

CONSTRUCTING SIMULTANEOUS DIOPHANTINE  
APPROXIMATIONS OF CERTAIN CUBIC NUMBERS

by  
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A Dissertation Submitted to the Faculty of the  
GRADUATE INTERDISCIPLINARY PROGRAM  
IN APPLIED MATHEMATICS

In Partial Fulfillment of the Requirements  
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

2014

THE UNIVERSITY OF ARIZONA  
GRADUATE COLLEGE

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## ACKNOWLEDGMENTS

Thank you to my advisor Marek Rychlik for sharing his knowledge and time over these last three and a half years, and for the stimulating conversations which led to the main ideas of this dissertation. Thank you to my committee members — Kirti Joshi, Dan Madden, and Dinesh Thakur — for taking the time to read through several drafts of this dissertation and for their many helpful suggestions. In addition, thank you to Kirti and Dinesh for all of their help over the years, and for the encouraging conversations regarding this work. Thank you to John Brillhart for the memorable stories and valuable conversations, many of which served as inspiration for ideas in this dissertation. Thank you to Yann Bugeaud for pointing me toward some results I would have otherwise missed.

Fall 2008, spring 2009, summer 2009, and spring 2012 research assistantships were funded by the University of Arizona Department of Mathematics NSF VIGRE (Vertical InteGration in Research and Education) grant. The remaining semesters of my PhD (through spring 2014) were funded by teaching assistantships. Fall 2014 was funded by a Thesis/Dissertation Tuition Scholarship.

Thank you to Marion Scheepers (my Master's thesis advisor at Boise State University), Uwe Kaiser, Justin Moore, and Stuart Zoble for their time and help during my years at BSU, and for inspiring me to want to be a mathematician. At a time when my math hobby had transformed into a full-blown obsession, their welcoming my never-ending questions certainly influenced my decision to eventually pursue a PhD. I am also grateful for Stuart for his being the topic about which my now-wife and I had our first few conversations.

Thank you to everyone else who has helped or inspired or encouraged me at any point over the years, notably: (at UA) Romyar Sharifi, Tom Kennedy, Jan Wehr, Klaus Lux, Alan Newell, Doug Pickrell, Bryden Cais, Sandy Sutton, Sylvia Anderson, Martin Leslie, Sarah Mann, Alex Tao, Aleks Petrov, John Kerl; (at BSU) Billy Hudson, Randall Holmes, Kathy Rohrig, Lorin McArthur, Scott Barron, Paula Smith, Nanna Hanchett; (at AIM) David Farmer, Matt Young, Brian Conrey; (at BHS) Debra Thorson, Dan Masters, Jon Hanson.

Thank you to my parents and Tim, Joanne, and Patty.

Most importantly, I thank my wife, Crystal. For EVERYTHING.

## DEDICATION

For Crystal and Nova.

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## ABSTRACT

For  $K$  a cubic field with only one real embedding and  $\alpha, \beta \in K$ , we show how to construct an increasing sequence  $\{m_n\}$  of positive integers and a subsequence  $\{\psi_n\}$  such that (for some constructible constants  $\gamma_1, \gamma_2 > 0$ )

$$\max\{\|m_n\alpha\|, \|m_n\beta\|\} < \frac{\gamma_1}{m_n^{1/2}}$$

and

$$\|\psi_n\alpha\| < \frac{\gamma_2}{\psi_n^{1/2} \log \psi_n}$$

for all  $n$ . As a consequence, we have

$$\psi_n \| \psi_n \alpha \| \| \psi_n \beta \| < \frac{\gamma_1 \gamma_2}{\log \psi_n}$$

for all  $n$ , thus giving an effective proof of Littlewood's conjecture for the pair  $(\alpha, \beta)$ . Our proofs are elementary and use only standard results from algebraic number theory and the theory of continued fractions.

CHAPTER 1  
INTRODUCTION

## 1.1 Background and Context

Littlewood conjectured that for real numbers  $\alpha$  and  $\beta$ ,

$$\liminf_{n \rightarrow \infty} n \|n\alpha\| \|n\beta\| = 0, \quad (1.1.1)$$

where  $\|x\| = |x - \lfloor x + \frac{1}{2} \rfloor|$  denotes the distance from  $x$  to the nearest integer. It is straightforward<sup>1</sup> to show this is true if  $1, \alpha, \beta$  are  $\mathbb{Q}$ -linearly dependent. From the theory of continued fractions, we know that the conjecture is true if either  $\alpha$  or  $\beta$  has unbounded partial quotients in its continued fraction expansion. Although almost all real numbers have unbounded partial quotients (with respect to Lebesgue measure – see Theorem 29 of [Kh]), we actually have very few examples<sup>2</sup> of common numbers which are known to have unbounded partial quotients. For instance, we don't know whether the partial quotients are bounded or unbounded for  $\pi$  or for any specific non-quadratic algebraic numbers.

Cassels and Swinnerton-Dyer had the first major result with their 1955 paper [CSD55]. They showed that the conjecture holds for the pair  $(\alpha, \beta)$  when  $\alpha, \beta \in \mathbb{R}$  are both in the same cubic field. Their proof involved showing that

$$\inf_{m, n \neq 0} |mn| \|m\alpha + n\beta\| = 0,$$

---

<sup>1</sup>See Remark 2.1.32 for example.

<sup>2</sup>Most of these involve values of the exponential function, for instance (see [Le18], [Sh92], [Th96], [RS])

$$\begin{aligned} e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots] = [2; \overline{1, 2n, 1}]_{n=1}^{\infty}, \\ e^{1/k} &= [1; \overline{(2n-1)k-1, 1, 1}]_{n=1}^{\infty}, \\ \coth(1/k) &= \frac{e^{2/k} + 1}{e^{2/k} - 1} = [k; 3k, 5k, 7k, 9k, \dots] \end{aligned}$$

and then showing that this implies (1.1.1).

Peck showed a slightly stronger result in 1961 with [P61]. He showed that if  $K \subset \mathbb{R}$  is an algebraic extension of  $\mathbb{Q}$  of degree  $n$ , and if  $\gamma_1, \dots, \gamma_n$  is a  $\mathbb{Q}$ -basis of  $K$ , then there exist infinitely many integer  $n$ -tuples  $(q_1, \dots, q_n)$  with  $q_1 > 0$  and  $\gcd(q_1, \dots, q_n) = 1$  such that

$$|q_1 \gamma_j - q_j \gamma_1| < \frac{C_1}{q_1^{1/(n-1)} \log q_1^{1/(n-2)}}$$

(for  $j = 2, \dots, n-2$ ), and

$$|q_1 \gamma_n - q_n \gamma_1| < \frac{C_2}{q_1^{1/(n-1)}}.$$

In particular, if  $[K : \mathbb{Q}] = 3$  and  $(\gamma_1, \gamma_2, \gamma_3) = (1, \alpha, \beta)$ , then there are infinitely many triples  $(q_1, q_2, q_3)$  with  $q = q_1 > 0$  such that

$$\|q\alpha\| = |q\gamma_2 - q_2| < \frac{C_1}{q^{1/2} \log q}, \quad (1.1.2)$$

$$\|q\beta\| = |q\gamma_3 - q_3| < \frac{C_2}{q^{1/2}}, \quad (1.1.3)$$

or

$$q\|q\alpha\|\|q\beta\| < \frac{C_3}{\log q}.$$

This implies the results from [CSD55].

Pollington and Velani showed [PV00] that if  $\alpha$  has bounded partial quotients, there exist uncountably many  $\beta$  with bounded quotients such that

$$q\|q\alpha\|\|q\beta\| < \frac{1}{\log q}$$

for infinitely many  $q \in \mathbb{N}$ .

In [dM03], de Mathan showed how to construct, for a given quadratic  $\alpha$ , a  $\mathbb{Q}$ -linearly independent irrational  $\beta$  with bounded partial quotients such that Littlewood's conjecture holds for the pair  $(\alpha, \beta)$ . This was the first explicit example of  $\mathbb{Q}$ -linearly independent pairs  $(\alpha, \beta)$  with (known) bounded partial quotients satisfying Littlewood's conjecture.

Einsiedler, Katok, Lindenstrauss [EKL06] proved that the set of counterexamples to Littlewood's conjecture has Hausdorff dimension 0.

Adamczewski and Bugeaud [AB06] showed how, given an  $\alpha$  with bounded partial quotients, to construct uncountably many  $\beta$  with bounded partial quotients such that the conjecture holds for  $(\alpha, \beta)$ .

Modifying the arguments of Peck in [P61], Bugeaud [Bu2] has recently shown, for  $K$  a cubic field with only one real embedding, and for  $1, \alpha, \beta$  a  $\mathbb{Q}$ -basis of  $K$ , how to construct a linear recursive sequence  $\{q_n\}$  of positive integers eventually satisfying

$$\max\{\|q_n\alpha\|, \|q_n\beta\|\} < \frac{C}{q_n^{1/2}}$$

for some  $C$ .

## 1.2 Our Results

The work of this dissertation is most similar to results in Peck [P61] and Bugeaud [Bu2]. What differs is that we effectively construct sequences whose terms satisfy Peck's inequalities (1.1.2) and (1.1.3). These inequalities motivate the following definition.

For the sake of stating our results more easily, we will call a sequence  $\{s_n\}$  of positive integers a **Peck sequence** for the pair  $(\sigma, \tau)$  if there are constants  $M_1, M_2$  and a subsequence  $\{\psi_n\}$  of  $\{s_n\}$  such that for all  $n$

$$\begin{aligned} \max\{\|s_n\sigma\|, \|s_n\tau\|\} &< \frac{M_1}{s_n^{1/2}}, \\ \psi_n\|\psi_n\sigma\| &< \frac{M_2}{\psi_n^{1/2} \log \psi_n}. \end{aligned}$$

In Theorem 1, we show that if  $\theta$  is the only real root of an irreducible cubic of the form  $x^3 - Px - Q \in \mathbb{Z}[x]$ , then we can construct a Peck sequence for the pair  $(\theta, \theta^2)$ . In Theorem 2, we show that if  $\alpha$  is the only real root of an irreducible cubic in  $\mathbb{Q}[x]$ , then we can construct a Peck sequence for the pair  $(\alpha, \alpha^2)$ . Finally, we show in Theorem 3

that if  $K$  is a real cubic field with only one real embedding, and if  $\alpha, \beta \in K$ , then we can construct a Peck sequence for the pair  $(\alpha, \beta)$ . As a consequence, we can construct  $n$  for which the Littlewood product  $n\|n\alpha\|\|n\beta\|$  is arbitrarily small, thus providing an effective proof of Littlewood's conjecture for the pair  $(\alpha, \beta)$ . Other than relying on Dirichlet's units theorem and the ability to produce a unit in a ring of cubic integers, our proofs are elementary and constructive.

Chapter 2 contains the proofs of our theorems. Chapter 3 is devoted to showing how to construct Peck sequences for several examples of  $(\alpha, \beta)$ . In the Appendix, we give some of the algorithms we've used in our constructions.

### 1.3 Overview and Motivating Example

This work arose from considering the cubic pell equation

$$x^3 + my^3 + m^2z^3 - 3mxyz = N_{K/\mathbb{Q}}(x + y\theta + z\theta^2) = 1 \quad (1.3.1)$$

(see Chapter 7 of [Ba03]), where  $m \in \mathbb{Z}$  is not a perfect cube,  $\theta = \sqrt[3]{m}$ , and  $K = \mathbb{Q}(\theta)$ . (We assume  $m > 0$ .) It is a tedious-but-not-difficult algebra exercise to verify that (1.3.1) factors as

$$N_{K/\mathbb{Q}}(x + y\theta + z\theta^2) = \frac{1}{2}(x + y\theta + z\theta^2)((x - y\theta)^2 + (x - z\theta^2)^2 + (y\theta - z\theta^2)^2). \quad (1.3.2)$$

From this, we see that

$$\max\{|x - y\theta|, |x - z\theta^2|, |y\theta - z\theta^2|\} \leq \left( \frac{2N_{K/\mathbb{Q}}(x + y\theta + z\theta^2)}{x + y\theta + z\theta^2} \right)^{1/2}. \quad (1.3.3)$$

So if  $|N_{K/\mathbb{Q}}(x + y\theta + z\theta^2)|$  is "small" compared to  $|x + y\theta + z\theta^2|$ , then  $|x - y\theta|$ ,  $|y\theta - z\theta^2|$ , and  $|x - z\theta^2|$  will also be "small". In this case we would have

$$\frac{y}{z} \approx \theta \quad \text{and} \quad \frac{x}{z} \approx \theta^2.$$

That is, we can find simultaneous Diophantine approximations to  $\theta$  and  $\theta^2$  by considering  $\zeta$  with small norm and large absolute value.

In particular, we consider units with large absolute value. Dirichlet's units theorem guarantees that we can produce infinite sequences  $\{\lambda^n\}$  of distinct units (where we can assume  $\lambda > 1$ ), so we have  $N_{K/\mathbb{Q}}(\lambda) = 1$  and  $\lambda^n \rightarrow \infty$ . If it is the case that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , and if we define<sup>3</sup>  $a_n$ ,  $b_n$ , and  $c_n$  as the coordinates of  $\lambda^n$  in the basis  $1, \theta, \theta^2$ , then we will see (using (1.3.3)) that eventually

$$\begin{aligned} \|c_n\theta\| &= \frac{|b_n\theta - c_n\theta^2|}{\theta} < \frac{1}{\theta} \cdot \frac{\sqrt{2}}{\lambda^{n/2}} < \frac{C}{c_n^{1/2}}, \\ \|c_n\theta^2\| &= |a_n - c_n\theta^2| < \frac{\sqrt{2}}{\lambda^{n/2}} < \frac{C}{c_n^{1/2}} \end{aligned}$$

for some constant  $C$  which is independent of  $n$ . In particular, the sequence

$$\{c_n \|c_n\theta\| \|c_n\theta^2\|\}$$

of Littlewood products is bounded. If we could make either  $c_n^{1/2} \|c_n\theta\|$  or  $c_n^{1/2} \|c_n\theta^2\|$  arbitrarily small, then we could make  $c_n \|c_n\theta\| \|c_n\theta^2\|$  arbitrarily small. That is, we would have a constructive proof of Littlewood's conjecture for the pair  $(\sqrt[3]{m}, \sqrt[3]{m^2})$ .

### 1.3.1 Example 1

Consider the pair  $(\theta, \theta^2)$  for  $\theta = \sqrt[3]{2}$ . Now  $\lambda = 1 + \theta + \theta^2$  is a unit greater than 1 ( $\lambda \approx 3.8473$ ), so we can use the sequence  $\{\lambda^n\}$  to produce a sequence of simultaneous approximations to  $(\theta, \theta^2)$ . Define  $a_n$ ,  $b_n$ , and  $c_n$  by

$$a_n + b_n\theta + c_n\theta^2 = \lambda^n = (1 + \theta + \theta^2)^n.$$

The first few  $\lambda^n$  are

$$\begin{aligned} \lambda^1 &= 1 + \theta + \theta^2, \\ \lambda^2 &= 5 + 4\theta + 3\theta^2, \\ \lambda^3 &= 19 + 15\theta + 12\theta^2, \\ \lambda^4 &= 73 + 58\theta + 46\theta^2, \\ \lambda^5 &= 281 + 223\theta + 177\theta^2, \end{aligned}$$

---

<sup>3</sup>We discuss how to compute these in Remark 2.1.1 and in the Appendix.

and first several  $c_n$  are:

1, 3, 12, 46, 177, 681, 2620, 10080, 38781, 149203, 574032, 2208486, 8496757.

By the previous discussion, we have that  $\{c_n^{1/2}\|c_n\theta\|\}$  and  $\{c_n^{1/2}\|c_n\theta^2\|\}$  (and therefore  $\{c_n\|c_n\theta\|\|c_n\theta^2\|\}$ ) are bounded. In Figure 1.2 we show  $c_n\|c_n\theta\|\|c_n\theta^2\|$  for  $n \leq 1000$ . To provide some context, we also show  $n\|n\theta\|\|n\theta^2\|$  for  $n \leq 1000$  in Figure 1.1.

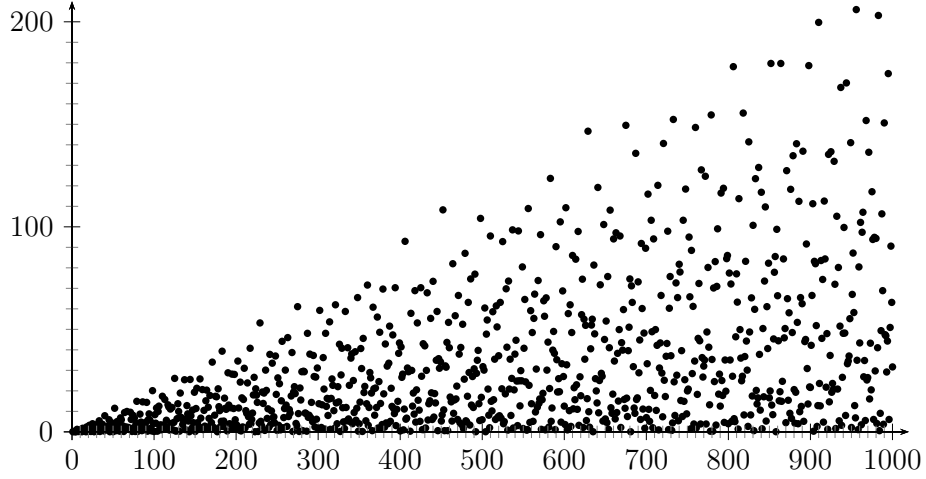


Figure 1.1:  $n\|n\sqrt[3]{2}\|\|n\sqrt[3]{4}\|$  for  $n \leq 1000$

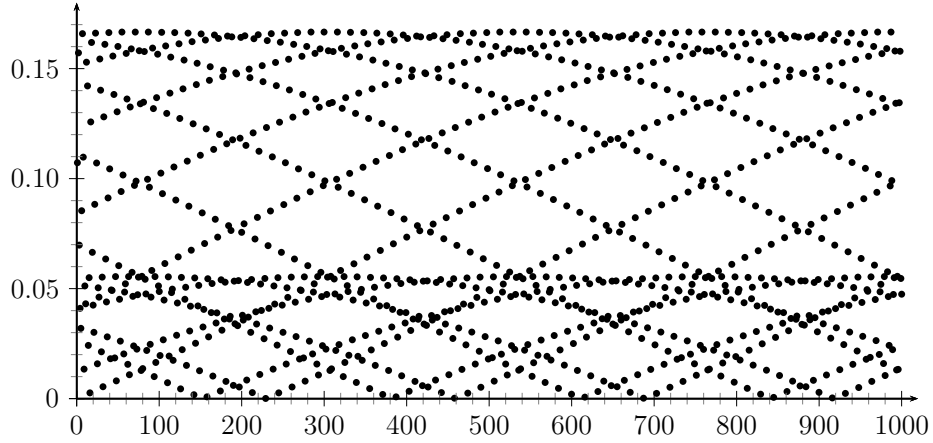


Figure 1.2:  $c_n\|c_n\sqrt[3]{2}\|\|c_n\sqrt[3]{4}\|$  for  $n \leq 1000$

Figure 1.3 shows  $c_n^{1/2}\theta\|c_n\theta\| = c_n^{1/2}|b_n\theta - c_n\theta^2|$  and  $c_n^{1/2}\|c_n\theta^2\| = c_n^{1/2}|a_n - c_n\theta^2|$  for  $n \leq 1000$ . Since the graphs are so similar, we consider the points  $(c_n^{1/2}\|c_n\theta\|, c_n^{1/2}\|c_n\theta^2\|)$ .

In Figure 1.4 we have the first 500 points  $(c_n^{1/2}\|c_n\theta\|, c_n^{1/2}\|c_n\theta^2\|)$ . From this graph, we see that  $c_n^{1/2}\|c_n\theta\|$  and  $c_n^{1/2}\|c_n\theta^2\|$  almost completely determine each other. In order



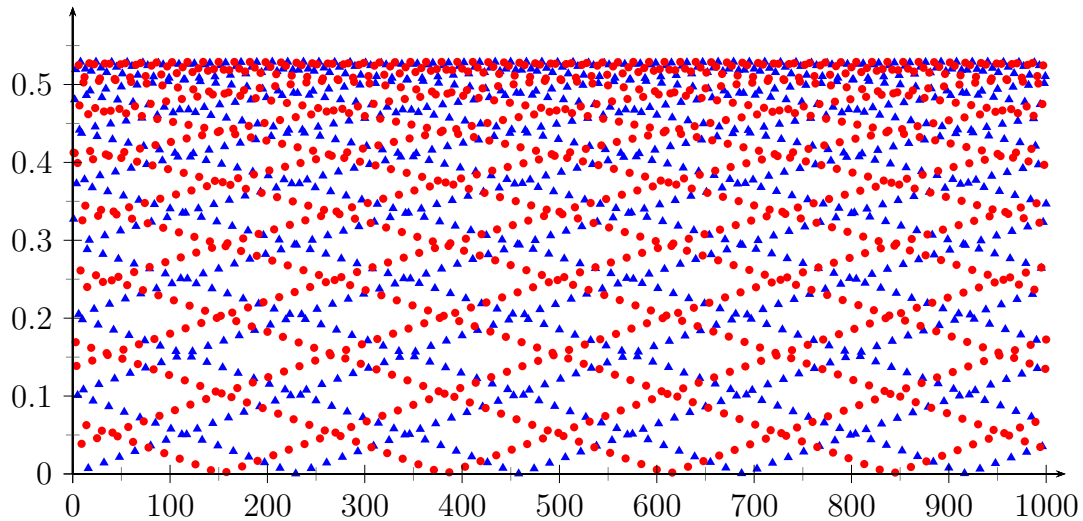


Figure 1.3:  $\theta \cdot c_n^{1/2} \|c_n \sqrt[3]{2}\|$  (triangles) and  $c_n^{1/2} \|c_n \sqrt[3]{4}\|$  (circles) for  $n \leq 1000$

to uncover more information about this relationship, we use the following notation.

**Definition 1.3.4.** For  $x \in \mathbb{R}$ , we define<sup>4</sup>  $\langle x \rangle = x - \lfloor x + \frac{1}{2} \rfloor$ . (So  $\|x\| = |\langle x \rangle|$ .)

By considering  $\langle c_n \theta \rangle$  and  $\langle c_n \theta^2 \rangle$ , we can better see the relationship between  $\|c_n \theta\|$  and  $\|c_n \theta^2\|$ . We plot the points  $(c_n^{1/2} \langle c_n \theta \rangle, c_n^{1/2} \langle c_n \theta^2 \rangle)$  for  $n \leq 500$  in Figure 1.5 and for  $n \leq 25$  in Figure 1.6. From these graphs, we see that the sequence  $\{(c_n^{1/2} \langle c_n \theta \rangle, c_n^{1/2} \langle c_n \theta^2 \rangle)\}$  appears to fill an ellipse. In Lemma 2.1.34 we will see how to derive that the equation of the ellipse in this example is

$$(\sqrt[3]{2} \cdot x - 2y)^2 + 3\sqrt[3]{4} \cdot x^2 = \frac{4}{3\sqrt[3]{4}},$$

and that the sequence is in fact asymptotic to the ellipse. In Table 1.1 we list  $c_n$ ,  $c_n^{1/2} \langle c_n \theta \rangle$ ,  $c_n^{1/2} \langle c_n \theta^2 \rangle$ , and  $c_n \|c_n \theta\| \|c_n \theta^2\|$  for the first several  $n$ .

From Figure 1.6, it appears that the points go around the ellipse at a fairly constant rate of rotation. For  $n > 1$ , going forward five terms in the sequence yields a point on the ellipse a little more than two full counterclockwise rotations away, so the angle between successive terms would be a little more than  $4\pi/5 \approx 2.51$ . Indeed, by

<sup>4</sup>This is more commonly denoted by  $\{\cdot\}$ . We use  $\langle \cdot \rangle$  to avoid confusion with sequence/set notation.

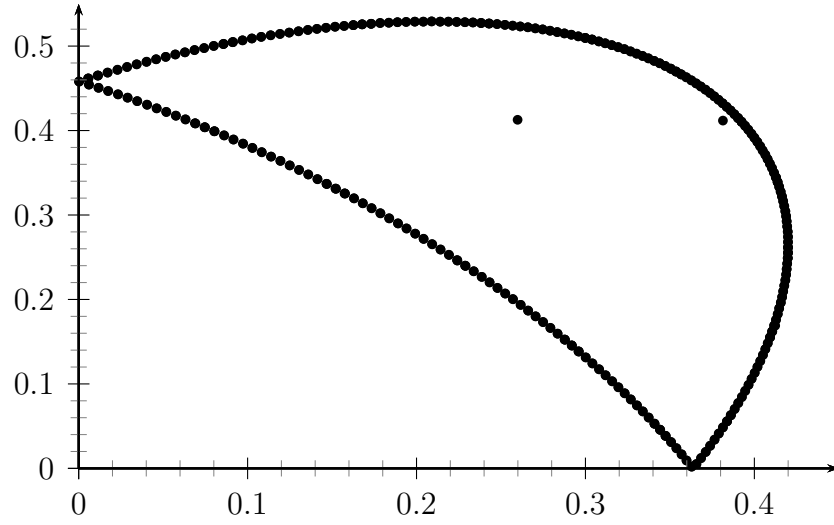


Figure 1.4: The points  $(c_n^{1/2} \|c_n \sqrt[3]{2}\|, c_n^{1/2} \|c_n \sqrt[3]{4}\|)$  for  $n \leq 500$

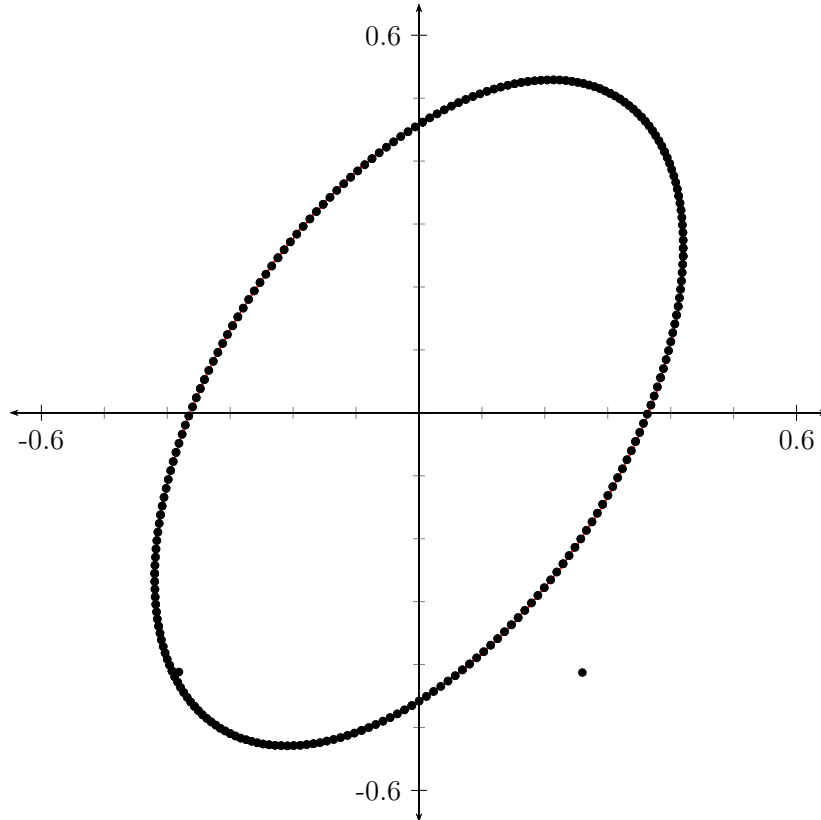


Figure 1.5:  $(c_n^{1/2} \langle c_n \sqrt[3]{2} \rangle, c_n^{1/2} \langle c_n \sqrt[3]{4} \rangle)$  for  $n \leq 500$ .

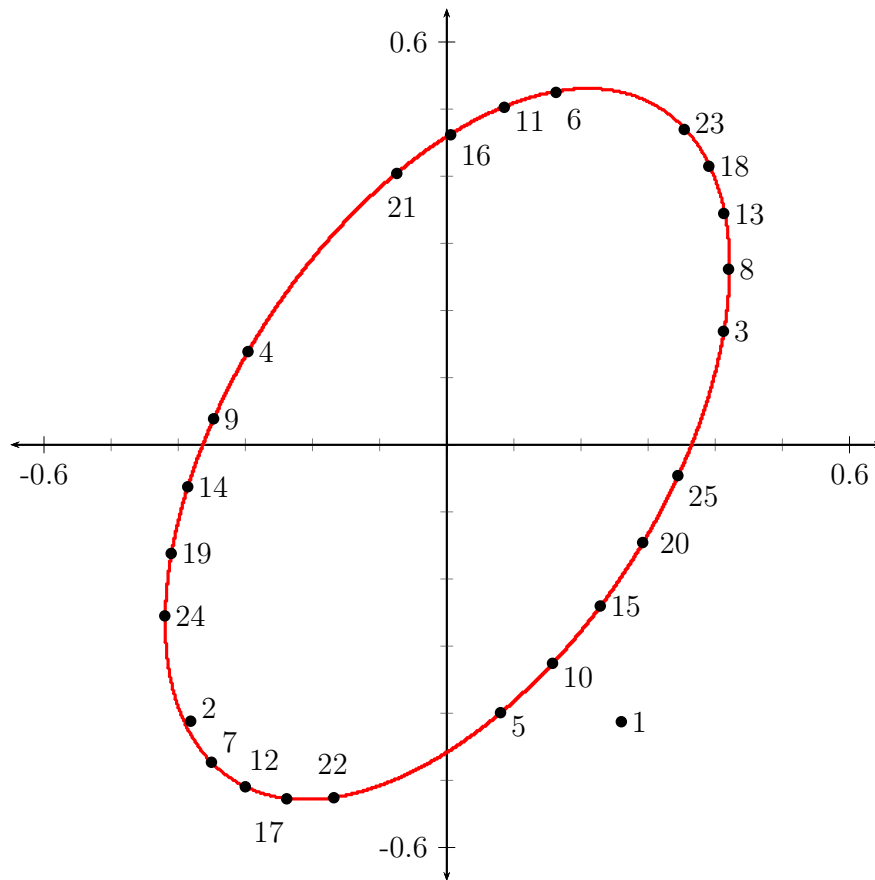


Figure 1.6:  $(c_n^{1/2} \langle c_n \sqrt[3]{2} \rangle, c_n^{1/2} \langle c_n \sqrt[3]{4} \rangle)$  for  $n \leq 25$ .

transforming the ellipse to a circle and applying the same transformation to the points  $\{(c_n^{1/2}\langle c_n\theta\rangle, c_n^{1/2}\langle c_n\theta^2\rangle)\}$ , we find that the angles between successive points seem to converge to roughly 2.551688241. If we knew the exact angle of rotation, then maybe we could use it to predict when either  $c_n^{1/2}\langle c_n\theta\rangle$  and  $c_n^{1/2}\langle c_n\theta^2\rangle$  gets arbitrarily close to zero.

We will see that by considering the sequence  $\{Q_n\}$  of denominators of convergents of  $\phi/\pi$ , where

$$\phi = \arctan\left(\frac{\sqrt{3}\theta^2 \cdot (b_1 - c_1\theta)}{2a_1 - b_1\theta - c_1\theta^2}\right) = \arctan\left(\frac{\sqrt{3}(\theta - \theta^2)}{2 - \theta - \theta^2}\right)$$

(and where  $\pi - \phi \approx 2.551688241$ ), we have

$$c_{Q_n}^{1/2}\|c_{Q_n}\theta\| < \frac{C}{Q_{n+1}}$$

for some constant  $C$ . Therefore

$$\lim_{n \rightarrow \infty} c_{Q_n}\|c_{Q_n}\theta\|\|c_{Q_n}\theta^2\| = \lim_{n \rightarrow \infty} \left(|c_{Q_n}^{1/2}\langle c_{Q_n}\theta\rangle| \cdot |c_{Q_n}^{1/2}\langle c_{Q_n}\theta^2\rangle|\right) = 0.$$

This dissertation is a generalization of these results to include all pairs  $(\alpha, \beta)$ , where  $\alpha, \beta \in K \subset \mathbb{R}$  and  $K$  is a cubic field with only one real embedding.

$n$	$a_n + b_n\theta + c_n\theta^2 = (1 + \theta + \theta^2)^n$	$c_n^{1/2}\langle c_n\theta \rangle$	$c_n^{1/2}\langle c_n\theta^2 \rangle$	$c_n\ c_n\theta\ \ c_n\theta^2\ $
1	$1 + \theta + \theta^2$	0.2599210	-0.4125989	0.1072432
2	$5 + 4\theta + 3\theta^2$	-0.3814614	-0.4118762	0.1571149
3	$19 + 15\theta + 12\theta^2$	0.4124103	0.1690919	0.0697352
4	$73 + 58\theta + 46\theta^2$	-0.2959246	0.1386877	0.04104111
5	$281 + 223\theta + 177\theta^2$	0.08016847	-0.3993077	0.03201189
6	$1081 + 858\theta + 681\theta^2$	0.1627079	0.5249569	0.0854146
7	$4159 + 3301\theta + 2620\theta^2$	-0.3505866	-0.4731548	0.1658817
8	$16001 + 12700\theta + 10080\theta^2$	0.4199639	0.2614234	0.1097884
9	$61561 + 48861\theta + 38781\theta^2$	-0.3473879	0.03867265	0.01343441
10	$236845 + 187984\theta + 149203\theta^2$	0.1573906	-0.3256969	0.0512616
11	$911219 + 723235\theta + 574032\theta^2$	0.08580665	0.5026315	0.04312913
12	$3505753 + 2782518\theta + 2208486\theta^2$	-0.3000002	-0.5096706	0.1529013
13	$13487761 + 10705243\theta + 8496757\theta^2$	0.4127900	0.3444347	0.1421792
$n$	$c_n$	$c_n^{1/2}\langle c_n\theta \rangle$	$c_n^{1/2}\langle c_n\theta^2 \rangle$	$c_n\ c_n\theta\ \ c_n\theta^2\ $
14	32689761	-0.386052	-0.0627756	0.0242346
15	125768040	0.228823	-0.240102	0.054941
16	483870160	0.00575037	0.461823	0.00265565
17	1861604361	-0.238380	-0.527441	0.125732
18	7162191603	0.390435	0.414778	0.161944
19	27555258052	-0.410517	-0.161915	0.066469
20	106013953326	0.291840	-0.145677	0.0425145
21	407869825737	-0.0745174	0.404029	0.0301072
22	1569206595241	-0.167993	-0.525814	0.088333
23	6037243216260	0.353720	0.469867	0.166202
24	23227219260240	-0.419885	-0.255100	0.107113
25	89362594024741	0.344124	-0.0458947	0.0157935
26	343806683071203	-0.152045	0.331376	0.050384
27	1322735050548072	-0.0914277	-0.504849	0.0461571
28	5088987794882566	0.303996	0.507676	0.154332

Table 1.1: Approximating  $\theta = \sqrt[3]{2}$  and  $\theta^2$  using units of  $\mathbb{Q}(\theta)$

## 1.4 Heuristics

The following is not anything resembling a proof, but it illustrates how often we might expect to encounter numbers  $n$  satisfying

$$n\|n\alpha\|\|n\beta\| < \varepsilon$$

or

$$n\|n\alpha\|\|n\beta\| < \frac{C}{\log n}$$

for a given  $\varepsilon$  or  $C$ .

Suppose  $1, \alpha, \beta$  are  $\mathbb{Q}$ -linearly independent, and consider the sequences  $\{\langle n\alpha \rangle\}$  and  $\{\langle n\beta \rangle\}$ . Define for  $T \geq 1$  the sets  $U_\varepsilon(T)$  and  $V_C(T)$  by

$$U_\varepsilon(T) = \{n \leq T : n\|n\alpha\|\|n\beta\| < \varepsilon\},$$

$$V_C(T) = \{n \leq T : n \log n\|n\alpha\|\|n\beta\| < C\}.$$

Supposing the terms act like independent uniformly distributed random variables, we estimate expected values for the sizes of  $U_\varepsilon(T)$  and  $V_C(T)$ .

If  $X, Y$  are independent and uniformly distributed over  $(-\frac{1}{2}, \frac{1}{2})$  and if  $0 < r < \frac{1}{4}$ , then

$$\begin{aligned} P(|X||Y| < r) &= 4 \iint_{\substack{0 < u, v < 1/2 \\ uv < r}} du dv \\ &= 4 \int_0^{2r} \frac{1}{2} du + 4 \int_{2r}^{1/2} \frac{r}{u} du \\ &= 4r - 4r \log 4r. \end{aligned}$$

So if we assume that  $\langle n\alpha \rangle$  and  $\langle n\beta \rangle$  act independent and uniform, and if  $\frac{\varepsilon}{n}, \frac{C}{n \log n} < \frac{1}{4}$ , then

$$\begin{aligned} P(n\|n\alpha\|\|n\beta\| < \varepsilon) &= P(\|n\alpha\|\|n\beta\| < \frac{\varepsilon}{n}) = \frac{4\varepsilon}{n} - \frac{4\varepsilon}{n} \log \left( \frac{4\varepsilon}{n} \right), \\ P(n\|n\alpha\|\|n\beta\| < \frac{C}{\log n}) &= P(\|n\alpha\|\|n\beta\| < \frac{C}{n \log n}) = \frac{4C}{n \log n} - \frac{4C}{n \log n} \log \left( \frac{4C}{n \log n} \right). \end{aligned}$$

### 1.4.1 Estimating $|U_\varepsilon(T)|$

Assume  $\varepsilon < \frac{1}{4}$ . Then  $\frac{\varepsilon}{n} < \frac{1}{4}$  for all  $n$ . We estimate  $|U_\varepsilon(T)|$  as an expected value:

$$E(|U_\varepsilon(T)|) = \sum_{n \leq T} \frac{4\varepsilon}{n} - \frac{4\varepsilon}{n} \log \left( \frac{4\varepsilon}{n} \right).$$

Since  $x(1 - \log x)$  is positive and decreasing for  $x \geq 1$ , this sum equals

$$\int_1^T \frac{4\varepsilon}{x} - \frac{4\varepsilon}{x} \log \left( \frac{4\varepsilon}{x} \right) dx + R$$

for some  $R$  with

$$|R| < \frac{4\varepsilon}{1} - \frac{4\varepsilon}{1} \log \left( \frac{4\varepsilon}{1} \right) = 4\varepsilon(1 - \log 4\varepsilon) < 1$$

(since  $4\varepsilon < 1$ ). So

$$\begin{aligned} E(|U_\varepsilon(T)|) &= \int_1^T \left( 4\varepsilon - 4\varepsilon \log \left( \frac{4\varepsilon}{x} \right) \right) \frac{dx}{x} + R \\ &= \int_0^{\log T} (4\varepsilon - 4\varepsilon \log 4\varepsilon + 4\varepsilon u) du + R \\ &= 2\varepsilon(\log T)^2 + 4\varepsilon(1 - \log 4\varepsilon) \log T + R. \end{aligned}$$

Define

$$F_\varepsilon(T) = 2\varepsilon(\log T)^2 + 4\varepsilon(1 - \log 4\varepsilon) \log T.$$

In Figures 1.7, 1.8, 1.9, and 1.10 we compare the graphs of  $F_\varepsilon(T)$  and  $|U_\varepsilon(T)|$  for the following examples.

1.  $(\sqrt{2}, \sqrt{3})$ : Both have bounded partial quotients, and it is unknown whether Littlewood's conjecture is true for this pair.
2.  $(e, \pi)$ : We know that  $e$  has unbounded partial quotients, so we know that Littlewood's conjecture is true for this pair.
3.  $(\sqrt[3]{2}, \sqrt[3]{4})$ : We do not know whether the partial quotients are bounded or unbounded. We *do* know that Littlewood's conjecture is true for this pair.

4.  $(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{65}}{5})$ : Like the pair  $(\sqrt{2}, \sqrt{3})$ , these have small partial quotients. The continued fractions are  $\frac{1+\sqrt{5}}{2} = [1; \overline{1}]$  and  $\frac{\sqrt{65}}{5} = [1; \overline{1, 1, 1, 1, 2}]$ . Since both numbers are “badly approximable” by rationals, it is worth considering how well they can be simultaneously approximated.

#### 1.4.2 Estimating $|V_C(T)|$

Assume  $C > 0$ , and let  $m_C$  be the first integer such that  $m_C \log m_C \geq 4C$ . (Also, assume  $T > m_C$ .) For all  $k < m_C$

$$\begin{aligned} k \log k &< 4C, \\ \frac{1}{4} &< \frac{C}{k \log k}, \\ \|k\alpha\| \|k\beta\| &\leq \frac{1}{4} < \frac{C}{k \log k}. \end{aligned}$$

That is,  $1, \dots, m_C - 1$  are all in  $V_C(T)$  (so  $|V_C(m_C - 1)| = m_C - 1$ ). Since  $\frac{C}{n \log n} < \frac{1}{4}$  for  $m_C \leq n \leq T$ , we estimate  $|V_C(T)|$  as

$$\begin{aligned} E(|V_C(T)|) &= |V_C(m_C - 1)| + \sum_{n=m_C}^T \frac{4C}{n \log n} - \frac{4C}{n \log n} \log \left( \frac{4C}{n \log n} \right) \\ &= (m_C - 1) + \sum_{n=m_C}^T \frac{4C}{n \log n} - \frac{4C}{n \log n} \log \left( \frac{4C}{n \log n} \right). \end{aligned}$$

Now

$$\frac{4C}{x \log x} \left( 1 - \log \left( \frac{4C}{x \log x} \right) \right)$$

is positive and decreasing for  $x \geq m_C$ . (To see this, note that for  $x \geq m_C$ ,  $\frac{4C}{x \log x}$  is in  $(0, 1]$  and is decreasing.) So

$$\sum_{n=m_C}^T \frac{4C}{n \log n} - \frac{4C}{n \log n} \log \left( \frac{4C}{n \log n} \right) = \int_{m_C}^T \frac{4C}{x \log x} - \frac{4C}{x \log x} \log \left( \frac{4C}{x \log x} \right) dx + R,$$



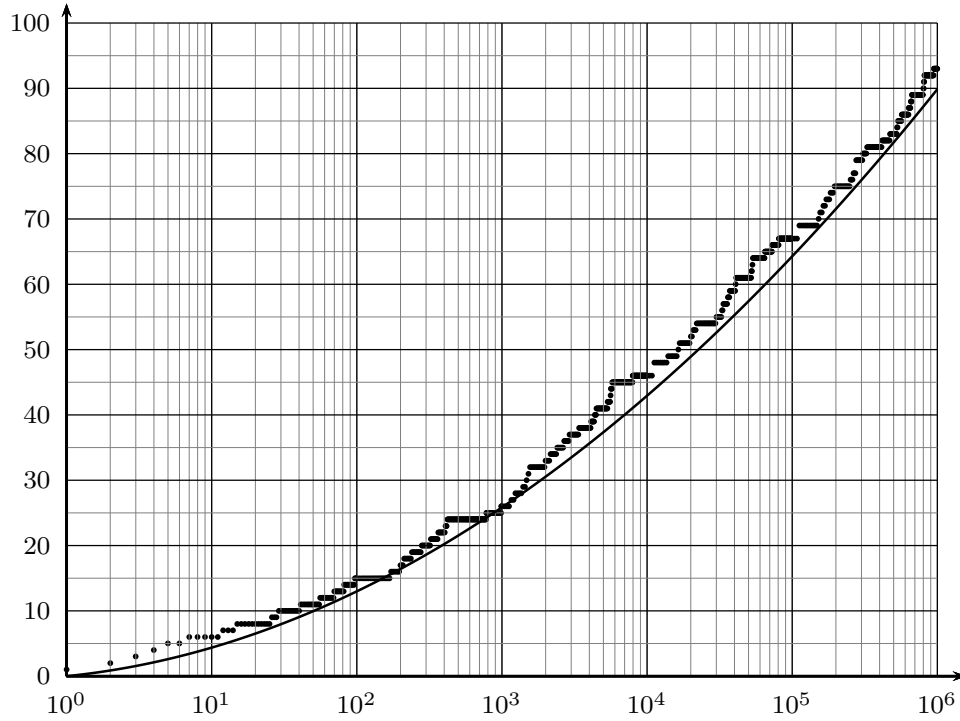


Figure 1.7:  $|U_\varepsilon(n)|$  and  $F_\varepsilon(n)$  for the pair  $(\sqrt{2}, \sqrt{3})$  (with  $\varepsilon = .2$ ).

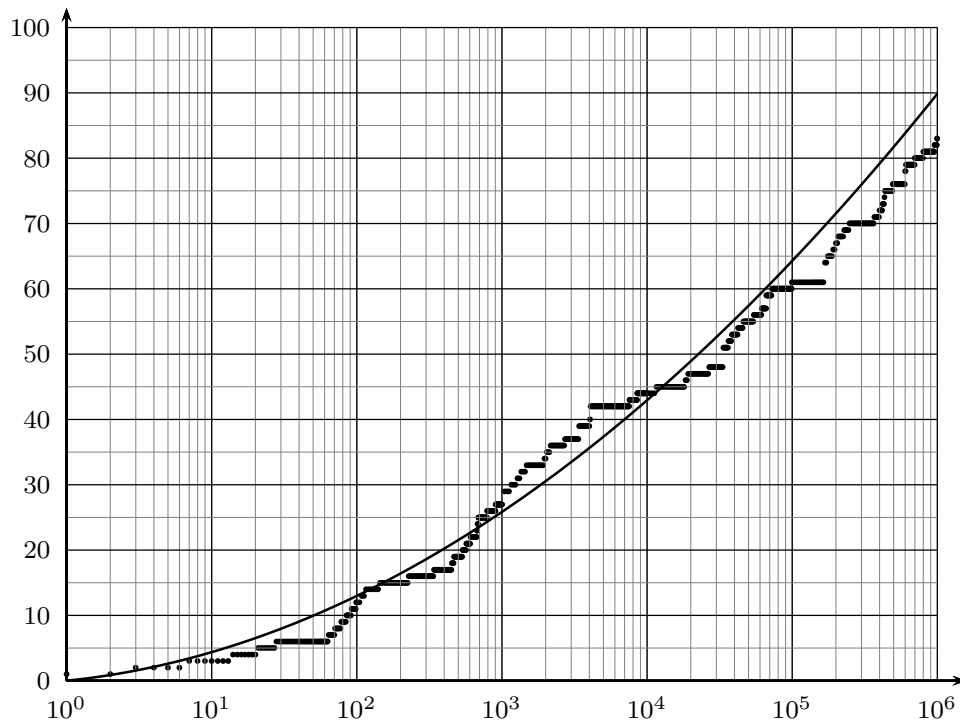


Figure 1.8:  $|U_\varepsilon(n)|$  and  $F_\varepsilon(n)$  for the pair  $(e, \pi)$  (with  $\varepsilon = .2$ ).

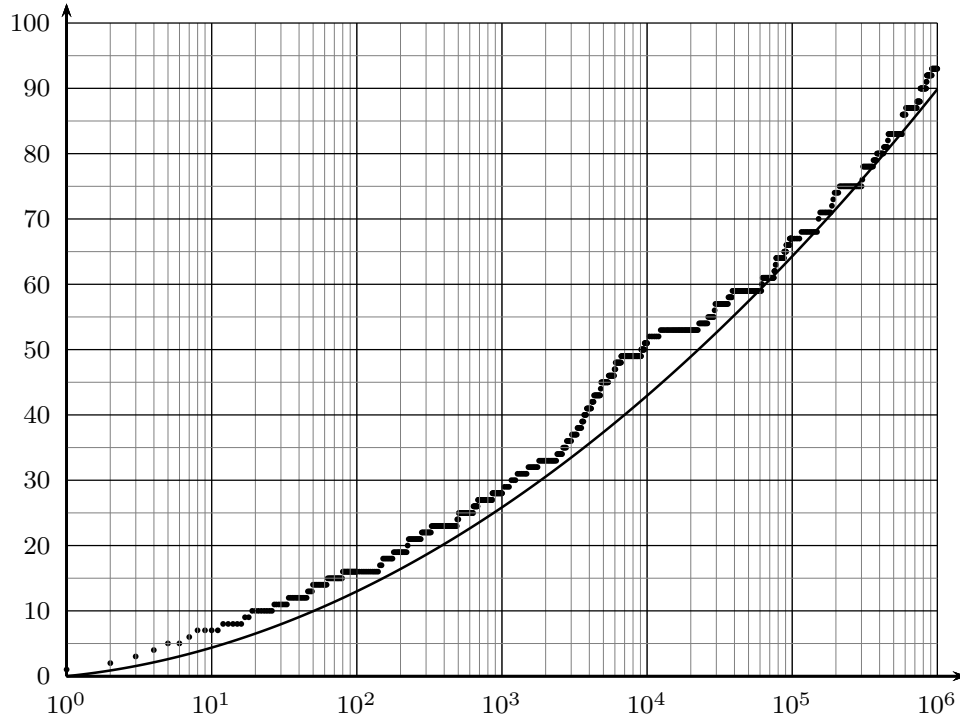


Figure 1.9:  $|U_\varepsilon(n)|$  and  $F_\varepsilon(n)$  for the pair  $(\sqrt[3]{2}, \sqrt[3]{4})$  (with  $\varepsilon = .2$ ).

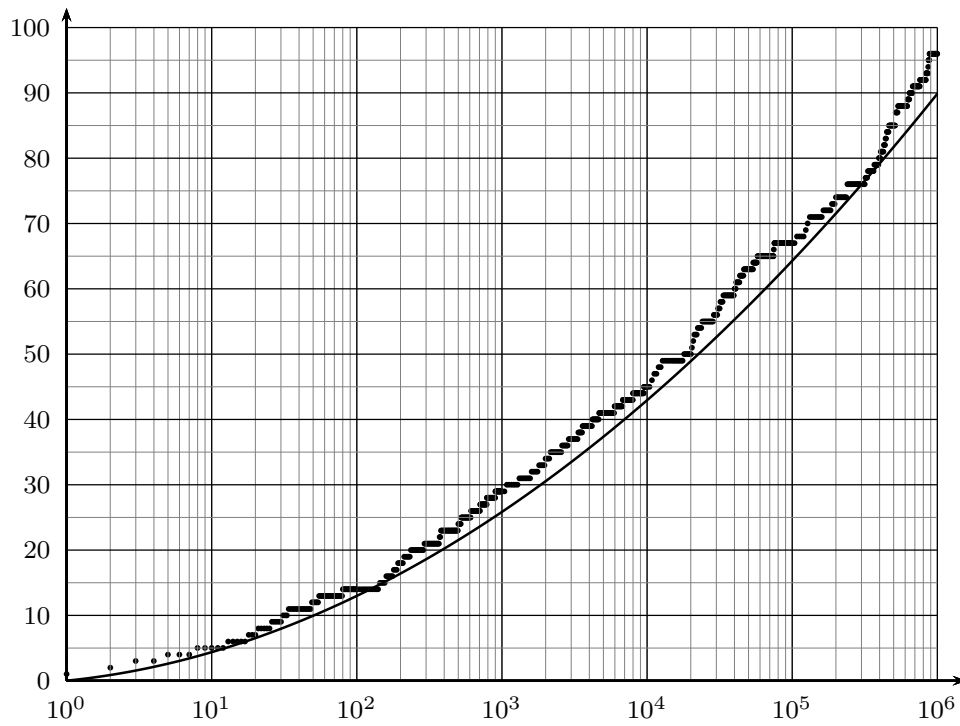


Figure 1.10:  $|U_\varepsilon(n)|$  and  $F_\varepsilon(n)$  for the pair  $(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{65}}{5})$  (with  $\varepsilon = .2$ ).

where  $|R| < \frac{4C}{m_C \log m_C} (1 - \log \frac{4C}{m_C \log m_C}) < 1$ . If we let  $I$  denote the integral, then

$$\begin{aligned}
 I &= \int_{m_C}^T \frac{4C}{\log x} - \frac{4C}{\log x} \log \left( \frac{4C}{x \log x} \right) \frac{dx}{x} \\
 &= 4C \int_{\log m_C}^{\log T} \left( \frac{1 - \log 4C}{x} + \frac{\log x}{x} + 1 \right) dx \\
 &= 4C \left( (1 - \log 4C) \log x + \frac{(\log x)^2}{2} + x \right) \Big|_{\log m_C}^{\log T} \\
 &= S_C(T) - S_C(m_C),
 \end{aligned}$$

where

$$S_C(x) = 4C \log x + 2C(\log \log x)^2 + 4C(1 - \log 4C) \log \log x.$$

Then

$$E(|V_C(T)|) = S_C(T) - S_C(m_C) + (m_C - 1) + R,$$

and  $|R| < 1$ . Define

$$G_C(T) = S_C(T) - S_C(m_C) + (m_C - 1).$$

In Figures 1.11, 1.12, 1.13, and 1.14 we compare the graphs of  $G_C(T)$  and  $|V_C(T)|$  for the same pairs as before.

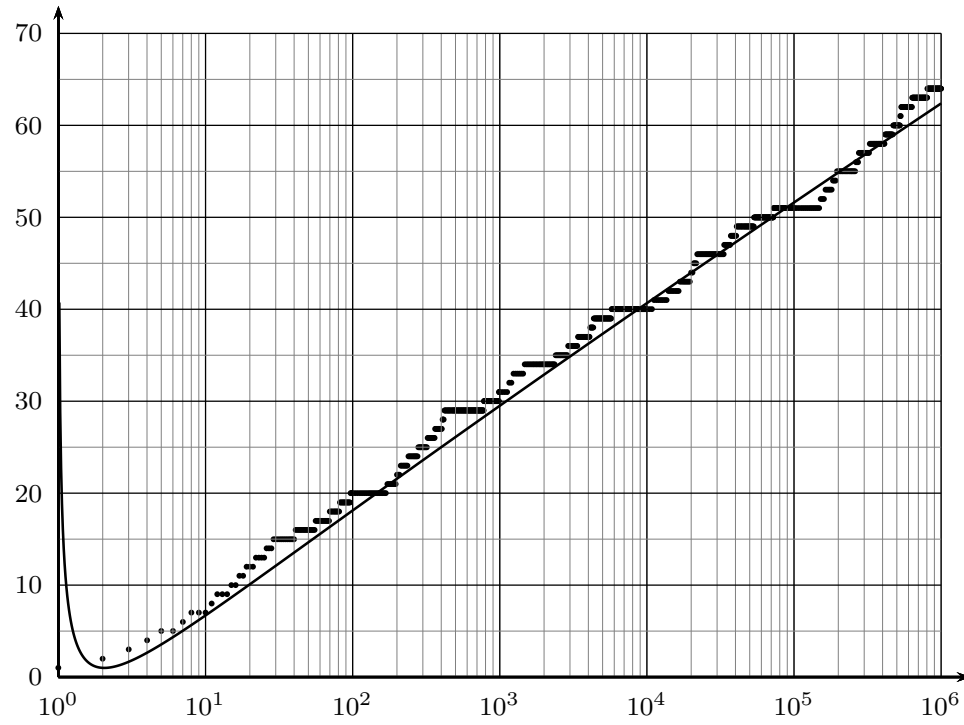


Figure 1.11:  $|V_C(n)|$  and  $G_C(n)$  for the pair  $(\sqrt{2}, \sqrt{3})$  (with  $C = 1$ ).

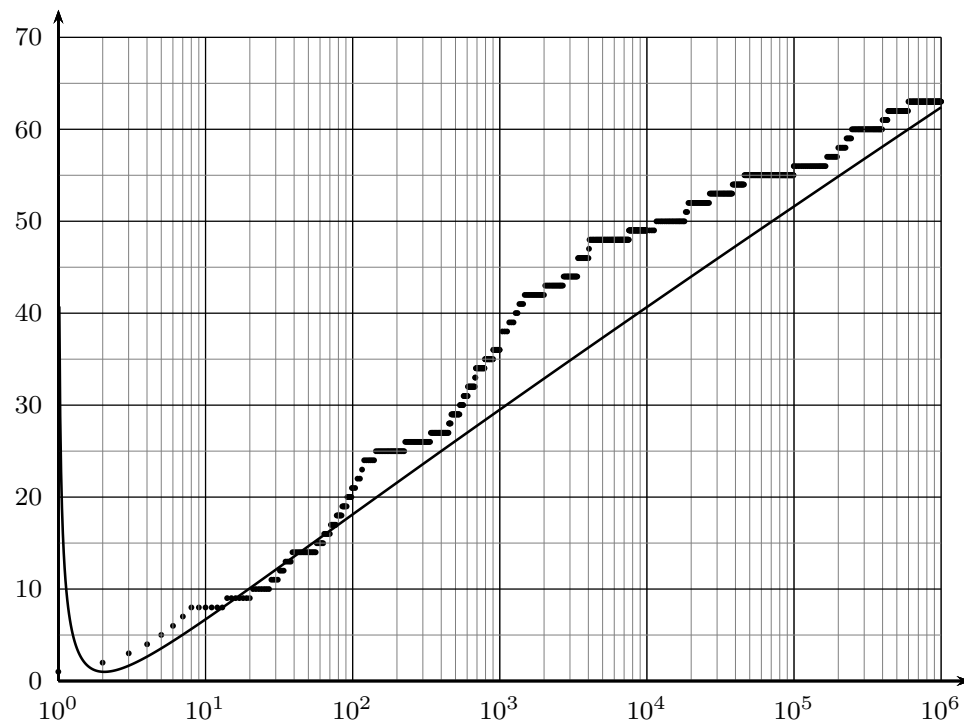


Figure 1.12:  $|V_C(n)|$  and  $G_C(n)$  for the pair  $(e, \pi)$  (with  $C = 1$ ).

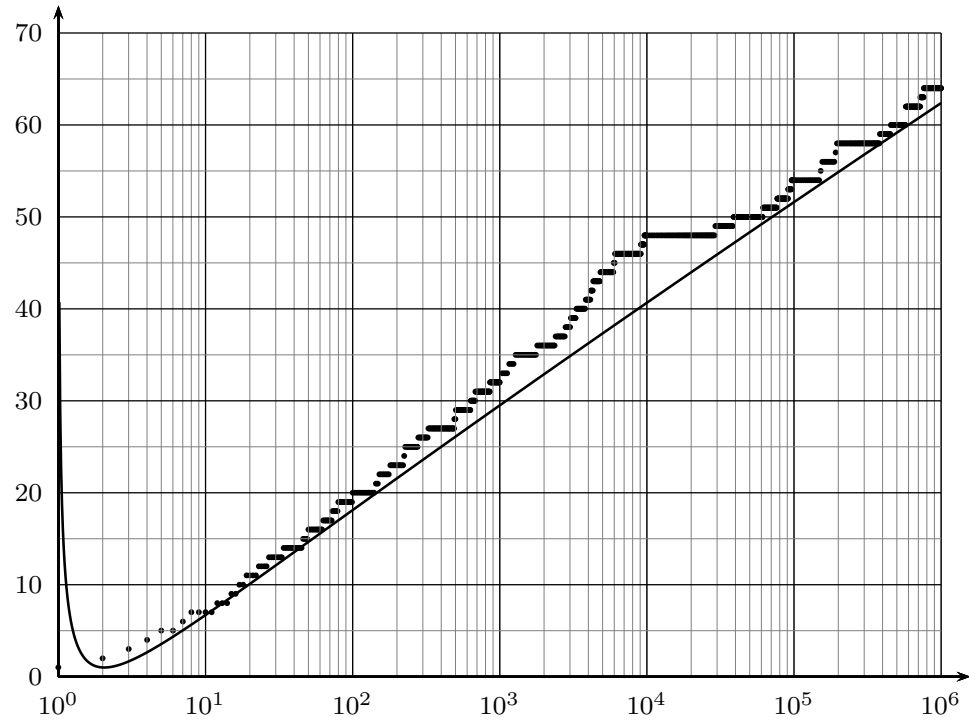


Figure 1.13:  $|V_C(n)|$  and  $G_C(n)$  for the pair  $(\sqrt[3]{2}, \sqrt[3]{4})$  (with  $C = 1$ ).

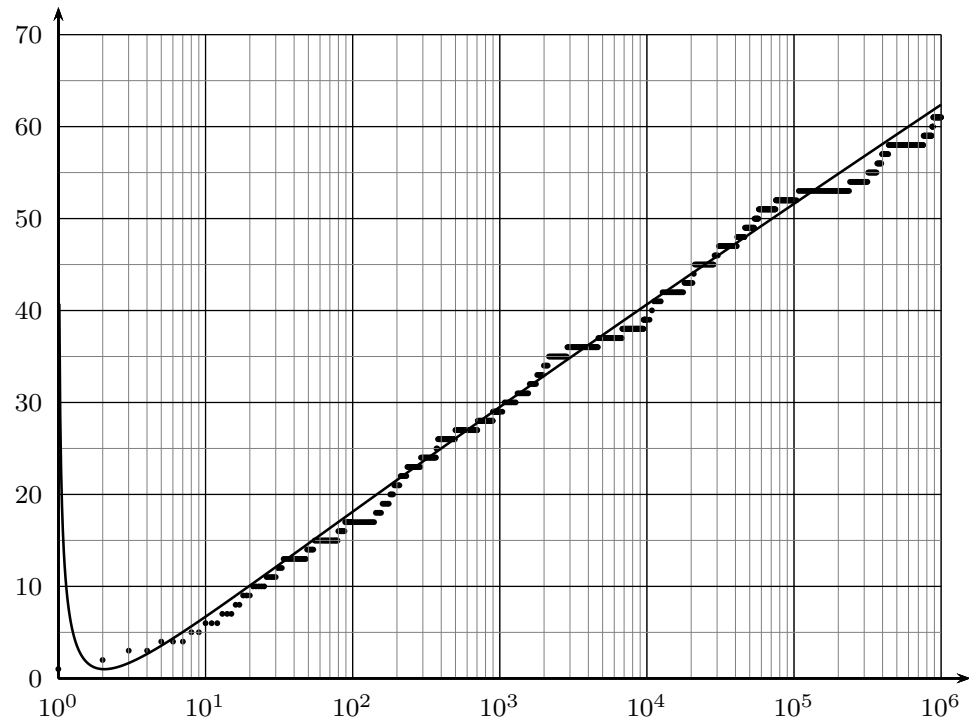


Figure 1.14:  $|V_C(n)|$  and  $G_C(n)$  for the pair  $(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{65}}{5})$  (with  $C = 1$ ).

CHAPTER 2  
MAIN RESULTS

In this chapter, we prove the following theorems:

**Theorem 1.** *Suppose  $\theta \in \mathbb{R}$  is the only real root of an irreducible polynomial of the form  $x^3 - px - q \in \mathbb{Z}[x]$ . Then we can construct a Peck sequence for the pair  $(\theta, \theta^2)$ .*

**Theorem 2.** *Suppose  $\alpha \in \mathbb{R}$  is the only real root of an irreducible cubic polynomial in  $\mathbb{Q}[x]$ . Then we can construct a Peck sequence for the pair  $(\alpha, \alpha^2)$ .*

**Theorem 3.** *Suppose  $\alpha, \beta \in K$ , where  $K \subseteq \mathbb{R}$  is a cubic field with only one real embedding. Then we can construct a Peck sequence for the pair  $(\alpha, \beta)$ .*

The following immediately follows from Theorem 3:

**Corollary. (Special Case of Littlewood's Conjecture)** *If  $\alpha, \beta$  are as in Theorem 3, then we can construct a sequence  $\{\psi_n\}$  of positive integers such that*

$$\lim_{n \rightarrow \infty} \psi_n \|\psi_n \alpha\| \|\psi_n \beta\| = 0.$$

## 2.1 Lemmas

### 2.1.1 Definitions and Notation used in Lemmas

Suppose  $\theta \in \mathbb{R}$  is the only real root of the irreducible cubic polynomial

$$x^3 - px - q \in \mathbb{Z}[x].$$

Let  $K$  denote  $\mathbb{Q}(\theta)$ , and let  $\sigma_1, \sigma_2, \sigma_3$  denote the embeddings of  $K$  in  $\mathbb{C}$ . (We assume  $\sigma_1$  is the real embedding. We will make an assumption about the choice of  $\sigma_2$  and  $\sigma_3$  in Remark 2.1.3.) For  $\zeta \in K$ , let  $\zeta_i$  denote  $\sigma_i(\zeta)$  for  $i = 1, 2, 3$  (so  $\zeta_1 = \zeta$ ). We use two standard results from number theory (see [Ma]):

- (i) There are positive integers  $d$  for which  $\mathcal{O}_K \subseteq \frac{1}{d}\mathbb{Z}[\theta]$ . In particular,

$$|\text{disc}(1, \theta, \theta^2)| = |4p^3 - 27q^2|$$

would work (see Theorem 9 of Marcus). In practice, we can find this by computing (in PARI/GP, for instance) an integral basis.

- (ii) If  $K$  is a cubic field with only one real embedding, then the group of units of  $\mathcal{O}_K$  has rank 1. (Dirichlet's units theorem)

Let  $\lambda \in \mathcal{O}_K$  be a unit of infinite order (i.e.,  $\lambda^n \neq 1$  for  $n > 0$ ). In particular  $\lambda \neq \pm 1$ , so we can assume WLOG that  $\lambda > 1$ . (If not, then replace  $\lambda$  with one of  $\pm\lambda^{\pm 1}$ .)

Define the sequences  $\{a_n\}, \{b_n\}, \{c_n\}$  of rational numbers by

$$a_n + b_n\theta + c_n\theta^2 = \lambda^n.$$

Since  $1, \theta, \theta^2$  is a  $\mathbb{Q}$ -basis of  $\mathbb{Q}(\theta)$ , each  $a_n, b_n, c_n$  is well-defined. Since  $\lambda \in \mathcal{O}_K \subseteq \frac{1}{d}\mathbb{Z}[\theta]$ , each  $a_n, b_n, c_n$  is in  $\frac{1}{d}\mathbb{Z}$ . Also note that  $\lambda_i^n = \sigma_i(\lambda^n) = a_n + b_n\theta_i + c_n\theta_i^2$ .

We define auxiliary sequences  $\{X_n\}, \{Y_n\}$ , and  $\{Z_n\}$  by:

$$X_n = a_n + pc_n - b_n\theta$$

$$Y_n = a_n + pc_n - c_n\theta^2$$

$$Z_n = b_n\theta - c_n\theta^2$$

(where  $p \in \mathbb{Q}$  is as in  $x^3 - px - q$ ).

Finally, define the sequence  $\{k_n\}$  by  $k_n = dc_n$ .

**Remark 2.1.1.** We can easily compute  $a_n, b_n, c_n$  for an arbitrary  $n$  once we know  $\lambda = a + b\theta + c\theta^2$ . One way is to work in  $\mathbb{Q}[x]/(x^3 - px - q)$  (making the identification  $\theta \leftrightarrow x$ ), and then (using repeated squaring)

$$a_n + b_n\theta + c_n\theta^2 = (a + b\theta + c\theta^2)^n \longleftrightarrow (a + bx + cx^2)^n \pmod{x^3 - px - q}.$$

Another way is to work in the matrix ring  $\mathbb{Q}[T]$ , where

$$T = \begin{pmatrix} 0 & 0 & q \\ 1 & 0 & p \\ 0 & 1 & 0 \end{pmatrix}$$

is the companion matrix of  $x^3 - px - q$ . In this ring we identify  $\theta$  and  $T$ , and so

$$a_n + b_n\theta + c_n\theta^2 \longleftrightarrow a_nI + b_nT + c_nT^2 = (aI + bT + cT^2)^n.$$

In particular,

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} a & qc & qb \\ b & a + pc & pb + qc \\ c & b & a + pc \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Again, this exponentiation can be done by repeated squaring.

We used this matrix method in Mathematica 8.0 to compute our examples. In the Appendix, we list some specific commands and functions we used.

### 2.1.2 Lemmas

**Lemma 2.1.2.**

$$\theta_2, \theta_3 = \frac{-\theta \pm \sqrt{4p - 3\theta^2}}{2}.$$

**Proof.** We have two expressions for the minimal polynomial of  $\theta$ :

$$x^3 - px - q$$



and

$$(x - \theta_1)(x - \theta_2)(x - \theta_3) = x^3 - (\theta_1 + \theta_2 + \theta_3)x^2 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)x - \theta_1\theta_2\theta_3.$$

Comparing coefficients, we see that  $\theta_1 + \theta_2 + \theta_3 = 0$  and  $\theta_1\theta_2\theta_3 = q$ . Then

$$(x - \theta_2)(x - \theta_3) = x^2 - (\theta_2 + \theta_3)x + \theta_2\theta_3 = x^2 - (-\theta)x + \frac{q}{\theta}.$$

Since  $\theta^3 = p\theta + q$ , we have  $\theta^2 - p = \frac{q}{\theta}$ . So

$$(x - \theta_2)(x - \theta_3) = x^2 + \theta x + (\theta^2 - p).$$

Then we use the quadratic formula and the fact that  $\theta_2$  and  $\theta_3$  are roots.  $\square$

**Remark 2.1.3.** WLOG we assume that  $\sigma_2$  and  $\sigma_3$  are such that  $\theta_2 = \frac{-\theta + \sqrt{4p - 3\theta^2}}{2}$  and  $\theta_3 = \frac{-\theta - \sqrt{4p - 3\theta^2}}{2}$ .

**Lemma 2.1.4.** *We have the following inequalities:*

$$3\theta^2 - 4p > 0 \tag{2.1.5}$$

$$\theta^2 - p > 0 \tag{2.1.6}$$

$$3\theta^2 - p > 0 \tag{2.1.7}$$

**Proof.** Because  $(x - \theta_2)(x - \theta_3)$  has complex roots, its discriminant  $4p - 3\theta^2$  is negative. That is,  $p < \frac{3\theta^2}{4}$ . So we have

$$p < \frac{3\theta^2}{4} < \theta^2 < 3\theta^2,$$

and therefore  $\theta^2 - p > 0$  and  $3\theta^2 - p > 0$  as well.  $\square$

The following identities generalize the factorization of  $N_{K/\mathbb{Q}}$  that we had in (1.3.2).

**Lemma 2.1.8.** *If  $\zeta = x + y\theta + z\theta^2 \in K^\times$ , then*

$$(2(x + pz) - y\theta - z\theta^2)^2 + (3\theta^2 - 4p)(y - z\theta)^2 = \frac{4N(\zeta)}{\zeta} \tag{2.1.9}$$

and

$$(x + pz - y\theta)^2 + (x + pz - z\theta^2)^2 + (1 - 2p/\theta^2)(y\theta - z\theta^2)^2 = \frac{2N(\zeta)}{\zeta}. \tag{2.1.10}$$

**Proof.** We start with  $\zeta_1\zeta_2\zeta_3 = N(\zeta)$ , or  $\zeta_2\zeta_3 = \frac{N(\zeta)}{\zeta}$ . We wish to find  $\zeta_2$  and  $\zeta_3$  in terms of  $\theta$ . Now (using Lemma 2.1.2)

$$\left(\frac{-\theta \pm \sqrt{4p - 3\theta^2}}{2}\right)^2 = \frac{\theta^2 \mp 2\theta\sqrt{4p - 3\theta^2} + 4p - 3\theta^2}{4} = p + \frac{-\theta^2 \mp \theta\sqrt{4p - 3\theta^2}}{2},$$

so

$$\begin{aligned} \zeta_2 &= x + y\theta_2 + z\theta_2^2 \\ &= x + y \left(\frac{-\theta + \sqrt{4p - 3\theta^2}}{2}\right) + z \left(p + \frac{-\theta^2 - \theta\sqrt{4p - 3\theta^2}}{2}\right) \\ &= \left((x + pz) - \frac{y\theta}{2} - \frac{z\theta^2}{2}\right) + \frac{\sqrt{4p - 3\theta^2}}{2}(y - z\theta) \\ &= \frac{1}{2}(2(x + pz) - y\theta - z\theta^2) + \frac{\sqrt{4p - 3\theta^2}}{2}(y - z\theta). \end{aligned}$$

Similarly we find that

$$\zeta_3 = \frac{1}{2}(2(x + pz) - y\theta - z\theta^2) - \frac{\sqrt{4p - 3\theta^2}}{2}(y - z\theta).$$

Then

$$\frac{N(\zeta)}{\zeta} = \zeta_2\zeta_3 = \frac{1}{4}(2(x + pz) - y\theta - z\theta^2)^2 - \frac{4p - 3\theta^2}{4}(y - z\theta)^2,$$

or

$$\frac{4N(\zeta)}{\zeta} = (2(x + pz) - y\theta - z\theta^2)^2 + (3\theta^2 - 4p)(y - z\theta)^2.$$

To get (2.1.10) we use the identity

$$(U + V)^2 + 3(U - V)^2 = 2U^2 + 2V^2 + 2(U - V)^2$$

with  $U = x + pz - z\theta^2$  and  $V = x + pz - y\theta$ :

$$\begin{aligned} \frac{4N(\zeta)}{\zeta} &= (2(x + pz) - y\theta - z\theta^2)^2 + 3(y\theta - z\theta^2)^2 - \frac{4p}{\theta^2}(y\theta - z\theta^2)^2 \\ &= (U + V)^2 + 3(U - V)^2 - \frac{4p}{\theta^2}(U - V)^2 \\ &= 2U^2 + 2V^2 + 2(U - V)^2 - \frac{4p}{\theta^2}(U - V)^2, \end{aligned}$$

or

$$\begin{aligned}\frac{2N(\zeta)}{\zeta} &= U^2 + V^2 + (1 - 2p/\theta^2)(U - V)^2 \\ &= (x + pz - z\theta^2)^2 + (x + pz - y\theta)^2 + (1 - 2p/\theta^2)(y\theta - z\theta^2)^2.\end{aligned}$$

□

**Definition 2.1.11.** We define the constants  $C_1$ ,  $C_2$ , and  $C_3$  by:

$$\begin{aligned}C_1 &= \max\left\{\sqrt{2}, \frac{\sqrt{2}|\theta|}{\sqrt{3\theta^2 - 4p}}\right\} \\ C_2 &= \frac{\sqrt{d}}{\sqrt{3\theta^2 - p}} \\ C_3 &= \max\left\{1, \frac{3C_2}{2}\right\}.\end{aligned}$$

We will use these constants throughout the lemmas, as well as in the proofs of the theorems and the constructions in Chapter 3.

**Lemma 2.1.12.**  $C_1$  also satisfies

$$C_1 = \begin{cases} \frac{\sqrt{2}|\theta|}{\sqrt{3\theta^2 - 4p}}, & \theta^2 - 2p < 0 \\ \sqrt{2}, & \theta^2 - 2p > 0. \end{cases}$$

**Proof.** Note that  $C_1 = \sqrt{2} \cdot \max\left\{1, \frac{|\theta|}{\sqrt{3\theta^2 - 4p}}\right\}$  and that

$$\begin{aligned}1 < \frac{|\theta|}{\sqrt{3\theta^2 - 4p}} &\Leftrightarrow 3\theta^2 - 4p < \theta^2 \\ &\Leftrightarrow 2\theta^2 - 4p < 0 \\ &\Leftrightarrow \theta^2 - 2p < 0.\end{aligned}$$

□

**Lemma 2.1.13.** For all positive integers  $n$ ,

$$|a_n + pc_n - b_n\theta| = |X_n| < \frac{C_1}{\lambda^{n/2}} \quad (2.1.14)$$

$$|a_n + pc_n - c_n\theta^2| = |Y_n| < \frac{C_1}{\lambda^{n/2}} \quad (2.1.15)$$

$$|b_n\theta - c_n\theta^2| = |Z_n| < \frac{2|\theta|}{\sqrt{3\theta^2 - 4p}} \cdot \frac{1}{\lambda^{n/2}} \leq \frac{\sqrt{2}C_1}{\lambda^{n/2}} \quad (2.1.16)$$

$$|\lambda^n - (3\theta^2 - p)c_n| < \frac{(1 + \sqrt{2})C_1}{\lambda^{n/2}} \quad (2.1.17)$$

**Proof.** (Third inequality) With  $\zeta = \lambda^n$  in (2.1.9),

$$\begin{aligned} (X_n + Y_n)^2 + (3\theta^2 - 4p) \left( \frac{Z_n}{\theta} \right)^2 &= \frac{4N(\lambda^n)}{\lambda^n} = \frac{4}{\lambda^n}, \\ (3\theta^2 - 4p) \left( \frac{Z_n}{\theta} \right)^2 &< \frac{4}{\lambda^n}. \end{aligned}$$

Then since  $3\theta^2 - 4p > 0$  (from (2.1.5)),

$$\begin{aligned} Z_n^2 &< \frac{4\theta^2}{3\theta^2 - 4p} \cdot \frac{1}{\lambda^n}, \\ |Z_n| &< \frac{2|\theta|}{\sqrt{3\theta^2 - 4p}} \cdot \frac{1}{\lambda^{n/2}} \leq \frac{\sqrt{2}C_1}{\lambda^{n/2}}. \end{aligned}$$

(First and second inequalities) By (2.1.10),

$$X_n^2 + Y_n^2 + \left( \frac{\theta^2 - 2p}{\theta^2} \right) Z_n^2 = \frac{2}{\lambda^n}. \quad (2.1.18)$$

(Case 1:  $\theta^2 - 2p < 0$ .) From (2.1.18) we have

$$X_n^2 + Y_n^2 = \frac{2p - \theta^2}{\theta^2} Z_n^2 + \frac{2}{\lambda^n}.$$

Since  $\frac{2p - \theta^2}{\theta^2} Z_n^2 > 0$ , and since  $Z_n^2 < \frac{4\theta^2}{3\theta^2 - 4p} \cdot \frac{1}{\lambda^n}$ ,

$$\begin{aligned} X_n^2 + Y_n^2 &< \frac{2p - \theta^2}{\theta^2} \left( \frac{4\theta^2}{3\theta^2 - 4p} \cdot \frac{1}{\lambda^n} \right) + \frac{2}{\lambda^n} \\ &= \left( \frac{4(2p - \theta^2)}{(3\theta^2 - 4p)} + 2 \right) \frac{1}{\lambda^n} \\ &= \frac{4(2p - \theta^2) + 2(3\theta^2 - 4p)}{3\theta^2 - 4p} \cdot \frac{1}{\lambda^n} \\ &= \frac{2\theta^2}{3\theta^2 - 4p} \cdot \frac{1}{\lambda^n}. \end{aligned}$$

So

$$|X_n|, |Y_n| < \frac{\sqrt{2}|\theta|}{\sqrt{3\theta^2 - 4p}} \cdot \frac{1}{\lambda^{n/2}}.$$

By Lemma 2.1.12, the RHS of this inequality is  $\frac{C_1}{\lambda^{n/2}}$ .

(Case 2:  $\theta^2 - 2p > 0$ .) Suppose  $\theta^2 - 2p > 0$ . Then

$$X_n^2 + Y_n^2 \leq X_n^2 + Y_n^2 + \left(\frac{\theta^2 - 2p}{\theta^2}\right) Z_n^2 = \frac{2}{\lambda^n}.$$

We claim that equality cannot hold with our choice of  $\lambda$ . Note that we would have equality iff  $b_n\theta - c_n\theta^2 = Z_n = 0$ . Since  $b_n, c_n \in \mathbb{Q}$  and  $\theta \notin \mathbb{Q}$ , this would mean  $b_n = c_n = 0$ , so  $\lambda^n = a_n$ . Then

$$1 = N_{K/\mathbb{Q}}(\lambda) = N_{K/\mathbb{Q}}(a_n) = \sigma_1(a_n)\sigma_2(a_n)\sigma_3(a_n) = a_n^3 = (\lambda^n)^3.$$

But  $\lambda > 1$  (by assumption), so  $\lambda^{3n} > 1$ . Therefore we have the strict inequality

$$X_n^2 + Y_n^2 < \frac{2}{\lambda^n},$$

and so

$$|X_n|, |Y_n| < \frac{\sqrt{2}}{\lambda^{n/2}}.$$

By Lemma 2.1.12, the RHS of this inequality equals  $\frac{C_1}{\lambda^{n/2}}$ .

(Fourth inequality) Since

$$\begin{aligned} \lambda^n - (3\theta^2 - p)c_n &= a_n + b_n\theta + c_n\theta^2 - 3c_n\theta^2 + pc_n \\ &= (a_n + pc_n - c_n\theta^2) + (b_n\theta - c_n\theta^2) \\ &= Y_n + Z_n, \end{aligned}$$

we have

$$|\lambda^n - (3\theta^2 - p)c_n| \leq |Y_n| + |Z_n| = (1 + \sqrt{2})\frac{C_1}{\lambda^{n/2}}.$$

□

**Recall:** We are using the notation  $\langle x \rangle$  to denote  $x - \lfloor x + \frac{1}{2} \rfloor$ , the signed distance from  $x$  to the nearest integer (so  $\|x\| = |\langle x \rangle|$ ).

**Lemma 2.1.19.**

(a) If  $\lambda^{n/2} > \frac{2\sqrt{2}dC_1}{|\theta|}$ , then  $\langle k_n\theta \rangle = -\frac{dZ_n}{\theta}$ .

(b) If  $\lambda^{n/2} > 2dC_1$ , then  $\langle k_n\theta^2 \rangle = -dY_n$ .

**Proof.** (a) Suppose  $\lambda^{n/2} > \frac{2\sqrt{2}dC_1}{|\theta|}$ . Then  $\frac{\sqrt{2}C_1}{\lambda^{n/2}} < \frac{|\theta|}{2d}$ , so by (2.1.16)

$$|db_n - dc_n\theta| = \left| \frac{dZ_n}{\theta} \right| < \frac{d}{|\theta|} \frac{\sqrt{2}C_1}{\lambda^{n/2}} < \frac{1}{2}.$$

So since  $db_n$  is an integer (by the choice of  $d$ ), it is the closest integer to  $dc_n\theta = k_n\theta$ .

That is,

$$\langle k_n\theta \rangle = k_n\theta - db_n = -\frac{dZ_n}{\theta}.$$

(b) The proof is identical to (a). If  $\lambda^{n/2} > 2dC_1$ , then  $\frac{C_1}{\lambda^{n/2}} < \frac{1}{2d}$ , and so by (2.1.15)

$$|da_n + pdc_n - dc_n\theta^2| = |dY_n| < \frac{dC_1}{\lambda^{n/2}} < \frac{1}{2}.$$

Now  $da_n + pdc_n$  is an integer (since  $p \in \mathbb{Z}$ , and since  $da_n, dc_n \in \mathbb{Z}$  by our choice of  $d$ ), so it is the closest integer to  $dc_n\theta^2 = k_n\theta^2$ . So

$$\langle k_n\theta^2 \rangle = k_n\theta^2 - (da_n + pk_n) = -Y_n.$$

□

**Lemma 2.1.20.**

(a) If  $\lambda^{3n/2} > (1 + \sqrt{2})C_1$ , then  $c_n > 0$ .

(b) If  $\lambda^{3n/2} > \frac{(1 + \sqrt{2})C_1}{\sqrt{\lambda}(\sqrt{\lambda} - 1)}$ , then  $c_n$  is increasing.

**Proof.** Put  $\eta = (1 + \sqrt{2})C_1$ .

(a) Suppose  $\lambda^{3n/2} > \eta$ . Then  $\lambda^n > \frac{\eta}{\lambda^{n/2}}$ , or

$$\lambda^n - \frac{\eta}{\lambda^{n/2}} > 0.$$

By (2.1.17), we have

$$\lambda^n - \frac{\eta}{\lambda^{n/2}} < (3\theta^2 - p)c_n.$$

So  $0 < (3\theta^2 - p)c_n$ . And  $3\theta^2 - p > 0$  (by 2.1.7), so  $c_n > 0$ .

(b) We use (2.1.17) again to get

$$\begin{aligned} (3\theta^2 - p)c_{n+1} &> \lambda^{n+1} - \frac{\eta}{\lambda^{(n+1)/2}}, \\ (3\theta^2 - p)c_n &< \lambda^n + \frac{\eta}{\lambda^{n/2}}. \end{aligned}$$

Then

$$(3\theta^2 - p)(c_{n+1} - c_n) > \lambda^n(\lambda - 1) - \frac{\eta}{\lambda^{n/2}} \left(1 + \frac{1}{\sqrt{\lambda}}\right),$$

which is positive if

$$\lambda^n(\lambda - 1) > \frac{\eta}{\lambda^{n/2}} \left(1 + \frac{1}{\sqrt{\lambda}}\right),$$

or

$$\lambda^{3n/2} > \frac{\eta(\sqrt{\lambda} + 1)}{\sqrt{\lambda}(\lambda - 1)} = \frac{\eta}{\sqrt{\lambda}(\sqrt{\lambda} - 1)}.$$

□

**Lemma 2.1.21.** *If  $\lambda^{3n/2} > 2(1 + \sqrt{2})C_1$ , then*

$$|\lambda^{n/2} - \sqrt{3\theta^2 - p} \cdot c_n^{1/2}| < \frac{\sqrt{2}C_1}{\lambda^n}. \quad (2.1.22)$$

**Proof.** Put  $\eta = (\sqrt{2} + 1)C_1$  as before, and suppose  $\lambda^{3n/2} > 2\eta$ . By Lemma 2.1.20,  $c_n$  is positive. Now

$$|\lambda^{n/2} - \sqrt{3\theta^2 - p} \cdot c_n^{1/2}| \cdot |\lambda^{n/2} + \sqrt{3\theta^2 - p} \cdot c_n^{1/2}| = |\lambda^n - (3\theta^2 - p)c_n| < \frac{\eta}{\lambda^{n/2}}$$

(by (2.1.17)), so

$$|\lambda^{n/2} - \sqrt{3\theta^2 - p} \cdot c_n^{1/2}| < \frac{\eta}{\lambda^{n/2}} \cdot \left(\lambda^{n/2} + \sqrt{3\theta^2 - p} \cdot c_n^{1/2}\right)^{-1}.$$

We rewrite  $\lambda^{3n/2} > 2\eta$  as  $\frac{\eta}{\lambda^{n/2}} < \frac{\lambda^n}{2}$ , and then (by (2.1.17) again)

$$\begin{aligned} (3\theta^2 - p)c_n &> \lambda^n - \frac{\eta}{\lambda^{n/2}} > \lambda^n - \frac{\lambda^n}{2} = \frac{\lambda^n}{2}, \\ \sqrt{3\theta^2 - p} \cdot c_n^{1/2} &> \frac{\lambda^{n/2}}{\sqrt{2}}, \\ \lambda^{n/2} + \sqrt{3\theta^2 - p} \cdot c_n^{1/2} &> \left(1 + \frac{1}{\sqrt{2}}\right) \lambda^{n/2}. \end{aligned}$$

So

$$\begin{aligned} |\lambda^{n/2} - \sqrt{3\theta^2 - p} \cdot c_n^{1/2}| &< \frac{\eta}{\lambda^{n/2}} \cdot \left( \left(1 + \frac{1}{\sqrt{2}}\right) \lambda^{n/2} \right)^{-1} \\ &= \frac{\eta\sqrt{2}}{\sqrt{2} + 1} \cdot \frac{1}{\lambda^n} \\ &= \frac{(\sqrt{2} + 1)C_1\sqrt{2}}{\sqrt{2} + 1} \cdot \frac{1}{\lambda^n} \\ &= \frac{\sqrt{2}C_1}{\lambda^n}. \end{aligned}$$

□

**Corollary 2.1.23.** *If  $\lambda^{3n/2} > 2(1 + \sqrt{2})C_1$ , then*

$$\frac{C_2}{2} \cdot \lambda^{n/2} < k_n^{1/2} < \frac{3C_2}{2} \cdot \lambda^{n/2}.$$

**Proof.** Suppose  $\lambda^{3n/2} > 2(1 + \sqrt{2})C_1$ . From Lemma 2.1.21, we have

$$\lambda^{n/2} - \frac{\sqrt{2}C_1}{\lambda^n} < \sqrt{3\theta^2 - p} \cdot c_n^{1/2} < \lambda^{n/2} + \frac{\sqrt{2}C_1}{\lambda^n}. \quad (2.1.24)$$

Now  $\lambda^{3n/2} > 2(1 + \sqrt{2})C_1 > 2\sqrt{2}C_1$ , so

$$\frac{\sqrt{2}C_1}{\lambda^n} < \frac{\lambda^{n/2}}{2}.$$

Combining this with (2.1.24) gives us

$$\frac{\lambda^{n/2}}{2} < \sqrt{3\theta^2 - p} \cdot c_n^{1/2} < \frac{3\lambda^{n/2}}{2}.$$

Now multiply through by  $C_2 = \frac{\sqrt{d}}{\sqrt{3\theta^2 - p}}$  (and recall that  $k_n = dc_n$ ). □



**Lemma 2.1.25.** *For all positive integers  $n$ ,*

$$\lambda_2^n = \frac{(X_n + Y_n)}{2} + \frac{\sqrt{4p - 3\theta^2} Z_n}{2\theta}.$$

**Proof.** As in the proof of Lemma 2.1.8, we use  $\theta_2 = \frac{-\theta + \sqrt{4p - 3\theta^2}}{2}$  (see Remark 2.1.3) to find  $\lambda_2^n$  in terms of  $\theta$ :

$$\begin{aligned} \lambda_2^n &= a_n + b_n\theta_2 + c_n\theta_2^2 \\ &= a_n + b_n \left( \frac{-\theta + \sqrt{4p - 3\theta^2}}{2} \right) + c_n \left( p + \frac{-\theta^2 - \theta\sqrt{4p - 3\theta^2}}{2} \right) \\ &= \left( a_n + pc_n - \frac{b_n\theta}{2} - \frac{c_n\theta^2}{2} \right) + \frac{\sqrt{4p - 3\theta^2}}{2} (b_n - c_n\theta) \\ &= \frac{X_n + Y_n}{2} + \frac{\sqrt{4p - 3\theta^2} Z_n}{2\theta}. \end{aligned}$$

□

**Corollary 2.1.26.** *For all positive integers  $n$ ,*

$$\operatorname{Im}(\lambda_2^n) = \frac{\sqrt{3\theta^2 - 4p} Z_n}{2\theta} = \frac{\sqrt{3\theta^2 - 4p}}{2} (b_n - c_n\theta)$$

**Proof.** This follows immediately from the previous lemma and the fact that (by Lemma 2.1.5)  $4p - 3\theta^2 < 0$ . □

**Lemma 2.1.27.** *The set  $\{(\sqrt{\lambda}\lambda_2)^n : n \in \mathbb{N}\}$  is a dense subset of the unit circle.*

**Proof.** It is enough to show that  $|\sqrt{\lambda}\lambda_2| = 1$  and that  $(\sqrt{\lambda}\lambda_2)^k \neq 1$  for  $k \in \mathbb{N}$ . Since  $\lambda_2$  and  $\lambda_3$  are complex conjugates and  $\lambda = \lambda_1$  is a unit,

$$|\sqrt{\lambda}\lambda_2|^2 = (\sqrt{\lambda}\lambda_2)(\sqrt{\lambda}\lambda_3) = \lambda_1\lambda_2\lambda_3 = N(\lambda) = 1.$$

Now suppose that  $k$  is a positive integer and  $(\sqrt{\lambda}\lambda_2)^k = 1$ . Then  $\operatorname{Im}((\sqrt{\lambda}\lambda_2)^m) = 0$  for every positive integer multiple  $m$  of  $k$ . Since

$$\operatorname{Im}((\sqrt{\lambda}\lambda_2)^m) = \lambda^{m/2} \operatorname{Im}(\lambda_2^m) = \lambda^{m/2} \frac{\sqrt{3\theta^2 - 4p} Z_m}{2\theta},$$

and since  $\lambda^{m/2}$  and  $\sqrt{3\theta^2 - 4p}$  are both nonzero, we would have

$$b_m - c_m\theta = \frac{Z_m}{\theta} = 0,$$

or  $b_m = c_m\theta$ . Since  $b_m, c_m \in \mathbb{Q}$  and  $\theta \notin \mathbb{Q}$ , this would imply  $b_m = c_m = 0$ . In particular, we would have  $c_m = 0$  for every positive multiple  $m$  of  $k$ . But by Lemma 2.1.20, eventually  $\{c_n\}$  is positive (and so there can be only finitely many  $m$  such that  $c_m = 0$ ). Therefore  $(\sqrt{\lambda}\lambda_2)^k \neq 1$ .  $\square$

**Definition 2.1.28.** Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 - 4p} \cdot (b_1 - c_1\theta)}{2(a_1 + pc_1) - b_1\theta - c_1\theta^2} \right)$$

**Lemma 2.1.29.**  $\sqrt{\lambda}\lambda_2 = \pm e^{i\phi}$

**Proof.** Since  $\sqrt{\lambda}\lambda_2$  is on the unit circle (by 2.1.27), it can be expressed as  $e^{iw}$  for some  $w \in [-\pi/2, 3\pi/2)$ . By Lemma 2.1.25, we have

$$e^{iw} = \sqrt{\lambda}\lambda_2 = \sqrt{\lambda} \left( \frac{2(a_1 + pc_1) - b_1\theta - c_1\theta^2}{2} + i \frac{\sqrt{3\theta^2 - 4p}}{2} (b_1 - c_1\theta) \right),$$

so

$$\tan w = \frac{\operatorname{Im} \sqrt{\lambda}\lambda_2}{\operatorname{Re} \sqrt{\lambda}\lambda_2} = \frac{\sqrt{3\theta^2 - 4p} \cdot (b_1 - c_1\theta)}{2(a_1 + pc_1) - b_1\theta - c_1\theta^2} = \tan \phi.$$

Therefore either  $w = \phi$  or  $w = \phi + \pi$ , so  $\sqrt{\lambda}\lambda_2 = e^{iw} = \pm e^{i\phi}$ .  $\square$

**Lemma 2.1.30.** Let  $\{P_n/Q_n\}$  be the sequence of convergents of  $\phi/\pi$ . For  $n \geq 1$ ,

$$\left| \operatorname{Im}(\sqrt{\lambda}\lambda_2)^{Q_n} \right| < \frac{\pi}{Q_{n+1}}.$$

**Proof.** By the theory of continued fractions (see [Kh], Theorem 9),

$$\left| \frac{\phi}{\pi} - \frac{P_n}{Q_n} \right| < \frac{1}{Q_n Q_{n+1}},$$

or

$$|Q_n \phi - P_n \pi| < \frac{\pi}{Q_{n+1}}.$$

Then

$$|\operatorname{Im}(\sqrt{\lambda}\lambda_2)^{Q_n}| = |\operatorname{Im} e^{iQ_n\phi}| = |\sin Q_n\phi| = |\sin(Q_n\phi - P_n\pi)| \leq |Q_n\phi - P_n\pi| < \frac{\pi}{Q_{n+1}}.$$

□

**Lemma 2.1.31.**

(a) If  $\|x_1\| + \cdots + \|x_k\| \leq \frac{1}{2}$ , then  $\langle x_1 + \cdots + x_k \rangle = \langle x_1 \rangle + \cdots + \langle x_k \rangle$ .

(b)  $\langle -x \rangle = -\langle x \rangle$

(c) If  $n \in \mathbb{Z}$  and  $|n|\|x\| < \frac{1}{2}$ , then  $\langle nx \rangle = n\langle x \rangle$ .

**Proof.** (a) Say  $x_i = n_i + \delta_i$  for each  $i$ , where  $n_i \in \mathbb{Z}$  and  $|\delta_i| \leq \frac{1}{2}$ . Then each  $\langle x_i \rangle = \delta_i$ .

Put

$$m' = n_1 + \cdots + n_k,$$

$$\delta' = \delta_1 + \cdots + \delta_k.$$

Then

$$\langle x_1 + \cdots + x_k \rangle = \langle (n_1 + \cdots + n_k) + (\delta_1 + \cdots + \delta_k) \rangle = \langle m' + \delta' \rangle.$$

Since

$$|\delta'| \leq |\delta_1| + \cdots + |\delta_k| = \|x_1\| + \cdots + \|x_k\| \leq \frac{1}{2},$$

we have

$$\langle x_1 + \cdots + x_k \rangle = \langle m' + \delta' \rangle = \delta' = \delta_1 + \cdots + \delta_k = \langle x_1 \rangle + \cdots + \langle x_k \rangle.$$

(b) Say  $x = n + \delta$ , where  $n \in \mathbb{Z}$  and  $|\delta| \leq \frac{1}{2}$ . Then  $\langle x \rangle = \delta$ . Now  $-x = -n + (-\delta)$ , where  $-n \in \mathbb{Z}$  and  $|-\delta| \leq \frac{1}{2}$ , so

$$\langle -x \rangle = -\delta = -\langle x \rangle.$$

(c) If  $n = 0$ , then  $\langle nx \rangle = n\langle x \rangle$ . If  $n \neq 0$  and  $|n|\|x\| < \frac{1}{2}$ , put  $x_1 = x_2 = \cdots = x_{|n|} = x$  in part (a) to get  $|n|\langle x \rangle = \langle |n|x \rangle$ . Therefore  $\langle nx \rangle = n\langle x \rangle$  if  $n = |n|$ . Now if  $n = -|n|$ , we apply part (b) to get

$$\langle nx \rangle = \langle -|n|x \rangle = -\langle |n|x \rangle = -|n|\langle x \rangle = n\langle x \rangle.$$

□

**Remark 2.1.32.** In the introduction, we mentioned that it is straightforward to show Littlewood's conjecture for the pair  $(\alpha, \beta)$  when  $1, \alpha, \beta$  is  $\mathbb{Q}$ -linearly dependent. If  $\alpha$  or  $\beta$  is rational (WLOG say  $\alpha = \frac{r}{s}$ , with  $r, s \in \mathbb{Z}$  and  $s > 0$ ), then for all  $k$ ,  $ks$  is an integer for which

$$ks\|ks\alpha\|\|ks\beta\| = ks \cdot 0 \cdot \|ks\beta\| = 0.$$

If  $\alpha$  and  $\beta$  are irrational, the following lemma gives us an increasing sequence  $\{k_n\}$  of integers for which  $k_n\|k_n\alpha\|\|k_n\beta\| < \frac{C}{k_n}$  (for some  $C$ ) for all  $n$ , and therefore that Littlewood's conjecture holds for the pair  $(\alpha, \beta)$ .

**Lemma 2.1.33.** *If  $\alpha, \beta \in \mathbb{R}$  are such that  $1, \alpha, \beta$  is  $\mathbb{Q}$ -linearly dependent, then we can construct an increasing sequence  $\{k_n\}$  of positive integers such that  $\{k_n\|k_n\alpha\|\}$  and  $\{k_n\|k_n\beta\|\}$  are both bounded.*

**Proof.** Let  $\{q_n\}$  be the sequence of denominators of convergents of  $\alpha$ . Suppose

$$\beta = \frac{r_1}{s}\alpha + \frac{r_2}{s},$$

where  $r_1, r_2, s \in \mathbb{Z}$  and  $s > 0$ . So  $s\beta = r_1\alpha + r_2$ . Put  $k_n = sq_n$ . For all  $n$ ,  $\|q_n\alpha\| < \frac{1}{q_n}$ . So eventually (say for  $n > n'$ ) both  $|r_1|\|q_n\alpha\|$  and  $s\|q_n\alpha\|$  are less than  $\frac{1}{2}$ . Suppose  $n > n'$ . Then by Lemma 2.1.31(c),  $\langle r_1q_n\alpha \rangle = r_1\langle q_n\alpha \rangle$  and  $\langle sq_n\alpha \rangle = s\langle q_n\alpha \rangle$ . Now

$$\langle k_n\alpha \rangle = \langle sq_n\alpha \rangle = s\langle q_n\alpha \rangle,$$

$$\langle k_n\beta \rangle = \langle sq_n\beta \rangle = \langle q_n \cdot s\beta \rangle = \langle q_n(r_1\alpha + r_2) \rangle = \langle r_1q_n\alpha + q_nr_2 \rangle = \langle r_1q_n\alpha \rangle$$

(this last equality due to  $q_n r_2$  being an integer). So

$$\begin{aligned} k_n \|k_n \alpha\| &= k_n \cdot s \|q_n \alpha\| < k_n \frac{s}{q_n} = k_n \frac{s^2}{k_n} = s^2, \\ k_n \|k_n \beta\| &= k_n \cdot |r_1| \|q_n \alpha\| < k_n \frac{|r_1|}{q_n} = s |r_1|. \end{aligned}$$

□

### 2.1.3 Equation of the Ellipse

Put  $u_n = k_n^{1/2} \langle k_n \theta \rangle$  and  $v_n = k_n^{1/2} \langle k_n \theta^2 \rangle$  (and assume that  $k_n > 0$ ), and consider the sequence  $\{(u_n, v_n)\}$ . We will see in the proof of Theorem 1 that  $u_n$  and  $v_n$  are bounded, but we can say more about the relationship between them:

**Lemma 2.1.34.** *The points  $(u_n, v_n)$  get arbitrarily close to the ellipse*

$$(\theta x - 2y)^2 + (3\theta^2 - 4p)x^2 = \frac{4d^3}{3\theta^2 - p}.$$

**Proof.** By Lemma 2.1.19, eventually  $\langle k_n \theta \rangle = -\frac{dZ_n}{\theta}$  and  $\langle k_n \theta^2 \rangle = -dY_n$ . By Lemma 2.1.8 (and by the definitions of  $a_n, b_n, c_n$ )

$$(X_n + Y_n)^2 + (3\theta^2 - 4p) \left( \frac{Z_n}{\theta} \right)^2 = \frac{4}{\lambda^n},$$

or (since  $X_n = Y_n - Z_n$ )

$$\begin{aligned} (2Y_n - Z_n)^2 + (3\theta^2 - 4p) \left( \frac{Z_n}{\theta} \right)^2 &= \frac{4}{\lambda^n}, \\ (2dY_n - dZ_n)^2 + (3\theta^2 - 4p) \left( \frac{dZ_n}{\theta} \right)^2 &= \frac{4d^2}{\lambda^n}, \\ (-2\langle k_n \theta \rangle + \theta \langle k_n \theta^2 \rangle)^2 + (3\theta^2 - 4p) \langle k_n \theta \rangle^2 &= \frac{4d^2}{\lambda^n}. \end{aligned}$$

Multiplying both sides by  $k_n$ , we have

$$(\theta u_n - 2v_n)^2 + (3\theta^2 - 4p)u_n^2 = 4d^2 \frac{k_n}{\lambda^n}.$$

By Lemma 2.1.13,

$$\frac{d\lambda^n}{3\theta^2 - p} - \frac{C'}{\lambda^{n/2}} < k_n < \frac{d\lambda^n}{3\theta^2 - p} + \frac{C'}{\lambda^{n/2}},$$

where  $C' = \frac{(1+\sqrt{2})C_1d}{3\theta^2-p}$ . So

$$\frac{4d^3}{3\theta^2-p} - \frac{4d^2C'}{\lambda^{3n/2}} < k_n \frac{4d^2}{\lambda^n} < \frac{4d^3}{3\theta^2-p} + \frac{4d^2C'}{\lambda^{3n/2}},$$

or

$$\frac{4d^3}{3\theta^2-p} - \frac{4d^2C'}{\lambda^{3n/2}} < (\theta u_n - 2v_n)^2 + (3\theta^2 - 4p)u_n^2 < \frac{4d^3}{3\theta^2-p} + \frac{4d^2C'}{\lambda^{3n/2}}.$$

Since  $\lambda > 1$ , this shows that the points  $\{(u_n, v_n)\}$  get arbitrarily close to

$$(\theta x - 2y)^2 + (3\theta^2 - 4p)x^2 = \frac{4d^3}{3\theta^2-p}.$$

Since  $3\theta^2 - 4p > 0$  (2.1.5), this is an ellipse. □

## 2.2 Theorem 1

**Theorem 1.** *Suppose  $\theta \in \mathbb{R}$  is the only real root of an irreducible polynomial of the form  $x^3 - px - q \in \mathbb{Z}[x]$ . Then we can construct a Peck sequence for the pair  $(\theta, \theta^2)$ .*

**Proof.** Suppose  $x^3 - px - q \in \mathbb{Z}[x]$  is irreducible and has only one real root  $\theta$ . Let  $K$  denote  $\mathbb{Q}(\theta)$ . Let  $d$  be a positive integer such that  $\mathcal{O}_K \subset \frac{1}{d}\mathbb{Z}[\theta]$ . We can find this with PARI/GP by finding an integral basis using `bnfinit(X^3-p*X-q).zk`, and then letting  $d$  be the largest denominator in this integral basis. We want to find an infinite-order unit  $\lambda$  satisfying

$$\lambda > \left( (1 + \sqrt{2})C_1 \right)^{2/3},$$

where (as in Definition 2.1.11)  $C_1 = \max\left\{\sqrt{2}, \frac{\sqrt{2}|\theta|}{\sqrt{3\theta^2 - 4p}}\right\}$ . We can produce such a  $\lambda$  as follows. We can find a fundamental unit  $\varepsilon_0$  of  $\mathcal{O}_K$  (for example with PARI/GP, using `bnfinit(X^3-p*X-q).fu`). By putting

$$\varepsilon' = \max \left\{ \pm\varepsilon_0, \pm\frac{1}{\varepsilon_0} \right\},$$

we have a unit with  $\varepsilon' > 1$ . Put

$$m' = \left\lceil \frac{2}{3} \log_{\varepsilon'}(1 + \sqrt{2})C_1 \right\rceil,$$

and finally

$$\lambda = (\varepsilon')^{m'}.$$

As before, we define  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ , and  $\{k_n\}$  by

$$a_n + b_n\theta + c_n\theta^2 = \lambda^n$$

and  $k_n = dc_n$ . We will show that  $\{k_n\}$  is a Peck sequence for the pair  $(\theta, \theta^2)$ .

We need to show that

$$\max\{\|k_n\theta\|, \|k_n\theta^2\|\} < \frac{M_{\theta_1}}{k_n^{1/2}} \quad (2.2.1)$$

for some constant  $M_{\theta_1}$ , and that  $\{k_n\}$  has a subsequence  $\{\psi_n\}$  such that

$$\|\psi_n \theta\| < \frac{M_{\theta_2}}{\psi_n^{1/2} \log \psi_n} \quad (2.2.2)$$

for some constant  $M_{\theta_2}$ .

Since  $\lambda > ((1 + \sqrt{2})C_2)^{2/3}$ , we have

$$\lambda^{3n/2} > (1 + \sqrt{2})C_1$$

for all  $n$ , and so each  $c_n$  (and therefore each  $k_n$ ) is positive. Put

$$N'_\theta = \max \left\{ \frac{2\sqrt{2}dC_1}{|\theta|}, 2dC_1, \left(2(1 + \sqrt{2})C_1\right)^{1/3}, \lambda^{1/2} \right\}$$

and  $n_\theta = 2 \log_\lambda N'_\theta$ . Suppose  $n > n_\theta$ . We note that  $n \geq 2$ , since  $n > n_\theta$  is an integer and  $n_\theta \geq 1$ . (We will use this later on.) Now

$$\lambda^{n/2} > \frac{2\sqrt{2}dC_1}{|\theta|},$$

$$\lambda^{n/2} > 2dC_1,$$

$$\lambda^{3n/2} > 2(1 + \sqrt{2})C_1,$$

so (by Lemma 2.1.19 and Corollary 2.1.23)

$$\|k_n \theta\| = \left| \frac{dZ_n}{\theta} \right|, \quad (2.2.3)$$

$$\|k_n \theta^2\| = |dY_n|, \quad (2.2.4)$$

and

$$k_n^{1/2} < \frac{3}{2}C_2 \lambda^{n/2} \quad (2.2.5)$$

(recall from Definition 2.1.11 that  $C_2 = \frac{\sqrt{d}}{\sqrt{3\theta^2 - p}}$ ). Combining (2.2.3) and (2.2.4) with Lemma 2.1.13 gives us

$$\|k_n \theta\| = \frac{d}{|\theta|} |Z_n| < \frac{d\sqrt{2}C_1}{|\theta|} \frac{1}{\lambda^{n/2}},$$

$$\|k_n \theta^2\| = d|Y_n| < \frac{dC_1}{\lambda^{n/2}}.$$



Together with (2.2.5), we have

$$\begin{aligned}\|k_n\theta\| &< \frac{d\sqrt{2}C_1}{|\theta|} \cdot \frac{3C_2}{2} \cdot \frac{1}{k_n^{1/2}}, \\ \|k_n\theta^2\| &< dC_1 \cdot \frac{3C_2}{2} \cdot \frac{1}{k_n^{1/2}}.\end{aligned}$$

Put

$$M' = \max_{i \leq n_\theta} \{k_i^{1/2}\|k_i\theta\|, k_i^{1/2}\|k_i\theta^2\|\}$$

and

$$M_{\theta_1} = \max\left\{M', \frac{3d\sqrt{2}C_1C_2}{2|\theta|}, \frac{3dC_1C_2}{2}\right\}.$$

Then for all  $n$ ,

$$\max\{\|k_n\theta\|, \|k_n\theta^2\|\} < \frac{M_{\theta_1}}{k_n^{1/2}}.$$

To find a subsequence of  $\{k_n\}$  satisfying (2.2.2), we first combine (2.2.3) with Corollary 2.1.26 to get that (for  $n > n_\theta$ )

$$\|k_n\theta\| = d \left| \frac{Z_n}{\theta} \right| = \frac{2d}{\sqrt{3\theta^2 - 4p}} |\operatorname{Im} \lambda_2^n|.$$

This, along with (2.2.5), gives us

$$k_n^{1/2}\|k_n\theta\| < \frac{3C_2}{2} \cdot \frac{2d}{\sqrt{3\theta^2 - 4p}} \cdot \lambda^{n/2} |\operatorname{Im} \lambda_2^n| = \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \left| \operatorname{Im}(\sqrt{\lambda}\lambda_2)^n \right|.$$

Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 - 4p}(b_1 - c_1\theta)}{2(a_1 + pc_1) - b_1\theta - c_1\theta^2} \right)$$

(so that  $\sqrt{\lambda}\lambda_2 = \pm e^{i\phi}$ ), and let  $\{P_n/Q_n\}$  be the sequence of convergents of  $\phi/\pi$ . Then (by Lemma 2.1.30)

$$k_{Q_n}^{1/2}\|k_{Q_n}\theta\| < \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \left| \operatorname{Im}(\lambda^{1/2}\lambda_2)^{Q_n} \right| < \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}} \quad (2.2.6)$$

for  $Q_n > n_\theta$ .

By Corollary 2.1.23,

$$k_{Q_{n+1}}^{1/2} < \frac{3C_2}{2} \lambda^{Q_{n+1}/2} \leq C_3 \lambda^{Q_{n+1}/2}$$

(where  $C_3 = \max\{1, \frac{3C_2}{2}\}$  as in Definition 2.1.11). Earlier we noted that  $n \geq 2$ , and so  $Q_{n+1} \geq n \geq 2$  (because  $\{Q_n\}$  is bounded below termwise by the Fibonacci sequence.) Then (since  $C_3 \geq 1$ )  $C_3 \leq C_3^{Q_{n+1}/2}$ , so

$$k_{Q_{n+1}}^{1/2} < (C_3\lambda)^{Q_{n+1}/2}. \quad (2.2.7)$$

Then

$$\frac{1}{Q_{n+1}} < \frac{\log C_3\lambda}{\log k_{Q_{n+1}}}.$$

So putting

$$M_{\theta_2} = \frac{3dC_2\pi}{2\sqrt{3\theta^2 - 4p}} \cdot \log C_3\lambda,$$

we have that for  $Q_n > n_\theta$ , the subsequence  $\{k_{Q_n}\}$  satisfies

$$k_{Q_n}^{1/2} \|k_{Q_n}\theta\| < \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}} < \frac{M_{\theta_2}}{\log k_{Q_{n+1}}}. \quad (2.2.8)$$

Since  $Q_{n+1} > Q_n$  for  $n \geq 2$ , this gives us

$$\|k_{Q_n}\theta\| < \frac{M_{\theta_2}}{k_{Q_n}^{1/2} \log k_{Q_n}} \quad (2.2.9)$$

when  $Q_n > n_\theta$ . We can shift the sequence  $\{k_{Q_n}\}$  to produce a sequence  $\{\psi_n\}$  such that

$$\|\psi_n\theta\| < \frac{M_{\theta_2}}{\psi_n^{1/2} \log \psi_n}$$

for all  $n$ . □

## 2.3 Theorem 2

**Theorem 2.** *Suppose  $\alpha \in \mathbb{R}$  is the only real root of an irreducible cubic polynomial in  $\mathbb{Q}[x]$ . Then we can construct a Peck sequence for the pair  $(\alpha, \alpha^2)$ .*

**Proof.** First, we make some reductions in order to use Theorem 1. By clearing denominators, we get an irreducible cubic

$$f(x) = Ax^3 + Bx^2 + Cx + D \in \mathbb{Z}[x]$$

(so  $A \neq 0$ ) for which  $\alpha$  is the only real root. WLOG we assume  $A$  is positive. By putting

$$g(x) = 27A^2 f\left(\frac{x-B}{3A}\right),$$

we transform  $f(x)$  into the form  $x^3 - px - q \in \mathbb{Z}[x]$ , where

$$p = 3A(B^2 - 3AC),$$

$$q = -2B^3 + 9ABC - 27A^2D.$$

Since  $f$  is irreducible, so is  $g$ . Recall that  $\alpha_i$  denotes  $\sigma_i(\alpha)$  for  $i = 1, 2, 3$  (where  $\sigma_1$  is the real embedding). Put

$$\theta_i = 3A\alpha_i + B.$$

Now  $\theta := \theta_1$  is real, and  $\theta_2, \theta_3$  are complex, and all three satisfy

$$g(\theta_i) = 27A^2 f(\alpha_i) = 0.$$

So  $\theta$  satisfies the hypothesis of Theorem 1.

Let  $\{k_n\}$ ,  $C_2$ ,  $C_3$ ,  $M_{\theta_1}$ ,  $M_{\theta_2}$ ,  $\{Q_j\}$ ,  $N'_\theta$ ,  $n_\theta$  be as in the proof of Theorem 1. Define the sequence  $\{\ell_n\}$  by

$$\ell_n = 9A^2 k_n.$$

To show that  $\{\ell_n\}$  is a Peck sequence for  $(\alpha, \alpha^2)$ , we first note that since  $\{k_n\}$  is an increasing sequence of positive integers, so is  $\{\ell_n\}$ . To show the other properties, we put

$$N'_\alpha = \max\{N'_\theta, 12AM_{\theta_1}, 16|B|M_{\theta_1}\}$$

and  $n_\alpha = 2 \log_\lambda N'_\alpha$ , and use the following:

**Lemma 2.3.1.** *For  $n > n_\alpha$ ,*

$$\langle \ell_n \alpha \rangle = 3A \langle k_n \theta \rangle$$

and

$$\langle \ell_n \alpha^2 \rangle = \langle k_n \theta^2 \rangle - 2B \langle k_n \theta \rangle.$$

**Proof of Lemma.** Our strategy is to use Lemma 2.1.31 to get  $\langle \ell_n \alpha \rangle$  and  $\langle \ell_n \alpha^2 \rangle$  in terms of  $\langle k_n \theta \rangle$  and  $\langle k_n \theta^2 \rangle$ . Let  $n > n_\alpha$ . Then

$$\lambda^{n/2} > 12AM_{\theta_1},$$

$$\lambda^{n/2} > 16|B|M_{\theta_1}.$$

Since  $n > n_\alpha > n_\theta$ , we also have  $k_n^{1/2} > \frac{\lambda^{n/2}}{2}$  (by Corollary 2.1.23). So

$$k_n^{1/2} > 6AM_{\theta_1},$$

$$k_n^{1/2} > 8|B|M_{\theta_1}.$$

Now

$$3A\alpha = \theta - B,$$

$$9A^2\alpha^2 = (3A\alpha)^2 = \theta^2 - 2B\theta + B^2,$$

so

$$\ell_n \alpha = 9A^2 k_n \alpha = 3A k_n (3A\alpha) = 3A k_n (\theta - B) = 3A k_n \theta - 3AB k_n,$$

$$\ell_n \alpha^2 = 9A^2 k_n \alpha^2 = k_n (3A\alpha)^2 = k_n (\theta - B)^2 = k_n \theta^2 - 2B k_n \theta + B^2 k_n.$$

Since  $3AB k_n$  and  $B^2 k_n$  are integers, we have

$$\langle \ell_n \alpha \rangle = \langle 3A k_n \theta - 3AB k_n \rangle = \langle 3A k_n \theta \rangle \quad (2.3.2)$$

and

$$\langle \ell_n \alpha^2 \rangle = \langle k_n \theta^2 - 2B k_n \theta + B^2 k_n \rangle = \langle k_n \theta^2 - 2B k_n \theta \rangle. \quad (2.3.3)$$

Now  $k_n^{1/2} > 6AM_{\theta_1} > 4M_{\theta_1}$  (the second inequality holding because  $A \geq 1$ ) and  $k_n^{1/2} > 8|B|M_{\theta_1}$ , so

$$3A \|k_n \theta\| < 3A \frac{M_{\theta_1}}{k_n^{1/2}} < 3A \frac{M_{\theta_1}}{6AM_{\theta_1}} = \frac{1}{2},$$

$$\|k_n \theta^2\| < \frac{M_{\theta_1}}{k_n^{1/2}} < \frac{M_{\theta_1}}{4M_{\theta_1}} < \frac{1}{4},$$

$$2|B| \|k_n \theta\| < 2|B| \frac{M_{\theta_1}}{k_n^{1/2}} < 2|B| \frac{M_{\theta_1}}{8|B|M_{\theta_1}} = \frac{1}{4}.$$

Since  $3A\|k_n\theta\| < \frac{1}{2}$  and  $2|B|\|k_n\theta\| < \frac{1}{2}$ , we have (by Lemma 2.1.31 (c))

$$\begin{aligned}\langle 3Ak_n\theta \rangle &= 3A\langle k_n\theta \rangle, \\ \langle 2Bk_n\theta \rangle &= 2B\langle k_n\theta \rangle.\end{aligned}$$

And since

$$\|k_n\theta^2\| + \|2Bk_n\theta\| = \|k_n\theta^2\| + 2|B|\|k_n\theta\| < \frac{1}{2},$$

we have (by Lemma 2.1.31 (b) and (c))

$$\langle k_n\theta^2 - 2Bk_n\theta \rangle = \langle k_n\theta^2 \rangle + \langle -2Bk_n\theta \rangle = \langle k_n\theta^2 \rangle - \langle 2Bk_n\theta \rangle = \langle k_n\theta^2 \rangle - 2B\langle k_n\theta \rangle.$$

Then

$$\begin{aligned}\langle \ell_n\alpha \rangle &= \langle 3Ak_n\theta \rangle = 3A\langle k_n\theta \rangle, \\ \langle \ell_n\alpha^2 \rangle &= \langle k_n\theta^2 - 2Bk_n\theta \rangle = \langle k_n\theta^2 \rangle - 2B\langle k_n\theta \rangle.\end{aligned}$$

□

From here, it is fairly straightforward (and similar to Theorem 1) to show that  $\{\ell_n\}$  is a Peck sequence for the pair  $(\alpha, \alpha^2)$ . For  $n > n_\alpha$

$$\ell_n^{1/2}\|\ell_n\alpha\| = (3Ak_n^{1/2})(3A\|k_n\theta\|) = 9A^2k_n^{1/2}\|k_n\theta\| < 9A^2M_{\theta_1}$$

and

$$\ell_n^{1/2}\|\ell_n\alpha^2\| \leq (3Ak_n^{1/2}) (\|k_n\theta^2\| + 2|B|\|k_n\theta\|) < 3AM_{\theta_1} (1 + 2|B|).$$

Put

$$M'_\alpha = \max_{i \leq n_\alpha} \{\ell_i^{1/2}\|\ell_i\alpha\|, \ell_i^{1/2}\|\ell_i\alpha^2\|\}$$

and

$$M_{\alpha_1} = \max\{M'_\alpha, 9A^2M_{\theta_1}, 3AM_{\theta_1}(1 + 2|B|)\}.$$

Then

$$\max\{\ell_n^{1/2}\|\ell_n\alpha\|, \ell_n^{1/2}\|\ell_n\alpha^2\|\} < M_{\alpha_1} \tag{2.3.4}$$

for all  $n$ .

Now consider the subsequence  $\{\ell_{Q_n}\}$ . In equations (2.2.6) and (2.2.7) in the proof of Theorem 1, we saw that

$$k_{Q_n}^{1/2} \|k_{Q_n} \theta\| < \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}}$$

and

$$k_{Q_{n+1}}^{1/2} < C_3 \lambda^{Q_{n+1}/2}$$

when  $Q_n > n_\theta$ . If  $Q_n > n_\alpha \geq n_\theta$ , then

$$\ell_{Q_n}^{1/2} \|\ell_{Q_n} \alpha\| = 9A^2 k_{Q_n}^{1/2} \|k_{Q_n} \theta\| < 9A^2 \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}}. \quad (2.3.5)$$

Since  $Q_{n+1}/2 \geq 1$  and  $3A \geq 1$ ,

$$\ell_{Q_{n+1}}^{1/2} = 3A k_{Q_{n+1}}^{1/2} < 3A (C_3 \lambda)^{Q_{n+1}/2} \leq (3AC_3 \lambda)^{Q_{n+1}/2}, \quad (2.3.6)$$

or

$$\frac{1}{Q_{n+1}} < \frac{\log(3AC_3 \lambda)}{\log \ell_{Q_{n+1}}}.$$

Then

$$\ell_{Q_n}^{1/2} \|\ell_{Q_n} \alpha\| < 9A^2 \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi \log(3AC_3 \lambda)}{\log \ell_{Q_{n+1}}}. \quad (2.3.7)$$

Putting

$$M_{\alpha_2} = 9A^2 \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \pi \log(3AC_3 \lambda),$$

and since  $Q_{n+1} > Q_n$  for  $n \geq 2$ , we have

$$\|\ell_{Q_n} \alpha\| < \frac{M_{\alpha_2}}{\ell_{Q_n}^{1/2} \log \ell_{Q_n}}$$

when  $Q_n > n_\alpha$ . By an appropriate shift of the sequence  $\{\ell_{Q_n}\}$ , we get a sequence  $\{\psi_n\}$  such that

$$\|\psi_n \alpha\| < \frac{M_{\alpha_2}}{\psi_n^{1/2} \log \psi_n}$$

for all  $n$ . □

## 2.4 Theorem 3

**Theorem 3.** *Suppose  $\alpha, \beta \in K$ , where  $K \subseteq \mathbb{R}$  is a cubic field with only one real embedding. Then we can construct a Peck sequence for the pair  $(\alpha, \beta)$ .*

**Proof.** Let  $\alpha, \beta \in K \subset \mathbb{R}$ , where  $K$  is a cubic field which has only one real embedding. As before, let  $\sigma_1$  be the real embedding and  $\sigma_2, \sigma_3$  the complex embeddings.

**Case 1:  $\alpha \in \mathbb{Q}$  or  $\beta \in \mathbb{Q}$ .** WLOG suppose  $\beta = \frac{u}{v}$ . If  $\alpha \in \mathbb{Q}$  (say  $\alpha = \frac{r}{s}$ ), then put  $m_n = |nsv|$ . If  $\alpha \notin \mathbb{Q}$ , then put  $m_n = |v|q_n$ , where  $\{q_n\}$  is the sequence of denominators of convergents of  $\alpha$ . In the first case,  $\|m_n\alpha\| = \|m_n\beta\| = 0$  for all  $n$ . In the second case,  $\|m_n\beta\| = 0$  for all  $n$ , and eventually  $\|m_n\alpha\| = |v|\|q_n\alpha\| < \frac{|v|}{q_n^2}$ . In either case,  $\{m_n^{1/2}\|m_n\alpha\|\}$ ,  $\{m_n^{1/2}\|m_n\beta\|\}$ , and  $\{m_n^{1/2} \log m_n \|m_n\alpha\|\}$  are all bounded.

**Case 2:  $\alpha, \beta \notin \mathbb{Q}$ .** Let  $F_\alpha(x)$  be the minimal polynomial (over  $\mathbb{Q}$ ) of  $\alpha$ . Since  $\alpha \notin \mathbb{Q}$ ,  $F_\alpha$  is a cubic. Clearing denominators, we get (for some  $A > 0$ )

$$f(x) := Ax^3 + Bx^2 + Cx + D := A \cdot F_\alpha(x) \in \mathbb{Z}[x],$$

where  $\gcd(A, B, C, D) = 1$ . Since  $f \in \mathbb{Z}[x]$  is an irreducible cubic with one real root  $\alpha$ , we use Theorem 2.

Let  $\{k_n\}$ ,  $\{\ell_n\}$ ,  $C_2$ ,  $C_3$ ,  $M_{\alpha_1}$ ,  $M_{\alpha_2}$ ,  $\{Q_j\}$ ,  $N'_\alpha$ ,  $n_\alpha$ , be as in the proofs of Theorems 1 and 2. Since  $1, \alpha, \alpha^2$  is a  $\mathbb{Q}$ -basis for  $K$ , we can write

$$\beta = \frac{r_0}{s} + \frac{r_1}{s}\alpha + \frac{r_2}{s}\alpha^2,$$

where  $r_0, r_1, r_2, s \in \mathbb{Z}$  and  $s > 0$ . (We note that, since  $\beta \notin \mathbb{Q}$ ,  $r_1$  and  $r_2$  are not both zero.) Put  $m_n = s\ell_n$ . We will show that  $\{m_n\}$  is a Peck sequence for the pair  $(\alpha, \beta)$ .

Put

$$N'_\beta = \max \left\{ \frac{8M_{\alpha_1}|r_1|}{3A}, \frac{8M_{\alpha_1}|r_2|}{3A}, \frac{4sM_{\alpha_1}}{3A}, N'_\alpha \right\}$$

and

$$n_\beta = 2 \log_\lambda N'_\beta.$$

As in the proof of Theorem 2, we define  $n_\beta$  so that we can use Lemma 2.1.31 to get bounds on  $\{m_n^{1/2} \|m_n \alpha\|\}$  and  $\{m_n^{1/2} \|m_n \beta\|\}$ . We do this with the following:

**Lemma 2.4.1.** *For  $n > n_\beta$*

$$\begin{aligned} \langle m_n \alpha \rangle &= s \langle \ell_n \alpha \rangle, \\ \langle m_n \beta \rangle &= r_1 \langle \ell_n \alpha \rangle + r_2 \langle \ell_n \alpha^2 \rangle. \end{aligned}$$

**Proof of Lemma.** Let  $n > n_\beta$  and note that

$$\lambda^{n/2} > N'_\beta \geq \frac{2}{3A} \max\{4M_{\alpha_1}|r_1|, 4M_{\alpha_2}|r_2|, 2sM_{\alpha_1}\},$$

and so (using Corollary 2.1.23 and that  $\ell_n = 9A^2k_n$ )

$$\ell_n^{n/2} = 3Ak_n^{n/2} > 3A \frac{\lambda^{n/2}}{2} > \max\{4M_{\alpha_1}|r_1|, 4M_{\alpha_2}|r_2|, 2sM_{\alpha_1}\}.$$

Then

$$|r_1| \frac{M_{\alpha_1}}{\ell_n^{1/2}} < \frac{1}{4}, \tag{2.4.2}$$

$$|r_2| \frac{M_{\alpha_2}}{\ell_n^{1/2}} < \frac{1}{4}, \tag{2.4.3}$$

$$s \frac{M_{\alpha_1}}{\ell_n^{1/2}} < \frac{1}{2}. \tag{2.4.4}$$

Now

$$\begin{aligned} s\beta &= r_0 + r_1\alpha + r_2\alpha^2, \\ m_n\beta &= (s\ell_n)\beta = \ell_n(s\beta) = \ell_n(r_0 + r_1\alpha + r_2\alpha^2). \end{aligned}$$

Since  $r_0\ell_n \in \mathbb{Z}$ ,

$$\langle m_n \beta \rangle = \langle r_1 \ell_n \alpha + r_2 \ell_n \alpha^2 \rangle.$$



Combining (2.3.4) (from Theorem 2) with (2.4.2) and (2.4.3), we have

$$|r_1| \|\ell_n \alpha\| < |r_1| \frac{M_{\alpha_1}}{\ell_n^{1/2}} < \frac{1}{4} < \frac{1}{2}$$

and

$$|r_2| \|\ell_n \alpha^2\| < |r_2| \frac{M_{\alpha_2}}{\ell_n^{1/2}} < \frac{1}{4} < \frac{1}{2},$$

so Lemma 2.1.31(c) yields

$$\begin{aligned} \langle r_1 \ell_n \alpha \rangle &= r_1 \langle \ell_n \alpha \rangle, \\ \langle r_2 \ell_n \alpha^2 \rangle &= r_2 \langle \ell_n \alpha^2 \rangle. \end{aligned}$$

Then since

$$\|r_1 \ell_n \alpha\| + \|r_2 \ell_n \alpha^2\| = |r_1| \|\ell_n \alpha\| + |r_2| \|\ell_n \alpha^2\| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

Lemma 2.1.31(a) gives us

$$\langle r_1 \ell_n \alpha + r_2 \ell_n \alpha^2 \rangle = \langle r_1 \ell_n \alpha \rangle + \langle r_2 \ell_n \alpha^2 \rangle.$$

So

$$\langle m_n \beta \rangle = \langle r_1 \ell_n \alpha \rangle + \langle r_2 \ell_n \alpha^2 \rangle = r_1 \langle \ell_n \alpha \rangle + r_2 \langle \ell_n \alpha^2 \rangle.$$

To show  $\langle m_n \alpha \rangle = s \langle \ell_n \alpha \rangle$ , we use (2.3.4) with (2.4.4) to get

$$s \|\ell_n \alpha\| < s \frac{M_{\alpha_1}}{\ell_n^{1/2}} < \frac{1}{2}.$$

Then (by Lemma 2.1.31(c))  $\langle m_n \alpha \rangle = \langle s \ell_n \alpha \rangle = s \langle \ell_n \alpha \rangle$ . □

With this lemma in hand, the rest of the proof of Theorem 3 is nearly identical to the arguments in the proofs of Theorems 1 and 2. The lemma, along with (2.3.4), gives us that for  $n > n_\beta$ ,

$$\begin{aligned} m_n^{1/2} \|m_n \alpha\| &= (s^{1/2} \ell_n^{1/2})(s \|\ell_n \alpha\|) = s^{3/2} \ell_n^{1/2} \|\ell_n \alpha\| < s^{3/2} M_{\alpha_1}, \\ m_n^{1/2} \|m_n \beta\| &= (s^{1/2} \ell_n^{1/2})(|r_1| \|\ell_n \alpha\| + |r_2| \|\ell_n \alpha^2\|) < s^{3/2} (|r_1| + |r_2|) M_{\alpha_1}. \end{aligned}$$

We noted earlier that  $r_1$  and  $r_2$  are not both zero, so  $|r_1| + |r_2| \geq 1$ , and therefore

$$m_n^{1/2} \|m_n \alpha\| < s^{3/2} (|r_1| + |r_2|) M_{\alpha_1}.$$

So with

$$M'_\beta = \max_{i \leq n_\beta} \{m_n^{1/2} \|m_n \alpha\|, m_n^{1/2} \|m_n \beta\|\}$$

and

$$M_{\beta_1} = \max\{M'_\beta, s^{3/2} (|r_1| + |r_2|) M_{\alpha_1}\},$$

we have

$$\max\{m_n^{1/2} \|m_n \alpha\|, m_n^{1/2} \|m_n \beta\|\} \leq M_{\beta_1}$$

for all  $n$ .

Consider the subsequence  $\{m_{Q_n}\}$ . In (2.3.5) and (2.3.6) from the proof of Theorem 2, we had

$$\ell_{Q_n}^{1/2} \|\ell_{Q_n} \alpha\| < 9A^2 \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}}$$

and

$$\ell_{Q_{n+1}}^{1/2} < (3AC_3\lambda)^{Q_{n+1}/2}$$

for  $Q_n > n_\alpha$ . With Lemma 2.4.1, we have that

$$m_{Q_n}^{1/2} \|m_{Q_n} \alpha\| = s^{3/2} \ell_{Q_n}^{1/2} \|\ell_{Q_n} \alpha\| < 9A^2 s^{3/2} \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi}{Q_{n+1}} \quad (2.4.5)$$

for  $Q_n > n_\beta \geq n_\alpha$ . Since  $Q_{n+1}/2 \geq 1$  and  $s \geq 1$ , we have

$$m_{Q_{n+1}}^{1/2} = s^{1/2} \ell_{Q_{n+1}}^{1/2} < s^{1/2} (3AC_3\lambda)^{Q_{n+1}/2} \geq (3As^{1/2}C_3\lambda)^{Q_{n+1}/2},$$

from which we get

$$\frac{1}{Q_{n+1}} \leq \frac{\log(3As^{1/2}C_3\lambda)}{\log m_{Q_{n+1}}}.$$

Then

$$m_{Q_n}^{1/2} \|m_{Q_n} \alpha\| < 9A^2 s^{3/2} \cdot \frac{3dC_2}{2\sqrt{3\theta^2 - 4p}} \cdot \frac{\pi \log(3As^{1/2}C_3\lambda)}{\log m_{Q_{n+1}}}. \quad (2.4.6)$$

Put

$$M_{\beta_2} = 9A^2 s^{3/2} \cdot \frac{3dC_2\pi}{2\sqrt{3\theta^2 - 4p}} \cdot \log(3As^{1/2}C_3\lambda).$$

Then since  $Q_{n+1} > Q_n$  for  $n \geq 2$ , we get

$$\|m_{Q_n}\alpha\| < \frac{M_{\beta_2}}{m_{Q_n}^{1/2} \log m_{Q_n}}$$

for  $n > n_\beta$ . By appropriately shifting the sequence  $\{m_{Q_n}\}$ , we have a sequence  $\{\psi_n\}$  such that

$$\|\psi_n\alpha\| < \frac{M_{\beta_2}}{\psi_n^{1/2} \log \psi_n}$$

for all  $n$ . □

## 2.5 Further Questions

### 2.5.1 Constructing Simultaneous Approximations

The main idea of this dissertation is to take an appropriate  $\theta$  with  $K = \mathbb{Q}(\theta)$ , find a unit  $\lambda \in \mathcal{O}_K$  with  $\lambda > 1$ , and then use the sequence  $\{\lambda^n\}$  to produce a sequence that satisfies Peck's inequalities for the power basis  $1, \theta, \theta^2$ .

1. Our most immediate question is whether this method could work when  $K$  is a totally real cubic field? (So far, our method has failed to produce Peck sequences in this case.)

Peck's proof in [P61] makes no distinction between the cases that  $K$  has one real embedding or three real embeddings. In the second case, the unit group has rank 2. So if this method were to work in that case, then perhaps finding an appropriate  $\lambda$  is a more delicate process than in the rank 1 case (where our main requirement was only that  $\lambda > 1$ ).

2. Another question is whether this method could generalize to find simultaneous approximations of pairs coming from higher-degree number fields?

For example, consider  $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Since  $K = \mathbb{Q}(\theta)$  for  $\theta = \sqrt{2} + \sqrt{3}$ , could we take powers of an appropriate unit to construct “good” simultaneous approximations to the basis  $1, \theta, \theta^2, \theta^3$ ? If so, that would also give us simultaneous approximations to the pair  $(\sqrt{2}, \sqrt{3})$ . How good would such approximations be? (I.e., would they satisfy Peck’s inequalities? Would they be sharp enough to show Littlewood’s conjecture for the pair  $(\sqrt{2}, \sqrt{3})$ ?)

### 2.5.2 Questions Based on Heuristics

Let  $f : \mathbb{N} \rightarrow (0, \infty)$ , and consider sequences of the form  $\{f(n)\|n\alpha\|\|n\beta\|\}$  for  $\alpha, \beta \in \mathbb{R}$ . Based on the probabilistic argument in Section 1.4, it seems natural to ask:

3. If  $\sum_{n=1}^{\infty} \frac{1}{f(n)}$  diverges, then is

$$\liminf_{n \rightarrow \infty} f(n)\|n\alpha\|\|n\beta\| = 0?$$

4. If  $\sum_{n=1}^{\infty} \frac{1}{f(n)}$  converges, then for almost all  $\alpha, \beta \in \mathbb{R}$  is

$$\liminf_{n \rightarrow \infty} f(n)\|n\alpha\|\|n\beta\| > 0?$$

We could also consider a subset  $S \subset \mathbb{N}$  and the set  $\{s\|s\alpha\|\|s\beta\| : s \in S\}$ . Based on the probabilistic argument again, it seems reasonable to ask:

5. If  $\sum_{s \in S} \frac{1}{s}$  diverges, then is

$$\inf_{s \in S} s\|s\alpha\|\|s\beta\| = 0?$$

6. If  $\sum_{s \in S} \frac{1}{s}$  converges, then for almost all  $\alpha, \beta$  is

$$\inf_{s \in S} s\|s\alpha\|\|s\beta\| > 0?$$

## CHAPTER 3

## LITTLEWOOD'S CONJECTURE (EXAMPLES)

If  $\alpha, \beta \in K$ , where  $K$  is a cubic field with only one real embedding, then as a result of the theorems, we are able to construct an eventually-increasing sequence  $\{\psi_n\}$  of positive integers satisfying

$$\psi_n \|\psi_n \alpha\| \|\psi_n \beta\| < \frac{C}{\log \psi_n}$$

(for some  $C > 0$ ), thus providing a constructive proof that Littlewood's conjecture holds for the pair  $(\alpha, \beta)$ . However, in the proofs of the theorems, we actually have a slightly sharper bound (and get the  $\frac{C}{\log \psi_n}$  bound as a consequence).

In this chapter, we will consider this sharper bound

$$\psi_n \|\psi_n \alpha\| \|\psi_n \beta\| < \frac{C_0}{Q_{n+1}},$$

and we will compare  $C_0/Q_{n+1}$  to the actual value of  $\psi_n \|\psi_n \alpha\| \|\psi_n \beta\|$  for several examples of pairs  $(\alpha, \beta)$ .

### 3.1 Constructing $\{\psi_n\}$ and Calculating the Bound

To bound the Littlewood product  $\psi_n \|\psi_n \alpha\| \|\psi_n \beta\|$ , we will consider separate bounds on  $\psi_n^{1/2} \|\psi_n \alpha\|$  and  $\psi_n^{1/2} \|\psi_n \beta\|$ , as we did in the proofs of the theorems. We are interested in *eventual* bounds, so we will slightly modify our setup in the following ways:

- (i) In the proof of Theorem 1, we wanted  $\lambda > ((1 + \sqrt{2})C_1)^{1/3}$ . For our constructions, we will just require  $\lambda > 1$ .
- (ii) The constants  $M_{\theta_1}$ ,  $M_{\alpha_1}$ , and  $M_{\beta_1}$  were defined so that certain inequalities held for all  $n$ . We will relax that requirement so that the corresponding inequalities

hold *eventually*, and then we will slightly modify the definitions of the constants. We will denote these re-defined constants by  $\widetilde{M}_\theta$ ,  $\widetilde{M}_\alpha$ , and  $\widetilde{M}_{\beta_1}$ . We will define  $\widetilde{N}$  so that the inequalities hold for  $\lambda^{n/2} > \widetilde{N}$ .

- (iii) In the proofs of the theorems, we took  $\{\psi_n\}$  to be an appropriately shifted subsequence (of  $\{k_{Q_n}\}$  or  $\{\ell_{Q_n}\}$  or  $\{m_{Q_n}\}$ ). Instead, we will take  $\{\psi_n\}$  to be  $m_{Q_n}$ , and construct a constant  $n_0$  for which  $\psi_n$  satisfies the inequalities when  $Q_n > n_0$ .

We summarize our results in the following algorithm to construct  $\{\psi_n\} = \{m_{Q_n}\}$  and to compute  $n_0, C_0$  such that

$$\psi_n \|\psi_n \alpha\| \|\psi_n \beta\| \leq \frac{C_0}{Q_{n+1}}$$

for  $Q_n > n_0$ . We will assume that  $\alpha, \beta \notin \mathbb{Q}$ , and that we have the polynomial  $f(x) = Ax^3 + Bx^2 + Cx + D \in \mathbb{Z}[x]$  for which:  $f(\alpha) = 0$ ,  $\gcd(A, B, C, D) = 1$ , and  $A > 0$ . In the case that  $B = 0$ , we can modify the algorithm to reduce  $C_0$  by a factor of 81. In each step of the algorithm, we will state any modifications for the case  $B = 0$ .

### 3.1.1 The Algorithm

---

**Given:**  $\alpha, \beta$ :  $\alpha$  is real root of  $f(x) = Ax^3 + Bx^2 + Cx + D$ ,  $\beta = \frac{r_0}{s} + \frac{r_1}{s}\alpha + \frac{r_2}{s}\alpha^2$

---

**Step 1.** Define  $g \in \mathbb{Z}[x]$ ,  $p, q \in \mathbb{Z}$ ,  $\theta \in K$  by:

$$g(x) = 27A^2 f\left(\frac{x-B}{3A}\right) = x^3 - px - q$$

$$p = 3(B^2 - 3AC)$$

$$q = -2B^3 + 9ABC - 27A^2D$$

$$\theta = 3A\alpha + B$$

(These are chosen so that  $\theta$  is the real root of  $g(x)$ .) Choose a positive integer  $d$  such that  $\mathcal{O}_K \subset \frac{1}{d}\mathbb{Z}[\theta]$  (this can be computed in PARI/GP<sup>1</sup>).

.....

If  $B = 0$ : Put  $\theta = A\alpha$ ,  $p = -AC$ ,  $q = -A^2D$ ,  $g(x) = A^2f(x/A) = x^3 - px - q$

---

**Step 2.** Find fundamental unit  $\varepsilon_0$  of  $\mathcal{O}_K$  (can be computed in PARI/GP<sup>2</sup>), and put

$$\lambda = \max\{\pm\varepsilon_0, \pm\varepsilon_0^{-1}\}$$

---

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{m_n\}$  by:

$$a_n + b_n\theta + c_n\theta^2 = \lambda^n$$

$$m_n = (9A^2sd)c_n$$

.....

If  $B = 0$ : Put  $m_n = (A^2sd)c_n$ .

---

**Step 4.** Define the constant  $\phi$  and the sequence  $\{Q_n\}$  by:

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 - 4p}(b_1 - c_1\theta)}{2(a_1 + pc_1) - b_1\theta - c_1\theta^2}\right)$$

And  $\{Q_n\}$  is the sequence of denominators of the *non-integer* convergents of  $\phi/\pi$

---

<sup>1</sup>By `bnfinit(X^3-p*X-q).zk`

<sup>2</sup>By `bnfinit(X^3-p*X-q).fu`

**Step 5.** Define constants  $C_1, C_2, \widetilde{M}_\theta, \widetilde{M}_\alpha, \widetilde{M}_{\beta_1}, \widetilde{M}_{\beta_2}, \widetilde{N}$  by:

$$\begin{aligned}
C_1 &= \max \left( \sqrt{2}, \frac{\sqrt{2}|\theta|}{\sqrt{3\theta^2 - 4p}} \right), \\
C_2 &= \frac{\sqrt{d}}{\sqrt{3\theta^2 - p}}, \\
\widetilde{M}_\theta &= \max \left\{ \frac{3d\sqrt{2}C_1C_2}{2|\theta|}, \frac{3dC_1C_2}{2} \right\} \\
\widetilde{M}_\alpha &= \max \left\{ 9A^2\widetilde{M}_\theta, 3A\widetilde{M}_\theta(1 + 2|B|) \right\}, \\
\widetilde{M}_{\beta_1} &= s^{3/2}(|r_1| + |r_2|)\widetilde{M}_\alpha \\
\widetilde{M}_{\beta_2} &= 9A^2s^{3/2} \cdot \frac{3dC_2\pi}{2\sqrt{3\theta^2 - 4p}} \\
\widetilde{N} &= \max \left\{ \lambda^{1/2}, \frac{2\sqrt{2}dC_1}{|\theta|}, 2dC_1, \left(2(1 + \sqrt{2})C_2\right)^{1/3}, 12A\widetilde{M}_\theta, \right. \\
&\quad \left. 16|B|\widetilde{M}_\theta, \frac{8\widetilde{M}_\alpha|r_1|}{3A}, \frac{8\widetilde{M}_\alpha|r_2|}{3A}, \frac{4s\widetilde{M}_\alpha}{3A} \right\}
\end{aligned}$$

.....  
If  $B = 0$ : Put

$$\begin{aligned}
\widetilde{M}_\alpha &= A^2\widetilde{M}_\theta, \\
\widetilde{M}_{\beta_2} &= A^2s^{3/2} \cdot \frac{3dC_2\pi}{2\sqrt{3\theta^2 - 4p}}, \\
\widetilde{N} &= \max \left\{ \lambda^{1/2}, \frac{2\sqrt{2}dC_1}{|\theta|}, 2dC_1, \left(2(1 + \sqrt{2})C_2\right)^{1/3}, 4A\widetilde{M}_\theta, \right. \\
&\quad \left. \frac{8\widetilde{M}_\alpha|r_1|}{A}, \frac{8\widetilde{M}_\alpha|r_2|}{A}, \frac{4s\widetilde{M}_\alpha}{A} \right\}.
\end{aligned}$$

---

**Step 6.** Define the sequence  $\{\psi_n\}$  and constants  $C_0, n_0$  by:

$$\begin{aligned}
\psi_n &= m_{Q_n} \\
C_0 &= \widetilde{M}_{\beta_1} \cdot \widetilde{M}_{\beta_2} \\
n_0 &= 2 \log_\lambda \widetilde{N}
\end{aligned}$$



---

**Result.** For  $Q_n > n_0$

$$\psi_n \|\psi_n \alpha\| \|\psi_n \beta\| \leq \frac{C_0}{Q_{n+1}}.$$


---

**Remark 3.1.1.** As we stated earlier, the purpose of modifying the algorithm for the case  $B = 0$  is to sharpen our bounds. It will reduce  $C_0$  by a factor of 81, while at worst it will increase  $n_0$  by up to  $\log_\lambda 9$ . (For instance, in Example 1 it reduces  $C_0$  from 87.46 to 1.08 and decreases  $n_0$  from 4.85 to 3.22.)

## 3.2 Examples

### 3.2.1 Example 1: $\theta^3 = 2$

In Example 1 from Chapter 1 (Section 1.3.1), we considered the pair  $(\theta, \theta^2)$  for  $\theta = 2^{1/3}$ . So we take  $\alpha = \theta$ ,  $f(x) = x^3 - 2$ , and  $\beta = \frac{0}{1} + \frac{0}{1}\theta + \frac{1}{1}\theta^2$ . Therefore we have  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ . (We remark that the discriminant of  $K$  is  $-108$ .)

**Step 1.** We already have  $\theta = \alpha = 2^{1/3}$ , and  $f(x)$  is already in the form  $x^3 - px - q$  (with  $p = 0$  and  $q = 2$ ). Using PARI/GP, we see that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Using PARI/GP, we find that a fundamental unit is  $\varepsilon_0 = \theta - 1 \approx 0.2599$ . Since  $0 < \varepsilon_0 < 1$ , we put

$$\lambda = \varepsilon_0^{-1} = 1 + \theta + \theta^2 \approx 3.8473.$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{m_n\}$  by

$$a_n + b_n \theta + c_n \theta^2 = \lambda^n$$

and  $m_n = (A^2 s d) c_n = c_n$ . (We saw the points  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  in Figures 1.5 and 1.6.)

**Step 4.** Put

$$\phi = \arctan \left( \frac{\sqrt{3} \cdot \theta(1 - \theta)}{2 - \theta - \theta^2} \right) \approx 0.5899.$$

The first few (non-integer) convergents of  $\phi/\pi \approx 0.18777$  are:

$$\frac{1}{5}, \frac{3}{16}, \frac{43}{229}, \frac{1551}{8260}, \frac{1594}{8489}, \frac{3145}{16749}, \frac{4739}{25238}, \frac{64752}{344843}, \frac{198995}{1059767}, \frac{263747}{1404610}$$

**Step 5.** We have the following values for the constants:

$$\begin{aligned} C_1 &= \sqrt{2} \approx 1.41421 \\ C_2 &= \frac{1}{\theta\sqrt{3}} \approx 0.458243 \\ \widetilde{M}_\theta &= \frac{\sqrt{3}}{\theta^2} \approx 1.09112 \\ \widetilde{M}_\alpha &= M_\theta \\ \widetilde{M}_{\beta_1} &= M_\alpha \\ \widetilde{M}_{\beta_2} &= \frac{\pi}{2\theta^2} \approx 0.989540 \end{aligned}$$

We also have

$$\begin{aligned} \widetilde{N} &= \max \left\{ \lambda^{1/2}, \frac{4}{\theta}, 2\sqrt{2}, \left( \frac{2(1 + \sqrt{2})}{\theta\sqrt{2}} \right)^{1/3}, \frac{4\sqrt{3}}{\theta^2}, 0, \frac{8\sqrt{3}}{\theta^2}, \frac{4\sqrt{3}}{\theta^2} \right\} \\ &\approx \max\{1.961, 3.175, 2.828, 1.303, 4.364, 0, 8.729, 4.364\} \\ &= 8.729. \end{aligned}$$

**Step 6.** We put

$$\begin{aligned} C_0 &= \frac{\sqrt{3} \cdot \pi}{4\theta} \approx 1.07971 \\ n_0 &= 2 \log_\lambda \frac{8\sqrt{3}}{\theta^2} \approx 3.216 \end{aligned}$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_5, m_{16}, m_{229}, m_{8260}, m_{8489}, m_{16749}, m_{25238}, m_{344843}, m_{1059767}, c_{1404610}.$$

**Result.** For  $Q_n \geq 4 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{1.07971}{Q_{n+1}}.$$

In Table 3.1 we list the first several values of  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  (Even with `$MaxExtraPrecision` set to  $10^{250,000,000}$ , Mathematica 8.0 was unable to calculate this for  $n = 12$ .)

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \theta\  \ \psi_n \theta^2\ $	$C_0/Q_{n+1}$
5	177	0.0320119	0.0674819
16	483870160	0.00265565	0.00471489
229	210 ... 617 (134 digits)	0.000071868	0.000130716
8260	539 ... 526 (4833 digits)	0.0000473553	0.000127189
8489	540 ... 581 (4967 digits)	0.0000245330	0.0000644642
16749	138 ... 317 (9801 digits)	0.0000228153	0.0000427811
25238	357 ... 401 (14768 digits)	$1.72115 * 10^{-6}$	$3.13102 * 10^{-6}$
344843	141 ... 112 (201788 digits)	$4.38682 * 10^{-7}$	$1.01882 * 10^{-6}$
1059767	109 ... 020 (620132 digits)	$4.05112 * 10^{-7}$	$7.6869 * 10^{-7}$
1404610	740 ... 203 (821919 digits)	$3.35689 * 10^{-8}$	$6.02682 * 10^{-8}$
17915087	405 ... 320 (10483166) digits	$2.28507 * 10^{-9}$	$4.2809 * 10^{-9}$
252215828	743 ... 566 (147586247) digits		$3.99699 * 10^{-9}$

Table 3.1:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$  for  $\theta = \sqrt[3]{2}$

### 3.2.2 Example 2: $\theta^3 = \theta + 1$

Let  $\theta \approx 1.32472$  be the real root of  $f(x) = x^3 - x - 1$ , and consider the pair  $(\theta, \theta^2)$ . (We chose this example to have a unit  $\lambda$  close to 1, so that we can calculate the Littlewood product  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for larger  $n$  than in other examples.) We take  $\alpha = \theta$  and  $\beta = \theta^2$ , so (as in Example 1) put  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ . The discriminant of  $K$  is  $-23$ .

**Step 1.** We already have  $\theta = \alpha$ , and  $f(x)$  is already in the form  $x^3 - px - q$  (with  $p = 1$  and  $q = 1$ ). Using PARI/GP, we see that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Now  $\theta$  is already a unit, since the constant term of  $f(x)$  is  $-1$ . So since  $\theta > 1$ , we will put

$$\lambda = \theta \approx 1.32472.$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{m_n\}$  by

$$a_n + b_n \theta + c_n \theta^2 = \lambda^n$$

and  $m_n = (A^2 s d) c_n = c_n$ . (We plot the points  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $n = 1$  to 200 in Figure 3.1.)

**Step 4.** Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 - 4}}{-\theta} \right) \approx -0.703858.$$

The first few convergents of  $\phi/\pi \approx -0.224045$  are:

$$-\frac{1}{4}, -\frac{2}{9}, -\frac{13}{58}, -\frac{41}{183}, -\frac{1038}{4633}, -\frac{1079}{4816}, -\frac{2117}{9449}, -\frac{15898}{70959}, -\frac{18015}{80408}, -\frac{63969148}{285519359}$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 1.66593 & C_2 \approx 0.484238 & \widetilde{M}_\theta \approx 1.29181 \\ \widetilde{M}_\alpha \approx 1.29181 & \widetilde{M}_{\beta_1} \approx 1.29181 & \widetilde{M}_{\beta_2} \approx 2.02917 \end{array}$$

and

$$\widetilde{N} \approx \max\{1.151, 3.557, 3.332, 1.327, 5.167, 0, 10.334, 5.167\} = 10.334$$

**Step 6.** We put

$$C_0 \approx 2.6213$$

$$n_0 \approx 16.6109$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_4, m_9, m_{58}, m_{183}, m_{4633}, m_{4816}, m_{9449}, m_{70959}, m_{80408}, m_{285519359}$$

**Result.** For  $Q_n \geq 17 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{2.6213}{Q_{n+1}}.$$

In Table 3.2 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

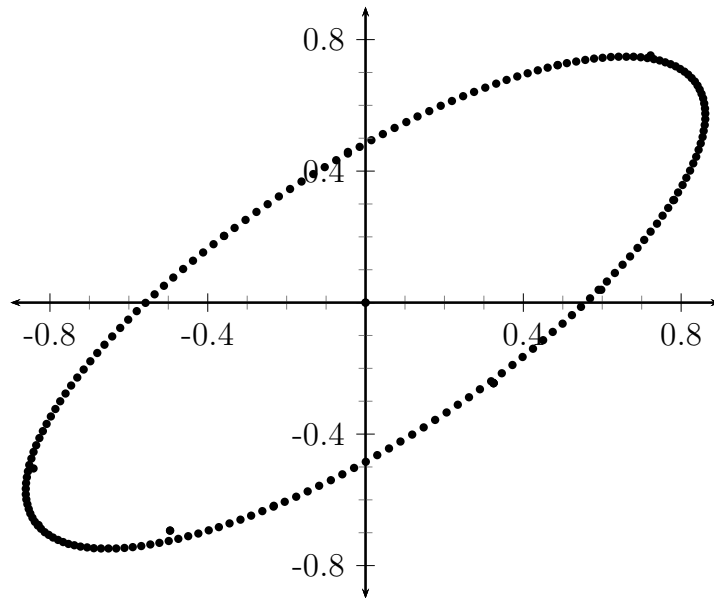


Figure 3.1:  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $\theta^3 = \theta + 1$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \theta\  \ \psi_n \theta^2\ $	$C_0/Q_{n+1}$
4	1	0.079596	0.291255
9	3	0.0205192	0.0451948
58	2839729	0.0072113	0.014324
183	5232446865180756766896	0.000276774	0.000565788
4633	147...145 (566 digits)	0.000146703	0.000544289
4816	328...896 (588 digits)	0.000130300	0.000277415
9449	206...593 (1154 digits)	0.0000162949	0.000036941
70959	133...906 (8666 digits)	0.0000162888	0.000032599
80408	117...377 (9820 digits)	$4.58789 * 10^{-9}$	$9.18080 * 10^{-9}$
285519359	891...001 (34868601 digits)	$2.53019 * 10^{-9}$	$9.17822 * 10^{-9}$
285599767	447...577 (34878421 digits)	$2.05770 * 10^{-9}$	$4.58976 * 10^{-9}$
571119126	169...901 (69747023 digits)	$4.72492 * 10^{-10}$	$1.01993 * 10^{-9}$

Table 3.2:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = \theta + 1$ 

### 3.2.3 Example 3: $\alpha^3 = 7\alpha^2 + 2$

Consider the pair  $(\alpha, \alpha^{-1})$ , where  $\alpha$  is the real root of  $f(x) = x^3 - 7x^2 - 2$ . This field has discriminant  $-2852$ . Since

$$\begin{aligned} \alpha^3 - 7\alpha^2 &= 2, \\ \alpha \left( \frac{\alpha^2 - 7\alpha}{2} \right) &= 1, \end{aligned}$$

we can write  $\alpha^{-1}$  in the basis  $1, \alpha, \alpha^2$  as

$$\alpha^{-1} = -\frac{7}{2}\alpha + \frac{1}{2}\alpha^2$$

(i.e., we have  $r_0 = 0$ ,  $r_1 = -7$ ,  $r_2 = 1$ ,  $s = 2$ ).

**Step 1.** We have  $g(x) = x^3 - 147x - 740$  (in particular,  $p = 147$ ), and we put  $\theta = 3\alpha - 7 \approx 14.121$ . Using PARI/GP, we find that  $\mathcal{O}_k \subset \frac{1}{9}\mathbb{Z}[\theta]$  (so we take  $d = 9$ ).

**Step 2.** Using PARI/GP, we find that a fundamental unit is

$$\varepsilon_0 = -\frac{4309}{9} - \frac{62}{9}\theta + \frac{26}{9}\theta^2 \approx -0.0000108758.$$

Then

$$\lambda = \max\{\pm\varepsilon_0, \pm\varepsilon_0^{-1}\} = -\varepsilon_0^{-1} = \frac{96109}{9} + \frac{25898}{9}\theta + \frac{1834}{9}\theta^2 \approx 91946.994$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before, and define  $\{m_n\}$  by  $m_n = (9sd)c_n = 162c_n$ . (We plot the points  $(m_n^{1/2}\langle m_n\theta\rangle, m_n^{1/2}\langle m_n\theta^2\rangle)$  for  $n = 1$  to 100 in Figure 3.2.)

**Step 4.** Put

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 - 588} \cdot (25898 - 1834 \cdot \theta)}{731414 - 25898 \cdot \theta - 1834 \cdot \theta^2}\right) \approx -0.253982$$

The first few convergents of  $\phi/\pi \approx -0.0808448$  are:

$$-\frac{1}{12}, -\frac{2}{25}, -\frac{3}{37}, -\frac{8}{99}, -\frac{19}{235}, -\frac{46}{569}, -\frac{111}{1373}, -\frac{1378}{17045}, -\frac{1489}{18418}, -\frac{7334}{90717}.$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 6.24920 & C_2 \approx 0.141231 & \widetilde{M}_\theta \approx 11.9149 \\ \widetilde{M}_\alpha \approx 536.169 & \widetilde{M}_{\beta_1} \approx 12132.1 & \widetilde{M}_{\beta_2} \approx 47.7138 \end{array}$$

and

$$\widetilde{N} \approx 10008.5$$

**Step 6.** We have

$$C_0 \approx 578869$$

$$n_0 \approx 1.6119$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_{12}, m_{25}, m_{37}, m_{99}, m_{235}, m_{569}, m_{1373}, m_{17045}, m_{18418}, m_{90717}, m_{381286}.$$

**Result.** For  $Q_n \geq 2 > n_0$ ,

$$\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^{-1}\| < \frac{C_0}{Q_{n+1}} \approx \frac{578869}{Q_{n+1}}.$$

In Table 3.3 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

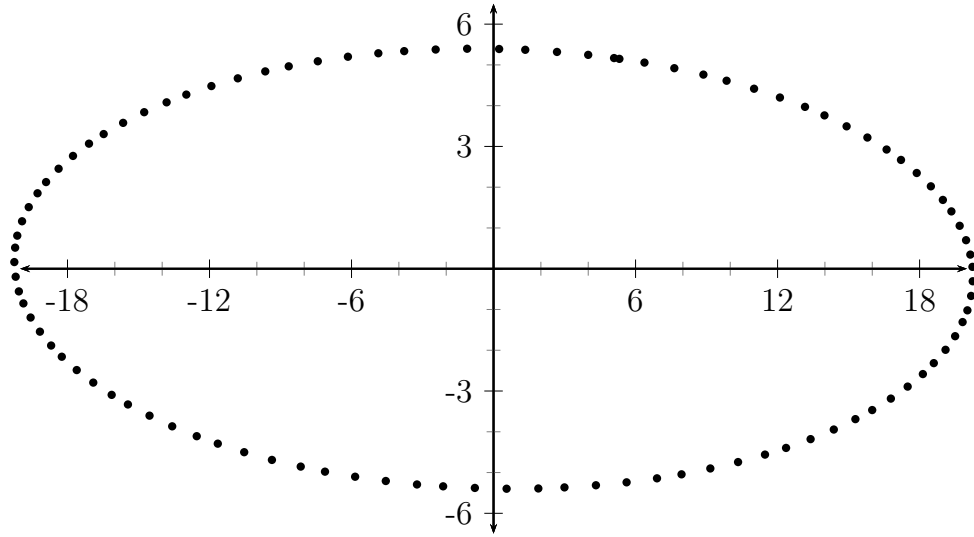


Figure 3.2:  $(m_n^{1/2} \langle m_n \alpha \rangle, m_n^{1/2} \langle m_n \alpha^{-1} \rangle)$  for  $\alpha^3 = 7\alpha^2 + 2$ ,  $n \leq 100$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \alpha\  \ \psi_n \alpha^{-1}\ $	$C_0/Q_{n+1}$
12	131 ... 120 (60 digits)	10.2213	23154.8
25	440 ... 692 (124 digits)	7.2066	15645.1
37	160 ... 332 (184 digits)	3.00062	5847.17
99	881 ... 988 (491 digits)	1.24736	2463.27
235	969 ... 892 (1166 digits)	0.50306	1017.35
569	642 ... 132 (2824 digits)	0.242022	421.609
1373	310 ... 052 (6815 digits)	0.0188969	33.9612
17045	112 ... 812 (84604 digits)	0.0152817	31.4295
18418	975 ... 496 (91418 digits)	0.00361497	6.38105
90717	615 ... 348 (450277 digits)	0.00082197	1.5182
381286	939 ... 264 (1892526 digits)	0.000327067	0.678398
853289	379 ... 748 (4235331 digits)	0.000167840	0.468882
1234575	894 ... 812 (6127858 digits)	0.000159226	0.277254
2087864	851 ... 344 (10363190 digits)	$8.6140 * 10^{-6}$	0.0149131

Table 3.3:  $\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^{-1}\|$  and  $C_0/Q_{n+1}$ , where  $\alpha^3 = 7\alpha^2 + 2$



### 3.2.4 Example 4: $\theta^2 = 147 \cdot \theta + 740$

Consider the pair  $(\theta, \theta^2)$ , where  $\theta^3 = 147 \cdot \theta + 740$  as in Example 3. (The field is still the same as in Example 3, so the discriminant is still  $-2852$ .) We put  $A = 1$  and  $B = 0$ , and (since we're taking  $\beta = \theta^2$ )  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ .

**Step 1.** We already have  $\theta \approx 14.121$ , and  $f(x)$  is already in the form  $x^3 - px - q$  (with  $p = 147$  and  $q = 740$ ). We already found in Example 3 that we can take  $d = 9$ .

**Step 2.** In the previous example, we found

$$\lambda = \frac{96109}{9} + \frac{25898}{9}\theta + \frac{1834}{9}\theta^2 \approx 91946.994$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = dc_n = 9c_n$ . (We plot the points  $(m_n^{1/2}\langle m_n\theta \rangle, m_n^{1/2}\langle m_n\theta^2 \rangle)$  for  $n = 1$  to 100 in Figure 3.3.)

**Step 4.** Since  $\lambda$  is the same as in Example 3, so is  $\phi$  (and therefore the sequence  $\{Q_n\}$ ).

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 6.24920 & C_2 \approx 0.141231 & \widetilde{M}_\theta \approx 11.9149 \\ \widetilde{M}_\alpha \approx 11.9149 & \widetilde{M}_{\beta_1} \approx 11.9149 & \widetilde{M}_{\beta_2} \approx 1.87438 \end{array}$$

and

$$\widetilde{N} \approx \max\{303.228, 11.27, 112.5, 0.8802, 47.66, 0, 95.32, 47.66\} = \lambda^{1/2}$$

**Step 6.** We have

$$\begin{aligned} C_0 &\approx 22.3329, \\ n_0 &= 2 \log_\lambda \widetilde{N} = 1. \end{aligned}$$

and we define  $\{\psi_n\}$  (as before) by  $\psi_n = m_{Q_n}$ .

**Result.** For  $Q_n > n_0 = 1$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{22.3329}{Q_{n+1}}.$$

In Table 3.4 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

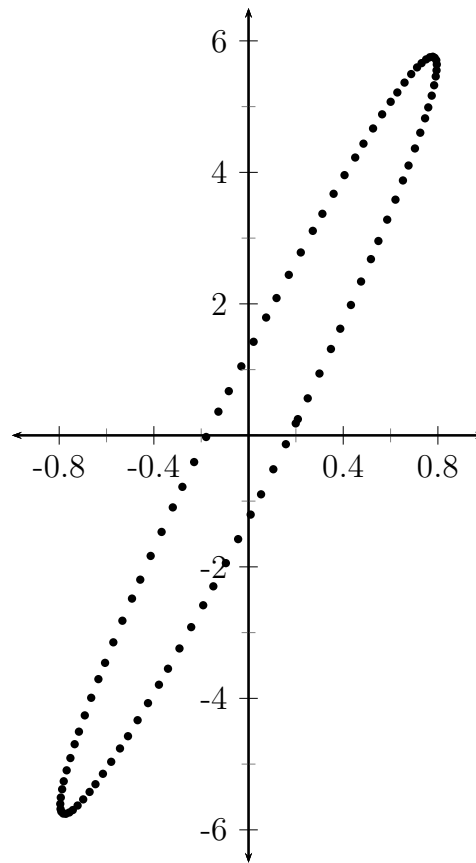


Figure 3.3:  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $\theta^3 = 147 \cdot \theta + 740$ ,  $n \leq 100$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \theta\  \ \psi_n \theta^2\ $	$C_0/Q_{n+1}$
12	728...840 (58 digits)	0.133515	0.893317
25	244...594 (123 digits)	0.0472532	0.603593
37	892...074 (182 digits)	0.0311235	0.225585
99	489...166 (490 digits)	0.0109711	0.0950337
235	538...494 (1165 digits)	0.00475198	0.0392494
569	357...674 (2823 digits)	0.00221919	0.0162658
1373	172...114 (6814 digits)	0.000175104	0.00131023
17045	626...434 (84602 digits)	0.000141411	0.00121256
18418	542...972 (91417 digits)	0.0000334768	0.000246182
90717	342...186 (450276 digits)	$7.6106 * 10^{-6}$	0.0000585726
381286	469...132 (1892526 digits)	$3.02844 * 10^{-6}$	0.0000261728
853289	189...874 (4235331 digits)	$1.55407 * 10^{-6}$	0.0000180896
1234575	447...406 (6127858 digits)	$1.47433 * 10^{-6}$	0.0000106965
2087864	425...672 (10363190 digits)	$7.9760 * 10^{-8}$	$5.75352 * 10^{-7}$

Table 3.4:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = 147 \cdot \theta + 740$

### 3.2.5 Example 5: $\theta^2 = 8 \cdot \theta + 10$

Let  $\theta$  be the real root of  $x^3 - 8x - 10$ . (This cubic (a “miracle cubic”, as D. H. Lehmer called it) was brought to my attention by John Brillhart, who discovered in 1964 that  $\theta$  has several unusually large partial quotients very early in its continued fraction expansion. For example,  $a_{17} = 22986$ ,  $a_{33} = 1501790$ , and  $a_{121} = 16467250$ . This is related to the fact that the discriminant of the polynomial is  $-652 = -4 \cdot 163$  and that  $\mathbb{Q}(\sqrt{-163})$  has class number 1. See [St71].)

We consider the pair  $(\theta, \theta^2)$ . As in Examples 1 and 2, we take  $\alpha = \theta$  and  $\beta = \theta^2$  (so  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ ).

**Step 1.** We already have  $\theta = \alpha$ , and  $f(x)$  is already in the form  $x^3 - px - q$  (with  $p = 8$  and  $q = 10$ ). Using PARI/GP, we see that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Using PARI/GP, we find that a fundamental unit is

$$\varepsilon_0 = \theta^2 - 11 \approx 0.0132932$$

Then (since  $0 < \varepsilon_0 < 1$ ) we put

$$\lambda = \varepsilon_0^{-1} = 9 + 10 \cdot \theta + 3 \cdot \theta^2 \approx 75.2262.$$

**Step 3.** Define  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = c_n$ . (We plot the points  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $n = 1$  to 1000 in Figure 3.4.)

**Step 4.** Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 - 32} \cdot (10 - 3 \cdot \theta)}{66 - 10 \cdot \theta - 3 \cdot \theta^2} \right) \approx -0.196350$$

Curiously, the continued fraction of  $\phi/\pi \approx -0.062499998136$  also has an early large partial quotient. The continued fraction of  $\phi/\pi$  is

$$[-1; 1, 15, 2095966, 30, 1, 2, 1, 1, 3, 1, 3, 1, 1, 1, \dots],$$

and the first few (non-integer) convergents of  $\phi/\pi$  are:

$$-\frac{1}{16}, -\frac{2095966}{33535457}, -\frac{62878981}{1006063726}, -\frac{64974947}{1039599183}, -\frac{192828875}{3085262092}, -\frac{257803822}{4124861275}.$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 4.60238 & C_2 \approx 0.199841 & \widetilde{M}_\theta \approx 1.37961 \\ \widetilde{M}_\alpha \approx 1.37961 & \widetilde{M}_{\beta_1} \approx 1.37961 & \widetilde{M}_{\beta_2} \approx 0.923493 \end{array}$$

and

$$\widetilde{N} \approx \max\{8.6733, 3.923, 9.205, 0.9882, 5.518, 0, 11.04, 5.518\} = 11.04$$

**Step 6.** We have

$$C_0 \approx 1.27406$$

$$n_0 \approx 1.11156$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_{16}, m_{33535457}, m_{1006063726}, m_{1039599183}, m_{3085262092}, m_{4124861275}, m_{7210123367}.$$

**Result.** For  $Q_n \geq 2 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{1.27406}{Q_{n+1}}.$$

In Table 3.5 we compute the bound  $\frac{C_0}{Q_{n+1}}$  for several values of  $n$  and compare it to the actual value of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for  $n = 1$  and 2. (Since  $Q_n$  is so large for  $n > 2$ , calculating more values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  requires more computing power than my circa-2009 computer has. On my desktop computer, Mathematica 8.0 takes about 45 minutes to calculate  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  when  $\psi_n$  has about 60 million digits, and reaches an overflow error when  $\psi_n$  has somewhere between 60 million and 147 million digits.)

In Figure 3.4 we have the points  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $n = 1$  to 1000, although there appear to be only 32 points. The reason is that since  $Q_2$  is so large,  $\phi$  (the angle of rotation of the sequence  $\{(\operatorname{Im}(\sqrt{\lambda} \lambda_2)^n, \operatorname{Re}(\sqrt{\lambda} \lambda_2)^n)\}$ ) is nearly  $-\frac{\pi}{16}$ . A transformation of the point  $(\operatorname{Im}(\sqrt{\lambda} \lambda_2)^n, \operatorname{Re}(\sqrt{\lambda} \lambda_2)^n)$  yields the point

$$(\lambda^{1/2} \langle m_n \theta \rangle, \lambda^{1/2} \langle m_n \theta^2 \rangle),$$

and

$$(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle) \approx C' (\lambda^{1/2} \langle m_n \theta \rangle, \lambda^{1/2} \langle m_n \theta^2 \rangle)$$

for some  $C'$ .

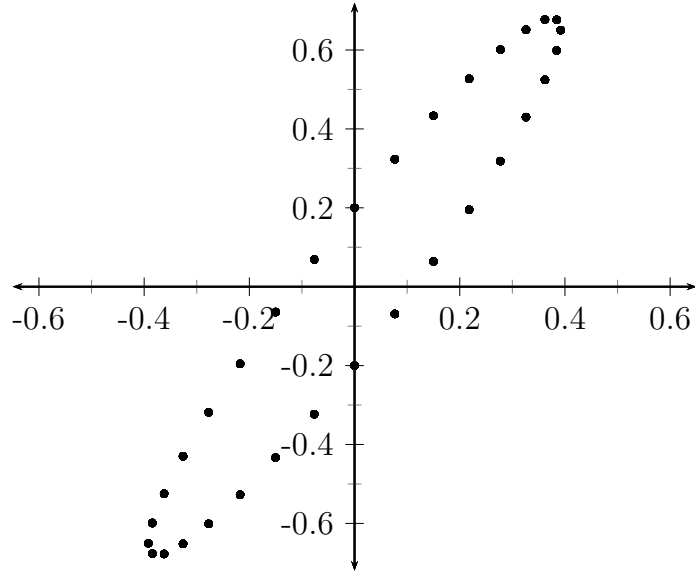


Figure 3.4:  $(m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle)$  for  $\theta^3 = 8\theta + 10$ ,  $n \leq 1000$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \theta\  \ \psi_n \theta^2\ $	$C_0/Q_{n+1}$
16	420...880 (29 digits)	$7.3376 * 10^{-9}$	$3.79915 * 10^{-8}$
33 535 457	(63 million digits)	$2.38861 * 10^{-10}$	$1.26638 * 10^{-9}$
1 006 063 726	(1.89 billion digits)		$1.22553 * 10^{-9}$
1 039 599 183	(1.95 billion digits)		$4.12951 * 10^{-10}$
3 085 262 092	(5.8 billion digits)		$3.08874 * 10^{-10}$
4 124 861 275	(7.7 billion digits)		$1.76705 * 10^{-10}$
7 210 123 367	(13.5 billion digits)		$4.94681 * 10^{-11}$
25 755 231 376	(48 billion digits)		$3.86485 * 10^{-11}$
32 965 354 743	(62 billion digits)		$1.0221 * 10^{-11}$
124 651 295 605	(234 billion digits)		$8.0833 * 10^{-12}$

Table 3.5:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = 8\theta + 10$

### 3.2.6 Example 6: $\theta^3 = -\theta + 1$

We consider the pair  $(\theta, \theta^2)$ , so  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = s = 0$ ,  $r_2 = s = 1$ . The discriminant of  $K$  is  $-31$ .

**Step 1.** We already have  $f(x) = x^3 + x - 1$  in the form  $x^3 - px - q$  (with  $p = -1$  and

$q = 1$ ). Using PARI/GP, we see that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Since  $\theta \approx 0.6823$  is already a unit, and since  $0 < \theta < 1$ , we put

$$\lambda = \theta^{-1} = \theta^2 + 1 \approx 1.46557.$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = c_n$ . (We plot the points  $\{m_n^{1/2}\langle m_n\theta \rangle, m_n^{1/2}\langle m_n\theta^2 \rangle\}$  for  $n = 1, \dots, 500$  in Figure 3.5.)

**Step 4.** Put

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 + 4}}{\theta}\right) \approx 1.28511,$$

The first few convergents of  $\phi/\pi \approx 0.409065$  are:

$$\frac{1}{2}, \frac{2}{5}, \frac{9}{22}, \frac{704}{1721}, \frac{7753}{18953}, \frac{16210}{39627}, \frac{202273}{494477}, \frac{825302}{2017535}, \frac{81907171}{200230442}, \frac{246546815}{602708861}$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 1.41421 & C_2 \approx 0.645940 & \widetilde{M}_\theta \approx 2.84001 \\ \widetilde{M}_\alpha \approx 2.84001 & \widetilde{M}_{\beta_1} \approx 2.84001 & \widetilde{M}_{\beta_2} \approx 1.31029 \end{array}$$

and

$$\widetilde{N} \approx \max\{1.21061, 5.862, 2.828, 1.461, 11.36, 0, 22.72, 11.36\} = 22.72$$

**Step 6.** We have

$$C_0 \approx 3.72125$$

$$n_0 \approx 16.3416$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_2, m_5, m_{22}, m_{1721}, m_{18953}, m_{39627}, m_{494477}, m_{2017535}, m_{200230442}, m_{602708861}$$

**Result.** For  $Q_n \geq 17 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{3.72125}{Q_{n+1}}.$$

In Table 3.6 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

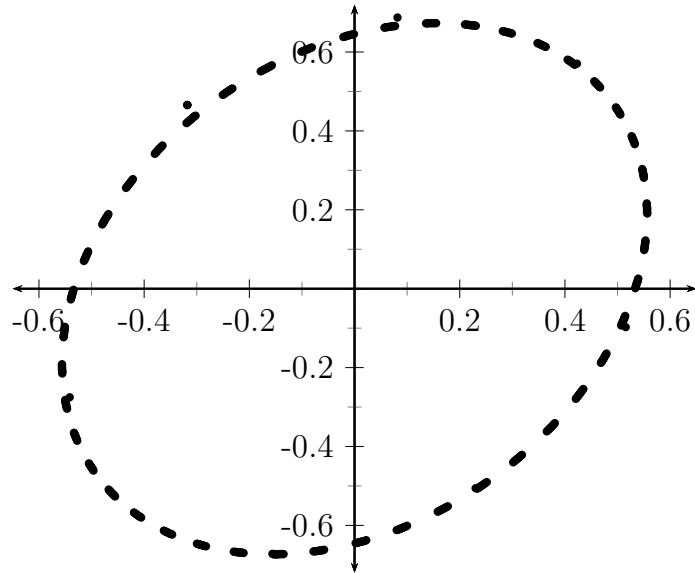


Figure 3.5:  $(\sqrt{m_n}\langle m_n\theta\rangle, \sqrt{m_n}\langle m_n\theta^2\rangle)$  for  $\theta^3 = -\theta + 1$ ,  $n \leq 500$

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n\theta\  \ \psi_n\theta^2\ $	$C_0/Q_{n+1}$
2	1	0.147899	0.744249
5	3	0.055917	0.169148
22	1873	0.00065464	0.00216226
1721	208...364 (286 digits)	0.000057056	0.000196341
18953	890...536 (3146 digits)	0.0000274071	0.0000939069
39627	948...318 (6578 digits)	$2.23749 * 10^{-6}$	$7.52562 * 10^{-6}$
494477	169...664 (82087 digits)	$5.5797 * 10^{-7}$	$1.84445 * 10^{-6}$
2017535	259...473 (334925 digits)	$5.6172 * 10^{-9}$	$1.85848 * 10^{-8}$
200230442	781...124 (33239641 digits)	$1.86084 * 10^{-9}$	$6.17420 * 10^{-9}$

Table 3.6:  $\psi_n \|\psi_n\theta\| \|\psi_n\theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = -\theta + 1$



**3.2.7 Example 7:**  $\alpha^3 = \alpha^2 + \alpha + 1$

Consider the pair  $(\alpha, \alpha^{-1})$ , where  $\alpha \approx 1.83929$  is the real root of  $f(x) = x^3 - x^2 - x - 1$ . (The discriminant of  $f$  is  $-44$ .) Note that  $\alpha(\alpha^2 - \alpha - 1) = 1$ , or  $\alpha^{-1} = \alpha^2 - \alpha - 1$ . So we have  $A = 1$ ,  $B = -1$ ,  $r_0 = s = 1$ ,  $r_1 = r_2 = -1$ .

**Step 1.** We transform  $f$  to

$$g(x) = 27f\left(\frac{x+1}{3}\right) = x^3 - 12x - 38,$$

which has  $\theta := 3\alpha - 1 \approx 4.51786$  as its real root. Using PARI/GP, we find that  $\mathcal{O}_K \subset \frac{1}{9}\mathbb{Z}[\theta]$  (so we put  $d = 9$ ).

**Step 2.** Note that  $\alpha = \frac{\theta+1}{3} \approx 1.83929$  is a unit. Since  $\alpha > 1$ , we take  $\lambda = \alpha$ .

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = 9A^2sdc_n = 81c_n$ . (We plot the points  $\{m_n^{1/2}\langle m_n\alpha \rangle, m_n^{1/2}\langle m_n\alpha^{-1} \rangle\}$  for  $n = 1, \dots, 200$  in Figure 3.6.)

**Step 4.** Put

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 - 48}}{2 - \theta}\right) \approx -0.965359,$$

The first few convergents of  $\phi/\pi \approx -0.307283$  are:

$$-\frac{1}{3}, -\frac{3}{10}, -\frac{4}{13}, -\frac{55}{179}, -\frac{59}{192}, -\frac{173}{563}, -\frac{578}{1881}, -\frac{13467}{43826}, -\frac{162182}{527793}, -\frac{175649}{571619}$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 1.75637 & C_2 \approx 0.427555 & \widetilde{M}_\theta \approx 10.1378 \\ \widetilde{M}_\alpha \approx 91.2398 & \widetilde{M}_{\beta_1} \approx 182.480 & \widetilde{M}_{\beta_2} \approx 44.8628 \end{array}$$

and

$$\widetilde{N} \approx \max\{1.356, 9.896, 31.61, 1.273, 121.7, 162.2, 243.3, 243.3, 121.7\} = 243.3$$

**Step 6.** We have

$$C_0 \approx 8186.54$$

$$n_0 \approx 18.0326$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_3, m_{10}, m_{13}, m_{179}, m_{192}, m_{563}, m_{1881}, m_{43826}, m_{527793}, m_{571619}, m_{1671031}$$

**Result.** For  $Q_n \geq 19 > n_0$ ,

$$\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^{-1}\| < \frac{C_0}{Q_{n+1}} \approx \frac{8186.54}{Q_{n+1}}.$$

In Table 3.7 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^{-1}\|$  for the first few  $n$ .

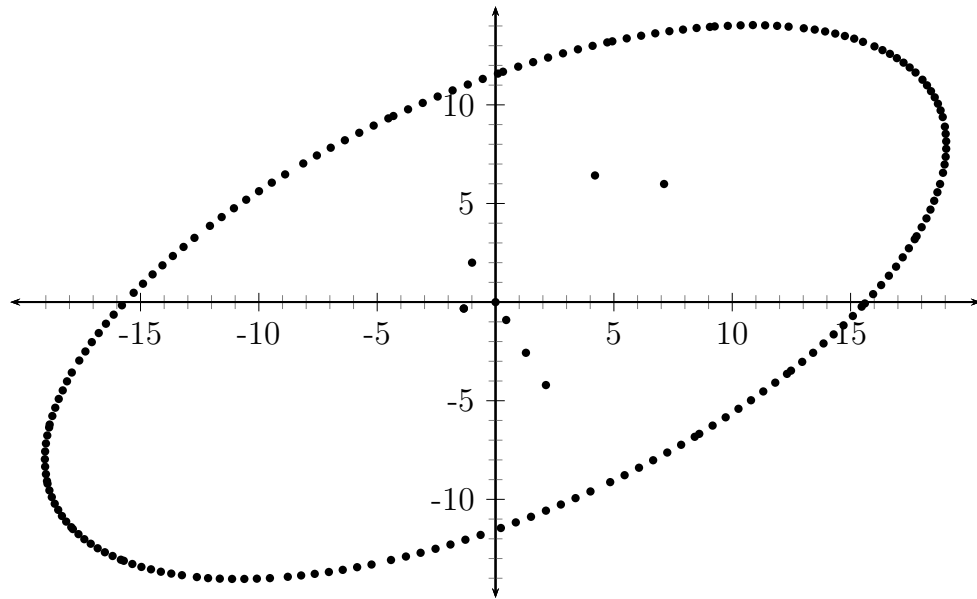


Figure 3.6:  $(\sqrt{m_n} \langle m_n \alpha \rangle, \sqrt{m_n} \langle m_n \alpha^{-1} \rangle)$  for  $\alpha^3 = \alpha^2 + \alpha + 1$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \alpha\  \ \psi_n \alpha^{-1}\ $	$C_0/Q_{n+1}$
3	9	0.429094	818.654
10	729	40.7299	629.734
13	4536	3.71296	45.7349
179	387...121 (48 digits)	2.54725	42.6382
192	106...568 (52 digits)	1.10673	14.5409
563	163...601 (150 digits)	0.361992	4.35223
1881	105...308 (499 digits)	0.0157040	0.186796
43826	553...553 (11599 digits)	0.00123216	0.0155109
527793	218...856 (139681 digits)	0.00091720	0.0143217
571619	735...889 (151279 digits)	0.000314968	0.00489910
1671031	436...357 (442238 digits)	0.000287266	0.00365039
2242650	195...217 (593517 digits)	0.0000277022	0.000339725
24097531	242...645 (6377399 digits)	0.0000102440	0.000162310
50437712	424...464 (13348313 digits)	$7.2142 * 10^{-6}$	0.000109834
74535243	625...773 (19725711 digits)	$3.02985 * 10^{-6}$	0.0000410336
199508198	613...799 (52799734 digits)	$1.15446 * 10^{-6}$	0.0000172875

Table 3.7:  $\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^{-1}\|$  and  $C_0/Q_{n+1}$ , where  $\alpha^3 = -\alpha^2 - \alpha - 1$

**3.2.8 Example 8:**  $\theta^3 = -2 \cdot \theta + 1$

Let  $\theta$  be the real root of  $f(x) = x^3 + 2x - 1$ , and consider the pair  $(\theta, \theta^2)$ . (The discriminant of  $f$  is  $-59$ .) We have  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ .

**Step 1.** We already have  $f$  in the form  $x^3 - px - q$ , with  $p = -2$  and  $q = 1$ . Using PARI/GP, we find that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Since  $\theta \approx 0.4534$  is a unit and  $0 < \theta < 1$ , we put

$$\lambda = \theta^{-1} = \theta^2 + 2 \approx 2.20557.$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and (since  $B = 0$  and  $d = s = 1$ ) define  $\{m_n\}$  by  $m_n = c_n$ . (We plot the points  $\{m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle\}$  for  $n = 1, \dots, 200$  in Figure 3.7.)

**Step 4.** Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 + 8}}{\theta} \right) \approx 1.41755,$$

The first few convergents of  $\phi/\pi \approx 0.45122$  are:

$$\frac{1}{2}, \frac{4}{9}, \frac{5}{11}, \frac{9}{20}, \frac{14}{31}, \frac{37}{82}, \frac{8265}{18317}, \frac{16567}{36716}, \frac{190502}{422193}, \frac{20971787}{46477946}, \frac{21162289}{46900139}, \frac{2243012132}{4970992541}$$

**Step 5.** We have

$$\begin{array}{llll} C_1 & \approx & 1.41421 & C_2 & \approx & 0.618191 & \widetilde{M}_\theta & \approx & 4.09039 \\ \widetilde{M}_\alpha & \approx & 4.09039 & \widetilde{M}_{\beta_1} & \approx & 4.09039 & \widetilde{M}_{\beta_2} & \approx & 0.992414 \end{array}$$

and

$$\widetilde{N} \approx \max\{1.48512, 8.822, 2.828, 1.440, 16.36, 0, 32.72, 16.36\} = 32.72$$

**Step 6.** We have

$$C_0 \approx 4.05936$$

$$n_0 \approx 8.81958$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_2, m_9, m_{11}, m_{20}, m_{31}, m_{82}, m_{18317}, m_{36716}, m_{422193}, m_{46477946}, m_{46900139}$$

**Result.** For  $Q_n \geq 9 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{4.05936}{Q_{n+1}}.$$

In Table 3.8 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

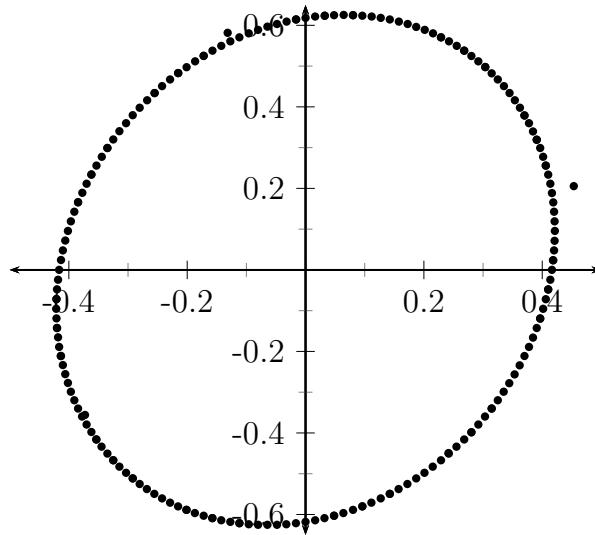


Figure 3.7:  $(\sqrt{m_n} \langle m_n \theta \rangle, \sqrt{m_n} \langle m_n \theta^2 \rangle)$  for  $\theta^3 = -2\theta + 1$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \alpha\  \ \psi_n \alpha^{-1}\ $	$C_0/Q_{n+1}$
2	2	0.076640	0.451040
9	472	0.050131	0.369032
11	2296	0.0291294	0.202968
20	2835694	0.0201199	0.130947
31	17036776865	0.0098903	0.0495043
82	563...769 (28 digits)	0.0000445638	0.000221617
18317	710...188 (6292 digits)	0.0000213130	0.000110561
36716	194...174 (12613 digits)	$1.93599 * 10^{-6}$	$9.61493 * 10^{-6}$
422193	427...544 (145032 digits)	$1.74428 * 10^{-8}$	$8.73394 * 10^{-8}$
46477946	495...762 (15966138 digits)	$1.72784 * 10^{-8}$	$8.65532 * 10^{-8}$
46900139	554...712 (16111170 digits)	$1.64421 * 10^{-10}$	$8.16609 * 10^{-10}$

Table 3.8:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = -2\theta + 1$

### 3.2.9 Example 9: $\theta^3 = 2 \cdot \theta + 2$

Let  $\theta$  be the real root of  $f(x) = x^3 - 2x - 2$ , and consider the pair  $(\theta, \theta^2)$ . (The discriminant of  $f$  is  $-76$ .) We have  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ .

**Step 1.** We already have  $f$  in the form  $x^3 - px - q$ , with  $p = 2$  and  $q = 2$ . Using PARI/GP, we find that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Using PARI/GP, we find that a fundamental unit is

$$\varepsilon_0 = \theta + 1 \approx 2.76929.$$

Since  $\varepsilon_0 > 1$ , we put  $\lambda = \varepsilon_0$ .

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = c_n$ . (We plot the points  $\{m_n^{1/2} \langle m_n \theta \rangle, m_n^{1/2} \langle m_n \theta^2 \rangle\}$  for  $n = 1, \dots, 200$  in Figure 3.8.)

**Step 4.** Put

$$\phi = \arctan \left( \frac{\sqrt{3\theta^2 - 8}}{2 - \theta} \right) \approx 1.37763,$$

The first few convergents of  $\phi/\pi \approx 0.438515$  are:

$$\frac{1}{2}, \frac{3}{7}, \frac{4}{9}, \frac{7}{16}, \frac{25}{57}, \frac{82}{187}, \frac{189}{431}, \frac{3484}{7945}, \frac{3673}{8376}, \frac{25522}{58201}, \frac{105761}{241180}, \frac{131283}{299381}, \frac{893459}{2037466}$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 2.12140 & C_2 \approx 0.367826 & \widetilde{M}_\theta \approx 1.17046 \\ \widetilde{M}_\alpha \approx 1.17046 & \widetilde{M}_{\beta_1} \approx 1.17046 & \widetilde{M}_{\beta_2} \approx 1.46957 \end{array}$$

and

$$\widetilde{N} \approx \max\{1.66412, 3.391, 4.243, 1.211, 4.682, 0, 9.364, 4.682\} = 9.364$$

**Step 6.** We have

$$C_0 \approx 1.72008$$

$$n_0 \approx 4.39202$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_2, m_7, m_9, m_{16}, m_{57}, m_{187}, m_{431}, m_{7945}, m_{8376}, m_{58201}, m_{241180}, m_{299381}$$

**Result.** For  $Q_n \geq 5 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{1.72008}{Q_{n+1}}.$$

In Table 3.9 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

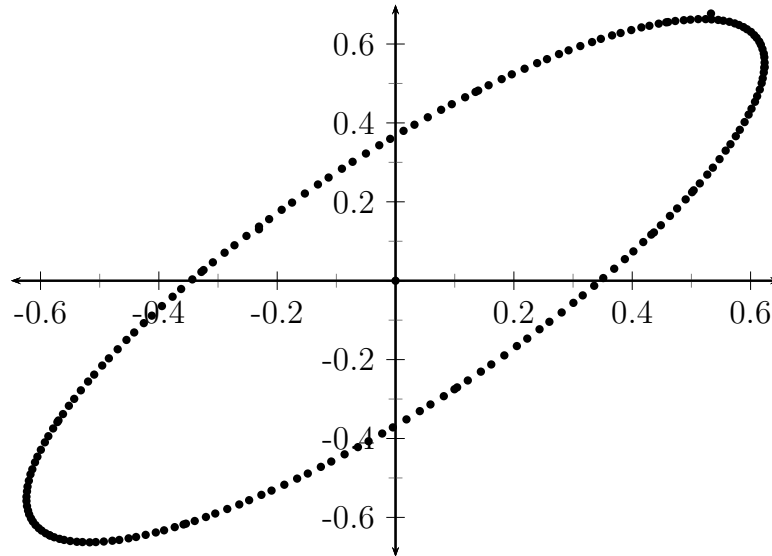


Figure 3.8:  $(\sqrt{m_n}\langle m_n\theta\rangle, \sqrt{m_n}\langle m_n\theta^2\rangle)$  for  $\theta^3 = 2\theta + 2$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n\theta\  \ \psi_n\theta^2\ $	$C_0/Q_{n+1}$
2	1	0.0300832	0.245725
7	169	0.064779	0.191120
9	1296	0.0281634	0.107505
16	1618776	0.0125768	0.0301768
57	2219769241218582281661888	0.00328181	0.00919827
187	714 ... 835 (82 digits)	0.00165204	0.00399089
431	619 ... 593 (190 digits)	0.000086617	0.000216498
7945	564 ... 776 (3514 digits)	0.000074675	0.000205358
8376	258 ... 036 (3705 digits)	0.0000120265	0.0000295541
58201	274 ... 936 (25746 digits)	$2.47362 * 10^{-6}$	$7.13192 * 10^{-6}$
241180	436 ... 500 (106690 digits)	$2.13313 * 10^{-6}$	$5.74544 * 10^{-6}$
299381	885 ... 886 (132436 digits)	$3.40429 * 10^{-7}$	$8.44223 * 10^{-7}$
2037466	342 ... 477 (901311 digits)	$9.0588 * 10^{-8}$	$2.68268 * 10^{-7}$
6411779	143 ... 251 (2836371 digits)	$6.8663 * 10^{-8}$	$2.03577 * 10^{-7}$
8449245	362 ... 526 (3737682 digits)	$2.19251 * 10^{-8}$	$5.41594 * 10^{-8}$
31759514	274 ... 581 (14049418 digits)	$2.88800 * 10^{-9}$	$7.45377 * 10^{-9}$

Table 3.9:  $\psi_n \|\psi_n\theta\| \|\psi_n\theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = 2\theta + 2$



**3.2.10 Example 10:**  $\alpha^3 = \alpha^2 - \alpha + 2$

Let  $\alpha \approx 1.35321$  be the real root of  $f(x) = x^3 - x^2 + x - 2$ , and consider the pair  $(\alpha, \alpha^2)$ . (The discriminant of  $f$  is  $-83$ .) We have  $A = 1$ ,  $B = -1$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ .

**Step 1.** We transform  $f$  to

$$g(x) = 27f\left(\frac{x+1}{3}\right) = x^3 + 6x - 47,$$

which has  $\theta := 3\alpha - 1 \approx 3.05963$  as its real root. Using PARI/GP, we find that  $\mathcal{O}_K \subset \frac{1}{9}\mathbb{Z}[\theta]$  (so we put  $d = 9$ ).

**Step 2.** Using PARI/GP, we find that a fundamental unit is

$$\varepsilon_0 = -\frac{2}{3} + \frac{1}{3}\theta \approx 0.35321.$$

Since  $0 < \varepsilon_0 < 1$ , we put

$$\lambda = \varepsilon_0^{-1} = \frac{10}{9} + \frac{2}{9}\theta + \frac{1}{9}\theta^2 \approx 2.83118$$

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = 9A^2sdc_n = 81c_n$ . (We plot the points  $\{m_n^{1/2}\langle m_n\alpha \rangle, m_n^{1/2}\langle m_n\alpha^2 \rangle\}$  for  $n \leq 200$  in Figure 3.9.)

**Step 4.** Put

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 + 24}(2 - \theta)}{8 - 2\theta - \theta^2}\right) \approx 0.796415,$$

The first few convergents of  $\phi/\pi \approx 0.253507$  are:

$$\frac{1}{3}, \frac{1}{4}, \frac{18}{71}, \frac{235}{927}, \frac{253}{998}, \frac{2259}{8911}, \frac{4771}{18820}, \frac{7030}{27731}, \frac{25861}{102013}, \frac{84613}{333770}, \frac{110474}{435783}, \frac{195087}{769553}$$

**Step 5.** We have

$$\begin{array}{llll} C_1 & \approx & 1.41421 & C_2 & \approx & 0.513861 & \widetilde{M}_\theta & \approx & 9.81058 \\ \widetilde{M}_\alpha & \approx & 88.2952 & \widetilde{M}_{\beta_1} & \approx & 88.2952 & \widetilde{M}_{\beta_2} & \approx & 27.1782 \end{array}$$

and

$$\tilde{N} \approx \max\{1.68261, 11.77, 25.46, 1.354, 117.7, 157.0, 0, 235.5, 117.7\} = 235.5$$

**Step 6.** We have

$$C_0 \approx 2399.70$$

$$n_0 \approx 10.4959$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_3, m_4, m_{71}, m_{927}, m_{998}, m_{8911}, m_{18820}, m_{27731}, m_{102013}, m_{333770}, m_{435783}$$

**Result.** For  $Q_n \geq 11 > n_0$ ,

$$\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{2399.70}{Q_{n+1}}.$$

In Table 3.10 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^2\|$  for the first few  $n$ .

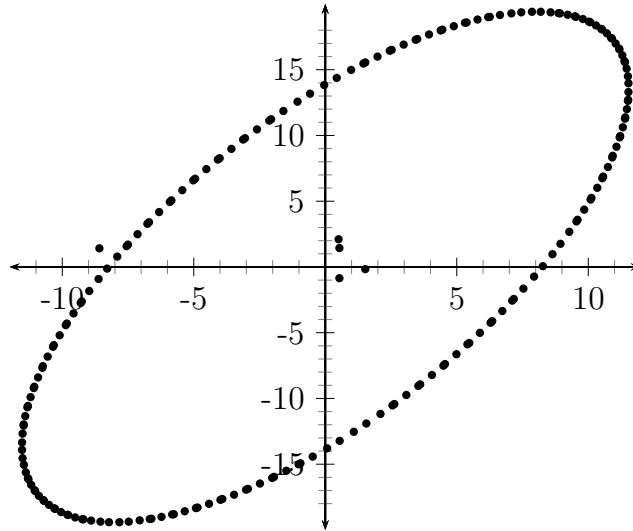


Figure 3.9:  $(\sqrt{m_n} \langle m_n \alpha \rangle, \sqrt{m_n} \langle m_n \alpha^2 \rangle)$  for  $\alpha^3 = \alpha^2 - \alpha + 2$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \alpha\  \ \psi_n \alpha^2\ $	$C_0/Q_{n+1}$
3	54	0.461096	599.925
4	153	1.07036	33.7986
71	292 ... 560 (33 digits)	0.50606	2.58867
927	223 ... 190 (420 digits)	0.454662	2.40451
998	274 ... 634 (452 digits)	0.054208	0.269296
8911	715 ... 385 (4028 digits)	0.0196042	0.127508
18820	248 ... 673 (8507 digits)	0.0150207	0.0865350
27731	748 ... 424 (12534 digits)	0.00457984	0.0235235
102013	777 ... 457 (46107 digits)	0.00128298	0.00718968
333770	262 ... 058 (150854 digits)	0.00073076	0.00550664
435783	859 ... 070 (196960 digits)	0.00055223	0.00311830
769553	949 ... 944 (347813 digits)	0.000178517	0.000874386
2744442	549 ... 968 (1240398 digits)	0.0000166865	0.0000850536
28213973	411 ... 832 (12751787 digits)	0.0000116516	0.0000775137
30958415	950 ... 680 (13992184 digits)	$5.0349 * 10^{-6}$	0.0000266246

Table 3.10:  $\psi_n \|\psi_n \alpha\| \|\psi_n \alpha^2\|$  and  $C_0/Q_{n+1}$ , where  $\alpha^3 = \alpha^2 - \alpha + 2$

**3.2.11 Example 11:**  $\theta^3 = \theta + 2$

Let  $\theta$  be the real root of  $f(x) = x^3 - x - 2$ , and consider the pair  $(\theta, \theta^2)$ . (The discriminant of  $f$  is  $-104$ .) We have  $A = 1$ ,  $B = 0$ ,  $r_0 = r_1 = 0$ ,  $r_2 = s = 1$ .

**Step 1.** We already have  $f$  in the form  $x^3 - px - q$ , with  $p = 1$  and  $q = 2$ . Using PARI/GP, we find that  $\mathcal{O}_K = \mathbb{Z}[\theta]$ , so we put  $d = 1$ .

**Step 2.** Using PARI/GP, we find that a fundamental unit is

$$\varepsilon_0 = 1 + \theta + \theta^2 \approx 4.83598.$$

Since  $\varepsilon_0 > 1$ , we put  $\lambda = \varepsilon_0$ .

**Step 3.** Define the sequences  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  as before and define  $\{m_n\}$  by  $m_n = c_n$ . (We plot the points  $\{m_n^{1/2}\langle m_n\theta \rangle, m_n^{1/2}\langle m_n\theta^2 \rangle\}$  for  $n \leq 200$  in Figure 3.10.)

**Step 4.** Put

$$\phi = \arctan\left(\frac{\sqrt{3\theta^2 - 4}(1 - \theta)}{4 - \theta - \theta^2}\right) \approx -1.38945,$$

The first few convergents of  $\phi/\pi \approx -0.442276$  are:

$$-\frac{1}{2}, -\frac{3}{7}, -\frac{4}{9}, -\frac{19}{43}, -\frac{23}{52}, -\frac{272}{615}, -\frac{11175}{25267}, -\frac{11447}{25882}, -\frac{22622}{51149}, -\frac{34069}{77031}, -\frac{90760}{205211}$$

**Step 5.** We have

$$\begin{array}{lll} C_1 \approx 1.41421 & C_2 \approx 0.410174 & \widetilde{M}_\theta \approx 0.870111 \\ \widetilde{M}_\alpha \approx 0.870111 & \widetilde{M}_{\beta_1} \approx 0.870111 & \widetilde{M}_{\beta_2} \approx 1.12656 \end{array}$$

and

$$\widetilde{N} \approx \max\{2.19909, 2.629, 2.828, 1.256, 3.480, 0, 6.961, 3.480\} = 6.961$$

**Step 6.** We have

$$C_0 \approx 0.980236$$

$$n_0 \approx 2.46219$$

and we define  $\{\psi_n\}$  by  $\psi_n = m_{Q_n}$ . The first several terms are:

$$m_2, m_7, m_9, m_{43}, m_{52}, m_{615}, m_{25267}, m_{25882}, m_{51149}, m_{77031}, m_{205211}, m_{282242}$$

**Result.** For  $Q_n \geq 3 > n_0$ ,

$$\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\| < \frac{C_0}{Q_{n+1}} \approx \frac{0.980236}{Q_{n+1}}.$$

In Table 3.11 we compute the bound  $\frac{C_0}{Q_{n+1}}$  and compare it to the actual values of  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  for the first few  $n$ .

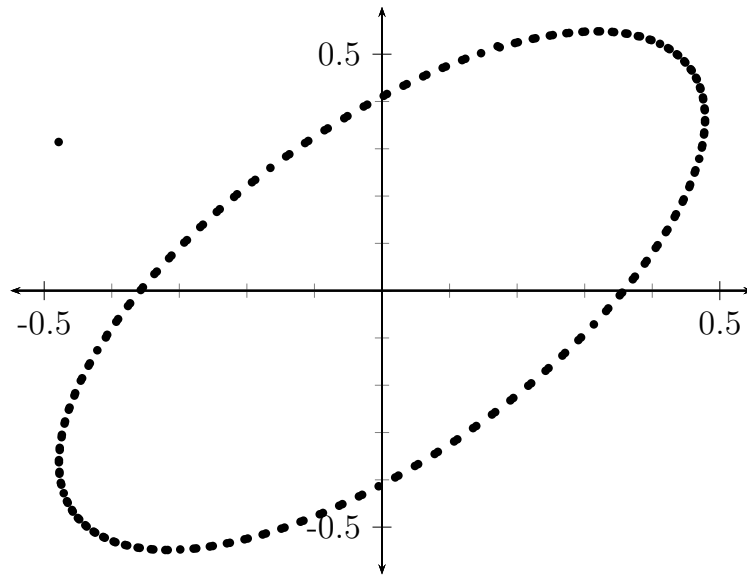


Figure 3.10:  $(\sqrt{m_n} \langle m_n \theta \rangle, \sqrt{m_n} \langle m_n \theta^2 \rangle)$  for  $\theta^3 = \theta + 2$ ,  $n \leq 200$ .

$Q_n$	$\psi_n (= m_{Q_n})$	$\psi_n \ \psi_n \theta\  \ \psi_n \theta^2\ $	$C_0/Q_{n+1}$
2	4	0.088387	0.140034
7	10407	0.0402688	0.108915
9	243385	0.0126439	0.0227962
43	455 ... 491 (29 digits)	0.0104498	0.0188507
52	659 ... 364 (35 digits)	0.00100429	0.00159388
615	152 ... 879 (421 digits)	0.0000240102	0.0000387951
25267	122 ... 779 (17295 digits)	0.0000152616	0.0000378733
25882	110 ... 180 (17716 digits)	$8.7519 * 10^{-6}$	0.0000191643
51149	802 ... 749 (35010 digits)	$6.5084 * 10^{-6}$	0.0000127252
77031	528 ... 079 (52726 digits)	$2.24402 * 10^{-6}$	$4.77672 * 10^{-6}$
205211	791 ... 131 (140463 digits)	$2.02017 * 10^{-6}$	$3.47303 * 10^{-6}$
282242	248 ... 196 (193190 digits)	$2.23897 * 10^{-7}$	$3.57048 * 10^{-7}$
2745389	266 ... 125 (1879175 digits)	$5.0764 * 10^{-9}$	$8.09582 * 10^{-9}$

Table 3.11:  $\psi_n \|\psi_n \theta\| \|\psi_n \theta^2\|$  and  $C_0/Q_{n+1}$ , where  $\theta^3 = \theta + 2$

## APPENDIX A

## COMMANDS AND ALGORITHMS

We construct our sequences  $\{c_n\}$  by taking powers of a unit  $\lambda$  in  $\mathbb{Q}[x]/(x^3 - px - q)$ , and taking their  $\theta^2$ -coordinates in the power basis  $1, \theta, \theta^2$  (where  $\theta$  is the real root of  $x^3 - px - q$ ). Two convenient ways of computing the  $c_n$ 's are by working with polynomials modulo  $x^3 - px - q$  (for which we used PARI/GP 2.7.2) and by working with matrices (for which we used Mathematica 8.0). We list some of the specific commands and algorithms we used.

**A.1 Using Polynomials Modulo  $x^3 - px - q$  in PARI/GP**

Let  $f = x^3 - px - q$  (for example, `f=x^3-2` or `f=x^3-147*x-740`), and initialize the number field: `k=bnfinit(f)`;

**A.1.1 Finding  $\lambda$** 

1. Find a fundamental unit  $\varepsilon_0$ : `ep=k.fu[1]`
2. Find the real value of  $\varepsilon_0$  (get value of  $\theta$  and plug it into polynomial for  $\varepsilon_0$ ):  
`t=polroots(f)[1]; subst(ep.pol,x,t)`
3. Depending on the value of  $\varepsilon_0$ , let  $\lambda$  be one of  $\pm\varepsilon_0^{\pm 1}$ . For example: `lam=ep^-1`  
or `lam=-ep`

**A.1.2 Computing  $c_n$** 

Assuming we've found a unit  $\lambda > 1$ , we can compute  $\lambda^n$  directly (as `lam^n`), and  $c_n$  by

```
polcoeff((lam^n).pol,2)
```

We can define a function

$$\text{cn}(1, n) = \text{polcoeff}((1^n) . \text{pol}, 2)$$

and then compute  $c_n$  as  $\text{cn}(\text{lam}, n)$ . If we also wanted to be able to find the  $a_n$ 's or  $b_n$ 's, we could define a function

$$\text{lcoeff}(1, n, j) = \text{polcoeff}((1^n) . \text{pol}, j)$$

where  $\text{lcoeff}(\text{lam}, n, j)$  finds the  $\theta^j$  coordinate of  $\lambda^n$  in the basis  $1, \theta, \theta^2$ . So

$$\text{lcoeff}(\text{lam}, n, 0)$$

would be  $a_n$  and

$$\text{lcoeff}(\text{lam}, n, 1)$$

would be  $b_n$ .

### A.1.3 Finding $m_n$ and Calculating $m_n \|m_n \alpha\| \|m_n \beta\|$

To find  $k_n$ , we find an integral basis with  $\mathbf{k.zk}$  and let  $d$  be the largest denominator in this integral basis. Then  $k_n = dc_n$ , and  $m_n$  is some integer multiple of  $k_n$ .

To evaluate  $n \|n \alpha\| \|n \beta\|$ , we can first define  $\langle x \rangle$  by

$$\text{mod1}(x) = x - \text{round}(x)$$

and then define

$$\text{LWprod}(n, a, b) = n * \text{abs}(\text{mod1}(n*a) * \text{mod1}(n*b))$$

For example, we could find  $k_n \|k_n \theta\| \|k_n \theta^2\|$  as

$$\text{LWprod}(d * \text{cn}(\text{lam}, n), t, t^2)$$



### A.1.4 Increasing Precision and Memory

To be able to compute this for large  $n$ , we may need to increase the precision and available memory. To increase the precision to  $N$  digits, we use

```
default(realprecision,N)
```

If we increase the precision, we need to re-calculate the values of  $\theta$  or  $\alpha$  or  $\beta$ . This may require increasing the available memory with `allocatemem()` several times. For instance, calculating that

$$c_n \|c_n \sqrt[3]{2}\| \|c_n \sqrt[3]{4}\| \approx 4.05112 * 10^{-7}$$

for  $n = 1059767$  requires around 1 million digits of precision (the default is 28 significant digits), and finding  $\sqrt[3]{2}$  to this precision requires a memory stack size of around 1024MB (the default is 4MB).

## A.2 Using Matrices

Given  $f(x) = x^3 - px - q$  with  $\theta$  as a root, we represent an element  $\zeta = x + y\theta + z\theta^2$  of  $K = \mathbb{Q}(\theta)$  by a vector<sup>1</sup>  $(x, y, z) \in \mathbb{Q}^3$  or as a matrix  $M_\zeta := xI + yT + zT^2 \in \mathbb{Q}[T]$ , where

$$T = \begin{pmatrix} 0 & 0 & q \\ 1 & 0 & p \\ 0 & 1 & 0 \end{pmatrix}$$

is the companion matrix of  $f$ . We add, subtract, and scalar-multiply componentwise. To multiply  $\zeta_1$  and  $\zeta_2$  we compute  $M_{\zeta_1}\zeta_2$ . To divide  $\zeta_1/\zeta_2$ , we compute  $M_{\zeta_2}^{-1}\zeta_1$ . (If  $\zeta_2 \neq 0$ , then  $M_{\zeta_2}$  is nonsingular.) To compute  $\zeta^n$  for  $n > 0$ , we can use repeated squaring on the matrix  $M_\zeta$ , and then

$$\zeta^n \longleftrightarrow M_\zeta^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We assume we already have  $\alpha$  and  $\beta$ , and have transformed the minimal polynomial of  $\alpha$  into the form  $g(x) = x^3 - px - q$ . We initialize the field by building the companion matrix  $T$  for  $g$ , building the vector  $(I, T, T^2)$  (which we call **TPowers**), setting up a vector of roots  $(\theta_1, \theta_2, \theta_3)$  of  $g$ , and putting  $\theta = \theta_1$ . We still use PARI/GP to find  $d$  and a unit  $\lambda > 1$ . We list some of the Mathematica functions we used.

### A.2.1 Computing in $K$

1. `KMat[v]` takes the vector  $\mathbf{v}=\{x, y, z\}$  and returns the matrix  $\mathbf{v} \cdot \mathbf{TPowers}$  (which is  $xI + yT + zT^2$ ).
2. `KN[a_] := Det[KMat[a]]`; (\* Norm of a \*)
3. `KTr[a_] := Tr[KMat[a]]`; (\* Trace of a \*)
4. `KMul[a_, b_] := KMat[a].b` (\* multiply a and b in K \*)

---

<sup>1</sup>Mathematica treats  $\{x, y, z\}$  as either a row vector or column vector, depending on the context.

```

5. KInv[a_] := Inverse[KMat[a]].UnitVector[3, 1]; (* get a^{-1} *)

6. KDiv[a_, b_] := Inverse[KMat[b]].a (* a/b in K *)

7. (* calculate a^n (n>0) in K by repeated squaring *)
   KExpPos[a_, n_] := Module[{digits, A, b, i},

      (* get bits of n, highest to lowest *)
      digits = IntegerDigits[n, 2];
      A = KMat[a];          (* matrix for a *)

      (* repeated squaring step *)
      b = a;  (* first bit is 1 *)
      For[i = 2, i <= Length[digits], i++,
         b = KMat[b].b;          (* squaring *)
         If[digits[[i]] == 1,    (* if ith bit is 1, multiply by a *)
            b = A.b;
         ];
      ];
      b
   ]

8. (* calculate a^n for any int n *)
   KExp[a_, n_Integer] :=
   If[n > 0, KExpPos[a, n],
      If[n == 0, UnitVector[3, 1],
         KExpPos[KInv[a], -n]
      ]
   ]

```

### A.2.2 Calculating $\langle x \rangle$ , $\|x\|$ , $n\|n\alpha\| \|n\beta\|$

9. `mod1[x_] := x - Round[x]` (this is  $\langle x \rangle$ )
10. `LWNorm[x_] := Abs[mod1[x]]` (this is  $\|x\|$ )
11. (\* find LW product to precision prec \*)  
`LWProd[n_, a_, b_, prec_] := Abs[N[n mod1[n a] mod1[n b], prec]]`

### A.2.3 Calculating Sequences of Powers

12. (\* get seq of  $a^i$  from m to n \*)  
`KExpSeq[a_, m_, n_] := Module[{A, ai, i, seq},  
 A = KMat[a];  
 ai = KExp[a, m];  
 seq = {ai};  
  
 For[i = m, i < n, i++,  
 ai = A.ai;  
 seq = Append[seq, ai];  
 ];  
 seq  
 ]`

### A.2.4 Calculating the Point $(n^{1/2}\langle n\alpha \rangle, n^{1/2}\langle n\beta \rangle)$

13. (\*  $(n^{1/2}\langle n\alpha \rangle, n^{1/2}\langle n\beta \rangle)$  \*)  
`SqrtMod1Pt[n_, list_, prec_] := N[Sqrt[n] mod1[n list], prec]`

### A.2.5 Calculating the $m_n$ 's

We let  $\mathbf{lam} = \{a, b, c\}$  be the vector for  $\lambda = a + b\theta + c\theta^2$  in the  $1, \theta, \theta^2$  basis. We can compute individual  $m_n$ 's or compute a sequence of them. To get an individual  $m_n$ , we put

```
mn = const KExp[lam, n]
```

(where `const` is some constant depending on  $\alpha$  and  $\beta$  and  $d$ ). To find  $m_n^{1/2} \langle m_n \alpha \rangle$  and  $m_n^{1/2} \langle m_n \beta \rangle$  we used

```
SqrtMod1[mn,  $\alpha$ ,  $\beta$ ]
```

and to find  $m_n \|m_n \alpha\| \|m_n \beta\|$  (to `prec` significant digits) we used

```
LWProd[mn,  $\alpha$ ,  $\beta$ , prec]
```

To get a sequences of these, we put (for  $m < n$ )

```
lamnseq = KExpSeq[lam, m, n];
```

and

```
mnseq = const #[[3]] & /@ lamnseq;
```

This picks the last entry of each vector for  $\lambda^n$  in `lamnseq`, and scales by `const`. For a sequence of points  $(m_n^{1/2} \langle m_n \alpha \rangle, m_n^{1/2} \langle m_n \beta \rangle)$ , we used

```
mnptseq = Map[SqrtMod1Pt[#,  $\alpha$ ,  $\beta$ , prec] &, mnseq];
```

### A.2.6 Increasing Precision

Since  $\{m_n\}$  grows exponentially, we soon need to increase `$MaxExtraPrecision` to calculate  $m_n^{1/2} \langle m_n \alpha \rangle$  and  $m_n^{1/2} \langle m_n \beta \rangle$  with any accuracy. (In Mathematica 8.0, the default is 50.) To compute these for a few thousand  $m_n$ 's, `$MaxExtraPrecision=10000` works. For some of the examples (like computing  $m_{Q_n} \|m_{Q_n} \alpha\| \|m_{Q_n} \beta\|$  when  $m_{Q_n}$  has tens of millions of digits), we used `$MaxExtraPrecision=10^250000000` (250 million).

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