinite series of spherical harmonic terms. For axial symmetry about the polar axis, and for the origin of a spherical-polar coordinate system at the earth's center of mass, the gravitational potential at any point exterior to the earth's surface may be written[1]

$$V = -(\mu/r)\left[1 - \sum_{n=2}^{\infty} J_n (R/r)^n P_n(\cos\Theta)\right] . \qquad (1)$$

In this expression, r is the geocentric distance, Θ is the polar angle measured from the north polar axis (i.e., the co-latitude), $\mu = GM$ is the product of the Newtonian gravitational constant and the mass of the earth, R is the mean equatorial radius of the earth, P_n is the Legendre coefficient of degree n, and the J_n are constants characteristic of the earth's mass distribution. Determination of these parameters by Kozai[2] from the study of satellite motion gives

$$J_2 = (1082.48 \pm 0.04) \times 10^{-6},$$

$$J_3 = (-2.566 \pm 0.012) \times 10^{-6},$$

$$J_4 = (-1.84 \pm 0.09) \times 10^{-6},$$

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[1] J. M. A. Danby, Fundamentals of Celestial Mechanics (New York: Macmillan Company, 1962), p. 111. Danby's J_n contain the factor R^n.

[2] Maurice Roy (ed.), Dynamics of Satellites (New York: Academic Press Incorporated, 1963), p. 71.

$$J_5 = (-0.063 \pm 0.019) \times 10^{-6},$$
$$J_6 = (0.39 \pm 0.09) \times 10^{-6},$$
$$J_7 = (-0.469 \pm 0.021) \times 10^{-6},$$
$$J_8 = (-0.02 \pm 0.07) \times 10^{-6},$$
$$J_9 = (0.114 \pm 0.025) \times 10^{-6}.$$

If there were no perturbing forces on the satellite, the size, shape, and orientation of the orbit in space could be specified in terms of a set of six elliptic orbit elements of classical celestial mechanics. (See Fig. 1). If the projection of the elliptic orbit on the celestial sphere is considered, the point on the orbit at which the satellite crosses the plane of the equator going northward is called the ascending node. Its position is specified by Ω, the right ascension of the node, where right ascension is the angular distance measured eastward along the equator from the vernal equinox.

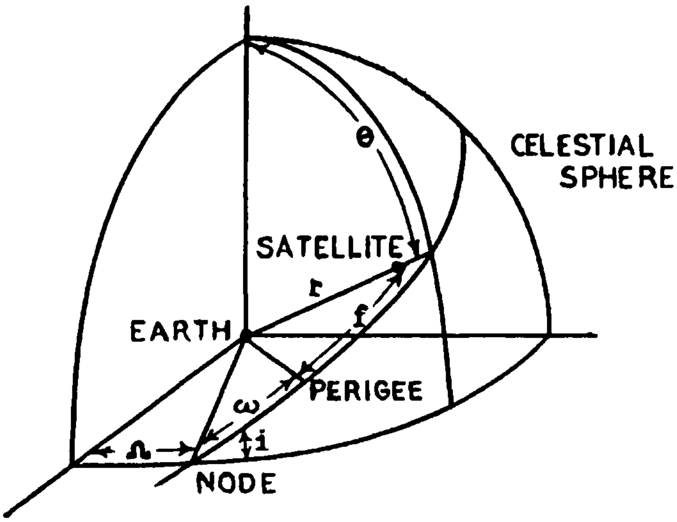

FIGURE 1

THE ORBIT IN SPACE

The plane of the orbit is specified by the inclination angle, i, where i is measured counterclockwise from the equatorial plane to the orbit plane at the ascending node. The other element needed to fix the ellipse in space is the argument of perigee, ω, which is the angular distance from the ascending node to the perigee. Here perigee is used to denote the point on the orbit at which the value of r is a minimum. The size and shape of the ellipse are determined respectively by the semi-major axis, a, and the eccentricity of the ellipse, e. The remaining element necessary to specify the motion is the time of perigee passage, t_p.

To locate the satellite in its orbit at some instant of time several additional elements are usually introduced. The first of these is the true anomaly, f, which is the angular distance from the perigee to the radius vector of the satellite. (See Fig. 2). The equation of the orbit is[3]

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \quad . \tag{2}$$

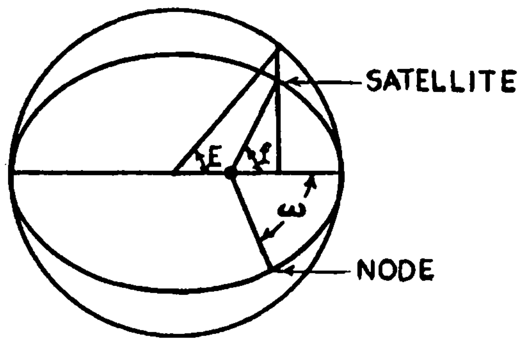

FIGURE 2

THE TRUE AND ECCENTRIC ANOMALIES

If a circle is constructed of radius a about the center of the ellipse and a line drawn through the satellite position perpendicular to the

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[3] W. M. Smart, Celestial Mechanics (London: Longmans, Green, and Company, 1953), p. 17.

major axis, then the extension of this line onto the circle forms an angle with the major axis at the center called the eccentric anomaly, E. It can be shown[4] that the relation between E and f is given by the equation

$$\tan f/2 = \sqrt{\frac{1 + e}{1 - e}} \tan E/2 \tag{3}$$

and that the distance r as a function of E is

$$r = a(1 - e \cos E). \tag{4}$$

Another element, called the mean anomaly, M, is defined as a linear function of time by the equation

$$M = n(t - t_p). \tag{5}$$

Here, n is the unperturbed mean angular velocity of the satellite and can be shown to have the value[5]

$$n = (\mu/a^3)^{1/2}, \tag{6}$$

and $(t - t_p)$ is the time elapsed since perigee passage. E and M are then related by Kepler's equation[6]

$$M = E - e \sin E.$$

Because of the small additional forces acting on the satellite, a closed elliptic orbit is no longer obtained, since the non-central forces cause the orbit elements to change in time. One method by which these changes may be calculated is by the use of Lagrange's planetary equations. These six equations are simultaneous first-order differential equations relating the rates of change of the orbit elements to the elements and to the derivatives of the disturbing function with respect to the elements. The disturbing function,

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[4] *Ibid.*, p. 18.

[5] *Ibid.*, p. 3.

[6] *Ibid.*, p. 18.

S, is the negative of the potential function, Eq. (1), including all but the first term, which is the potential of a point mass. Setting $\Lambda = -nt_p$, these six equations can be written[7]

$$\frac{da}{dt} = (2/na)\frac{\partial S}{\partial \Lambda},$$

$$\frac{de}{dt} = (1/na^2e)\left[(1-e^2)\frac{\partial S}{\partial \Lambda} - \sqrt{1-e^2}\,\frac{\partial S}{\partial \omega}\right],$$

$$\frac{d\Lambda}{dt} = -\frac{1-e^2}{na^2e}\frac{\partial S}{\partial e} - (2/na)\frac{\partial S}{\partial a}, \tag{8}$$

$$\frac{d\Omega}{dt} = (na^2\sqrt{1-e^2}\sin i)^{-1}\frac{\partial S}{\partial i},$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial S}{\partial e} - \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial S}{\partial i},$$

$$\frac{di}{dt} = (na^2\sqrt{1-e^2})^{-1}\left[\cot i\frac{\partial S}{\partial \omega} - \csc i\frac{\partial S}{\partial \Omega}\right].$$

The method of obtaining approximate solutions to these equations consists of first expressing S explicitly in terms of the desired orbit elements and then expanding S in terms of series involving powers of e and either the mean anomaly or the true anomaly. The elements a, e, and i undergo only periodic variations about their mean values a_o, e_o, and i_o, if there is no drag on the satellite. Hence, as a first approximation, these values may be used in the right-hand side of the equations. On the other hand, Ω, ω, and Λ are each the sum of a constant term, a secular term (one which varies as a linear function of time), and various periodic terms. To a first approximation, these may be represented by the sum of a constant terms plus the product of the mean rate of change of the variable times the time. The planetary equations written explicitly as a function of the true or mean anomaly can then be integrated directly to give the first-order changes in the variables over short intervals

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[7] Ibid., p. 69.

of time.

Although all of the orbit elements undergo periodic variations, the most important changes are secular perturbations resulting from the earth's oblateness. If Λ is replaced by the mean anomaly, the secular perturbations show up as terms in the equations for Ω, ω, and M. Including only the J_2 terms, these secular terms are[8]

$$\frac{d\Omega}{dt} = -\frac{3}{2} J_2 n (R/a)^2 (1 - e^2)^{-2} \cos i, \tag{9a}$$

$$\frac{d\omega}{dt} = \frac{3}{4} J_2 n (R/a)^2 (1 - e^2)^{-2} (4 - 5 \sin^2 i), \tag{9b}$$

$$\frac{dM}{dt} = n \left[1 + \frac{3}{4} J_2 (R/a)^2 \frac{(-1 + 3 \cos^2 i)}{(1 - e^2)^{3/2}}\right]. \tag{9c}$$

From Eq. (9a), it can be seen that the node regresses at a rate proportional to cos i. Therefore, for $i = 90^\circ$ (a polar orbit), the orbit plane is fixed in space, while for orbits nearly in the plane of the equator, the rate is most rapid. On the other hand, from Eq. (9b), the perigee can be seen to advance for $0^\circ \leqq i < 63.^\circ4$, to remain at a constant latitude for $i = 63.^\circ4$ ($\sin^2 i = 4/5$), and to regress for $63.^\circ4 < i \leqq 90^\circ$.

Two special cases will now be investigated in detail: circular orbits of zero inclination and closed elliptic polar orbits. It will be seen that these two cases do not require the use of the Lagrange planetary equations, but may be attacked in a more elementary manner using mathematical methods more familiar to the engineer or physicist.

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[8]Y. Kozai, "The Motion of a Close Earth Satellite," Astronomical Journal, 64:367, November, 1959.

CHAPTER III

CIRCULAR ORBITS OF ZERO INCLINATION

If the earth is assumed to be axially symmetric, the only class of circular orbits will be those of zero inclination. This is because the nearly oblate-spheroidal shape of the earth leads to non-central forces, which would cause a satellite to deviate from a circular orbit for any inclination angle other than zero. However, it will be shown that the presence of odd harmonic terms in the potential function causes the plane of the orbit to lie slightly to the south of the equator rather than in the equatorial plane itself.

The forces per unit mass on the satellite in the r-direction and in the Θ-direction can be found from the potential function, Eq. (1), and are

$$F_r = -\frac{\partial V}{\partial r} = -\frac{\mu}{r^2}\left[1 - \sum_{n=2}^{\infty} J_n(n+1)\left(\frac{R}{r}\right)^n P_n(\cos\Theta)\right], \qquad (10a)$$

$$F_\Theta = -\frac{1}{r}\frac{\partial V}{\partial \Theta} = \frac{\mu}{r^2}\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n'(\cos\Theta)\sin\Theta \quad , \qquad (10b)$$

where the prime denotes differentiation with respect to cos Θ. Now, a circular orbit will exist only if the magnitudes of the radius vector and the polar angle remain constant. To insure constant latitude, a necessary dynamical condition is that the total force on the satellite normal to the plane of the orbit is zero. Following the method of Blitzer,[9] this requirement is specified as

$$F_N = F_r \cos\Theta - F_\Theta \sin\Theta = 0 \qquad (11)$$

or

[9]Leon Blitzer, "Circular Orbits in an Axially Symmetric Gravitational Field," American Rocket Society Journal, 32:1102, July, 1962.

$$\cos\Theta\left[1-\sum_{n=2}^{\infty}J_n(n+1)\left(\frac{R}{r}\right)^n P_n(\cos\Theta)\right]+$$

$$\sin^2\Theta\sum_{n=2}^{\infty}J_n\left(\frac{R}{r}\right)^n P_n'(\cos\Theta)=0. \tag{12}$$

Because of the small magnitudes of the J_n, the co-latitude Θ will be very nearly $\pi/2$; hence, setting $\Theta = \pi/2 - \delta$, where δ is the latitude, one may write $\cos\Theta = \delta$. Then, letting r be equal to a constant value a, Eq. (12) can be written

$$\delta\left[1-\sum_{n=2}^{\infty}J_n(n+1)\left(\frac{R}{a}\right)^n P_n(\delta)\right]+\sum_{n=2}^{\infty}J_n\left(\frac{R}{a}\right)^n P_n'(\delta)=0. \tag{13}$$

By use of the expansion of the Legendre polynomial[10]

$$P_n(\delta)=\sum_{j=0}^{m}\frac{(-1)^j(2n-2j)!\,\delta^{n-2j}}{2^n j!(n-j)!(n-2j)!}\quad, \tag{14}$$

where

$$m=\begin{cases} n/2, & \text{for n even,}\\ (n-1)/2, & \text{for n odd,}\end{cases}$$

to first-order terms in δ, the $P_n(\delta)$ and $P_n'(\delta)$ may be approximated by

$$P_n(\delta)\cong\frac{(-1)^{n/2}n!}{2^n\left[(n/2)!\right]^2}\quad,\ \text{n even,} \tag{15a}$$

$$P_n(\delta)\cong\frac{(-1)^{(n-1)/2}(n+1)!\,\delta}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\ ,\ \text{n odd,} \tag{15b}$$

$$P_n'(\delta)\cong-\frac{(-1)^{n/2}\,n(n+1)!\,\delta}{2^n\left[(n/2)!\right]^2}\ ,\ \text{n even,} \tag{16a}$$

$$P_n'(\delta)\cong\frac{(-1)^{(n-1)/2}\,(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\ ,\ \text{n odd.} \tag{16b}$$

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[10] R. V. Churchill, Fourier Series and Boundary Value Problems (New York: McGraw-Hill Book Company, 1941), p. 177.

After utilizing these approximations in Eq. (13) and retaining only linear terms in δ, it is found that

$$\delta = \frac{\sum_{n\ odd} A_n J_n \left(\frac{R}{a}\right)^n}{1 - \sum_{n\ even} B_n J_n \left(\frac{R}{a}\right)^n} , \tag{17}$$

where

$$A_n = \frac{(-1)^{(n+1)/2}\,(n+1)!}{2^n \left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} ,$$

$$B_n = \frac{(-1)^{n/2}\,(n+1)(n+1)!}{2^n \left[(n/2)!\right]^2} .$$

The smallness of the J_n allows Eq. (17) to be approximated by

$$\delta \cong \sum_{n\ odd} A_n J_n \left(\frac{R}{a}\right)^n$$

$$= \frac{3}{2} J_3 \left(\frac{R}{a}\right)^3 - \frac{15}{8} J_5 \left(\frac{R}{a}\right)^5 + \frac{35}{16} J_7 \left(\frac{R}{a}\right)^7 - \ldots \quad . \tag{18}$$

Because of the dominance of the negative J_3 term over the other terms in the earth's potential function, δ must be negative; hence the plane of the orbit lies in the southern hemisphere. As examples, for $(a/R) = 1$, $\delta = -3.4 \times 10^{-6}$ radians = $-0.00020°$ (75 feet south of the equatorial plane), and for $(a/R) = 3$, $\delta = -0.096 \times 10^{-6}$ radians = $-0.000008°$ (6 feet south of the equatorial plane).

The angular velocity about the polar axis, W, may be obtained by setting the component of centripetal acceleration directed along the radius vector equal to $-F_r$:

$$aW^2 \cos^2 \delta = \frac{\mu}{a^2}\left[1 - \sum_{n=2}^{\infty} J_n (n+1) \left(\frac{R}{a}\right)^n P_n(\sin\delta)\right] , \tag{19}$$

or

$$W = \left(\frac{\mu}{a^3}\right)^{1/2} \left[1 - \sum_{n=2}^{\infty} J_n (n+1) \left(\frac{R}{a}\right)^n P_n(\delta)\right]^{1/2} . \tag{20}$$

The period is thus easily found to be

$$T = 2\pi\frac{a^{3/2}}{\mu^{1/2}}\left[1 - \sum_{n\,\text{even}} J_n\left(\frac{R}{a}\right)^n \frac{B_n}{n+1}\right]^{-1/2}$$

$$= 2\pi\frac{a^{3/2}}{\mu^{1/2}}\left[1 - \frac{3}{4}J_2\left(\frac{R}{a}\right)^2 + \ldots\right], \qquad (21)$$

where use has been made of Eq. (15) and all second-order product terms in $J_n\delta$ have been ignored. The period can be seen to be slightly less than that for an orbit at the same distance from a spherical object of the same mass as the earth, the factor being 0.9992 for $(a/R) = 1$. This can be explained by noting from Eq. (10a) that the radial force on the satellite in a near-equatorial orbit is slightly larger than it would be in an orbit about a spherical earth; hence, the satellite must move at a slightly greater speed in order to remain at the same radial distance.

These circular orbits can be shown to be stable. That is, a small disturbance of the satellite from its equilibrium orbit results only in small bounded oscillations about the circular equilibrium orbit. To demonstrate this, three difference coordinates are introduced. These are defined by

$$d = r - a,$$
$$\lambda = \theta - \theta_o, \qquad (22)$$
$$\alpha = \phi - Wt,$$

where θ_o and a are the equilibrium coordinates of the circular orbit, and W is the angular velocity when $r = a$ and $\theta = \theta_o$.

The Lagrangian function in r, θ, and ϕ is

$$L = (1/2)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r,\theta). \qquad (23)$$

The method of Goldstein[11] is followed in which the Lagrangian is
The method of Goldstein[11] is followed in which the Lagrangian is

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[11] Herbert Goldstein, Classical Mechanics (Reading: Addison-Wesley Publishing Company, 1959), pp. 318-20.

expanded in Taylor's series about the equilibrium orbit. Since $\dot{a}$ and $\dot{\theta}_o$ are zero and $\dot{\phi}$ is equal to $\dot{\alpha} + W$, Eq. (23) becomes

$$L = (1/2)\Big[\dot{d}^2 + (a+d)^2\dot{\lambda}^2 + (a+d)^2(\dot{\alpha}+W)^2(\sin^2\theta_o + \sin 2\theta_o \lambda + \cos 2\theta_o \lambda^2 + \ldots)\Big] - V(a,\theta_o) - V_1(a,\theta_o)d - V_2(a,\theta_o)\lambda - V_{12}(a,\theta_o)\lambda d - (1/2)V_{11}(a,\theta_o)d^2 - (1/2)V_{22}(a,\theta_o)\lambda^2 + \ldots \quad (24)$$

In this expression, the subscripts 1 and 2 denote the partial derivatives with respect to r and θ, respectively. Also, at equilibrium there exists the conditions

$$V_1 = aW^2\sin^2\theta_o , \quad V_2 = a^2W^2\sin \theta_o \cos \theta_o , \quad L_z = a^2W \sin^2\theta_o , \quad (25)$$

where L_z is the constant value of the angular momentum about the polar axis. Lagrange's equations of motion for the generalized coordinates q_i can be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 . \quad (26)$$

Utilizing Eqs. (24) and (25), and neglecting all but linear terms in d, α, and λ, it is found that Lagrange's equations of motion for d, ϕ, and λ are

$$\ddot{d} - 2aW \sin^2\theta_o \dot{\alpha} + (V_{12} - aW^2\sin 2\theta_o)\lambda + (V_{11} - W^2\sin^2\theta_o)d = 0, \quad (27)$$

$$\dot{\alpha} = - \frac{W \sin 2\theta_o}{\sin^2\theta_o} \lambda - \frac{2W}{a} d, \quad (28)$$

$$a^2\ddot{\lambda} - a^2W \sin 2\theta_o \dot{\alpha} + (V_{22} - a^2W^2\cos 2\theta_o)\lambda + (V_{12} - aW^2\sin 2\theta_o)d = 0. \quad (29)$$

Eq. (28) can be used to eliminate $\dot{\alpha}$ from Eqs. (27) and (29) with the result that two coupled linear differential equations are obtained. These equations can be written in the form

$$\ddot{d} + Ad + C\lambda = 0, \tag{30}$$

$$\ddot{\lambda} + B\lambda + Dd = 0, \tag{31}$$

where

$$\begin{aligned} A &= V_{11} + 3W^2\sin^2\theta_o, \\ B &= a^{-2}V_{22} + (2 + \cos 2\theta_o)W^2, \\ C &= V_{12} + aW^2\sin 2\theta_o, \\ D &= a^{-2}C. \end{aligned} \tag{32}$$

To evaluate these expressions, Eq. (20) is used to eliminate W^2 and Eq. (18) is used to eliminate δ. These expressions to order J_3 then become

$$\begin{aligned} A &= \frac{\mu}{a^3}\left[1 + \sum_{n=2}^{\infty} J_n\left(\frac{R}{a}\right)^n (n^2-1)P_n(\delta)\right] \\ &\cong \frac{\mu}{a^3}\left[1 - \frac{3}{2} J_2\left(\frac{R}{a}\right)^2\right], \\ B &= \frac{\mu}{a^3}\left[1 + \sum_{n=2}^{\infty} J_n\left(\frac{R}{a}\right)^n \left[P_n''(\delta) - P_n'(\delta)\delta - (n+1)P_n(\delta)\right]\right] \\ &\cong \frac{\mu}{a^3}\left[1 + \frac{9}{2} J_2\left(\frac{R}{a}\right)^2\right], \\ C &= \frac{\mu}{a^2}\left[2\delta + \sum_{n=2}^{\infty} J_n\left(\frac{R}{a}\right)^n \left[(n+1)\, P_n'(\delta) - 2P_n(\delta)\delta\right]\right] \\ &\cong - \frac{3\mu}{a^2} J_3\left(\frac{R}{a}\right)^3, \\ D &\cong - \frac{3\mu}{a^4} J_3\left(\frac{R}{a}\right)^3. \end{aligned} \tag{33}$$

By noting that A, B, C, and D are all positive quantities, and by setting the initial values of $\dot{d}$ and $\dot{\lambda}$ equal to zero, the solutions to Eqs. (30) and (31) can readily be obtained by the use of Laplace transforms:[12]

$$d = E \cos W_1 t + G \cos W_2 t, \tag{34}$$

$$\lambda = H \cos W_1 t + I \cos W_2 t. \tag{35}$$

The characteristic frequencies are

$$W_1 = \left[(1/2)\left[A + B - \sqrt{(A + B)^2 - 4(AB - CD)}\right]\right]^{1/2} \tag{36a}$$

$$\cong B^{1/2} \cong \left(\frac{\mu}{a^3}\right)^{1/2}\left[1 + \frac{9}{4} J_2\left(\frac{R}{a}\right)^2\right],$$

$$W_2 = \left[(1/2)\left[A + B + \sqrt{(A + B)^2 - 4(AB - CD)}\right]\right]^{1/2} \tag{36b}$$

$$\cong A^{1/2} \cong \left(\frac{\mu}{a^3}\right)^{1/2}\left[1 - \frac{3}{4} J_2\left(\frac{R}{a}\right)^2\right],$$

where the product CD is ignored because of its small magnitude in comparison with $(A - B)^2$. The remaining constants in Eqs. (34) and (35) are

$$E = \frac{W_1^2 d_o - B d_o + \lambda_o C}{W_1^2 - W_2^2} = \frac{\lambda_o C}{W_1^2 - W_2^2}, \tag{37a}$$

$$G = \frac{B d_o - C\lambda_o - W_2^2 d_o}{W_1^2 - W_2^2} = d_o - \frac{\lambda_o C}{W_1^2 - W_2^2}, \tag{37b}$$

$$H = \frac{W_1^2 \lambda_o - A\lambda_o + D d_o}{W_1^2 - W_2^2} = \lambda_o + \frac{D d_o}{W_1^2 - W_2^2}, \tag{37c}$$

$$I = \frac{A\lambda_o - D d_o - W_2^2 \lambda_o}{W_1^2 - W_2^2} = -\frac{D d_o}{W_1^2 - W_2^2}, \tag{37d}$$

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[12] N. W. McLachlan, Theory of Vibrations (New York: Dover Publications, Incorporated, 1959), p. 63.

where the subscript o denotes the initial values of λ and d. Since $W = (W_1 + W_2)/2$, Eqs. (34) and (35) can be written as

$$d = d_o \cos W_2 t + \frac{2\lambda_o J_3 R}{J_2} \sin (1/2)(W_1 - W_2)t \sin Wt, \tag{38}$$

$$\lambda = \lambda_o \cos W_1 t + \frac{2d_o J_3 R}{J_2 a^2} \sin (1/2)(W_1 - W_2)t \sin Wt. \tag{39}$$

By substituting the solutions for λ and d into Eq. (28), integrating, and ignoring very small terms, the solution for α can be found to be

$$\alpha \cong - \frac{2Wd_o}{W_2 a} \sin W_2 t. \tag{40}$$

The solutions therefore show the motion to be oscillatory with periods

$$T_1 = \frac{2\pi}{W_1} \cong 2\pi\left(\frac{a^3}{\mu}\right)^{1/2}\left[1 - \frac{9}{4} J_2\left(\frac{R}{a}\right)^2\right], \tag{41}$$

$$T_2 = \frac{2\pi}{W_2} \cong 2\pi\left(\frac{a^3}{\mu}\right)^{1/2}\left[1 + \frac{3}{4} J_2\left(\frac{R}{a}\right)^2\right]; \tag{42}$$

and, hence, the orbits are stable.

If the J_n are all set equal to zero, $W_1 = W_2 = W$, and the solutions to Eqs. (28), (30), and (31) become

$$\begin{aligned} d &= d_o \cos Wt, \\ \lambda &= \lambda_o \cos Wt, \\ \alpha &= - \frac{2d_o}{a} \sin Wt. \end{aligned} \tag{43}$$

Allowing a to become very large has the same effect on the orbit as setting the $J_n = 0$, as would be expected, since at very large distances the earth appears to the satellite to be only a point mass.

Since T_1 and T_2 both differ in magnitude from the equilibrium period of the satellite, Eq. (21), by the small quantity

$$3\pi\left(\frac{a^3}{\mu}\right)^{1/2} \cdot J_2\left(\frac{R}{a}\right)^2,$$

until the two cosine terms in Eqs. (38) and (39) become appreciably out of phase, the oscillations about the equilibrium orbit will be closely approximated by Eqs. (43), and the oscillations will be nearly in phase with the orbital motion itself.

CHAPTER IV

CLOSED POLAR ELLIPTICAL ORBITS

Since the earth's figure is very nearly that of an ellipse rotated about its minor axis, the question arises as to the possible existence of a class of closed polar elliptical orbits. A satellite in an orbit of this type would follow the path of an ellipse lying in a plane through the earth's axis and would return to the same point in space above the earth on each successive orbital revolution. It will be shown here that such orbits do exist, if the earth's potential is approximated by that of an oblate spheroid, and that their centers lie at the center of the earth. Furthermore, it will be shown that the major axis of such an orbit remains fixed in the equatorial plane.

A plane elliptic-hyperbolic coordinate system is used, since a constant value of the coordinate ξ designates a closed ellipse in this system in the same way that a constant value of r is the locus of a circle in a plane polar coordinate system. The equation of an ellipse in Cartesian coodinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{44}$$

where a is the value of the semi-major axis and b is the value of the semi-minor axis. If c is the distance from the origin to either focus, a quantity ξ can be defined by means of

$$\xi = b/c. \tag{45}$$

Therefore, in terms of ξ and c, a can be expressed as

$$a = (b^2 + c^2)^{1/2} = (\xi^2 + 1)^{1/2}c; \tag{46}$$

and Eq. (44) takes the form

$$\frac{x^2}{c^2(\xi^2+1)} + \frac{y^2}{c^2\xi^2} = 1. \tag{47}$$

For a given value of c, different values of ξ give a family of confocal ellipses.

In a similar manner, the equation of a family of confocal hyperbolas is

$$\frac{x^2}{c^2(1-\eta^2)} - \frac{y^2}{c^2\eta^2} = 1. \tag{48}$$

The foci are again at $x = \pm\, c$, and the semi-axes of the hyperbola are

$$\begin{aligned} b' &= \eta c, \\ a' &= (c^2 - b'^2)^{1/2} = (1 - \eta^2)^{1/2} c. \end{aligned} \tag{49}$$

Since the angle, ϕ, between the major axis of the hyperbola and the asymptotes of the hyperbola is found from

$$\tan \phi = \frac{b'}{a'} = \frac{\eta}{\sqrt{1-\eta^2}} = \frac{\sin \phi}{\sqrt{1-\sin^2\phi}}\,, \tag{50}$$

it can be seen that η is the sine of this angle.

For a given value of c, a point in the coordinate system used is then specified by the intersection of a hyperbola, given by the coordinate η, and an ellipse, given by ξ. (See Fig. 3). It can be seen that in two dimensions this coordinate system is not single-valued in that given values of ξ and η can denote two points on opposite sides of the y-axis. The family of ellipses can be shown to be everywhere orthogonal to the family of hyperbolas. Finally, by elimination of first y and then x between Eqs. (47) and (48), the transformation equations can be shown to be

$$x = \pm\, c\sqrt{(\xi^2+1)(1-\eta^2)}\,, \tag{51}$$

$$y = c\xi\eta. \tag{52}$$

It may be noted briefly that in three dimensions the third variable would be the azimuthal angle, thereby forming a system, a point in which is specified by the intersection of an oblate spheroid, a hyperboloid of one sheet, and a plane.

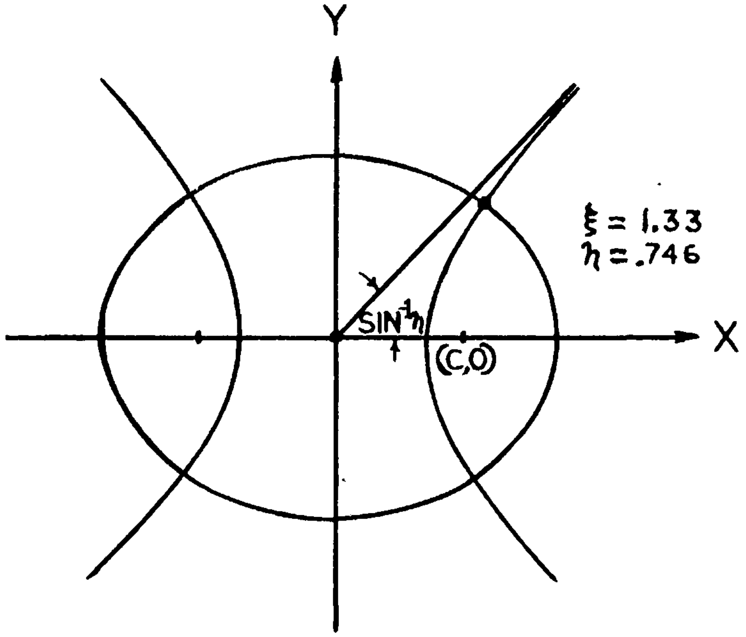

FIGURE 3

AN ELLIPTIC-HYPERBOLIC COORDINATE SYSTEM

Distance from the origin is

$$r = \sqrt{x^2 + y^2} = c\sqrt{\xi^2 + 1 - \eta^2}\ , \tag{53}$$

and the polar angle measured positively from the x-axis is

$$\delta = \sin^{-1}\left(\frac{y}{r}\right) = \sin^{-1}\frac{\xi\eta}{\sqrt{\xi^2 - \eta^2 + 1}}\ . \tag{54}$$

The differential element of length is given by

$$(ds)^2 = (dx)^2 + (dy)^2 = h_1{}^2(d\xi)^2 + h_2{}^2(d\eta)^2, \tag{55}$$

where

$$h_1{}^2 = c^2\frac{\xi^2+\eta^2}{1+\xi^2}\ ,$$

$$h_2^{\,2} = c^2\frac{\xi^2+\eta^2}{1-\eta^2} .$$

Therefore, the velocity of a satellite in terms of ξ and η will be

$$v^2 = c^2(\xi^2 + \eta^2)\left[\frac{\dot{\xi}^2}{1+\xi^2} + \frac{\dot{\eta}^2}{1-\eta^2}\right] . \tag{56}$$

In the present analysis, use is made of a potential function developed by Vinti[13] for an axially symmetric ellipsoid of revolution. For the origin at the center of mass, it is given by

$$V = - \frac{\mu}{c}\left[\frac{\xi}{\xi^2 + \eta^2}\right] . \tag{57}$$

Vinti shows that this expression can be expanded in the form

$$V = \frac{b_0 \xi}{\xi^2 + \eta^2} = b_0 c r^{-1}\Big[\, 1 - c^2 r^{-2} P_2(\sin \delta) + c^4 r^{-4} P_4(\sin \delta) - \ldots\Big] . \tag{58}$$

Comparing this expression with that for the earth's potential written in the usual form,

$$V = - \mu r^{-1}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin \delta)\right] , \tag{59}$$

where δ is now the latitude, it can be seen that

$$b_0 c = - \mu, \tag{60a}$$

$$c^2 = J_2 R^2. \tag{60b}$$

Eq. (57), therefore, contains all of the even harmonic terms of the

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[13] John P. Vinti, "New Method of Solution For Unretarded Satellite Orbits," Journal of Research of the National Bureau of Standards, 63B:105-16, October, 1959.

potential function of an oblate spheroid, where the 0th- and 2nd-order terms have been made to fit those of the earth. Using the values R = 6378.15 km. and $J_2 = 1082.48 \times 10^{-6}$, c is found to be 209.847 km. The coefficient of the fourth harmonic term of Eq. (58) then turns out to be $c^4 = J_2^2R^4 = (1.17 \times 10^{-6})R^4$, as compared with $-J_4R^4 = (1.84 \times 10^{-6})R^4$ in Eq. (59). Use of this function assumes equatorial symmetry of the earth, of course, since none of the odd terms is present.

From Eq. (47) it is clear that the condition which must be satisfied in order to have a closed elliptic orbit is that ξ equal a constant, or that $\dot{\xi}$ and $\ddot{\xi}$ are zero. It will be shown that a necessary and sufficient condition for this to be true is that

$$\dot{\eta}^2 = \left(\frac{\mu}{c^3}\right)\frac{(1-\eta^2)(\xi^2-\eta^2)}{\xi(\xi^2+\eta^2)^2} \,. \tag{61}$$

Lagrange's equation of motion for ξ is

$$\frac{d}{dt}\left[\left(\frac{\xi^2+\eta^2}{1+\xi^2}\right)\dot{\xi}\right] + \left(\frac{\mu}{c^3}\right)\frac{\xi^2-\eta^2}{(\xi^2+\eta^2)^2} - \left[\frac{\dot{\xi}^2(1-\eta^2)}{(1+\xi^2)^2} + \frac{\dot{\eta}^2}{1-\eta^2}\right]\xi = 0. \tag{62}$$

Setting $\dot{\xi} = \ddot{\xi} = 0$ in this equation and solving for $\dot{\eta}^2$ yields Eq. (61). On the other hand, to show that <u>if</u> $\dot{\eta}$ is given by Eq. (61) $\dot{\xi}$ and $\ddot{\xi}$ will always be zero, Eq. (61) can be substituted into Lagrange's equation of motion for η,

$$\frac{d}{dt}\left[\left(\frac{\xi^2+\eta^2}{1-\eta^2}\right)\dot{\eta}\right] + \left(\frac{\mu}{c^3}\right)\frac{2\eta\xi}{(\xi^2+\eta^2)^2} - \left[\frac{\dot{\eta}^2(1+\xi^2)}{(1-\eta^2)^2} + \frac{\dot{\xi}^2}{1+\xi^2}\right]\eta = 0, \tag{63}$$

which will yield

$$\left[\left(\frac{\mu}{c^3}\right)^{1/2} \frac{\xi^2 + \eta^2}{2\left[(\xi^2 - \eta^2)(1 - \eta^2)\xi\right]^{1/2}} - \frac{\eta \dot{\xi}}{1 + \xi^2}\right]\dot{\xi} = 0. \tag{64}$$

For arbitrary values of η, the above equation implies that $\dot{\xi} = 0$. Substituting $\dot{\xi} = 0$ and Eq. (61) into Eq. (62) then gives

$$\frac{\xi^2 + \eta^2}{1 + \xi^2}\ddot{\xi} = 0, \tag{65}$$

which shows that $\ddot{\xi} = 0$. Hence, once a satellite is put into an elliptical polar orbit such that the rate of change of η is given by Eq. (61), it will remain in this same orbit indefinitely, providing that the other small perturbing forces can be neglected.

In the elliptical orbit with $\dot{\xi} = 0$, the velocity of the satellite as a function of ξ and η is, from Eqs. (56) and (61),

$$v^2 = \left(\frac{\mu}{b}\right)\frac{\xi^2 - \eta^2}{\xi^2 + \eta^2}. \tag{66}$$

The total energy per unit mass of the satellite then becomes

$$E = \frac{1}{2}v^2 + V = -\frac{\mu}{2b}. \tag{67}$$

It should be noted that in this case the total energy is inversely proportional to the semi-minor axis b, while for a Kepler orbit about a spherical mass distribution the total energy is inversely proportional to the semi-major axis of the ellipse.

The expression for η as a function of time is obtained by integration of Eq. (61):

$$t - t_o = \left(\frac{\xi c^3}{\mu}\right)^{1/2}\int_{\eta_o}^{\eta}\frac{(\xi^2 + \eta^2)\, d\eta}{\sqrt{(\xi^2 - \eta^2)(1 - \eta^2)}}. \tag{68}$$

The right-hand side of this expression cannot be integrated explicitly in terms of η. However, by making the substitutions $k = 1/\xi$ and $\eta = \sin\phi$, the integral can be written in terms of elliptic integrals of the first and second kind with parameter k:

$$\int_{\eta_o}^{\eta} \frac{(\xi^2 + \eta^2)\ d\eta}{\sqrt{(\xi^2 - \eta^2)(1-\eta^2)}} \tag{69}$$

$$= \xi\left[2\int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} - \int_0^{\phi}\sqrt{1-k^2\sin^2\phi}\ d\phi\right] -$$

$$\xi\left[2\int_0^{\phi_o} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} - \int_0^{\phi_o}\sqrt{1-k^2\sin^2\phi}\ d\phi\right]$$

$$= \xi\left[\left[2F(k,\phi) - E(k,\phi)\right] - \left[2F(k,\phi_o) - E(k,\phi_o)\right]\right] .$$

Expanding the integrands by means of the binomial theorem and integrating termwise, the quantity 2F - E becomes

$$2F - E = \phi + 3k^2\left(\frac{\phi}{4} - \frac{1}{8}\sin 2\phi\right) + \tag{70}$$

$$\frac{21}{16}k^4\left(\frac{\phi}{4} - \frac{1}{8}\sin 2\phi - \frac{1}{6}\sin^3\phi\cos\phi\right) +$$

$$\frac{495}{576}k^6\left(\frac{\phi}{4} - \frac{1}{8}\sin 2\phi - \frac{1}{6}\sin^3\phi\cos\phi -\right.$$

$$\left.\frac{2}{15}\sin^5\phi\cos\phi\right) + \ldots ,$$

where $0 \leqq \phi \leqq \pi/2$. Substituting into Eq. (70) the expression for k, namely

$$k = RJ_2^{1/2}/b, \tag{71}$$

and integrating Eq. (68) over a complete cycle, it is found that the period of the satellite is

$$T = 4\,\frac{(\xi c)^{3/2}}{\mu^{1/2}}\left[2F(k,\pi/2) - E(k,\pi/2)\right] \tag{72}$$

or

$$T = 2\pi\left(\frac{b^3}{\mu}\right)^{1/2}\left[1 + \frac{3}{4} J_2\left(\frac{R}{b}\right)^2 + \frac{21}{64} J_2^{\,2}\left(\frac{R}{b}\right)^4 + \frac{495}{3304} J_2^{\,3}\left(\frac{R}{b}\right)^6 + \ldots\right] . \qquad (73)$$

The period of a Kepler orbit of the same size and shape about a spherical mass distribution equal to that of the earth can be written

$$T = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2} . \qquad (74)$$

By use of Eqs. (46) and (60b) this expression can be expanded into

$$T = 2\pi\left(\frac{b^3}{\mu}\right)^{1/2}\left[1 + \frac{3}{4} J_2\left(\frac{R}{b}\right)^2 - \frac{3}{32} J_2^{\,2}\left(\frac{R}{b}\right)^4 + \frac{5}{128} J_2^{\,3}\left(\frac{R}{b}\right)^6 - \ldots\right] , \qquad (75)$$

where the J_2R^2 in this expression is interpreted here only as the square of the distance from the center of the ellipse to either focus. Therefore, the period given by Eq. (73) is very slightly larger than that for a Kepler orbit of the same size and shape in an inverse square law force field where the force is along the radius vector.

The velocity of the satellite as a function of the latitude can be found by first solving Eq. (54) for η, which yields

$$\eta^2 = \frac{(\xi^2+1)\sin^2\delta}{\xi^2 + \sin^2\delta} , \qquad (76)$$

and then substituting this result into Eq. (66):

$$v^2 = \frac{\mu}{b}\left[\frac{b^4 - c^4\sin^2\delta}{b^4 + (2b^2c^2 + c^4)\sin^2\delta}\right]. \qquad (76)$$

This equation can then be expanded using the binomial series to obtain

$$v^2 \cong \frac{\mu}{b}\left[1 - 2\left(\frac{R}{b}\right)^2 J_2 \sin^2\delta\right], \tag{78}$$

where only terms of order J_2 have been retained. The velocity is thus at a maximum at the equator. Note also that for $J_2 = 0$ (spherical earth) this equation reduces to that for a circular orbit of radius b.

The geocentric distance in terms of the latitude can be obtained from Eqs. (52) and (76) by writing

$$r^2\sin^2\delta = y^2 = \frac{b^2(\xi^2+1)\sin^2\delta}{\xi^2 + \sin^2\delta} . \tag{79}$$

Solving for r and multiplying the numerator and denominator by c yields

$$r = \frac{ab}{\sqrt{b^2 + R^2 J_2 \sin^2\delta}} . \tag{80}$$

Although the maximum values of r occur at the equator, the height of the satellite above the earth's surface is actually less at the equator than at other points in the orbit because of the earth's oblateness.

A numerical example was calculated. For R = 6378.15 km., c = 209.847 km., and $\mu = 3.98603 \times 10^5$ km.^{3}/sec.^{2}, if ξ is set equal to 32.000, which corresponds to an orbit at a height of approximately 211.4 miles above the mean equatorial radius, the results below are found:

a = 6718.38 km.,
b = 6715.10 km.,
e = 0.0312348,
T = 5480.32 sec.
= 91.3386 min.,
v_{max} = 7.7044 km./sec.,
v_{min} = 7.6969 km./sec.

CHAPTER V

CONCLUSIONS

In Chapter II it was pointed out that any orbit within the most general class of satellite orbits about the earth has two main features: (1) as a first approximation it can be considered as a Kepler ellipse with one focus at the earth's center of mass; and (2) because of small additional forces, the largest of which is due to the oblate shape of the earth, all of the orbit elements vary by small amounts as a function of time. The most important variations of a drag-free satellite orbit were seen to be secular changes in the orbit orientation consisting of a regression of the node for all but polar orbits and a regression or advance of the perigee, depending on whether the inclination angle is greater than or less than $63\overset{\circ}{.}4$, respectively. In addition, there is a small change in the secular variation of the mean anomaly and periodic variations in all of the orbit elements.

In Chapters III and IV, however, two special classes of orbits were investigated which do not come within the general class of orbits mentioned above. These two classes consist of the circular orbits of zero inclination and the closed polar elliptical orbits with their centers at the center of mass of the earth. Neither case exhibits any secular changes nor any periodic variations in any of the equilibrium orbit elements.

It was found that one class of circular orbits which can exist about the earth are those of zero inclination, and because of the existence of the negative J_3 term in the potential function, these orbits lie in a plane slightly to the south of the equatorial plane. These orbits are similar to Kepler orbits only in the sense that the angular momentum of the satellite is conserved. This is because the earth's potential function was assumed to be axially symmetric.

It was shown that if the satellite is given a small displacement out of its circular orbit, it will oscillate about the circular orbit position. Hence, the orbit is said to be stable. To calculate the exact motion, however, the additional perturbing forces would need to be taken into consideration. For example, longitudinal variations in the potential function would cause the satellite to deviate from a perfectly circular orbit, and the gravitational attractions of the sun and moon acting nearly in the plane of the ecliptic would be expected to cause some additional effects.

A class of closed elliptical orbits in the polar plane was shown to exist if the earth's potential function is approximated by that of an oblate spheroid. One of the most interesting features of these orbits is that the center of mass of the main attracting body lies at the center of the ellipse rather than at one focus as in ordinary two-body motion. Also, even though the ellipse is closed, it is clear that the angular momentum of the satellite is not a constant because of the non-central force produced by the earth's oblateness. An additional point of interest is that the major axis of the ellipse remains constantly in the plane of the equator, unlike that of the general polar orbit. The direction of the orbit plane remains fixed in space also, of course; but this is a general property of all polar orbits.

The question of stability of the polar orbits was not studied because the equations of motion in a three-dimensional oblate-spheroidal coordinate system become extremely complicated. Another point which was not investigated is the effect that the odd J_n terms in the potential function of the actual earth would have on the closed elliptical orbits.

The above results are in agreement with the general solution of Vinti's dynamical equations obtained by Izsak[14] for orbits of very

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[14] I. Izsak, "On Satellite Orbits With Very Small Eccentricities," Astronomical Journal, 66:129, April, 1961.

small eccentricity. Setting $\rho = c\xi$, Izsak defines an eccentricity by

$$e = \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}} . \tag{81}$$

For $\rho(t)$ = constant (equivalent to making ξ a constant), $e = 0$. It should be noted that $e = 0$ does not imply a circular orbit except for the case in which the inclination angle is zero. For these special cases in which $e = 0$, Izsak finds orbits with two apogees with the maximal values of r at the ascending and descending nodes (i.e., on the equator) and two perigees with the minimal values of r at the points on the orbit which have the largest value of latitude north and south of the equator. These orbits are nearly ellipses that do not rotate in the orbit plane and whose centers are at the center of the earth. For the special case of $i = 90^o$, $r_{max} = a$ (in the direction of the line of nodes), and $r_{min} = b$.

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