A SOLUTION FOR THE GENERAL POLYNOMIAL
BY AN ANALOG TECHNIQUE

by

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A Thesis

submitted to the faculty of the

Department of Electrical Engineering

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in the Graduate College, University of Arizona

1956

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ACKNOWLEDGMENT

Grateful acknowledgment is made to Dr. Paul E. Russell for his supervision and helpful suggestions on the research leading to and the preparation of this thesis.

Graditude is also expressed for the financial assistance of the Experiment Station of the College of Engineering, University of Arizona.

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INTRODUCTION

By definition, a polynomial is a function of a complex variable that is regular throughout the complex plane and whose power series expansion requires only a finite number of terms. This type of function is widely encountered with the dominant area of application being the study of linear, lumped parameter systems. The polynomial can also be used as an approximation for more complicated functions. A particular application may require numerical evaluation of the polynomial for variations in the complex variable. This requires tedious calculations if done by longhand. In other situations it is necessary to determine values of the complex variable which make the polynomial zero. For polynomials of degree four or less, these roots can be obtained from appropriate formulas, but higher degree polynomials require the tedious techniques of numerical evaluation. It is the purpose of this thesis to describe a machine capable of reducing the labor required to solve polynomials of any degree.

The concept of machine solution of polynomials is not new. Various analog relationships have been tried with varying degrees of success.[1] Unfortunately the solution of a general polynomial by these methods is usually too slow, too inaccurate or the machine is too expensive and complicated. Digital techniques offer a fast and accurate solution but the

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1) For a discussion on some of these methods see: Walter W. Soroka, Analog Methods in Computation and Simulation; McGraw-Hill, 1954, Chapter 4.

initial cost of the machine is prohibitive. The machine described herein is analog in nature, combining relatively low cost with speed and accuracy.

The particular analog relationship used throughout the following work has been used before. The harmonic synthesizers use a special case of it.[2] The general principle has been partially applied to at least two working models.[3] Nevertheless, the full capabilities of simplicity and accuracy inherent in the basic concept have not been exploited heretofore. This presentation includes many new ideas enabling the construction of a simpler, yet more effective computer.

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2) Ibid

3) Lofgren, Lars "Analog Computer for the Roots of Algebraic Equations"; Proceedings of the IRE, July, 1953.

Chapter 1

THEORY OF OPERATION

A general polynomial, P_z, of the nth degree, can be written

$$P_z = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{1.1}$$

or

$$P_z = \sum_{k=0}^{n} a_k z^k \tag{1.2}$$

The coefficients, a_k, can assume any value, real or complex. A technique for the solution of the polynomial fixed by any set of a_k's is to be presented. The specialized solution involving only the polynomial zeros will be called a root solution to differentiate it from the complete solution which evaluates the polynomial as a function of the complex variable, z.

(1.1) Basic Formulation

The complex variable, z, can be represented in polar form. Let

$$\text{magnitude of } z = \exp(\pm\sigma t - \beta) \tag{1.3}$$

$$\text{argument of } z = \omega t \tag{1.4}$$

Thus

$$z = \exp(\pm\sigma t - \beta + j\omega t) \tag{1.5}$$

These definitions place no limitations on the range of z provided sufficient variations in σ, ω, β, and t are allowed.

Since the coefficient, a_k, may be a complex number, it can also be represented in polar form. Let

$$a_k = |a_k| \exp(j\theta_k) \tag{1.6}$$

The product, $a_k z^k$, now becomes

$$a_k z^k = |a_k| \exp\left[\pm k\sigma t - k\beta + j(k\omega t + \theta_k)\right] \quad (1.7)$$

or

$$a_k z^k = |a_k| \exp(\pm k\sigma t - k\beta) \left[\cos(k\omega t + \theta_k) + j\sin(k\omega t + \theta_k)\right] \quad (1.8)$$

Substitution of equation 1.8 into the equation of the polynomial, equation 1.2, gives

$$P_z = \sum_{k=0}^{n} |a_k| \exp(\pm k\sigma t - k\beta) \cos(k\omega t + \theta_k) +$$
$$j \sum_{k=0}^{n} |a_k| \exp(\pm k\sigma t - k\beta) \sin(k\omega t + \theta_k) \quad (1.9)$$

For simplicity of nomenclature, define

$$U = \sum_{k=0}^{n} |a_k| \exp(\pm k\sigma t - k\beta) \cos(k\omega t + \theta_k) \quad (1.10)$$

$$V = \sum_{k=0}^{n} |a_k| \exp(\pm k\sigma t - k\beta) \sin(k\omega t + \theta_k) \quad (1.11)$$

Thus

$$P_z = U + jV \quad (1.12)$$

The advantages in the definitions of equations 1.3 and 1.4 can now be seen. Consider t as representing time, ω as a frequency in radians per second, $1/\sigma$ as a time constant, and β as a dimensionless number. When viewed in this light, U and V become the sums of exponentially varying cosine and sine waves respectively. This forms the basis of the mechanization as an analogy is established between the terms of the polynomial and physical quantities of a realizable network.

A complete solution of the polynomial is obtained when U and V are recorded. A root solution is determined by a simultaneous zero of U and V. In this special case, either of the following two expressions may be substituted for the separate study of both U and V.

$$U^2 + V^2 = 0 \tag{1.13}$$

$$|U| + |V| = 0 \tag{1.14}$$

The basis of the mechanization is in the definitions of equations 1.3 and 1.4. These definitions placed no limitations on the parameters; σ, ω, β, and t. It will now be shown that the time variation coupled with a finite variation in β is sufficient to cover all possible values of z.

It may be seen from the defining equations that the time variation covers all values of the magnitude of z as well as all values of the argument. Unfortunately, all combinations of magnitude and argument are not covered. This additional variation is to be provided by variations in β.

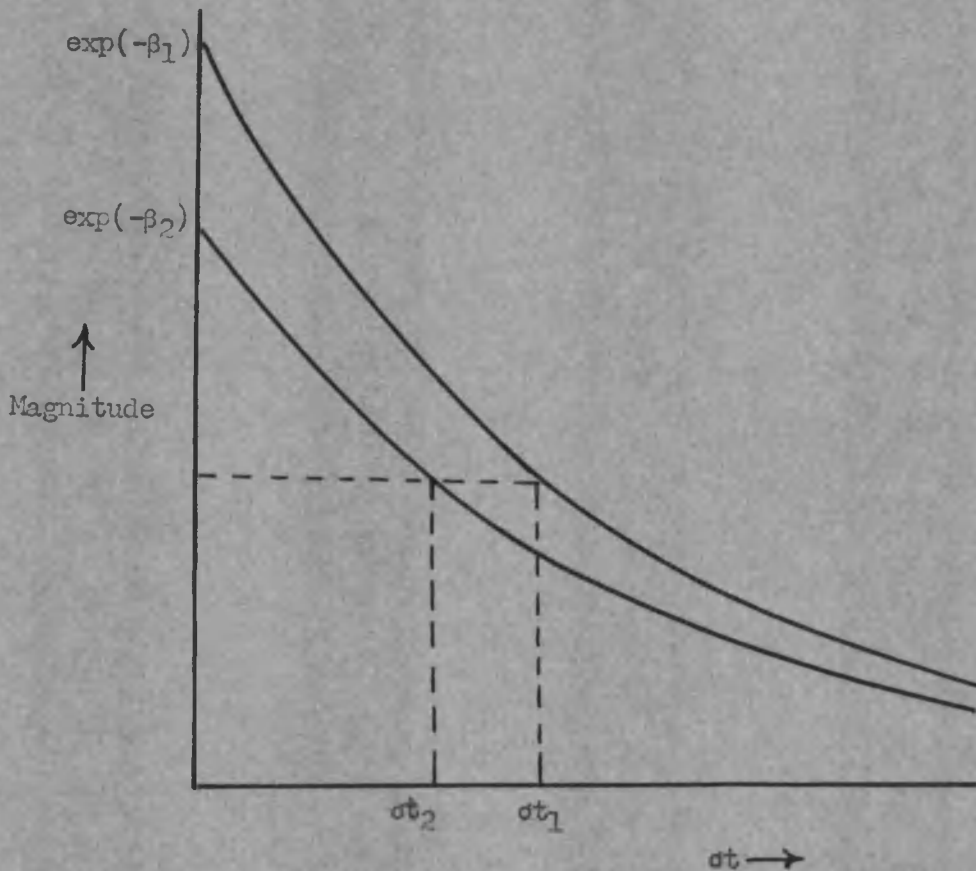
Figure 1.1 is a plot of the magnitude of z, as given by equation 1.3, versus time for two different values of β, β_1 and β_2. Although the figure is drawn for the degenerative case of the exponential variation, the discussion applies equally as well to the regenerative case. The two curves parallel each other and, except for the range between $\exp(-\beta_1)$ and $\exp(-\beta_2)$, they sweep through the same range of magnitudes. A line of constant magnitude is shown in the figure. Equating the magnitudes of the points of interception gives

$$\exp(-\sigma t_1 - \beta_1) = \exp(-\sigma t_2 - \beta_2) \tag{1.15}$$

or

$$t_2 - t_1 = (\beta_1 - \beta_2)/\sigma \tag{1.16}$$

The variation between β_1 and β_2 will be sufficient to cover all values of z if the time interval, $t_2 - t_1$, is large enough to allow



Magnitude of z versus Time

For Two Different Values of β

FIGURE 1.1

the argument a variation of 2π radians. This means

$$t_2 - t_1 = 2\pi/\omega \qquad (1.17)$$

The required β variation is obtained by equating the time intervals of equations 1.16 and 1.17. This gives

$$\beta_1 - \beta_2 = \Delta\beta = 2\pi\sigma/\omega \qquad (1.18)$$

Equation 1.18 indicates a possible approximation. If $\omega \gg 2\pi\sigma$, the required β variation becomes small and can be neglected. This will introduce an intrinsic error which can be made as small as desired by choice of σ and ω.

(1.2) Repeated Roots

The roots of the polynomial are fixed by the simultaneous zeros of its real and imaginary parts. Unfortunately, these zeros do not furnish a positive indication of the order of the root. This problem is solved by the following analysis.

Suppose the polynomial, $P_z = U + jV$, contains a repeated root. Then dP_z/dz contains the same root with degree decreased by one. A repeated root is therefore indicated by simultaneous roots in both P_z and dP_z/dz.

The complex variable is defined, $z = \exp(\pm\sigma t - \beta + j\omega t)$. If β is assumed constant,

$$dz/dt = (\pm\sigma + j\omega)\ z \qquad (1.19)$$

Therefore

$$dP_z/dz = (dP_z/dt)\ (dt/dz) = \left[z(\pm\sigma + j\omega)\right]^{-1} \left[dU/dt + j dV/dt\right] \qquad (1.20)$$

A root of dP_z/dz is determined when

$$|dP_z/dz| = \left[|z|\ |\pm\sigma + j\omega|\right]^{-1} |dU/dt + j dV/dt| = 0 \qquad (1.21)$$

This expression can be simplified. The term, $|\pm\sigma + j\omega|$ is a constant and can be neglected as a condition for $dP_z/dz = 0$. The term, $|z|$, is time varying but it too can be dropped as it does not affect the zeros of dP_z/dz. Therefore the condition for a root of dP_z/dz reduces to

$$|dU/dt + j dV/dt| = 0 \tag{1.22}$$

Equation 1.22 indicates the mechanization for the isolation of repeated roots. U and V are smoothly varying functions of time and differentiation with respect to time can be easily performed.

The requirements for a repeated root can be expressed in two forms analogous to equations 1.13 and 1.14.

$$U^2 + V^2 = 0 \quad \text{and} \quad (dU/dt)^2 + (dV/dt)^2 = 0 \tag{1.23}$$

$$|U| + |V| = 0 \quad \text{and} \quad |dU/dt| + |dV/dt| = 0 \tag{1.24}$$

Equations 1.23 and 1.24 indicate only the presence of a repeated root, not its degree. The root degree could be obtained by repeating the differentiation processes until the root no longer appears.

Chapter 2

SCALING AND NORMALIZATION

The preceding chapter forms a theoretical basis for the construction of a polynomial solving machine. It is now necessary to consider some of the practical problems that arise by representing the complex variable with a physical variation. Physical variables are subject to bounds which form upper and lower limits to their allowable ranges. If the required variation in z does not correspond to these bounds, it is necessary to resort to scaling. There is an analogous problem with the coefficients, a_k, as an upper limit on their amplitudes must usually be introduced. This requires normalization procedures.

(2.1) Determination of Region of Interest, Root Bounds

Scaling of the entire z-plane into the finite range of a physical variable is impossible. It is necessary to define a finite region of z-plane interest. This is not difficult with the complete type solution as the areas of interest are usually fixed by external conditions. However, in the case of root evaluation, the approximate root locations may not be known and it is necessary to determine bounds on the roots from the coefficients of the polynomial. The following development is devoted to this purpose. Similiar presentations may be found elsewhere.[4,5]

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4) Heinrich Burkhardt, Theory of Function of a Complex Variable, Translated by S. E. Rasor, D. C. Heath and Company

5) Lars Lofgren, "Analog Computer for the Roots of Algebraic Equations", Proceedings of the IRE, July, 1953

Evaluation of an upper bound on the magnitude of the roots will be considered first. For convenience, the work on upper root bounds will assume that the coefficient, a_n, is unity. This does not effect the root values.

The equation for a general polynomial is

$$P_z = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots + a_n z^n \qquad (2.1)$$

Define a number, G, such that

$$G \geq |a_k| \qquad \text{for } k = 0, 1, 2, \cdots, n-1 \qquad (2.2)$$

Since $a_n = 1$, it is seen from equations 2.1 and 2.2 that

$$|P_z - z^n| \leq G(1 + |z| + |z|^2 + \cdots + |z|^{n-1}) \qquad (2.3)$$

Application of the formula for a geometric progression gives

$$|P_z - z^n| \leq G(|z|^n - 1)/(|z| - 1) \qquad (2.4)$$

For all values of z with absolute value greater than or equal to G + 1, equation 2.4 becomes

$$|P_z - z^n| \leq |z|^n \qquad (2.5)$$

Therefore by denoting an upper limit on root amplitudes by M, equation 2.5 shows that

$$M = G + 1 \qquad (2.6)$$

The upper limit indicated by equation 2.6 may be the smallest one available but a second and usually better formula can be developed through the use of it.

Substitute z = hw into the equation for the general polynomial. w is considered a new variable and h a scale factor. After division by h^n, the following is obtained.

$$w^n + (a_{n-1}w^{n-1})/h + \cdots + (a_k w^k)/h^{n-k} + \cdots + a_0/h^n = 0 \quad (2.7)$$

None of the coefficients of equation 2.7 are greater than one if h is chosen so that

$$h \geq \sqrt[n-k]{a_k} \qquad \text{for } k = 0, 1, 2, \cdots, n-1 \qquad (2.8)$$

Since equation 2.8 imposes an upper limit of unity on the coefficients, the conclusion of equation 2.6 can be applied to place an upper limit on the variable w. This limit is 2. The corresponding upper limit, M, on z is

$$M = 2 \left. \sqrt[n-k]{a_k} \right/ \text{maximum for variations in } k$$

Two equations have been developed to determine bounds on the maximum root magnitude. Similar equations can be developed for the lower bounds by applying the upper bound conditions to the polynomial resulting when the original variable, z, is replaced by 1/z. The upper bounds on the variable of this new equation are the reciprocals of the lower bounds of z in the original polynomial. It is assumed, of course, that all zero roots have been eliminated by factoring. In evaluating lower bounds it is found to be convenient to set the coefficient, a_0, equal to unity rather than a_n.

Table I summarizes the formulas for evaluation of root bounds. In all cases, the lower upper bound and the upper lower bound are to be used for the purposes of scaling.

(2.2) Scaling

The region of interest in the z-plane has been confined to a finite area. It is now necessary to consider the scaling which will match the machine variable to z. Different techniques of machine setup will result in various allowable variational ranges but no generality is lost

	Upper Limit, M, on Maximun Root Amplitudes	Lower Limit, m, on Minimun Root Amplitudes
Conditions of Evaluation	$a_n = 1$	$a_0 = 1$ No zero roots
Formulas:	$M = G + 1$ where $G \geq \lvert a_k \rvert$ for $k = 0, 1, 2, \cdots, n-1$	$m = 1/g+1$ where $g \geq \lvert a_k \rvert$ for $k = 1, 2, 3, \cdots, n$
Choose results giving least upper limit and greatest lower limit.	$M \geq 2 \sqrt[n-k]{\lvert a_k \rvert}$ for $k = 0, 1, 2, \cdots, n-1$	$m \leq 1/\ 2 \sqrt[k]{\lvert a_k \rvert}$ for $k = 1, 2, 3, \cdots, n$

Evaluation of Root Bounds for a General Polynomial

$$P_z = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots + a_n z^n = 0$$

TABLE I

in assuming an upper bound on the machine variable of unity.

Define the machine variable to be z'. This variable has an upper bound of unity. Let its lower bound be denoted by R. These limits define an annular ring in the z'-plane.

Since a finite region of z-plane interest is assumed, the least upper bound, M, and the greatest lower bound, m, are fixed. These values define an annular ring in the z-plane.

It is desired to scale the z-plane annular ring into the z' ring. The following discussion will show that, in general, more than one scaling may be required.

Divide the z-plane region of interest into L sub-rings. This is illustrated in figure 2.1 in which $R_L = M$ and $R_o \leq m$. Each of the separate sub-regions of the z-plane are to be scaled into the annular ring of the z'-plane.

The desired scaling requires matching at two boundaries. Therefore two equations are required. For the special case of the sub-region between R_L and R_{L-1} the equations are

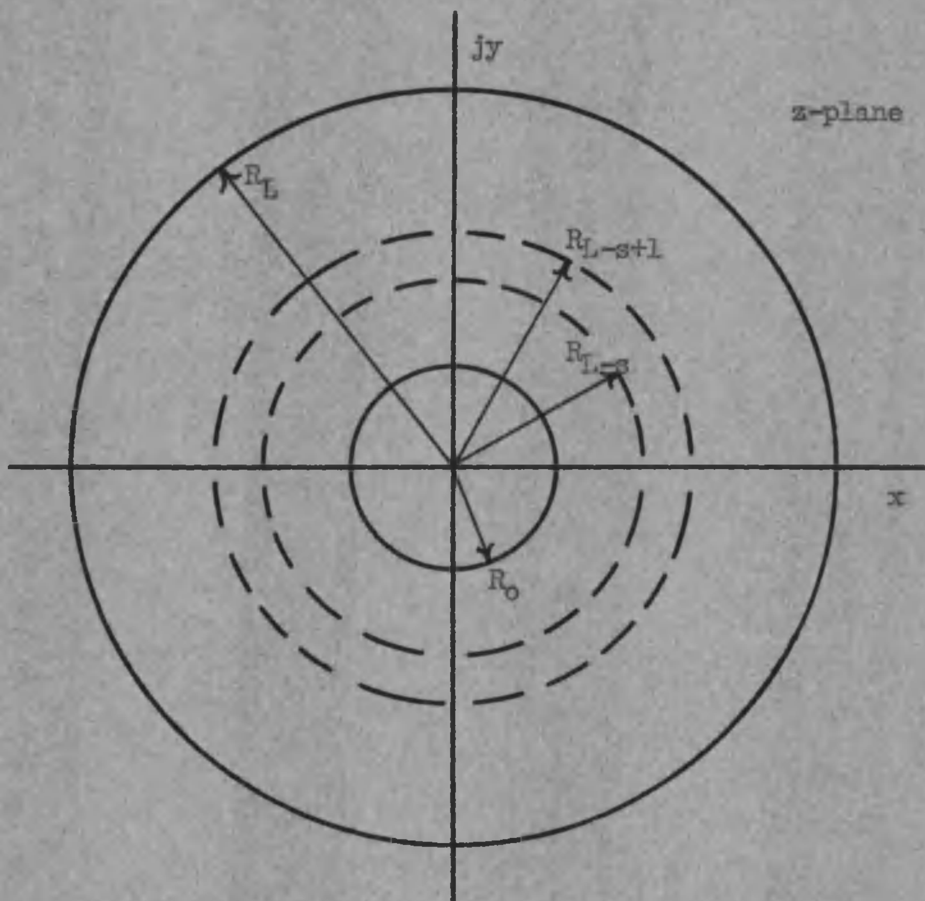
$$z/R_L = z'/1 \qquad \text{and} \qquad z/R_{L-1} = z'/R \tag{2.10}$$

Rewriting these equations in a more convenient form gives

$$z = R_L\, z' \qquad \text{and} \qquad R = R_{L-1}/R_L \tag{2.11}$$

Equation 2.11 represents one special scaling. The general case is obtained by the repetitious application of the technique to successive sub-regions. This results in the following

1^{st} Scaling	$z = R_L\, z'$	$R_L = M$
2^{nd} Scaling	$z = R_{L-1}\, z'$	$R_{L-1} = R\, M$



Sub-Division of z-plane Region of Interest

$R_L = M \quad R_o \leq m$

$s = 1, 2, 3, \cdots, L$

FIGURE 2.1

3^{rd} Scaling $\quad z = R_{L-2}\ z' \qquad R_{L-2} = R^2 M$

s^{th} Scaling $\quad z = R_{L-s+1}\ z' \qquad R_{L-s+1} = R^{s-1} M \qquad (2.12)$

Equation 2.12 fixes the scale factor for each scaling. The sketch in figure 2.1 shows the region of the s^{th} scaling. The parameter s varies from 1 to L to cover the required region of the z-plane.

Equation 2.12 can also be used to evaluate L, the total number of scalings. Although the scaling procedure requires an s variation of only 1 to L, there is no reason to prevent the evaluation of equation 2.12 for s = L + 1. This gives

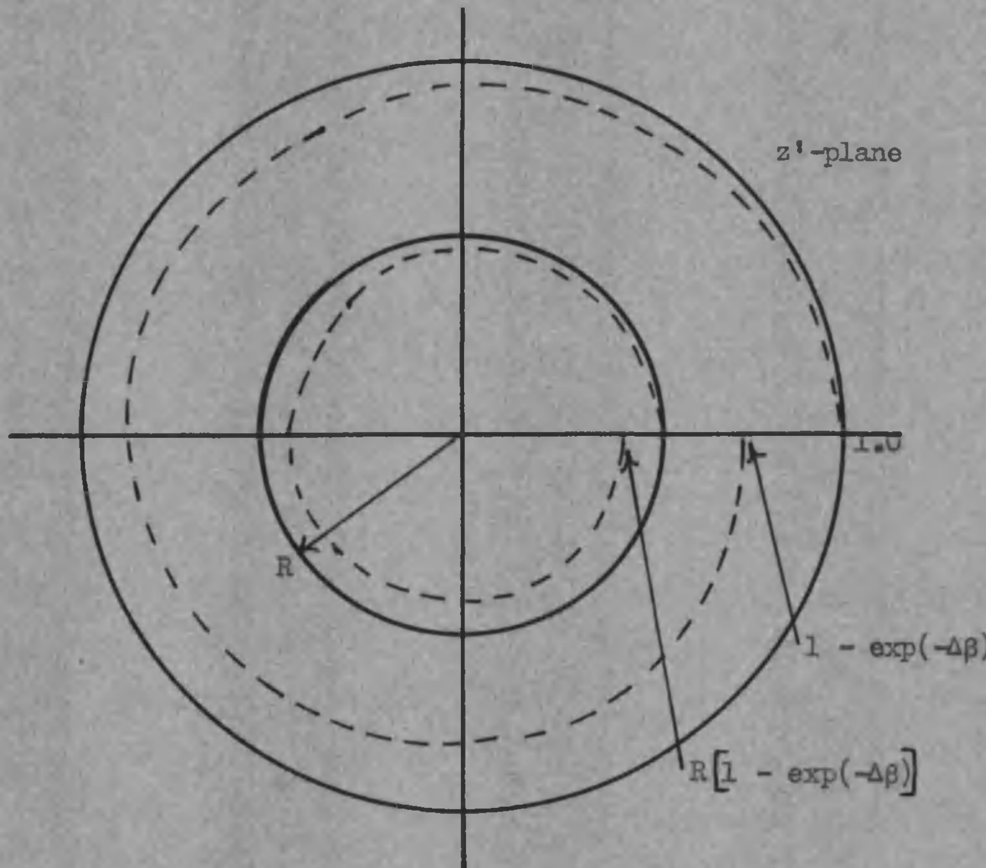
$$R^L = R_o/R_L \leq m/M \qquad (2.13)$$

The inequality sign in equation 2.13 can be dropped if L is always rounded off to the next highest integer. Thus

$$L = \log(m/M)/\log R \qquad (2.14)$$

Equation 2.14 is the fundamental equation evaluating the number of scalings required for a polynomial in terms of the lower bound, R, of the machine variable.

The preceding work on scaling has assumed that the machine variable covers the z'-plane with perfect annular rings. This is not exactly correct when the β variation is used. Figure 2.2 compares the idealized annular ring to the actual region covered by the variable. The region determined by $\exp(-\Delta\beta) < |z| < 1$ is not completely covered while certain values of z' outside of the ideal ring are included. Except for the first scaling, this presents no problems as the missing values are included in the extra values of the preceding scaling. To assure a complete coverage at the first scaling, the upper bound, M, should be



Comparison of Annular Ring to Region
Actually Covered by Machine Variable

Annular Ring Bounded by ————
Machine Coverage Bounded by - - - - -

FIGURE 2.2

increased by $M \exp(\Delta\beta)$.

(2.3) Normalization

The polynomial has been scaled to fit the machine variable. The problem of normalization of coefficients must now be considered.

The general polynomial of equation 1.2 can be rewritten in terms of the machine variable by the use of equation 2.12.

$$P_{z'} = \sum_{k=0}^{n} a_k \left(R_{L-s+1}\right)^k z'^k \qquad \text{where } s = 1, 2, \cdots, L \tag{2.15}$$

The factor, R_{L-s+1}, can be removed with the use of equation 2.12.

$$P_{z}' = \sum_{k=0}^{n} a_k M^k R^{ks-k} z'^k \tag{2.16}$$

Define

$$b_k = |a_k| M^k \tag{2.17}$$

This gives

$$P_{z}' = \sum_{k=0}^{n} b_k R^{ks-k} \exp(j\theta_k) z'^k \tag{2.18}$$

For any particular scaling, $b_k R^{ks-k}$ represents the coefficients which are to be normalized. It is desired to normalize them to unity.

Unity normalization is accomplished by dividing equation 2.18 by the maximum value of $b_k R^{ks-k}$ with respect to variations in k. This normalization process does not affect the root values but the factor must be included in the complete type solutions. Define C_k, the normalized coefficient, as

$$C_k = (b_k R^{ks-k})/(b_k R^{ks-k})_{\text{maximum}} \tag{2.19}$$

The polynomial can now be expressed in a final form, suitable for machine solution.

$$P_{z}' = \sum_{k=0}^{n} C_k \exp(j\theta_k) z'^k \tag{2.20}$$

Equation 2.20 can also be expressed in the expanded form analogous to equation 1.9.

$$P_z' = \sum_{k=0}^{n} C_k \exp(\pm k\sigma t - k\beta) \cos(k\omega t + \theta_k) +$$
$$j \sum_{k=0}^{n} C_k \exp(\pm k\sigma t - k\beta) \sin(k\omega t + \theta_k) \qquad (2.21)$$

Unfortunately the evaluation of the coefficients, C_k, can prove tedious as separate calculations must be made for each value of s. The amount of this labor can be reduced by a graphical solution as follows:

Express $b_k R^{ks-k}$ in logarithmic form.

$$\log(b_k R^{ks-k}) = (ks-k) \log(R) + \log(b_k) \qquad (2.22)$$

A plot of $b_k R^{ks-k}$ versus s on semi-logarithmic paper results in a family of straight lines of slope k log(R) and vertical intercept $\log(b_k)$ - K log(R). The family parameter is k. The normalized coefficients can be read directly from the plot using dividers.

Chapter 3

MECHANIZATION CONSIDERATIONS

The means by which the general polynomial can be reduced to a scaled and normalized form suitable for mechanization in a physically realizable machine have been detailed in Chapter 2. It is now logical to consider some general aspects of the machine itself. The final choice of design will be based on many factors: type of solution desired, physical variable to be used, speed of operation and degree of accuracy required, justifiable investment, available equipment, number of machines to be constructed, and even the experience of the designer. The wide range of these considerations renders any attempt at designation of optimum design impossible but a few of the general requirements and considerations can be discussed.

(3.1) Basic Operations

The basic operations of the polynomial solving machine can be divided into three main divisions: function generation, addition, and read out.

The function generators provide the exponentially varying sine and cosine waves of equation 2.21. The generators must respond to an easily obtained stimulus and have provision for programing the coefficient amplitudes, C_k, and when necessary, the coefficient phases, Θ_k, and the β variation. Mechanization of the coefficient phases is not always required as many polynomials have only real coefficients. If repetitious solutions are desired, repetitious excitation and provision for the resetting of zero initial conditions are necessary. It is naturally desired that the programing of the coefficients and the β variation be as simple as possible. Thus one setting of the coefficient parameters should be sufficient to program both the sine and cosine generator outputs. Similarly, the β variation is ideally mechanized in ganged assembly so that one control affects all function generators the required amount.

The addition of the function generator outputs is straightforward mechanization requiring little comment. When provision is made for the checking of repeated roots, the required differentiators may be built into or around the adder circuit.

The read out system converts the time varying outputs of the adders into a usable form of solution. The various types of read out systems fall into two categories depending on whether the complete or only the root solution is required.

The simplest method of read out for the complete solution records the adder outputs as functions of time. Thus the polynomial is evaluated in rectangular form by two recordings. If desired, a complete solution may be obtained from a single recording if U and V, the adder outputs, are plotted against each other before recording. This gives a plot of the complex plane of the polynomial. Both of these systems must be calibrated with respect to time if the polynomial is to be calibrated with respect to z. This calibration may be difficult in the complex plane form of read out.

The complete solutions also include the root values but the specialized root solutions deserve separate attention because of the simplified read out mechanization that is possible.

It was mentioned in Chapter 1 that either of two forms of the equation, $P_z = 0$, can be considered instead of the study of the separate zeros of U and V. These forms are

$$U^2 + V^2 = 0 \qquad (3.1)$$

$$|U| + |V| = 0 \qquad (3.2)$$

Either of the two quantities, $U^2 + V^2$ or $|U| + |V|$, can be recorded against calibrated time, the instants of waveform zeros fixing the root values.

All of the preceding systems require external computation involving scale factors and equation 1.5 to obtain the actual values of z. A wide variety of additional mechanization can be included to provide a direct reading output. The possibilities in this area are almost unlimited.

(3.2) <u>Speed and Accuracy of Operation</u>

Fast speed of operation and high accuracy are both desired properties. Unfortunately, one will usually be obtained at the expense of the other. There are four main factors bearing on the problem: the time base of the machine, the machine's noise level, the decision regarding use of the β sweep, and the choice of read out system. Although these factors are related, they will be discussed separately.

The time base of the machine actually refers to two quantities: the time required for the machine variable to cover its given range and the choice between repetitious or "single shot" operation. The time consumed as the variable covers its given range is not limiting in an electronic mechanization because of the relatively high operating frequencies but with other setups, such as mechanical mechanizations, this factor can limit the possible speed of operation. The choice between repetitious and "single shot" solutions is an important speed of operation consideration when the β sweep is used. In order to study the effects of the β sweep the machine must present a separate time variation for each value of β. Unless the function generators have a fairly high repetition rate, this can become time consuming. Theoretically the accuracy should not be affected by the time base but practical considerations will usually invalidate this relationship. Unfortunately the effects of these factors will vary from machine to machine and no general conclusions can be drawn.

The discussion of scaling presented in Chapter 2 introduced a factor, R, which determined the number of scalings required. The minimum acceptable value of R is fixed by the noise level of the machine. This is an important consideration as each scaling requires additional computation and setting of coefficients, thereby increasing the time of solution. In certain simpler types of read out systems, however, a large R increases accuracy. It is also possible that the time for the computation of coefficients can be reduced by the use of an auxiliary computer. These considerations can reduce the advantage of a low noise level but noise remains the one factos whose reduction always improves operational characteristics.

A very important design consideration concerns the ratio of σ to ω. When $2\pi\sigma \ll \omega$, the β variation can be neglected at the expense of an intrinsic mechanization error which can be made as small as desired. Thus it seems that the extra equipment and operating time required by the β sweep is not justifiable. There are, however, at least three other factors to be considered. Before discussing these points a few words about the extra time and equipment are in order. The β sweep requires repetitious types of solution but this method of mechanization has many other advantages and will often be chosen independent of β sweep considerations. In electronic mechanizations of this type, the only extra equipment needed might be a set of ganged, non-linear potentiometers. With ganged assembly the extra time of operation can be considered small, especially compared to the time necessary for the computation and setting of the scaled coefficients.

The first factor that might make the β sweep advantageous is the mere presence of an extra variable. The output waveforms are simpler when β is included and this can result in simpler read out systems and easier analysis of results. Of course the presence of an extra variable increases the chance of error and may therefore decrease the accuracy.

A second factor, slightly analogous to the first, appears in the evaluation of the argument of the machine variable. This angle is given by $\omega t_o - N2\pi$ where t_o denotes the time of interest and N is an integer fixing the angle, $\omega t_o - N2\pi$, between 0 and 2π radians. If $\omega \gg 2\pi\sigma$, ωt_o and N can become very large. When a desired result is the difference between large numbers of about the same magnitude, both numbers must be very accurately determined if the difference is to be accurate. Thus the exclusion of the β variation requires greater accuracy in fixing the frequencies of the function generators and in determining the time calibration of the output waveforms.

A final consideration bearing on the choice of the σ to ω ratio is the type of available equipment. With certain types of equipment it may be impossible to generate the functions under the condition that $\omega \gg 2\pi\sigma$. This was the case in the experimental computer described in the next chapter.

The fourth factor listed as applying to the accuracy and speed of operation is the choice of read out system. After a class of read out systems has been fixed by the type of desired solution, there remains a variety of possible designs furnishing different degrees of simplicity, accuracy, and speed of read out. The systems discussed in the preceding section are basic ideas which can be expanded to meet different requirements. These basic systems are slow in operation as the answer is not displayed in the most convenient form. The speed of operation can be increased with some type of direct answer display. This extra mechanization may decrease accuracy because of extra complexity or it may increase accuracy by removing the necessity of waveform analysis. The actual effect of various read out systems is an important consideration that needs further study.

The effect of four factors, time base, noise level, β sweep, and read out system have been discussed in this section. The discussions have been, of necessity, very general and few, if any, conclusions were reached. It seems, however, that for a majority of cases, the predominate determent to high speed of operation is the time required for the computation and setting of the scaled coefficients. There appears to be no major accuracy factor although the accuracy of the read out system could easily become a limiting consideration.

Chapter 4

THE EXPERIMENTAL COMPUTER

The design and construction of a working computer is dependent upon many factors, a few of which were listed at the beginning of the preceding chapter. The predominate design consideration of the pilot computer about to be described was the availability of equipment and simplicity of construction. The main purpose of the pilot machine was the provision of experimental verification of the theory. The computer's development also furnished data to guide its expansion.

(4.1) The Computer in General

The pilot computer was designed to handle a third degree polynomial with real coefficients. This was deemed sufficient to fulfill the computer's requirements. Provision for programing complex coefficients was not considered necessary to the general theory of operation.

The computer was constructed around analog computing equipment built by G. A. Philbrick Inc. This equipment included operational amplifiers, multipliers, variable gain amplifiers, and adders. Auxiliary equipment included two square wave generators, one oscilloscope, one pulse generator, one oscillographic camera, decade capacitor boxes, fixed resistors, diodes, and 12AU7 vacuum tubes.

The computer provided the repetitious type solution. Excitation was obtained from a square wave generator with frequencies of the order of 10 cps. Solutions were obtained over one half of the cycle, the second half of the wave being used to activate the clamping circuits, thereby resetting the necessary zero initial conditions.

The operating values of α and ω were chosen to match available equipment. σ was set at 20 sec^{-1} and ω was fixed at 200 radians per second. These values required a β variation of 0 to 0.628.

The circuitry required for the checking of repeated roots was incorporated in the adder circuits. A switch was provided to change both of the adders into adder-differentiators.

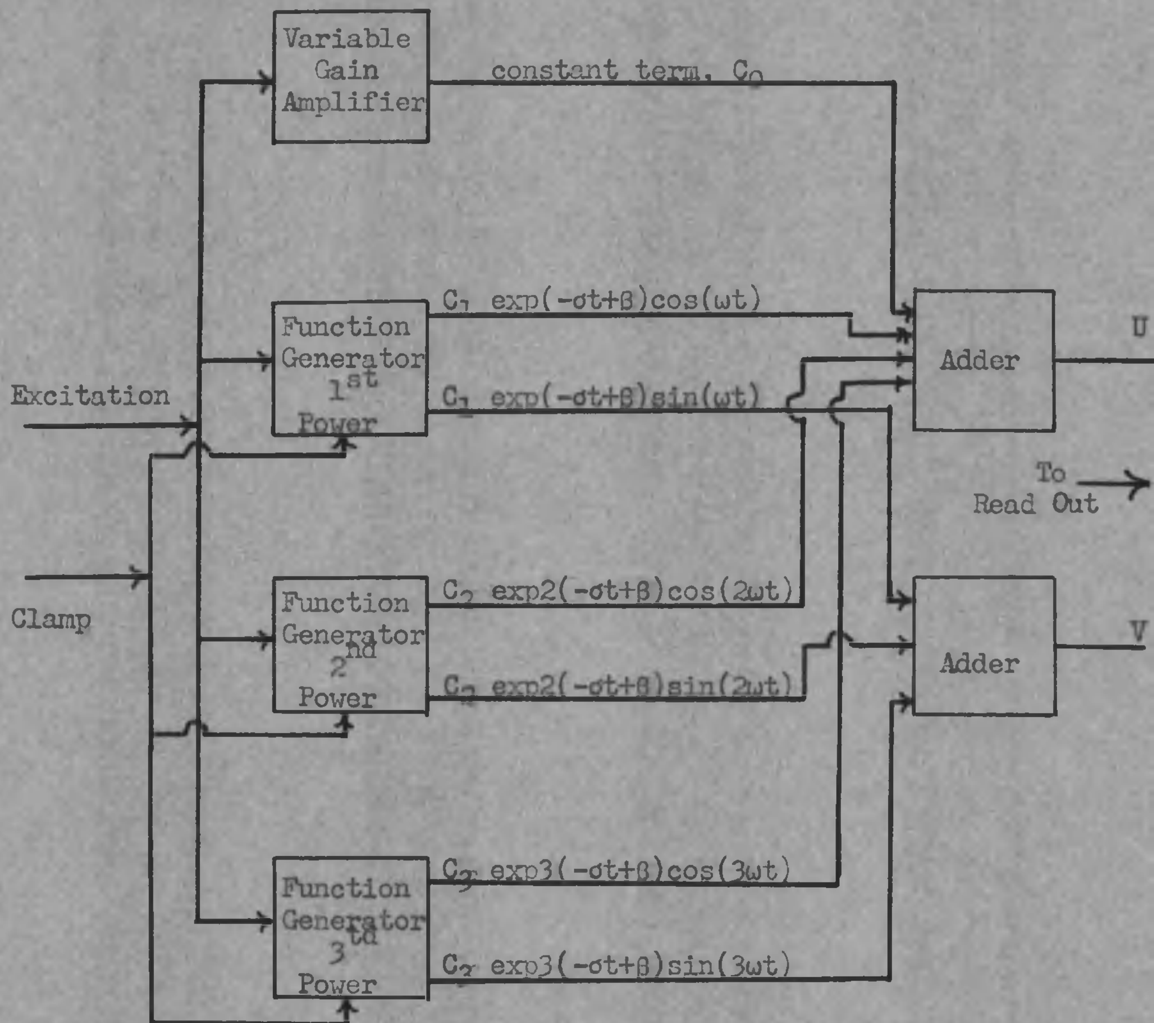
Section 3.1 discussed four basic methods of read out. All of these systems were mechanized in the computer although no attempt was made to introduce ramifications to increase ease of read out. All of the systems used the simplest vehicle for presenting the outputs: the oscilloscopic trace. Permanent recordings were obtained with the oscillographic camera.

Figure 4.1 presents the general block diagram of the computer up to the point of read out. The coefficients C_0, C_1, C_2, and C_3 were set manually into the function generators and the variable gain amplifier. The β variation was set in the same manner as the coefficients.

Figure 4.8 illustrates the various read out systems in block diagram form.

(4.2) Function Generation

The function generator should be considered the heart of the computer, all other operations being necessary, but subordinate. Each of the three function generators consisted of two d-c operational amplifiers, Philbrick model K2W, with associated input and feed back circuitry.



Block Diagram of Computer

FIGURE 4.1

The Laplace transforms of the required outputs are[6]

$$\mathcal{L}\exp(-\sigma t-\beta)\cos(\omega t) = (s+\sigma)/(s^2+2\sigma s+\sigma^2+\omega^2) \tag{4.1}$$

$$\mathcal{L}\exp(-\sigma t-\beta)\sin(\omega t) = (\omega)/(s^2+2\sigma s+\sigma^2+\omega^2) \tag{4.2}$$

Equations 4.1 and 4.2 represent the response of the function generators to a unit step input with zero initial conditions. Therefore the function generator had to have the following transfer functions.

$$s(s+\sigma)/(s^2+2\sigma s+\sigma^2+\omega^2) \tag{4.3}$$

$$(s\omega)/(s^2+2\sigma s+\sigma^2+\omega^2) \tag{4.4}$$

Due to the similarity of equations 4.3 and 4.4 the following mechanization procedure was chosen. The transfer function of equation 4.3 was mechanized with one operational amplifier and associated circuitry. This network was followed by a second operational amplifier circuit which modified the output into the form of equation 4.2. The transfer function of the second system was then

$$(\omega)/(s+\sigma) \tag{4.5}$$

Both of the operational amplifier circuits had the basic form shown in figure 4.2. Since the operational amplifiers had a very large open loop gain, the transfer function of figure 4.2 can be written in terms of the general impedances Z_1 and Z_2 as[7]

$$E_{out}/E_{in} = Z_2 / Z_1 \tag{4.6}$$

The main problem in the synthesis of equation 4.3 was the introduction of the required complex poles. This problem was complicated

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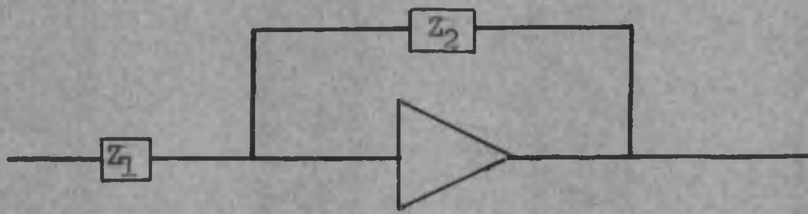
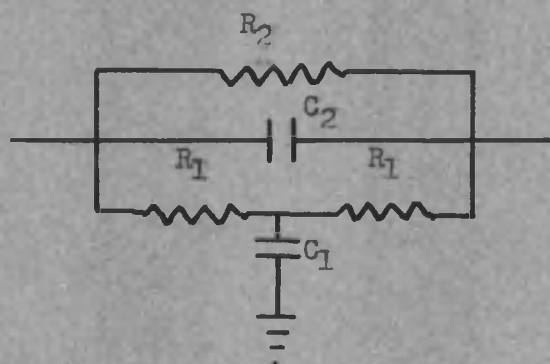
6) Murray F. Gardner and John L. Barnes, Transients in Linear Systems, John Wiley and Sons, New York, vol. 1, 1942

7) Walter W. Soroka, Analog Methods in Computation and Simulation, McGraw-Hill, 1954, page 45



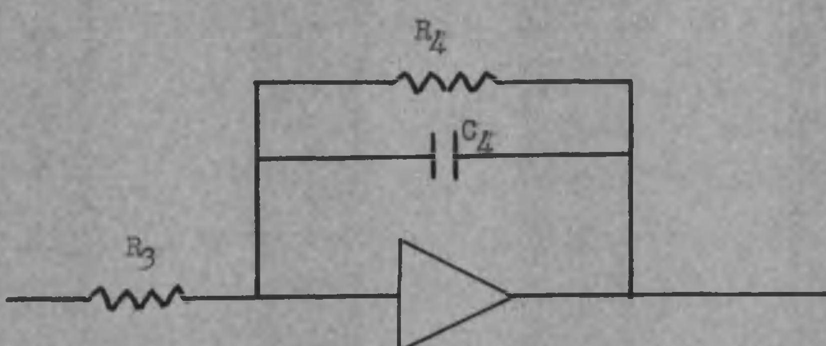
Basic Form of Operational Amplifier Circuitry

FIGURE 4.2



Circuit to Provide Complex Poles

FIGURE 4.3



Second Half of Function Generator

FIGURE 4.4

by the necessity of avoiding the use of inductors because of the difficulty in obtaining them with the required values and accuracy. Figure 4.3 shows the circuit chosen to provide the complex poles. The following development presents the design equations for this circuit.

The impedance of the circuit of figure 4.3 may be found in the literature as[8]

$$Z = (AT_3)(s + 1/T_3) \;/\; (T_1T_2)(s^2 + s/T_2 + 1/T_1T_2) \qquad (4.7)$$

The factor, A, is arbitrary and can be assigned any convenient value.

Comparison of equation 4.7 with equation 4.3 shows that the following condition must be imposed.

$$2T_2 = T_3 \qquad (4.8)$$

Under the condition of equation 4.8 the following relations can be shown to hold.[9]

$$R_1 = (2T_2A)/(4T_2 - T_1) \qquad (4.9)$$

$$R_2 = (4T_2A)/(T_1) \qquad (4.10)$$

$$C_1 = 2(4T_2 - T_1)/(A) \qquad (4.11)$$

$$C_2 = (T_1)/(2A) \qquad (4.12)$$

Comparison of equation 4.7 with equation 4.3 also shows that the following must hold.

$$T_2 = 1/(2\sigma) \qquad (4.13)$$

$$T_1 = (2\sigma)/(\omega^2 + \sigma^2) \qquad (4.14)$$

---

8) F. R. Bradley and R.McCoy, "Driftless D-c Amplifier", Electronics 25(4):144-148

9) Ibid

Equations 4.9 through 4.14 are sufficient for design but more convenient forms can be obtained by defining a new parameter, r.

$$r = \omega/\sigma \tag{4.15}$$

With this new parameter, T_2 and T_1 can be related. From equations 4.13 and 4.14

$$4T_2 = T_1(r^2 + 1) \tag{4.16}$$

Application of equation 4.16 to equations 4.9 through 4.12 gives the final design equations.

$$R_2 = A(r^2 + 1) \tag{4.17}$$

$$R_1 = R_2/(2r^2) \tag{4.18}$$

$$C_2 = T_1/(2A) \tag{4.19}$$

$$C_1 = 4r^2\, C_2 \tag{4.20}$$

The impedance of equation 4.7 includes a constant coefficient, $(AT_3)/(T_1T_2)$. After application of equation 4.8 and 4.19 this factor becomes $1/C_2$.

As result of the preceding work, equation 4.7 can be written

$$Z = (s + \sigma)/(C_2)(s^2 + 2\sigma s + \sigma^2 + \omega^2) \tag{4.21}$$

The Z of equation of 4.21 is to be the Z_2 of the operational amplifier of figure 4.2. Comparison of the transfer function, equation 4.6, of this circuit with the required transfer, equation 4.3, shows that Z_1 must be a capacitor: Denoting this capacitance by C_v,

$$Z_1 = 1/sC_v \tag{4.22}$$

Equations 4.21 and 4.22 fix the circuitry of the first operational amplifier. The transfer function of the second system is defined by equation 4.5. The circuitry shown in figure 4.4 was used to develop this function. The transfer function of figure 4.4 is easily seen to be

$$1/(R_3C_4)(s + 1/R_4C_4) \tag{4.23}$$

A comparison of equation 4.5 and 4.23 gives the design equations for the second operational amplifier circuitry.

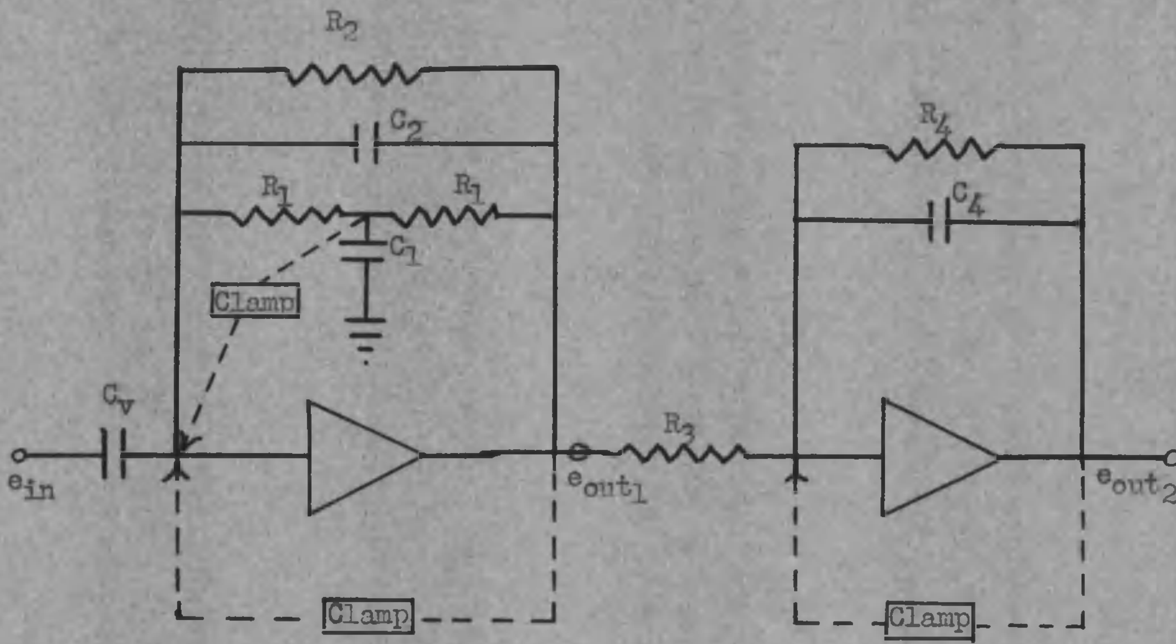
$$1/(R_4C_4) = \sigma \tag{4.24}$$

$$(R_4)/(R_3) = r \tag{4.25}$$

Figure 4.5 shows one complete function generator. In addition to the discussed circuitry, three clamping actions are indicated. The desired output was obtained only when unit step excitation was applied to a generator with zero initial conditions. Since repetitious solutions were desired, provision had to be made to reset these zero conditions after each solution. The clampers fulfilled this requirement.

The coefficient, C_k, was programed into the function generators by variations in the capacitor, C_v. Unfortunately the β variation had to be programed using the same capacitor. This was theoretically sound but rather inconvenient as separate settings were required for each generator. A ganged assembly would have increased speed of operation.

The function generators were designed for $\sigma = 20\ \mathrm{sec}^{-1}$ and $\omega =$ 200 radians per second. These values were chosen as they resulted in easily available resistors and capacitors. A somewhat larger ratio of ω to σ probably would have been chosen if the components had been on hand as the range of the β sweep was relatively large. It is worth mentioning, however, that the circuit of figure 4.5 combined with the K2W operational amplifiers is incapable of generating an output that would remove the necessity of the β sweep. When ω/σ becomes very large



Function Generator

e_{in} = Square Wave

$$e_{out_1} = [-C_v/C_2]\,[\exp(-\sigma t)\ \cos(\omega t)]$$

$$e_{out_2} = [C_v/C_2]\,[\exp(-\sigma t)\ \sin(\omega t)]$$

FIGURE 4.5

the resulting circuit values load the operational amplifier beyond its capacity. Although different circuitry probably exists, the author was unable to develop it for application to the K2W amplifiers.

Table 2 lists the values actually used in the various function generators.

(4.3) Clamping, Addition, and Read Out

The clamping action shown in figure 4.5 provided the necessary resetting of zero initial conditions. The circuit for one of the clampers is given in figure 4.6. The clamping signal was the excitation square wave, inverted and amplified to 100 volts peak to peak. The negative portion of this clamp signal was sufficient in amplitude to cut off the tube. The various capacitors discharged through the tube when the clamping signal was positive.

An adder and repeated root checker are incorporated in the circuit of figure 4.7. The switches of both adders were ganged together. When the switch was open, differentiation was closely approximated for the frequencies of interest. The switch was kept closed until a root had been determined. The switch was then opened for checking repeated roots. This operation correspondes to the conditions of equations 1.23 and 1.24.

The four read out systems of section 3.1 were mechanized. The simplest mechanization recorded U and V as functions of time. The block diagrams of the other three set ups are shown in figure 4.8.

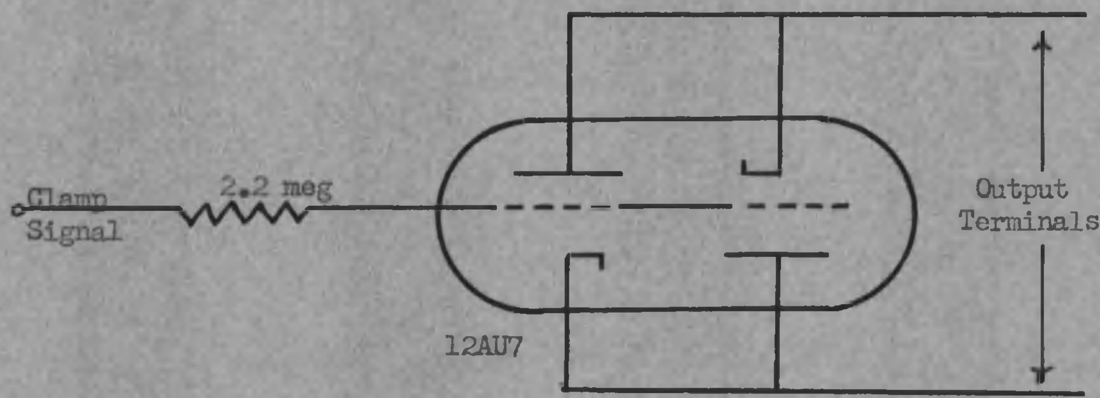
Figure 4.8a represents the system for presenting a complete solution as a single plot. The result of applying U and V to the vertical and horizontal deflection plates was an oscillographic trace representing

	First Power	Second Power	Third Power
C_1	2.0 μf	1.0	0.667
C_2	0.005 μf	0.0025	0.00167
C_4	0.05 μf	0.025	0.0167
R_1	50K Ω	50K	50K
R_2	10 meg Ω	10 meg	10 meg
R_3	0.1 meg Ω	0.1 meg	0.1 meg
R_4	1.0 meg Ω	1.0 meg	1.0 meg

Circuit Values for Function Generators

See figure 4.5

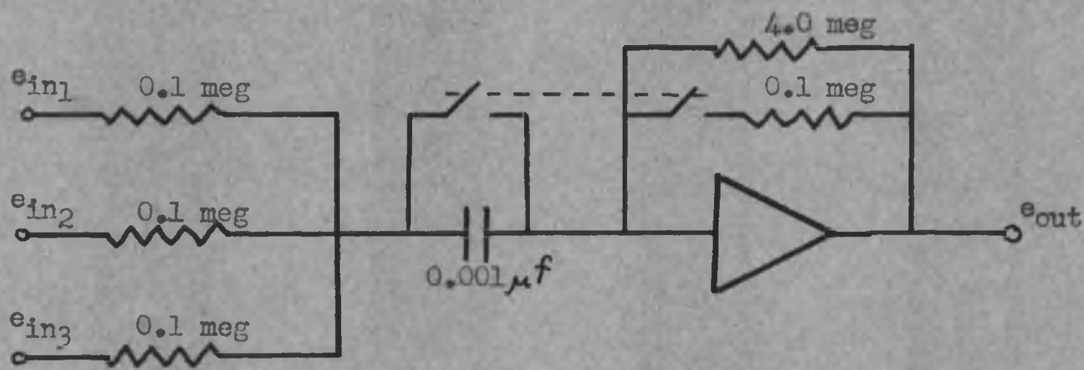
TABLE II



Clamping Circuit

Provides Low Impedance Output With Positive Clamp Signal

FIGURE 4.6

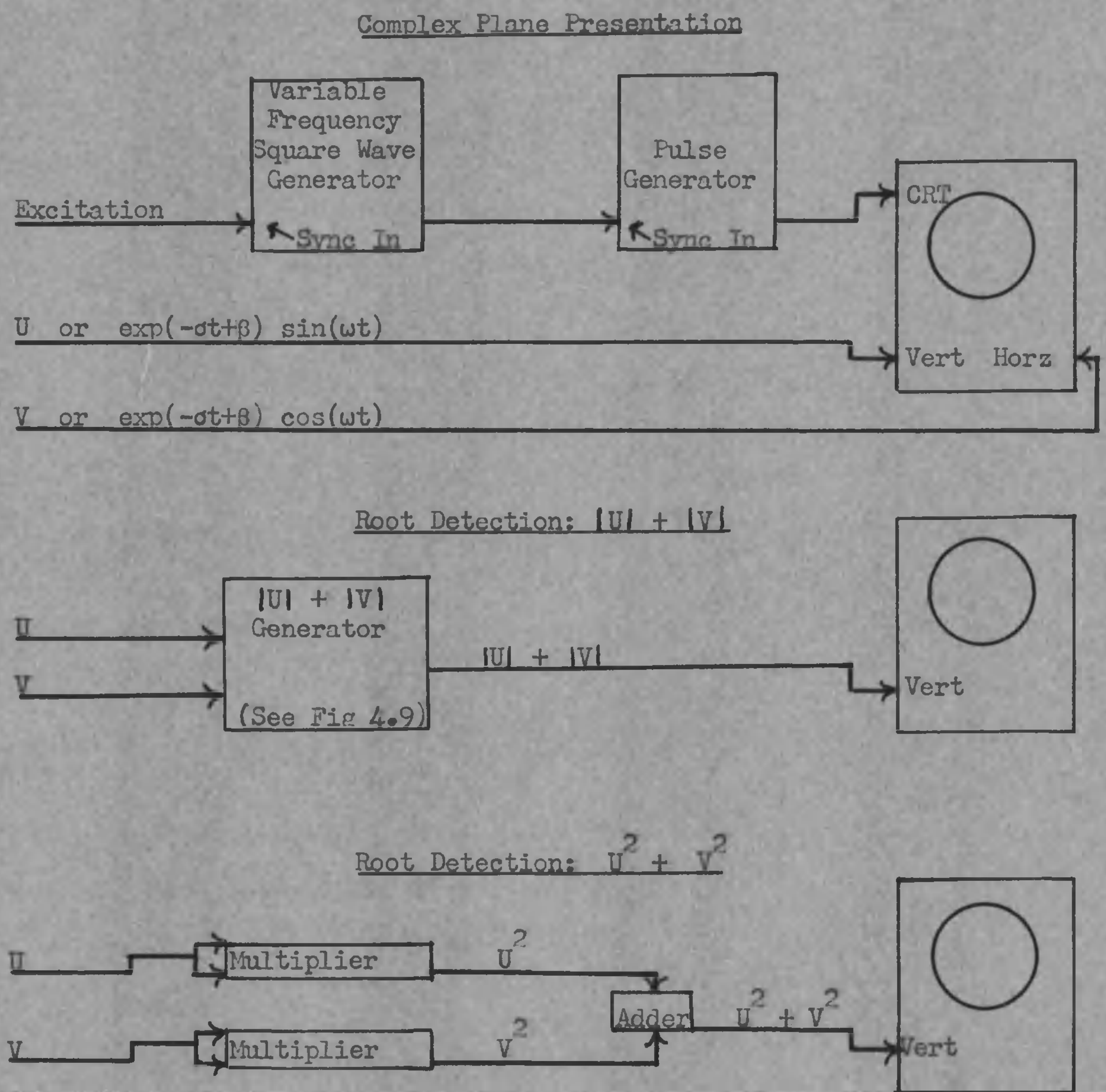


Adder and Repeated Root Checker

Switch Closed: $e_{out} = (e_{in_1} + e_{in_2} + e_{in_3})$

Switch Open: $e_{out} = 40\ d/dt(e_{in_1} + e_{in_2} + e_{in_3})$

FIGURE 4.7



Read Out Systems

FIGURE 4.8

the complex plane of the polynomial. The representation of the machine variable, z', in the complex plane was similarly obtained by applying $\exp(-\sigma t-\beta)\cos(\omega t)$ and $\exp(-\sigma t-\beta)\sin(\omega t)$ instead of U and V. The variable frequency square wave generator and pulse generator shown in the figure were used to amplitude modulate the scope trace with a time synchronized pulse to provide time calibration for the trace. This pulse was used to relate the complex planes of the machine variable and the polynomial. Unfortunately the pulse was not synchronized unless the frequency of the variable square wave generator was set on an integral multiple of the excitation square wave.

Figures 4.8b and 4.8c are the block diagrams of the read out systems capable of root detection only. The methods were essentially the same, the quantity $|U|+|V|$ or the quantity $U^2 + V^2$ being plotted against time. The times of waveform zeros fixed the root values.

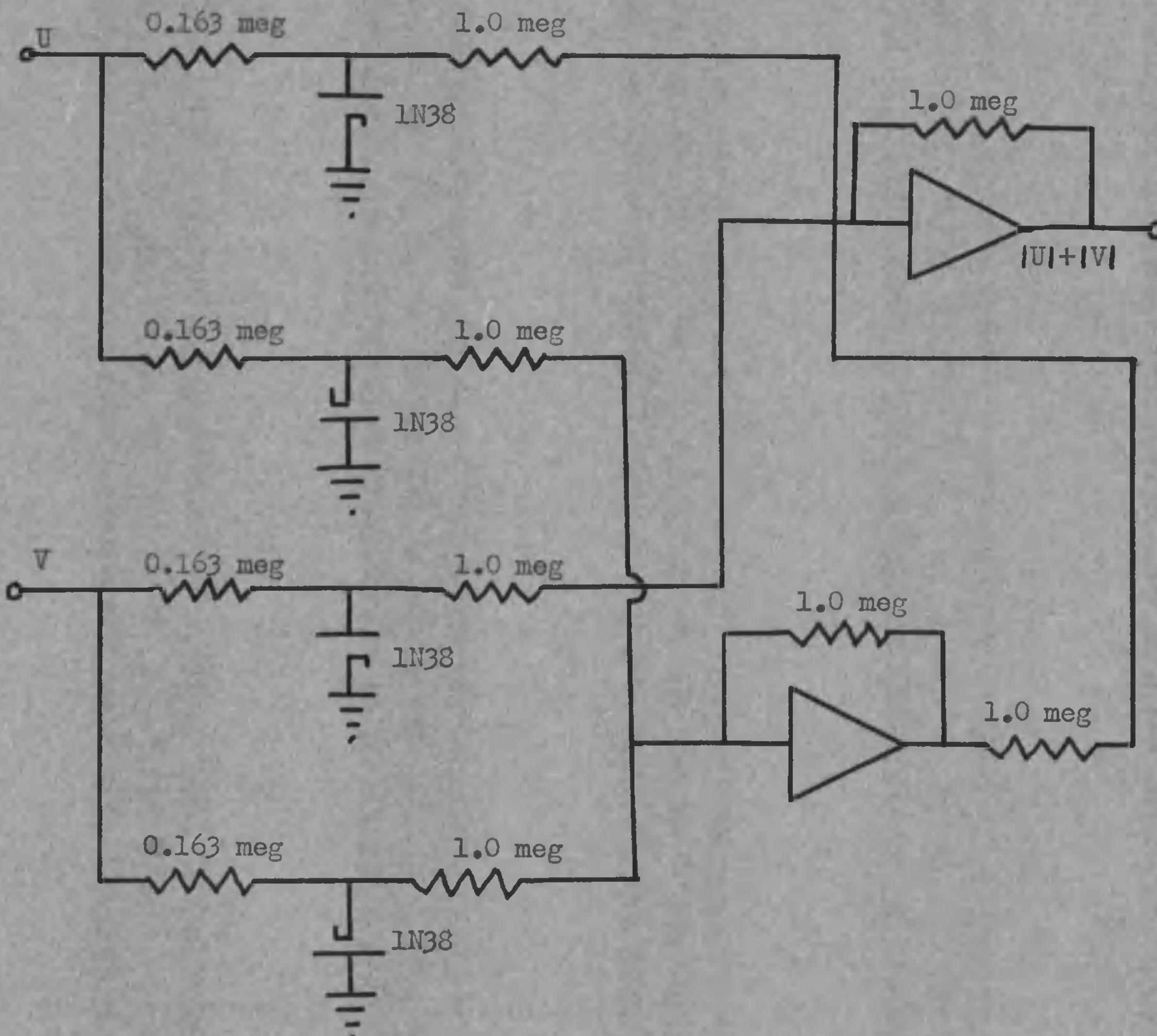
Figure 4.9 gives the details of the $|U|+|V|$ generator. This circuit can be considered as two full wave rectifiers with summed output.

The mechanization of the computer was completed with three Philbrick variable gain amplifiers. These provided an inverted function generator input to provide for the sign variations in the coefficients, a variable amplitude of the excitation square wave to represent the polynomial's constant term, and the amplification required to derive the clamping signal from the excitation signal.

Figure 4.10 presents two photographs of the actual computer.

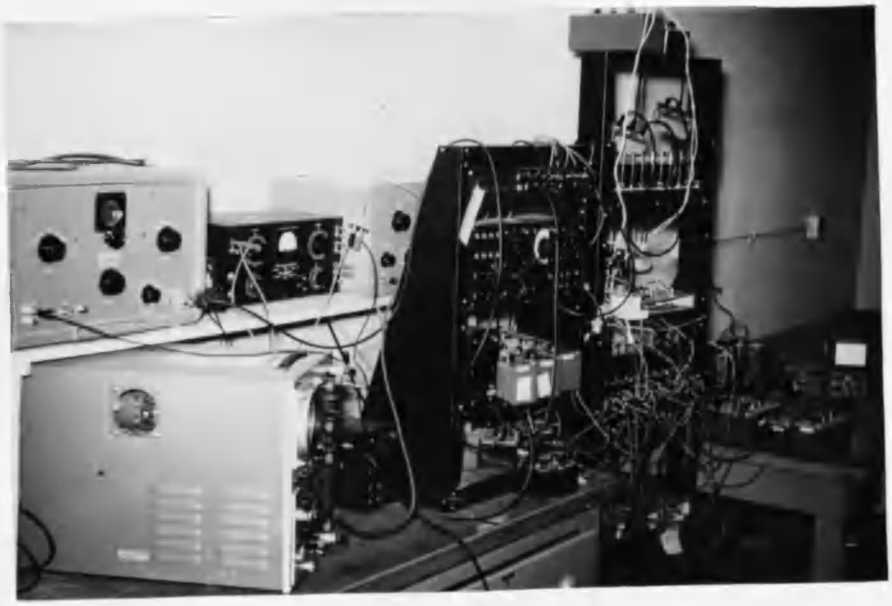
(4.4) Computer Wave Forms

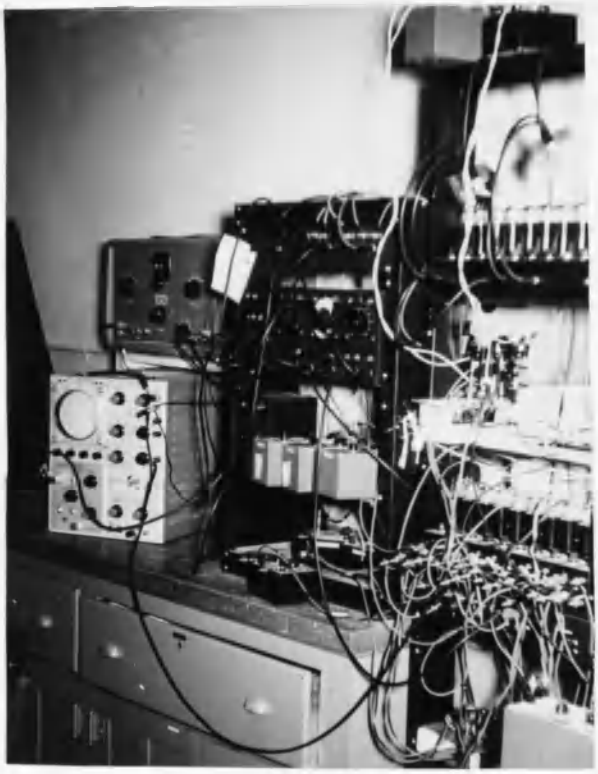
This section is devoted to the presentation of photographs of



Generation of |U| + |V|

FIGURE 4.9





Photographs of Experimental Computer

FIGURE 4.10

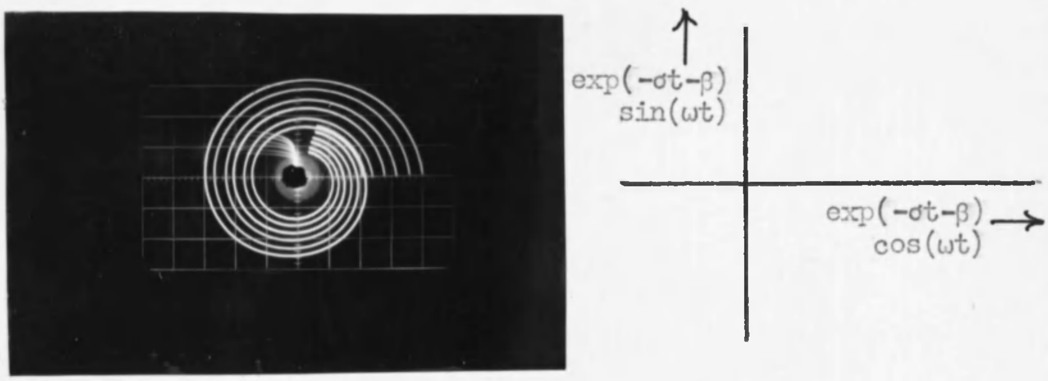
the computer wave forms. They provide verification of the theory and illustrate the type of results that can be expected.

Figure 4.11 is a multiexposure, intended to demonstrate the effect of the β variation. Each trace resulted from the application of $\exp(-\sigma t-\beta)\cos(\omega t)$ and $\exp(-\sigma t-\beta)\sin(\omega t)$ to respective horizontal and vertical deflection plates. Therefore each spiral represents a complex plane plot of the values covered by the machine variable for a specific value of β. The parameter of the family of traces is β which was varied in discrete steps over the required range of variation, 0 to 0.628. The lower bound of the machine variable, R, was set at about 0.47 by the action of the clamping circuits.

Figures 4.12, 4.13, 4.14, and 4.15 compare the four read out systems. The same equation was programed in all cases. This equation is

$$z^3 + 6z^2 - 32z + 80 = (z + 10)(z - 2 - j2)(z - 2 + j2) = 0 \tag{4.26}$$

The equation was programed in the computer with a scale factor of 5. Thus $z = 5z'$ where z' was the machine variable $\exp(-\sigma t-\beta+k\omega t)$. The lower limit, R, on the machine variable was set at 0.47. No special significance can be placed on these values as they were chosen for convenience and provided waveforms to a satisfactory scale. Since the factored form of the equation was available, it was not necessary to use the complete scaling procedures of Chapter 2. In all of the various solutions of equation 4.26, β was set at 0.025. For this setting the machine swept through a range of values that included the polynomial root, $z = 2 - j2$.



Complex Plane of Machine Variable;

Effect of Variation in β

FIGURE 4.11

Figure 4.12 gives the complex planes of both the polynomial and the machine variable. The traces resulted from the read out system of figure 4.8a. The amplitude modulation related the two planes at only one instant of time and in this sense the solution was not of the complete type. However, the modulating pulse is variable in time position and the planes can be related at any chosen instant of time. From this point of view, figure 4.12 represents a complete type of solution.

Figure 4.13 presents the simplest method of read out: the separate recordings of both U and V versus time. This is, of course, a complete type of solution.

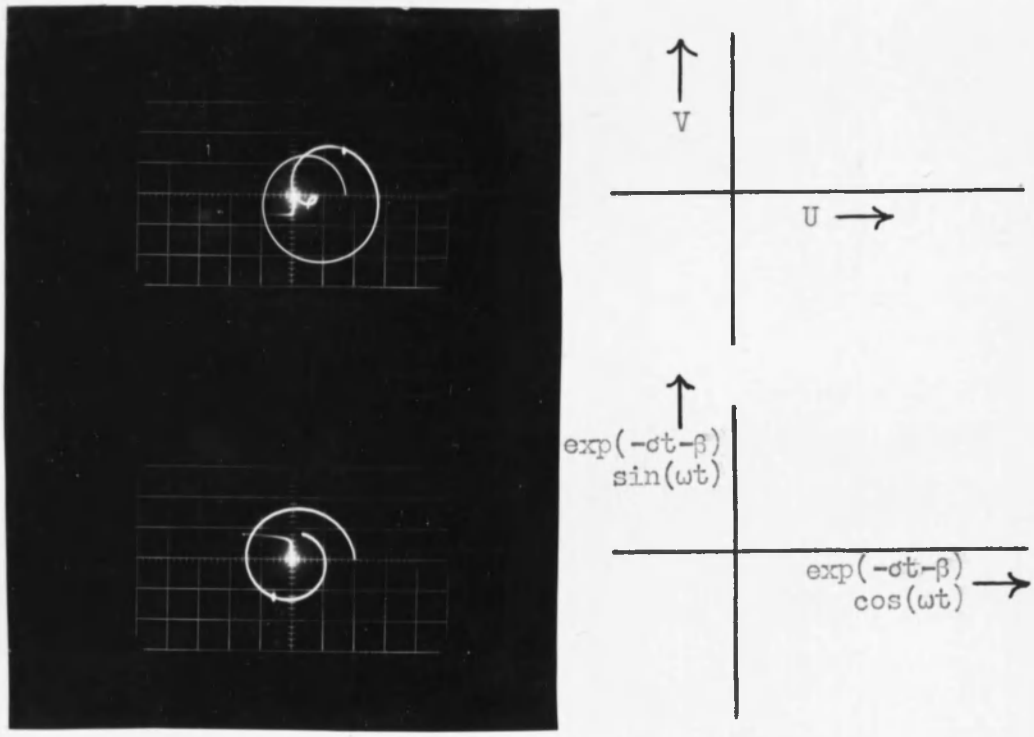
Figure 4.14 offers the oscillographic plots corresponding to the read out system of figure 4.8b. This plot of $|U|+|V|$ versus time gives only the root type solution, the wave form having no meaning except when it is zero. The two traces are the same wave forms; the second shows the effect of a recorder magnification of 50.

Figure 4.15 is analogous to figure 4.14 except that $U^2 + V^2$ is plotted against time. In this case, however, the waveforms have meaning at all instants of time as $U^2 + V^2$ is the magnitude of the polynomial.

Figures 4.12, 4.13, 4.14 and 4.15 have presented solutions of equation 4.26. Figure 4.16 illustrates the effect of the repeated root checker and results from programing the following equation.

$$z^3 + z^2 - 8z - 12 = (z + 2)(z + 2)(z - 3) = 0 \qquad (4.27$$

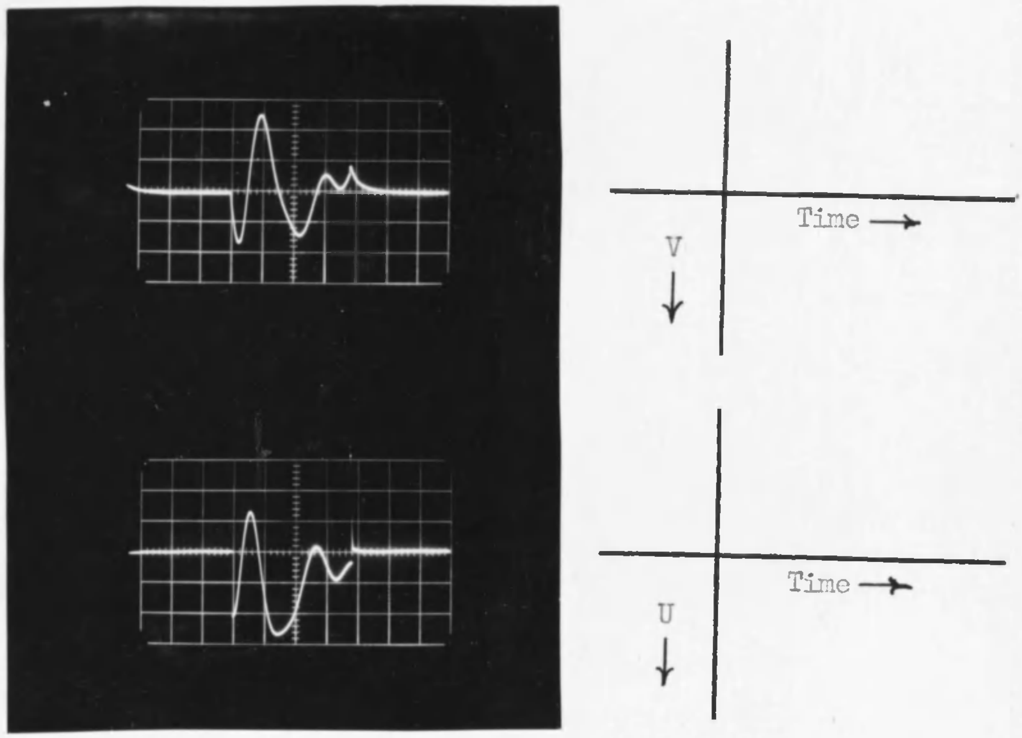
This equation was programed with a scale factor of 3 or with $z = 3z'$. This factor, like the scaling of equation 4.26, was chosen entirely for



Complex Planes of the Polynomial

and the Complex Variable

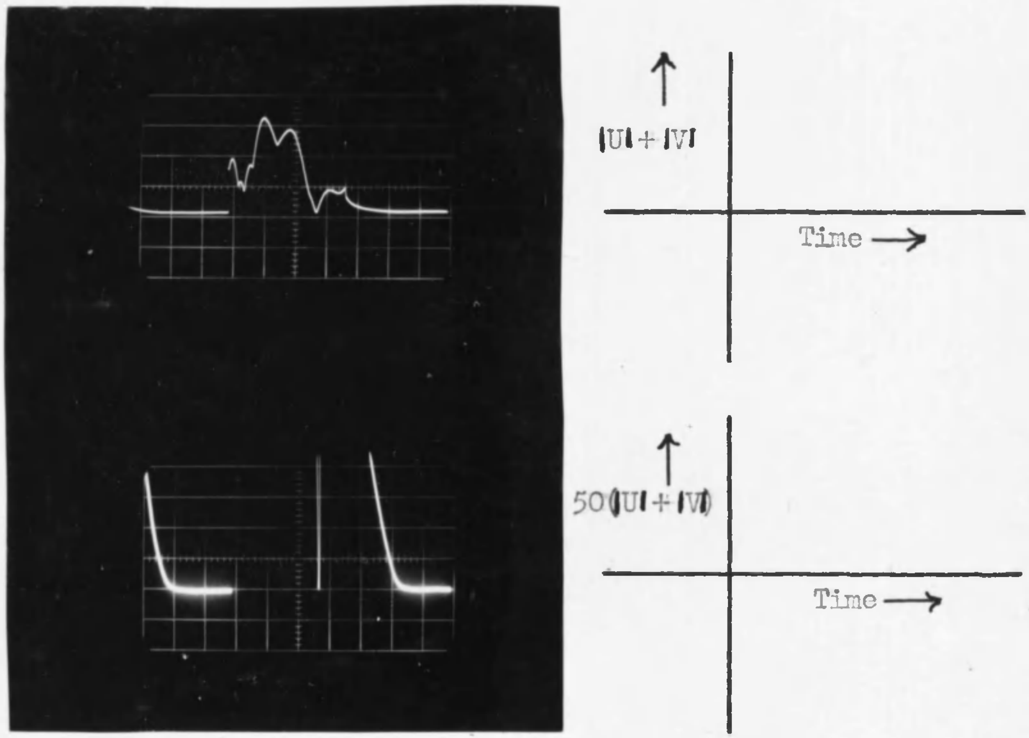
FIGURE 4.12



U versus Time; V versus Time

0.01 seconds per cm

FIGURE 4.13

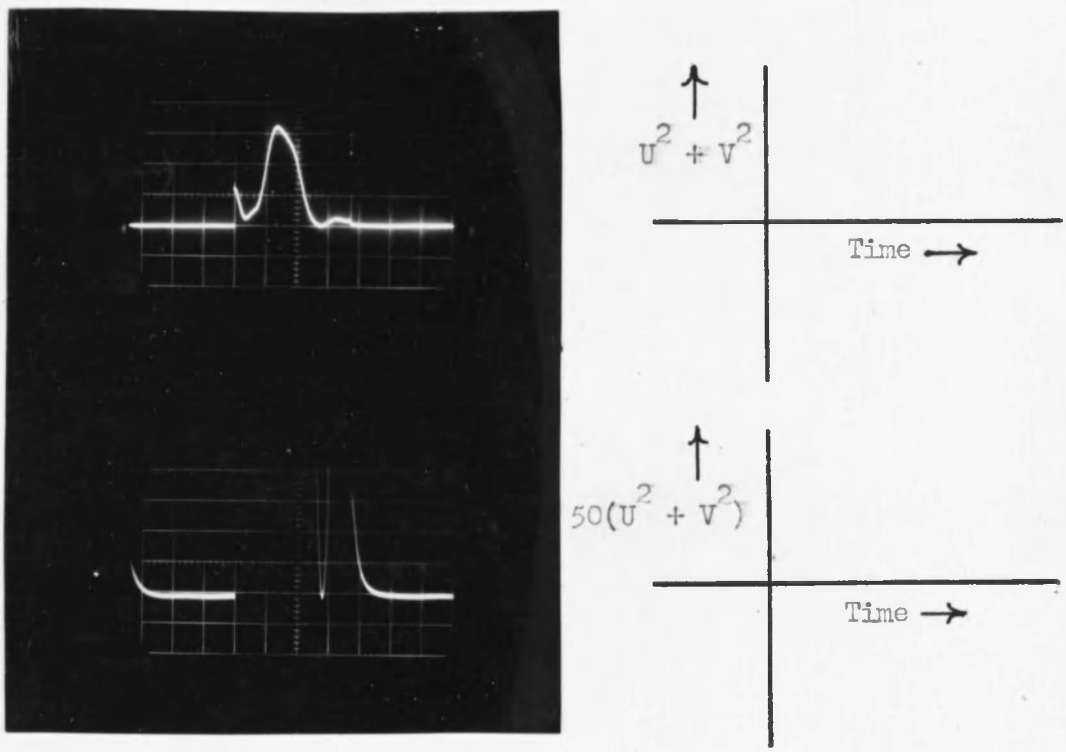


|U| + |V| versus Time

Second Trace Shows Effect of Magnification by 50

0.01 seconds per cm

FIGURE 4.14



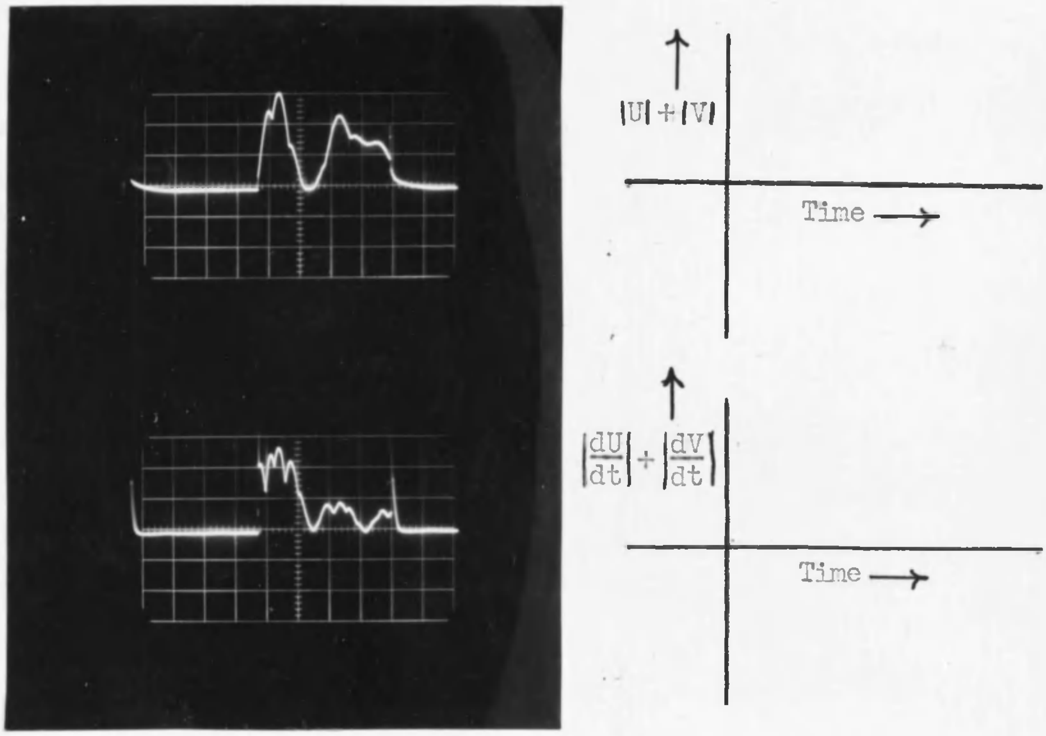
$U^2 + V^2$ versus Time

Second Trace Shows Effect of Magnification by 50

0.01 seconds per cm

FIGURE 4.15

Univ. of Arizona Library



$|U| + |V|$ versus Time; $|dU/dt| + |dV/dt|$ versus Time

0.01 seconds per cm

FIGURE 4.16

convenience and ease of presentation. Both of the traces of figure 4.16 are shown for the same value of β, 0.09. For this value the root at $z = -2$ is evaluated by the computer. The read out system chosen is the one used to obtain figure 4.14: the absolute value circuit of figure 4.9.

A repeated root is indicated by figure 4.16 as both $|U| + |V|$ and $|dU/dt| + |dV/dt|$ have zeros at the same instant of time. A second zero apparently exists in the differentiated trace but it has no meaning.

(4.5) Computer Accuracy

A computer design which includes the β variation has no theoretical accuracy limit. The computer under discussion was limited as it was constructed on a "breadboard" basis with equipment that was satisfactory but which lacked many desired features. In particular, the G. A. Philbrick components which formed the heart of the computer were relatively low accuracy equipments with no d-c stabilization and poor loading characteristics.

The photographs of figures 4.12 to 4.15 illustrate the effect of various read out systems, all of which were fed by the same basic computer. Figures 4.14 and 4.15 are the best illustrations of the accuracy potentialities of the machine. The magnified traces show exceedingly sharp definition of wave form zero. The time of the zero was obtained from the calibrated trace of figure 4.14 as 27.5 milliseconds. This value is an averaged value obtained from several readings. Substitution of this value of time into equations 1.3 and 1.4 and applying the scale factor of 5 gives the value of the root as $z = 2.81 \angle 316°$

The actual value of the root as fixed by equation 4.26 is $z = 2 - j2$, or in polar form, $z = 2.83 \angle 315°$.

The experimental root agrees with the actual value with an accuracy of better than 1% in both argument and amplitude. This is indicative of possible accuracy but an accuracy figure of 2 to 5% is probably more realistic for the present computer as the value of 27.5 milliseconds is an averaged value; neither the photograph nor the original trace as presented on the oscilloscope's screen could be read on any one reading to an accuracy of more than one millisecond. The accuracy obtainable from figures 4.12 and 4.13 was far below the 5% figure as recorder magnification was not possible. This was the fault of the read out system, not the basic computer.

In addition to the above discussion on accuracy it would be desirable to present figures on the speed of operation. Unfortunately, representative values cannot be given. It was necessary to program β into each separate function generator. This was a very time consuming process which would usually be eliminated in other than a pilot model computer. Furthermore, data is not available on the time required for the computation and setting of the scaled and normalized coefficients. The graphical technique mentioned at the end of section 2.3 was checked and found to be relatively fast but it was not applied to the equations used above. The author is certain, however, that the machine is faster than long hand methods, even for third degree equations which have analytical solutions. For high degree polynomials there is no question about the advantage of the machine method of solution.

(4.6) Recommendations for Further Mechanization

This section is devoted to the presentation of proposed improvements in the pilot model computer. Although equal or superior methods of mechanization undoubtedly exist, the following are ideas that would have been realized if the time and equipment had been available.

The most obvious change required for a general purpose computer is an increase in capacity. The pilot computer was designed for third degree polynomials. This should be increased to handle at least fifth or sixth degree equations. There are no difficulties foreseen to hinder this expansion.

The method of β sweep should be changed to a ganged assembly so only one external setting has to be made for a specific value of β. Commercially available potentiometers of the required nonlinear design could produce this mechanization.

The setting of the coefficients of the function generators by variable capacitors should be replaced by linear potentiometers as the present mechanization is slower than required.

The d-c drift of the operational amplifiers should be corrected with long time d-c stabilization. This is important as correct d-c voltage levels are necessary to the proper operation of the computer. The maintenance of these levels is very difficult with the present computer. The required equipment is commercially available.

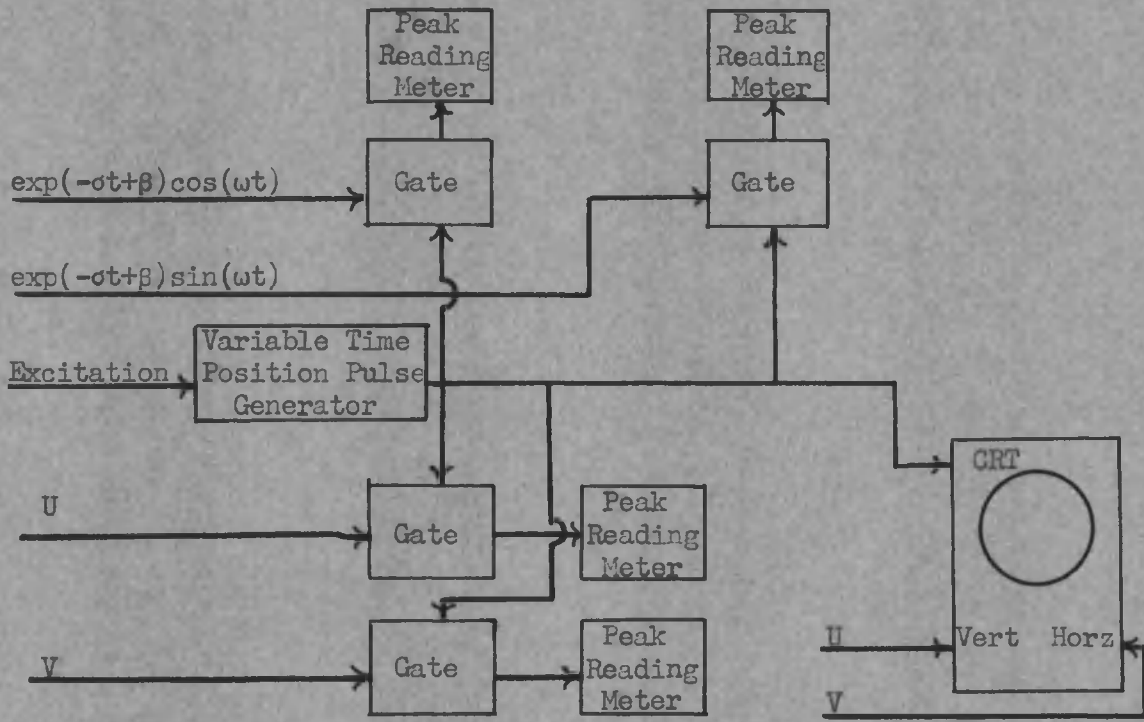
The breadboard type construction must be eliminated. The use of haphazard, unshielded wiring resulted in a large quantity of 60 cycle hum. This hum was very annoying.

There are two features that would be desirable but not necessary to correct. 1) The circuitry of the function generators loads the operational amplifiers to a small degree. 2) The clamping action is not perfect. Neither of these faults introduce noticeable error provided a correct voltage level is maintained.

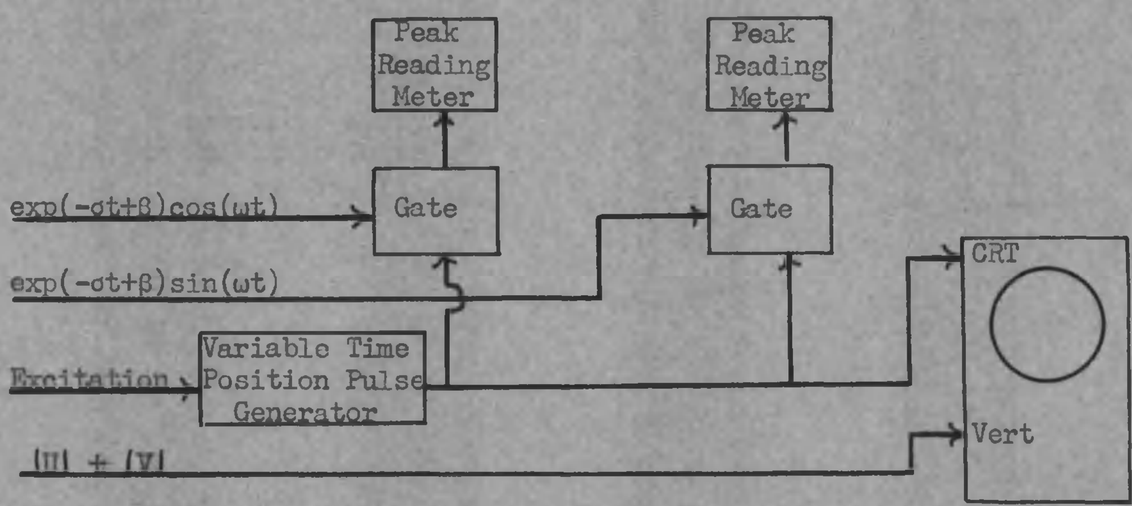
The final modification of the computer should be an increase in the ratio of ω to σ. It was previously mentioned that this introduces increased loading problems but they can be solved, especially if better operational amplifiers are used. Further work is required to determine the exact value of ω/σ as the author has been unable to arrive at an optimum figure.

The choice of read out system depends on the type of solution required. Figure 4.17 presents two read out systmes, one for the complete and one for the root type solution. The root solution system is probably deserving of the most attention. Both systems combine the advantages of a direct reading output without losing the overall picture that is presented by an oscillographic trace. The gates pass a pulse of height corresponding to the value of the input at the time of the synchronizing pulse. The peak reading meters are calibrated to record the amplitude of this height. Both of the systems of figure 4.17 appear to be practical; however, no actual laboratory work has been attempted in the area.

Complete Read Out



Root Read Out



Proposed Read Out Systems

FIGURE 4.17

SUMMARY

The object of this thesis is the description of a machine technique designed to fill the need for a fast, accurate, and economical method of solving general polynomials of any degree. In the opinion of the author, this object has been fulfilled.

The accuracy potentialities of the technique are excellent. The basic theory of operation need not include any approximations. However, in many applications an approximation is practical which will simplify the mechanization. The complete elimination of mechanization errors is impossible but they need not be large as the function generation, addition, and read out systems can be developed without excessive difficulty. These predications of good accuracy are borne out by the pilot computer. In spite of makeshift construction, the experimental machine has an accuracy of around 2%.

In all types of polynomial solving machines, the basic idea is the evaluation of the polynomial for variations in the complex variable. The machine presented in this thesis used time to provide most or all of these variations. Thus a fast solution is indicated, especially in electronic mechanizations. Speed of operation is reduced by the need for scaling and programing but this is common to all types of computers. The required calculations are usually negligible compared to the long hand method of solution.

The economics of mechanization cannot be accurately evaluated because of the wide variety of possible designs and applications. In

general, however, analog mechanizations are considered to be relatively inexpensive. Also, with the possible exception of a set of commercially available, nonlinear potentiometers, the computer requires no special or unconventional equipment. Although an extensive economic study has not been made, it seems that commercial production may be economically feasible. It is certain that a well equipped computer laboratory could construct a polynomial solving machine almost entirely from available equipment.

The basic mechanizational technique of the pilot computer is satisfactory but further work in the field is indicated and hoped for.

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