THEORY OF AXIAL COLLISIONAL HEATING IN LINEAR MAGNETIC FUSION SYSTEMS

by

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APPROVAL BY THESIS DIRECTOR

This thesis has been approved on the date shown below:

Richard L. Morse — Dec 15 '58

R. L. Morse — Date

Professor of Nuclear Engineering

Dedicated to My Parents

Mr. Ahmad H. Salehi

and

Mrs. Raf'at A. Salehi

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ABSTRACT

A simple method has been proposed for performing work on the long θ-pinch system where the plasma columns have a large ratio of length to radius by using the external magnetic field as a pumping device in order to squeeze the plasma radially with a spatial dependence that is sinusoidal along the plasma column.

The method is studied by the use of the fluid approximation. In order to simplify the solution of the fluid equations, it will be assumed that the temperatures of the electrons and ions in plasma are highly coupled so $T = T_e = T_i$. By coupling the pressure balance condition at the surface of the plasma with the fluid approximation and using a perturbation analysis, an analytical equation for work will be obtained. This analytical expression will be examined for different cases and will be solved numerically.

Results for relative energy increase have been plotted for different cases of β the ratio of plasma pressure to the external magnetic pressure as a function of a unitless frequency $\frac{\omega}{kc_s}$ for different values of $\lambda_{mfp}\ k_p$, where λ_{mfp} is the mean free path and k_p is the axial wave number of the pumping fields. These results of the relative energy increase for optimum pumping frequencies are in qualitative agreement with expectations. The results obtained seem to support considerably the utility of axial collisional heating in long linear magnetic fusion systems.

CHAPTER 1

INTRODUCTION

Consider a long θ-pinch system where the plasma columns have a large ratio of length to radius [Ref. 1]. Figure 1.1 illustrates the long θ-pinch of interest. A θ-pinch is a device for producing a controlled nuclear fusion reaction, in which plasma in a long skinny tube is confined by a magnetic field produced by current-carrying coils, and is shock-heated and compressed by pulses in this field to produce the high temperatures at which fusion reactions take place; the magnetic field is then sustained in order to maintain the plasma confinement. It is known that usual radial shock heating requires rapidly rising fields (i.e., high frequency) and is, therefore, expensive. However, lower frequency heating can be done by axial periodic radial compressional pumping which drives waves along the axis. As a result, the heating occurs when these axial waves are damped by thermal conductivity and viscosity of the charged particles in a plasma. Consequently, the background plasma temperature will increase until the necessary conditions for the thermonuclear reaction (i.e., D-T reaction) are satisfied. The purpose of this thesis is to present an analytical expression for work which can be done on the system as a result of locally adiabatic radial compressions giving rise to a non-adiabatic axial behavior.

It seems that the process depends on the pumping period (T_p) and the pumping wave length (λ_p) in order to optimize the heating. In

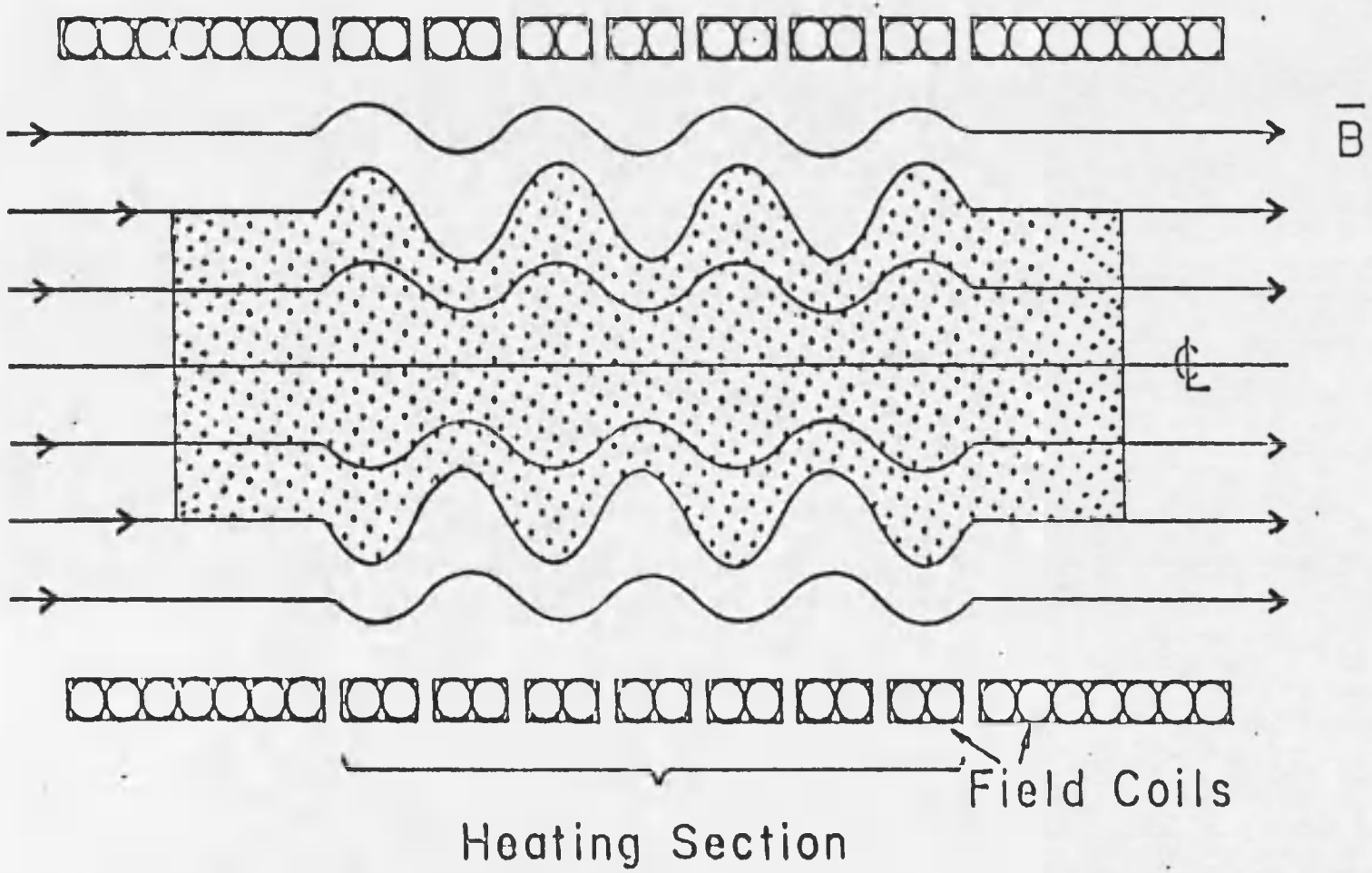

Fig. 1.1 Axial Collisional Heating of Linear Magnetic Fusion System

order to simplify the analysis, it is assumed that the plasma behaves as a fluid. Under this condition, the mean free paths of the ionized particles (λ_{mfp}) are small compared with the pumping length, λ_p. In addition, the plasma collision time (t_c) is small compared with the characteristic time [pumping time (t_p)] for the fluid approximation. In order to simplify the solution of the fluid equations, a region is considered in which the pumping period (T_p) is large compared with the electron-ion equilibration time (τeq). As a result, one-temperature fluid approximation applies to our calculation, that is $T = T_e = T_i$. In this case, the heat generated in the electrons as a consequence of collisions with ions (Q_e) could be removed from the energy equation.

For clarity, suppose the externally imposed magnetic field is made to change as a function of time (t) and the axial length (Z) so that the radius of the plasma column is [Ref. 1]

$$r = r_0 - r_1 \sin\left(\frac{2\pi t}{T_p}\right) \cos\left(\frac{2\pi Z}{\lambda_p}\right), \quad (0<r_1<r_0)$$

It is obvious to see when $\sin \frac{2\pi t}{T_p} = +1$, the plasma is most compressed where $Z = 0, \lambda_p, 2\lambda_p, \ldots$, and decompressed where $Z = \lambda_p/2, 3\lambda_p/2, \ldots$, and the situation reverses when $\sin \frac{2\pi t}{T_p} = -1$. If the acoustic transit time (λ_p/C_s) between the maxima and minima of compression is of the order of the pumping period (T_p), and if $r_0 \ll \lambda_p$, then the radial compression will be locally adiabatic but the wave motion along the Z-axis can be made non-adiabatic (irreversible) by thermal conductivity and viscosity, and the radial compression is adiabatic because there is

very little radial thermal conduction across the internal magnetic B-field. In addition, the radial shocks do not form because the pumping period (T_p) is so long compared to the radial acoustic transit time (r/C_s). By linearizing the fluid and heat flow equations and under the conditions mentioned above, a heating expression is obtained which is second order in wave amplitude and proportional to the sine of the phase shift between pump and axial waves caused by dissipation. Numerical evaluation of the analytic result shows the largest heating to occur near resonance between pump frequency and axial waves.

In Chapter 2 the fusion system and its requirements are discussed, and it will be a brief review. In Chapter 3 simple analytical expressions for work and relative energy increase are derived by introducing the fluid equations, the pressure balance condition at the surface of the plasma, and the concept of the perturbation analysis to the solution of the fluid equations. In addition, the many cases will be discussed for the one-temperature radial fluid approximation (no plasma motion along the magnetic field) and the one-temperature fluid approximation. In Chapter 4 the numerical results which are obtained for relative energy increase will be discussed as a function of a unitless frequency ($\frac{\omega}{kC_s}$), and the conclusion for this thesis will then be discussed.

CHAPTER 2

FUSION SYSTEMS AND ITS REQUIREMENT

2.1 Introduction

The large amount of energy which can possibly be achieved by using an external magnetic field, in the near future can be used to raise a plasma to very high temperatures in order for the thermonuclear reactions to occur in the system. Since the fuel of this reaction is the heavy hydrogen in seawater, the realization of this goal means achieving a limitless energy source which can be a very important aspect for future generations.

2.2 Thermonuclear Reactions

In order to initiate a thermonuclear reaction, it is necessary to have two light elements brought close enough together for the short-range attractive nuclear forces to overcome the electrostatic repulsion. The cross section of many fusion reactions which are known, and in the literature, indicates that the easiest are to those including the heavy isotope of hydrogen, deuterium and tritium. These reactions are [Ref. 2, Eqs. (4.1), (4.2)]:

$$
\begin{aligned}
&{}_1D^2 + {}_1D^2 \rightarrow {}_1T^3\ (1.01) + {}_1H^1\ (3.03) \\
&{}_1D^2 + {}_1D^2 \rightarrow {}_2He^3\ (0.82) + {}_0^1n\ (2.45)
\end{aligned}
\qquad (2.1)
$$

and

$$
{}_1D^2 + {}_1T^3 \rightarrow {}_2He^4\ (3.52) + {}_0n^1\ (14.06)
$$

The number in parenthesis shows the energy carried away by each of the reaction product particles in Mev. The following reaction can take place, but relatively slow [Ref. 2, Eq. (4.3)]:

$$_1D^2 + {}_2He^3 \rightarrow {}_2He^4\ (3.67) + {}_1H^1\ (14.67)$$

Deutrium can be extracted cheaply from water, but tritium is scarce and very radioactive. However, we could produce tritium from lithium by a reaction involving a neutron [Ref. 2, Eq. (4.4)]:

$$_3Li^6 + {}_0n^1 \rightarrow {}_2He^4 + {}_1T^3 + 4.8\ \text{Mev}$$

Thus the neutrons which are produced by D-D and D-T reactions could be used to generate tritium in a lithium blanket surrounding the reaction region.

It is possible to use the charged reaction products in a fusion reactor. If the ignition temperatures can be achieved, these kinds of reactions involving the charged reaction products are very desirable. They are expressed as [Ref. 3, Eq. (9.5)]:

$$D + {}^3He \rightarrow {}^4He + p + 18.3\ \text{Mev}$$

$$p + {}^6Li \rightarrow {}^4He + {}^3He + 4.0\ \text{Mev}$$

$$p + {}^{11}B \rightarrow 3\ {}^4He + 8.7\ \text{Mev}$$

The number of reactions per unit time and volume between two different nuclei, one with number density n_{11} and the other with number density n_{12} per unit volume is given by [Ref. 2, Eq. (4.5)]

$$R = n_{12} \cdot n_{12} \langle \sigma v \rangle$$

Here $\langle\sigma v\rangle$ is the product of the relative velocity v with the (velocity dependent) reaction cross sections (σ) averaged over the velocity distribution. The rate of released energy per unit time is given by [Ref. 2, Eq. (4.7)]

$$P_N = R\ Q$$

Where Q is the average nuclear energy released per reaction.

Values of $\langle\sigma v\rangle$ for the different reactions are shown as functions of temperature in Fig. 2.1 [Ref. 4, pp. 201-2].

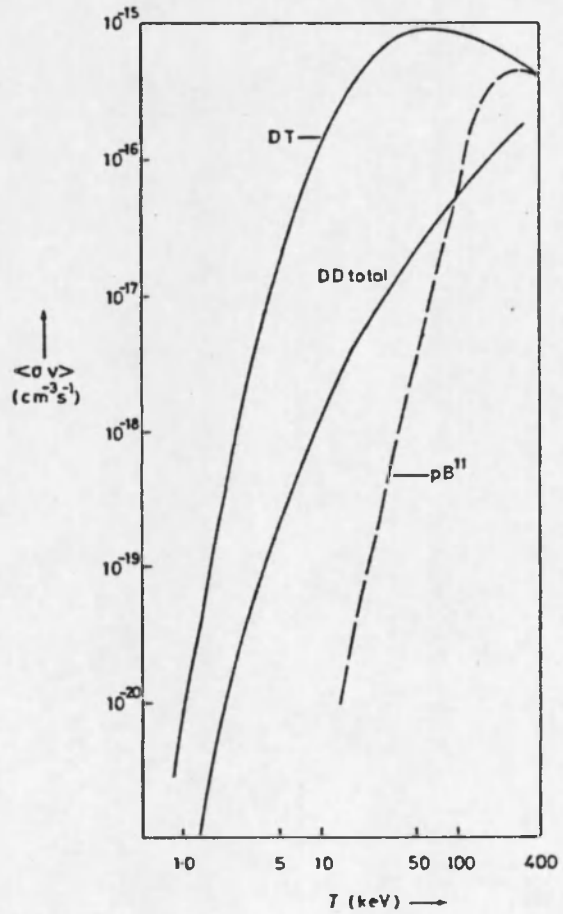

Fig. 2.1 Thermonuclear Reaction Rates. -- values of $\langle\sigma v\rangle$ for D-T, D-D and $p\text{-}B^{11}$ reactions for Maxwellian velocity distributions at temperature T [Ref. 4].

The D-D curve is the sum of the two reactions described by equations (2.1). From Fig. 2.1, it is easily seen that the probability of a D-T reaction at the lower temperature is higher and this reaction is more desirable.

2.3 Ignition Temperatures

The power which is produced by the D-T reactions per unit volume (cm^3) in each reaction is [Ref. 3]:

$$P_r = n_D n_T \langle \sigma v \rangle W$$

where $\langle \sigma v \rangle$ is the averaged cross section for a D-T reaction, and W is the 17.6 Mev of energy which is released in each reaction. To maintain the plasma temperature, this power must exceed that which is lost. When the electrons make elastic collisions with the ions, the radiation, called Bremstrahlung, is emitted due to the accelerated electrons. The Bremstrahlung power is given by [Ref. 3, Eq. (9.7)]:

$$P_b = 5 \times 10^{-31} Z^2 n^2 (KTe)^{\frac{1}{2}}_{kev}$$

P and P_b are both functions of n^2, but P_r increases more rapidly than P_b does with KT. In order to initiate the reaction, it is obvious that P_b would be equal to P_r, and this equality will give us the ignition temperature assuming that there is enough time to transfer the energy from the produced ions to the other ions and the electrons by coulomb collisions, so that all the temperature of the charged particles are equal. For example, the ignition temperature for D-T reaction and D-D reaction are about 4 Kev and 35 Kev, respectively [Ref. 3]. In

addition, the higher temperature is required for the high-Z reactions of Eq. (2.2).

2.4 The Lawson Criterion

To produce more energy by fusion than is required to heat the plasma and supply the radiation losses imposes a condition on plasma n and confinement time (t_c), as well as on the temperature. It turns out that n and t_c occur only in the product $n\tau$. This is called the Lawson Criterion [Ref. 5].

The product of the density and the life time of the plasma (t_c) should exceed a value determined only by the temperature, the efficiency of energy conversion (H), and the proportion of nuclear energy retained in the plasma (K). Figure 2.2 below shows nt_c versus the temperature which satisfies the Lawson Criterion for the D-T, D-D and P - B^{11} reactions [Ref. 6].

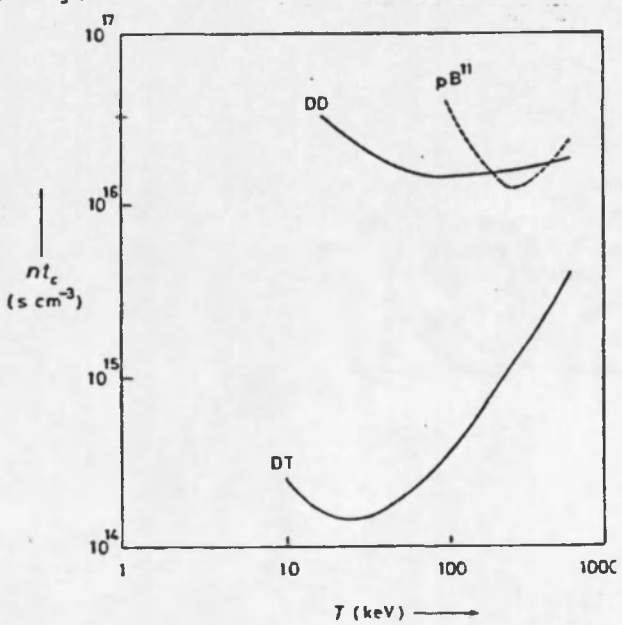

Fig. 2.2 Lawson Criterion. -- values of nt_c satisfying the Lawson Criterion for D-T, D-D and p-B^{11} reactions with K=½ as functions of temperature (n is total density of nuclei) [Ref. 6].

One can see from the above figure that the minimum value of nt_c for the D-T reaction is about 10^{14} scm^{-3} and occurs at a temperature of about 24 Kev. For the D-D and P-B^{11} reactions the minimum values are one hundred times greater and occur at temperatures greater than 100 Kev. Thus the D-T reaction is a more desirable reaction, in the laboratory, than any other reactions; it is more practical. It is possible to lower these figures by combining beams with plasmas, as by more sufficient energy recovery, such as direct conversion to electricity.

2.5 Magnetic Containment

Charged particles move in a helical orbit around magnetic lines of force. Although they are moving freely along the magnetic lines of force, they are restrained from moving across them. Thus, the exerting field of pressure at right angles to the lines of force are equal to $B^2/8\pi$ in order to balance the kinetic pressure of the plasma. Although the very high temperatures are needed for thermonuclear reactions to take place at a useful rate, only the relatively low densities ($n \leq 10^{17}$) can be restrained in this way with available fields ($\sim 10^5$ G). In order to satisfy the Lawson Criterion, one needs long containment times ($\tau_c \sim$ millisecond or longer) for the thermonuclear reactions to take place. But it turns out that the containment times are limited by diffusion of particles across the magnetic field lines, and by the growth of instability. Many fusion systems are proposed, such as [Refs. 2, 3]:

1. External helical conductor: Stellarators
2. Internal plasma current: Tokamaks

3. Internal conductor: Multipoles

4. Internal particle beams: The Astron

5. Pinches

2.5.1 Pinches

The simplest fusion devices, which are proposed thus far, are the pinches. There are two different geometries: the Z-pinch and the θ-pinch (Fig. 2.3). In pinches, a plasma carrying a current is confined by the magnetic field which is produced by its own current. An increase in current forces the magnetic field to become more dense as to compress the plasma and to heat it. Therefore, the confinement and heating occur together. In general, there are magnetic mirrors at both ends of the pinch so as to reduce the end losses. A θ-pinch shows a remarkable degree of stability, as can be seen from Fig. 2.4 [Ref. 3, p. 306]. The circular fringes indicate the constant density, however, the plasma does not migrate to the walls as it decays [Ref. 3]. This observed when the collision rate is so high that mirror confinement is not efficient. Thus, long-linear θ-pinches have been constructed to increase the time of flight of ions to the ends. The long θ-pinch density is considered for our calculation, but the external magnetic field around the plasma column varies sinusoidally with the condition that the plasma columns of interest, have a large ratio of length to radius. The external magnetic field varies as a function of time and direction and it is given by:

$$B_{est} = |B_1| e^{i(kZ-\omega t)}$$

where k is the wave number and ω is the frequency of the plane wave.

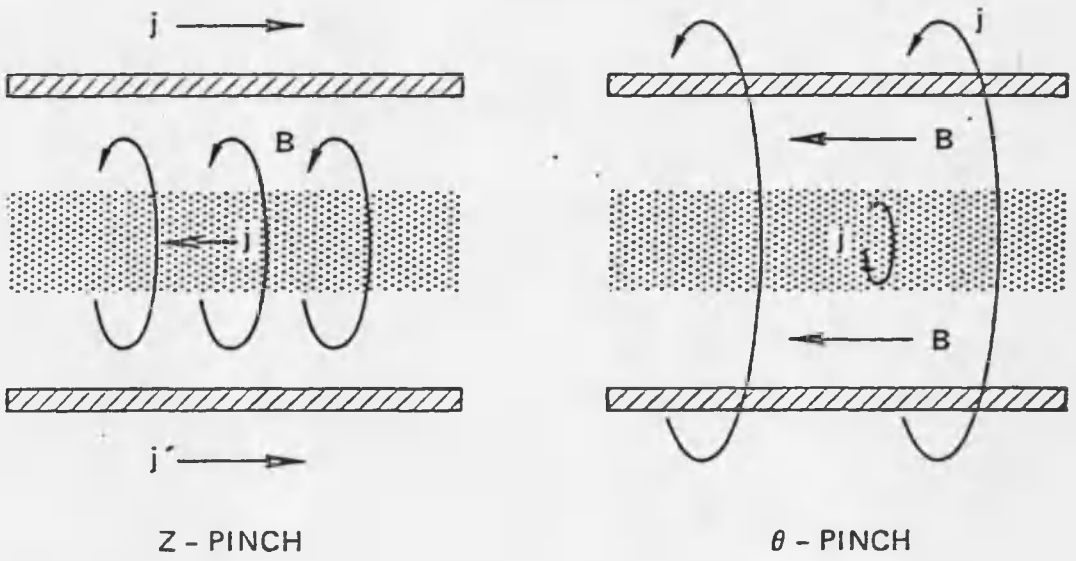

Fig. 2.3 Pinch Configurations. -- geometry of a z-pinch (left) and a θ-pinch (right) [Ref. 3, 305].

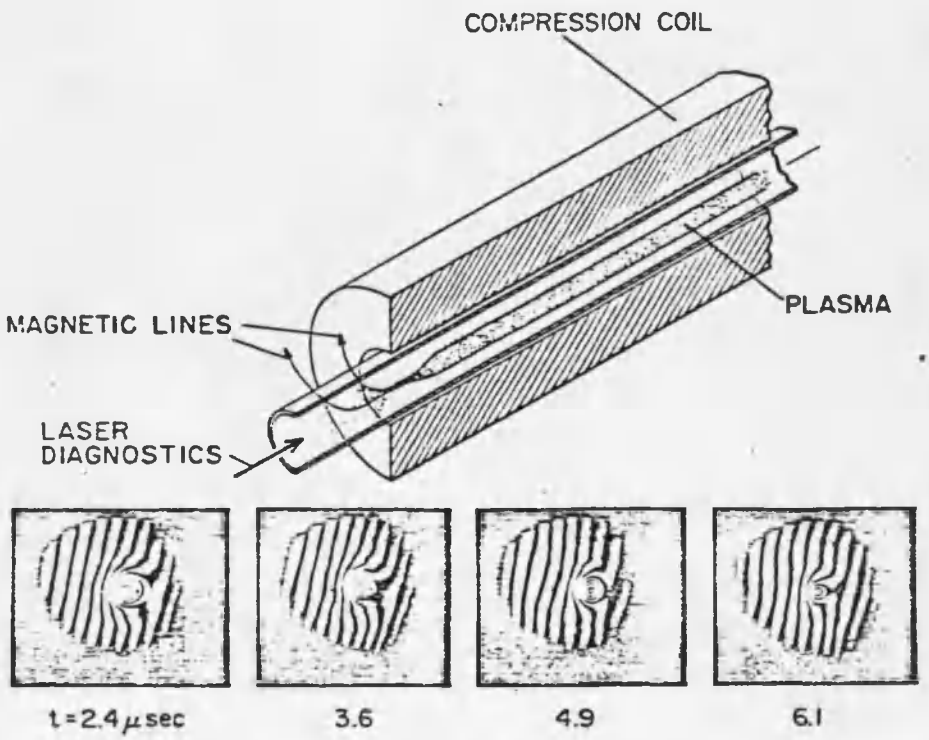

Fig. 2.4 Diagram of a Linear θ-pinch. -- the laser interferograms below it show that the plasma decays in density with time but remains stably confined [Ref. 3, p. 305].

CHAPTER 3

ANALYTICAL METHODS FOR THE DERIVATION OF THE THEORY OF AXIAL HEATING

3.1 Introduction

The primary concept analyzed in the present work is the use of the equations of continuity, motion, and energy for later use. In this chapter we will develop an expression for the work which is produced in the long-linear θ-pinch (fusion system) by using the external magnetic field (due to the locally adiabatic radial compressions). In developing this expression the use of perturbation analysis and non-linear heating theory (to calculate the PdV work) is used in our calculations.

3.2 Explanation of Theory

This section will deal with the theory that applies to our calculation. As noted in Chapter 2, the long-linear θ-pinch is considered as a fusion device, and the plasma columns in this fusion device have a large ratio of length to radius [Ref. 1]. Because of this condition, non-adiabatic plasma flows can occur along the length of the plasma column as a result of locally adiabatic radial compressions. For example, the external magnetic field changes as a function of time t, and axial length Z. The radius of the plasma column is $r = r_0 - r_1 \sin(2\pi t/T_p) \cos(2\pi Z/\lambda_p)$ where $0<r_1<r_0$. We assume that the plasma is radially, azimuthally homogeneous everywhere along the length of the plasma column, therefore, there is no restriction on β, the ratio of

plasma to magnetic pressure. The maximum column radius r_0 can be close to the tube radius, as we see from the expressions below:

if $\sin(2\pi t/T_p) = 1$ such that $Z = 0, \lambda_p, 2\lambda_p \ldots$, the plasma is most compressed.

if $\sin(2\pi t/T_p) = 1$ such that $Z = \lambda_p/2, 3\lambda_p/2, \ldots$, the plasma is most uncompressed.

if $\sin(2\pi t/T_p) = -1$ such that $Z = 0, \lambda_p, 2\lambda_p, \ldots$, the plasma is most uncompressed.

if $\sin(2\pi t/T_p) = -1$ such that $Z = \lambda_p/2, 3\lambda_p/2, \ldots$, the plasma is most compressed.

Suppose C_s is the ion sound speed along the field lines, if $\lambda_p/C_s \cong T_p$ time between maxima and minima of compression is of the order of the primary period and if $r_0 << \lambda_p$ as we assumed before, then the radial compressions will be locally adiabatic. As a result, the "sloshing" motion along the column in the Z direction will be near XX resonance ψ; therefore of large amplitude, and the heating due to dissipative processes will be seen to be correspondingly maximized. The process primarily depends on T_p and λ_p in order to optimize the heating; i.e., T_p would need to be decreased (or λ_p increased) as the plasma is heated and C_s increases. There are two ranges of λ_p and T_p with respect to the electron thermal conductivity as follows:

a) If $\lambda_p >> L$ (the length of the plasma column), then a simple one-temperature fluid approximation applies.

b) If $\lambda_p << L$, and if thermal conduction is fast enough and $T_p >> t_{eq}$, then the electron temperature may remain independent of Z. Thus, in this case, the electrons do not participate in the

non-adiabatic heating but add to the pressure which must be overcome in compressing the plasma column.

c) If $\lambda_p \simeq L$, the electrons do participate in the non-adiabatic heating, but there is some thermal conduction in Z; $T_p \simeq t_{eq}$ (the electron - ion energy equalization time) and a single temperature model is inapplicable.

A full dynamic simulation is required to determine the axial heating rate for the above cases. However, the case b) above can be modified analytically and gives a reasonable estimate of the effectiveness of the method and the scope of this thesis presents the analytical model for the estimation of the axial heating rate along the plasma column.

3.3 Thermal Conductivity

The thermal conductivity which will be used in the fluid equations for the electrons and ion heat fluxes parallel and perpendicular to a magnetic field can be easily determined from Braginskii [Ref. 7] and Spitzer [Ref. 8]. It should be noted that in motion across a magnetic field ($\omega\tau >> 1$), a particle is displaced an order of the Larmor radius rather than the usual mean free path [Ref. 7, p. 228]:

$$(\omega x)_1 \sim r \sim V/\omega$$

where

r = Larmor radius (cm)

V = velocity of electron and ion (cm/sec.)

ω = the cyclotron frequency of electron as ion ($sec.^{-1}$)

Therefore, the thermal conductivity perpendicular to the direction of motion is given by [Ref. 7, p. 228]

$$K_{\perp} \sim nr^2/\tau \sim nT/m\omega^2\tau \qquad (3.3.1)$$

where

n = no. density of electron or ion ($\#/cm^3$)

r = Larmor radius (cm)

τ = the electron or ion collision time (sec)

ω = the cyclotron frequency of electron or ion (sec^{-1})

On the other hand, as a particle moves freely along the field, the displacement of a particle is

$$(\Delta x)_{\parallel} \sim \lambda_{mfp} \sim V\tau$$

where

λ_{mfp} = the mean free path of electron or ion (cm)

V = the velocity of electron or ion (cm/sec.)

τ = the electron or ion collision time (sec.)

so that the thermal conductivity in the direction of motion is givey by [Ref. 7, p. 28]

$$K_{\parallel} \sim n\lambda_{mfp}^2/\tau \sim nT\tau/m \qquad (3.3.2)$$

where

n = no. density of electron or ion ($\#/cm^3$)

λ_{mfp} = the mean free path of electron or ion (cm)

τ = the electron or ion collision time (sec.)

T = the temperature of electron or ion (K^o)

m = the mass of electron or ion (gas)

$$K_{\parallel}/K_{\perp} \sim (\omega\tau)^2 >> 1 \qquad (3.3.3)$$

These estimates apply for both ions and electrons. It seems that K is much greater than $K_{\perp}$ for both electron and ion for strong plasma ($\omega t >> 1$). We assume that $T_e \cong T_i$, therefore, the electron thermal conductivity in the direction of the field is greater than the ion conductivity, and it is expressed as [Ref. 7]:

$$K^e_{\parallel}/K^i_{\parallel} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} >> 1 \qquad (3.3.4)$$

where

m_i = the ion mass (gm)

m_e = the electron mass (gm)

On the other hand, the ion thermal conductivity is greater in the transverse direction as follows [Ref. 7]:

$$K^e_{\perp}/K^i_{\perp} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} << 1 \qquad (3.3.5)$$

We assume that the motion of the waves inside the pinch is in the direction of the field, therefore, Eq. (3.3.4) shows that the ion thermal conductivity could be ignored with respect to the electron thermal conductivity, and it can be expressed as:

$$K^e_{\parallel} >> K^{ion}_{\parallel} \qquad (3.3.6)$$

In our calculation, whenever we use the thermal conductivity, this is the electron thermal conductivity. As for a Lorentz gas (the ions are treated as infinitely massive during collisions) the value of K is [Ref. 8, Eq. (5.4)]

$$K_L = 20 \left(\frac{2}{\pi}\right)^{3/2} \frac{(kT)^{5/2}\, k}{m e^{1/2} e^4\, Z \ln\Lambda}$$

$$= 4.67 \times 10^{-12} \frac{T^{5/2}}{Z \ln\Lambda} \quad \frac{CaL}{sec.\ deg.\ cm}$$

where

k = Boltzman constant

T = the electron or the ion temperature (K^o)

Z = the no. of proton or electron in the atom

$\ln\Lambda$ = Debye logarithm

Values of $\ln\Lambda$ (Debye Logarithm) for an electron and proton gas are given by Spitzer [Ref. 8]. In our case, we assume $\ln\Lambda = 10$. While ε (the correction to the coefficient of conductivity in the presence of the secondary electric field) is 0.40. For an actual gas K becomes [Ref. 8, Eq. (5.45)]

$$K = \delta_T \; K_L$$

Values of δ_T and ε are given by Spitzer [Ref. 8] for different values of Z in Table 3.1.

Table 3.1 Values of δ_T and ε

Z	1	2	4	16	∞
δ_T	0.225	0.356	0.513	0.791	1.000
ε	0.419	0.410	0.401	0.396	0.400

In our case Z = 1, therefore, the value of δ_T is 0.225.

3.4 Viscosity

The viscosity of a plasma in a magnetic field is complicated since it is a tensor quantity; but making some basic assumptions we could make the viscosity of a plasma a scalar quantity along the magnetic field which can be used in our problem. It is easily shown that this tensor vanishes if the plasma undergoes a uniform isotopic compression, that is to say, this tensor vanishes if the volume element of the plasma are not deformed.

First assume that the velocity of an electron or ion is along the Z-axis, and it has a derivative in the same direction; the momentum flux associated with the electron or ion velocity is of the same order as in the absence of the field, and is given by:

$$\Pi_{ZZ} \sim -\eta_{\parallel} \frac{\partial V_Z}{\partial_Z}$$

where η = viscosity along the field

$\frac{\partial V_Z}{\partial_Z}$ = the plasma gradient along the field

In addition, the viscous stress is expressed as:

$$F_Z = (\partial/\partial_Z)\, \eta_{\parallel} \left(\frac{\partial V_Z}{\partial_Z} \right)$$

Since the longitudinal momentum is freely transported along the magnetic field.

In contrast with the case of thermal conductivity, where the electron conductivity is greater along the field (for $\omega_i \tau_i >> 1$), when

$T_e \sim T_i$ the ion viscosity is always much larger than the electron viscosity [Ref. 7, p. 231]:

$$\eta_{\parallel}^{i} \sim \left(\frac{m_i}{m_e}\right)^{1/2} \eta_{\parallel}^{e} \qquad (3.4.1)$$

For this reason the viscosity of a plasma is determined essentially by the ions.

Suppose now that the velocity is changing in a direction perpendicular to the magnetic field. In this case momentum is transferred across the magnetic field where a particle is displaced as an order of Larmor radius ($\Delta x \sim r$). The following relation, therefore, is obtained [Ref. 7, p. 231]:

$$\frac{\eta_{\parallel}}{\eta_{\perp}} \sim (\omega\tau)^2 >> 1$$

$$\eta_{\perp}^{i} << \eta^{i}$$

For this reason the momentum flux along the other directions perpendicular to the magnetic field is generally negligible with respect to the momentum flux along the field.

The viscosity of a fully ionized gas has been analyzed by Braginskii [Ref. 7]. In the absence of a magnetic field the coefficient if viscosity is given by [Ref. 8, Eq. (5-54)

$$\eta = \frac{0.406\, m_i^{1/2} (kT)^{5/2}}{Z^4 e^4 \ln \Lambda}$$

$$= 2.2.1 \times 10^{-15} \frac{T^{5/2} A_i^{1/2}}{Z^4 \ln \Lambda} \quad \frac{gm}{cm\text{-}sec}$$

where A_i is the atomic weight of the positive ions and T is in Kelvin. The viscosity is due primarily to positive ions; viscous stresses are due to electrons and are generally negligible.

3.5 The Equation of Continuity, Motion and Energy

The three equations of fluid which are the equations of continuity, momentum and energy can be written in cylindrical geometry as follows [Ref. 9]:

The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial_r} (\rho r V_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho V_\theta) + \frac{\partial}{\partial_Z} (\rho V_Z) = 0 \qquad (3.5.1)$$

The equation of motion including viscosity (η) along the Z-axis is:

$$\rho \frac{\partial V_Z}{\partial t} + V_r \frac{V_Z}{\partial_r} + \frac{V_\theta}{r} \frac{\partial V_Z}{\partial \theta} + V_Z \frac{\partial V_Z}{\partial Z} = - \frac{\partial P}{\partial Z} + \eta \left[\frac{1}{r} \frac{\partial}{\partial_r} \left(r \frac{\partial V_Z}{\partial_r} \right) + \frac{1}{r^2} \frac{\partial^2 V_Z}{\partial \theta^2} + \frac{\partial^2 V_Z}{\partial_Z{}^2} \right] \qquad (3.5.2)$$

The energy equation is:

$$\rho \frac{\partial E}{\partial t} = \left\{ - \rho RT + \eta \left[\frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{\partial V_Z}{\partial Z} \right] \right\} \cdot \left[\frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{\partial V_Z}{\partial Z} \right] + \left[\frac{1}{r} \frac{\partial}{\partial r} (rK \nabla T) + \frac{\partial}{\partial Z} (K \nabla T) \right] \qquad (3.5.3)$$

In order to make the equation of fluid easier to solve, we will make some basic assumptions (for our case). First, assume that there is no motion in θ-direction ($V_\theta \to 0$), as a result, the continuity equation is written as:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{\partial}{\partial Z}(\rho V_Z) = 0 \tag{3.5.4}$$

Second, assume that the fluid velocity along the Z-axis does not vary with radius and angle ($V_Z \neq f(r,\theta)$). As a result, the equation of motion is written as:

$$\rho\left(\frac{\partial V_Z}{\partial t} + V_Z\frac{\partial V_Z}{\partial Z}\right) = -\frac{\partial P}{\partial Z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial V_Z}{\partial r} + \frac{\partial^2 V_Z}{\partial Z^2}\right] \tag{3.5.5}$$

The temperature gradient in three dimension is

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{\partial T}{\partial Z}\hat{Z} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\phi}$$

For simplicity, we assume that there is just an axial heating rate along the Z-direction, therefore, the fluid temperature does not vary with radius and angle ($T \neq f(r,\theta)$). Using this assumption in Eq. (3.5.3), the energy equation reduces to the following expression:

$$\rho\frac{\partial E}{\partial t} = \left\{-\rho RT + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_Z}{\partial Z}\right]\right\}\cdot\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_Z}{\partial Z}\right] + \frac{\partial}{\partial Z}\left(K\frac{\partial T}{\partial Z}\right) \tag{3.5.6}$$

It is frequently more convenient to use a single-fluid model for the plasma. In this case, the two equations of motion (for the ions and electrons) are replaced by a single equation of motion for the plasma as a whole. In addition, the single-fluid hydrodynamic model is very useful to describe the low frequency phenomena since it is valid to neglect electron inertia and to assume that the plasma remains neutral; this does not mean zero space charge in the plasma, but it means that the space charge is small compared with the quantity en_e, so that the difference $Zn_i - n_e$ can be neglected compared with $n = n_e$. Therefore, the electric force $Fe = e(\neq n_i - n_e)\vec{F}$ is neglected in the equation of motion. We assume that all processes are slow so that the equations of continuity for the electrons and ions for the conservation of electric charge can be neglected, therefore, div $J = 0$. Also the displacement of the electric field can be neglected in Maxwell's equations (div $\vec{E} = 0$) because of the neutrality condition. However, if the thermal coupling between the electrons and ions is strong, the relative temperature difference is small $|T_e - T_i| \ll T$ and one can use the combined equation, writing $T_e = T_i = T$. In addition, the equation of the fluid could be simplified further, if we use the linear theory which imposes all the other terms with the second order as above will vanish.

Applying all the assumptions which we have made, and using the results of the conductivity and viscosity sections, the fluid equations for a plasma could be written as:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\dfrac{1}{r}\dfrac{\partial}{\partial r}\ (\rho r V_r) - \dfrac{\partial}{\partial Z}\ (\rho V_Z) \\ \\ \rho\left[\dfrac{\partial V_Z}{\partial t} + V_Z \dfrac{\partial V_Z}{\partial t}\right] = -\dfrac{\partial P}{\partial Z} + \eta^i_{\parallel}\left[\dfrac{\partial^2 V_Z}{\partial Z^2}\right] \\ \\ \rho\left[\dfrac{dE}{dt}\right] = -\rho RT\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\ (rV_r) + \dfrac{\partial V_Z}{\partial Z}\right] + K^e_{\parallel}\left[\dfrac{\partial^2 T}{\partial Z^2}\right] \end{cases}$$

where

η^i = the ion viscosity along the magnetic field $(\frac{gm}{cm\text{-}sec})$

K^e = the electron thermal conductivity in the direction of the field (erg/Sec-dez-cm)

$T \simeq T_e \simeq T_i$ (K^o)

P = the plasma pressure (erg/cm^3)

V_Z = the motion of a plasma along the Z-axis (cm/sec)

ρ = the plasma density (gm/cm^3)

The above equations are the most simplified relations which are used in our calculation.

3.6 Solutions to the Fluid Equations Using the Perturbation Theory

The fluid equations can easily be solved by the procedure of lineraization. Here the amplitude of oscillation is small, and the terms containing higher powers of amplitude factors can be neglected. We first separate the dependent variables into two terms; an "equilibrium" term indicated by a subscript 0, and a "perturbation" term indicated by a prime:

$$\rho = \rho_0 + \rho'$$

$$T = T_0 + T' \qquad (3.6.1)$$

$$r = r_0 + r'$$

where the oscillating quantities are assumed to behave sinusoidally:

$$\rho' = \rho_1 \, e^{i(kZ-wt)}$$

$$T' = T_1 \, e^{i(kZ-25)} \qquad (3.6.2)$$

$$r' = r_1 \, e^{i(kZ-wt)}$$

The fluid equations obtained from the last section are givey by:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\dfrac{1}{r}\dfrac{\partial}{\partial r}(\rho r V_r) - \dfrac{\partial}{\partial Z}(\rho V_Z) \\ \rho\left(\dfrac{\partial V_Z}{\partial t} + V_Z \circ \dfrac{\partial V_Z}{\partial Z}\right) = -\dfrac{\partial P}{\partial Z} + \eta_{\parallel}^{i}\left[\dfrac{\partial^2 V_Z}{\partial Z^2}\right] \\ \rho\left(\dfrac{dE}{dt}\right) = -R\rho T\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(rV_r) + \dfrac{\partial V_Z}{\partial Z}\right] + K_{\parallel}^{e}\left[\dfrac{\partial^2 T}{\partial Z^2}\right] \end{cases} \qquad (3.6.3)$$

For simplicity, assume that the radial motion [V(r,t)] is proportional to the radius by the following equation:

$$V(r,t) = V_{r_0}(t) \circ \frac{r}{r_0} \quad (0<r<r_0) \qquad (3.6.4)$$

where

$$V_{r_0}(t) = V_r \, e^{-i\omega t}$$

r_0 = the plasma radius (cm)

Introducing Eq. (3.6.4) in the continuity equation, we get

$$\frac{\partial \rho}{\partial t} = -2\left[\frac{\rho}{r_0}\right] V_{r_0}(t) - \frac{\partial}{\partial Z}(\rho V_Z) \qquad (3.6.5)$$

Using the following relations

$$\rho = \rho_0 + \rho'$$

$$V_{r_0}(t) = V_{a0} + V_a' \tag{3.6.6}$$

$$V_Z = V_{Z0} + V_Z'$$

in Eq. (3.6.5), the equation is written as:

$$\frac{\partial(\rho_0+\rho')}{\partial t} = -\frac{2}{r_0}\left[(\rho_0+\rho') \circ (V_{a0}+V_a')\right] - \frac{\partial}{\partial Z}\left[(\rho_0+\rho') \circ (V_{Z0}+V_Z')\right] \tag{3.6.7}$$

In equilibrium, when the external magnetic field is not applied on the system, there is no motion in the radial and the Z-axis. This means $V_{Z0} = 0$, $V_{a0} = 0$, and $\frac{\partial \rho_0}{\partial t} = 0$. In addition, the time derivative $(\partial/\partial t)$ of the perturbed terms can be replaced by $-i\omega$ and the gradient by ik where

ω = the frequency of the plasma wave (sec^{-1})

k = the wave number of the plasma motion (cm^{-1})

ω and k are the same as ω_p (the pumping frequency) and k_p (the pumping wave length), but for simplicity, the subscript p has been droped in this chapter. Because of the linearity, the following terms approach zero,

$$\rho' V_a' \to 0$$

$$\rho' V_Z' \to 0$$

Thus Eq. (3.6.7) becomes

$$- i\omega\rho' = - 2 \left(\frac{\rho_0}{r_0}\right) V_a' - (ik\ \rho_0)\ V_Z' \tag{3.6.8}$$

In addition, the perturbed velocity in the radial direction is expressed as:

$$V_a' = \frac{dr'}{dt} = - i\omega r' \tag{3.6.9}$$

Now, we introduce Eq. (3.6.9) in Eq. (3.6.8), the following relation can be obtained:

$$\omega\rho' = - 2 \left(\frac{\rho_0}{r_0}\right)(\omega)\ r' + k\rho_0\ V_Z' \tag{3.6.10}$$

It is easier to work with the dimensionless equations, therefore, the relative perturbed continuity equation could be written as:

$$\frac{\rho'}{\rho_0} = - 2 \left(\frac{r'}{r_0}\right) + \left(\frac{k}{\omega}\right) V_Z' \tag{3.6.11}$$

Consider now the equation of motion,

$$\rho \left(\frac{\partial V_Z}{\partial t} + V_Z \cdot \frac{\partial V_Z}{\partial Z}\right) = - \frac{\partial P}{\partial Z} + \eta_{\parallel}^{i} \left[\frac{\partial^2 V_Z}{\partial v^2}\right] \tag{3.6.12}$$

where

P = the plasma pressure (erg/cm^3)

= R ρ T

R = the gas constant (erg/gm. K^o)

Applying the perturbation analysis of Eq. (3.6.12), we obtain the following expression,

$$(\rho_0+\rho') \cdot \left[\frac{\partial V_Z'}{\partial t} + V_Z' \cdot \frac{\partial V_Z'}{\partial Z}\right] = - R \cdot \frac{\partial}{\partial Z} \left[(\rho_0+\rho') \cdot (T_0+T')\right] + \eta_{\parallel}^{i} \cdot \left[\frac{\partial^2}{\partial Z^2} (V_Z')\right] \tag{3.6.13}$$

Using the linear theory and rearranging, the following result is obtained:

$$(-i\omega\rho_0+k^2\eta_{\parallel}^{i}) \cdot V_Z' = - (ikR\rho_0T_0) \cdot \left[\frac{T'}{T_0} + \frac{\rho'}{\rho_0}\right] \tag{3.6.13a}$$

The relative perturbed momentum equation is:

$$\left[\frac{\omega^2}{k^2RT_0} + \frac{i\eta^{i}\omega}{R\rho_0T_0}\right] \left(\frac{k}{\omega}\right) V_Z' = \left(\frac{T'}{T_0}\right) + \left(\frac{\rho'}{\rho_0}\right) \tag{3.6.14}$$

We now consider the energy equation which is written as follows:

$$\rho \cdot \left(\frac{dE}{dt}\right) = - (R\rho T) \cdot \left[\frac{1}{r} \frac{\partial}{\partial r} (rV_r) + \frac{\partial V_Z}{\partial Z}\right] + K_{\parallel}^{e} \left(\frac{\partial^2 T}{\partial Z^2}\right) \tag{3.6.15}$$

where

$$E = C_V T \text{ (erg)}$$

C_V = the specific heat at constant volume (erg/gm. K^o)

Introducing Eq. (3.6.4) in Eq. (3.6.15), the following equation is obtained:

$$\rho C_V \left(\frac{dT}{dt}\right) = - R\rho T \left[2\left(\frac{V_{r_0}(t)}{r_0} + \frac{\partial V_Z}{\partial Z}\right)\right] + K_{\parallel}^{e} \frac{\partial^2 T}{\partial Z^2} \tag{3.6.16}$$

Applying the linear theory of Eq. (3.6.16), we obtain the following expression,

$$(-i\omega C_V \rho_0)T' = -\left[\frac{2R\rho_0 T_0}{r_0}\right] \cdot V_a' - (ikR\rho_0 T_0)V_Z' - (k^2 K^e) \cdot T' \tag{3.6.17}$$

where

$$V_a' = -i\omega r'$$

or

$$(i\omega\rho C_V - k^2 K_{\parallel}^e) \cdot \left(\frac{T'}{T_0}\right) = -(2i\omega R\rho_0) \cdot \left(\frac{r'}{r_0}\right) + (i\omega R\rho_0)\left(\frac{k}{\omega}\right) \cdot V_Z' \tag{3.6.18}$$

The relative perturbed energy equation is written as follows:

$$\left(\frac{C_V}{R} + \frac{ik^2 K^e}{R\omega\rho_0}\right) \cdot \left[\frac{T'}{T_0}\right] = -2\left(\frac{r'}{r_0}\right) + \left(\frac{k}{\omega}\right) \cdot V_Z' \tag{3.6.19}$$

All the perturbed fluid equations which are dimensionless can be written as follows:

$$\begin{cases} \left(\dfrac{\rho'}{\rho_0}\right) = -2\left(\dfrac{r'}{r_0}\right) + \left(\dfrac{k}{\omega}\right) \cdot V_Z' \\ \left(\dfrac{\omega^2}{k^2 R T_0} + \dfrac{i\eta_{\parallel}^i \omega}{R\rho_0 T_0}\right)\left(\dfrac{k}{\omega}\right) V_Z' = \left(\dfrac{T'}{T_0}\right) + \left(\dfrac{\rho'}{P_0}\right) \\ \left(\dfrac{C_V}{R} + \dfrac{ik^2 K_{\parallel}^e}{R\omega\rho_0}\right) \cdot \left(\dfrac{T'}{T_0}\right) = -2\left(\dfrac{r'}{r_0}\right) + \left(\dfrac{k}{\omega}\right) V_Z' \end{cases} \tag{3.6.20}$$

We now want to find the relations between the relative perturbed temperature and the relative perturbed density as a function of the relative perturbed radius.

For simplicity, we define

$$\begin{cases} X = \dfrac{\omega^2}{k^2 RT_0} + i \cdot \left[\dfrac{\eta_{\parallel}^{i}\,\omega}{R\rho_0 T_0}\right] \\ Y = \dfrac{C_V}{R} + i\left[\dfrac{k^2 K_{\parallel}^{e}}{R\omega\rho_0}\right] \end{cases} \tag{3.6.21}$$

The new set of equations can be written as follows:

$$\begin{cases} \dfrac{\rho'}{\rho_0} = -2\left[\dfrac{r'}{r_0}\right] + \left[\dfrac{k}{\omega}\right] V_Z' & (3.6.22a) \\ (X)\left[\dfrac{k}{\omega}\right] V_Z' = \left[\dfrac{T'}{T_0}\right] + \left[\dfrac{\rho'}{P_0}\right] & (3.6.22b) \\ (Y)\left[\dfrac{T'}{T_0}\right] = -2\left[\dfrac{r'}{r_0}\right] + \left[\dfrac{k}{\omega}\right] V_Z' & (3.6.22c) \end{cases}$$

Combining Eq. (3.6.22a) with Eq. (3.6.22b) and the result for the relative perturbed density is given as:

$$(X-1)\left[\frac{\rho'}{\rho_0}\right] = -2\,(X)\left[\frac{r'}{r_0}\right] + \left[\frac{T'}{T_0}\right] \tag{3.6.23}$$

Similarly, combining Eq. (3.6.22a) with Eq. (3.6.22c), the following result is obtained:

$$(Y)\left[\frac{T'}{T_0}\right] = \left[\frac{\rho'}{\rho_0}\right] \tag{3.6.24}$$

From Eq. (3.6.23) and Eq.(3.6.24), we obtain the following equation,

$$\left(X - \frac{1}{Y} - 1\right)\left[\frac{\rho'}{P_0}\right] = -2(X)\left[\frac{r'}{r_0}\right] \tag{3.6.25}$$

Upon substitution of Eq. (3.6.24) into Eq. (3.6.25), we obtain the following expression,

$$(X - \frac{1}{Y} - 1)\left(\frac{T'}{T_0}\right) = -2\left(\frac{X}{Y}\right)\left(\frac{r'}{r_0}\right) \tag{3.6.26}$$

where X and Y are complex number. As can be seen, Eq. (3.6.25) and Eq. (3.6.26) indicate the direct relationship between both the relative perturbed plasma density and temperature as a function of the relative perturbed plasma radius which is inhibited as a result of locally adiabatic radial compressions.

3.7 Solution to the Pressure Balance Using the Perturbation Theory

Consider now the long-linear θ-pinch system, and the external magnetic field imposed outside the system which are made to change as a function of time and axial length. Assume that there is also present a uniform axial magnetic field inside the plasma and the condition of a presure balance at the surface of the plasma that requires, therefore,

$$P_{ext} = P_{plasma} + P_{int} \tag{3.7.1}$$

where

P_{ext} = the external magnetic pressure (erg/cm^3)

P_{plasma} = the plasma pressure (erg/cm^3)

P_{int} = the internal magnetic pressure (erg/cm^3)

The external magnetic pressure is given by the following expression:

$$P_{ext} = \left(\frac{1}{8\pi}\right) B_{ext}^2 \tag{3.7.2}$$

where

B_{ext} = the external magnetic field (gauss)

In addition, the internal magnetic pressure is,

$$P_{int} = \frac{1}{8\pi} (B_{int})^2 \tag{3.7.3}$$

where

$$B_{int} = \frac{\Phi_{int}}{\pi r^2} \quad 0<r<r_0) \tag{3.7.4}$$

Φ_{int} = the internal flux in plasma $[(erg\text{-}cm)^{\frac{1}{2}}]$

When the internal magnetic pressure is known, the internal flux can be obtained as:

$$\Phi_{int} = (8\pi)^{\frac{1}{2}} \circ (P_{int})^{\frac{1}{2}} \circ (\pi r^2) \quad (0<r<r_0) \tag{3.7.5}$$

Applying the perturbation analysis in the external magnetic pressure, one can obtain the following expression,

$$P_{ext0} + P_{ext}' = \frac{1}{8\pi} (B_0 + B')^2 \tag{3.7.6}$$

Using the linear theory in Eq. (3.7.6), the following expressions are obtained:

$$\begin{cases} P_{ext0} = \frac{1}{8\pi} B_0^2 & \text{(unperturbed external magnetic pressure)} \\ P_{ext}' = \left(\frac{1}{4\pi}\right) \left(B_0 \circ B'\right) & \text{(perturbed external magnetic pressure)} \end{cases} \tag{3.7.7}$$

Similarly, the following expression for the internal magnetic pressure is true (as long as the internal magnetic flux is kept constant in a plasma):

$$P_{int0} + P_{int}' = \left(\frac{1}{8\pi}\right) \circ \left(\frac{\Phi_{int0}}{\pi r_0^2}\right)^2 \circ \left[\frac{1}{1+\frac{2r'}{r_0}}\right]^2 \tag{3.7.8}$$

For simplicity if $X = \frac{2r'}{r_0}$, then the following relation is true:

$$\left(\frac{1}{1+X}\right)^2 = 1 - 2X + 3X^2 - 4X^3 + 5X^4 - \ldots$$

when $X \ll 1$, similarly,

$$\left(\frac{1}{1+\frac{2r'}{r_0}}\right)^2 = 1 - 4\left(\frac{r'}{r_0}\right) + 12\left(\frac{r'}{r_0}\right)^2 - \ldots \qquad (3.7.9)$$

Using the linear theory and Eq. (3.7.9), therefore, Eq. (3.7.8) can be separated as follows:

$$\begin{cases} P_{int0} = \left(\frac{1}{8\pi}\right) \cdot \left[\frac{\Phi_{int0}}{\pi r_0^{\,2}}\right] & \text{(unperturbed internal magnetic pressure)} \qquad (3.7.10a) \\ P_{int}' = -4 \cdot (P_{int0}) \cdot \left(\frac{r'}{r_0}\right) & \text{(perturbed internal magnetic pressure)} \qquad (3.7.10b) \end{cases}$$

Using the same perturbation procedure in the plasma pressure, we get:

$$P_{pla0} + P_{pla}' = R(\rho_0 T_0 + \rho_0 T' + T_0 \rho') \qquad (3.7.11)$$

Similarly, Eq. (3.7.11) can be separated as:

$$\begin{cases} P_{pla0} = R\rho_0 T_0 & \text{(unperturbed plasma pressure)} \\ P_{pla}' = P_{pa0} \cdot \left(\frac{T'}{T_0}\right) + \left(\frac{\rho'}{\rho_0}\right) & \text{(perturbed plasma pressure)} \end{cases} \qquad (3.7.12)$$

The condition of the perturbed or the unperturbed pressure balance at the surface of the plasma that requires, therefore,

$$\begin{cases} P_{ext0} = R\rho_0 T_0 + \left(\frac{1}{8\pi}\right)\left[\frac{\Phi_{int}}{\pi r_0^2}\right]^2 & (3.7.13a) \\ P_{ext}' = P_{pla0} \cdot \left[\left(\frac{T'}{T_0}\right) + \left(\frac{\rho'}{\rho_0}\right)\right] - 4 \cdot (P_{int0}) \cdot \left(\frac{r'}{r_0}\right) & (3.7.13b) \end{cases}$$

One can define β as:

$$\beta = \frac{P_{pla0}}{P_{ext0}} \qquad (3.7.14)$$

therefore,

$$1 - \beta = \frac{P_{int0}}{P_{ext0}} \qquad (3.7.15)$$

Upon substitution of Eq. (3.7.15) into Eq. (3.7.10b), the relation below is obtained:

$$P_{int}' = -4(1-\beta) \cdot P_{ext0} \cdot \left(\frac{r'}{r_0}\right) \qquad (3.7.16)$$

Hence from the substitution of Eq. (3.7.14) and Eq. (3.7.16) into Eq. (3.7.13b), the relative external perturbed pressure can be expressed as:

$$\frac{P_{ext}'}{P_{ext0}} = \beta \cdot \left(\frac{T'}{T_0} + \frac{\rho'}{\rho_0}\right) - 4(1-\beta) \cdot \left(\frac{r'}{r_0}\right) \qquad (3.7.17)$$

3.8 Relation Between the Relative Perturbed Radius and the External Pressure

The three equations which were obtained in the previous sections, are as follows:

$$\begin{cases} \left(x - \frac{1}{y} - 1\right)\left(\frac{\rho'}{\rho_0}\right) = -2(x)\left(\frac{r'}{r_0}\right) & (3.8.1) \\ x - \frac{1}{y} - 1\left(\frac{T'}{T_0}\right) = -2\left(\frac{x}{y}\right)\left(\frac{r'}{r_0}\right) & (3.8.2) \\ \frac{P'_{ext}}{P_{ext0}} = \beta\left[\frac{T'}{T_0} + \frac{\rho'}{\rho_0}\right] - 4(1-\beta) \cdot \left(\frac{r'}{r_0}\right) & (3.8.3) \end{cases}$$

where

$$x = \frac{\omega^2}{k^2 R T_0} + \frac{i\eta_{\parallel}^{i}\omega}{R\rho_0 T_0}$$

$$y = \frac{C_v}{R} + \frac{ik^2 K_{\parallel}^{e}}{R\omega\rho_0}$$

One could see the relative perturbed temperature, density, and external magnetic pressure are all related to the relative perturbed radius. Therefore, from substitution of Eq. (3.8.1) and Eq. (3.8.2) into Eq. (3.8.3), we obtain:

$$\frac{P'_{ext}}{P_{ext0}} = \left\{-2 \cdot x \cdot \beta\left[\frac{1+y}{xy-y-1}\right] - 4(1-\beta)\right\} \cdot \left(\frac{r'}{r_0}\right) \qquad (3.8.4)$$

Define α as:

$$\alpha = -2 \cdot x \cdot \beta\left[\frac{1+y}{xy-y-1}\right] - 4(1-\beta) \qquad (3.8.5)$$

Eq. (3.8.4) can then be written as:

$$\frac{P'_{ext}}{P_{ext0}} = (\alpha) \cdot \left(\frac{r'}{r_0}\right) \qquad (3.8.6)$$

α should be defined as a complex number since x and y are complex numbers. Moreover, Eq. (3.8.6) is a very useful relation for defining how much work is done on the system. α being a complex number, P'_{ext} and r' are phase dependent as follows:

$$\begin{cases} P'_{ext} = |P_1|\, e^{i(kZ-\omega t + \theta_p)} & (3.8.7) \\ r' = |r_i|\, e^{i(kZ-\omega t + \theta_r)} & (3.8.8) \end{cases}$$

Upon substitution of Eq. (3.8.7) and Eq. (3.8.8) in Eq. (3.8.6), we obtain:

$$\exp\left[i(\theta_p-\theta_r)\right] = \left(\frac{\alpha}{r_0}\right) \cdot \left(\frac{|r_1|}{|P_1|}\right) \cdot P_{ext0} \qquad (3.8.9)$$

where α is a complex number, therefore,

$$\alpha = \varepsilon + \delta \qquad (3.8.10)$$

where ε and δ are related to the electron conductivity and the ion viscosity. In exponential form α is

$$\alpha = (\varepsilon^2 + \delta^2)^{\frac{1}{2}} \cdot \exp\ i\tan^{-1}(\delta/\varepsilon) \qquad (3.8.11)$$

$(\theta_p-\theta_r)$ and $|r_1|$ can be expressed by the following expressions:

$$\begin{cases} \theta_p-\theta_r = \tan^{-1}(\delta/\varepsilon) & (3.8.12) \\ |r_1| = |p_1| \cdot r_0 \cdot |\partial|^{-1} \cdot (P_{ext0})^{-1} & (3.8.13) \end{cases}$$

where

$$P_1 = \left(\frac{1}{4\pi}\right)(B_0 \cdot B')$$

$$P_{ext0} = \left(\frac{1}{8\pi}\right)(B_0^2)$$

In fact, B' is known from the initial condition, and $|\alpha|$ can be calculated from Eq. (3.8.5). Equation (3.8.12) and Eq. (3.8.13) are very important because the difference of the phase angle $(\theta_p-\theta)$ and the magnitude of the perturbed radius (r_1) could be directly calculated from the complex number α, where the values of $(\theta_p-\theta)$ and $|r_1|$ will be used for the calculation of work which is done on the system as a result of locally adiabatic radial compressions. In other words, the electron conductivity and ion viscosity in one-temperature fluid approximation have exhibited the change of the phase angle between the external perturbed pressure and the perturbed radius $(\theta_p-\theta_r)$ which imposes work will be done on the system by the locally adiabatic radial compression.

3.9 Non-Linear Heating Theory

Our interest in systems of fusion have a large ratio of length to radius; because of this condition, non-adiabatic plasma flows can occur along the length of plasma as a result of locally adiabatic radial compressions. However, the acoustic transit time (λ_p/C_s) between the maximum and minimum of compression is of the order of the pumping period (T_p), and if $r_0 \ll \lambda_p$, then the radial compressions will be locally adiabatic but the resulting "sloshing" back and forth along the Z-axis will be nonadiabatic because of the wave damped by the electron thermal conductivity and the ion viscosity (and thus irreversible), and heating will occur.

The change of work is expressed as follows:

$$\Delta W = - P_{ext} \cdot \Delta V \qquad (3.9.1)$$

where

$$p_{ext} = \text{external magnetic pressure } (\text{erg/cm}^3)$$

$$\Delta_v = \pi r^2 \, \Delta Z$$

If one applies the perturbation analysis in Eq. (3.9.1), then the $P_{ext0}\Delta V'$ and $P_{ext0}\Delta V'_2$ terms are zero because of the integration over a cycle. Thus this equation will be

$$\Delta W = - P'_{ext} \cdot \Delta V' \tag{3.9.2}$$

where

$$p'_{ext} = p_1 \sin (k_p Z) \cdot \sin (\omega_p - \theta_p)$$

$$v' = v_1 \sin (k_p Z) \cdot \sin (\omega_p t - \theta_r)$$

Upon integration of Eq. (3.9.2) over one cycle of the plasma column, the work on the system is found to be

$$W = -2\pi r_0 \cdot p_1 \cdot \omega_p \left[\int_0^{r_1} dr' \cdot \int_0^T \sin(\omega_p t - \theta_p) \sin(\omega_p t - \theta_r) dt \right] \cdot \left[\int_0^L \sin^2(k_p Z) dZ \right]$$

which includes only the leading correction to ΔV. After expanding the sine and cosine and using the relation below,

$$\int (\sin^2 ax) \, dx = \frac{1}{2} x - \frac{1}{4a} \sin 2ax$$

the work on the system is:

$$W = + \frac{1}{2} \pi r_0 \cdot p \cdot \omega_p \cdot r_1 \cdot T \cdot L \cdot \sin (\theta_p - \theta_r)$$

where

$$T = \frac{2\pi}{\omega_p} \text{ (sec)}$$

L = the length of the plasma column (cm)

Thus the work is:

$$W = \pi^2 r_0 \, |p_1| \, |r_1| \cdot L \cdot \sin(\theta_p - \theta_r) \tag{3.9.3}$$

Where (results from the previous sections)

$$\alpha = \varepsilon + i\delta$$

$$\theta_p - \theta_r = \tan^{-1}(\delta/\varepsilon)$$

$$|r_1| = |p_1| \, (r_0) \, |\partial|^{-1} \cdot (P_{ext0})^{-1}$$

$$P_1 = P_{ext1} = \left(\frac{1}{4\pi}\right) (B_0 \cdot B')$$

$$P_{ext0} = \left(\frac{1}{8\pi}\right) (B_0^2)$$

The quantity $\sin(\theta_p - \theta_r)$ must be positive if work is to be done on the system. If the total internal plasma energy is $E_T = (3/2) \, P_{pla0} \, (\pi r_0^2) \cdot (L)$, then the relative energy increase is expressed as follows:

$$\frac{\delta E}{E} = (2 \cdot 1) \cdot (|\alpha|^{-1}) \cdot (|P_{ext1}|^2) \cdot (P_{pla0})^{-1} \cdot (P_{ext0})^{-1} \cdot \sin(\theta_p - \theta_r) \tag{3.9.4}$$

The work which is initiated by the resulting "sloshing" back and forth along the Z-axis is obtained by the locally adiabatic radial compressions (dissipative energy is neglected along the radial

direction). Likewise, one can say that the electron thermal conductivity and the ion viscosity cause a phase shift between the external perturbed pressure and the perturbed plasma radius ($\theta_p - \theta_r$) which gives us the work on the system. Moreover, Eq. (3.9.3) and Eq. (3.9.4) which are derived theoretically by using the linear theory, are used in our numerical calculation.

3.10 Considerations of Different Cases in One-Temperature Radial Fluid Approximation

In this section, we assume that there is a radial motion, and the plasma freezes along the magnetic field ($v_Z=0$) when the phase velocity approaches infinity; this means $(\frac{k}{\omega})V_Z'$ could be neglected. Therefore, Eq. (3.6.11) becomes

$$\frac{\rho'}{\rho_0} = -2\left(\frac{r'}{r_0}\right) \tag{3.10.1}$$

Based on our assumption above, the momentum equation along the magnetic field is trivial; this means that there will be no dissipation of energy due to the ion viscosity. In addition, the relative perturbed equation [Eq. (3.6.19] is written as follows:

$$\left(\frac{C_V}{R} + \frac{ik^2 K_{e\parallel}}{R\omega\rho_0}\right)\left(\frac{T'}{T_0}\right) = -2\left(\frac{r'}{r_0}\right) \tag{3.10.2}$$

where

$$\frac{C_V}{R} = \frac{1}{\gamma-1}$$

and $\gamma = 5/3$ is the adiabaticity index for the monatomic gas. Equation (3.10.2) could be rewritten as:

$$1 + \left(\frac{ik^2K^e}{\omega\rho_0 C_V}\right)\left(\frac{T'}{T_0}\right) = -2(\gamma-1)\left(\frac{r'}{r_0}\right) \tag{3.10.3}$$

Besides, the relative external perturbed pressure [Eq. (3.7.17)] can be expressed as:

$$\frac{P'_{ext}}{P_{ext0}} = \beta\left(\frac{T'}{T_0} + \frac{\rho'}{\rho_0}\right) - 4(1-\beta)\left(\frac{r'}{r_0}\right) \tag{3.10.4}$$

where

$$\beta = \frac{P_{pla0}}{P_{ext0}}$$

Upon substitution of Eq. (3.10.1) and Eq. (3.10.3) into Eq. (3.10.4), we obtain the relation as follows:

$$\frac{P'_{ext}}{P_{ext0}} = -(4-2\beta) + \frac{2\beta(\gamma-1)}{1 + i\dfrac{k^2K^e}{\omega C_V\rho_0}}\left(\frac{r'}{r_0}\right) \tag{3.10.5}$$

For $V_Z = 0$, α can be expressed as:

$$\alpha = -(4-2\beta) + \frac{2\beta(\gamma-1)}{1 + i\dfrac{k^2K^e}{\omega C_V\rho_0}} \tag{3.10.6}$$

The phase difference between the external magnetic pressure is given by Eq. (3.8.12) and the magnitude of the perturbed radius is

$$|r_1| = P_{ext1} \cdot r_0 \cdot \left(\frac{1}{P_{ext0}}\right) \cdot (|\alpha|)^{-1}$$

Hence, the work done per one cycle and the negative energy increase are expressed as:

$$W = \pi^2 r_0 |P_{ext1}| \; |r_1| \; L \cdot \sin(\theta_p - \theta_r) \tag{3.10.7}$$

$$\frac{\partial E}{E} = (2.1) \cdot (|\alpha|)^{-1} \cdot (|P_{ext1}|)^2 \cdot (P_{pla0})^{-1} \cdot \sin(\theta_p - \theta_r) \cdot (P_{ext0})^{-1} \tag{3.10.8}$$

In fact, this work is just derived for the radial motion if the plasma freezes along the magnetic field ($V_Z = 0$).

Consider now the extreme cases as follows:

a) if $\frac{k^2 K^e}{\omega \rho_0 C_V} \ll 1$, then α can be written as:

$$\alpha \simeq \{(4-2\beta) + 2\beta(\gamma-1)\}$$

For this case α is a real number, therefore, $\theta_p - \theta_r = 0$. Thus there is no work done on the system.

b) if $\frac{k^2 K^e_{\parallel}}{\omega \rho_0 C_V} \gg 1$, then α can be expressed as:

$$\alpha \simeq - \left\{ (4-2\beta) - \frac{i\, 2\beta(\gamma-1)}{k^2 K^e / \omega \rho_0 C_V} \right\}$$

if $|Im(\alpha)| \ll |Re(\alpha)|$, then there will be very little work done on the system which is true in most cases.

c) if $\frac{k^2 K^e_{\parallel}}{\omega_{CV} \rho_0} = 1$, then the work will be done on the system and this will be the region of interest.

It seems that work depends upon the magnitude of the unitless number $\left(\frac{k^2 K^e}{\omega \rho_0 C_V} \right)$ in the plasma for the radial motion if the plasma freezes

along the magnetic field; when there will be no dissipation of energy due to the ion viscosity.

3.11 Considerations of Different Cases in One-Temperature Fluid Approximation

Before considering and analyzing different cases in one-temperature fluid approximation, one should obtain a very simple relation for a complex number α. The equation of perturbed motion which was obtained in the previous sections Eq. (3.6.20) is written as:

$$\left[\frac{\omega^2}{k^2RT_0} + i\,\frac{\eta_{\parallel}^{i}\omega}{R\rho_0 T_0}\right]\left(\frac{k}{\omega}\right) V_Z' = \left(\frac{T'}{T_0}\right) + \left(\frac{\rho'}{\rho_0}\right) \tag{3.11.1}$$

Eq. (3.11.1) can be rearranged as:

$$\left[1 + i\,\frac{k^2\eta_{\parallel}^{i}}{\omega\rho_0}\right]\left(\frac{k}{\omega}\right) V_Z' = \left(\frac{k}{\omega}\right)^2 \cdot \left[\frac{P_0}{\rho_0}\right] \cdot \left[\frac{\rho'}{\rho_0} + \frac{T'}{T_0}\right] \tag{3.11.2}$$

where

$$\rho_1' = \rho_r' + \rho_Z'$$

$$\text{and} \quad \frac{\rho_r}{\partial_0} = -2\left[\frac{r}{r_0}\right]$$

$$\frac{\rho_Z'}{\rho_0} = \left(\frac{k}{\omega}\right) V_Z'$$

ρ_r' and ∂_0 are defined as above because of the simplicity to the calculation. Equation (3.11.2) can be rearranged as follows:

$$\left[1 - \frac{i\omega\eta^{i}}{P_0} - \left(\frac{\omega}{k}\right)^2\left(\frac{\rho_0}{P_0}\right)\right] \cdot \left(\frac{\rho_Z'}{\rho_0}\right) = -\frac{\rho_r'}{\rho_0} - \frac{T'}{T_0} \tag{3.11.3}$$

If we combine

$$c_{s0}^2 = \gamma \frac{P_0}{\rho_0}$$

and $$\frac{\rho_r'}{\rho_0} = -2 \frac{r'}{r_0}$$

with Eq. (3.11.3), then we obtain:

$$\frac{\rho_Z'}{\rho_0} = \frac{2\frac{r'}{r_0} - \frac{T'}{T_0}}{1 - \frac{i\omega\eta_{\parallel}^i}{P_0} - \left(\frac{\omega}{kc_{s0}}\right)^2 \gamma} \tag{3.11.4}$$

In addition, the relative perturbed balance pressure equation is expressed as:

$$\frac{P'_{est}}{P_{ext0}} = -(4-2\beta)\left(\frac{r'}{r_0}\right) + \beta \frac{T'}{T_0} + \frac{\rho_Z'}{\rho_0} \tag{3.11.5}$$

The relative perturbed energy equation, which was derived previously, is:

$$\left(\frac{C_V}{R} + \frac{ik^2 K_{\parallel}^e}{R\omega\rho_0}\right)\left(\frac{T'}{T_0}\right) = -2\left(\frac{r'}{r_0}\right) + \left(\frac{k}{\omega}\right) V_Z' \tag{3.11.6}$$

In fact, Eq. (3.11.6) can be rewritten as:

$$\left(1 + \frac{ik^2 K_{\parallel}^e}{\rho_0 \omega C_V}\right)\left(\frac{T'}{T_0}\right) = (\gamma-1)\left(\frac{\rho_r'}{\rho_{r0}} + \frac{\rho'}{\rho_0}\right) \tag{3.11.7}$$

For simplicity, let

$$A = 1 - \frac{i\omega\eta_{\parallel}^i}{P_0} - \left(\frac{\omega}{kc_{s0}}\right)^2 \gamma$$

$$B = 1 + i\frac{k^2 K_{\parallel}^e}{\rho_0 \omega C_V}$$

Combining Eq. (3.11.4) and Eq. (3.11.7), one obtains

$$\frac{\rho_z'}{\rho_0} = -(D/C)\frac{\rho_r'}{\rho_0} \tag{3.11.8}$$

and

$$\frac{T'}{T_0} = (1-D/C)\left(\frac{\gamma-1}{B}\right)\frac{\rho_r'}{\rho_0} \tag{3.11.9}$$

where we let

$$C = 1 + \frac{\gamma-1}{AB}$$

$$D = \frac{1}{A} + \frac{\gamma-1}{AB}$$

Upon substitution of Eq. (3.11.8) and Eq. (3.11.9) into Eq. (3.11.5), the relative external perturbed magnetic pressure can be written as:

$$\frac{P'_{ext}}{P_{ext0}} = \{2\beta(D/C-F) - (4-2\beta)\}\left(\frac{r'}{r_0}\right) \tag{3.11.10}$$

where we let

$$F = (1-D/C)\left(\frac{\gamma-1}{B}\right)$$

Assume α can be expressed as:

$$\alpha = 2\beta(D/C-F) - (4-2\beta) \tag{3.11.11}$$

Thus

$$\frac{P'_{ext}}{P_{ext0}} = (\alpha)\cdot\left(\frac{r'}{r_0}\right)$$

α defined from Eq. (3.8.5) is exactly the same as α which is defined from Eq. (3.11.11).

Consider the cases when the ion viscosity approaches zero. For example, if T = 5 Kev, n = 5×10^{16}, and L = 1 Km, then $\frac{\omega\eta_{\parallel}^{i}}{P_0} \simeq .043$ which is much smaller than unity, is negligible in our calculation. As

a result, there will be very little energy dissipation due to the ion viscosity, and the calculated work on relative energy increase is based on the electron thermal conductivity and the radial expansions and compressions assuming process is not adiabatic axially. In addition, we pointed out that the ion radial viscoisty can be neglected in our calculation in a very strong plasma ($\omega t >> 1$).

The extreme cases will be discussed as follows:

a) If $\frac{\omega}{kc_{s0}} <<$, then $A \cong 1$, $F \cong 0$, and $D/C \cong 1$. α can be expressed as $\alpha \cong 4(\beta-1)$ which shows that α is a real number, therefore, $\theta_p - \theta_r = 0$. Thus, there is no work done on the system.

b) If $\left(\frac{\omega}{kc_{s0}}\right)^2 \gamma \cong 1$, then $A \cong 0$, $D/C \cong 1 + \frac{B}{\gamma-1}$, and $F \cong -1$. α can be expressed as:

$$\alpha \cong 2\beta \left[2 + \frac{1}{\gamma-1} + i\,\frac{k^2 K_{\parallel}^e / \rho_0 \cdot C_V \omega}{\gamma-1}\right] - (4-2\beta) \qquad (3.11.12)$$

Let $\beta = .5$ and $\partial = 5/3$, α is

$$\alpha \cong \frac{1}{2} + i\,(3/2) \cdot \left[\frac{k^2 K_{\parallel}^e}{\rho_0 \omega C_V}\right]$$

Now we consider the following cases:

1) If $\frac{k^2 K_{\parallel}^e}{\rho_0 C_V \omega} << 1$, then $\alpha \cong 1/2$. As a result, there is no work done on the system by the external magnetic field.

2) If $\frac{k^2 K_{\parallel}}{\rho_0 \omega C_V} >> 1$, then $\alpha \cong i(3/2)\,\frac{k^2 K_{\parallel}}{\rho C_V \omega}$, as a result, $\sin(\theta_p - \theta_r) = 1$. As can be seen, the work is proportional to the inverse of $|\alpha|$. If $|\alpha| \to \infty$, then $w \to 0$.

3) If $(3)\,\frac{k^2 K_{\parallel}}{\rho C_V \omega} \cong 1$, then $\sin(\theta_p - \theta_r) = \frac{\sqrt{2}}{2}$. As a result, there will be work done on the wystem.

c) If $\frac{\omega}{kc_{s0}} >> 1$, then $D \cong 0$, $C \cong 1$, and $F \cong \frac{\gamma-1}{B}$. α can be expressed as:

$$\alpha \cong -2\beta\left[\frac{\gamma-1}{1+i\,\dfrac{k^2K_{\parallel}^{e}}{\rho_0 C_V \omega}}\right] - (4-2\beta) \qquad (3.11.13)$$

Consider now the following cases:

1) If $\frac{k^2K_{\parallel}}{\rho_0\omega C_V} \ll 1$, then α is a real number, therefore, sin $(\theta_p - \theta_r) \to 0$. As a result, there is no work done on the system.

2) If $\frac{k^2K_{\parallel}}{\rho_0\omega C_V} \gg 1$, then α es expressed as:

$$\alpha \cong 2\beta\left[\frac{i\,(\gamma-1)}{k^2K_{\parallel}^{e}/\rho_0\omega C_V}\right] - (4-2\beta) \qquad (3.11.14)$$

one can see that the imaginary part approaches zero. Thus, very little work is done on the system by the external magnetic field.

3) If $\frac{k^2K_{\parallel}^{e}}{\rho_0\omega C_V} \cong 1$, we have the maximum work that can be done on the system.

It seems the demand of work depends upon the magnitude of the unitless number $\frac{k^2K_{\parallel}}{\rho C_V \omega}$ and $\frac{\omega}{kC_{s0}}$ in the plasma for one-temperature fluid approximation. Furthermore, the numerical results for work will be discussed in the next chapter.

3.12 Conclusion

In this chapter simple analytical expressions were derived for work (W) which is done on the system as a result of locally adiabatic radial compressions and the relative energy increase $(\frac{\partial E}{E})$. In addition, the different cases (of work) for one-temperature fluid radial approximation and one-temperature fluid approximation were discussed.

CHAPTER 4

DISCUSSIONS AND RESULTS

4.1 Introduction

The purpose of this chapter is to explore and to evaluate the heating rate in linear magnetic fuison systems caused by the radial compressions due to an external magnetic field. In the previous chapter an equation was derived for the heating rate. In this chapter the numerical results which were obtained for the heating rate will be discussed and applied. A small code was written to evaluate the analytical expression for relative energy increase Eq. (3.9.4). Furthermore, following the condition that the mean free path is small compared to the characteristic length, in this case $(\lambda_p/2\pi)$, the fluid equations can be applied with the addition of electron thermal conductivity and ion viscosity.

4.2 The Resonance Region

Before anything is said about the numerical results for the relative energy per cycle as a function of the unitless frequency $(\frac{\omega}{kc_s})$, one should consider the resonance region which is important for our calculations. In order to identify the resonance region, consider the case typical when T = 5 Kev and n = 5 x 10^{16} cm^{-3}; then $\frac{\omega\eta^i}{P_0} \cong .043$ which is much smaller than unity, is negligible in our calculation. As a result, there is very little energy dissipation due to the ion viscosity. The relative energy increase is caused by the electron thermal

conductivity and locally adiabatic radial expansions and compressions. As mentioned previously, when viscosity is negligible and thermal conduction is small ($p \ll 1$, see below), resonance is reached near $\frac{\omega}{kc_{s0}} \cong 1$ for $\beta \ll 1$, and the relative energy increase per cycle has its largest value. The region near resonance is, therefore, very important.

4.3 Numerical Results for the Relative Energy Increase

Consider now the relative energy increase as a function of $\frac{\omega}{kc_s}$ for different values of the parameter P which occurs in Eq. (4.3.3) and is defined as:

$$P = \frac{k_p K_{\parallel}^e}{\rho_0 C_v C_s} \tag{4.3.1}$$

In order to relate the electron mean free path or ion mean free path to P, the following expressions from Spitzer [Ref. 8] are used:

$$\begin{cases} \lambda_e = 1.896 \times 10^4 \dfrac{T^2}{Z^2 n_i} \quad \text{(cm)} \\ \lambda_i = 1.896 \times 10^4 \dfrac{T^2}{Z^4 n_i} \quad \text{(cm)} \end{cases}$$

where

T = the plasma temperature (K°)

n_i = the ion number density (cm^{-3})

and

$$\frac{K_{\parallel}^e}{\rho_0 C_v C_s} = 1.806 \times 10^6 \frac{T^2}{Z(1+Z)n_i} \quad \text{(cm)}$$

The relation between $\frac{K^e_{\parallel}}{\rho_0 C_v C_s}$ and the mean free path for Z = 1 is, therefore,

$$\frac{K^e_{\parallel}}{\rho_0 C_v C_s} = (47.62)\ \lambda_{mfp} \tag{4.3.2}$$

Consequently, P can be expressed as:

$$P = (47.62) \cdot k_p \cdot \lambda_{mfp} \tag{4.3.3}$$

where

$$k_p = \frac{2\pi}{\lambda_p}$$

λ_p = the pumping wave length

Values of the relative energy increase per cycle have been calculated as a function of ω/kC_s for different values of P. These results are plotted for different values of P, starting with Fig. 4.1 with the different curves labeled with their values of λk instead of P, through the use of Eq. (4.3.3).

From Fig. 4.1 for $\lambda k \ll 1$, one sees that the relative energy increase around the resonance regions has the highest values, and $(\frac{\omega}{kC_s})$ approaches .96, as it was expected. When β = .1 the maximum relative energies vary between 1 to 8 for different vlaues of $k_p\ \lambda_{mfp}$. In addition, notice that when β increases, the peak will be shifted to the lower values of $(\frac{\omega}{kC_s})$, as is expected; see Figs. 4.1, 4.2, 4.3, 4.4 4.5. Furthermore, when increases, the relative energy also increases at the resonance. From the engineering standpoint, a reasonable value for $\delta P_B/P_B$ which can be indicated on the figures, is 1/3 in order not to let the plasma reach to the walls.

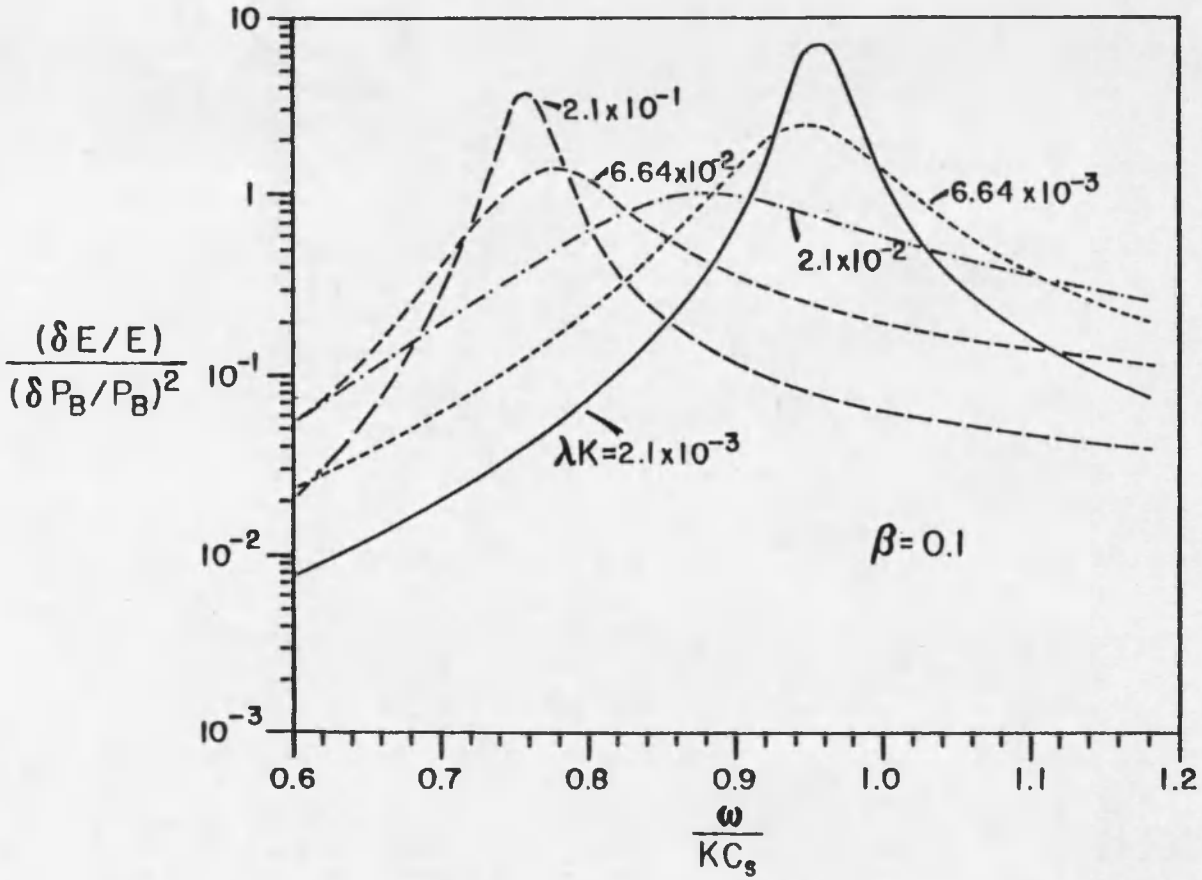

Fig. 4.1 Relative Energy Increase as a Function of Unitless Pumping Frequency for β = 0.1

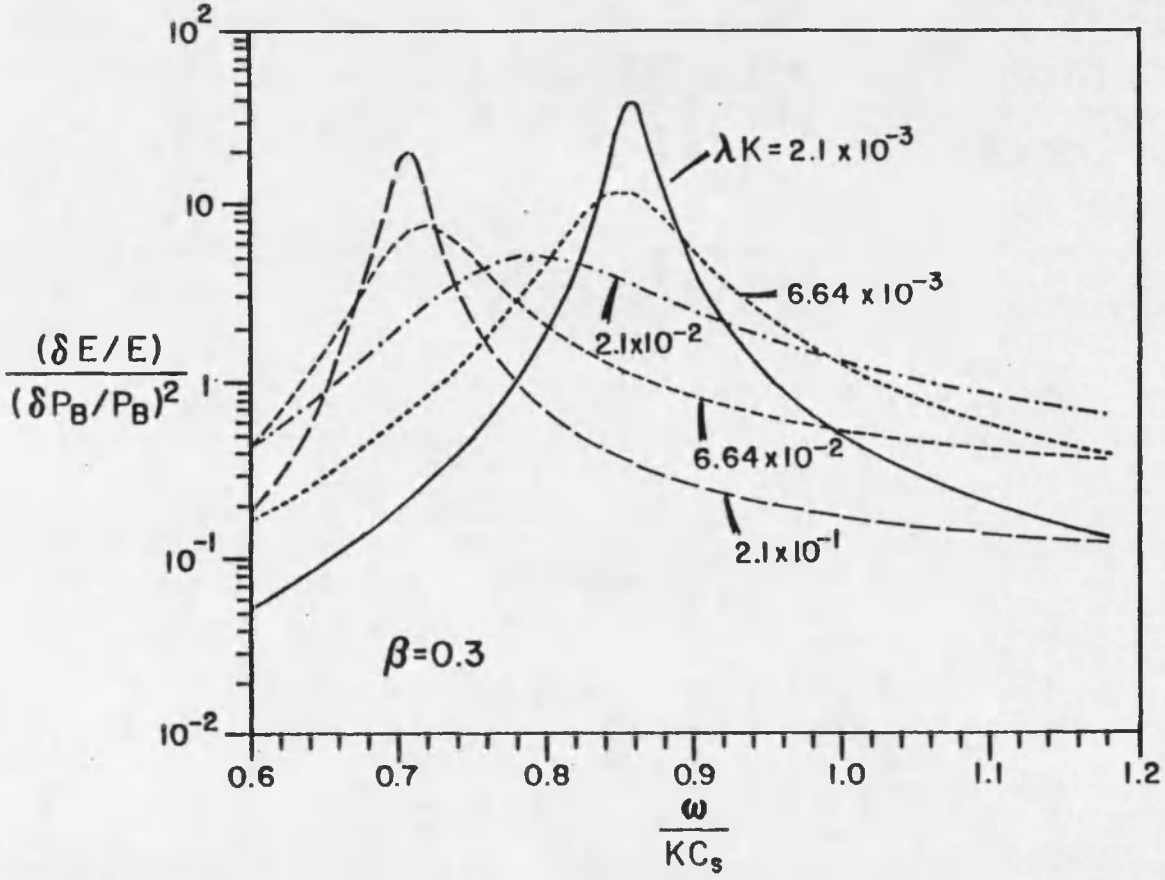

Fig. 4. 2 Relative Energy Increase as a Function of Unitless Pumping Frequency for $\beta = 0.3$

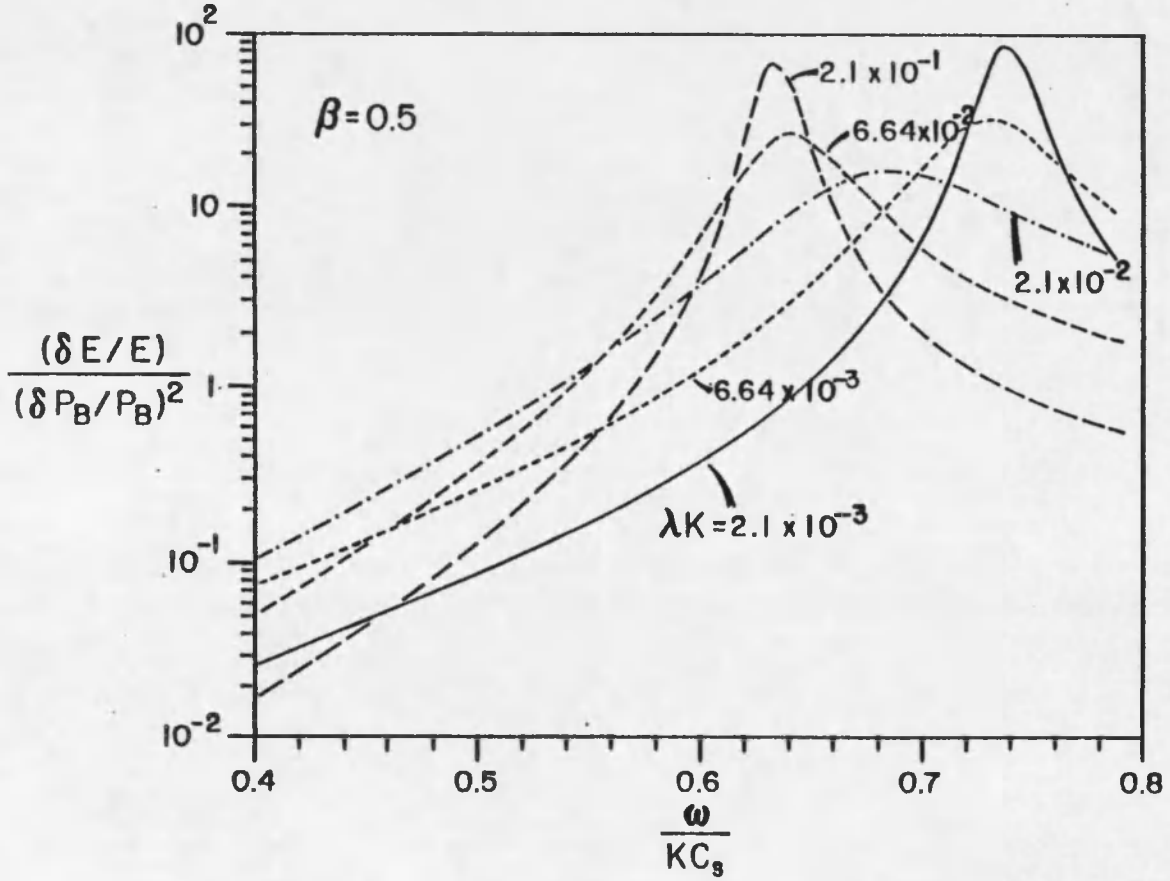

Fig. 4.3 Relative Energy Increase as a Function of Unitless Pumping Frequency for $\beta = 0.5$

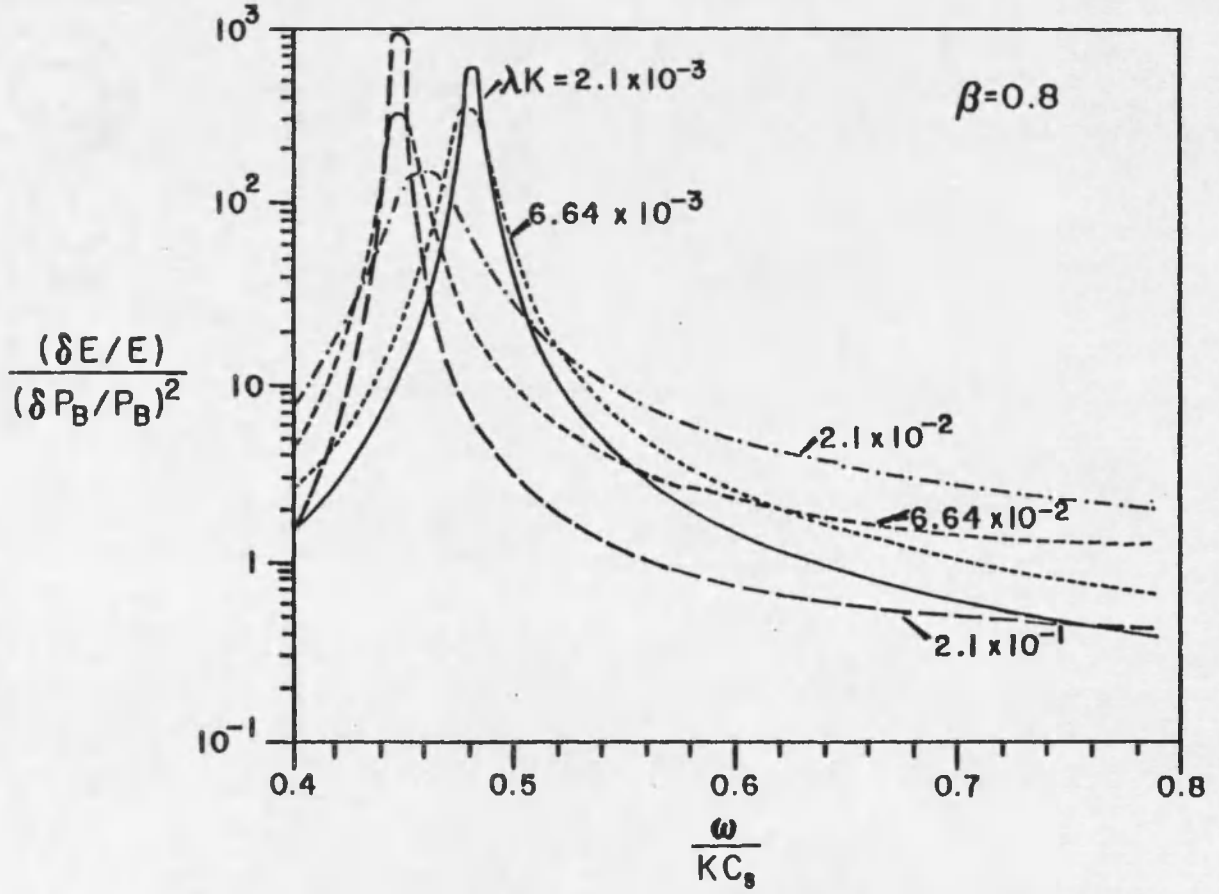

Fig. 4.4 Relative Energy Increase as a Function of Unitless Pumping Frequency for $\beta = 0.8$

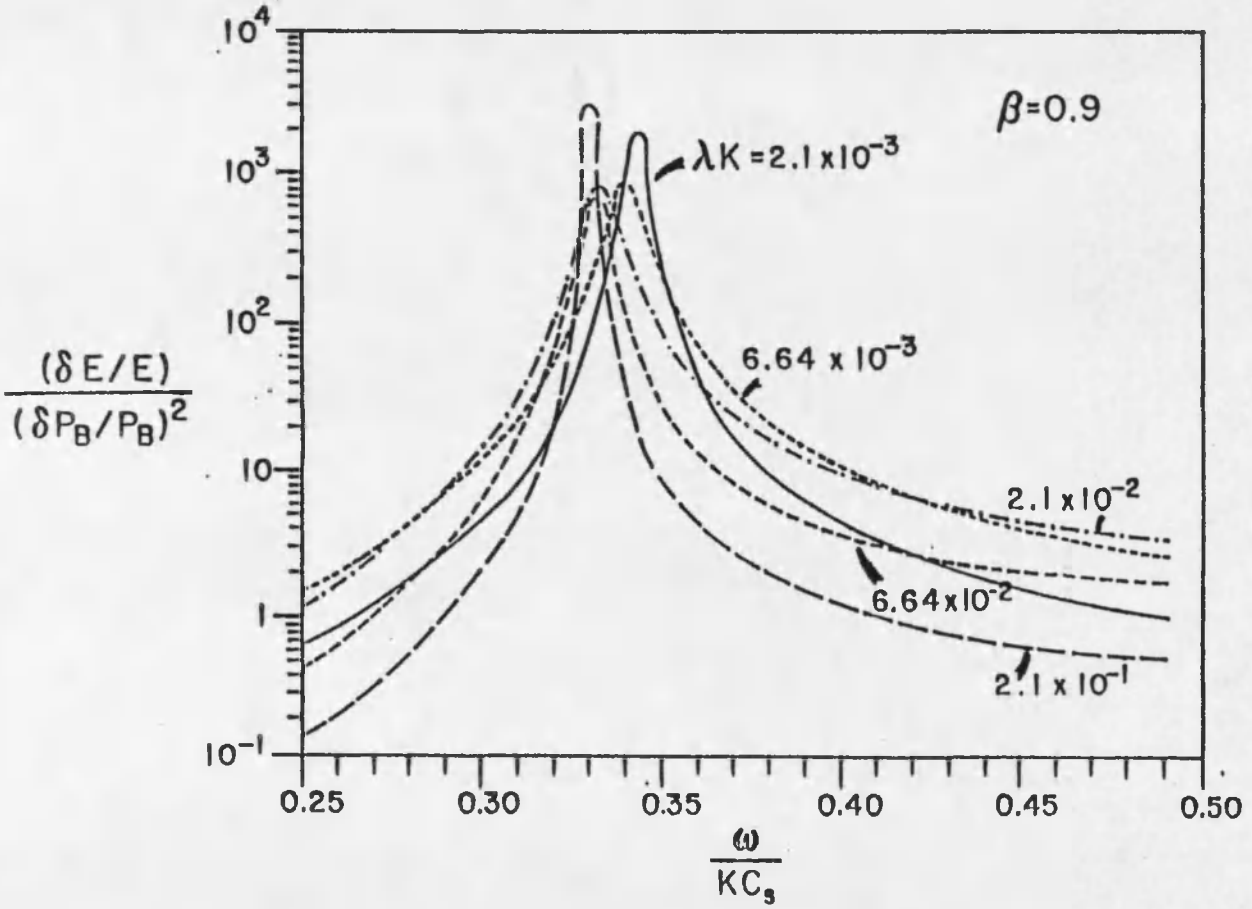

Fig. 4.5 Relative Energy Increase as a Function of Unitless Pumping Frequency for β = 0.9

For example, the examples below will support the theory of axial collisional heating in linear magnetic fusion system and show how our work can be used to estimate design parameters for a linear magnetic fusion reactor.

Example 1:

In this example, the maximum relative energy is evaluated per cycle as a function of an optimum pumping frequency (ω_p) when $\beta = .1$, $T = 500$ ev, and $n = 5 \times 10^{16}$ cm^{-3}. Under these conditions, $\lambda_{mfp} k_p$ is 2.1×10^{-3}. Then from Eq. (4.3.3), $P = 0.1$ and Fig. 4.1 shows that the maximum $(\delta E/E)/(\delta P_B/P_B)^2 \cong 8$ occurs when $\frac{\omega}{kC_s} = .97$. From Eq. (4.3.3), the pumping wave length is 3.8×10^4 cm. The optimum frequency for exhibiting the maximum relative energy on the system by the locally adiabatic compression is, therefore.

$$\omega_p = .97 \cdot k_p \cdot C_s$$

or

$$\omega_p = 4.52 \times 10^3 \ (sec^{-1})$$

If $\frac{\delta P_B}{P_B} = 1/3$, then the maximum relative energy increase, $\delta E/E$, for the optimum frequency is 0.89. That is, the energy increases by a factor of 1.89 per cycle.

Example 2:

In this example, the maximum relative energy is evaluated per cycle at the optimum pumping frequency (ω_p) when $\beta = .5$, $T = 5$ Kev, and $n = 5 \times 10^{16}$ cm^{-3}. Under these conditions, if k_p (the wave number of pumping wave) is kept the same number as in Example 1

$(1.64 \times 10^{-4}\ cm^{-1})$, then $\lambda_{mfp}\, k_p = .21$ and from Eq. (4.3.3) $P = 10.0$. Figure 4.1 shows that the maximum $(\delta E/E)/(\delta P_B/P_B)^2 \simeq 70$ occurs when $\frac{\omega}{k C_s} = 0.63$. From Eq. (4.3.3), the pumping frequency for exhibiting the maximum relative energy on the system by the locally adiabatic compression is therefore,

$$\omega_p = .63 \cdot k_p \cdot C_s$$

or

$$\omega_p = 9.28 \times 10^3\ (sec^{-1})$$

If $\frac{\delta P_B}{P_B} = 1/3$, then the maximum relative energy increase for the optimum pumping frequency is 7.77. That is, the energy increases by a factor of 8.77 per cycle.

Comparing the results of the examples above, the following results are obtianed:

1. Frequency (ω_p) increases for same λ_p (the pumping wave length) with increasing the plasma temperature (T).
2. The maximum relative energy increase with β.
3. Energy increase in both examples is sufficiently large to be useful.

CHAPTER 5

CONCLUSION

A simple method was proposed for performing work on the long θ-pinch system. In this system the plasma columns have a large ratio of length to radius. By using the external magnetic field as a pumping device it is possible to squeeze the plasma radially with a spatial dependence that is sinusoidal along the plasma column.

The fluid equations were used to describe the plasma dynamics under the condition that the temperatures of the electrons and ions in plasma are highly coupled. By coupling the pressure balance condition at the surface of the plasma with the fluid equations and using a perturbation analysis, an analytical expression for the relative energy increase was obtained, and it was examined for different values of β numerically. β is defined as the ratio of plasma pressure to the external magnetic pressure.

Values of the maximum relative energy for two different optimum pumping frequencies (corresponding to two different temperatures and values of β) were obtained, and the results compared to each other. The primary conclusion is that the pumping frequency (ω_p) increases with the plasma temperature and also the maximum relative energy increases with β, for a given λ_p (the pumping wave length). The pumping frequency corresponds to the resonance region. The main results obtained is that energy increase is sufficiently large to be useful.

The results obtained are in qualitative agreement with out anticipations. As a result, it is possible that the background plasma temperature could be increased unitl the necessary conditions for the thermonuclear reactions (i.e., D-T reaction) are satisfied. The results obtained seem to support the utility of axial collisional heating in long linear magnetic fusion systems.

REFERENCES

1. Morse, Richard L., "Electron Temperature and Thermal Conduction in High-Energy θ-Pinch," The Physics of Fluids, 16, No. 4, 1973.

2. Hughes, T. P., Plasma and Laser Light, New York: John Wiley and Sons, 1975.

3. Chen, Francis F., Introduction to Plasma Physics, New York: Plenum Press, 1974.

4. Tuck, J. L., "Thermonuclear Reaction Rates," Nuclear Fusion, 1, p. 201-2, 1961.

5. Lawson, J. D., "Some Criteria for a Power Producing Thermonuclear Reactor," Proc. Phys. Soc., 870, p. 6-10, 1957.

6. Weaver, T., Zimmerman, G. and Wood, L., "Prospect for Exotic Fuel Usage in CTR Systems I," Lawrence Livermore Laboratory Phys., 44, p. 2311-4, 1973.

7. Braginskii, J. D., "Transport Processes in a Plasma," Review of Plasma Physics, New York: Consultants Bureau, 1, p. 205, 1965.

8. Spitzer, Lyman, Jr., Physics of Fully Ionized Gases, New York: Interscience Publishers, Inc., p. 65-94, 1956.

9. Whitaker, Stephen, Introduction to Fluid Mechanics, New Jersey: Princeton-Hill, Inc., 1968.