

THE TIME DEPENDENT BEHAVIOR  
OF THE RADIOACTIVE SERIES

by

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ABSTRACT: A development of the mathematics of radioactive series decay using Laplace transforms to derive a modified form of the Bateman Equation in both the real time plane and the transformed or s-plane, with a brief discussion of certain special cases of the general equations. These general equations are then used as the basis for the development of a digital computer solution for radioactive series decay problems for series of from one to forty-nine members. The complementary analog computer solution, using the transformed equations as the mathematical model, is presented to provide maximum analytical flexibility in the study of the time dependent behavior of the radioactive series.

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## CHAPTER 1

### BASIC EQUATIONS

#### (1.1) INTRODUCTION

Calculation of the number of nuclides of a particular radioactive isotope in a given decay series which remain after a time lapse,  $t$ , or the determination of the rate of decay, termed "activity," of such an isotope is a familiar problem in nuclear science. Slow flux measurements by gold foil activation and reactor poison concentration calculations are two practical examples in which computations involving radioactive series decay play an important role.

The recent development of reliable electronic computing equipment provides us with an invaluable analytical tool for swift and accurate investigation of the behavior of a radioactive series as a function of time. We will develop and analyze the equations appropriate to the problem of series decay and present the means for their solution using digital and analog computers.

#### (1.2) BASIC EQUATIONS

(a) The Radioactive Decay Constant. The probability that a particular nuclide will disintegrate or decay in unit time is called the "radioactive decay constant," the "disintegration constant," or simply the "decay constant" in the jargon of the nuclear scientist. Some nuclides decay in more than one manner and with each decay mode is associated a specific probability of occurrence. Axiomatically, we

realize that there may exist a number of "partial decay constants" in addition to the "total decay constant" for any particular nuclide.

Certain fundamental relationships exist between the total decay constant and the partial decay constants of any radioactive nuclide, and between the partial decay constants and the "branching ratios" of that nuclide. Figure 1.1 is an example of a decay scheme. In this instance we have used Copper 64 which, having three alternate decay modes, possesses a total decay constant, three partial decay constants, and three branching ratios.

The total natural decay constant of a nuclide is the sum of all the partial decay constants of that nuclide. In mathematical form, this may be stated

$$\lambda_R = \lambda_{R^1} + \lambda_{R^2} + \dots + \lambda_{R^n} \quad (1.1)$$

The term "natural decay constant" is used to distinguish between the probability of natural decay in unit time and the probability of artificial or induced decay (neutron bombardment, e.g.) in unit time. Following the logic expressed in Eq. (1.1) we may state that the total decay constant (-the probability of decay in unit time by both natural and induced means) is the sum of all the natural partial decay constants and the induced decay constant.

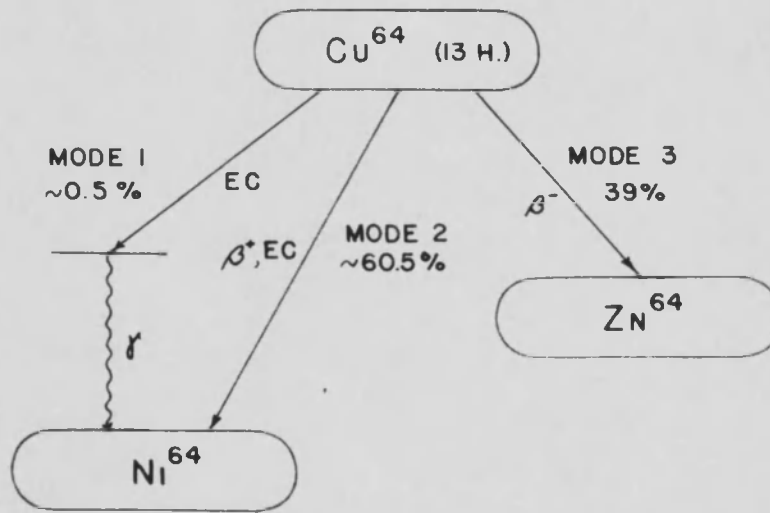
$$\lambda_R^* = \lambda_R + \phi \sigma_{\text{abs}} \quad (1.2)$$

The symbols used in Eqs. (1.1) and (1.2) are defined below and further explained in Figure 1.2.

$\lambda_R$  - the total natural decay constant of Nuclide R

$\lambda_R^*$  - the total decay constant of Nuclide R

## COPPER-64 DECAY SCHEME



$$\lambda_{\text{C}} = \frac{0.693}{(13)(3600)} = 0.148 \cdot 10^{-4} \text{ sec}^{-1}$$

$$\lambda_{\text{C}^{\text{I}}} = 0.005 \cdot \lambda_{\text{C}} = 0.0074 \cdot 10^{-5} \text{ sec}^{-1}$$

$$\lambda_{\text{C}^{\text{II}}} = 0.605 \cdot \lambda_{\text{C}} = 0.8954 \cdot 10^{-5} \text{ sec}^{-1}$$

$$\lambda_{\text{C}^{\text{III}}} = 0.390 \cdot \lambda_{\text{C}} = 0.5772 \cdot 10^{-5} \text{ sec}^{-1}$$

FIGURE 1.1

- $\lambda_{R1}$  - the partial decay constant of Nuclide R in Mode 1  
 $\lambda_{R''}$  - the partial decay constant of Nuclide R in Mode 2  
 .  
 .  
 $\lambda_{Rn}$  - the partial decay constant of Nuclide R in Mode n  
 $\phi$  - the area time density of bombarding particles per  
 (cm<sup>2</sup>)(sec). The track length.  
 $\sigma_{\text{abs}}$  - the effective cross section in cm<sup>2</sup> of Nuclide R for an  
 interaction with a bombarding particle.

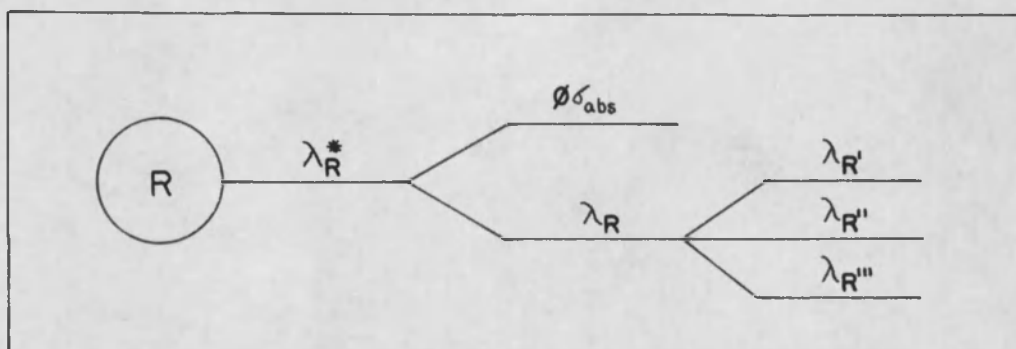


Figure 1.2. Schematic representation of the decay constants of Nuclide R described in Eqs. (1.1) and (1.2).

The branching ratio of a radioactive nuclide may be defined as the fraction of the total disintegrations of the parent nuclide which result from a particular mode of decay, such as alpha emission, beta emission, positron emission, electron capture, neutron absorption, etc. Of particular interest in series decay computations is the fraction of the total disintegrations of the parent nuclide which contribute to the growth of a given daughter nuclide. This latter statement may be thought of as a special concept of branching in the decay of a series member and is discussed in Section (1.3).

If Nuclide R, whose total probability of natural decay in unit

time is  $\lambda_R$ , decays to Nuclide S in Mode 1 with a probability of  $x$ , decays to Nuclide S in Mode 2 with a probability of  $y$ , and decays to Nuclide T in Mode 3 with a probability of  $z$ ; where the sum of  $x$ ,  $y$ , and  $z$ , is unity since we have included all the natural decay modes of Nuclide R, we may write

$$\begin{aligned}\lambda_{R'} &= x \lambda_R \\ \lambda_{R''} &= y \lambda_R \\ \lambda_{R'''} &= z \lambda_R\end{aligned}\tag{1.3}$$

(b) Radioactive Decay Laws. We will briefly review the basic radioactive decay laws as they are cogent to the treatment of series decay.

The rate of decay of any given nuclide is the rate at which the atoms comprising a sample of that nuclide are lost from the sample. Since "decay" implies a loss of atoms, the rate of change of the number of atoms in the sample is always negative (we are not considering any effects resulting from the decay of a precursor) and is equal to the probability of decay in unit time multiplied by the number of atoms present in the sample at time  $t_0$ , the start of the observation. Denoting  $N$  as the number of atoms present in the sample at any time  $t$ , and  $\lambda$  as the total decay constant of the nuclide in question, we have

$$-\frac{dN}{dt} = \lambda N\tag{1.4}$$

Integrating, we obtain

$$\int_{N_0}^N \frac{dN}{N} = -\lambda \int_0^t dt$$

or

$$N = N_0 e^{-\lambda t} \quad (1.5)$$

where  $N_0$  denotes the number of atoms present in the sample at time  $t_0$ .

Since ordinary laboratory instruments measure the intensity of the emitted radiation which is proportional to the number of disintegrations of a nuclide in unit time, i.e., the "activity" of a nuclide, we often find it convenient to work directly with this function. Extending Eq. (1.4), we readily see that the activity,  $A$ , at any time  $t$ , of any given nuclide is described by the relation

$$A = \left[ \begin{array}{c} \text{prob. of decay} \\ \text{in unit time} \end{array} \right] \left[ \begin{array}{c} \text{number of nuclides} \\ \text{risking decay} \end{array} \right] \quad (1.6)$$

or,

$$A = \lambda N \quad (1.7)$$

which, when substituted into Eq. (1.5), yields

$$A = A_0 e^{-\lambda t} \quad (1.8)$$

The probability of decay of a nuclide is usually expressed in terms of another function, the half-period or half-life of a nuclide. Denoted by  $T_{\frac{1}{2}}$ , the half-period is the time required for one-half of the number of atoms present at time  $t_0$  in a pure sample to disintegrate. Setting  $N/N_0 = \frac{1}{2}$ , and  $t = T_{\frac{1}{2}}$  in Eq. (1.5), we obtain

$$T_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} \quad (1.9)$$

The quantities  $\lambda$ ,  $N$ ,  $A$ , and  $T_{\frac{1}{2}}$  are average values for a large number of nuclides and are sufficiently accurate for most computational purposes since we are normally concerned with samples of milligram

mass. The mean life,  $\bar{T}$ , of a large number of atoms is an absolute quantity, and, by definition, a mean value. Eq. (1.5) states that the number of atoms,  $N$ , remaining after a time  $t$  from an initial stock of  $N_0$  atoms is  $N_0 e^{-\lambda t}$ . Each of these  $N$  atoms has lived for a time  $t$  after  $t_0$  and all will decay sometime between  $(t_0 + t)$  and time infinity. However, a few of the atoms will decay in a small increment of time,  $dt$ , between  $t$  and  $(t + dt)$ . These few atoms number  $N \lambda(dt)$  and have a specific lifetime of  $t$ . The mean lifetime of all the atoms in the sample is then given by the familiar mean value equation.

$$\bar{T} = \frac{\int_0^{\infty} t N \lambda dt}{\int_0^{\infty} N \lambda dt} = \frac{N_0 \int_0^{\infty} t \lambda e^{-\lambda t} dt}{N_0 \int_0^{\infty} \lambda e^{-\lambda t} dt} = \frac{1}{\lambda} \quad (1.10)$$

The expression of the decay constant,  $\lambda$ , in terms of the mean life,  $\bar{T}$ , will facilitate the discussion of the analog computer simulation of the Bateman Equation in Chapter 5.

### (1.3) THE GENERALIZED RADIOACTIVE SERIES

A simple radioactive series may be generalized by the following schematic:

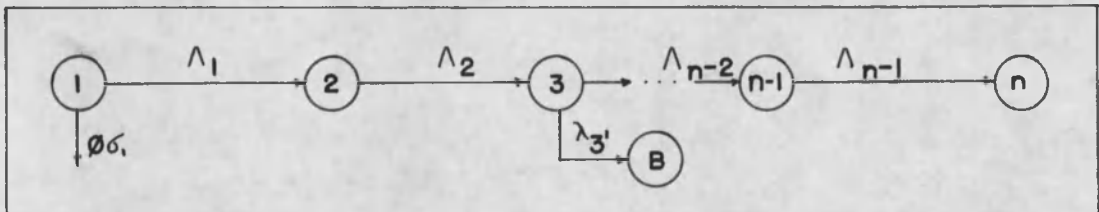


Figure 1.3

A brief perusal of Figure 1.3 should indicate that one need be concerned only with the partial decay constants of the precursors of any given nuclide in the series when discussing the rate of growth of that nuclide; hence, the introduction of the symbol  $\Lambda_j$  denoting the partial decay constant of the  $j^{\text{th}}$  nuclide in a given decay series in the direction of the perpetuation of that series. However, when discussing the rate of decay of the  $j^{\text{th}}$  nuclide, the total decay constant  $\lambda_j^*$  must be considered. This differentiation may seem minute at first glance, but it is quite important to our discussion of radioactive series decay in Chapter 2.

## CHAPTER 2

### THE GENERAL BATEMAN EQUATION

#### (2.1) GENERAL

The basic decay laws and the terms associated with radioactive decay discussed in the preceding chapter are the fundamental tools required to obtain a mathematical model for the growth, decay, activity, etc. of any given nuclide in a decay series.

#### (2.2) THE THREE MEMBER DECAY CHAIN

Let us consider the simple case of a three member decay chain wherein a radioactive parent decays to a radioactive daughter which in turn decays to a granddaughter (whose stability is of no consequence to the problem). Consider the decay chain represented by Figure 2.1.

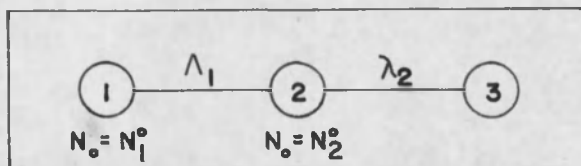


Figure 2.1

In word form, the balance equation describing the rate of change of the number of atoms of Nuclide 2 remaining at any time  $t$  after  $t_0$  is

$$\begin{aligned}
 \left( \begin{array}{l} \text{Rate of change of} \\ \text{the number of} \\ \text{daughter atoms} \end{array} \right) &= \left( \begin{array}{l} \text{Rate of growth} \\ \text{of the daughter} \\ \text{nuclide} \end{array} \right) - \left( \begin{array}{l} \text{Rate of decay of} \\ \text{the daughter} \\ \text{nuclide} \end{array} \right) \\
 &= \left( \begin{array}{l} \text{Rate of decay of} \\ \text{the parent nuclide} \\ \text{to the daughter} \end{array} \right) - \left( \begin{array}{l} \text{Rate of decay of} \\ \text{the daughter} \\ \text{nuclide} \end{array} \right)
 \end{aligned}$$

and the mathematical expression is

$$\frac{dN_2}{dt} = \Lambda_1 N_1 - \lambda_2 N_2 \quad (2.1)$$

$$= \Lambda_1 N_1^0 e^{-\lambda_1 t} - \lambda_2 N_2 \quad (2.2)$$

where  $N_1^0$  denotes the initial number of atoms of Nuclide 1 present at time  $t_0$ . Rearranging Eq. (2.2) we obtain

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \Lambda_1 N_1^0 e^{-\lambda_1 t} \quad (2.3)$$

Taking the Laplace transform of Eq. (2.3),

$$s n_2(s) - N_2^0 + \lambda_2 n_2(s) = \frac{\Lambda_1 N_1^0}{s + \lambda_1} \quad (2.4)$$

and collecting terms, we get

$$n_2(s) = \frac{\Lambda_1 N_1^0}{(s + \lambda_1)(s + \lambda_2)} + \frac{N_2^0}{(s + \lambda_2)} \quad (2.5)$$

where the final term in Eq. (2.5) results from the initial condition on Nuclide 2 at time  $t_0$ .

The inverse transform of the above equation may be obtained by any of a number of means, but the "residue method" is chosen here as being the most facile, especially when a particularly complicated equation is involved. Employing the residue method on the second term in Eq. (2.5) and direct inverse transformation on the last term yields

$$N_2 = R_{(-\lambda_1)} + R_{(-\lambda_2)} + N_2^0 e^{-\lambda_2 t} \quad (2.6)$$

and the residue terms are identified as follows:

$$\begin{aligned}
 R_{(-\lambda_1)} &= (s + \lambda_1) \frac{\Lambda_1 N_1^0}{(s + \lambda_1)(s + \lambda_2)} e^{st} \Big|_{s = -\lambda_1} \\
 &= \frac{\Lambda_1 N_1^0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}
 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
 R_{(-\lambda_2)} &= (s + \lambda_2) \frac{\Lambda_1 N_1^0}{(s + \lambda_1)(s + \lambda_2)} e^{st} \Big|_{s = -\lambda_2} \\
 &= \frac{\Lambda_1 N_1^0}{\lambda_1 - \lambda_2} e^{-\lambda_2 t}
 \end{aligned} \tag{2.8}$$

Therefore,

$$N_2 = N_1^0 \left( \frac{\Lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\Lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t} \right) + N_2^0 e^{-\lambda_2 t} \tag{2.9}$$

This latter equation is deliberately not reduced by the common factor  $(\Lambda_1 / \lambda_2 - \lambda_1)$  as we eventually wish to deduce a general expression for the coefficients of the exponential terms.

Notice that the last term in Eq. (2.9), resulting from the initial condition specifying the number of atoms of Nuclide 2 present at time  $t_0$ , is merely the exponential decay equation describing the number of those atoms remaining at time  $t$  after  $t_0$ . We can ignore this type of initial condition in the transformed equation, e.g. Eq. (2.4), by separately computing its contribution to the number of atoms of Nuclide 2 remaining after a lapse of time,  $t$ , and adding the result to the final expression. Each of these computations is considered as a separate problem since the precursors of a nuclide have no effect on the rate of

decay of an initial stock of atoms of that nuclide. This last statement does not necessarily apply (upon deletion of the words "of an initial stock") to a daughter nuclide whose existence depends solely on the decay of her parent. Consider the very real possibility of a parent nuclide whose half-period is extremely long compared to the half-period of the daughter. An excellent case in point is the alpha decay of Radon 222 to Polonium 218. The half-periods of the parent and daughter are, respectively, 3.82 days and 3.05 minutes. A short time after a pure sample of the parent is assembled the rate of decay of the daughter is governed by the rate at which the daughter atoms are formed. This condition is termed secular equilibrium and is discussed in more detail in Chapter 3.

The procedure for separating the basic equation from the effects arising from various initial conditions discussed in the preceding paragraph obviously cannot apply to the prime parent (the first nuclide in the series) as the entire equation would then disappear. It applies only to initial conditions on the descendants of the prime parent. Hereafter we will use this more convenient method to develop the basic equation for an n-member series.

It should be noted that the several total decay constants appear everywhere in the equation but in the numerator of the ratios of the functions of the several decay constants.

### (2.3) THE FOUR MEMBER DECAY CHAIN

We will now consider a four member radioactive series from which

we may deduce the general expression for any n-member series.

Consider the series represented in Figure 2.2.

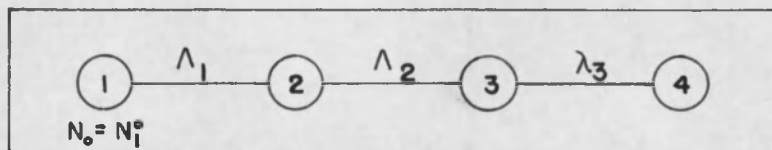


Figure 2.2

We are interested in the number of atoms of Nuclide 3 remaining at time  $t$  after  $t_0$ . Writing the balance equation in the manner previously described, we have

$$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3 \quad (2.10)$$

It is again imperative to note that the rate of growth of Nuclide 3 is dependent only on the directional rate of decay of Nuclide 2.

Substituting the expression obtained for  $N_2$  from Eq. (2.9), but disregarding the last term of that expression, into Eq. (2.10), we get, after rearranging terms,

$$\frac{dN_3}{dt} + \lambda_3 N_3 = \frac{\lambda_1 \lambda_2 N_1^0}{\lambda_2 - \lambda_1} \left[ e^{-\lambda_1 t} - e^{-\lambda_2 t} \right] \quad (2.11)$$

Taking the Laplace transform,

$$s n_3(s) + \lambda_3 n_3(s) = \frac{\lambda_1 \lambda_2 N_1^0}{\lambda_2 - \lambda_1} \left[ \frac{1}{(s + \lambda_1)} - \frac{1}{(s + \lambda_2)} \right] \quad (2.12)$$

and solving for  $n_3(s)$ , yields

$$n_3(s) = \frac{\lambda_1 \lambda_2 N_1^0}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)} \quad (2.13)$$

Note the development of a pattern for the transformed equation by comparing Eq. (2.13) with Eq. (2.5), disregarding the last term in the latter equation.

We will now assume without formal proof a general s-plane equation for the number of atoms of any nuclide ( $N_j$ ) remaining at time  $t$  after  $t_0$  in a radioactive decay series.

$$n_j(s) = N_1^0 \left[ \frac{\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_{(j-1)}}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3) \cdots (s + \lambda_j)} \right] \quad (2.14)$$

Similarly, the general transformed equation for the activity of Nuclide  $j$  is

$$a_j(s) = \lambda_j n_j(s) \quad (2.15)$$

Eqs. (2.14) and (2.15) form the basis for the development of an analog computer simulation of the general Bateman Equation and will be treated at greater length in Chapter 5.

Returning now to Eq. (2.13), the solution, by the residue method, is

$$N_3 = N_1^0 \left[ R_{(-\lambda_1)} + R_{(-\lambda_2)} + R_{(-\lambda_3)} \right] \quad (2.16)$$

where

$$\begin{aligned} R_{(-\lambda_1)} &= (s + \lambda_1) \frac{\lambda_1 \lambda_2}{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)} e^{st} \Big|_{s = -\lambda_1} \\ &= \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 t} \end{aligned} \quad (2.17)$$

and similarly,

$$R_{(-\lambda_2)} = \frac{\Lambda_1 \Lambda_2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} e^{-\lambda_2 t} \quad (2.18)$$

$$R_{(-\lambda_3)} = \frac{\Lambda_1 \Lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t} \quad (2.19)$$

Substituting Eqs. (2.17), (2.18), and (2.19) into Eq. (2.16), we obtain

$$N_3 = N_1^0 \left[ \frac{\Lambda_1 \Lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 t} + \frac{\Lambda_1 \Lambda_2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} e^{-\lambda_2 t} + \frac{\Lambda_1 \Lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t} \right] \quad (2.20)$$

If we were to factor out of the brackets the  $\Lambda_1$  term we would have  $\Lambda_1 N_1^0$  as a common multiplier. This term obviously represents the partial initial activity of the prime parent and should not be confused with initial activity, written  $\lambda_1 N_1^0$ . In addition, one should realize that the directional decay constant,  $\Lambda_j$ , may often be identical to the total natural decay constant,  $\lambda_j$ , but will rarely be identical to the total decay constant,  $\lambda_j^*$ , for those nuclides which decay naturally and are exposed to the additional risk of disintegration by particle bombardment.  $\Lambda_j$  will always be equal to the total decay constant,  $\phi \sigma_{\text{abs}}$ , for those stable nuclides subjected to particle bombardment.

#### (2.4) THE GENERAL BATEMAN EQUATION

The general expression for the number of atoms of any Nuclide  $j$

of an n-member decay chain remaining at time t after  $t_0$  may be assumed from a comparison between Eqs. (1.5), (2.9), and (2.20). Eq. (2.21) is a modified form of the general equation developed in 1910 by Bateman<sup>1</sup>.

$$N_j = N_1^0 \left[ K_1 e^{-\lambda_1 t} + K_2 e^{-\lambda_2 t} + K_3 e^{-\lambda_3 t} + \dots + K_j e^{-\lambda_j t} \right] \quad (2.21)$$

where

$$K_1 \text{ of } j = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \dots \lambda_{(j-1)}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) \dots (\lambda_j - \lambda_1)}$$

$$K_2 \text{ of } j = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \dots \lambda_{(j-1)}}{(\lambda_1 - \lambda_2) (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2) \dots (\lambda_j - \lambda_2)}$$

$$K_3 \text{ of } j = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \dots \lambda_{(j-1)}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) (\lambda_4 - \lambda_3) \dots (\lambda_j - \lambda_3)}$$

⋮  
⋮

$$K_i \text{ of } j = \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \dots \lambda_{(j-1)}}{(\lambda_1 - \lambda_i) \dots (\lambda_{i-1} - \lambda_i) (\lambda_{i+1} - \lambda_i) \dots (\lambda_j - \lambda_i)}$$

and wherein we always omit the one term (denoted by an asterisk) in the succession of terms of the denominator which would allow the coefficient to become arbitrarily large.

Further generalization of Eq. (2.21) leads to

$$N_j = N_1^0 \sum_{i=1}^j K_i e^{-\lambda_i t} + \sum_{k=2}^j C_k^0 \quad (2.22)$$

---

<sup>1</sup>H. Bateman, "The Solution of a System of Differential Equations Occurring in the Theory of Radio-active Transformations," Proc. Cambridge Phil. Soc., Vol. 16, 1910, p. 423

where

$$C_k^0 = \text{Contribution to } N_j \text{ resulting from an initial condition on the } k^{\text{th}} \text{ precursor.} \quad (2.23)$$

$K_i$  is defined as before in Eq. (2.21)

The equation describing the activity of the  $j^{\text{th}}$  member nuclide follows directly from Eq. (2.22).

$$A_j = \lambda_j N_j = \lambda_j N_1^0 \sum_{i=1}^j K_i e^{-\lambda_i t} + \lambda_j \sum_{k=2}^j C_k^0 \quad (2.24)$$

where  $A_j$  is the activity of the  $j^{\text{th}}$  member nuclide and all other terms are defined in Eq. (2.23).  $A_j$  may also be expressed as a function of initial activities.

$$\begin{aligned} A_j &= \lambda_j \left[ \frac{\lambda_1 N_1^0}{\lambda_1} \sum_{i=1}^j \epsilon_i + \sum_{k=2}^j \frac{\lambda_k N_k^0}{\lambda_k} \epsilon_k \right] \\ &= \lambda_j \left[ \frac{A_1^0}{\lambda_1} \sum_{i=1}^j \epsilon_i + \sum_{k=2}^j \frac{A_k^0}{\lambda_k} \epsilon_k \right] \end{aligned} \quad (2.25)$$

where

$$\begin{aligned} \epsilon_i &= K_i e^{-\lambda_i t} \\ \epsilon_k &= K_k e^{-\lambda_k t} \end{aligned} \quad (2.26)$$

Eqs. (2.25) and (2.26) form the basis for the digital computer solution to radioactive series decay computations, given a discrete time interval, developed in Chapter 4.

## CHAPTER 3

### SPECIAL CASES OF THE GENERAL EQUATION

#### (3.1) GENERAL

The interrelation of the decay constants of certain parent-daughter decay combinations gives rise to special cases of the basic equation describing the behavior of the radioactive series with time.

In some instances one might wish to determine the rate of build-up or production of a stable end product in a decay series. Other problems might require knowledge of the time lapse to maximum activity of the daughter; or perhaps the time interval required for the activities of the parent and daughter nuclides to become approximately equivalent is desired.

These interesting problems are all special cases of the general equation developed in Chapter 2 and will be treated in the following sections.

The basic reference used in this chapter is Evans<sup>1</sup> supplemented by Kaplan<sup>2</sup>. The three member decay chain is used as the primary vehicle for explanation of these special cases.

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<sup>1</sup>R. D. Evans, The Atomic Nucleus, McGraw-Hill Book Co., Inc., 1955, Chap. 15

<sup>2</sup>I. Kaplan, Nuclear Physics, Addison-Wesley Publishing Co., Inc., 1955, Chap. 10

(3.2) THE STABLE END PRODUCT

The stable end product presents no particular complication in the solution to the general equation. The solution follows immediately after the substitution of  $\lambda_j = 0$  in the general equation. This substitution obviously indicates that there is no loss of atoms from the  $j^{\text{th}}$  nuclide (hence the activity,  $A_j$ , is zero) and the solution for  $N_j$  yields the total number of atoms of Nuclide  $j$  produced during the time interval of the problem. Note that  $\Lambda_j$  does not appear in the equation.

(3.3) EQUILIBRIUM CONDITIONS

If in the equation describing the activity of the first daughter nuclide, (2), at any time  $t$  after  $t_0$ ,

$$A_2 = \lambda_2 N_2 = \frac{\lambda_2 \Lambda_1 N_1^0}{\lambda_2 - \lambda_1} \left( e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \quad (3.1)$$

we substitute the exponential expression, Eq. (1.5), describing the decay of the parent, Nuclide 1, during the same time interval,

$$A_2 = \frac{\lambda_2 \Lambda_1}{\lambda_2 - \lambda_1} \left( N_1 e^{+\lambda_1 t} \right) \left( e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \quad (3.2)$$

and write the result as an expression describing the ratio of the activity of the daughter to the partial activity of the parent,

$$\frac{A_2}{A_{1p}} = \frac{\lambda_2 N_2}{\Lambda_1 N_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1} \left( 1 - e^{-(\lambda_2 - \lambda_1)t} \right) \quad (3.3)$$

we can describe two types of equilibrium by presupposing certain conditions on the interrelation of the decay constants of the parent and daughter nuclides.

(a) Transient Equilibrium. If  $\lambda_1 < \lambda_2$  ( $T_{1/21} > T_{1/22}$ , i.e., the daughter is shorter lived than the parent) then

$$\frac{\lambda_2}{\lambda_2 - \lambda_1} > 1 \quad (3.4)$$

and  $[1 - e^{-(\lambda_2 - \lambda_1)t}]$  approaches its maximum value, unity, as  $t$  becomes arbitrarily large. Therefore,

$$\lim_{t \rightarrow \infty} \left( \frac{A_2}{A_{1p}} \right) = \frac{\lambda_2}{\lambda_2 - \lambda_1} > 1 \quad (3.5)$$

and this condition is termed "transient equilibrium."

Properly, transient equilibrium exists only after the ratio of activities becomes significantly greater than unity and is approaching an asymptotic value. It is readily apparent that many parent-daughter combinations possess this necessary relationship between their decay constants and that the subsequent attainment of transient equilibrium will be governed only by the time interval,  $t$ . Figure 3.1 illustrates, schematically, the approach to transient equilibrium.

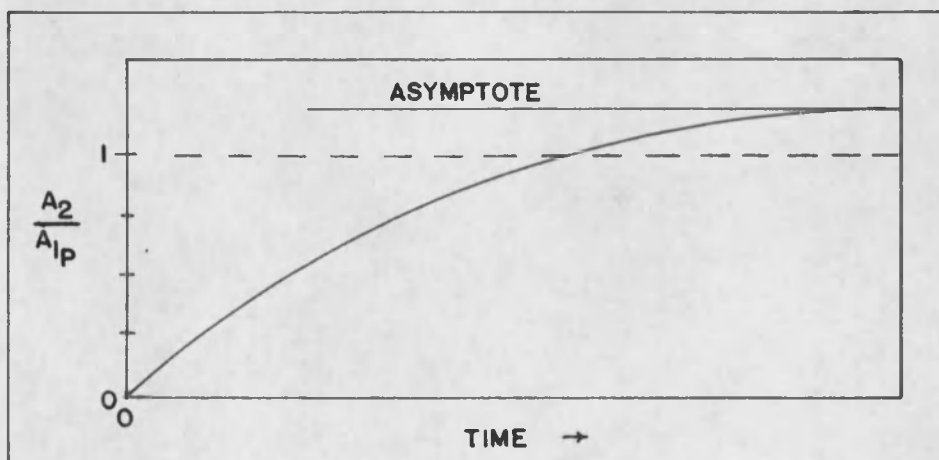


Figure 3.1. The approach to transient equilibrium.

(b) Secular Equilibrium. If  $\lambda_1 \ll \lambda_2$  ( $T_{\frac{1}{\lambda_1}} \gg T_{\frac{1}{\lambda_2}}$ , i.e., the daughter is much shorter lived than the parent) then

$$\frac{\lambda_2}{\lambda_2 - \lambda_1} \sim 1 \quad (3.6)$$

and the exponential again approaches zero as  $t$  is allowed to become arbitrarily large. Therefore,

$$\lim_{t \rightarrow \infty} \left( \frac{A_2}{A_{1P}} \right) = \frac{\lambda_2}{\lambda_2 - \lambda_1} \sim 1 \quad (3.7)$$

and the approach of the activity ratio to the asymptote of approximate unit value is spoken of as "secular equilibrium." Figure 3.2 is a schematic illustration of the phenomenon.

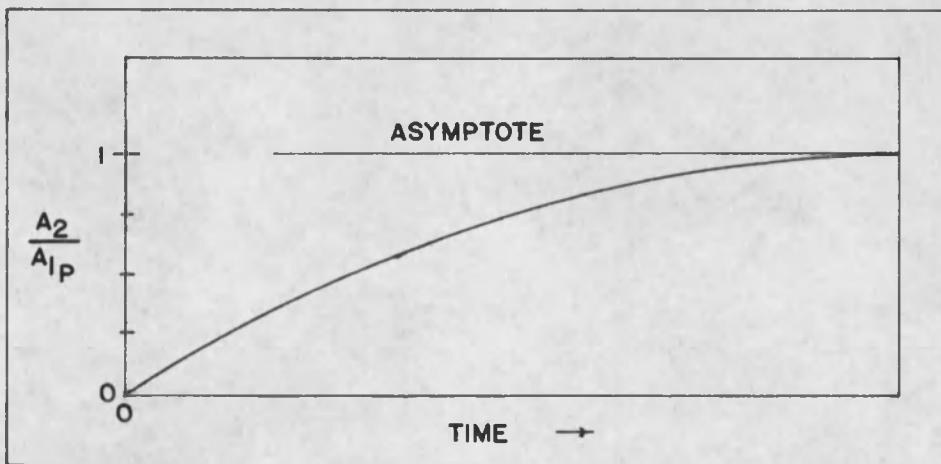


Figure 3.2. The approach to secular equilibrium.

Since the semantic difference between "less than" and "much less than" is questionable, it is desirable to differentiate between the two types of equilibrium, transient and secular, through strict adherence to the value of the asymptotic limit of the activity ratio.

(3.4) MAXIMUM ACTIVITY OF THE DAUGHTER NUCLIDE

A plot of Activity vs. Time ( $A_2$  vs.  $t$ ) from Eq. (3.1) would indicate that the activity at the terminal points of the time dimension  $[0, \infty)$  is zero and that its value becomes positive and non-zero, reaches a maximum value, and declines somewhere between these two extremes of time. Since the activity is never negative, differentiation of  $A_2$  with respect to time, with the resulting expression set equal to zero, will allow us to solve for the time,  $t_m$ , of maximum activity.

$$\frac{dA_2}{dt} = 0 = \frac{\lambda_2 \lambda_1 N_1^0}{\lambda_2 - \lambda_1} \left[ \lambda_2 e^{-\lambda_2 t} - \lambda_1 e^{-\lambda_1 t} \right] \quad (3.8)$$

Solving for  $t = t_m$ ,

$$e^{(\lambda_2 - \lambda_1)t_m} = \frac{\lambda_2}{\lambda_1}$$

$$t_m = \frac{\ln \lambda_2 - \ln \lambda_1}{\lambda_2 - \lambda_1}, \quad (3.9)$$

we see that the time interval to maximum activity of the daughter nuclide is a function of the decay constants (or half-periods) of the parent and daughter nuclides. A transcendental equation in the time variable arises when more than two nuclides are represented in the equation.

CHAPTER 4  
DIGITAL COMPUTER SOLUTION TO THE  
BATEMAN TYPE EQUATION

(4.1) GENERAL

Eq. (2.25), which describes the activity of the  $j^{\text{th}}$  member nuclide at time  $t$  after  $t_0$ , becomes quite unwieldy for numerical computation if a number of nuclides comprise the series. Obtaining a solution by manual computation with as few as three or four member series is tedious, to say the least. However, as a pattern has been established in Eq. (2.21) and repeated in Eq. (2.25), these equations readily lend themselves to solution by a digital computer for a discrete time interval.

A digital computer solution to the transcendental equation arising when the time interval to a particular activity is desired is feasible but beyond the scope of this thesis. The transcendental solution is readily obtained (to graphic accuracy) from the analog computer solution discussed in Chapter 5.

(4.2) CAPABILITIES AND LIMITATIONS OF THE SOLUTION

The solution, as presented, is programmed in SOAP X<sup>1</sup> (Symbolic Optimizing Assembly Program X) for the IBM 650 Data Processing System

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<sup>1</sup>Staff, Numerical Analysis Laboratory, College of Engineering, The University of Arizona, SOAP X, Tucson, Arizona.

having Indexing Registers, Immediate Access Storage, and an Alphabetic Attachment. The program may be easily converted for use with more basic systems. Single problems are limited to samples of, at most, three isotopes in a common radioactive series consisting of a maximum of 49 members. These limitations are not considered overly restrictive as most decay chains consist of less than 25 members and an unstable sample of three isotopes is usually the most complex arrangement presented by the average problem. More important, solution by addition of sub-solutions to any convenient combination of sub-samples or sub-series is easily obtainable:

- (1) For samples consisting of more than three unstable isotopes
- (2) When the sample consists of nuclides from several distinct radioactive series
- (3) When more than one mode is available for the decay of any given isotope in the series

#### (4.3) THE SOLUTION

The following discussion is limited to natural radioactive decay. Induced decay resulting from particle bombardment is similarly treated but involves somewhat more complicated decay chains among the heavy elements (usually a combination of several natural decay series whose prime parents are the members of the artificial series produced by the bombardment).

Two major characteristics, whose mere existence greatly simplifies the computer solution, common to all natural decay series are illustrated by a study of three representative natural radioactive series;

by prime parent, Uranium 238 (UI), Neptunium 237, and Plutonium 239, represented in Figures 4.1, 4.2, and 4.3, respectively.

The first characteristic is simply that when a branch occurs in the series (due to alternate modes of decay) the resulting companion series<sup>1</sup> are eventually rejoined, generally without any sub-branching, or branching of a branch member. The descendants of  $\text{Ac}^{227}$  in the Plutonium 239 series (see Figure 4.3) are striking exceptions to the latter statement concerning sub-branching.

Secondly, all branches are characteristically symmetric. That is, an equal number of distinct nuclides are formed in each of the companion branches. Isomeric pairs are not considered separately in the above sense since the half-periods of the excited daughter nuclides are negligible in comparison to the half-periods of the parent nuclides.

If we consider a radioactive series as a combination of  $k$  sub-series, each of which constitutes one of the  $k$  possible routes which might be taken from the prime parent nuclide to the  $n^{\text{th}}$  member of the series, we can deal separately with each of these simplified series and add their respective solutions for the complete solution to the entire series. Figures 4.4, 4.5, and 4.6 illustrate such a combination of sub-series for each of the complete series represented in Figures 4.1, 4.2, and 4.3.

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<sup>1</sup>Companion series or companion branches are defined here to be partial series with common antecedent and descendant nuclides. For example,  $\text{Bi}^{213} \rightarrow \text{Po}^{213} \rightarrow \text{Pb}^{209}$  and  $\text{Bi}^{213} \rightarrow \text{Tl}^{209} \rightarrow \text{Pb}^{209}$  in Figure 4.2 are companion series.

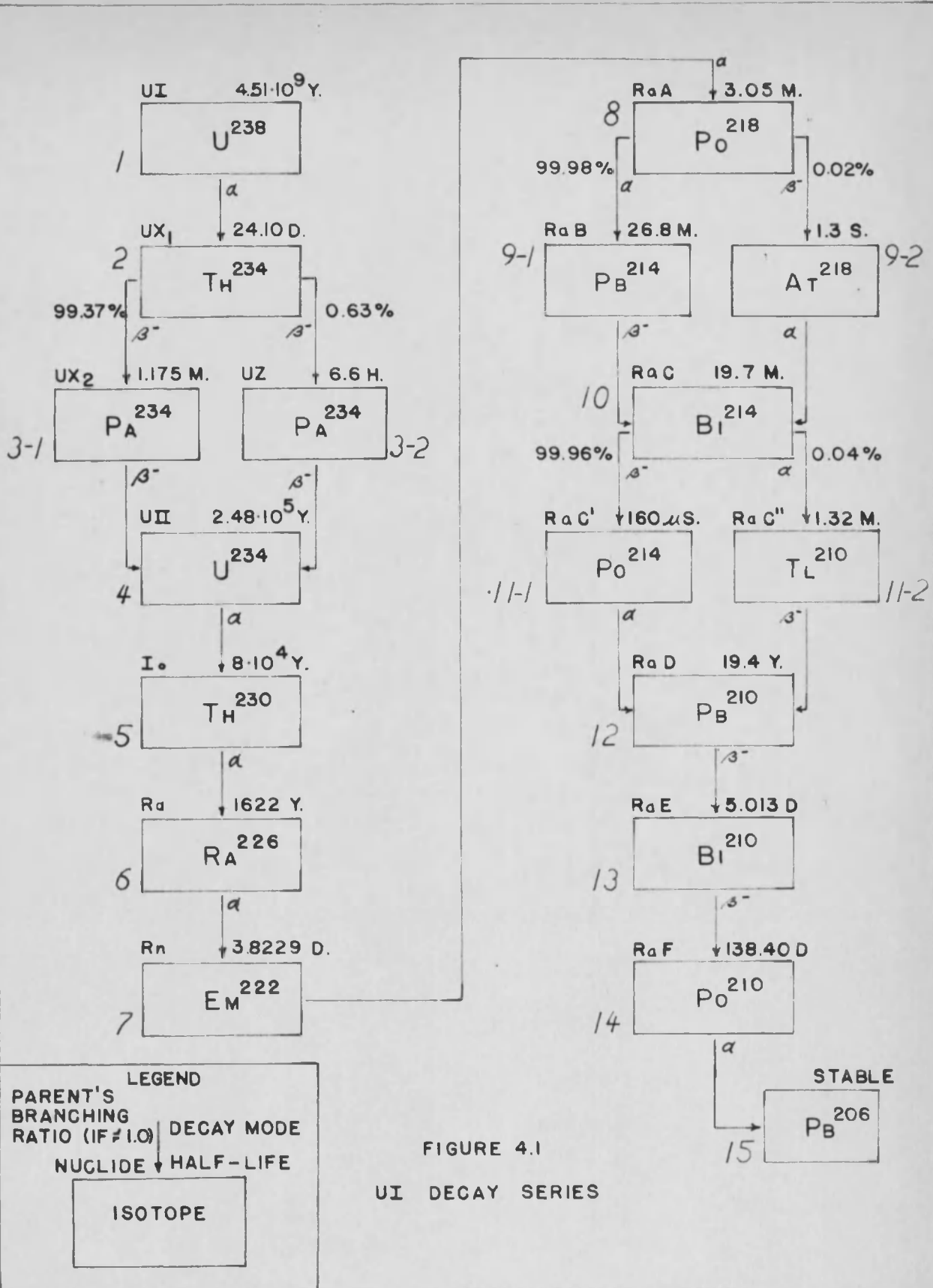


FIGURE 4.1

UI DECAY SERIES

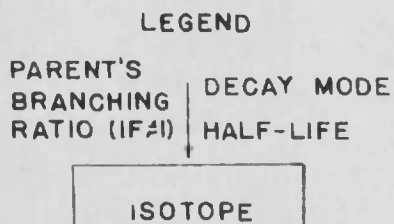
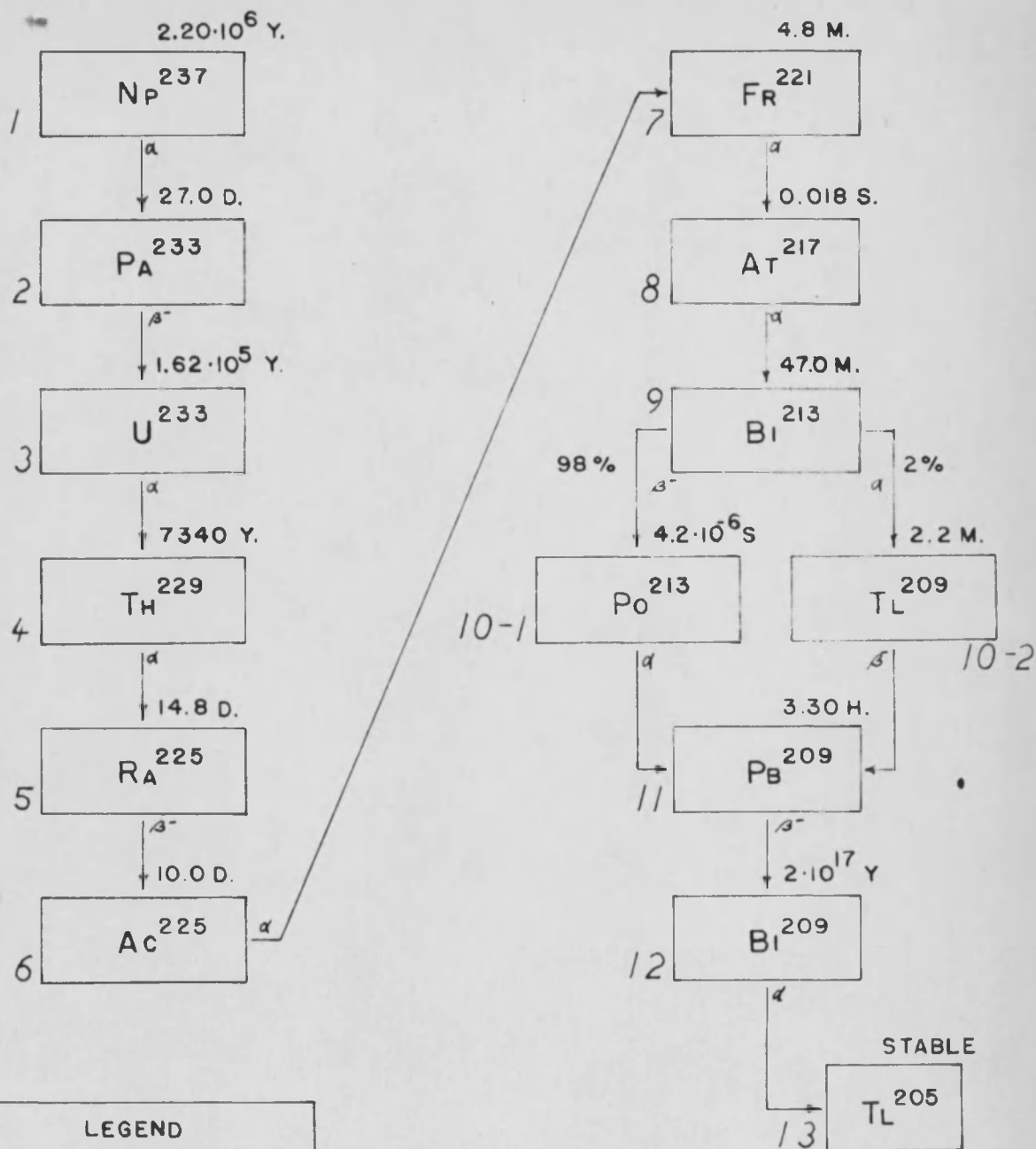
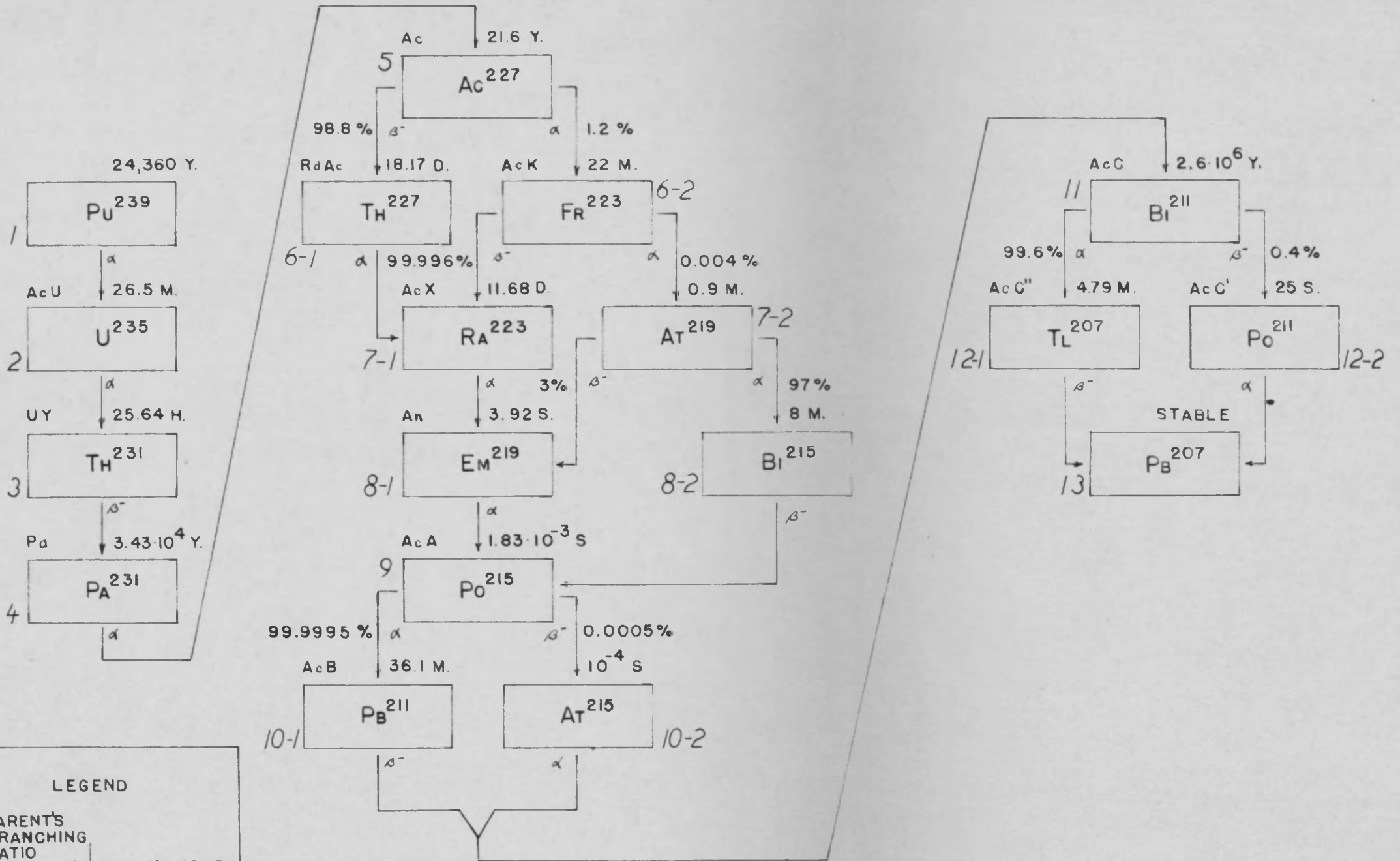


FIGURE 4.2  
 $\text{Np}^{237}$   
 DECAY SERIES



**LEGEND**

PARENT'S BRANCHING RATIO (IF $\neq 1.0$ )	DECAY MODE
NUCLIDE	HALF-LIFE
ISOTOPE	

FIGURE 4.3  
 $Pu^{239}$   
DECAY SERIES

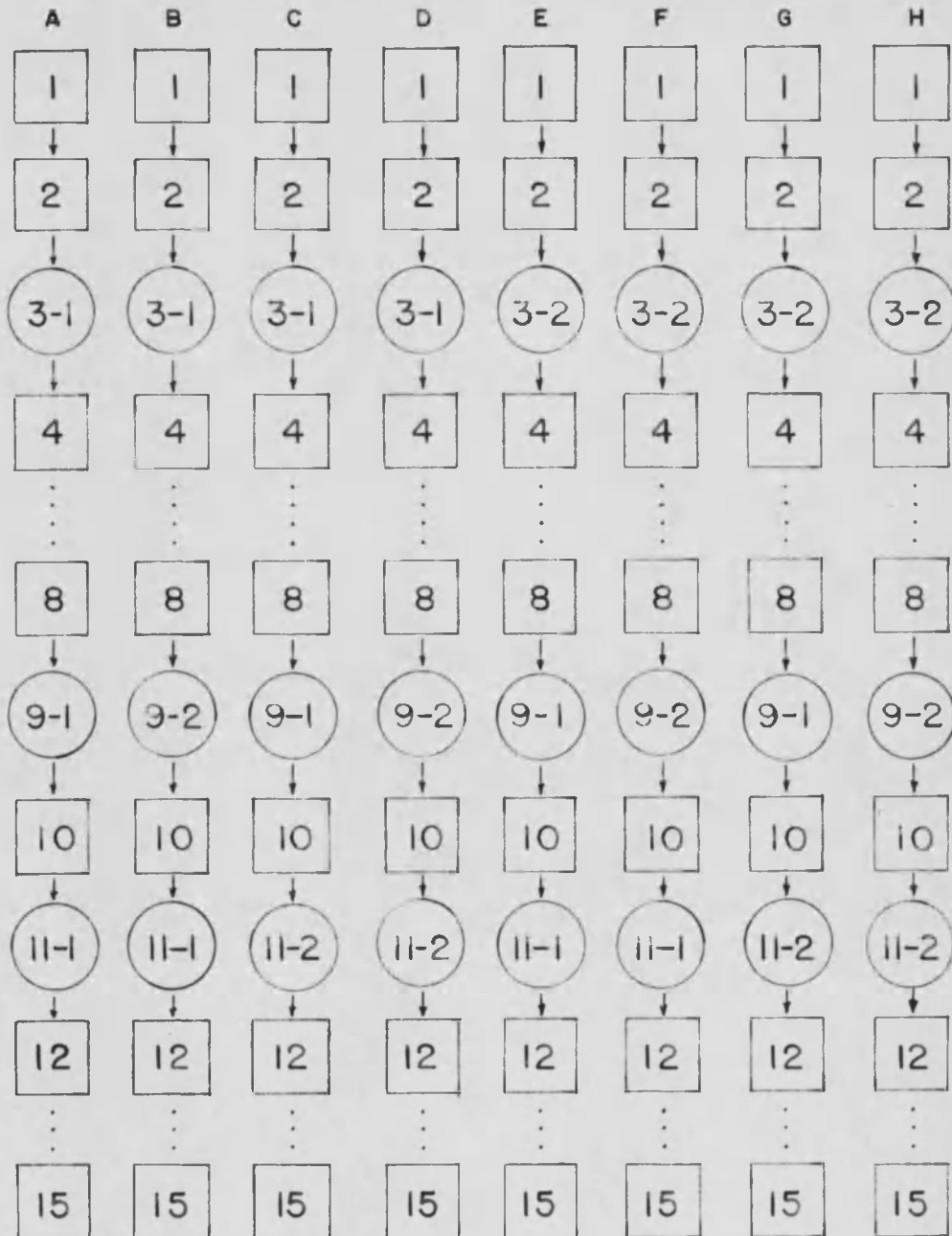
SUB-SERIES OF THE U<sup>238</sup> SERIES

FIGURE 4.4

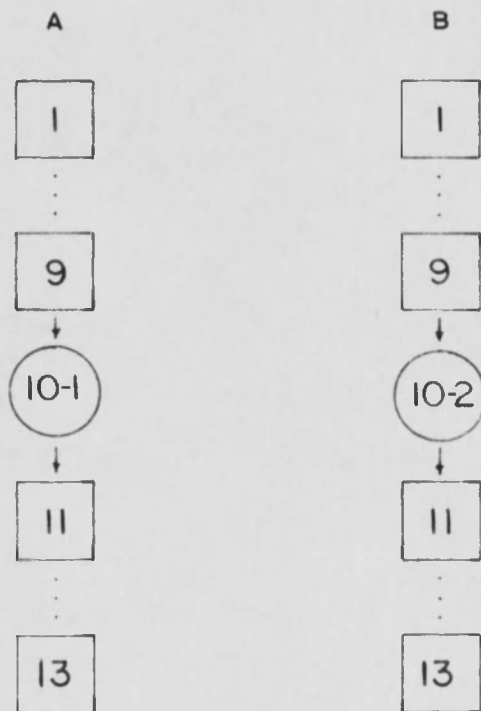
SUB-SERIES OF THE  $\text{Np}^{237}$  SERIES

FIGURE 4.5

SUB-SERIES OF THE Pu<sup>239</sup> SERIES

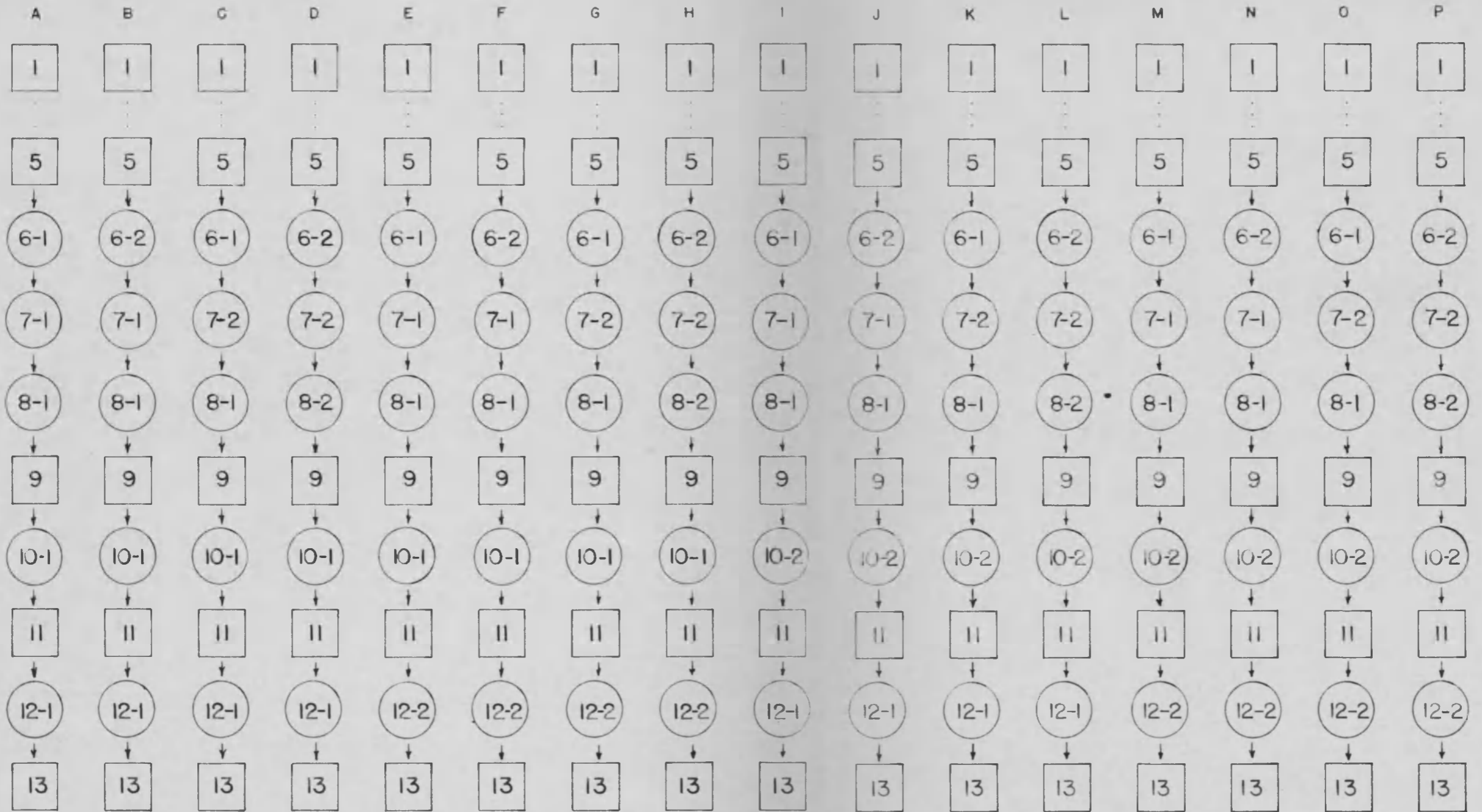


FIGURE 4.6

Some of these sub-series are insignificant for most practical purposes in that their use probability is negligible. Route H of Figure 4.4 has a use probability of about  $0.504 \cdot 10^{-9}$ . Of course, the entire series computation is seldom required. More often, analysis of partial series such as the Rn to RaD series (a partial series of the Uranium 238 series) is all that is required.

The computer solution developed here allows these sub-series or partial series to be solved singly or in pairs, depending upon the information entered on the data cards comprising the library of sub-series, shown in Figure 2 of Appendix II. Generally, two sub-series are represented on each set of data cards. If no branching occurs between the sample nuclide and the  $n^{\text{th}}$  member nuclide, a built-in program idiosyncrasy requires the computer to solve the same problem twice and deliver both solutions and their sum. This eccentric behavior occurs only when there are two sub-series per set of data cards.

The quantitative initial condition on the problem is always given in microcuries of activity as this is generally more easily determined than are the initial number of atoms in the sample. The time interval is given in seconds. The solution is tendered, either printed or on punched cards, in microcuries of activity and in number of atoms present at the termination of the time interval (see Figure 4, Appendix II). Each individual solution requires approximately one to two minutes of computer time. Quite a long electronic process!

The complete program listing, accompanied by detailed flow charts, is included as Appendix I. Appendix II consists of the necessary

technical instructions for the preparation of data and general operation of the program.

A digital computer solution to the  $R_n$  to RaD partial series is included in Chapter 6 for comparison with the analog computer solution to that series.

CHAPTER 5  
ANALOG COMPUTER SIMULATION OF  
RADIOACTIVE SERIES DECAY

(5.1) GENERAL

The digital computer solution discussed in Chapter 4 is a "brute force" method of solution involving much costly equipment, construction of series libraries, and relatively long computing times. In addition, discrete time solutions, although invaluable for certain problems, must be performed in quantity for complete analysis of the behavior of a nuclide or series of nuclides within a given time frame. The analog computer provides a means for circumventing most of the disadvantages of the digital computer. One must realize, however, that the two methods of solution, digital and analog, are complementary analytical tools allowing thorough mathematical investigation and possessing innate cross-checking characteristics.

(5.2) SIMULATION EQUATIONS

The analog computer solution for simulating radioactive series decay is not developed in the sense that the digital computer solution was developed. Rather, one is required merely to recognize the potential of the s-plane equations (Eqs. (2.14) and (2.15)) describing the behavior of a series member, introduce a few simplifications, and proceed with a simulation diagram and the solution.

The technique for solving similar equations is well known and discussed in most analog computer texts (see Bibliography).

The general first order linear differential equation for a sum of exponentials with time as the independent variable in the exponent, transformed to the s-plane, is further generalized for analog computer simulation as

$$f(s) = \frac{K}{(T_1s + 1)(T_2s + 1) \cdots (T_ns + 1)} \quad (5.1)$$

where K is a constant or gain factor and T is a time constant.

Recall Eq. (2.14), restated here as

$$n_j(s) = \frac{\Lambda_1 \Lambda_2 \cdots \Lambda_{(j-1)}}{(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_j)} N_1^0 \quad (5.2)$$

Multiplying numerator and denominator by  $(1/\lambda_k) = T_k$ , ( $k = 1, 2, \dots, j$ ), immediately translates Eq. (5.2) into the general form of Eq. (5.1).

$$n_j(s) = \frac{|\Lambda_1 \Lambda_2 \cdots \Lambda_{(j-1)}| |T_1 T_2 \cdots T_j|}{(T_1s + 1)(T_2s + 1) \cdots (T_js + 1)} N_1^0 \quad (5.3)$$

Association of terms of like subscript forms a succession of products,  $\Lambda_i T_i$ , which are the branching ratios of the (j-1) parents of Nuclide j. Thus

$$0 < K \leq T_j N_1^0 \quad (5.4)$$

One may be interested only in the behavior of Nuclide j and the

simulated solution to Eq. (5.3) would suffice, but let us require the capability (perhaps "flexibility" is the more appropriate term) for simultaneous solutions for each nuclide, 1 through  $n$ , in the series.

Consider the schematic series in Figure 5.1.

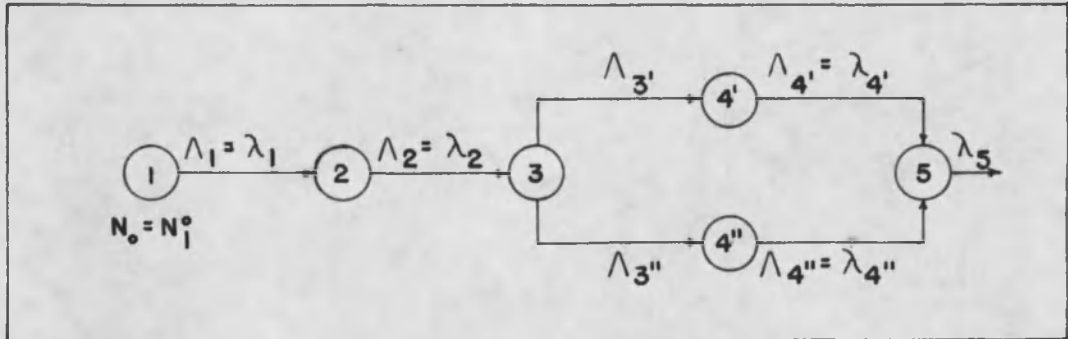


Figure 5.1

Using Eq. (5.3) as the mathematical model, we can obtain expressions for  $n_j(s)$  in terms of  $n_{j-1}(s)$ .

$$\begin{aligned}
 n_1(s) &= N_1^0 \frac{\tau_1}{(\tau_1 s + 1)} \\
 n_2(s) &= N_1^0 \frac{\tau_1}{(\tau_1 s + 1)} \cdot \frac{\Lambda_1 \tau_2}{(\tau_2 s + 1)} = n_1(s) \cdot \frac{\lambda_1 \tau_2}{(\tau_2 s + 1)} \\
 n_3(s) &= N_1^0 \frac{\tau_1}{(\tau_1 s + 1)} \cdot \frac{\Lambda_1 \tau_2}{(\tau_2 s + 1)} \cdot \frac{\Lambda_2 \tau_3}{(\tau_3 s + 1)} \\
 &= n_2(s) \cdot \frac{\lambda_2 \tau_3}{(\tau_3 s + 1)} \\
 n_{4'}(s) &= N_1^0 \frac{\tau_1}{(\tau_1 s + 1)} \cdot \frac{\Lambda_1 \tau_2}{(\tau_2 s + 1)} \cdot \frac{\Lambda_2 \tau_3}{(\tau_3 s + 1)} \cdot \frac{\Lambda_3' \tau_4'}{(\tau_4' s + 1)}
 \end{aligned} \tag{5.5}$$

$$= n_3(s) \cdot \frac{\Lambda_{3'} T_{4'}}{(\tau_{4'} s + 1)}$$

similarly,

$$n_{4''}(s) = n_3(s) \cdot \frac{\Lambda_{3''} T_{4''}}{(\tau_{4''} s + 1)}$$

and

$$\begin{aligned} n_5(s) &= n_3(s) \left[ \frac{\Lambda_{3'} \Lambda_{4'} T_{4'}}{(\tau_{4'} s + 1)} + \frac{\Lambda_{3''} \Lambda_{4''} T_{4''}}{(\tau_{4''} s + 1)} \right] \frac{T_5}{(\tau_5 s + 1)} \\ &= n_{4'}(s) \cdot \frac{\lambda_{4'} T_5}{(\tau_5 s + 1)} + n_{4''}(s) \cdot \frac{\lambda_{4''} T_5}{(\tau_5 s + 1)} \end{aligned}$$

A similar but slightly more simplified expression can be obtained for the s-plane activity,  $a_j(s)$ , in terms of  $a_{j-1}(s)$ .

$$a_1(s) = \lambda_1 n_1(s) = A_1^0 \frac{T_1}{(\tau_1 s + 1)}$$

$$a_2(s) = \lambda_2 n_2(s) = \lambda_1 n_1(s) \cdot \frac{\lambda_2 T_2}{(\tau_2 s + 1)} = a_1(s) \cdot \frac{1}{(\tau_2 s + 1)}$$

similarly,

$$a_3(s) = \lambda_3 n_3(s) = a_2(s) \cdot \frac{1}{(\tau_3 s + 1)}$$

(5.6)

and

$$\begin{aligned} a_{4'}(s) = \lambda_{4'} n_{4'}(s) &= \Lambda_{3'} n_3(s) \cdot \frac{\lambda_{4'} T_{4'}}{(\tau_{4'} s + 1)} \\ &= a_{3'}(s) \cdot \frac{1}{(\tau_{4'} s + 1)} = a_{3'}(s) \cdot \frac{x_{3'}}{(\tau_{4'} s + 1)} \end{aligned}$$

similarly,

$$a_{4''}(s) = \lambda_{4''} n_{4''}(s) = a_{3''}(s) \cdot \frac{1}{(\tau_{4''} s + 1)} = a_{3''}(s) \cdot \frac{x_{3''}}{(\tau_{4''} s + 1)}$$

and,

$$a_5(s) = \lambda_5 n_5(s) = a_{4'}(s) \cdot \frac{1}{(\tau_5 s + 1)} + a_{4''}(s) \cdot \frac{1}{(\tau_5 s + 1)}$$

where

$$a_{jp}(s) = \frac{\Lambda_j}{\lambda_j} a_j(s) = x_j \cdot a_j(s) \quad (5.7)$$


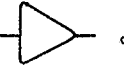
The term  $x_j = (\Lambda_j/\lambda_j)$  is, of course, the branching ratio of the parent nuclide.

We now have an equation for the complete series of nuclides, 1 through 5, such that each individual term corresponds to the respective nuclide in the series.

$$a_5(s) = A_1^0 \frac{1}{(\tau_1 s + 1)} \cdot \frac{1}{(\tau_2 s + 1)} \cdot \frac{1}{(\tau_3 s + 1)} \left[ \frac{x_{3'}}{(\tau_{4'} s + 1)} + \frac{x_{3''}}{(\tau_{4''} s + 1)} \right] \frac{1}{(\tau_5 s + 1)} \quad (5.8)$$

Eq. (5.8) is a special case of Eq. (2.15).

### (5.3) SIMULATION TECHNIQUE

A high gain, chopper stabilized, dc amplifier with a feedback resistor is an electronic summing device used in analog computers. This same amplifier, with the resistor replaced by a capacitor, is capable of electronic integration. The integrator, denoted by , performs a summing operation on its input prior to the integration operation. The summer is indicated by the symbol .

The output from the summer is minus the sum of the individual inputs, each amplified by a selected gain constant (usually 1 or 10).

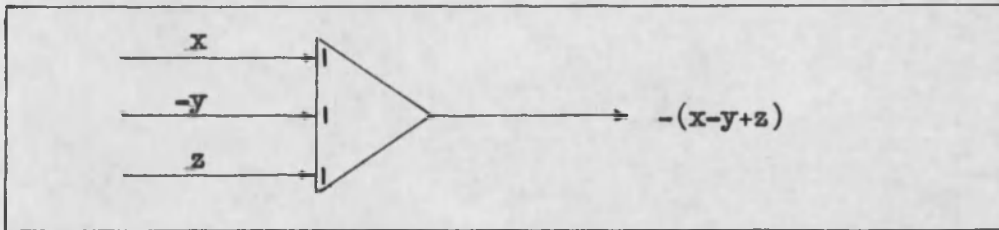


Figure 5.2. Schematic representation of the operation of the summing amplifier.

The output of the integrator is minus the sum of the integrals of the inputs amplified by a gain constant. The integration is carried out over real time.

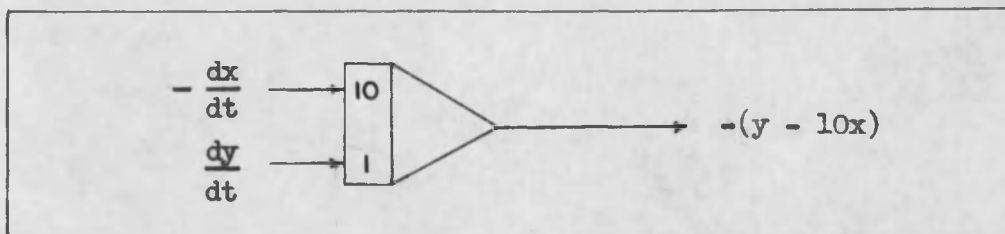


Figure 5.3. Schematic representation of the operation of an integrating amplifier.

The potentiometer, denoted by the symbol  $\bigcirc$ , is a gain attenuator. It is usually able to be set to an accuracy of 0.01 or 0.001 volt. The setting on the potentiometer determines the fraction of the input voltage appearing as the output.

The differential equation

$$\frac{dx}{dt} = -ax, \quad x = x_0 \text{ at } t_0 \quad (5.9)$$

has the solution

$$x = x_0 e^{-at} \quad (5.10)$$

and can be simulated as shown in Figure 5.4.

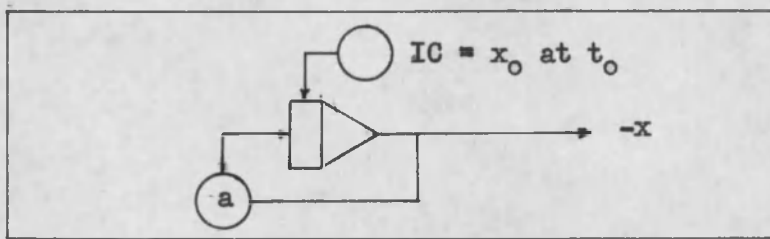


Figure 5.4

As one further example, the differential equation

$$\frac{dy}{dt} = cx - by, \quad y = y_0 \text{ at } t_0 \quad (5.11)$$

$x$  defined as in previous  
example

has the solution

$$y = \frac{cx_0}{(b-a)} (1 - e^{-(b-a)t}) + y_0 e^{-bt} \quad (5.12)$$

as we found in Eq. (2.9), the solution to Eq. (2.3), and is simulated as shown in Figure 5.5.

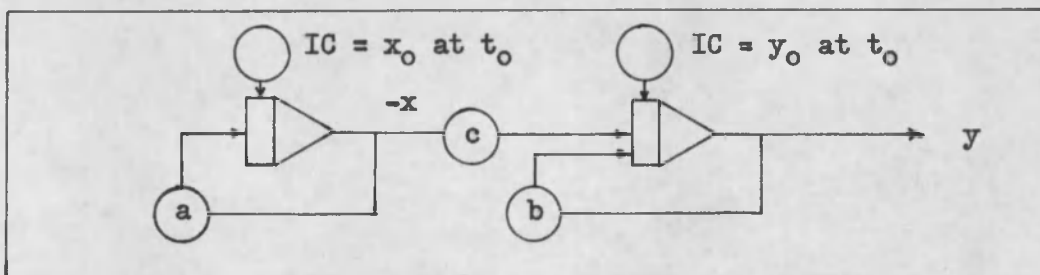


Figure 5.5

The time dependent output,  $y$ , in Figure 5.5, can be observed on an oscilloscope or fed through a recording device and the tape retained for further study.

It is much easier to work with these differential equations if they are transformed to the s-plane. A particular transform of Eq.(5.11) is given by the second equation of the set of Eqs. (5.5). It is a fairly simple matter to determine the pot settings for the simulation circuit of Figure 5.5 as

$$\begin{aligned}
 IC_x &= N_1^0 \\
 a &= \frac{1}{T_1} = \lambda_1 \\
 b &= \frac{1}{T_2} = \lambda_2 \\
 c &= \frac{\lambda_1 T_2}{T_2} = \lambda_1 \\
 IC_y &= 0
 \end{aligned}
 \tag{5.13}$$

Since the output from the computer is a function of real or problem time, it is often desirable to speed up or slow down the computation. The larger the coefficients of s, the longer the problem. It is desirable to make these coefficients as small as possible in order to reduce the computing time to a convenient interval and also reduce the number of components required to simulate the equation, yet stay within the response capabilities of the equipment and provide maximum ease of analysis of the solution. Time scaling may be accomplished as follows:

$$\begin{aligned}
 \text{Let} \\
 P &= \alpha'_t s, \\
 &\text{where } \alpha'_t \text{ is the time scale factor}
 \end{aligned}
 \tag{5.14}$$

then

$$\alpha_t = \frac{\text{Problem Time}}{\text{Computer Time}} \quad (5.15)$$

Assume we desire to speed up the computation by a factor of  $10^3$ . Then

$$\alpha_t = \frac{10^3 \text{ sec.}}{1 \text{ sec.}} = 10^3 \quad (5.16)$$

And if we desire to slow down the computation by  $10^3$ ,  $\alpha_t$  is obviously given by

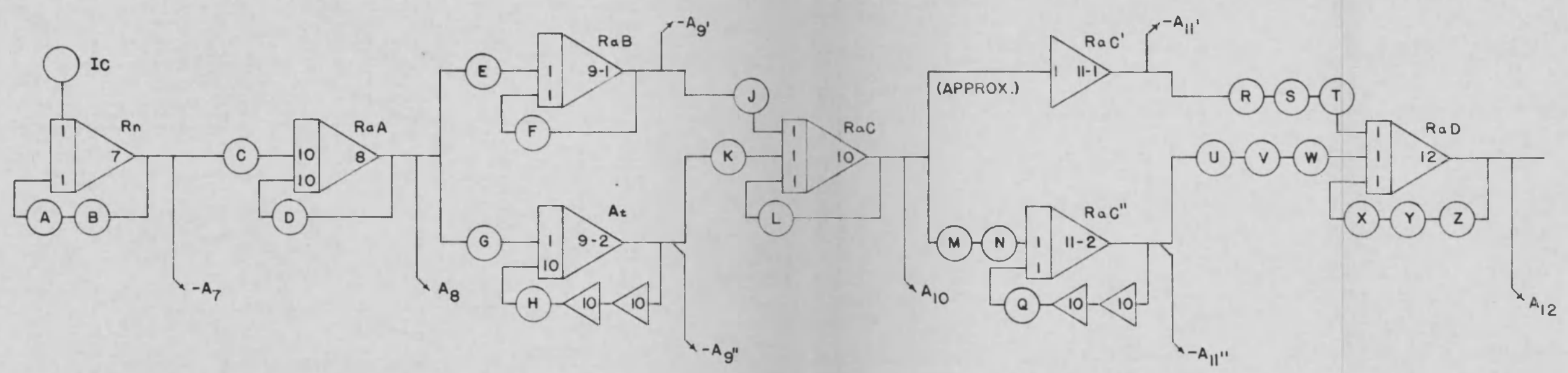
$$\alpha_t = \frac{1 \text{ sec.}}{10^3 \text{ sec.}} = 10^{-3} \quad (5.17)$$

The above examples have been concerned exclusively with the number of atoms of Nuclide  $j$  remaining at time  $t$  after  $t_0$ , but, as in the digital computer solution and as evidenced by a comparison of Eqs. (5.5) with Eqs. (5.6) and Eq. (5.8), there is more facility in dealing with activities than with number of nuclides.

A complete simulation diagram for the activity of the Rn to RaD series (see Figure 4.1) is given in Figure 5.6. The solution to this partial series is presented in Chapter 6 for comparison with the digital computer solution.

Recall that series branching presented quite a ponderous problem in the digital computer solution. Eq. (5.8) leads us to a very simple method for handling these branches and it is illustrated in Figure 5.6. Of particular interest is the striking similarity between the general features of the simulation diagram and the schematic representation of the actual series.

ANALOG COMPUTER SIMULATION OF R<sub>N</sub> - R<sub>A</sub>D SERIES



$$\frac{A_7^0 T_7}{T_7^{P+1}}$$

$$\frac{1}{T_8^{P+1}}$$

$$\frac{x'}{T_9^{P+1}}$$

$$\frac{1}{T_{10}^{P+1}}$$

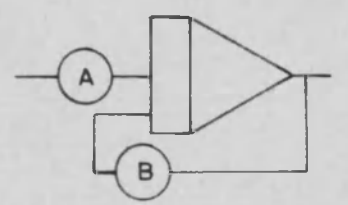
$$\frac{y'}{T_{11}^{P+1}} \approx y' \approx 1$$

$$\frac{1}{T_{12}^{P+1}}$$

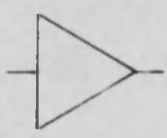
$$\frac{x''}{T_9^{P+1}}$$

$$\frac{y''}{T_{11}^{P+1}}$$

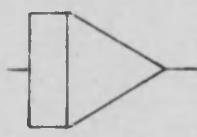
LEGEND



POTENTIOMETER



SUMMER



INTEGRATOR

$P = \frac{S}{d_t}$   
 $\alpha_t = \text{TIME SCALE FACTOR} = \frac{\text{PROBLEM TIME}}{\text{COMPUTER TIME}}$

$$\frac{K}{T^{P+1}} = \frac{K}{T \frac{S}{d_t} + 1}$$

$$A = KB$$

$$B = \frac{d_t}{T} = \lambda \alpha_t$$

FIGURE 5.6

(REFERENCE: FIGURE 4.1)

POTENTIOMETER SETTINGS  
 $(\alpha_t = 1.2 \cdot 10^3)$

IC = A <sub>7</sub> <sup>0</sup> = 1 = 100 V.	J = .704
AB = .252 · 10 <sup>-2</sup>	K = J
C = .455 · (10)	L = J
D = C	MN = .358 · 10 <sup>-2</sup>
E = .517	Q = .105 · (10 <sup>2</sup> )
F = E	RST = .136 · 10 <sup>-5</sup>
G = .095	UVW = RST
H = .475 · (10 <sup>3</sup> )	XYZ = RST

CONSTANTS

$$\lambda = \frac{1}{T}$$

- $\lambda_7 = .210 \cdot 10^{-5}$
- $\lambda_8 = .3787 \cdot 10^{-2}$
- $x' = .9998$
- $x'' = .0002$
- $\lambda_9 = .431 \cdot 10^{-3}$
- $\lambda_9'' = .396$
- $\lambda_{10} = .5864 \cdot 10^{-3}$
- $y' = .9996$
- $y'' = .0004$
- $\lambda_{11} = .433 \cdot 10^4$
- $\lambda_{11}'' = .876 \cdot 10^{-2}$
- $\lambda_{12} = .113 \cdot 10^{-8}$

Notice the approximation of the RaC' component of the simulation circuit in Figure 5.6 as a simple summer. The justification for this approximation should be intuitive from a brief consideration of the respective half-periods of parent and daughter. RaC has a half-period of 19.7 minutes while that of RaC' is 160 microseconds. Therefore the decay rate of RaC' is completely dominated by its rate of production. A mathematical justification is quickly determined by observing the effect of the product of terms

$$\frac{1}{(\tau_{10}^P + 1)} \cdot \frac{y'}{(\tau_{11}^P + 1)} = \frac{1}{(1710P + 1)} \cdot \frac{0.9996}{(0.000231P + 1)} \quad (5.18)$$

$$\approx \frac{0.9996}{(1710P + 1)} \approx \frac{1}{(\tau_{10}^P + 1)}$$

As inferred in Figure 5.5, if the sample contains more than one species of atoms in the same decay series, their presence at  $t_0$  is reflected in the simulation circuit by the simple process of tapping an initial condition relay into the appropriate integrator.

Since we are able to observe simultaneously the behavior of each nuclide represented in the simulation circuit, we are also able to observe, by summing these individual solutions, the behavior of the complete series as an integral unit.

It is apparent that there exist an infinite number of series decay problems and that each of them can be reduced, by the methods presented here, to a practical system for analog computer simulation.

## CHAPTER 6

### CONCLUSIONS

#### (6.1) EXAMPLE SOLUTION

The Rn to RaD partial series of the Uranium 238 series is submitted as a comparative example of the two methods of solution.

Table I gives the results of solving successively for the activities of each member of the partial series translated from machine language to standard mathematical notation. Comparison between the entries for RaC and RaC' justifies our approximation for the RaC' components in the analog computer solution.

Figure 6.1 is a reproduction of the results of the analog computer simulation of the decay of the nuclides within this same partial series. Figure 5.6 serves as the simulation diagram for this solution.

Both solutions reflect the effects of a normalized initial condition of one unit of activity of Rn.

Comparative accuracy is estimated at  $\pm 3-5\%$ , which is approximately the graphic accuracy of the recorded output.

Notice that all the significant daughter products of Rn attain secular equilibrium with their prime parent, Rn, within three hours after the start of the decay process. The activities of At<sup>218</sup>, RaC", and RaD are comparatively insignificant and consequently not shown.

#### (6.2) COMPARISON OF THE DIGITAL AND ANALOG COMPUTER SOLUTIONS

We have succeeded in developing both a digital and an analog

NUCLIDE	ACTIVITY	TIME (HRS)	NUCLIDE	ACTIVITY	TIME (HRS)
Rn	(1.00000000)	(0)	RaC	(0.00000000)	(0)
	0.99246800	1		0.49372330	1
	0.98499267	2		0.85014536	2
	0.97757375	3		0.95161357	3
	0.97021067	4		0.97039639	4
RaA	(0.00000000)	(0)	RaC'	(0.00000000)	(0)
	0.99301856	1		0.49352573	1
	0.98553911	2		0.84980525	2
	0.97811607	3		0.95123290	3
	0.97074892	4		0.97000812	4
RaB	(0.00000000)	(0)	RaC''	(0.00000000)	(0)
	0.75749066	1		0.00019026	1
	0.93927931	2		0.00033756	2
	0.97192692	3		0.00038007	3
	0.97302541	4		0.00038807	4
At <sup>218</sup>	(0.00000000)	(0)	RaD	(0.00000000)	(0)
	0.00019860	1		0.00082088	1
	0.00019711	2		0.00368751	2
	0.00019624	3		0.00740767	3
	0.00019415	4		0.01133036	4

TABLE I

DIGITAL COMPUTER SOLUTION TO THE  
Rn TO RaD PARTIAL SERIES

RESULTS OF ANALOG COMPUTER SIMULATION  
OF  $R_N - R_{AD}$  SERIES

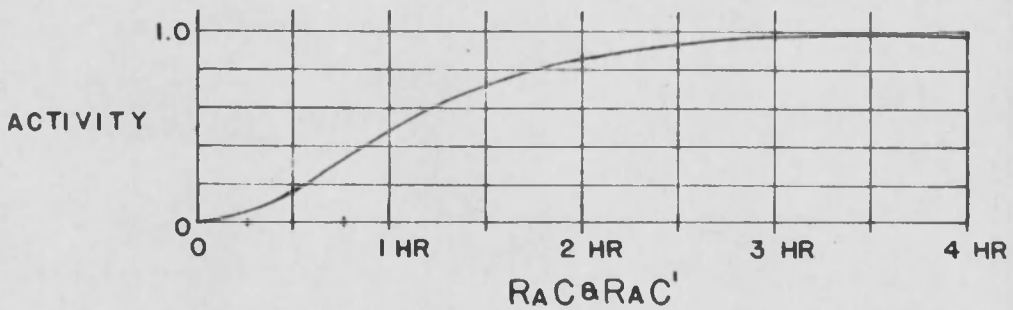
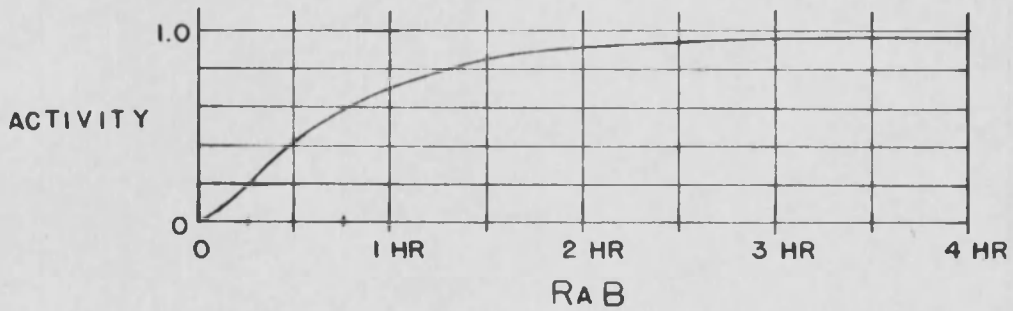
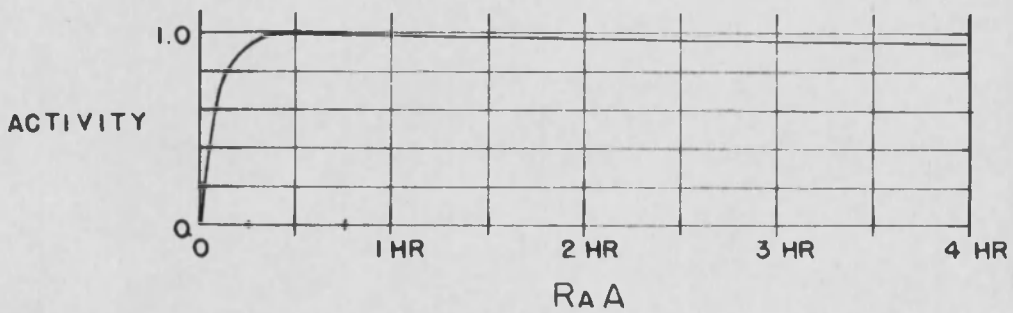
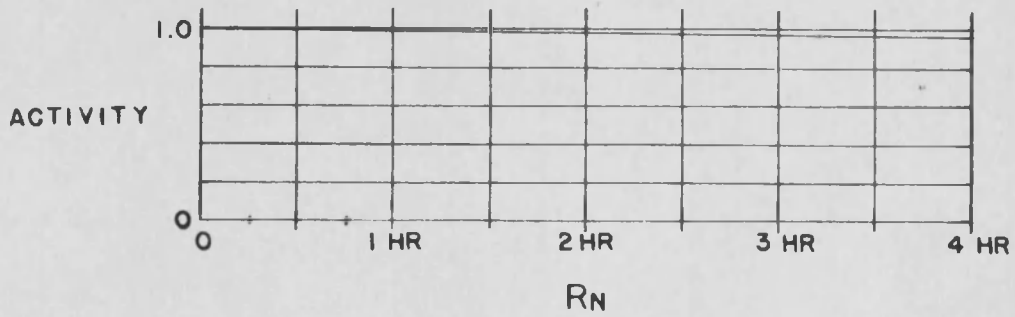


FIGURE 6.1

computer solution to radioactive series decay problems.

The digital computer solution is extremely accurate, producing results to eight significant figures through ninety-nine orders of magnitude ( $10^{-49}$  to  $10^{49}$ ). If this method of solution were used extensively, it is reasonable to expect that one would eventually accumulate a fairly complete card library of radioactive series.

The analog computer solution can be rapidly determined since it will produce a graph of the over-all behavior of the series, a capability not practically available in the digital solution. However, analog computers are not as accurate as digital computers, being subject to component limitations resulting in drift. Attached recording devices introduce another possible source of error. The time constants in the equations simulating series decay may often vary to the extreme between individual terms, resulting in a difficult choice for a time scale factor. Even so, large gain factors may still be present which can quickly deplete the stock of available amplifiers and therefore require an approximation for certain complexes. However, the versatility of the analog computer for this type of work cannot be denied.

APPENDIX I

0001		PRINT MIX						
0002	1							
0003	1	ROBERT W BLUM						
0004	1							
0005	1	DIGITAL COMPUTER SOLUTION						
0006	1	TO THE BATEMAN TYPE EQUATION						
0007	1	FOR A RADIOACTIVE SERIES						
0008	1	OF FROM 1 TO 49 MEMBERS						
0009	1							
0010	1	CONVERSION FACTORS FLOATING PT						
0011	1	YEARS 3155688658						
0012	1	DAYS 8640000055						
0013	1	HOURS 3600000054						
0014	1	MINS 6000000052						
0015	1	SECS 1000000051						
0016	1							
0017	1	CONSTANTS AND REGIONS						
0018		REG A0000	0049		LMD A TOT			
0019		REG B0050	0099		LMD B TOT			
0020		REG C0100	0149		LMB A PTL			
0021		REG D0150	0199		LMD B PTL			
0022		REG T0300	0309		ANSWERS			
0023		REG X1990	1999		PTL ANS			
0024		REG Y0400	0449		ZEROS			
0025		REG Z0463	0472					
0026		BLR 0200	0299		DATA HOLD			
0027		BLR 1750	1799		EXPONENTIA			
0028		BLR 1830	1849		HEADING			
0029		BLR 1850	1899		LOAD RTN			
0030		BLR 1950	1989		INPUT			
0031		ONEFL 10	0000	0051		0330	10	0000 0051+
0032		LN2 69	3150	0050		0331	69	3150 0050+
0033		CURIE 37	0000	0055	MICROCURIE	0332	37	0000 0055+
0034		TERM 00	0000	0000		0333	00	0000 0000+
0035		CHECK 00	0000	0000		0334	00	0000 0000+
0036		AUDIT 00	0000	0000		0335	00	0000 0000+
0037		SNOOP 00	0000	0000		0336	00	0000 0000+
0038	1							
0039	1	LOADING ROUTINE						
0040		1977 SET	9000			1977	27	9000 0382+
0041		LDI 1850	9000			0382	09	1850 9000+
0042		1851 DRC	9001					
0043		1850 LOD	1958			1850	69	1958 9001+
0044		RAA 8001			IDENT	1851	80	8001 9002+
0045		RAU 1951			HALF LIVES	1852	60	1951 9003+
0046		SCT 0008				1853	36	0008 9004+
0047		BOV 8000				1854	47	8000 9005+
0048		RAU LN2				1855	60	0331 9006+

0049		FDV	1951		HALF LIFE	1856	34	1951	9007+	
0050		FDV	1952		TIME CONV	1857	34	1952	9008+	
0051		LOD	1957			1858	69	1957	9009+	
0052		BD1		2F		1859	91	9010	9011+	
0053		STU	A0001	A		1860	21	2000	9012+	
0054		LOD	1957			1862	69	1957	9013+	
0055		BD2	3F		2F	1863	92	9014	9011+	
0056	2	STU	B0001	A	3F	1861	21	2050	9014+	
0057	3	STU	LMBDA			1864	21	9015	9016+	
0058		FMP	1953		PTL DECAY	1866	39	1953	9017+	
0059		RAL	8003			1867	65	8003	9018+	
0060		AUP	C0001	A		1868	10	2100	9019+	
0061		LOD	8002			1869	69	8002	9020+	
0062		SCT	0008			1870	36	0008	9021+	
0063		BOV			4F	1871	47	9022	9023+	
0064		STD	C0001	A	4F	1872	24	2100	9023+	
0065	4	RAU	LMBDA			1873	60	9015	9024+	
0066		FMP	1954			1874	39	1954	9025+	
0067		RAL	8003			1875	65	8003	9026+	
0068		AUP	D0001	A		1876	10	2150	9027+	
0069		LOD	8002			1877	69	8002	9028+	
0070		SCT	0008			1878	36	0008	9029+	
0071		BOV			8000	1879	47	9030	8000+	
0072		STD	D0001	A	8000	1880	24	2150	8000+	
0073		DRC								
0074	1									
0075	1	GENERAL PROGRAM								
0076		START	LOD	1953	ACTIVITY	0325	69	1953	0356+	
0077			STD	IC		0356	24	0359	0312+	
0078			LOD	1951		0312	69	1951	0354+	
0079			STD	T0007	I SUB 0	0354	24	0306	0459+	
0080			STD	IIII	BEGIN FIRST NUC	0459	24	0362	0315+	
0081		BEGIN	LOD	IIII		0315	69	0362	0365+	
0082			RAA	8001	I	0365	80	8001	0321+	
0083			RAC	8001	1F J	0321	88	8001	0327+	
0084	1		LOD	ONEFL		0327	69	0330	0383+	
0085			STD	DENOM		0383	24	0386	0339+	
0086			LOD	1952		0339	69	1952	0355+	
0087			STD	T0010	N	0355	24	0309	0462+	
0088			RAB	8001	LAST NUC	0462	82	8001	0318+	
0089			RAU	A0001	C LAMBDA J	0318	60	6000	0455+	
0090			FMP	1954	TIME	0455	39	1954	0454+	
0091			LOD	XXXX	PWRE	0454	69	0357	0310+	
0092		XXXX	STU	EXPON	2F	EXPON DEC	0357	21	0512	0515+
0093	2		RAU	A0001	B	LAMBDA X	0515	60	4000	0505+
0094			FSB	A0001	C		0505	33	6000	0377+
0095			NZU		3F		0377	44	0381	0482+
0096			FMP	DENOM			0381	39	0386	0486+
0097			STU	DENOM	3F		0486	21	0386	0482+
0098	3		SXB	0001	A		0482	53	2001	0389+
0099			BMB	4F			0389	43	0342	0343+

0100		AXB	0000	A	2B			0343	52	2000	0515+
0101	4	RAU	EXPON					0342	60	0512	0317+
0102		FDV	DENOM					0317	34	0386	0536+
0103		FAD	TERM					0536	32	0333	0509+
0104		STU	TERM					0509	21	0333	0586+
0105		AXC	0001					0586	58	0001	0392+
0106		LOD	1952					0392	69	1952	0555+
0107		RAB	8001					0555	82	8001	0311+
0108		SXB	0000	C				0311	53	6000	0368+
0109		BMB			1B			0368	43	0371	0327+
0110		AXB	0000	C			N	0371	52	6000	0328+
0111		RAU	A0001	B			LMBDA N	0328	60	4000	0605+
0112		NZU			5F			0605	44	0559	0360+
0113		FDV	A0001	A			LMBDA I	0559	34	2000	0350+
0114		FMP	TERM					0350	39	0333	0483+
0115		STU	TERM		5F			0483	21	0333	0360+
0116	5	SXB	0000	A				0360	53	2000	0367+
0117		NZB	6F					0367	42	0320	0521+
0118		RAU	ONEFL		9F			0521	60	0330	0385+
0119	6	AXB	0000	A				0320	52	2000	0477+
0120		SXB	0001	A				0477	53	2001	0384+
0121		NZB	7F					0384	42	0337	0338+
0122		RAU	C0001	A	9F			0338	60	2100	0385+
0123	7	AXB	0001	A				0337	52	2001	0344+
0124		SXB	0001				N - 1	0344	53	0001	0450+
0125		RAU	C0001	B	8F		LMBDA N-1	0450	60	4100	0655+
0126	8	FMP	C0000	B				0655	39	4099	0349+
0127		SXB	0001	A				0349	53	2001	0456+
0128		NZB			9F			0456	42	0609	0385+
0129		AXB	0000	A	8B			0609	52	2000	0655+
0130	9	FMP	IC					0385	39	0359	0659+
0131		FMP	TERM					0659	39	0333	0533+
0132		BOV			1F			0533	47	0636	0388+
0133		RAU	Y0001		1F			0636	60	0400	0388+
0134	1	RAL	8003					0388	65	8003	0345+
0135		AUP	CHECK					0345	10	0334	0489+
0136		NZU	2F					0489	44	0393	0394+
0137		RAC	0001		3F			0394	88	0001	0500+
0138	2	RAC	0003		3F			0393	88	0003	0500+
0139	3	STL	X0000	C				0500	20	7989	0492+
0140		STD	X0001	C	LINK			0492	24	7990	0493+
0141	1										
0142	1	LINK	ROUTINE								
0143		LINK	LOD	1952			N	0493	69	1952	0705+
0144		RAB	8001					0705	82	8001	0361+
0145		RAU	A0001	B				0361	60	4000	0755+
0146		NZU	1F					0755	44	0709	0460+
0147		LOD	Y0001					0460	69	0400	0353+
0148		STD	X0001	C				0353	24	7990	0543+
0149		RAU	X0000	C				0543	60	7989	0593+
0150		FDV	A0001	A	2F			0593	34	2000	0550+

0151		RAU X0000 C			0709	60	7989	0643+
0152	1	FDV A0001 B 2F			0643	34	4000	0550+
0153		FMP CURIE			0550	39	0332	0532+
0154	2	FAD T0000 C			0532	32	6299	0375+
0155		STU T0000 C		ATOMS A-B	0375	21	6299	0352+
0156		RAU X0001 C			0352	60	7990	0395+
0157		FAD T0001 C			0395	32	6300	0527+
0158		STU T0001 C		ACT A OR B	0527	21	6300	0453+
0159		RAU T0001			0453	60	0300	0805+
0160		FAD T0003			0805	32	0302	0329+
0161		STU T0005			0329	21	0304	0457+
0162		RAU T0002			0457	60	0301	0855+
0163		FAD T0004			0855	32	0303	0379+
0164		STU T0006		TOTAL ACT	0379	21	0305	0358+
0165		LOD Y0001			0358	69	0400	0503+
0166		STD TERM			0503	24	0333	0686+
0167		SET 9000			0686	27	9000	0341+
0168		LIB Y0001			0341	08	0400	0562+
0169		SET 9000			0562	27	9000	0517+
0170		SIB X0001		REGX TO 0	0517	28	1990	0452+
0171		RAU CHECK			0452	60	0334	0539+
0172		NZU 3F	1F		0539	44	0693	0494+
0173	1	RAU 1955		I SUB 2	0494	60	1955	0759+
0174		NZU	PATHB		0759	44	0313	0314+
0175		BDO	PATHB		0313	90	0567	0314+
0176		RAU AUDIT			0567	60	0335	0589+
0177		NZU 2F			0589	44	0743	0544+
0178		LOD ONEFL		DO I SUB 2	0544	69	0330	0583+
0179		STD AUDIT			0583	24	0335	0488+
0180		LOD 1955		I SUB 2A	0488	69	1955	0458+
0181		STD IIII			0458	24	0362	0565+
0182		LOD 1956	ROUTE	ACT 2B	0565	69	1956	0809+
0183	ROUTE	STD IC	BEGIN		0809	24	0359	0315+
0184	2	RAU 1957		I SUB 3	0743	60	1957	0461+
0185		NZU	PATHB		0461	44	0615	0314+
0186		BDO	PATHB		0615	90	0319	0314+
0187		RAU SNOOP			0319	60	0336	0391+
0188		NZU PATHB			0391	44	0314	0346+
0189		LOD ONEFL		DO I SB 2A	0346	69	0330	0633+
0190		STD SNOOP			0633	24	0336	0639+
0191		LOD 1957		I SUB 3A	0639	69	1957	0510+
0192		STD IIII			0510	24	0362	0665+
0193		LOD 1958	ROUTE	ACT SUB 3A	0665	69	1958	0809+
0194	3	RAU AUDIT			0693	60	0335	0689+
0195		NZU 5F	4F		0689	44	0793	0594+
0196	4	RAU 1955		I SUB 2	0594	60	1955	0859+
0197		NZU	WRITE		0859	44	0363	0364+
0198		STU IIII		I SUB 2	0363	21	0362	0715+
0199		LOD ONEFL			0715	69	0330	0683+
0200		STD AUDIT			0683	24	0335	0538+
0201		LOD 1956	ROUTE	ACT SUB 2B	0538	69	1956	0809+

0202		5	RAU SNOOP			0793	60	0336	0491+
0203			NZU WRITE			0491	44	0364	0396+
0204			RAU 1957		I SUB 3B	0396	60	1957	0511+
0205			NZU	WRITE		0511	44	0765	0364+
0206			STU IIII		I SUB 3B	0765	21	0362	0815+
0207			LOD ONEFL			0815	69	0330	0733+
0208			STD SNOOP			0733	24	0336	0739+
0209			LOD 1958	ROUTE	ACT SUB 3B	0739	69	1958	0809+
0210		1							
0211		1	PATH B SET UP						
0212			PATHB RAU CHECK			0314	60	0334	0789+
0213			NZU WRITE			0789	44	0364	0644+
0214			RAU B0001	A		0644	60	2050	0905+
0215			NZU	WRITE		0905	44	0909	0364+
0216			LOD ONEFL		PATH B PRI	0909	69	0330	0783+
0217			STD CHECK			0783	24	0334	0387+
0218			SET 9000			0387	27	9000	0542+
0219			LDI 0000		REGION A	0542	09	0000	0502+
0220			SET 9000			0502	27	9000	0507+
0221			STI 0200		HOLD REG A	0507	29	0200	0552+
0222			SET 9000			0552	27	9000	0557+
0223			LDI 0050		REGION B	0557	09	0050	0602+
0224			SET 9000			0602	27	9000	0607+
0225			STI 0000		B TO A	0607	29	0000	0652+
0226			SET 9000			0652	27	9000	0657+
0227			LDI 0100		REGION C	0657	09	0100	0702+
0228			SET 9000			0702	27	9000	0707+
0229			STI 0250		HOLD REG C	0707	29	0250	0752+
0230			SET 9000			0752	27	9000	0757+
0231			LDI 0150		REGION D	0757	09	0150	0802+
0232			SET 9000			0802	27	9000	0807+
0233			STI 0100	START	D TO C	0807	29	0100	0325+
0234		1	POWER OF E SUB RTN						
0235			PWRE STD EXIT			0310	24	0513	0316+
0236			SET 9000			0316	27	9000	0571+
0237			LDI 1750	9000		0571	09	1750	9000+
0238			1751 DRC 9001						
0239			1750 STU SAVE			1750	21	9001	9002+
0240			STU SAVEZ			1752	21	9003	9004+
0241			LOD ONEFL			1754	69	0330	9005+
0242			STD FACT			1755	24	9006	9007+
0243			STD HOLD			1757	24	9008	9009+
0244			FAD ONEFL			1759	32	0330	9010+
0245			STU EXPON	1F		1760	21	0512	9011+
0246		1	RAU SAVE			1761	60	9001	9012+
0247			FMP SAVEZ			1762	39	9003	9013+
0248			STU SAVEZ			1763	21	9003	9014+
0249			RAU FACT			1764	60	9006	9015+
0250			FAD ONEFL			1765	32	0330	9016+
0251			STU FACT			1766	21	9006	9017+
0252			FMP HOLD			1767	39	9008	9018+

0253		STU HOLD			1768	21	9008	9019+
0254		FMP EXPON			1769	39	0512	9020+
0255		FAD SAVEZ			1770	32	9003	9021+
0256		FDV HOLD			1771	34	9008	9022+
0257		FSB EXPON			1772	33	0512	9023+
0258		NZU	2F		1773	44	9024	9025+
0259		FAD EXPON			1774	32	0512	9026+
0260		STU EXPON	1B		1776	21	0512	9011+
0261	2	FAD EXPON			1775	32	0512	9027+
0262		BOV	EXIT		1777	47	9028	0513+
0263		RAU Y0001	EXIT		1778	60	0400	0513+
0264		DRC						
0265	1							
0266	1	WRITE ROUTINE						
0267		WRITE SET	9000		0364	27	9000	0369+
0268		LIB T0001			0369	08	0300	0612+
0269		RAL	8000		0612	65	8000	0519+
0270		BMI		1F	0519	46	0322	0323+
0271		WR2	9000	RESET	0322	74	9000	0485+
0272	1	LOD	9009		0323	69	9009	0479+
0273		STD	9007		0479	24	9007	0535+
0274		PCH	9000	RESET	0535	71	9000	0485+
0275	1							
0276	1	RESET ROUTINE						
0277	1	NO CARD - STOP						
0278	1	NO IMPULSE - RESET						
0279		RESET RCD	1951		0485	70	1951	0959+
0280		SET	9000		0959	27	9000	0514+
0281		LIB Y0001			0514	08	0400	0662+
0282		SET	9000		0662	27	9000	0617+
0283		SIB T0001		REG T TO 0	0617	28	0300	0712+
0284		SET	9000		0712	27	9000	0667+
0285		SIB X0001		REG X TO 0	0667	28	1990	0852+
0286		STD AUDIT		AUDIT RSTD	0852	24	0335	0588+
0287		STD SNOOP		SNOOP RSTD	0588	24	0336	0839+
0288		RAU CHECK			0839	60	0334	0889+
0289		NZU		START	0889	44	0843	0325+
0290		SET	9000		0843	27	9000	0348+
0291		LDI	0000	REGION B	0348	09	0000	0902+
0292		SET	9000		0902	27	9000	0857+
0293		STI	0050	REG B RSTD	0857	29	0050	0952+
0294		SET	9000	RESTR REGS	0952	27	9000	0907+
0295		LDI	0200		0907	09	0200	1002+
0296		SET	9000		1002	27	9000	0957+
0297		STI	0000	REG A RSTD	0957	29	0000	1052+
0298		SET	9000		1052	27	9000	1007+
0299		LDI	0100	REGION D	1007	09	0100	1102+
0300		SET	9000		1102	27	9000	1057+
0301		STI	0150	REG D RSTD	1057	29	0150	1152+
0302		SET	9000		1152	27	9000	1107+
0303		LDI	0250	REGION C	1107	09	0250	1202+

0304		SET	9000			1202	27	9000	1157+
0305		STI	0100		REG C RSTD	1157	29	0100	1252+
0306		LOD	Y0001			1252	69	0400	0553+
0307		STD	CHECK	START	CHECK RSTD	0553	24	0334	0325+
0308	1								
0309	1	DRUM	TRACE						
0310		EQU	311B	Z0009					
0311		EQU	311C	Z0010					
0312		311TR	STD	Z0003	TRACE	0561	24	0465	0518+
0313		LOD	8000	1F		0518	69	8000	0324+
0314		Z0001	LOD	8000	1F	0463	69	8000	0324+
0315	1	STD	Z0002			0324	24	0464	0717+
0316		STL	Z0005			0717	20	0467	0370+
0317		STU	Z0004			0370	21	0466	0569+
0318		SRT	0005			0569	30	0005	0481+
0319		SRT	0005			0481	30	0005	0893+
0320		STL	311B			0893	20	0471	0374+
0321		LOD	8005			0374	69	8005	0380+
0322		STD	Z0006			0380	24	0468	0621+
0323		LOD	8006			0621	69	8006	0577+
0324		STD	Z0007			0577	24	0469	0372+
0325		LOD	8007			0372	69	8007	0378+
0326		STD	Z0008			0378	24	0470	0373+
0327		WR2	Z0001			0373	74	0463	0585+
0328		RAL	311B			0585	65	0471	0475+
0329		SLO	8002			0475	16	8002	0833+
0330		AML	Z0002			0833	17	0464	0619+
0331		LOD	Z0001			0619	69	0463	0366+
0332		SDA	311C			0366	22	0472	0525+
0333		SLT	0004			0525	35	0004	0635+
0334		SDA	Z0001			0635	22	0463	0516+
0335		SLO	8002			0516	16	8002	0575+
0336		AML	7F			0575	17	0478	0883+
0337		LOD	Z0002			0883	69	0464	0767+
0338		SIA	311E			0767	23	0671	0474+
0339		SDA	7F			0474	22	0478	0531+
0340		SRT	0004			0531	30	0004	0541+
0341		ALO	5F			0541	15	0694	0399+
0342		NZU	2F			0399	44	0603	0504+
0343		ALO	9F			0504	15	1207	0611+
0344		NZU	3F			0611	44	0865	0566+
0345		ALO	5F			0566	15	0694	0499+
0346		NZU	2F	3F		0499	44	0603	0865+
0347	3	RAU	311B			0865	60	0471	0625+
0348		ALO	Z0005			0625	15	0467	0721+
0349		LOD	Z0003	311E		0721	69	0465	0671+
0350	2	LOD	7F			0603	69	0478	0581+
0351		STD	311E	3B		0581	24	0671	0865+
0352	6	STD	Z0003	Z0001		0351	24	0465	0463+
0353	4	LOD	311C			1302	69	0472	0675+

0354		STD	Z0001		0675	24	0463	0616+
0355		LOD	Z0003	Z0001	0616	69	0465	0463+
0356	5	10	0000	0000	0694	10	0000	0000+
0357	7	00	4B	6B	0478	00	1302	0351+
0358	9	40	0000	0000	1207	40	0000	0000+

0359	1								
0360	1	INITIALIZING							
0361		INITL	SET	9000	0560	27	9000	0915+	
0362		LDI	Y0001		0915	09	0400	1352+	
0363		SET	9000		1352	27	9000	1257+	
0364		STI	A0001		1257	29	0000	1402+	
0365		SET	9000		1402	27	9000	1307+	
0366		STI	B0001		1307	29	0050	1452+	
0367		SET	9000		1452	27	9000	1357+	
0368		STI	C0001		1357	29	0100	1502+	
0369		SET	9000		1502	27	9000	1407+	
0370		STI	D0001		1407	29	0150	1552+	
0371		SET	9000		1552	27	9000	1457+	
0372		STU	0200		1457	21	0200	0653+	
0373		SET	9000		0653	27	9000	0508+	
0374		STI	0250		0508	29	0250	1602+	
0375		SIB	T0001		1602	28	0300	0762+	
0376		STD	AUDIT		0762	24	0335	0638+	
0377		STD	SNOOP		0638	24	0336	0939+	
0378		STD	CHECK	8000	0939	24	0334	8000+	

0379	1								
0380	1	HEADING DATA							
0381		1830	ALF	ATOMS	1830	61	8376	7482+	
0382		1840	ALF	A	1840	00	0061	0000+	
0383		1831	ALF	ACTMC	1831	61	6383	7463+	
0384		1841	ALF	A	1841	00	0061	0000+	
0385		1832	ALF	ATOMS	1832	61	8376	7482+	
0386		1842	ALF	B	1842	00	0062	0000+	
0387		1833	ALF	ACTMC	1833	61	6383	7463+	
0388		1843	ALF	B	1843	00	0062	0000+	
0389		1834	ALF	TOTAL	1834	83	7683	6173+	
0390		1844	ALF	ATOMS	1844	61	8376	7482+	
0391		1835	ALF	TOTAL	1835	83	7683	6173+	
0392		1845	ALF	ACTMC	1845	61	6383	7463+	
0393		1839	ALF	YYYYY	1839	88	8888	8888+	
0394		1849	ALF	YYYYY	1849	88	8888	8888+	
0395			PAT						
0396			PST						
0397		EQU	HOLD	9008	-				
0398		EQU	AUDIT	0335	-				
0399		EQU	PWRE	0310	-				
0400		EQU	TERM	0333	-				
0401		EQU	WRITE	0364	-				
0402		EQU	EXIT	0513	-				
0403		EQU	START	0325	-				
0404		EQU	ONEFL	0330	-				

0405	EQU CURIE	0332	-
0406	EQU LN2	0331	-
0407	EQU CHECK	0334	-
0408	EQU RESET	0485	-
0409	EQU I III	0362	-
0410	EQU XXXX	0357	-
0411	EQU DENOM	0386	-
0412	EQU BEGIN	0315	-
0413	EQU INITL	0560	-
0414	EQU LINK	0493	-
0415	EQU SAVEZ	9003	-
0416	EQU FACT	9006	-
0417	EQU SAVE	9001	-
0418	EQU EXPON	0512	-
0419	EQU 311B	0471	-
0420	EQU 311C	0472	-
0421	EQU 311E	0671	-
0422	EQU ROUTE	0809	-
0423	EQU PATHB	0314	-
0424	EQU 311TR	0561	-
0425	EQU SNOOP	0336	-
0426	EQU LMBDA	9015	-
0427	EQU IC	0359	-

LOADING ROUTINE

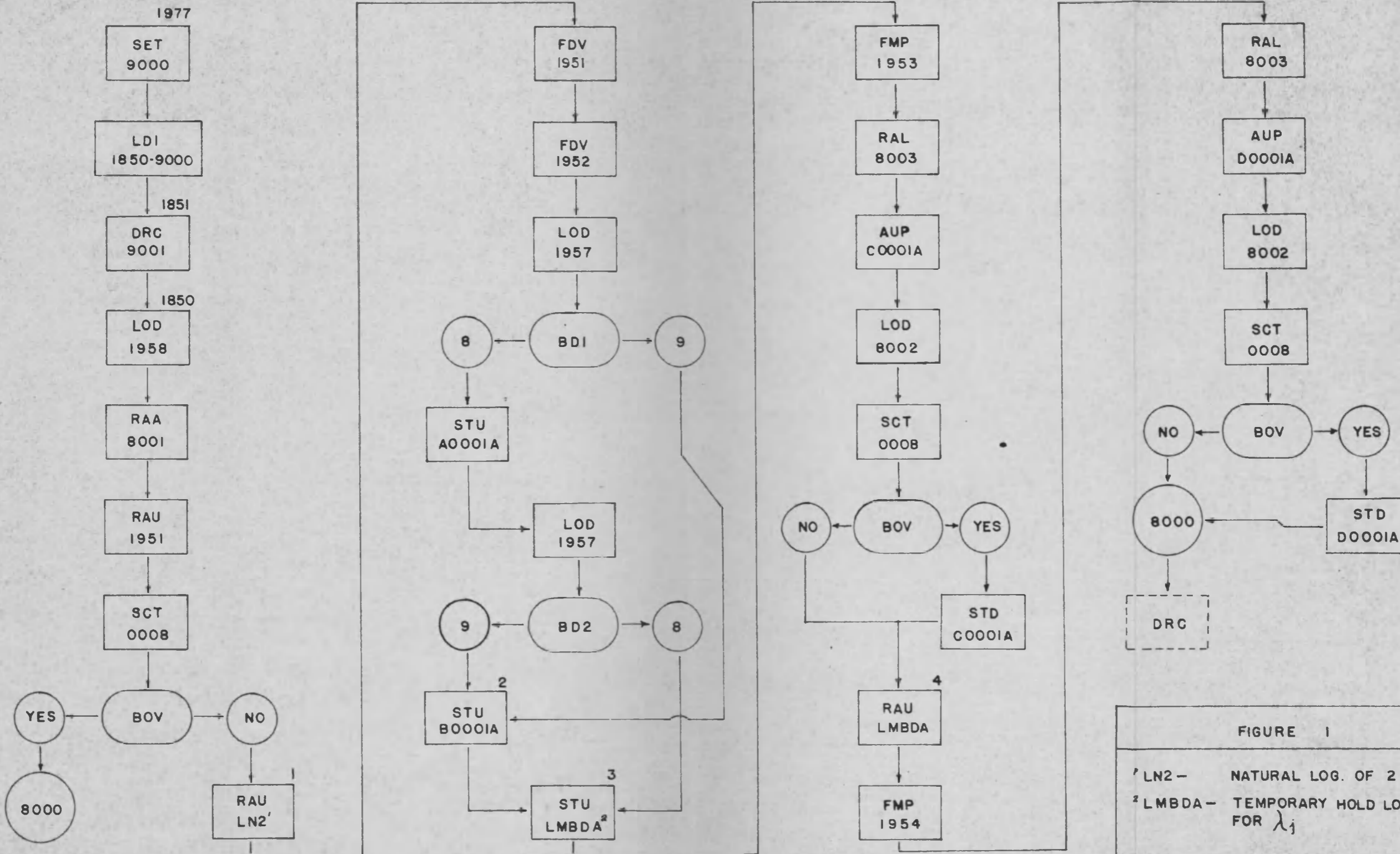


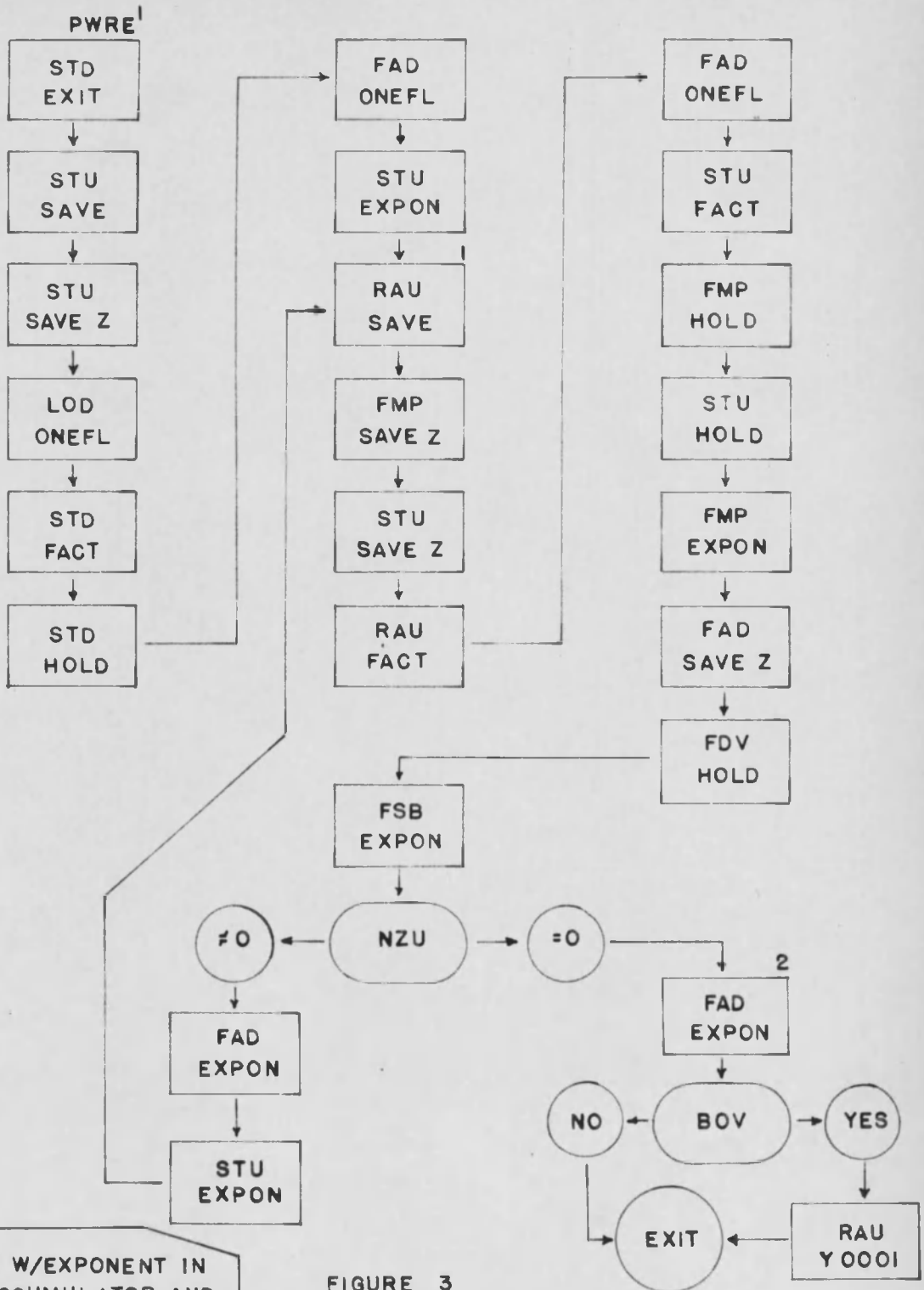
FIGURE 1

<sup>1</sup> LN2 - NATURAL LOG. OF 2  
<sup>2</sup> LMBDA - TEMPORARY HOLD LOCATION FOR  $\lambda_j$





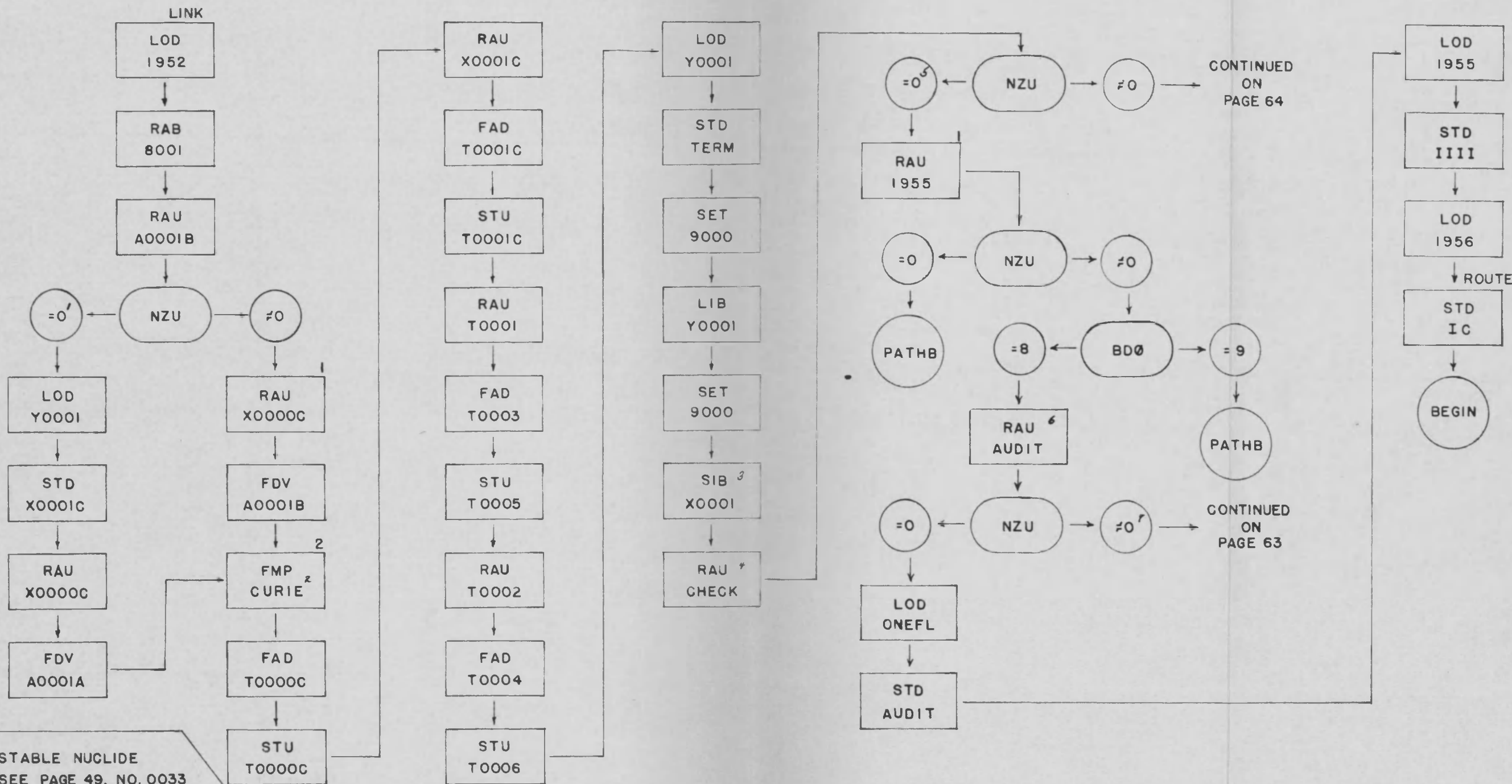
# EXPONENTIAL FUNCTION GENERATOR



ENTERED W/EXPONENT IN UPPER ACCUMULATOR AND COMMAND AFTER SUB-ROUTINE IN DISTRIBUTOR.

FIGURE 3

LINK ROUTINE



CONTINUED ON PAGE 64

CONTINUED ON PAGE 63

<sup>1</sup> STABLE NUCLIDE  
<sup>2</sup> SEE PAGE 49, NO. 0033  
<sup>3</sup> REG. X ZEROED  
<sup>4</sup> SEE PAGE 49, NO. 0035  
<sup>5</sup> PATH A PRIMARY COMPLETED  
<sup>6</sup> SEE PAGE 49, NO. 0036  
<sup>7</sup> PATH A SECONDARY COMPLETED

FIGURE 4

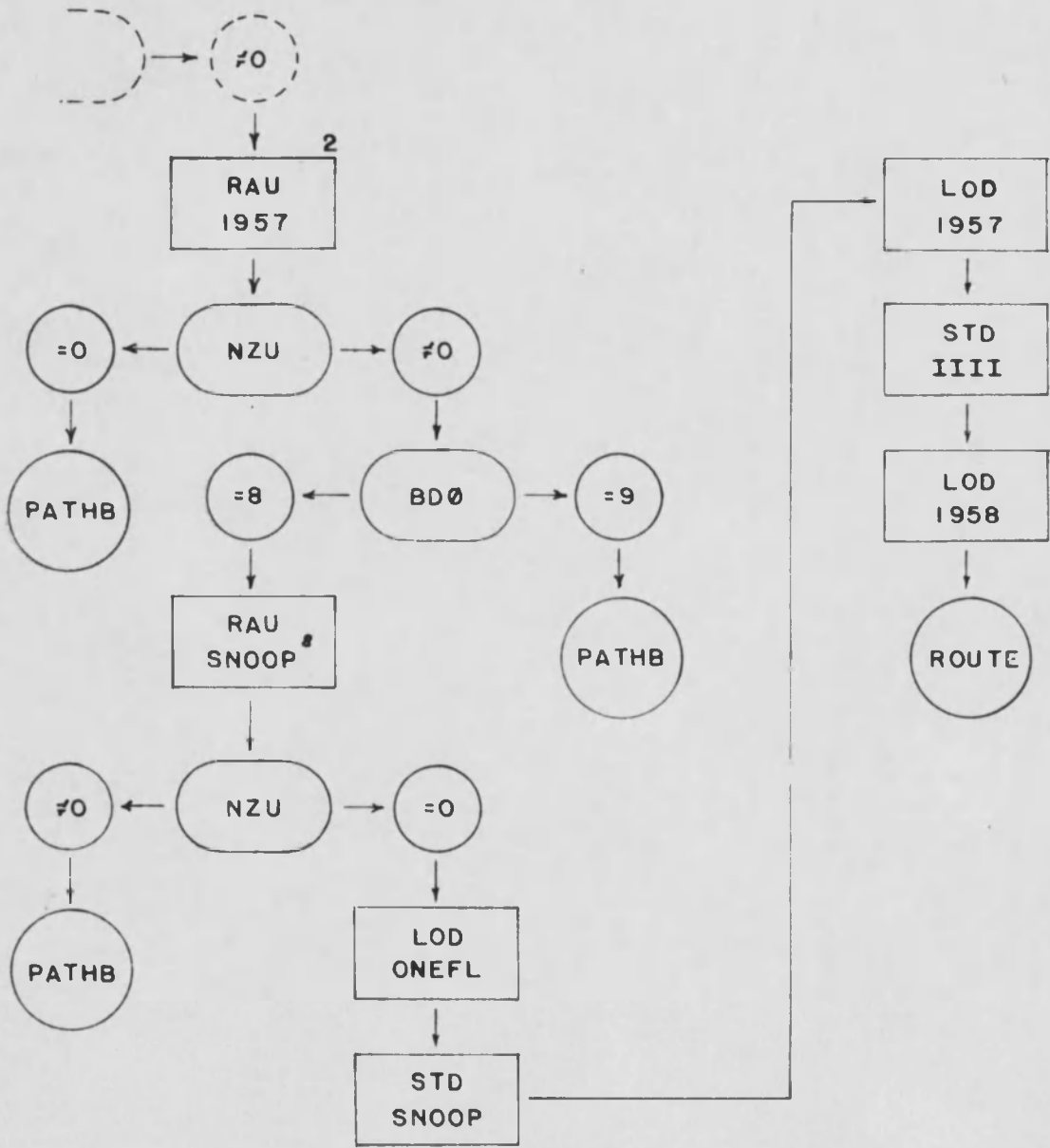


FIGURE 4  
LINK ROUTINE  
(CONT'D)

SEE PAGE 49, NO. 0037

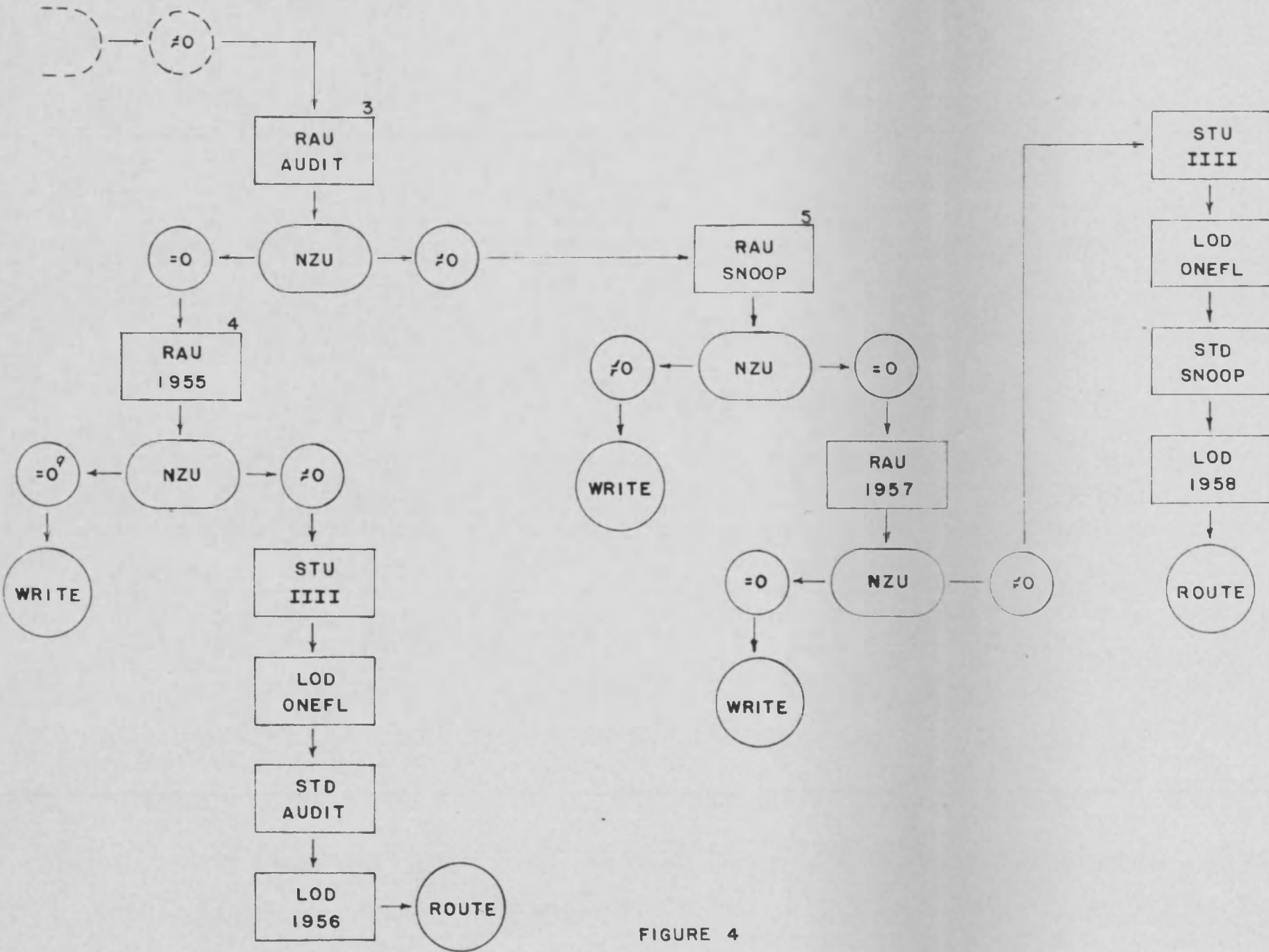
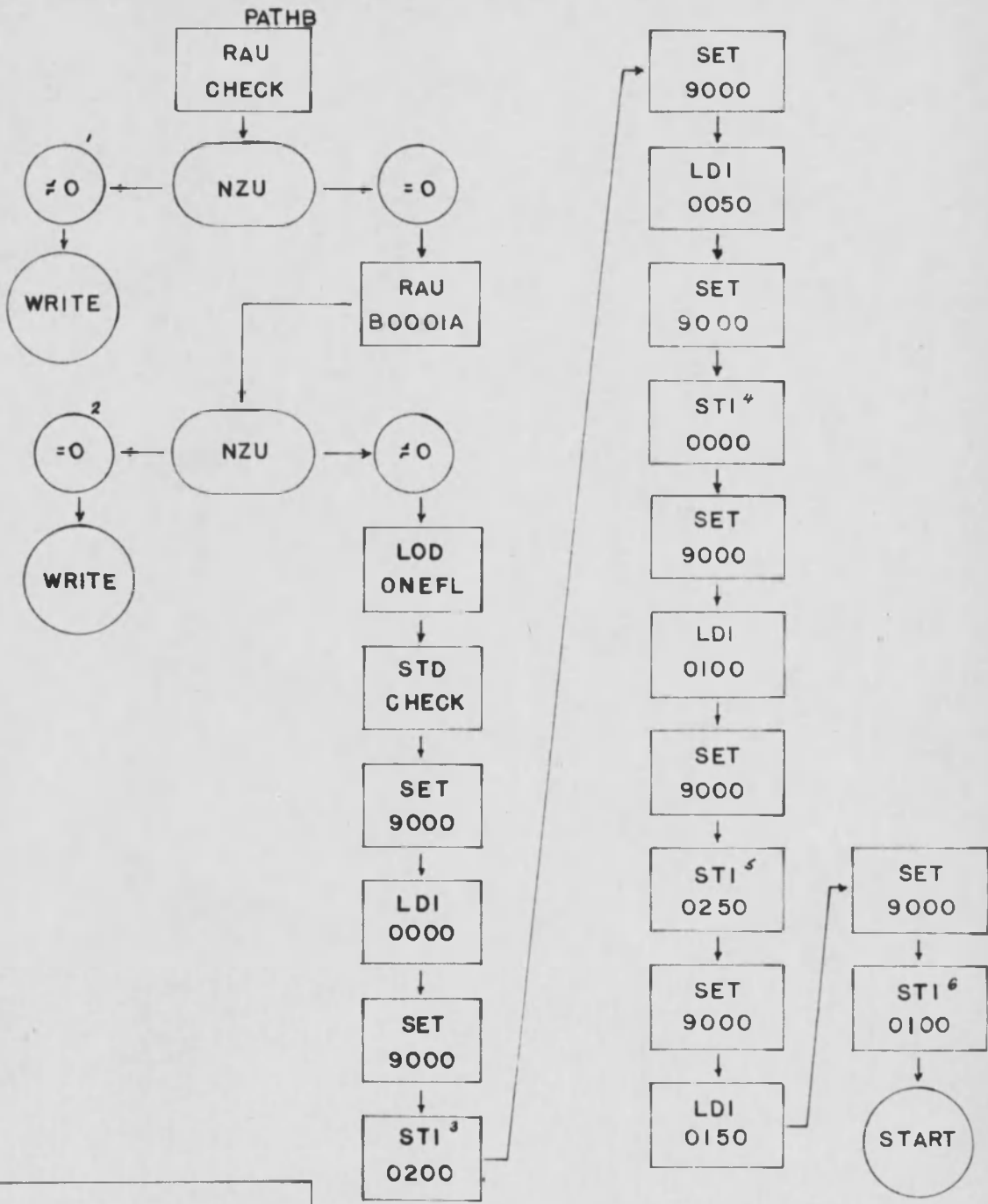


FIGURE 4  
LINK ROUTINE  
(CONT'D)

<sup>9</sup> ALL INITIAL CONDITIONS  
CONSIDERED

SHEET 3 OF 3

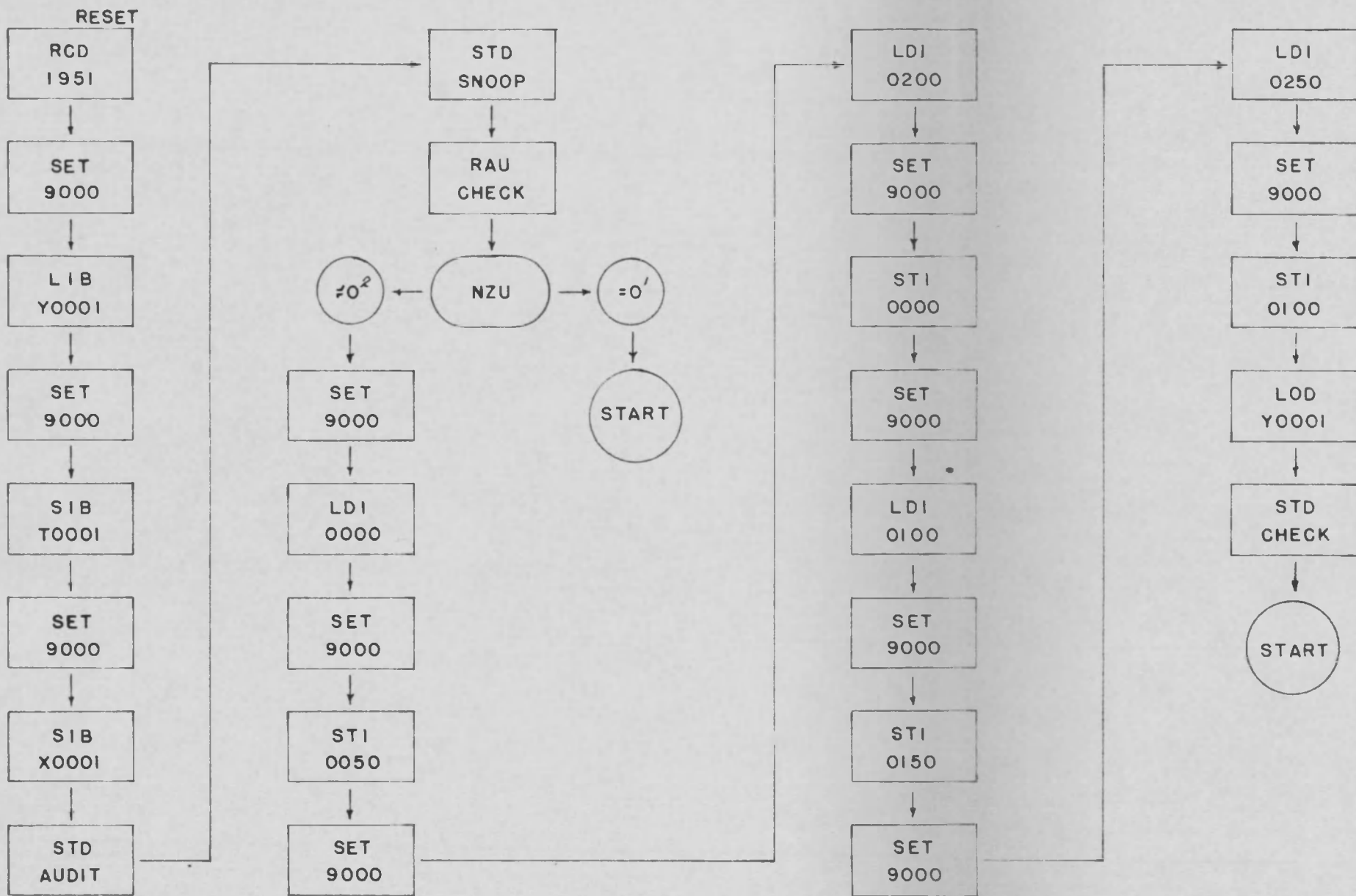
### PATH B SET-UP



- <sup>1</sup> PATH B COMPLETED
- <sup>2</sup> NO PATH B
- <sup>3</sup> HOLD REGION A
- <sup>4</sup> REGION B RELOCATED
- <sup>5</sup> HOLD REGION C
- <sup>6</sup> REGION D RELOCATED

FIGURE 5

## RESET ROUTINE



/ LAST OPERATION IN PATH A  
 2 " " " " PATH B,  
 RESTORE REGIONS AND  
 TRAFFIC CONSTANTS TO  
 PATH A CONDITION.

FIGURE 6

## WRITE ROUTINE

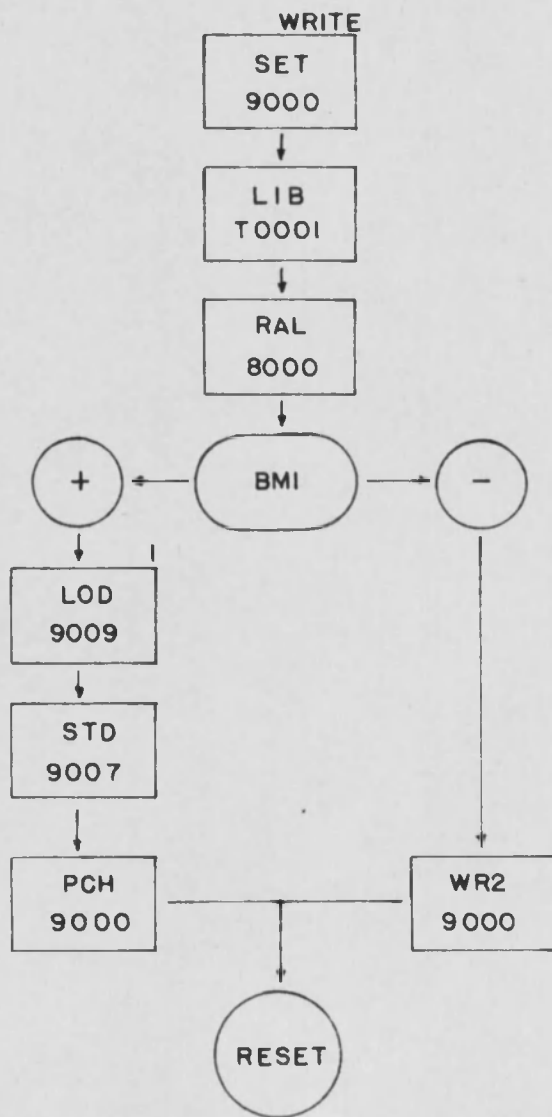


FIGURE 7

## INITIALIZING ROUTINE'

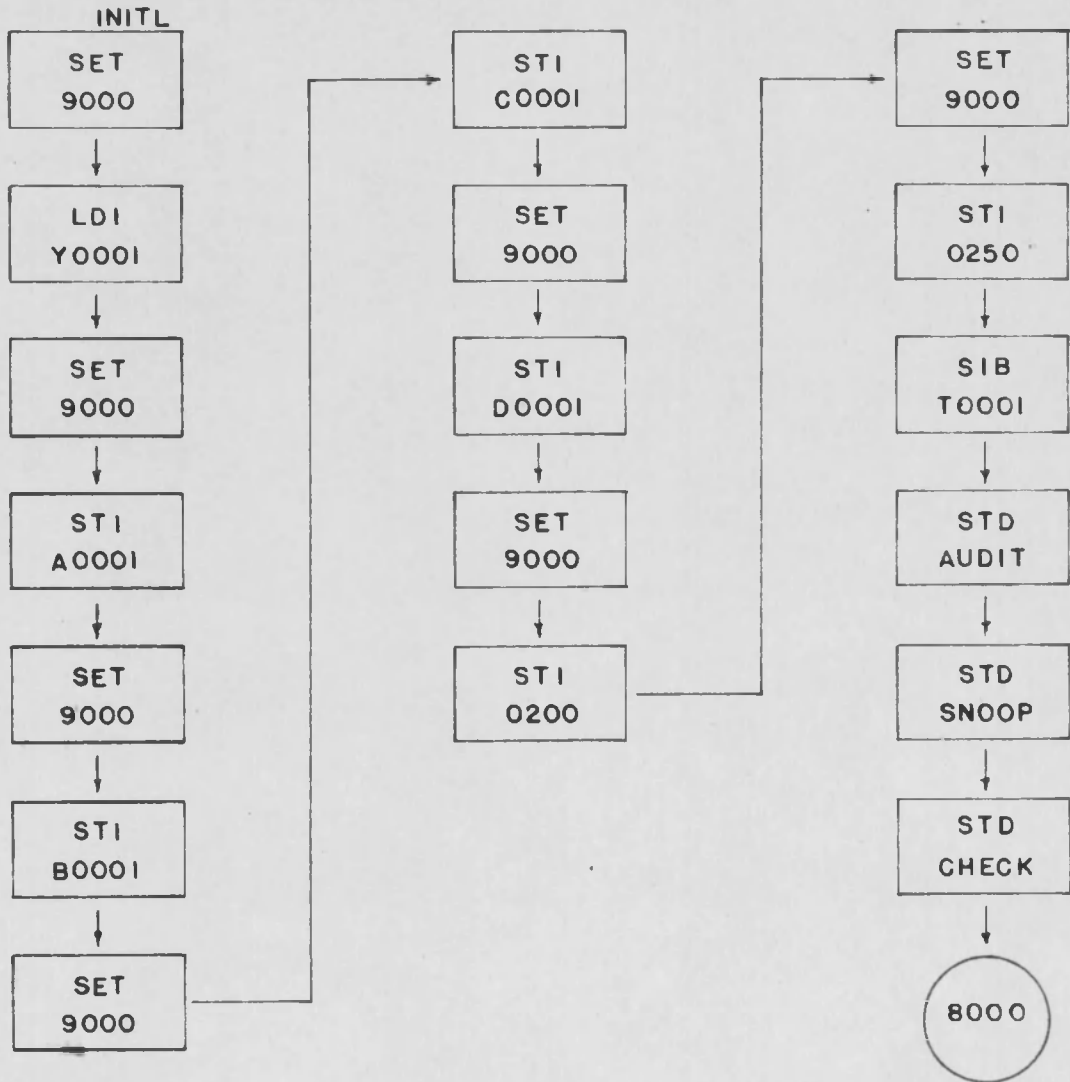


FIGURE 8

ZEROS ALL REGIONS IN  
PREPARATION FOR NEW  
SERIES DECK



APPENDIX II  
OPERATING INSTRUCTIONS  
AND DATA PREPARATION

A. OPERATING INSTRUCTIONS

1. Assemble deck as indicated in Figure 1. (If one instruction per card program deck is utilized in place of the five instruction per card deck, eliminate the five instruction per card loader and the blank card following the five per card program deck.)

2. Set IBM 650 Console switches as follows:

a. Storage Entry - 70 1951 9999 ( $\pm$ ). (+) causes punched card output, (-) causes printed output when IBM 407 Accounting Machine with standard 80-80 or 80-100 Control Panel is connected on-line with IBM 650.

b. Programmed - RUN

c. Half-Cycle - RUN

d. Address Selection - 8000

e. Control - RUN

f. Display - Optional

g. Overflow - SENSE

h. Error - STOP

3. Load assembled deck into IBM 533 Card Read Punch hopper with standard 80-80 Control Panel in place.

4. Press COMPUTER RESET key and PROGRAM START key on Console.

5. Press START key on IBM 533 when the operating lights on the Console read 70 1951.

## DECK COMPOSITION

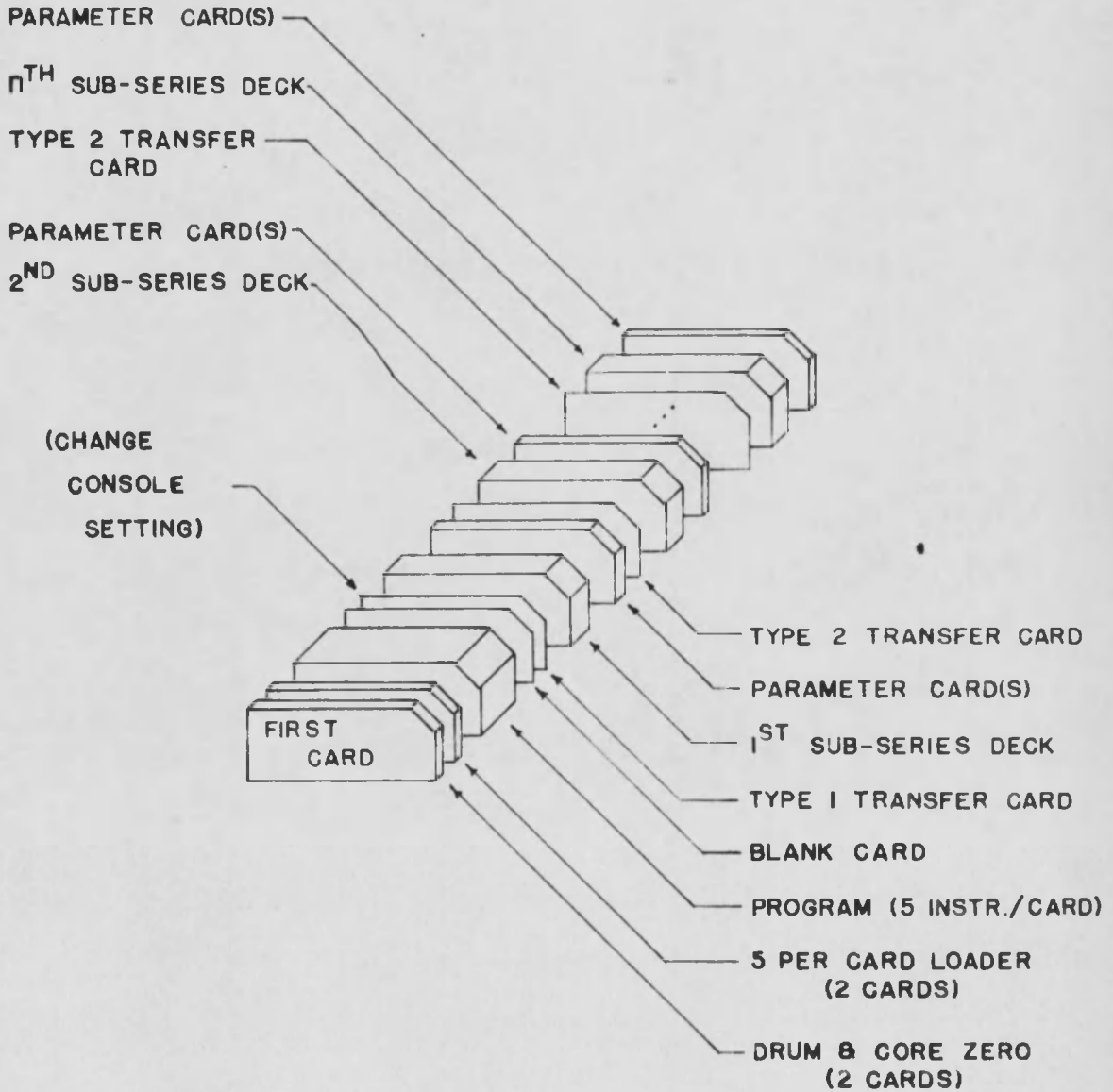


FIGURE 1

6. When the program stops:

a. In 5 per card mode -

- 1) Set Control Switch to MANUAL
- 2) Press ERROR RESET key
- 3) Press TRANSFER key
- 4) Set Control Switch to RUN
- 5) Press PROGRAM START key
- 6) When program stops again with 9999 in address lights,

proceed as in b. below.

b. In 1 per card mode -

- 1) Set Control Switch to MANUAL
- 2) Set Storage Entry Switches to 70 1977 0325 ( $\pm$ )
- 3) Press TRANSFER key
- 4) Set Control Switch to RUN
- 5) Press PROGRAM START key
- 6) Press START key on IBM 533 when the operating lights on

the Console read 70 1977.

B. DATA PREPARATION

Figure 2 illustrates the format of the data cards containing the necessary information for the computer on the decay series. Figure 3 illustrates the format of the IBM card containing the parameters which describe the particular problem to the computer. Figure 4 describes the output data format for both punched card and printed modes.<sup>1</sup>

---

<sup>1</sup>A great deal of information in the computer input and output

Figures 1 through 9 of Appendix I are detailed flow charts of the program with appropriate explanatory notes. A Drum Trace is included in the program (see Program Listing) which allows the operator to make a step-by-step check of a computation suspected of being in error. However, one usually finds the error(s) in the input information as an incorrect card punch rather than the result of a computer malfunction.

---

involves 10-digit numbers in Automatic Floating-Decimal Arithmetic. A brief explanation is in order. The right-most two digits represent the characteristic of the number and symbolize an exponent of 10. The characteristic 50 is equivalent to  $10^0$ . Digits three through ten represent the mantissa to a maximum of eight significant digits in parts of ten million. The decimal lies to the left of digit 10.

For example:

$$\begin{aligned} \text{xxxxxxxx52 (+)} &= +0.\text{xxxxxxxx} \cdot 10^2 \\ \text{yyyyyyyy99 (-)} &= -0.\text{yyyyyyyy} \cdot 10^{49} \\ \text{zzzzzzzz44 (-)} &= -0.\text{zzzzzzzz} \cdot 10^{-6} \end{aligned}$$

Interested readers are directed to the appropriate IBM 650 DPS Bulletin (see Bibliography) for further information.

# DATA CARD FORMAT

+WORD 1 +	WORD 2 +	WORD 3 +	WORD 4 +	WORD 5	WORD 6	WORD 7 +	WORD 8 +
$T_{1/2}^{1,2}$	TIME CONVER- SION FACTOR <sup>3</sup> TO SECS.	BRANCH- ING RATIO TO PATH A <sup>4</sup>	BRANCH- ING RATIO TO PATH B <sup>4</sup>	(BLANK)	(BLANK)	IDENT. OF $T_{1/2}^4$	IDENT. OF NUCLIDE <sup>5</sup>

<sup>1</sup> IN FLOATING DECIMAL FORM

<sup>2</sup> ALL ZEROS IF STABLE

<sup>3</sup> SEE APPENDIX I FOR FACTORS

<sup>4</sup> [ 0-088 FOR PATH A  
0-009 FOR PATH B  
0-098 FOR PATHS A & B

<sup>5</sup> [ X0-OJJ (JJ = 01 TO N) (X OPTIONAL AS  
DECAY IDENT., 1- $\alpha$ , 2- $\beta$ , 3- $\alpha+\beta$ , ETC.)

FIGURE 2

# PARAMETER CARD FORMAT

WORD 1 +	WORD 2 +	WORD 3 +	WORD 4	WORD 5 +	WORD 6 +	WORD 7 +	WORD 8 +
$i_1^1$	$n^2$	INITIAL ACTIVITY OF $i_1$ IN $\mu\text{CURIES}^3$	TIME INTERVAL IN SECS. <sup>3</sup>	$i_2^{4,5}$	INITIAL ACTIVITY OF $i_2$ IN $\mu\text{CURIES}^3$	$i_3^{4,5,6}$	INITIAL ACTIVITY OF $i_3$ IN $\mu\text{CURIES}^3$

<sup>1</sup> 0-0ii (WHERE TO START)

<sup>2</sup> 0-0nn (WHERE TO STOP)

<sup>3</sup> IN FLOATING DECIMAL FORM

<sup>4</sup> [ 80 0ii FOR PATH A

  [ 90 0ii FOR PATH B

<sup>5</sup> FOR DI- & TRI- ATOMIC SAMPLES

<sup>6</sup> IF WORDS 5 & 6 ARE ZERO, WORDS

  7 & 8 MUST BE ZERO

FIGURE 3

# OUTPUT DATA FORMAT

WORD 1	WORD 2	WORD 3	WORD 4	WORD 5	WORD 6	WORD 7	WORD 8
NO. OF ATOMS OF PATH A NUCLIDE	ACTIVITY OF PATH A NUCLIDE IN $\mu\text{C}$ .	NO. OF ATOMS OF PATH B NUCLIDE	ACTIVITY OF PATH B NUCLIDE IN $\mu\text{C}$ .	TOTAL NO. OF ATOMS	TOTAL ACTIVITY IN $\mu\text{C}$ .	IDENT. OF $i_1$	IDENT. OF $n$

(a) PUNCHED OUTPUT (+ CONSOLE SETTING)

ATOMS A	ACTMG A	ATOMS B	ACTMG B	TOTAL ATOMS	TOTAL ACTMG	IDENT. OF $i_1$	IDENT. OF $n$
							88-88
							88-88

(b) PRINTED OUTPUT (- CONSOLE SETTING)

FIGURE 4

ANSWERS ARE IN  
FLOATING DECIMAL FORM

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