ANALYSIS AND DESIGN OF AN ON-OFF FEEDBACK CONTROL SYSTEM WITH SAMPLING

By

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A Thesis Submitted to the Faculty of the
DEPARTMENT OF ELECTRICAL ENGINEERING
In Partial Fulfillment of the Requirements For the Degree of
MASTER OF SCIENCE
In the Graduate College
UNIVERSITY OF ARIZONA

1959
STATEMENT BY AUTHOR

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ACKNOWLEDGMENTS

This thesis was written while the author was employed by the Hughes Aircraft Company under their Master of Science program, and they deserve a great deal of credit for their assistance.

The author is also grateful to Professor L. E. Weaver for the assistance and suggestions he gave during the writing of this thesis, to John Hartman for his assistance with the analog computer simulation, and to Frank Perran who programmed the settling time equations on the IBM 650 computer.
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CHAPTER 1
INTRODUCTION

1.1 Description of System

The block diagram of the feedback control system under consideration is shown in Fig. 1.1. Both the system input, \( r(t) \), and output, \( c(t) \), are continuous functions of time. The error channel contains a constant rate sampling and holding device, designated by the switch, whose output is a stairstep, or jump function. The system nonlinearity is an ideal relay with dead zone. The sampler output, \( E^s(t) \), is the relay input. The output of the relay is either zero or pulses of amplitude \( \pm K \) whose periods are variable and equal to integral multiples of the sampling period. The relay output is applied to an integrator and first order lag.

1.2 Statement of Problem

The object of this thesis is to show that the phase plane methods can be used to analyze a system such as that described in section 1.1.

The response of the system to a step function input will be investigated, and settling time, maximum overshoot, and steady state error will be found.

1.3 Previous Work on the Problem

Previous studies of this system by Mullin and Jury\(^1\) indicated that the system could be analyzed using phase plane methods. Their

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Figure 1.1 Block Diagram of System
investigated only the stability of the system.

Izawa and Weaver\textsuperscript{1} investigated a system containing both a sampling and holding device and a relay. They introduced a new approach to the problem, and showed that sampling can be considered as a variable time delay. This approach allowed them to completely describe the system's performance.

1.4 Method of Attack

This thesis shows how the variable time delay concept of Izawa and Weaver can be applied to the system shown in Fig. 1.1. This concept, when used with standard phase plane methods, permits the system's response to a step function to be completely described. Settling time, maximum overshoot, and steady state error are calculated, and stability criterion are established.

Results are presented in a normalized form, so that they are readily applicable to all similar systems.

\textsuperscript{1}K. Izawa and L. E. Weaver, "Relay Type Feedback Control Systems With Dead Time and Sampling," AIEE Applications and Industry, May, 1959, pp. 49-58.
CHAPTER 2
DIFFERENTIAL EQUATIONS OF THE SYSTEM

The differential equations describing the behavior of the system can be written from Fig. 1.1.

\[ E(t) = r(t) - c(t) \quad (2.1) \]

The value of \( m(t) \) is dependent upon the position of the relay, which, in turn, is dependent upon the value of \( E^*(t) \).

\[
m(t) = \begin{cases} 
K & \text{if } E^*(t) > \Delta \\
0 & \text{if } -\Delta < E^*(t) < \Delta \\
-K & \text{if } E^*(t) < -\Delta 
\end{cases} \quad (2.2)
\]

\( m(t) \) is related to \( c(t) \) by the following differential equation.

\[ K, m(t) = T \ddot{c}(t) + \dot{c}(t) \quad (2.3) \]

where \( \dot{c}(t) \) is the time derivative of \( c(t) \).

By combining equations 2.2 and 2.3, an equation describing the system output is obtained.

\[
T \ddot{c}(t) + \dot{c}(t) = \begin{cases} 
K K_1 & \text{if } E^*(t) > \Delta \\
0 & \text{if } -\Delta < E^*(t) < \Delta \\
-K K_1 & \text{if } E^*(t) < -\Delta 
\end{cases} \quad (2.4)
\]

Since the error is a more convenient indicator of the system's
performance than the output, it is desirable to express equation 2.4 in terms of error.

From equation 2.1,

\[ \dot{C}(t) = \dot{r}(t) - \dot{e}(t) \]

\[ \ddot{C}(t) = \ddot{r}(t) - \ddot{e}(t) \]

Substituting these values of \( \ddot{c}(t) \) and \( \dot{c}(t) \) into equation 2.4,

\[ T \dddot{r}(t) + \ddot{r}(t) - T \dddot{e}(t) - \ddot{e}(t) = \begin{cases} K \eta, & \text{if } \dddot{e}(t) > \Delta \\ 0, & \text{if } -\Delta < \dddot{e}(t) < \Delta \\ -K \eta, & \text{if } \dddot{e}(t) < -\Delta \end{cases} \quad (2.5) \]

Since the response of the system to a step input is to be investigated, let \( r(t) \) be a step function applied at time \( t = 0 \). Then at any time \( t > 0 \), \( \dddot{r}(t) = \ddot{r}(t) = \dot{r}(t) = 0 \).

Equation 2.5 then becomes,

\[ T \dddot{e}(t) + \ddot{e}(t) = \begin{cases} K \eta, & \text{if } \dddot{e}(t) > \Delta \\ 0, & \text{if } -\Delta < \dddot{e}(t) < \Delta \\ -K \eta, & \text{if } \dddot{e}(t) < -\Delta \end{cases} \quad (2.6) \]

Equation 2.6 gives three differential equations describing the system. These equations could be solved and a relationship between \( (t) \) and \( t \) could be found. It would be very difficult to study these equations, however, because of the parameters \( T_s, K_\eta \), and \( K_\perp \). It would be
desirable to eliminate these parameters and find an equation that would be applicable to all systems, regardless of the values of $T$, $K_s$ and $K_f$.

This can be done if the equations are normalized by letting

$$t = T \tau$$

Then

$$\frac{dT}{d\tau} = \frac{dT}{d\tau} (\tau) = T$$

or, using operator notation,

$$\frac{d}{d\tau} = T \frac{d}{dt}$$

Then,

$$\frac{d^2}{d\tau^2} = T \frac{d}{dt} \left[ T \frac{d}{dt} \right] = T^2 \frac{d^2}{dt^2}$$

$$\frac{d^2 t}{d\tau^2} = T^2$$

Equation 2.6 then becomes

$$\frac{1}{T} \frac{d^2 \varepsilon}{d\tau^2} + \frac{1}{T} \frac{d\varepsilon}{d\tau} = \begin{cases} 
-KK_1, & \text{if } \varepsilon^{\Phi}(t) > \Delta \\
0, & \text{if } -\Delta < \varepsilon^{\Phi}(t) < \Delta \\
KK_1, & \text{if } \varepsilon^{\Phi}(t) < -\Delta 
\end{cases}$$

Let

$$\frac{\varepsilon(t)}{TKK_1} = \chi(\tau)$$

$$\frac{\Delta}{TKK_1} = a$$
Then

\[ x''(\tau) + x'(\tau) = \begin{cases} 
-1 & \text{if } x^*(\tau) > a \\
0 & \text{if } -a \leq x^*(\tau) < a \\
1 & \text{if } x^*(\tau) \leq -a 
\end{cases} \quad (2.7) \]

These are the normalized system equations. They will be studied in Chapters 3 and 4, and the results of these studies will be applicable to any system of the type shown in Fig. 1.1, irregardless of the values of \( T \), \( K \), and \( K_1 \). Thus, normalization has greatly simplified the system equations, and there has been no loss of generality.
CHAPTER 3

PHASE PLANE ANALYSIS

3.1 Introduction

The phase plane is a plot of system error rate, $x^1$, as a function of system error, $x$. Trajectories in the phase plane indicate the time variations of $x$ and $x^1$. The particular phase trajectory that the system follows is determined uniquely by its initial conditions, $x_0$ and $x^1_0$, and the totality of the phase trajectories is known as the phase portrait of the system. The phase portrait completely describes the system's behavior.

3.2 Effects of Sampling

From equation 2.7 it is seen that the system has three modes of operation, each described by a separate differential equation. The mode in which the system operates is controlled by the amplitude of the sampled error signal, $x^*(t)$.

Since the system has three modes of operation, the phase plane is divided into three regions, each region describing one mode of operation. If there were no sampling, the boundary lines between the three regions would be described by the equation $x=\pm a$, as shown in Fig. 3.1. Each time a phase trajectory reached one of these boundary lines, (called decision lines), the relay would operate and the system would switch to

---

Figure 3.1 Phase Trajectory With No Switching
the next mode of operation.

The insertion of the sampling and holding device in the error channel, however, may introduce a delay in the system such that the relay will not operate at the instant a phase trajectory crosses a decision line.

This can be clarified by considering Fig. 3.2. If the sampling period in real time is $L$, the normalized sampling time can be denoted as $\gamma = \frac{1}{L}$, and the sampling points can be shown as dots on the phase trajectories. Since the value of sampled error is constant between sampling instances, the relay can not operate until the first sampling point after the phase trajectory has crossed a decision line.

Trajectory #1 (Fig. 3.2) shows a case where sampling occurs at the instant the phase trajectory crosses the decision line. In this case the decision line and switching line are identical. In trajectory #3, however, sampling occurs at the instant before the trajectory crosses the decision line. Since the relay can not operate until the next sampling instance, there is a delay of $\gamma$ between the time the trajectory crosses the decision line and the time the relay operates. The switching line determined by this delay shows the extreme boundary between adjacent modes of operation.

Thus the switching lines have two extreme limits, determined by $\gamma$, and switching may occur at any point between these two lines, as illustrated in trajectory #2, Fig. 3.2.
Figure 3.2 Phase Trajectories Showing the Effects of Sampling
3.3 Derivation of Switching Line Equations

The system's three modes of operation will be called the positive mode, the negative mode, and the off mode. These modes of operation are described by the following differential equations:

Positive mode

\[ \chi''(\tau) + \chi'(\tau) = 1 \]  \hspace{1cm} (3.1)

Negative mode

\[ \chi''(\tau) + \chi'(\tau) = -1 \]  \hspace{1cm} (3.2)

Off mode

\[ \chi''(\tau) + \chi'(\tau) = 0 \]  \hspace{1cm} (3.3)

The positive and negative modes can be combined with a ± sign, \n
\[ \chi''(\tau) + \chi'(\tau) = \pm 1 \]  \hspace{1cm} (3.4)

Taking the Laplace transform of equation 3.4

\[ s^2 \chi - s \chi_0 - \chi'_0 + s \chi - \chi_0 = \pm \frac{1}{s} \]

\[ (s^2+s) \chi = \pm \frac{1}{s^2} + (s+1) \chi_0 + \chi'_0 \]

\[ \chi = \pm \frac{1}{s^2(s+1)} + \frac{\chi_0}{s} + \frac{\chi'_0}{s(s+1)} \]

Taking the inverse Laplace transform

\[ \chi(\tau) = \pm (e^{-\tau} + \tau - 1) + \chi_0 + (1 - e^{\tau}) \chi'_0 \]  \hspace{1cm} (3.5)

Taking the derivative of equation 3.5

\[ \chi'(\tau) = e^{-\tau} \chi'_0 \pm (1 - e^{-\tau}) \]  \hspace{1cm} (3.6)
Solving for \( x'_{o} \)

\[
x'_{o} = e^{\tau} x'(\tau) \pm (1 - e^{\tau})
\]

Substituting this value for \( x'_{o} \) in equation 3.5

\[
x(\tau) = x_{o} + (e^{\tau} - 1) x'(\tau) \pm (1 + \tau - e^{\tau})
\]  

(3.7)

Now considering the off mode

\[
x''(\tau) + x'(\tau) = 0
\]

Taking the Laplace transform

\[
S^{2} \hat{x} - S x_{o} - x'_{o} + S \hat{x} - x_{o} = 0
\]

\[
( S^{2} + s ) \hat{x} = ( S + 1 ) x_{o} + x'_{o}
\]

\[
\hat{x} = \frac{x_{o}}{S} + \frac{x'_{o}}{S(S+1)}
\]

Taking the inverse Laplace transform

\[
x(\tau) = x_{o} + (1 - e^{\tau}) x'_{o}
\]  

(3.8)

Taking the derivative of equation 3.8

\[
x'(\tau) = e^{\tau} x'_{o}
\]  

(3.9)

Solving for \( x'_{o} \)

\[
x'_{o} = e^{\tau} x'(\tau)
\]

Putting this value of \( x'_{o} \) in equation 3.8

\[
x(\tau) = x_{o} + (e^{\tau} - 1) x'(\tau)
\]  

(3.10)
Referring to Fig. 3.1a, it can be seen that the system switches from the negative mode to the off mode at the first sampling point after the negative trajectory has crossed the decision line, \( x = a \). The maximum time which can elapse between crossing the decision line and switching is \( \gamma \). Therefore, if the initial conditions of equation \( 3.7 \) are; \( x_0 = a \) and \( \gamma = 0 \); the equation of the extreme switching line may be obtained by setting \( \gamma = \gamma \). (The minus sign must be used in equation \( 3.7 \) to indicate negative mode).

\[
\lambda = a + (e^\gamma - 1)x' - (1 + \gamma - e^\gamma)
\]

To get the equations of all possible switching lines between the decision line and the extreme switching line, a new parameter will be defined;

\[
0 \leq \lambda \leq \gamma
\]  

(3.11)

Thus the equation of the negative-off switching line is

\[
\lambda = a + (e^\gamma - 1)x' - (1 + \gamma - e^\gamma)
\]  

(3.12)

This can be looked upon as a normalized time delay due to sampling which varies randomly within the limits imposed by equation \( 3.11 \). The positive-off line can be similarly derived

\[
\lambda = -a + (e^\gamma - 1)x' + (1 + \lambda - e^\gamma)
\]  

(3.13)

The off-positive switching line takes a slightly different form. Starting with equation \( 3.10 \), let \( x_0 = -a \) when \( \gamma = 0 \). Then let \( \gamma = \lambda \) to get the equation of the off-positive line.
\[ x = -\alpha + (e^{a-1})x' \]  
(3.14)

The off-negative line is similar.
\[ x = \alpha + (e^{a-1})x' \]  
(3.15)

The switching lines divide the phase plane into three regions, as shown in Fig. 3.3. The delay time \( \lambda \) (Fig. 3.3), is the time necessary for the phase trajectory to travel the distance \( d_1 \). \( \lambda_2 \) and \( d_2 \) are similarly related, but \( \lambda_2 \) is not necessarily equal to \( \lambda_1 \). Likewise, \( \lambda_3 \) and \( \lambda_4 \) are related to one another only in that they all satisfy equation 3.11.

### 3.4 Derivation of Phase Trajectories

In the positive mode,
\[ x'' + \kappa' = 1 \]

let
\[ x' = y \]
then
\[ y' + y = 1 \]

Dividing by \( x' \)
\[ \frac{y'}{x'} + \frac{y}{x'} = \frac{1}{x'} \]  
(3.16)

since
\[ y' = \frac{dy}{d\tau} \]
and
\[ x' = y = \frac{dx}{d\tau} \]
Figure 3.3 Phase Trajectory Showing Switching Lines
Then

\[ \frac{y'}{x'} = \frac{dy}{dx} \]

and

\[ \frac{y'}{x'} = 1 \]

Equation 3.16 then becomes

\[ \frac{dy}{dx} + 1 = \frac{1}{y} \]

Separating variables,

\[ dx = \left( \frac{y}{1-y} \right) dy \]

Integrating both sides of the equation,

\[ \chi = -\ln(1-y) + (1-y) + C_1 \quad (3.17) \]

where \( C_1 \) is a constant integration.

Equation 3.17 is the equation of the positive mode phase trajectory.

In the negative mode,

\[ \chi'' + \chi' = -1 \]

Following through as in the case of the positive mode, the phase trajectory of the negative mode can be found.

\[ \chi = \ln(1+y) - (1+y) + C_2 \quad (3.18) \]
In the off mode

\[ \chi'' + \chi' = 0 \]

Let \( x^i = y \) and divide by \( x^i \).

\[ \frac{y'}{x'} + 1 = 0 \]

Separating variables and integrating,

\[ dx = -dy \]

\[ \chi = -y + C_3 \] \hspace{1cm} (3.19)

The system is now completely described in the phase plane, with equations for the trajectories of all three modes, and equations for all four switching lines.
CHAPTER 4
SYSTEM PERFORMANCE

4.1 Stability

It is evident that the off trajectory (equation 3.19) is a straight line with a slope of -1. The positive and negative modes have more complex trajectories, as shown in Fig. 4.1.

The positive trajectories approach the line y = a asymptotically as x tends to infinity, and the negative trajectories approach the line y = -a as x tends to negative infinity.

If the system operates in each of the three modes in sequence, as shown in Fig. 3.3, the phase portrait is a sort of spiral. Since x and y are both decreasing in magnitude with increasing time, the system will come to rest when the phase trajectory reaches the x axis between the decision lines. This corresponds to the system velocity being reduced to zero while the relay is in its dead zone. Since the system can come to a stop at any point within the relay dead zone, the magnitude of the steady state error can be as large as ± a.

In Fig. 4.2, however, the value of γ is so large that the extreme value of the positive-off switching line has intersected the off-negative decision line. The positive phase trajectory has crossed the off-negative decision line before switching, and has switched directly from the positive mode to the negative mode. If the system should continue in like manner, switching directly from negative mode to positive mode, etc., it is possible that a limit cycle would be established. Since a limit
Figure 4.1 Typical Phase Trajectories of Positive and Negative Modes
Figure 4.2 Phase Plane Plot Showing Limit Cycle
cycle represents sustained oscillations, which are usually undesirable, conditions necessary for limit cycles to exist should be found so that they can be avoided in the design of similar systems.

Since the positive trajectory approaches the line $y = 1$ asymptotically, the value of $y$ at its intersection with the off-negative decision line must certainly be less than 1. Therefore, if the point of intersection of the extreme positive-off switching line and the off-negative decision line has an ordinate greater than 1, a limit cycle cannot occur.

The equation of the positive-off switching line is

$$x = a + (e^x - 1)y + (1 + \lambda - e^x)$$

If this line intersects the line $x = a$,

$$a = -a + (e^x - 1)y + (1 + \lambda - e^x)$$

$$y = \frac{2a + e^x - 1 - \lambda}{e^x - 1}$$

if $y > 1$

$$\frac{2a + e^x - 1 - \lambda}{e^x - 1} > 1$$

$$2a + e^x - 1 - \lambda > e^x - 1$$

$$2a > \lambda$$
Thus as long as $\lambda > 2a$, there is no possibility of a limit cycle being established.

4.2 Maximum Settling Time

Fig. 4.3 shows a phase portrait of the response of a system to a step function input. In this phase portrait switching occurs on the extreme switching lines. Even though this is very unlikely, it does give a means for calculating maximum settling time.

Since the system has three modes of operation, the settling time must be calculated piecemeal. For example, in Fig. 4.3 the time necessary for the phase trajectory to move from point 0 to point 1 would be calculated first. Then the time from point 1 to point 2 would be calculated. This would continue until the system came to rest.

Since the switching lines vary randomly between the decision lines and extreme switching lines, it would be difficult to calculate the exact settling time. Therefore, maximum settling time will be calculated by assuming that all switching occurs on the extreme switching lines, as shown in Fig. 4.3. This corresponds to setting $\lambda = \gamma$ in all switching line equations.

Equations expressing $x$ and $y$ as functions of time were found in Chapter 3. The time necessary for the phase trajectory to travel between any two points can be found from the time equations of $x$ and $y$ and the switching line equations.

For example, the phase trajectory in Fig. 4.3 starts in the negative mode at point 0. The time from point 0 to point 1 can be found from the time equations of the negative trajectory.
Figure 4.3 Phase Plane Plot Showing System Response to Step Function Input
\[ x = x_0 + (1 - e^{-\tau})y_0 - (e^{-\tau} + \tau - 1) \]  \hspace{1cm} (3.5) (4.1)

\[ y = e^{-\tau}y_0 - (1 - e^{-\tau}) \]  \hspace{1cm} (3.1) (4.2)

and the equation of the extreme negative-off switching line,

\[ x = a + (e^\gamma - 1)y - (1 + \gamma - e^\gamma) \]  \hspace{1cm} (3.12) (4.3)

Since \( y_0 = 0 \) (point 0, Fig. 4.1) equation 4.1 and 4.2 become

\[ x = x_0 - (e^{-\tau} + \tau - 1) \]

\[ y = e^{-\tau} - 1 \]

then at point 1,

\[ x_1 = x_0 - (e^{-\tau_1} + \tau_1 - 1) \]  \hspace{1cm} (4.4)

\[ y_1 = e^{-\tau_1} - 1 \]  \hspace{1cm} (4.5)

Because point 1 is on the extreme negative-off switching line,

\[ x_1 = a + (e^\gamma - 1)y_1 - (1 + \gamma - e^\gamma) \]

Substituting \( x_1 \) and \( y_1 \) from equations 4.4 and 4.5,

\[ x_1 = (e^{-\tau_1} + \tau_1 - 1) = a + (e^\gamma - 1)(e^{-\tau_1} - 1) - (1 + \gamma - e^\gamma) \]

\[ e^{-\tau_1 + \tau_1} = x_0 + 1 + \gamma - a \]  \hspace{1cm} (4.6)
This equation can be solved for $T_1$.

After $T_1$ is found, $x_1$ and $y_1$ can be found from equations 4.04 and 4.55. $x_1$ and $y_1$ are then used as initial conditions for the off mode which follows the negative mode.

In the off mode, from point 1 to point 2

$$x_2 = x_1 + (1 - e^{-T_2}) y_1 \quad (3.06) \quad (4.07)$$

$$y_2 = y_1 e^{-T_1} \quad (3.04) \quad (4.08)$$

Since point 2 is on the extreme off-positive switching line,

$$x_2 = -a + (e^x - 1) y_2 \quad (3.07) \quad (4.09)$$

Substituting $x_2$ and $y_2$ from equations 4.07 and 4.08 and solving for $T_2$.

$$x_1 + (1 - e^{-T_2}) y_1 = -a + (e^x - 1) y_1 e^{-T_2}$$

$$T_2 = \gamma + \ln \left[ \frac{y_1}{x_1 + y_1 + a} \right] \quad (4.10)$$

This procedure can be continued until the system comes to rest.

It is evident that the system must be operating in the off mode when it comes to rest. For instance, the example shown in Fig. 4.03 comes to rest after the ninth switching point. The time equation for the error rate in the off mode is

$$y = y_0 e^{-\gamma} \quad (4.08)$$
where \( y_0 \) is the value of \( y \) at the beginning of the off mode. This equation shows that it takes an infinite time for the error rate to reach zero. Thus, if defined as the time for the system to come to a complete stop, the settling time will always be infinite.

A more useful definition of settling time is; the time necessary for the error to come within, and stay within, the band \( 0 \pm 2a \). This definition will allow a finite settling time because it does not require that the system come to a complete stop.

Fig. 4.4 illustrates the settling time of the system whose phase trajectory is shown in Fig. 4.3. This figure is a plot of error versus time taken from Fig. 4.3.

Figs. 4.5 through 4.9 show the settling time (\( \tau_s \)) versus step input magnitude for different values of \( a \) and \( \gamma \). These results were obtained by using a digital computer to solve the equations developed in this section.

Given values for \( a \) and \( \gamma \), upper and lower boundaries can be placed on the settling time for any particular \( x_0 \). The lower boundary is the curve for \( \gamma = 0 \) since there is no delay in the system when the sampling period is zero. The upper boundary is dependent on the value of \( \gamma \), and is the maximum settling time that is possible for any given \( x_0 \). The actual settling time of a system may be at any point between these two boundaries.

For example, if \( x_0 = 0.2 \), \( a = 0.01 \), and \( \gamma = 0.02 \), Fig. 4.5 gives the minimum settling time as 2.79 and the maximum settling time as 3.09.

The discontinuities in the curves can be explained with the aid of Fig. 4.4. The settling time will increase gradually as \( x_0 \) increases.
Figure 4.1 Plot of Error versus Time
Figure 4.5
$T_s$ vs $X_0$ with $a = 0.01$
Figure 4.6

$T_s$ vs. $X_0$ with $\alpha = 0.06$
Figure 4.7
$\tilde{T}_s$ vs. $x_0$ with $a = 0.100$

$X_0$ (normalized step input)

$\tilde{T}_s$ (normalized settling time)
Figure 4.8
$T_s$ vs $X_o$ with $\alpha = 0.200$
Figure 4.9
$T_s$ vs $X_0$ for all $a \geq 0.4$
until that value of \( x_0 \) is reached which will cause the curve to go out of the \( 0 \pm 2a \) band near point \( 4 \). Since settling time is defined as the time necessary for \( x \) to settle within the band \( 0 \pm 2a \), \( \tau_s \) would jump from about 1.5 to a value somewhat over 2.0. If \( x_0 \) were made large enough, \( \tau_s \) would make another jump when the overshoot near point 6 became less than \(-2a\).

\[4.3 \text{ Sample Design Problem} \]

The following sample design is presented to demonstrate the use of the curves developed in section 4.2.

Consider the feedback control system shown in Fig. 1.1. Let \( T = 2.0 \) and \( K_1 = 0.5 \). The output is required to settle within \( \pm 10^\circ \) of its steady state value within 12 seconds after a step input of \( 180^\circ \) is applied.

The system's error detector has a sensitivity of 1 volt per degree, and the maximum allowable sampling rate is 20 cps.

The first step in the design is to determine what limitations are imposed by the restriction in sampling rate. Since the sampling rate must be less than 20 cps, the sampling period must be greater than 0.05.

\[ L \geq 0.05 \]

but

\[ L = \frac{\gamma T}{\pi} \]

then

\[ \frac{\gamma T}{\pi} \geq 0.05 \]
\[ \gamma \geq \frac{0.05}{2} = 0.025 \]

since

\[ 0 \leq \gamma \leq 2a \]

\[ a \geq 0.0125 \]

Let \( a = 0.06 \), since the inequality is satisfied for this value. If

\[ \varepsilon = 10^\circ \]

then

\[ \kappa = \frac{\varepsilon}{\gamma_Kk} = \frac{10}{K} \]

Since \( \gamma_s \) is defined as the time necessary for the normalized error to settle within the limits \( 0 \pm 2a \), the following inequality must hold:

\[ \frac{10}{K} \geq 2a \]

\[ K \leq \frac{10}{2a} \]

but \( a = 0.06 \)

\[ K \leq 8.3 \]

Thus it is necessary for \( K \) to be less than 8.3 if the system is to settle within the specified limits.
The following things have now been established:

1. \( a = 0.06 \)
2. \( 0.025 \leq \gamma \leq 0.12 \)
3. \( K \leq 83 \)

Since \( t = T \tau_s \), 12 seconds in real time are equivalent to 6 seconds in normalized time. Thus \( \tau_s \) must be 6 or less.

Because \( a = 0.06 \), Fig. 4.6 gives the proper relationship between \( x_0, \gamma \), and \( \tau_s \).

Let \( \gamma = 0.03 \)

Then (from Fig. 4.6) if \( x_0 \leq 2.4, \tau_s \leq 6.0 \)

It was required that the settling time be less than 12 seconds when a step input of \( 180^\circ \) was applied. Since the error detector has a sensitivity of 1 volt per degree, a step input of \( 180^\circ \) causes an initial error of 180 volts. In order to use Fig. 4.6, the normalized error must be found.

\[
X_0 = \frac{e}{TKK_1} = \frac{180}{(2)(0.5)(K)} = \frac{180}{K}
\]

From Fig. 4.6 the normalized error must be less than or equal to 2.4.

\[
\frac{180}{K} \leq 2.4
\]

\[
K \leq \frac{180}{2.4} = 75
\]

Thus all requirements can be met if \( K \) is between 75 and 83.

If \( K \) is set at 80,

\[
\Delta = aTKK_1 = (0.06)(2)(0.5)(80) = 4.8
\]

Now all of the system parameters are determined.

\[
\Delta = 4.8, \quad K = 80, \quad \tau_s = 0.06
\]
When a step input of 180° is applied to the system, the error will settle within 0 ± 9.6° in 11.4 seconds or less. The sampling rate is 16.67 cps, and all specifications have been met.

4 Maximum Overshoot

Suppose a positive step input is applied to the system. The system will then start in the negative mode, switch to the off mode, and then to the positive mode (see Fig. 4.3). The point where the positive mode crosses the x axis represents the maximum value of the first overshoot. Because of the spiral nature of the phase trajectory, it is larger than the maximum value of any following overshoot. Thus it is the maximum overshoot of the system when a step function input is applied.

Since the maximum overshoot is a function of the magnitude of the step function input, x₀, the sampling period, T, and the dead zone, a, its exact value could be shown only by a series of graphs, such as those used to show settling time.

A more direct approach is to find the maximum possible overshoot of the system. This can be done by assuming that input step function is infinite in amplitude. If the system input were a positive step function of infinite magnitude, the corresponding negative phase trajectory would intersect the negative-off line at y = -1, because the negative trajectory approaches the line y = -1 asymptotically. Thus the off trajectory could start at any point between -a and (assuming that y ≥ 2a) on the line y = -1. Since the maximum possible overshoot is desired, it is evident that the off trajectory should start at the point (-a, -1). The positive trajectory must then start from some point on this off trajectory.

Consider Fig. 4.10. Point 2 is the intersection of the off tra-
Figure 4.10 Phase Trajectory Showing Maximum Possible Overshoot
jectory and the off-positive switching line. This intersection can be made to fall at any point on the off trajectory by varying $\lambda$. For example, point 1 is the intersection of the off trajectory and the off-positive switching line when $\lambda = 0$.

Off trajectory

$$X = -y + x_0 + y_0$$

Since the off trajectory starts at $(-a, -1)$,

$$X = -y - a - 1 \tag{4.11}$$

Off-positive switching line.

$$X = -a + (e^2 - 1) y \tag{4.12}$$

Solving equations 4.11 and 4.12 for $y$,

$$y = -e^{-\lambda} \tag{4.13}$$

and

$$X = e^{-\lambda} - a - 1 \tag{4.14}$$

Positive trajectory

$$X = -\ln (1-y) + (1-y) + \zeta$$

$$X = -\ln (1-y) + (1-y) + x_0 + \ln (1-y_0) - (1-y_0)$$

Since the maximum overshoot occurs when $y = 0$,

$$X_m = x_0 + y_0 + \ln (1-y_0) \tag{4.15}$$
Using the values of \( x \) and \( y \) from equations 4.13 and 4.14 for initial conditions in equation 4.15

\[
X_m = -(1 + a) + \ln (1 + e^{-\lambda})
\]

Since \( x_m \) is negative, the term \( \ln(1 + e^{-\lambda}) \) must be made as small as possible subject to the restriction \( 0 \leq \lambda \leq \gamma \). This term is obviously smallest when \( \lambda = \gamma \).

Therefore, the maximum possible overshoot of the system is expressed by

\[
X_v = -\left(1 + a\right) + \ln \left(1 + e^{-\gamma}\right) \quad (4.16)
\]

subject only to the restriction \( 0 \leq \gamma \leq 2a \).

4.5 Effect of Dead Time Delay

Dead time delay is a fixed delay that is present in the error channel at all times. It affects the system in the same way as does the delay due to sampling.

The variable delay due to sampling \( (\lambda) \) enters only into the switching line equations, and \( \lambda \) can vary between zero and the normalized sampling period.

\[
0 \leq \lambda \leq \frac{1}{T}
\]

The dead time delay shifts the range of \( \lambda \) by a constant amount. Thus if the dead time delay (in real time) is \( T_0 \), then in normalized time

\[
\frac{T_0}{T} \leq \lambda \leq \frac{T_0 + \lambda}{T}
\]

Longer settling time and larger overshoots will result from the
addition of dead time delay; and, if its addition causes $\gamma$ to exceed $2a_0$, a limit cycle may be established.

The curves in Figs. 4.5 through 4.9 can also be applied to a system with dead time delay. For example, if $a = 0.100$, $\gamma = 0.100$, the settling time for a step input of $x_0 = 3.00$ is between 5.18 and 7.10 seconds (Fig. 4.7). If a dead time delay of $\frac{T_0}{1} = 0.05$ is placed in the system, $T_s$ is bounded by the curves for $\gamma = 0.05$ and $\gamma = 0.15$. The settling time for a step input of 3.00 will then be between 5.55 and 8.80 seconds.
CHAPTER 5

EXPERIMENTAL STUDY

5.1 Experimental Set-Up

The system shown in Fig. 1.1 was simulated on an analog computer. The flow diagram of the computer is shown in the Appendix. The following parameters were used:

1. \( V = 5 \) volts
2. \( K = 10 \) volts
3. \( K_1 = 1 \)
4. \( T = 5 \) seconds

A step input of 25 volts was applied, and runs were made for each of five sampling periods. The five sampling periods were:

1. \( L = 0 \)
2. \( L = 0.25 \)
3. \( L = 0.50 \)
4. \( L = 0.75 \)
5. \( L = 1.00 \)

The normalized system had the following parameters:

1. \( a = \frac{A}{TKK_1} = 0.10 \)

2. \( \gamma = \frac{L}{T} = 0, 0.05, 0.10, 0.15, 0.20 \)

The step input caused an initial error of 0.5 in the normalized system.
\[ X_o = \frac{Y_o(t) - C_o(t)}{T \times K_i} = \frac{25 - 0}{(5)(10)(1)} = 0.50 \]

5.2 Experimental Results

Phase trajectories of four of these runs are shown in Fig. 5.1. Both actual and normalized scales are given in order to illustrate the results more clearly.

Figs. 5.2 and 5.3 show the error and error rate for two different sampling periods as they were recorded on a Sanborn recorder. Again, both the actual and normalized scales are given.

Fig. 5.4 shows the action of the sampling and holding device.

The four phase trajectories in Fig. 5.1 show the effect of sampling very clearly, as the switching points can be seen as sharp breaks in the trajectories. In the case of \( Y = 0 \), all switching occurs on the decision lines. For other values of \( Y \), however, switching may take place at some point after the phase trajectory crosses the decision lines. It is seen that sampling increases settling time, overshoot, and error rate.

Quantitative results can be found from Figs. 5.2 and 5.3. In these figures the settling time and overshoot can be measured directly.

In Table I, the results of these five runs are compared with the theoretical results that were found in Chapter 4.

5.3 Comparison of Theoretical and Experimental Results

Due to the variable delay time caused by sampling, the settling time cannot be calculated exactly, but can be predicted to fall within certain limits. This is more fully explained in section 4.2.

When there is no sampling (\( Y = 0 \)), however, the graphs in Figs.
FIGURE 5.1 - PHASE TRAJECTORIES SHOWING THE EFFECT OF DIFFERENT SAMPLING PERIODS

ACTUAL SCALE
x axis - 5 v/cm
y axis - 2 v/cm

NORMALIZED SCALE
x axis - 0.1 v/cm
y axis - 0.2 v/cm
\[ \gamma_s = \text{Settling Time} \]

**FIGURE 5.2 - ERROR AND ERROR RATE VS TIME WITH ZERO SAMPLING PERIOD**
<table>
<thead>
<tr>
<th>Scale Type</th>
<th>Actual Scale</th>
<th>Normalized Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>TOP - Error</td>
<td>1.0 v/cm 50 mm/sec</td>
<td>0.2 v/cm 250 mm/sec</td>
</tr>
<tr>
<td></td>
<td>L = 1.00 sec</td>
<td>γ = 0.20 sec</td>
</tr>
<tr>
<td>BOTTOM - Sampled Error</td>
<td>1.0 v/cm 50 mm/sec</td>
<td>0.2 v/cm 250 mm/sec</td>
</tr>
<tr>
<td></td>
<td>L = 1.00 sec</td>
<td>γ = 0.20 sec</td>
</tr>
</tbody>
</table>

**FIGURE 5.4—PLOTS OF ERROR AND SAMPLED ERROR VS TIME**
4.5 through 4.9 give exact settling times, because the variable delay has been eliminated.

The results in Table I bear this out, as the experimental and theoretical settling times agree very closely when $\gamma = 0$. For all other values of $\gamma$ the experimental settling times fall within the predicted limits.

Thus the experimental study seems to verify the theoretical results.
### TABLE I

**COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS**

<table>
<thead>
<tr>
<th>Run</th>
<th>$\gamma$</th>
<th>Experimental Settling Time</th>
<th>Maximum Settling Time (Theoretical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.86 sec.</td>
<td>0.88 sec.</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.92 sec.</td>
<td>2.388 sec.</td>
</tr>
<tr>
<td>3</td>
<td>0.10</td>
<td>1.92 sec.</td>
<td>2.565 sec.</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
<td>2.08 sec.</td>
<td>4.138 sec.</td>
</tr>
<tr>
<td>5</td>
<td>0.20</td>
<td>3.6 sec.</td>
<td>5.980 sec.</td>
</tr>
</tbody>
</table>

*For all runs, $a = 0.1$, $x_0 = 0.5$. All values are normalized.*
CHAPTER 6

CONCLUSIONS

The phase plane technique can be used for the analysis of second order on-off feedback control systems. The sampling and holding device used in conjunction with the system considered here can be treated as variable delay time.

The phase plane analysis shows that the system may be unstable for very large sampling periods. Also, the limits of settling time, overshoot, and steady state error may be determined from the phase plane analysis.

The effects of dead time delay can also be determined by the phase plane method.
APPENDIX
Analog Computer Set-up for Experimental Study
BIBLIOGRAPHY

K. Izawa and L. E. Weaver, "Relay Type Feedback Control Systems with Dead Time and Sampling." AIEE Applications and Industry, May, 1959, pp. 49-54.

