THE GARMAN KOLHAGEN MODEL FOR FOREIGN EXCHANGE OPTION PRICING:

DERIVATION AND APPLICATION

By

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With Honors in
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Abstract

This paper explores the limitations of traditional option pricing models with regard to foreign exchange options. The prevalent foreign exchange option-pricing model will be derived using stochastic calculus and partial differential equations. Then this model will be presented in MATLAB software, and its validity empirically tested using actual foreign exchange option quotes.
Introduction

In the past several decades, use of options for hedging or speculative purposes has become widespread. Firms can use options to mitigate price fluctuations in physical commodities and securities, as well as interest rate risk. Another useful application of options is for foreign exchange, in which the underlying instrument is a fixed amount of a foreign currency. The use of foreign exchange options (referred to henceforth as “forex” options) is prevalent among firms who have significant operations outside of the country that rely on currency exchange rates. Further, forex options can be used as a part of an investment portfolio as a speculative instrument.

Perhaps the most prominent option pricing method is the Nobel Prize winning Black-Scholes model (1973), developed by Myron Scholes and Fischer Black. The underlying instrument in this model is a non-dividend paying stock with returns that follow a geometric Brownian motion. This model is not sufficient for forex options, however, because the distribution only accounts for the domestic risk-free interest rate. In currency exchange, both the domestic and foreign risk-free rates need to be accounted for, thus making the Black-Scholes model inappropriate for pricing. Interest rate parity dictates that the forward premium must equal the interest rate differential, which is reflected in the Garman-Kohlhagen forex option-pricing model (1983) [1].

The Garman-Kohlhagen model will be derived and then tested empirically with MATLAB simulation. Current forex option quotes will prove whether or not forex options are actually priced on exchanges using this model.
Statement of Problem

The Black-Scholes model is ineffective for the pricing of forex options because it only includes the domestic riskless rate in its calculation of the forward premium. A few alterations to the model make this a viable method for the pricing of forex options.

The following variables are needed to model prices of forex options:

\[ S \] Spot price* of deliverable currency
\[ K \] Exercise price of option
\[ T \] Time remaining until maturity of option
\[ C(S,T) \] Price of call option
\[ r_d \] Domestic riskless interest rate
\[ r_f \] Foreign riskless interest rate
\[ \sigma \] Volatility of spot currency price
\[ \mu \] Drift of spot currency price
\[ N(.) \] Cumulative normal distribution function
\[ \alpha \] Expected rate of return on a security
\[ \delta \] Standard deviation of the security rate of return

*All prices are in domestic units per foreign units

Additionally, several assumptions have been made:

1. Spot prices follow a geometric Brownian motion; spot price movements are represented differentially by \((1)\) \[ dS = \mu S \, dT + \sigma S \, dz \], where \(z\) is the standard Wiener process
2. Option prices include only one stochastic variable, \(S\)
3. Markets are frictionless
4. Interest rates, both foreign and domestic, are constant

This paper will focus on the pricing of European call options, which can only be exercised at maturity. To fully understand the role of the two riskless rates, ownership of a forex option can be compared to ownership of the underlying currency. In an arbitrage-free, continuous-time economy, excess returns of securities must equal

\[ \frac{\alpha - r_d}{\delta} = \lambda \]  

for all securities. Ownership of the underlying currency yields returns

\[ \frac{(\mu + r_f) - r_d}{\sigma} = \lambda \]  

in which the excess return is represented by the drift of the exchange rate plus the riskless return of assets in the foreign currency. The denominator, \( \sigma \) is the volatility of the spot currency price, or the standard deviation of the return on the currency.

Next, assuming \( C = C(S, T) \), the price of a call option, with time \( T \) to maturity,

\[ \frac{\alpha_C - r_d}{\delta_C} = \lambda \]

The derivative of \( C(S, T) \) then corresponds to the change in call price per unit of time, \( dT \). This differential can be found using a Taylor-expansion of the call price \(^2\), according to Ito’s Lemma.

\[ dC = \frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + \frac{\partial C}{\partial S} \frac{\partial C}{\partial T} dSdT + \frac{1}{2} \frac{\partial^2 C}{\partial T^2} dT^2 + \ldots \]
Recall (1), the differential of the spot price, included \(dz\), the standard Wiener process in which
\[dz = wv \sqrt{dT}.\]
The variable \(w\) represents a standard normal random variable, with \(v\) as its standard deviation. So,
\[
(6)
\]
\[dS = \mu S \, dT + \sigma S \, (wv \sqrt{dT})
\]
Substituting this value into (5),
\[
dC = \frac{\partial C}{\partial T} \, dT + \frac{\partial C}{\partial S} \, [\mu S \, dT + \sigma S \, (wv \sqrt{dT})] + \frac{1}{2} \left( \frac{\partial^2 C}{\partial T^2} \right) \, dT^2 + \frac{1}{2} \left( \frac{\partial^2 C}{\partial S^2} \right) \, [\mu S \, dT + \sigma S \, (wv \sqrt{dT})]^2
\]
\[+ \frac{\partial^2 C}{\partial S \, \partial T} \, dT [\mu S \, dT + \sigma S \, (wv \sqrt{dT})] + \ldots
\]
\[= \frac{\partial C}{\partial T} \, dT + \frac{\partial C}{\partial S} \, \mu S \, dT + \frac{\partial C}{\partial S} \, \sigma S \, (wv \sqrt{dT}) + \frac{1}{2} \left( \frac{\partial^2 C}{\partial T^2} \right) \, dT^2
\]
\[+ \frac{1}{2} \left( \frac{\partial^2 C}{\partial S^2} \right) \left[ \mu^2 S^2 dT^2 + 2 \mu S \, dT \, \sigma S \, wv \sqrt{dT} + \sigma^2 S^2 v^2 w^2 dT \right]
\]
\[+ \frac{\partial^2 C}{\partial S \, \partial T} \left[ \mu S \, dT^2 + \sigma S \, (wv \sqrt{dT})^2 \right] + \ldots
\]
The Brownian motion governing spot prices is continuous-time stochastic, so the change in time, \(dT\), is an infinitesimal quantity. Thus terms containing powers of \(dT\) greater than one effectively vanish, yielding
\[
(7)
\]
\[dC = \frac{\partial C}{\partial T} \, dT + \frac{\partial C}{\partial S} \, \mu S \, dT + \frac{\partial C}{\partial S} \, \sigma S \, (wv \sqrt{dT}) + \frac{1}{2} \left( \frac{\partial^2 C}{\partial S^2} \right) \sigma^2 S^2 (v^2 w^2 dT)
\]
As a standard normal random variable, the expected values of \( w^2 \) and \( v^2 \) are one. The differential \( dC \) can then be separated into the deterministic component,

\[
\frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S} \mu S dT + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dT
\]

and the stochastic component,

\[
\frac{\partial C}{\partial S} \sigma S dz
\]

summing to:

(8)

\[
dC = \frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S} \mu S dT + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dT + \frac{\partial C}{\partial S} \sigma S dz
\]

Now consider the delta-hedge portfolio, \( \Pi \). This portfolio is hedged to a risk-neutral position by holding a short position on the forex call option and a long position on the underlying currency. The value of this portfolio is then

(9)

\[
\Pi = -C + \frac{\partial C}{\partial S} S
\]

The total profit or loss from the delta-hedged portfolio is represented by

(10)

\[
d\Pi = -dC + \frac{\partial C}{\partial S} dS
\]

Substituting (6) and (8) into (10),

(10')
\[ d\Pi = -\frac{\partial C}{\partial T}dT - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dT \]

Notice the Wiener process has disappeared completely. Thus the “random walk” component has been removed, rendering the delta-hedged portfolio effectively risk-free. It follows that this portfolio will have a return during time \(dT\) equal to that of other riskless assets in an arbitrage-free economy. Therefore, \(r\Pi dT = d\Pi\).

(11)

\[ r \left[ -C + \frac{\partial C}{\partial S} S \right] dT = -\frac{\partial C}{\partial T}dT - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dT \]

Note that excess returns on a security relate directly to the domestic risk-free interest rate, whereas excess returns on a long position of the underlying currency are governed by the interest rate parity principle. So excess returns on the currency are estimated by the interest rate differential. Thus,

(12)

\[ \frac{\partial C}{\partial T} = \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 + \frac{\partial C}{\partial S} S (r_d - r_f) - r_d C \]

This PDE is similar to the Samuelson-Merton model for stocks with constant dividends. As applied to stocks, the equation assumes that a firm will consistently pay a proportional dividend. This rather large assumption makes the model ineffective for the pricing of stock options. However, the proportionality assumption holds for foreign exchange. The “adjustment of dividends” occurs naturally during conversion from foreign to domestic currency.

The solution to (12) can be obtained easily, yielding

(13)
\[ C(S, T) = e^{-\tau T} S N(x + \sigma \sqrt{T}) - e^{-\tau d T} K N(x) \]

where

\[
  x = \frac{\ln \left( \frac{S}{K} \right) + \left( r_d - r_f - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]
**Method**

Simple examples can be used to illustrate the results of this model.


<table>
<thead>
<tr>
<th>USD</th>
<th>JPY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riskless rate, r</td>
<td>0.5%</td>
</tr>
<tr>
<td>Spot price, S</td>
<td>102</td>
</tr>
<tr>
<td>Exercise price, K</td>
<td>90</td>
</tr>
<tr>
<td>Volatility, σ</td>
<td>4%</td>
</tr>
<tr>
<td>Time to maturity, T</td>
<td>3 months</td>
</tr>
<tr>
<td>Price of call option</td>
<td>$11.86</td>
</tr>
</tbody>
</table>

This call option is considered “in the money” because its exercise price, K, is less than its spot price, S. An option’s moneyness is reflected in its price— options further in the money carry higher prices and vice versa.

**Results**

The Garman-Kohlhagen model can be empirically tested using real option pricing data and MATLAB software [3]. A simple function can allow for the relevant inputs and draw returns from a normal distribution.

```matlab
function fxoptions( S0, X, rd, rf, T, vol, style)
    mynormcdf=@(x) quadgk( @(t) exp(-t.^2 / 2), -10, x)/sqrt(2*pi);
    if strcmp(style, 'E')||strcmp(style, 'e')
        F=S0*exp((rd-rf).*T);
        d1=log(F./X)./(vol.*sqrt(T))+vol.*sqrt(T)/2;
        d2=log(F./X)./(vol.*sqrt(T))-vol.*sqrt(T)/2;
        European_call = exp(-rd.*T).*(F.*mynormcdf(d1)-X.*mynormcdf(d2));
        European_put = European_call+(X-F)*exp(-rd.*T);
        fprintf('European Call: $%0.5g \n',European_call)
        fprintf('European Put:  $%0.5g \n\n',European_put)
    end
```
Example 1:
The following is a call option listed on NASDAQ’s option chain on May 4, 2014 [5].

- US Dollar- British Pound
- Expiration on June 21, 2014
- Assumptions:
  - $r_d$ is the two-month US Treasury rate as of May 2014, 0.025%
  - $r_f$ is the two-month LIBOR rate as of May 2014, 0.505%
  - $\sigma$ is measured at 4% per annum

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>GBP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riskless rate, $r$</td>
<td>0.025%</td>
<td>0.505%</td>
</tr>
<tr>
<td>Spot price, $S$</td>
<td>$169$</td>
<td></td>
</tr>
<tr>
<td>Exercise price, $K$</td>
<td>$152$</td>
<td></td>
</tr>
<tr>
<td>Volatility, $\sigma$</td>
<td>4%</td>
<td></td>
</tr>
<tr>
<td>Time to maturity, $T$</td>
<td>2 months</td>
<td></td>
</tr>
<tr>
<td>Current Bid-Ask</td>
<td>$15.30$-$17.90$</td>
<td></td>
</tr>
<tr>
<td>$fxoptions(\ )$</td>
<td>$16.93$</td>
<td></td>
</tr>
</tbody>
</table>

The USD-GBP call option will likely be priced between its current bid and ask at the time of maturity. The Garman-Kohlhagen model yields a price within that range.

Example 2:
The following is a call option listed on NASDAQ’s option chain on May 4, 2014.
- US Dollar- Japanese Yen
- Expiration on March 20, 2015
- Assumptions:
  - $r_d$ is the one-year US Treasury rate as of May 2014, 0.1000%
  - $r_f$ is the one-year JPY LIBOR rate as of May 2014, 0.3442%
  - $\sigma$ is measured at 6% per annum
<table>
<thead>
<tr>
<th>USD</th>
<th>JPY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riskless rate, $r$</td>
<td>0.1000%</td>
</tr>
<tr>
<td>Spot price, $S$</td>
<td>$102$</td>
</tr>
<tr>
<td>Exercise price, $K$</td>
<td>$89$</td>
</tr>
<tr>
<td>Volatility, $\sigma$</td>
<td>6%</td>
</tr>
<tr>
<td>Time to maturity, $T$</td>
<td>12 months</td>
</tr>
<tr>
<td>Current Bid-Ask</td>
<td>$8.80$-$10.65$</td>
</tr>
<tr>
<td>$\text{fxoptions( )}$</td>
<td>$11.74$</td>
</tr>
</tbody>
</table>

The USD-JPY call option will likely be priced between its current bid and ask as well. The Garman-Kohlhagen model did not yield a value in this range for this option. Perhaps this is due to the one-year maturity of the option in Example 2, as opposed to the two-month maturity in Example 1. Because the model is a function of time-varying volatility, one would expect it to yield more accurate results when closer to maturity.
Conclusion and Discussion

The Garman-Kohlhagen model is a direct consequence of Ito's Lemma for time-dependent stochastic processes. Unlike its close relative, the Black-Scholes option pricing model, Garman-Kohlhagen allows the user to price forex options, which pose the additional challenge of two interest rates. Adjustments to return structure allow forex options to seemingly be priced accurately and completely.

An empirical test of the model in MATLAB, however, yielded mixed results. For options with maturities in the near-term, the Garman-Kohlhagen model as written in MATLAB produced fairly accurate prices compared to the current quotes on identical call options [4]. The software model did not produce accurate prices for call options with maturities in one year or greater, however. This is likely due to the increased volatility experienced by the holder of the longer-term option.

From this analysis, it is clear that forex options currently being quoted in exchanges are not being priced according to the Garman-Kohlhagen formula. The model appears to be somewhat clumsy due to its strict assumptions about the economy. Therefore, it is likely that currently listed options are priced using multivariate stochastic calculus or a combination of such models.

In the future, an extensive database of historical option quotes could be obtained and the MATLAB model could be tested in a more comprehensive manner. Access to such data limited the analysis in this paper. Additionally, it may be worth exploring the relationship between the “random walk” Brownian motion and the normal distribution that governs price fluctuations.
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References


