THE RESPONSE OF A UNIFORMLY LOADED CANTILEVER BEAM TO A SUDDEN RELEASE OF CONSTRAINTS

by

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ABSTRACT

A theoretical and experimental analysis is made of the response of a uniformly loaded cantilever beam to a sudden release of constraints. This is an idealization of the response of a rocket vehicle, subjected to a steady ground wind, to the sudden release of constraints at launching.

Elementary beam theory predicts that the flexural vibrations resulting from the sudden release of a uniform cantilever beam will result in bending stresses that may exceed the initial static bending stresses. The stress magnification will be large and is unbounded for an instantaneous release, if damping is not considered. The stress magnification is not so large for a delayed release, i.e., a release requiring a finite time for completion.

The experimental results are in good agreement with those predicted by the theory.

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1. INTRODUCTION

1.1 Basis for the Problem

Consider the launching of a rocket vehicle subjected to a steady ground wind. If the effects of vortex shedding are neglected and the rocket is constrained at its lower end, the rocket is subjected to a steady lateral loading. At launch, the constraints are suddenly removed and the rocket experiences flexural vibrations. There are indications that the bending moments experienced by the rocket may be dangerously high.

1.2 Idealization of the Problem

Consider an idealized problem in which the rocket is replaced by a uniform beam of circular cross-section, subjected to a steady, uniform lateral loading, and constrained as a cantilever. The bending moment response of the beam subsequent to a sudden release of constraints will be studied.

1.3 Existing Literature

The theoretical solution for the forced flexural vibrations of a beam is well known. Reference 1 presents a development of the theoretical solution. A reduced and somewhat modified version is presented in Appendix B. The natural frequencies and mode shapes for the free vibrations of beams with various end conditions have been computed and tabulated. Reference 2 gives a typical tabulation for the first five modes.

A number of mathematical analyses have been made of the response of beams to impulsive and other forms of rapid loading. Reference 3 presents a summary and provides a good bibliography on the subject. However, the writer does not know of any publication regarding the analysis of the response of a uniformly loaded cantilever beam to a sudden release of constraints or of a comparison of theoretical and experimental results of this application.

1.4 Scope of the Problem

It can be expected that the rapidity of release will have an effect on the response of the beam. Further, there will be some delay inherent in any mechanical or electromechanical release mechanism. The effects of the time of action of the release mechanism will be considered.

In the problem being considered, before release the maximum stresses occur in the cantilever beam at the constraints. When the constraints are released, the stored strain energy will be redistributed throughout the beam. It can be expected that there will be an increase in stress at some points along the beam. The ratio of the stresses resulting from the dynamic loading (flexural vibrations) to those resulting from the static loading in various portions of the beam will be investigated.

The amplitudes of vibration in any structure tend to diminish with time due to structural damping. Aerodynamic forces also resist the motion and will damp the vibrations. Quantitative effects of damping are not considered here; only the general qualitative effects are considered.

2. THEORETICAL ANALYSIS

2.1 Natural Mode Shapes and Frequencies

The fundamental equations for the forced flexural motion of a slender beam are developed in Appendix B as well as in several texts. In this development, the bending deflections in a beam are

$$w_{(x,t)} = \sum_{n=1}^{\infty} \phi_{n(x)} q_{n(t)}, \qquad (2.1)$$

where ϕ_n is the n-th natural mode shape and q_n is a generalized coordinate giving the amplitude of the n-th mode as a function of time.

For an unrestrained (free-free end) beam the natural mode shapes are given by

$$\phi_{n(x)} = \cosh(\beta_n x) + \cos(\beta_n x) - \alpha_n(\sinh(\beta_n x) + \sin(\beta_n x)) \qquad (2.2)$$

where α_n and β_n are characteristic of the n-th mode. Their values appear in Table I, below. Further, the natural frequencies are given by

$$\omega_n^2 = (\beta_n^4)(EI/\mu) \qquad (2.3)$$

The nomenclature used in equations (2.1) through (2.3) is summarized below. Any consistent set of units may be used for the physical quantities; however, for convenience, the units of such quantities are noted in terms of the inch-pound-second system.

x = distance along the axis of the beam, inches.

w = lateral deflection of the beam, inches.

ℓ = length of the beam, inches.

t = time, seconds.

E = modulus of elasticity of beam material, pounds/inch^2.

I = moment of inertia of a cross-section of the beam about its neutral axis, inches^4.

n = 1, 2, 3, . . . = the mode number.

μ = mass per unit length of the beam, pound-second^2/inch^2.

ω_n = natural frequency of the n-th mode, radians/second.

α_n = a characteristic number.

β_n = a characteristic number.

$\phi_n(x)$ = shape of the n-th natural mode.

Derivatives of ϕ_n with respect to $\beta_n x$ are denoted by prime marks, thus

$$\phi_n' = \frac{1}{\beta_n} \frac{d\phi_n}{dx},$$

$$\phi_n'' = \frac{1}{\beta_n^2} \frac{d^2\phi_n}{dx^2},$$

$$\phi_n''' = \frac{1}{\beta_n^3}\,\frac{d^3\phi_n}{dx^3}.$$

The functions, ϕ_n, have the property of orthogonality. The tabulated values which will be used in this analysis are chosen so that

$$\int_0^\ell \phi_n\phi_m dx = 0, \qquad m \neq n,$$

$$\int_0^\ell \phi_n^2 dx = \ell.$$

The characteristic numbers and natural mode shapes have been tabulated for at least the first five modes. The values appearing below were extracted from reference 2.

TABLE I

Values of α_n and $\beta_n\ell$

n	α_n	$\beta_n\ell$
1	0.9825 0221 58	4.7300 408
2	1.0007 7731 1	7.8532 046
3	0.9999 6645 01	10.9956 078
4	1.0000 0145 0	14.1371 655
5	0.9999 9993 73	17.2787 596

For $n>5$, $\alpha_n = 1.0$, $\beta_n\ell = (2n + 1)\ \pi/2$

The representative values of x/ℓ in Table II, below, were chosen because they are close to the values used in the experimental portion of the study.

TABLE II

Natural Mode Shapes and Derivatives

Free-free Beam

First Mode

x/ℓ	ϕ_1	ϕ_1'	ϕ_1''	ϕ_1'''
0.00	2.00000	− 1.96500	0.00000	0.00000
0.26	− 0.27180	− 1.53085	0.91124	1.01113
0.34	− 0.77704	− 1.11735	1.26141	0.80774
0.50	− 1.21565	0.00000	1.58815	0.00000
0.66	− 0.77704	1.11735	1.26141	− 0.80774
1.00	2.00000	1.96500	0.00000	0.00000

Second Mode

x/ℓ	ϕ_2	ϕ_2'	ϕ_2''	ϕ_2'''
0.00	2.00000	− 2.00155	0.00000	0.00000
0.26	− 1.21876	− 0.56977	1.47245	0.30410
0.34	− 1.28181	0.36246	1.40914	− 0.51224
0.50	0.00000	1.37532	0.00000	− 1.45420
0.66	1.28181	0.36246	− 1.40914	− 0.51224
1.00	− 2.00000	− 2.00155	0.00000	0.00000

Third Mode

x/ℓ	ϕ_3	ϕ_3'	ϕ_3''	ϕ_3'''
0.00	2.00000	- 1.99993	0.00000	0.00000
0.26	- 1.18165	0.62425	1.29690	- 0.73833
0.34	- 0.24051	1.36606	0.28949	- 1.41222
0.50	1.42238	0.00000	- 1.40600	0.00000
0.66	- 0.24051	- 1.36606	0.28949	1.41222
1.00	2.00000	1.99993	0.00000	0.00000

Fourth Mode

x/ℓ	ϕ_4	ϕ_4'	ϕ_4''	ϕ_4'''
0.00	2.00000	- 2.00000	0.00000	0.00000
0.26	- 0.32640	1.34442	0.37700	- 1.39515
0.34	1.09776	0.89319	- 1.08159	- 0.90972
0.50	0.00000	- 1.41592	0.00000	1.41251
0.66	- 1.09776	0.89319	1.08159	- 0.90972
1.00	- 2.00000	- 2.00000	0.00000	0.00000

Fifth Mode

x/ℓ	ϕ_5	ϕ_5'	ϕ_5''	ϕ_5'''
0.00	2.00000	- 2.00000	0.00000	0.00000
0.26	0.76897	1.18287	- 0.74658	- 1.20525
0.34	1.31772	- 0.52341	- 1.31208	0.51781
0.50	- 1.41386	0.00000	1.41457	0.00000
0.66	1.31772	0.52341	- 1.31208	- 0.51781
1.00	2.00000	2.00000	0.00000	0.00000

2.2 Bending Moment and Stress

From elementary beam theory, the bending moment, M, at a cross-section of a beam is given by

$$M = EI(\partial^2 w/\partial x^2).$$

Substituting the expression for w from equation (2.1),

$$M = EI \sum_{n=1}^{\infty} q_n (d^2 \phi_n / dx^2)$$

$$= EI \sum_{n=1}^{\infty} q_n \beta_n^2 \phi_n'' . \quad (2.4)$$

Similarly, the maximum bending stress is given by

$$\sigma = Mr/I = (r/I)(EI) \sum_{n=1}^{\infty} q_n \beta_n^2 \phi_n'' \quad (2.5)$$

where r is the radius of the beam.

2.3 Pre-Release Static Deflections.

Before release ($t < 0$), the beam is subjected to a uniformly distributed load p per unit length and, at one end, a restraining shear force S and bending moment M.

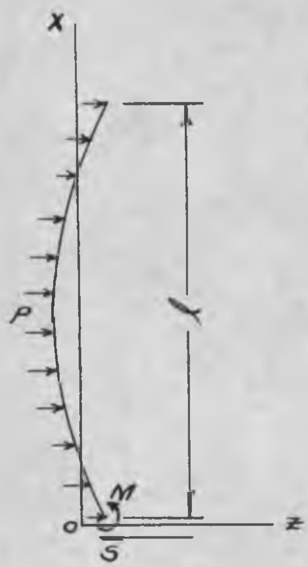


Figure 2.1 Beam Before Release

It will be convenient to express the deformations of the beam resulting from this static loading in terms of the free-free natural modes. The equation for the forced motion of the n-th mode, developed in Appendix B, is

$$\overline{M}_n \ddot{q}_n + \overline{M}_n \omega_n^2 q_n = Q_n, \qquad (2.6)$$

where $\overline{M}_n$ is a generalized mass given by

$$\overline{M}_n = \int_0^{\ell} \mu\, \phi_n^2 dx, \qquad (2.7)$$

and Q_n is a generalized force. This force is equal to the virtual work done by the external forces in a virtual displacement of the generalized coordinate q_n.

$$Q_n = \delta W / \delta q_n,$$

$$\delta W = \delta q_n \left(-S\phi_n(o) - M(d\phi_n/dx)_{(o)} + p \int_0^{\ell} \phi_n dx \right)$$

The integral above vanishes. Therefore,

$$Q_n = -\left(S\phi_n(o) + M\beta_n \phi_n'(o) \right) \qquad (2.8)$$

where S and M, the shearing force and bending moment on the end of the beam, have respective magnitudes of $p\ell$ and $\frac{1}{2}p\ell^2$.

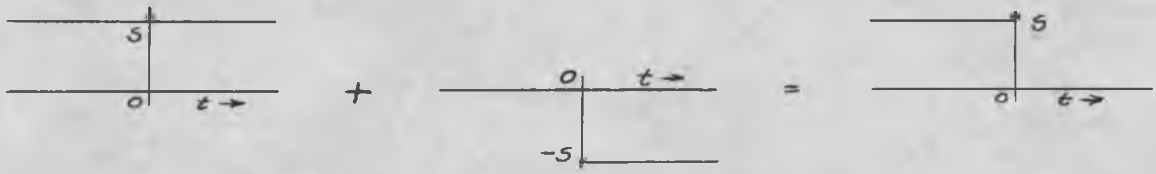
Since the beam is in a static condition, $\ddot{q}_n = \dot{q}_n = 0$, and

$$q_n = Q_n / (\overline{M}_n \omega_n^2).$$

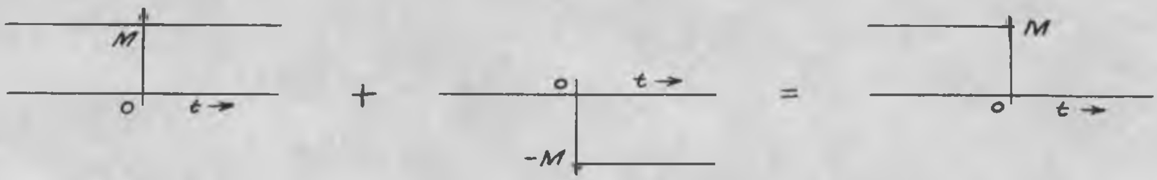
$$= -\tfrac{1}{2}p\ell(2\phi_n(o) + \beta_n \ell \phi_n'(o)) / (\overline{M}_n \omega_n^2). \qquad (2.9)$$

2.4 Release Conditions

At the time t = 0, the constraints are released and the shear and moment on the end of the beam go to zero. If the constraints are released instantaneously, the time history can be represented by a sustained action plus a negative step of both the shear and moment applied at t = 0.



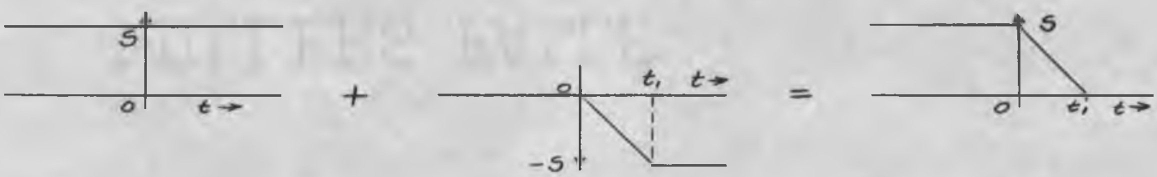
(a) Shear Time History



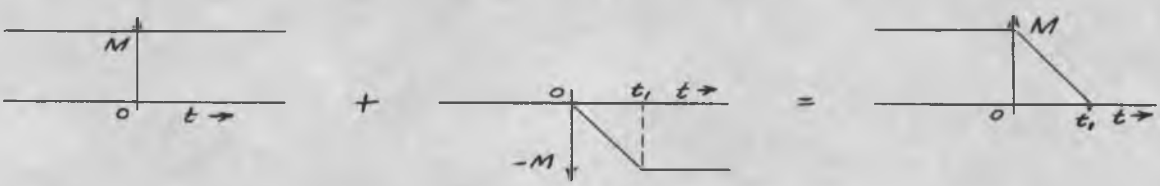
(b) Moment Time History

Figure 2.2 Instantaneous Release Time History

There will be some delay in the action of any electro-mechanical or mechanical release system. When delay is considered, it will be assumed that the time history can be represented by a sustained action plus a negative ramp-platform of both the shear and moment applied at t = 0.



(a) Shear Time History



(b) Moment Time History

Figure 2.3 Delayed Release Time History

2.5 Post-Release Response

The generalized force for the negative ramp-platform function is

$$Q_n = (S\phi_n(o) + MB_n\phi_n'(o))(t/t_1) \quad 0 \leqq t \leqq t_1 \quad (2.10a)$$

$$Q_n = S\phi_n(o) + MB_n\phi_n'(o) \qquad t \geqq t_1 \qquad (2.10b)$$

where t_1 is the time to completion of the release. Obviously, only equation (2.10b) applies for the limiting case $t_1 = 0$ (instantaneous release).

The response of the generalized coordinate q_n to the generalized force Q_n can be found through the use of Duhamel's integral:

For $t_1 = 0$,

$$q_n = (S\phi_n(o) + MB_n\phi_n'(o))/(\bar{M}_n\omega_n)\int_o^t \sin\omega_n(t - \tau)d\tau$$

$$= (S\phi_n(o) + MB_n\phi_n'(o))/(\bar{M}_n\omega_n^2)\ (1 - \cos\omega_n t) \qquad (2.11a)$$

For $t_1 \neq 0$, $0 \leqq t \leqq t_1$;

$$q_n = (S\phi_n(o) + MB_n\phi_n'(o))/(\bar{M}_n\omega_n)\int_o^t (\tau/t_1)\sin\omega_n(t - \tau)d\tau$$

$$= \frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}\left\{\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right\} \qquad (2.11b)$$

For $t_1 \neq 0$, $t \geqq t_1$;

$$q_n = \frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}\left\{\frac{1}{\omega_n t_1}\left[\sin\omega_n(t - t_1) - \sin\omega_n t\right]\right.$$
$$\left. + \cos\omega_n(t - t_1) + \omega_n \int_{\tau_1}^{t} \sin\omega_n(t - \tau)d\tau \right\}$$
$$= \frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}\left\{1 + \frac{\sin\omega_n(t - t_1) - \sin\omega_n t}{\omega_n t_1}\right\} \quad (2.11c)$$

The response of the generalized coordinates is the sum of the responses to the sustained shear and moment and to the negative step or ramp-platform function of shear and moment, equations (2.8) and (2.10).

For $t_1 = 0$;

$$q_n = \frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}(-\cos\omega_n t), \quad (2.12a)$$

For $t_1 \neq 0$ $\quad 0 \leqq t \leqq t_1$;

$$q_n = -\frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}\left\{1 - \frac{t}{t_1} + \frac{\sin\omega_n t}{\omega_n t_1}\right\} \quad (2.12b)$$

For $t_1 \neq 0$, $t \geqq t_1$;

$$q_n = \frac{S\phi_n(o) + MB_n\phi_n'(o)}{\bar{M}_n\omega_n^2}\left\{\frac{\sin\omega_n(t - t_1) - \sin\omega_n t}{\omega_n t_1}\right\} \quad (2.12c)$$

It should be noted that the time dependent portion of equation (2.12c) goes to $-\cos\omega_n t$ as t_1 goes to zero. This agrees with equation (2.12a).

The time dependent portion of equation (2.12c) can be expressed in a more convenient form,

$$\frac{\sin\omega_n(t - t_1) - \sin\omega_n t}{\omega_n t_1} = A_n \sin(\omega_n t + \varphi_n), \qquad (2.13)$$

where

$$A_n = -\sqrt{2(1 - \cos\omega_n t_1)}/(\omega_n t_1), \qquad (2.13a)$$

and

$$\varphi_n = \arctan(\sin\omega_n t_1/(1 - \cos\omega_n t_1)). \qquad (2.14b)$$

A_n has been plotted for values of $\omega_n t_1$ up to 10π (see Graph 1, Appendix C).

2.6 Magnification of Stresses

In the pre-release condition, the bending moment at a point x along the beam is given by

$$M_s = \tfrac{1}{2}p\ell^2(1 - (x/\ell))^2. \qquad (2.15)$$

For the dynamic response, the bending moment is given by equation (2.4). Substituting the values for q_n from equations (2.12) gives

$$M_d = EI \sum_{n=1}^{\infty} \beta_n^2 \phi_n''(S\phi_n(o) + M\beta_n\phi_n'(o))f_n(t)/(\overline{M}_n\omega_n^2),$$

where

$$S = p\ell,$$

$$M = \tfrac{1}{2}p\ell^2,$$

$$f_n(t) = -\cos\omega_n t, \qquad (t_1 = 0)$$

$$f_n(t) = \frac{\ }{\ }\left\{\left(1 - \frac{t}{t_1}\right) + \frac{\sin\omega_n t}{\omega_n t_1}\right\} \quad (t_1 \neq 0,\ 0 \leqq t \leqq t_1)$$

$$f_n(t) = A_n \sin(\omega_n t + \varphi_n) \quad (t_1 \neq 0,\ t \geqq t_1)$$

$$\overline{M}_n = \int_0^{\ell} \mu\, \phi_n^2 dx = \mu\ell = m.$$

Rearranging and simplifying gives the following expression for M_d:

$$M_d = \tfrac{1}{2} p \ell^2 \sum_{n=1}^{\infty} \left\{ \frac{2\phi_n(o)}{(\beta_n \ell)^2} + \frac{\phi_n'(o)}{\beta_n \ell} \right\} \phi_n'' f_n(t) \qquad (2.16)$$

The ratio of the dynamic bending moment to the static bending moment at a point x/ℓ is given by

$$\frac{M_d}{M_s} = \frac{1}{(1 - (x/\ell))^2} \sum_{n=1}^{\infty} \left\{ \frac{2\phi_n(o)}{(\beta_n \ell)^2} + \frac{\phi_n'(o)}{\beta_n \ell} \right\} \phi''_{n(x/\ell)} f_n(t). \qquad (2.17)$$

The maximum stresses in the outermost fibers of a beam are directly proportional to the bending moment at the point. The stresses can be found by substituting the proper physical quantities into

$$\sigma = (Mr)/I.$$

Therefore, equation (2.17) applies equally well to the ratio of maximum dynamic to static stresses, σ_d/σ_s.

A plot of the ratio of maximum dynamic (summation of five modes) to maximum static stress versus x/ℓ and $\omega_1 t_1$ appears in Graph 2 (Appendix C). The minimum value of the ratio is one.

2.7 Discussion of Theoretical Analysis

In the expression for the dynamic bending moment

$$M_d = \tfrac{1}{2}p\ell^2 \sum_{n=1}^{\infty} \left\{ \frac{2\phi_n(o)}{(\beta_n \ell)^2} + \frac{\phi_n'(o)}{\beta_n \ell} \right\} \phi_n'' f_{n(t)},$$

the terms in the series vary with n. The order of magnitude of the quantities can be established:

$$\phi_n(o),\ \phi_n'(o),\ \phi_{n(x)}'' \sim 1.$$

$$\beta_n \sim n.$$

$$f_{n(t)} \sim 1, \quad t_1 = 0.$$

$$f_{n(t)} \sim 1/n^2, \ t_1 > 0.$$

Thus, the order of magnitude of the terms of the series is:

$$(M_d)_n \sim 1/n, \ t_1 = 0.$$

$$(M_d)_n \sim 1/n^3, \ t_1 > 0.$$

It may be concluded that the bending moments resulting from an instantaneous release, $t_1 = 0$, may become very large and may, in fact, be unbounded. For the case of a delayed release, the bending moments can be expected to be smaller and are bounded. An examination of the results shown in Graph 2 bears this out. It can be seen that the stress magnification can become large, particularly near the initially free end of the beam. Further, it is evident that a delayed release reduces the stress magnification. A consideration of damping would serve to reduce the response further.

3. CONDUCT OF EXPERIMENT

3.1 Release Mechanism

It was necessary to select or devise a release mechanism that would:

a. Restrain a beam in a cantilevered position.

b. Be capable of a sudden release (small value of t_1).

c. Not impart motion, translatory, rotational or vibrational, to the beam.

d. Provide uniform action through several trials.

It was desirable that the constraints be in the form of pure moment and shear. However, if the release time were such that the higher modes would not be excited measurably, a pair of force constraints close together would duplicate a moment and shear constraint satisfactorily.

The release mechanism chosen utilized two solenoids parallel wired to a standard 110 volt, 60 cycle source. The solenoids are those used in the Maytag washing machine. These particular solenoids were chosen because they have a square piston action and they are sturdy enough to maintain the beam in a cantilevered position.

An aluminum block, 3 x 1 x 0.750 inches, was drilled through the center (9/32" diameter) and milled to cut through the vertical diameter of the hole. The two halves of the block were tapped and bolted to the pistons of the solenoids.

Coil springs were attached to the pistons of each of the solenoids to hold them in a closed position while the beam was suspended. A tension of 1½ pounds was found to be necessary on each spring when the mechanism was in a closed position.

The mechanism was tilted at the proper angle to place the beam in a horizontal position when constrained. In effect, the beam is supported at two points as shown in Figure 3.1.

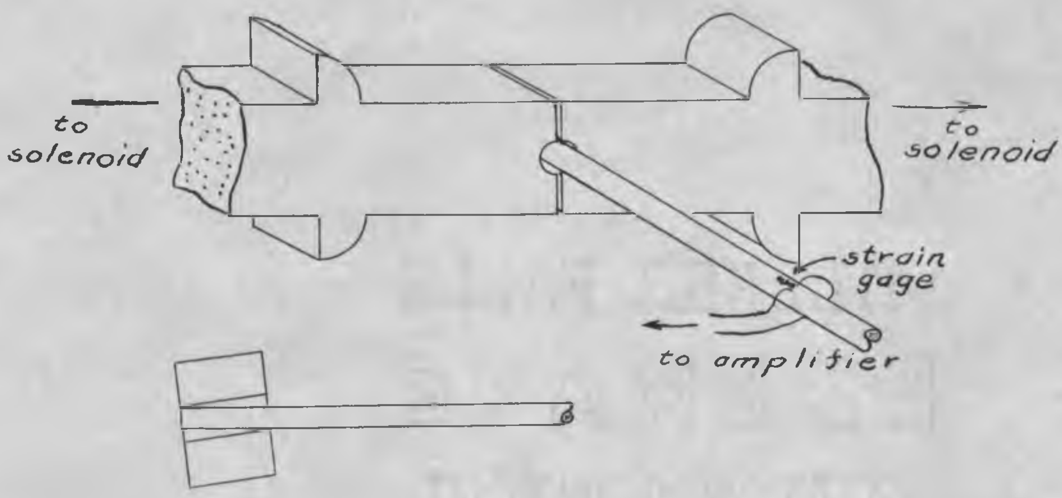


Figure 3.1 Release Mechanism

3.2 Beam Selection

A four-foot length of $\frac{1}{4}$-inch diameter aluminum rod was chosen. The weight of the rod was small enough to permit the use of the release mechanism selected and the flexural rigidity was low enough to provide easily measurable strains. The circular cross-section was favorable for a sudden release since any lateral motion of the mechanism tends to move the constraining elements away from the beam.

The dimensions and weight of the rod were measured carefully. The values are shown in Appendix A, Symbols and Units. The modulus of elasticity was found from the fundamental frequency of the rod in the cantilevered position.

3.3 Stress Measurement

SR-4 strain gages were affixed to the rod at five points. The x/ℓ ratios of these points are:

a. 0.0254.

b. 0.2482.

c. 0.3365.

d. 0.5016.

e. 0.6674.

Point a was useful only as an indicator of release time; the effects of flexural vibration are negligible this close to the end of the rod. The remaining four points, b through e, are referred to as points 1 through 4 respectively.

The strain gages were connected through a General Electric strain gage amplifier to an oscilloscope. Lead wires from the strain gages were varnished coil-winding wire of the smallest gage that could be obtained locally.

The total weight of the five strain gages plus two lead wires was found to be less than 0.1 percent of the weight of the rod. The effects of this small difference on the loading of the rod are ignored.

The release mechanism was mounted so that the rod was 72 inches from the floor when suspended. Continuous strain gage readings were taken from immediately before release through release and free fall. The readings were repeated several times at each of the four points. The traces were recorded by the use of a Polaroid camera on an oscilloscope mount. A typical trace appears below.

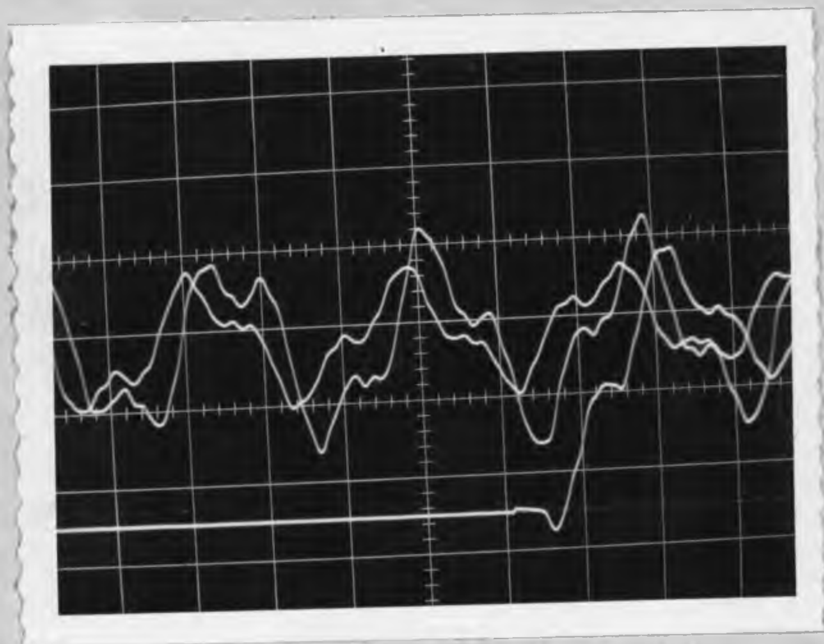


Figure 3.2 Typical Photographic Trace

There is a distortion of two percent over the width of the photograph due to camera astigmatism. Additionally, the camera does not record the full width of the grid. There is a time blank of 0.025 seconds from the right edge to the left edge of the photograph (beam travel is from left to right).

The solenoid pistons do not begin to move immediately with the closing of the switch. There is a reaction to the initial power surge that is evident in the first minor changes in the trace of the oscilloscope beam. Time zero has been taken as the beginning of the first major change in position rather than the initial reaction to the power surge.

3.4 Estimation of Release Time

A means was devised to determine directly the time history of the movement of a solenoid piston. As shown in Figure 3.3, a strain gage was fixed to a thin bar. A point contact was added to the free end of the bar and the other end was placed in a vise. In the pre-release condition, the bar was in a deflected position with its point in contact with a protrusion on the solenoid piston. The relation between piston position and measured strain in the bar was determined experimentally.

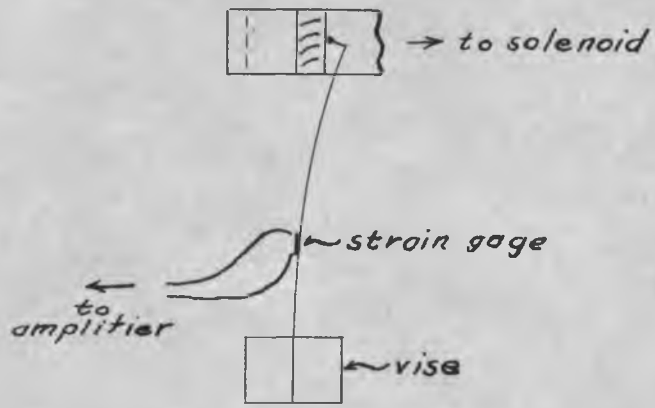


Figure 3.3 System for Determining Release Time

The maximum possible release time is the time required for the piston to move a distance equal to the radius of the beam. This time, determined to be 0.085 seconds, is a bounding value of the release time.

It was not possible to determine directly the piston displacement required for completion of release. The time history of the strain at the point $x/\ell = 0.0254$ was studied. Replacement of the actual strain-time curve by an equivalent ramp results in an effective release time of 0.014 seconds, as shown in the figure below.

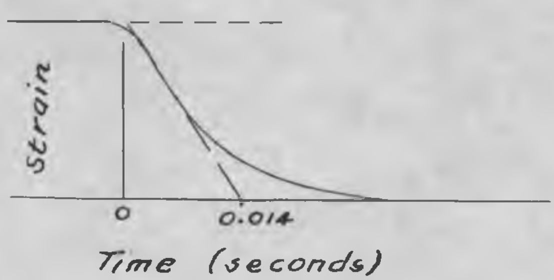


Figure 3.4 Strain-Time History at $x/\ell = 0.0254$

4. COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

4.1 Beam Suspension

The expression for the generalized forces, Q_n, in the static condition, equation (2.8), was developed for constraints in the form of pure moment and shear. The experimental portion of this study was conducted with the beam, in its suspended state, constrained by a pair of forces close together. Equation (2.8), when adjusted to reflect the actual constraints, becomes

$$Q_n = (R_1 \phi_n(o) - R_2 \phi_n(\Delta)) \tag{4.1}$$

where Δ is the distance between the constraints. This equation can be put in the form

$$Q_n = (R_1 - R_2)\phi_n(o) - R_2(\phi_n(\Delta) - \phi_n(o))$$

$$= - S\phi_n(o) - R_2\Delta(\phi_n(o) - \phi_n(\Delta))/\Delta.$$

In the limit, as Δ approaches zero, the later expression approaches

$$Q_n = - S\phi_n(o) - M(d\phi_n/dx)_{(o)}.$$

This expression for Q_n, static, is the same as that in equation (2.8).

The values of Q_n from equation (2.8) and from equation (4.1), determined for the physical properties of the rod, are tabulated below for comparison.

TABLE IV

Comparison of Generalized Forces
(pounds)

n	Q_n (Eq(2.8))	Q_n (Eq(4.1))
1	0.6405	0.6379
2	1.4080	1.4083
3	2.1616	2.1627
4	2.9165	2.9163
5	3.6715	3.6713

The maximum difference is 0.4 percent, within the accuracy of plotting and measurement.

4.2 Frequencies and Amplitudes of Vibration

Graph 3, Appendix C, is a plot of stress versus time comparing the experimental results with the theoretical solution. A plot is made for each of the four points described in Section 3.3. The theoretical plot is limited to a summation of the first three modes for points 1 and 2 and to the first two modes for point four. Point three is located almost at the point of zero amplitude for modes two and four. Nothing more than a trace of the fourth mode appears in the experimental results at points 1 and 2 and of the third mode at point 4. The fourth and fifth modes were not included in the theoretical plot. Their amplitudes are small and they would have complicated the comparison without adding anything to it.

The elementary theory of beam vibrations predicted the frequencies found experimentally within the errors of measurement.

Agreement of the stress magnitudes between the theoretical and experimental results is very good with one exception. The observed amplitude of the second mode is greater than the predicted value at points two and four. This is attributed to the difference between the theoretical and experimental release time history.

4.3 Stress Magnification

The theoretical solution predicts and the experimental results verify a stress magnification over most of the length of the beam. It should be noted that the initial reaction is the most severe. Later reactions may exceed the static stress but they never achieve the amplitude of the initial reaction. A curve connecting the results for the four points on the beam is included in Graph 2. This curve does not follow the form of the theoretical curves, probably because of a combination of the effects of structural damping and the action of the release mechanism.

5. CONCLUSIONS

Elementary beam theory predicts that the bending stresses resulting from the sudden release of a uniformly loaded cantilever beam may exceed the bending stresses due to the static loading. For an instantaneous release, the magnification will be large and, if damping is not considered, is unbounded. For a delayed release, the magnification is reduced below that in the instantaneous case. Thus, a delayed release would be desirable if it were necessary to limit the stresses.

The experimental results reinforce this conclusion. The experimental results were in good agreement with those predicted by the theory.

Appendix A

SYMBOLS AND UNITS

The nomenclature used is summarized below. The units used for the physical quantities are in terms of the inch-pound-second system.

Symbols and Units	Experimental Values
d = diameter of circular cross-section, inches.	0.252 in.
E = modulus of elasticity of beam material, pounds/inch^2.	10.460×10^6 lb/in^2.
I = moment of inertia of a cross section of a beam about its neutral axis, inches^4.	1.9796×10^{-4} in^4.
ℓ = length of beam, inches.	47.850 in.
M = bending moment at a cross section of the beam, inch-pounds.	
M_d = bending moment due to dynamic loading (vibrations).	
M_s = bending moment due to static loading.	
$\overline{M}_n$ = generalized mass of the n-th mode,	

Symbols and Units	Experimental Values
$\overline{M}_n = \int_0^\ell \mu \phi_n \phi_m \, dx = 0, \quad n \neq m,$ $= \mu \ell, \quad n = m.$	
$n = 1, 2, 3, \ldots$ = any integer, a subscript used to indicate the order of the mode of vibration.	
p = uniform lateral load per unit length, pounds/inch.	5.0225×10^{-3} lb/in.
q_n = generalized coordinate, a function of t.	
$Q_n = \int_0^\ell Z \phi_n \, dx$ = generalized force of the n-th mode.	
r = radius of cross section, inches.	0.126 in.
$R_{1,2}$ = forces of reaction of two-point constraints, pounds.	R_1 = 7.4260 lb. R_2 = 7.6663 lb.
S = shearing force, pounds.	
t = time, seconds.	
t_1 = release reaction time, seconds.	0.014 sec.
w = lateral deflection of beam axis, inches.	
W = weight of beam, pounds.	0.24031 lb.
x = distance along longitudinal axis of beam, inches.	

Symbols and Units	Experimental Values
z = distance perpendicular to longitudinal axis (undeflected), inches.	
$Z_{(x,t)}$ = general forcing function.	
α_n = a parameter occurring in the expression for the characteristic functions.	(See Section 2.1)
β_n = the characteristic number; related to the frequency, ω_n.	(See Section 2.1)
Δ = distance between the two-point constraints, inches.	0.750 in.
μ = mass per unit length of the beam, pound-second^2/inch^2.	1.2977×10^{-5} lb-sec^2/in^2.
$\phi_{n(x)}$ = characteristic function describing the n-th mode shape.	(See Section 2.1)
$\omega_n = \beta_n^2 \sqrt{EI/\mu}$ = natural frequency of n-th mode, radians/second.	

n	ω_n (rad/sec)
1	122
2	336
3	659
4	1090
5	1628

Appendix B

DEVELOPMENT OF THE EQUATIONS OF LATERAL VIBRATION

B.1 Free Vibrations of Beams.

The development that appears below is essentially the same as that in reference 1, pages 331-333.

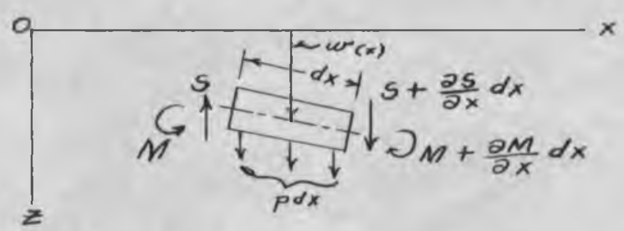


Figure B.1

Consider a free body diagram of a segment of a beam as shown in Figure B.1. The lateral deflection is w, the bending moment at any cross section is M, the shearing force is S, and the distributed load per unit length is p. The direction of the axes and the positive directions of bending moments and shearing forces are as shown.

The differential equation of the bending deflections is

$$EI(\partial^2 w/\partial x^2) = M, \qquad \text{(B.1)}$$

where E is the modulus of elasticity, I is the cross-sectional moment of inertia and the product EI is the flexural rigidity.

A consideration of the equlibrium of the beam segment leads to

$$(\partial/\partial x)(EI\partial^2 w/\partial x^2) = \partial M/\partial x = -S \qquad (B.2)$$

$$(\partial^2/\partial x^2)(EI\partial^2 w/\partial x^2) = -\partial S/\partial x = p \qquad (B.3)$$

Applying D'Alembert's principle, the inertia force per unit length is given by

$$p = -\mu\partial^2 w/\partial t^2, \qquad (B.4)$$

where μ is the mass of the beam per unit length.

Substituting equation (B.4) for p into equation (B.3), the general equation for the free vibration of the beam becomes

$$(\partial^2/\partial x^2)(EI\,\partial^2 w/\partial x^2) = -\mu\partial^2 w/\partial t^2. \qquad (B.5)$$

This is the well-known Bernoulli-Euler equation.

In the particular case of a uniform beam, EI is constant along the length of the beam and equation (B.5) becomes

$$EI\partial^4 w/\partial x^4 + \mu\partial^2 w/\partial t^2 = 0,$$

or

$$\partial^4 w/\partial x^4 + a^2(\partial^2 w/\partial t^2) = 0, \qquad (B.6)$$

in which

$$a^2 = \mu/(EI)$$

B.2 Normal Modes of Vibration

When a beam performs a normal mode of vibration, the deflection at any location varies harmonically with time and can be represented as follows:

$$w = \phi(A\cos\omega t + B\sin\omega t) \tag{B.7}$$

where ϕ is a function of the coordinate x and represents the shape of the normal mode under consideration; ω is the natural frequency corresponding to the particular mode shape. Substituting equation (B.7) into equation (B.6), we obtain,

$$\partial^4\phi/\partial x^4 - a^2\omega^2\phi = 0. \tag{B.8}$$

By using the notation

$$a^2\omega^2 = \mu\,\omega^2/(EI) = k^4, \tag{B.9}$$

it can be shown that sin kx, cos kx, sinh kx and cosh kx will be solutions of equation (B.8). The general solution of this equation will be

$$\phi = C_1\sin kx + C_2\cos kx + C_3\sinh kx + C_4\cosh kx, \tag{B.10}$$

in which C_1, C_2, C_3 and C_4 are constants to be determined in a particular case from the conditions at the ends of the beam. Applications of the four end or boundary conditions permit the determination of the relative magnitudes of the arbitrary constants of the general solution (B.10) and the frequency equation in k or ω. The modes of natural vibration and their frequencies will then be established. By superimposing all possible normal vibrations, the general expression for free lateral vibrations becomes

$$w = \sum_{n=1}^{\infty} \phi_n (A_n \cos\omega_n t + B_n \sin\omega_n t). \qquad (B.11)$$

B.3 Forced Vibrations

It is convenient to express the forced motion of a beam as a superposition of the motion in the normal modes ϕ_n, each having an amplitude q_n, referred to as a generalized coordinate.

$$w = \sum_{n=1}^{\infty} q_n \phi_n, \qquad (B.12)$$

where ϕ_n is a function of x and q_n is a function of t.

The differential equation for forced vibrations is

$$(\partial^2/\partial x^2)(EI\partial^2 w/\partial x^2) + \mu\partial^2 w/\partial t^2 = Z, \qquad (B.13)$$

where Z is a function of x and t. Substituting the value for w from equation (B.12) into this expression, we obtain

$$\sum_{n=1} q_n (EI\phi_n'')'' + \mu \sum_{n=1}^{\infty} \phi_n \ddot{q}_n = Z \qquad (B.14)$$

where the primes indicate derivatives with respect to x and the dots indicate derivatives with respect to t.

From the homogeneous portion of equation (B.14), by separation of variables, we get

$$\ddot{T} + \omega^2 T = 0,$$

$$(EI\phi_n'')'' - \mu\omega_n^2 \phi_n = 0.$$

Making use of the later identity, equation (B.14) can be written in the form

$$\mu \sum_{n=1}^{\infty} q_n \omega_n^2 \phi_n + \mu \sum_{n=1}^{\infty} \phi_n \ddot{q}_n = Z.$$

Multiplying both sides by ϕ_m and integrating with respect to x from zero to ℓ gives

$$\sum_{n=1}^{\infty} \omega_n^2 q_n \int_0^{\ell} \mu \phi_n \phi_m dx + \sum_{n=1}^{\infty} \ddot{q}_n \int_0^{\ell} \mu \phi_n \phi_m dx = \int_0^{\ell} Z \phi_m dx. \quad (B.15)$$

The functions representing the normal modes of vibration are orthogonal. The values of ϕ_n as tabulated in various references are chosen to give, for a uniform beam,

$$\int_0^{\ell} \phi_n \phi_m dx = 0 \qquad n \neq m$$

$$\int_0^{\ell} \phi_n \phi_m dx = \ell \qquad n = m.$$

All values of the summation reduce to zero when n is not equal to m. For n equal to m, Equation (B.15) becomes

$$\omega_m^2 q_m \int_0^{\ell} \mu \phi_m^2 dx + \ddot{q}_m \int_0^{\ell} \mu \phi_m^2 dx = \int_0^{\ell} Z \phi_m dx$$

or

$$\overline{M}_m \ddot{q}_m + \overline{M}_m \omega_m^2 q_m = Q_m, \quad (B.16)$$

where

$$\overline{M}_m = \int_0^{\ell} \mu \phi_m^2 dx = \text{generalized mass},$$

$$Q_m = \int_0^{\ell} Z \phi_m dx = \text{generalized force}.$$

B.4 Assumptions and Their Effects

The expressions for bending moments and stresses resulting from flexural vibration developed in the preceding sections are based on the usual assumptions of elementary beam theory. These assumptions have proven valid for most engineering applications. They are:

a. The deflections are small.

b. The effect of shearing force is negligible.

c. The effect of rotary inertia is negligible.

One of the two implications of small deflections is that the slopes are small. The bending moment at a cross section on the beam is then proportional to the radius of curvature at that point.

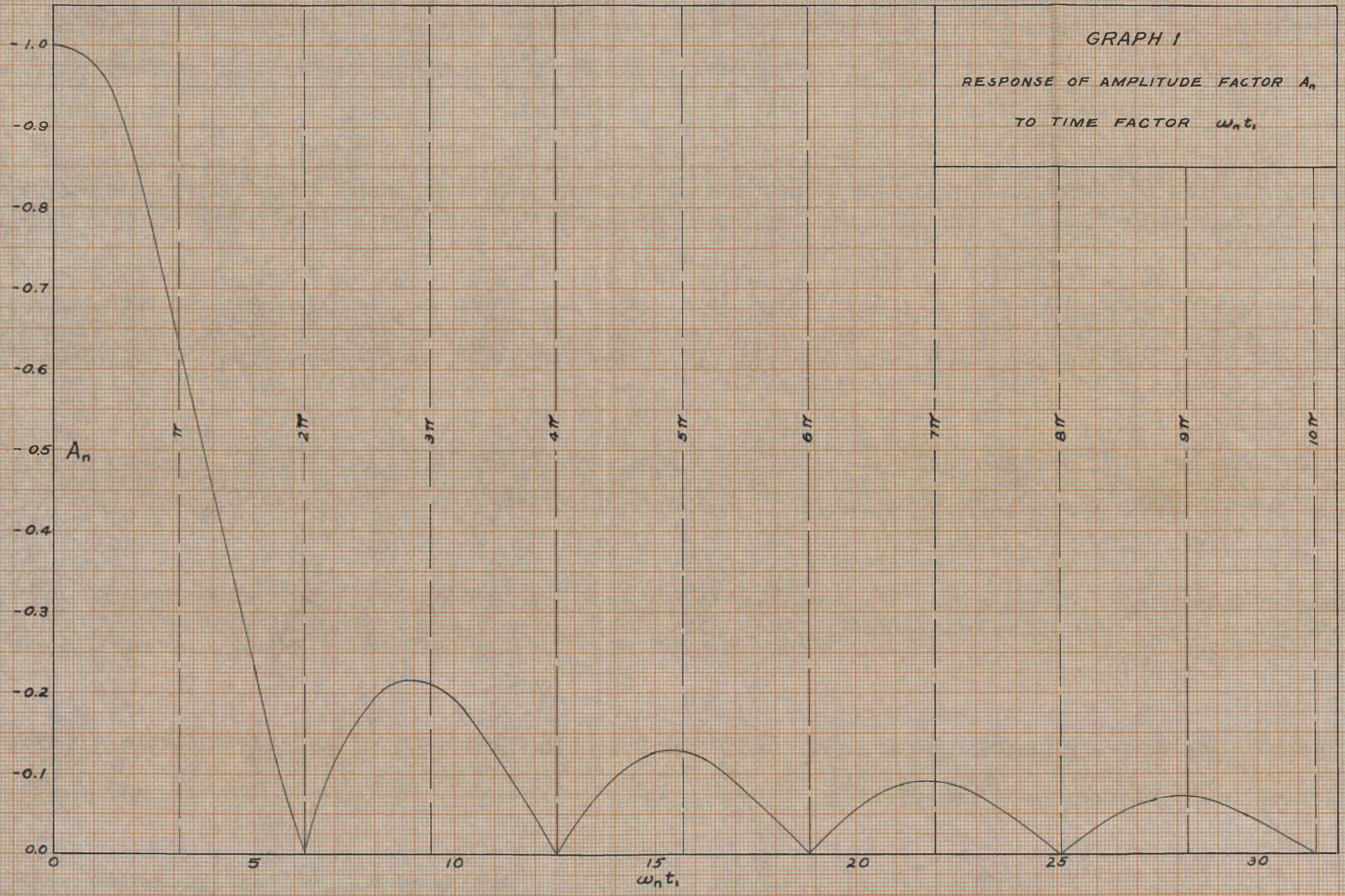
The second implication of small deflections is that the stresses are below the elastic limit and that no plastic deformation occurs; the material obeys Hooke's Law. If the stresses exceed the elastic limit, the energy losses will be high and the predicted mode shapes and frequencies will not be valid.

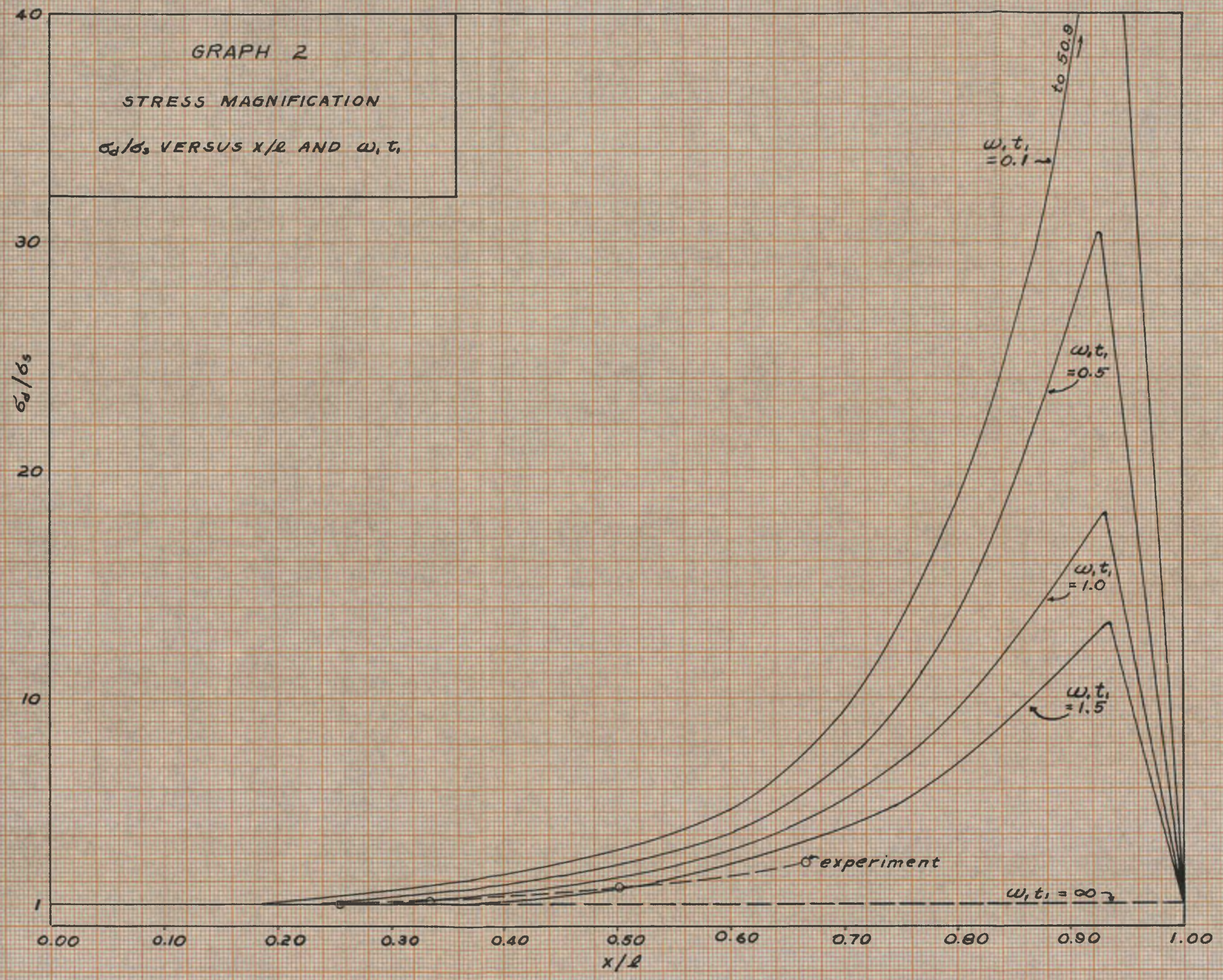
Corrections for shearing force and rotary inertia of elements of the beam become important in studying the modes of higher frequencies. At the higher frequencies the beam is divided by nodal cross sections into comparatively short portions.

The assumptions listed above are well within the accuracies of this study and no corrections have been made for plastic deformation, shearing forces, or rotary inertia.

Appendix C

GRAPHICAL REPRESENTATIONS


GRAPH I
RESPONSE OF AMPLITUDE FACTOR A_n
TO TIME FACTOR $\omega_n t_1$
A_n
$\omega_n t_1$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30
π
2π
3π
4π
5π
6π
7π
8π
9π
10π

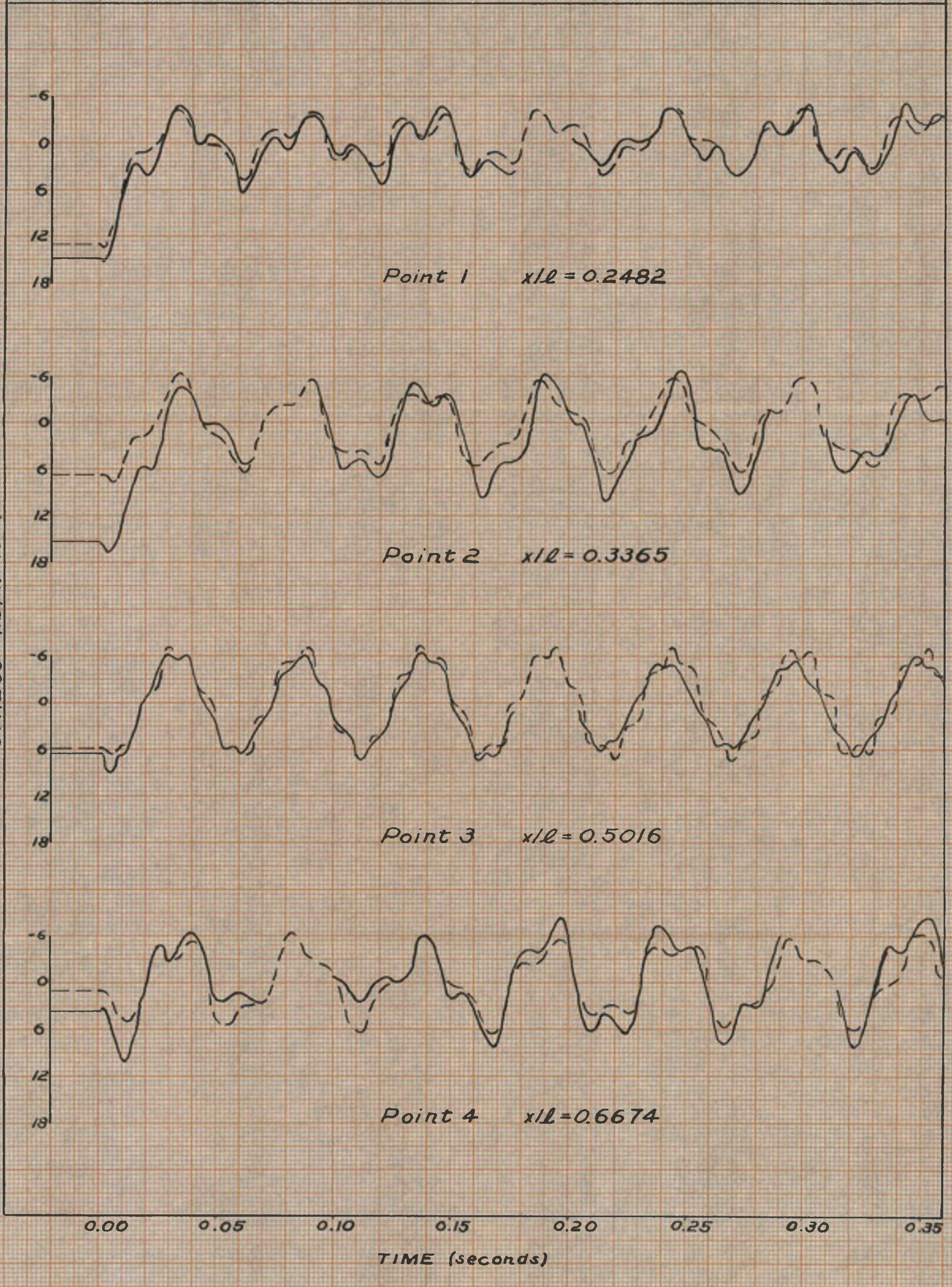


GRAPH 2
STRESS MAGNIFICATION
σ_d/σ_s VERSUS x/ℓ AND $\omega_1 t_1$
σ_d/σ_s
40
30
20
10
1
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
x/ℓ
to 50.9
$\omega_1 t_1 = 0.1$
$\omega_1 t_1 = 0.5$
$\omega_1 t_1 = 1.0$
$\omega_1 t_1 = 1.5$
$\omega_1 t_1 = \infty$
experiment

GRAPH 3

COMPARISON OF THEORY WITH EXPERIMENT

Theory - - - -   Experiment ——



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1. Timoshenko, S., Vibration Problems in Engineering, D. Van Nostrand Co., 1937, Second Edition, Sections 54-58.

2. Young, D., and Felgar, R. P., Tables of Characteristic Functions Representing Normal Modes of Vibration of Beams, University of Texas, Engineering Research Series Number 44, 1949.

3. Leonard, Robert W., On Solutions for the Transient Response of Beams, NASA Technical Report R-21, 1959.