DESIGN OF AN AXISYMMETRIC, HYPERSONIC NOZZLE,
UTILIZING THE METHOD OF CHARACTERISTICS

by

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STATEMENT BY AUTHOR

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CHAPTER I

INTRODUCTION

There has been much research, both theoretical and experimental, conducted in the field of nozzle flow. The greater part of this work has been devoted to the conventional two dimensional nozzle since this type lends itself more easily to design and construction. In addition, where such nozzles are used to test air foils and models, the two dimensional nozzle is readily adaptable for experimental observation. In the high supersonic and hypersonic flow ranges, the two dimensional nozzle concept yields unrealistic results. Modern applications have made nozzle flow of primary importance resulting, therefore, in increased research interest in the three dimensional axisymmetric flow nozzle.

In nozzle design, one difficulty is created by the concept of area ratio between test section and throat area. In order to achieve supersonic flow in the test section of a convergent-divergent nozzle, the area ratio is defined by the following equation (ref. 1),

\[
\frac{Y*W}{Y*W} = \frac{A_t}{A_*} = \frac{1}{M} \left[ \frac{\gamma}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} M^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}}
\] (1.1)
where \( A^*, A, \) and \( M \), are the throat area, the test section area and the test section Mach number respectively. In the case of a two dimensional nozzle, one of the channel dimensions is common to both sections for a given sized test section. When the test section flow is hypersonic, the throat section is reduced to a narrow slit. For example, if the test section Mach number is eight, the area ratio is 194.5. For a nozzle with 5" x 5" exit and plane parallel side walls, a throat height of 0.026" is required.

If an axisymmetric nozzle with circular cross sections were utilized, the area ratio would be

\[
\frac{\pi y_i^2}{\pi y_*^2} = \frac{A_i}{A^*} \quad \text{or} \quad \frac{y_i}{y_*} = \left(\frac{A_i}{A^*}\right)^{\frac{1}{2}} \tag{1.2}
\]

where \( y_i \) and \( y_* \) are the test section and throat radii, respectively. The throat area is now a circular aperture rather than a narrow slit and has more realistic dimension.

Another major problem in the design of a supersonic nozzle is the determination of the contour which assures that parallel uniform flow exists in the test section. Several methods are available to achieve this result. The authors of some papers propose an analytical approach (ref. 4), while others suggest a mathematical-graphical procedure incorporating what is known as the "method of characteristics" (ref. 2) to achieve the proper nozzle
design. A few of these methods will be discussed more fully in Chapter III.

The present method used in designing the hypersonic nozzle will include both analytical and graphical procedures. The "lattice-point" method of characteristics as described in ref. 2 will be utilized only in the transition region of the nozzle. In the transition region, the flow is transformed into uniform and parallel streamlines by the time it reaches the test section. The throat region will be designed analytically and the region between the throat and transition will be designed by assuming uniform radial flow.
CHAPTER II

THE METHOD OF CHARACTERISTICS

2.1 Two Dimensional Isentropic Flow

In deriving the equations of what we call a characteristic line in a flow field, we start with the full non-linear equation of motion for two dimensional, non-viscous, irrotational flow (ref. 1):

\[
(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y = 0
\]

(2.1)

\[
u_x - u_y = 0
\]

(2.2)

These two differential equations are special cases of the general expression

\[
L_1 = A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0
\]

\[
L_2 = A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0
\]

where \(A_1, A_2, B_1, B_2,\) etc. are functions of \(x, y, u, v.\)

We assume that all the functions are continuous and that they have continuous derivatives and that the equations are homogeneous \((E_1 = E_2 = 0).\) It is required that \(L_1\) and \(L_2\) be linearly independent; hence, the condition \(A_1 B_1 - A_2 B_2 - C_1 D_1 - C_2 D_2\) must not exist.
Before solving the differential equations of motion, let us examine first the case of one dependent variable \( f(x, y) \). Given a curve in parametric form where \( x \) is a function of \( \sigma \) and \( y \) is a function of \( \sigma \), then \( a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} \) is the derivative of \( f(x, y) \) along the curve \( x(\sigma) \), \( y(\sigma) \). The slope of the curve will be given by
\[
\frac{dy}{dx} = \frac{b}{a} = \frac{y_\sigma}{x_\sigma}.
\]

For the curve of diagram 2-1
\[
\frac{df}{d\sigma} = \frac{\partial f}{\partial x} \frac{dx}{d\sigma} + \frac{\partial f}{\partial y} \frac{dy}{d\sigma} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b.
\]

The slope of the curve in the \( \sigma \) direction equals zero. Hence
\[
\frac{b}{a} = \frac{y_\sigma}{x_\sigma} = \frac{dy}{dx}.
\]

Let us now consider the case of two dependent variables \( u = f(x, y) \) and \( v = g(x, y) \). We would like to combine \( L_1 \) and \( L_2 \) in such a way that the derivatives are taken along some curve whose slope is
\[
\frac{dy}{dx} = \frac{y(\sigma)}{x(\sigma)} = \frac{b}{a}.
\]
First, we form a linear combination so that \( L = \lambda_1 L_1 + \lambda_2 L_2 \).

From the general form of the differential equations in the physical plane we now have

\[
L = \left( \lambda_1 A_1 + \lambda_2 A_2 \right) u_x + \left( \lambda_1 B_1 + \lambda_2 B_2 \right) u_y + \left( \lambda_1 C_1 + \lambda_2 C_2 \right) v_x + \left( \lambda_1 D_1 + \lambda_2 D_2 \right) v_y = 0
\]

or

\[
L = a u_x + b u_y + c v_x + d v_y = 0
\]

In order to differentiate along some characteristic direction, the following condition must exist:

\[
a = c = \frac{x_x}{y_x} = \frac{dx}{dy}
\]

or

\[
\frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2} = \frac{x_x}{y_x} = \frac{dx}{dy}
\]

The above condition enables us to generate two equations

\[
\left( \lambda_1 B_1 + \lambda_2 B_2 \right) x_x - \left( \lambda_1 A_1 + \lambda_2 A_2 \right) y_x = 0
\]

\[
\left( \lambda_1 D_1 + \lambda_2 D_2 \right) x_x - \left( \lambda_1 C_1 + \lambda_2 C_2 \right) y_x = 0
\]

which can be written in the form

\[
\lambda_1 \left( A_1 y_x - B_1 x_x \right) + \lambda_2 \left( A_2 y_x - B_2 x_x \right) = 0
\]

\[
\lambda_1 \left( C_1 y_x - D_1 x_x \right) + \lambda_2 \left( C_2 y_x - D_2 x_x \right) = 0
\]

For \( \lambda_1 \) and \( \lambda_2 \) to be other than the trivial value of zero, the determinant of their coefficients must equal zero.
Thus
\[(A_1 y_r - B_1 x_r)(C_2 y_r - D_2 x_r) - (C_1 y_r - D_1 x_r)(A_2 y_r - B_2 x_r) = 0\]

collecting terms
\[(A, C_z - A_2 C_1)\frac{y_r^2}{x_r^2} - (B, C_z + A_1 D_z - D_1 A_z - C_1 B_z)x_r y_r + (B, D_z - D_1 B_z)x_r^2 = 0\]

we divide by \(x_r^2\)
\[
(A, C_z - A_2 C_1)\frac{y_r^2}{x_r^2} - (B, C_z + A_1 D_z - D_1 A_z - C_1 B_z)\frac{y_r}{x_r} + (B, D_z - D_1 B_z) = 0
\]

(2.3)

Now by comparing the differential equations of motion (2.1 and 2.2) with the general form equation

\((L_1 \text{ and } L_2)\) the values of \(A_1, A_2 \ldots D_2\) are obtained

\[
A_1 = (c^2 - u^2) \quad A_2 = 0
\]
\[
B_1 = -uv \quad B_2 = -1
\]
\[
C_1 = -uv \quad C_2 = 1
\]
\[
D_1 = (c^2 - v^2) \quad D_2 = 0
\]

Substituting these values into equation (2.3) gives

\[
(c^2 - u^2)\left(\frac{y_r}{x_r}\right)^2 + 2uv\frac{y_r}{x_r} + (c^2 - v^2) = 0
\]
Solving this equation for $\frac{y}{x'}$

$$\frac{y}{x'} = \frac{dy}{dx} = \frac{-uv \pm \sqrt{u^2v^2 - (c^2-u^2)(c^2-v^2)}}{c^2-u^2}$$

or reducing further

$$\frac{dy}{dx} = \frac{uv \pm \sqrt{c^2(u^2+v^2) - c^4}}{u^2 - c^2} \quad (2.4)$$

The above equation is now the equation for the slope of a characteristic line in the physical plane.

In a similar manner we can derive the equation for the characteristic line in the velocity plane. The velocity (or hodograph) plane is that plane in which the velocity components are the coordinates or independent variables. However, it is first necessary to interchange the roles of the dependent ($u$, $v$) and independent ($x$, $y$) variables. This can be accomplished and the resulting equations will be linear and amenable to solution if:

a) equations (2.1) and (2.2) are homogeneous

b) $A_1$, $A_2$, ..., $D_2$ are functions of $u$ and $v$ only

c) the Jacobian "$j$" is non zero

The use of the Jacobian permits us to transform our equations between the physical ($x$, $y$) plane and the hodograph ($u$, $v$) plane. If $x = f(u, v)$ and $y = g(u, v)$, it can be shown through partial differentiation that
\[ \frac{\partial u}{\partial x} = u_x = (u_x v_y - u_y v_x) y_v = j y_v \]

where \( u_x v_y - u_y v_x = "J" \) by definition. In a similar manner it can be shown that \( u_y = -j x_v, \quad v_x = -j y_u, \quad v_y = j x_u \).

To go from the hodograph plane to the physical plane we define the Jacobian \( "J" = \chi_u y_v - \chi_v y_u \), such that \( j J = 1 \).

If the transformation is made using \( "J" \), the general form of the differential equations in the hodograph plane is

\[ L_1 = A_1 y_v - B_1 x_v - C_1 y_u + D_1 x_u = 0 \]
\[ L_2 = A_2 y_v - B_2 x_v - C_2 y_u + D_2 x_u = 0 \]

The above equations may be analyzed and the characteristic equations in the hodograph plane may be developed in the same manner as those in the physical plane. The equation for the characteristic line in the velocity (hodograph) plane is

\[ \frac{du}{dv} = \frac{-u v \pm \sqrt{c^2(u^2 + v^2) - c^4}}{u^2 - c^2} \quad (2.5) \]

In examining these characteristic equations, it is evident that when \( (u^2 + v^2) - c^2 < 0 \), or the flow is subsonic, the slope of the line is complex. The flow is
termed elliptic and characteristic curves cannot be found in the physical plane. In the case of supersonic flow where 
\((u^2 + v^2) - c^2 > 0\), the flow is termed hyperbolic, and two different characteristic directions exist. It is readily seen that the use of characteristic equations is more adaptable to supersonic than subsonic flow.

2.2 The Physical Plane and the Hodograph Plane

In the physical plane, the characteristics which form a positive angle with the stream flow direction are denoted as the \(C^+\) characteristic and the characteristics which form a negative angle are denoted as the \(C^-\) characteristic.

\[
C^+ : \quad \frac{dy}{dx} = \frac{u \nu + [c^2(u^2+v^2)-c^2]^\frac{1}{2}}{u^2 - c^2}
\]

\[
C^- : \quad \frac{dy}{dx} = \frac{u \nu - [c^2(u^2+v^2)-c^2]^\frac{1}{2}}{u^2 - c^2}
\]

In the hodograph plane, these characteristic lines are denoted by the symbols \(\Gamma^+\) and \(\Gamma^-\) where

\[
\Gamma^+ : \quad \frac{du}{dv} = -u \nu + \left[\frac{c^2(u^2+v^2)-c^2}{u^2 - c^2}\right]^\frac{1}{2}
\]

\[
\Gamma^- : \quad \frac{du}{dv} = -u \nu - \left[\frac{c^2(u^2+v^2)-c^2}{u^2 - c^2}\right]^\frac{1}{2}
\]
It is evident from these equations that the \( C \) characteristics depend on the flow since both the independent \((x, y)\) and dependent \((u, v)\) variables are involved in the equation. However, the \( \Gamma^+ \) characteristics are determined only from the velocity and hence are two fixed families of curves. Also, in examining these equations, it is noted that if \( u, v \) and \( x, y \) are represented in the same coordinate plane, the curves of the \( C^+ \) and \( \Gamma^- \) characteristics will be orthogonal; as will the \( C^- \) and \( \Gamma^+ \) characteristics. That is,

\[
\begin{align*}
\frac{du}{dv} & = -\frac{dy}{dx}, \\
\text{and} \\
\frac{du}{dv} & = -\frac{dy}{dx}.
\end{align*}
\]

(2.8)

2.3 Characteristics in the Physical Plane

An alternative expression for the slope of the \( C \) characteristics can be derived, as described in ref. 3, by introducing the flow angle, \( \Theta \), and the angle \( \beta \) between the flow direction and the \( C \) characteristics. In diagram 2-2 let \( \Theta \) be the angle between the flow \( q \) and the \( x \) axis. By denoting a new set of axes \( y', x' \) with \( x' \) parallel to the direction of \( q \), \( u' = q \) and \( v' = 0 \). Hence, by equation (2.6),

\[
\frac{dy'}{dx'} = \pm \frac{(c^2 q^2 - c^4)^{1/2}}{q^2 - c^2} = \pm \frac{1}{(M^2 - 1)^{1/2}} = \pm \tan \beta
\]

(2.9)
from geometric consideration  \[ \beta = \sin^{-1} \frac{1}{M} \]  

Equation (2.9) becomes 

\[
\frac{dy}{dx} = \tan(\Theta \pm \beta)
\]  \hspace{1cm} (2.11)

At any point \( x, y \) the \( C^+, C^- \) characteristics make the same angle \( \beta \), the Mach angle, with respect to a streamline at that point. Also, since \( \sin \beta = \frac{1}{M} = \frac{\xi}{\theta} \), the velocity component normal to the direction of a \( C \) characteristic is equal to the speed of sound.

2.4 The Prandtl-Meyer Function

In the hodograph (velocity) plane it is convenient to express the velocity components in polar coordinates. Thus

\[ u = q \cos \Theta \quad du = dq \cos \Theta - q \sin \Theta d\Theta \]
\[ v = q \sin \Theta \quad dv = dq \sin \Theta + q \cos \Theta d\Theta \]
By using these transformations with equations (2.8), (2.10), (2.11), the relation between the increase in the speed \( dq \) and the change in the flow direction \( d\theta \) is given by the differential form of the hodograph characteristic equation

\[
\pm \frac{dq}{s} = \tan \beta \frac{d\theta}{s} = \frac{d\theta}{(M^2 - 1)^{1/2}}
\]

This equation may be arranged and solved for \( \Theta \) where

\[
\Theta = \int (M^2 - 1)^{1/2} \frac{dq}{s} = \sqrt[4]{(M)}
\]

(2.12)

\( \sqrt[4]{(M)} \) is called the Prandtl-Meyer function.

A supersonic Mach number is always associated with a definite value of the function \( \sqrt[4]{(M)} \).

2.5 Two Dimensional Natural Coordinates

It is usually more convenient for both derivation and application to use the natural coordinate system as described in ref. 1. In natural coordinates, the velocity is expressed in terms of its magnitude and direction \( (\omega_s, \Theta) \) and the independent variables are the stream-wise and normal coordinates \((s, \eta)\).
The equations of motion are:

**continuity** \( \rho w \Delta n = \text{constant} \)

**s-momentum** \( \rho \frac{dw}{ds} = -\frac{dp}{ds} \)

**n-momentum** \( \rho \frac{w^2}{R} = -\frac{dp}{dn} = \rho w^2 \frac{d\theta}{ds} \)

**irrotation** \( \frac{dw}{dn} - w \frac{d\theta}{ds} = 0 \)

By differentiating the continuity equation, dividing through by \( \rho w \Delta n \) and noting the geometry of diagram 2-3, we obtain

\[
\frac{1}{\rho} \frac{d\rho}{ds} + \frac{1}{w} \frac{dw}{ds} + \frac{d\theta}{dn} = 0 \quad (2.13)
\]
and rewriting the $s$-momentum equation as
\[
\frac{\omega \frac{d\omega}{ds}}{\omega} = -\frac{1}{\rho} \frac{d\rho}{ds} \frac{\partial \rho}{\partial \rho} = -\frac{c^2}{\rho} \frac{\partial \rho}{\partial s}
\]

substitute in equation (2.13) and get
\[
-\frac{\omega}{c^2} \frac{d\omega}{ds} + \frac{1}{\omega} \frac{d\omega}{ds} + \frac{\partial \theta}{\partial n} = 0
\]

which can be written in the form
\[
(M^2-1) \frac{1}{\omega} \frac{d\omega}{ds} - \frac{\partial \theta}{\partial n} = 0 \quad (2.14)
\]

Equations (2.1) and (2.2) in natural coordinates are now
\[
\frac{\cot^2 \gamma}{\omega} \frac{d\omega}{ds} - \frac{\partial \theta}{\partial \eta} = 0 \quad (2.15)
\]
\[
\frac{1}{\omega} \frac{d\omega}{\partial \eta} - \frac{\partial \theta}{\partial s} = 0 \quad (2.16)
\]

where the characteristic, or Mach direction, is expressed by \( \cot^2 \gamma = M^2 - 1 \).

2.6 **Relation between Prandtl-Meyer Function ($V'$) and the Flow Angle ($\Theta$).**

Equations (2.15) and (2.16) lend themselves naturally to the introduction of the Prandtl-Meyer function $V'$, which is defined by equation (2.12)
\[
V'(m) = \int (M^2 - 1)^{\frac{1}{2}} \frac{d\eta}{\eta} = \int \cot \gamma \frac{d\omega}{\omega}
\]
making use of this relationship for \( \nu' \), equations (2.15) and (2.16) become

\[
\frac{d\nu'}{ds} - \tan \gamma \frac{d\theta}{dn} = 0 \tag{2.17}
\]

\[
\tan \gamma \frac{d\nu'}{dn} - \frac{d\theta}{ds} = 0 \tag{2.18}
\]

Finding compatibility relations between \( \nu' \) and \( \theta \) is facilitated by writing equations (2.17) and (2.18) in a new non-orthogonal coordinate system \((\xi, \eta)\) oriented with respect to the \( C^+ \) and \( C^- \) characteristics.

As depicted in diagram 2-4, the coordinates are in fact Mach lines inclined at the angle \( \pm \gamma \) to the streamlines.

To write derivatives in the new coordinate system, the change in any function \( f \) in going from \( P \) to \( P' \) is

\[
\Delta f = \frac{df}{d\eta} \Delta \eta
\]
This change in $f$ may also be calculated in the natural coordinate plane as

$$
\Delta f = \frac{df}{ds} \Delta s + \frac{df}{dn} \Delta n = \left( \frac{df}{ds} + \frac{df}{dn} \frac{\Delta n}{\Delta s} \right) \Delta s
$$

Relating these two equations results in

$$
\frac{df}{dn} \frac{\Delta n}{\Delta s} = \frac{df}{ds} + \frac{df}{dn} \frac{\Delta n}{\Delta s}
$$

which may be written in the form

$$
\sec \gamma \frac{df}{dn} = \frac{df}{ds} + \tan \gamma \frac{df}{dn}
$$

(2.19)

Similarly, the $f$ derivative may be calculated as

$$
\sec \gamma \frac{df}{df} = \frac{df}{ds} - \tan \gamma \frac{df}{dn}
$$

(2.20)

Equations (2.19) and (2.20) give the relations for the derivatives of any function $f$, in the two coordinate systems. If now, we add and subtract equations (2.17) and (2.18) the following equations are formed:

$$
\frac{\partial}{\partial s} (\gamma' - \Theta) + \tan \gamma \frac{\partial}{\partial n} (\gamma' - \Theta) = 0
$$

$$
\frac{\partial}{\partial s} (\gamma' + \Theta) - \tan \gamma \frac{\partial}{\partial n} (\gamma' + \Theta) = 0
$$

Comparing these two equations with the rules for derivatives, equations (2.19) and (2.20), we see that

$$
\sec \gamma \frac{\partial}{\partial \eta} (\gamma' - \Theta) = 0 \quad \text{and} \quad \sec \gamma \frac{\partial}{\partial \tilde{f}} (\gamma' + \Theta) = 0
$$
or, by integrating

\[ \sqrt{'} \phi = R \text{ constant along an } \eta \text{- characteristic} \quad (2.21) \]

\[ \sqrt{'} + \theta = Q \text{ constant along a } \xi \text{- characteristic} \quad (2.22) \]

Equations (2.21) and (2.22) are the compatibility relations between \( \sqrt{'} \) and \( \theta \).

### 2.7 Characteristic Net Computation

In computing a characteristic net, it is convenient to use \( \sqrt{'} \) instead of \( u' \). The values of \( \sqrt{'} \) may be converted into values of \( M, \gamma, u'/c^* \), or any other related supersonic variable.

The method of computation is illustrated in diagram 2-5.
From a known data curve, two characteristics can be drawn through point 3, one of the \( f \) family from point 1, and one of the \( \eta \) family from point 2. The values of \( v' \) and \( \Theta \) are known at points 1 and 2, thus \( Q \) and \( R \) are known. From the compatibility relation, we may write \( Q_3 = Q_1 \) and \( R_3 = R_2 \). Therefore

\[
\begin{align*}
\gamma'_3 + \Theta_3 &= \gamma'_1 + \Theta_1 \\
\gamma'_3 - \Theta_3 &= \gamma'_2 - \Theta_2
\end{align*}
\]

The values of \( \gamma' \) and \( \Theta \) at point three are

\[
\begin{align*}
\gamma'_3 &= \frac{1}{2} (\gamma'_1 + \gamma'_2) + \frac{1}{2} (\Theta_1 - \Theta_2) = \frac{1}{2} (Q + R) \\
\Theta_3 &= \frac{1}{2} (\gamma'_1 - \gamma'_2) + \frac{1}{2} (\Theta_1 + \Theta_2) = \frac{1}{2} (Q - R)
\end{align*}
\]

(2.23) 

(2.24)

The accuracy of this type solution depends upon the mesh size. Numerical (graphical) solutions may be calculated by approximating the curved characteristics with straight lines. Also, the computation proceeds outward from some data curve.

2.8 Three Dimensional, Axially Symmetric Flow

For axially symmetric flow, the continuity equation in natural coordinates is

\[
\rho \omega (2 \pi R \Delta \eta) = \text{constant}
\]

where \( R \) is the distance from the axis of symmetry and \( \Delta \eta \) is
the distance between stream-lines in a meridian plane. By combining this equation with Euler's equation,

$$\omega \frac{d\omega}{ds} = -\frac{c^2}{\xi} \frac{d\xi}{ds}$$

as in two dimensional flow, we derive the three dimensional equations of motion

$$\cot \gamma \frac{d\omega}{ds} - \frac{d\theta}{d\eta} = \frac{\sin \theta}{R} \quad (2.25)$$

$$\frac{1}{\omega} \frac{d\omega}{d\eta} - \frac{d\theta}{ds} = 0 \quad (2.26)$$

The first equation differs from equation (2.15) only in the last term. When the distance from the axis of symmetry, R, is large, this term is small and the flow is almost two dimensional. The equation of irrotationality is unchanged in three dimensional flow.

Equations (2.25) and (2.26) may be written in the form

$$\tan \gamma \frac{d\psi}{d\eta} - \frac{d\theta}{ds} = \tan \gamma \frac{\sin \theta}{R}$$

$$\tan \gamma \frac{d\psi}{d\eta} - \frac{d\theta}{ds} = 0$$

As was done in the two dimensional case, these equations can be transformed to characteristic coordinates. Thus

$$\frac{d}{d\eta} (\psi' - \theta) = \sin \gamma \frac{\sin \theta}{R} \quad (2.27)$$
\[ \frac{d}{df}(\nu + \theta) = \sin \eta \frac{\sin \theta}{R} \]  

(2.28)

Since the geometry of the field is now involved through the variable \( R \), it is not possible to integrate these equations as before. However, a numerical integration, in conjunction with the construction of the characteristic net may be accomplished.

Diagram 2-6

Diagram 2-6(b) illustrates a typical mesh element in which point 3 is to be solved from known data at points 1 and 2. From equations (2.27) and (2.28) we integrate along the characteristic segments

\[ \int_{1}^{3} d(\nu + \theta) = \int_{1}^{3} (\sin \eta \frac{\sin \theta}{R}) df \]

\[ \int_{2}^{3} d(\nu - \theta) = \int_{2}^{3} (\sin \eta \frac{\sin \theta}{R}) d\eta \]
If the mesh is small, the integrand on the right side of the above equations may be assumed to be approximately constant over the interval of integration. The result is

\[(\gamma_3' + \theta_3') - (\gamma_1' + \theta_1') = \sin \psi_1 \frac{\sin \theta_1}{R_1} \Delta \xi_{13} \]

\[(\gamma_3' - \theta_3') - (\gamma_2' - \theta_2') = \sin \psi_2 \frac{\sin \theta_2}{R_2} \Delta \eta_{23} \]

where \(\Delta \xi_{13}\) and \(\Delta \eta_{23}\) are the segment lengths along the characteristics and the values at 1 and 2 are known. The solution of these equations is

\[\gamma_3' = \frac{1}{2} (\gamma_1' + \gamma_2') + \frac{1}{2} (\theta_1' - \theta_2') \]

\[+ \frac{1}{2} \left( \sin \psi_1 \frac{\sin \theta_1}{R_1} \Delta \xi_{13} + \sin \psi_2 \frac{\sin \theta_2}{R_2} \Delta \eta_{23} \right) \quad (2.29) \]

\[\theta_3' = \frac{1}{2} (\gamma_1' - \gamma_2') + \frac{1}{2} (\theta_1' + \theta_2') \]

\[+ \frac{1}{2} \left( \sin \psi_1 \frac{\sin \theta_1}{R_1} \Delta \xi_{13} - \sin \psi_2 \frac{\sin \theta_2}{R_2} \Delta \eta_{23} \right) \quad (2.30) \]

These differ from the two dimensional equations (2.23) and (2.24) only in the additional terms which depend on the geometry of the problem. The radial distances \(R_1\), \(R_2\) and the lengths of the mesh segments \(\Delta \xi_{13}\), \(\Delta \eta_{23}\) must be obtained graphically from the flow field.
2.9 **Iteration Procedure**

When equations (2.29) and (2.30) are solved, they will have been evaluated only on the basis of the known values at points 1 and 2. A more exact evaluation would take into account the value of \( \gamma \) and \( \Theta \) at point 3 also. However, until point 3 has been located, this is impossible. Once equations (2.29) and (2.30) have been solved, an iteration procedure can be employed whereby the average value of \( \gamma \), \( \Theta \) and R are computed between points 1, 3 and 2, 3. This average value then replaces the known values in the third terms of equations (2.29) and (2.30). The first two terms on the right side in each equation remain constant during the iteration procedure since they do not change with the geometry of the flow. Based on these average values of \( \gamma \) and \( \Theta \) a new mesh is drawn from points 1 and 2, \( \Delta f'_{13} \) and \( \Delta n'_{23} \) are drawn and point 3 is located. \( \gamma'_{3} \) and \( \Theta'_{3} \) are again evaluated from equations (2.29), (2.30) where now the third term on the right side is written for the average \( \gamma \), \( \Theta \) and R between points.

\[
\gamma'_{3} = \frac{1}{2} \left( \gamma'_{1} + \gamma'_{2} \right) + \frac{1}{2} \left( \Theta'_{1} - \Theta'_{2} \right)
+ \frac{1}{Z} \left( \sin \gamma_{avg} \sin \Theta_{avg} \frac{\Delta f'_{13}}{R'_{13}} + \sin \gamma_{avg} \sin \Theta_{avg} \frac{\Delta n'_{23}}{R'_{23}} \right)
\]

(2.31)
\[ \Theta_3' = \frac{1}{2} (\gamma_1' - \gamma_2') + \frac{1}{2} (\Theta_1 + \Theta_2) \]
\[ + \frac{1}{2} \left( \sin \eta_{3m} \sin \theta_{3m} \frac{\Delta \gamma_3'}{R_{13}} - \sin \eta_{3m} \sin \theta_{3m} \frac{\Delta \gamma_{23}'}{R_{23}} \right) \]  \quad (2.32)

This iteration procedure may be utilized several times in order to more closely approximate the unknown point. If the location of point \( 3' \) and the flow variables at \( 3' \) do not differ appreciably from the location of \( 3 \) and its flow variables to the accuracy desired, the procedure may be stopped. Otherwise point \( 3'' \) may be located. Point \( 3 \) is the initial location of the unknown point and \( 3', 3'', \) etc. are the locations on successive iterations.

Although arrived at separately, the procedure described above is similar to the iteration method suggested in ref. 9.
CHAPTER III

METHODS OF AXIALLY SYMMETRIC NOZZLE DESIGN

3.1 General

Several methods have been proposed for the design of three dimensional axisymmetric nozzles. Most of these are similar in that the method of characteristics is utilized for certain portions of the nozzle. Most methods also assume radial flow in a part of the flow field to reduce the computation time. The method of design as suggested by Foelsch (ref. 4) is an analytical approach to the problem. The methods described by references 5, 6, 7, 8, 9 and 11 propose mainly a numerical and graphical solution for the nozzle design.

3.2 Radial Source Flow

Diagram 3-1 depicts the four regions of flow in an axisymmetric nozzle:

I - Throat,
II - Radial Source Flow,
III - Transition,
IV - Uniform Parallel Flow.
When radial flow is assumed in a portion of the nozzle, the flow appears to originate from a source upstream of the throat area. This assumption is valid and is discussed in more detail by Pinkel (ref. 10). The region of radial flow could be plotted graphically by using the method of characteristics but the assumption in this case reduces computing time. The equations for three dimensional radial flow are based on the assumption of spherical symmetry. These equations are derived from the basic equations for a non-viscous, adiabatic flow as described in ref. 5. If the fluid moves radially, the velocity and speed of sound are functions of $r$ only (where $r$ is the distance from a fixed point). The continuity equation for this flow is

$$\frac{d(1-M^2)}{dt} + 2 \frac{V}{r} = 0$$

where $V$ is in the radial direction.

From the adiabatic energy equation the Mach number
and velocity are related,
\[ \frac{dV}{V} = \frac{1}{1 + \frac{Y-1}{2} M^2} \frac{dM}{M} \]  
(3.2)

Combining equations (3.1) and (3.2) gives
\[ \frac{dr}{r} = \frac{1}{2} \frac{M^2 - 1}{1 + \frac{Y-1}{2} M^2} \frac{dM}{M} \]  
(3.3)

Integrating this equation between the limits \( r = r^* \) to \( r \), and \( M = 1 \) to \( M \) gives
\[ \frac{r}{r^*} = \left[ \frac{1}{M} \left( \frac{2}{Y+1} \left( 1 + \frac{Y-1}{2} M^2 \right) \right) \right]^{\frac{M+1}{2(Y-1)}} \]  
(3.4)

From diagram 3-2 the characteristic equations can be determined.
For the $C^+$ Mach line

$$\frac{dR}{dx} = \tan (\Theta + \eta) \quad \text{and} \quad \frac{dr}{d\Theta} = \cot \eta \quad (3.5)$$

Combining equations (3.3) and (3.5) gives

$$d\Theta = \frac{1}{2} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{M^2}} \frac{dM}{M}$$

Integration of this equation from $\Theta = 0$ to $\sqrt{\gamma}$ and $M = 1$ to $M$ gives

$$\sqrt{\gamma} = \frac{1}{2} \left[ \frac{\gamma + 1}{\gamma - 1} \right]^{1/2} \tan^{-1} \left[ \frac{\gamma - 1}{\gamma + 1} (M - 1) \right]^{1/2} - \frac{1}{2} \cos^{-1} \frac{1}{M} \quad (3.6)$$

For the $C^-$ Mach line $\frac{dR}{dx} = \tan (\Theta - \eta)$ and

$$\sqrt{\gamma} = \frac{1}{2} \left[ \frac{\gamma + 1}{\gamma - 1} \right]^{1/2} \tan^{-1} \left[ \frac{\gamma - 1}{\gamma + 1} (M - 1) \right]^{1/2} + \frac{1}{2} \cos^{-1} \frac{1}{M} \quad (3.7)$$

Here $\sqrt{\gamma}$, which is the Prandtl-Meyer function in three dimensional flow is seen to be equivalent to one half the Prandtl-Meyer function in two dimensional flow ($\frac{1}{2} \sqrt{\gamma}$).

Equations (3.4) and (3.6) have been tabulated in ref. 5 for $\sqrt{\gamma} = 0$ to 53.375°. Sample tabulations are shown in Table I.

With these tabulated values of $r/r*$ the Mach lines BC and CD (diag. 3-1) may be constructed as follows: If
the desired test section Mach number is taken as $M_D = 8.03$, then $\gamma_D = 47.875 \text{ from the sample Table I.}$ The flow variables $M$ and $r/r^*$ for any point on line CD may be found by computing $\gamma = \gamma_D - \Theta$, where $\gamma$ is the Prandtl-Meyer angle of the point and $\Theta$ is the local flow angle of the point. Thus if $\Theta = 3^\circ$, $\gamma = 47.875^\circ - 3^\circ = 44.875^\circ$ and from Table I, $M = 6.77, r/r^* = 9.49$.

From any point along the upstream bounding Mach line, BC, $\gamma = \gamma_C - (\omega - \Theta) = \gamma_D - (2\omega - \Theta)$. The coordinates of each point may also be calculated since

$$\frac{X}{r^*} = \frac{r \cos \Theta}{r^*} \text{ and } \frac{R}{r^*} = \frac{r \sin \Theta}{r^*}$$

In references 5 and 11, radial source flow was assumed and the region BCD was designed utilizing tabulated values of three dimensional radial flow variables.

3.3 Design of the Throat Region

The portion of the nozzle between the minimum area section and the radial flow Mach line BC is termed the throat region. In reference 5, the flow in this region was computed by the method of characteristics with initial conditions of a known flow distribution along the Mach line BC and an arbitrary distribution along line AB. This arbitrary distribution was taken as a straight line with slope given by the radial flow at point B. For high Mach number
nozzles, this simplifying assumption satisfies the flow conditions because of the small dimensions of the minimum area section.

In plotting the flow net in this region, the characteristic equations were used in a slightly different form. The velocity parameter was replaced by the Mach number. Sample calculations indicated that this reduced numerical error when working in a range of high Mach numbers. When this is done, the characteristic equations are:

\[ d\Theta = \pm \frac{\sqrt{M^2-1}}{M \left(1 + \frac{\gamma - 1}{2} M^2 \right)} dM + \frac{1}{R \left(\sqrt{M^2 - 1} \cot \Theta \mp 1 \right)} d\chi \quad (3.8) \]

\[ dR = \tan (\Theta \pm \chi) d\chi \quad (3.9) \]

Since the last term in equation (3.8) becomes indeterminate at the axis of symmetry (as \( R \) and \( \Theta \rightarrow 0 \)), the limit of this term was taken as \( R \rightarrow 0 \). Equation (3.8) then reduces to

\[ d\Theta = \pm \frac{\sqrt{M^2-1}}{M \left(1 + \frac{\gamma - 1}{2} M^2 \right)} dM \mp \frac{\sqrt{M^2-1}}{2M \left(1 + \frac{\gamma - 1}{2} M^2 \right)} \left(\frac{\partial M}{\partial \chi}\right)_{R=0} d\chi = 0 \quad (3.10) \]

Thus when \( \left(\frac{\partial M}{\partial \chi}\right)_{R=0} \) is specified along the center line, equation (3.10) may be used with (3.8) and (3.9) to compute the flow for points close to the axis of symmetry.

However, in ref. 5, an alternative method for com-
puting the flow in the vicinity of the axis was used. Equation (3.10) was reduced by dividing through by \( dX \) and using the relation for \( \frac{dM}{dX} \) along a Mach line,

\[
\frac{dM}{dX} = \frac{\partial M}{\partial \chi} + \frac{\partial M}{\partial R} \frac{dR}{dX}.
\]

Since in axisymmetric isentropic flow \( \left( \frac{\partial M}{\partial R} \right)_{R=0} = 0 \),

\[
\frac{dM}{dX} = \left( \frac{\partial M}{\partial \chi} \right)_{R=0}.
\]

Hence the characteristic equations evaluated at the axis become

\[
d\theta = \pm \frac{1}{2} \sqrt{M^2 - 1} \frac{dM}{M(1 + \gamma - 1 \frac{M^2}{2})} \quad (3.11)
\]

At points near, but not on, the axis, equations (3.9) and (3.11) were used to compute the flow. Since equation (3.11) is of the same form as the corresponding equation for radial flow, equation (3.11) was integrated to determine the hodograph in the region of the axis. This method was assumed accurate enough providing that \( \partial M/\partial R \) is small. In this manner, the points adjacent to the axis in the throat region were computed by equation (3.11). The remainder of the flow field in region I was then computed by making use of equations (3.8) and (3.9).

In ref. 11, a linear axial Mach number distribution between point B and the throat was also arbitrarily chosen with a slope equal to that at B. From continuity consider-
ions, the throat and exit radii were expressed in terms of the critical radius \( \gamma^* \) and the nozzle expansion angle \( \omega \). The total mass flow through the nozzle is that flowing through the hemispherical cap of radius \( OC \) which is also equal to the mass which would flow through the area of the sonic arc of radius \( \gamma^* \) subtending an angle \( \omega \) at \( O \) (see diag. 3-1). This is true despite the fact that the region of radial source flow is terminated by the Mach line \( BC \) before the sonic arc is reached. Since the density and velocity have the same critical values at the nozzle throat and on the sonic arc, an equal mass flow through these sections is produced if their areas are equal. Thus

\[
\Pi (\gamma_r)^2 = \Pi (2\gamma^* \sin \omega/2) \quad \text{or} \quad \gamma_r = 2\gamma^* \sin \omega/2
\]

and from the 1 dimensional area relation (eqn 1.2)

\[
\gamma_e = 2\gamma^* \sin \omega/2 \sqrt{A/A^*}
\]

With the Mach number distribution and the throat height known, the position of the minimum area section was located. As in ref. 5, a form of the characteristic equations was used in plotting the flow field in the throat region. In ref. 11, these equations took the form of finite difference equations in the hodograph plane which were then
plotted in the physical plane in the throat region of the nozzle.

In ref. 8, the problem was to design a nozzle shorter in length than the ideal conical nozzle but with the same high performance capabilities. The method of characteristics was again used in establishing the flow field. From past experience, the expansion curve of the nozzle wall was selected as one initial boundary line. By means of approximate mathematical and transonic calculation methods, the Mach 1.15 line was drawn in the throat region and used as the second known curve. It was assumed that Prandtl-Meyer flow existed in the region of the wall. The form of the characteristic equations used in the $x, y$ plane were

$$\frac{dy}{dx} = \tan(\theta \pm \psi) \tag{3.12}$$

The corresponding equations for the $M, \Theta$ plane were

$$\frac{dM}{M} + \frac{d\Theta}{\sin \Theta \sin \psi} \cdot \frac{\tan \psi}{\cos(\Theta \pm \psi)} \cdot \frac{dx}{y} = 0 \tag{3.13}$$

From these two equations, difference equations were obtained between two known points and a third unknown point. The difference equations established the location of the third point.

In ref. 4, the author suggests an analytical approach
to the design of the throat region. Spherical three dimensional flow was converted into plane circular flow by bending the position of the nozzle wall adjacent to the throat into a smooth convex curve. This curve, QTF (diag. 3-4) is drawn tangent to the downstream portion of the wall at F and also is required to have zero slope at T.

For continuity, the cross-section area at the throat must be equal to the area of the spherical section

\[ \pi S^2 = \pi (2 r^* \sin \omega/2)^2 \]

The change in geometry of the sonic flow section will affect the source character of the flow in region II (see diagram 3-1). By inserting a straight line into the nozzle contour ahead of the transition curve, source flow can be preserved.
in region II (diagram 3-4).

3.4 Flow in the Transition Region

Apart from the analytical approach of Foelsch in ref. 4, the transition region (region III - diagram 3-1) for most axisymmetric nozzles has been designed by utilizing the method of characteristics. The characteristic equations are used in several forms which usually can be broken down into finite difference equations in the $x$ and $y$ or $u$ and $v$ directions.

In ref. 5, the flow in region III was computed by equations (3.8) and (3.9) from the initial conditions of a radial distribution along the Mach line CD and parallel uniform flow along the Mach line DE. However, analytical expressions for the coordinates of the transition streamlines were also derived in a manner similar to Foelsch. The general concept of this analytical approach is to start with an equation for the stream function and, making use of the known data curves (Mach lines CD and DE), write an expression for the coordinates of any point in the transition region. One of the intermediate steps makes use of the conservation of mass in determining the length of a transition Mach line. The final equations are
The points in the flow field are identified in diagram 3-5.

In designing the Mach number seven nozzle (ref. 11) a semi-graphical method was used in computing the flow in the transition region. Starting with the velocity potential equation for axisymmetric flow, the characteristic equations were obtained in the following finite difference form:

For a displacement parallel to $\eta$ axis

$$\Delta V_f \approx \frac{\sin^2 \eta}{y} \cdot v \Delta \eta$$  \hspace{1cm} (3.14)
For a displacement parallel to $\xi$ axis

$$\Delta V_{\eta} = \frac{\sin^2 \eta}{y} \cdot \nu \Delta \xi$$

(3.15)

In order to apply these equations to find the flow velocity at each point in the transition region, a grid of intersecting Mach lines $(\xi, \eta)$ was drawn. This was possible by using the radial flow boundary Mach line CD and the transition boundary Mach line DE. On these lines, the flow direction and Mach number distribution are known. The determination of flow parameters along line CD has already been discussed in section 2.2. Since region IV is a region of parallel flow, the velocity vector at any point along this line is parallel to the nozzle axis. In addition, this Mach line is drawn at an angle $(\eta + \theta)$ which, in this case, is the angle $\eta$ since $\theta = 0$. Mach line DE is, therefore, a straight line. A first approximation to the velocity at the center of each quadrilateral of the grid formed by the Mach lines was found by graphically adding successive velocity increments, calculated by equations (3.14) and (3.15), to the velocities along the boundary. This was done on a diagram in the hodograph plane (diag. 3-6). In this method, drawings of the Mach net and hodograph are placed side by side to permit ease of transferring the angles between the two drawings.
After several iterative procedures, a net of characteristics defines the flow in the transition region.

The axisymmetric characteristic equations used in ref. 7 are similar to equations (3.12) and (3.13) except that the velocity parameter is used instead of the Mach number. The form of these equations associated with each point in the flow field is

\[
\frac{dy}{dx} = \tan (\theta \pm \eta)
\]

\[
\frac{dV}{V} = \pm \tan \eta \, d\theta + \frac{\tan \eta \sin \eta \sin \theta}{\cos (\theta \pm \eta)} \, \frac{dx}{y}
\]

(\(\eta\) indicates an \(\eta\) line, \(-\) indicates a \(\xi\) line)

In ref. 7, the authors suggest using a finite difference form of equations (3.16) and (3.17). The method suggested for the design of the exhaust nozzle is a point-to-point calculation of the flow properties throughout the flow.
In diagram 3-7, if the distance between points is kept small, the Mach lines can be assumed straight. By numerical integration of equation (3.16), point C can be located by the following finite difference equations

\[
\frac{y_C - y_A}{X_C - X_A} = \tan(\theta_A + \eta_A)
\]

\[
\frac{y_C - y_B}{X_C - X_B} = \tan(\theta_B - \eta_B)
\]

These can be solved for \(X_C\) and \(y_C\)

\[
X_C = \frac{y_A - y_B + X_B \tan(\theta_B - \eta_B) - X_A \tan(\theta_A - \eta_A)}{\tan(\theta_B - \eta_B) - \tan(\theta_A + \eta_A)} \tag{3.18}
\]

\[
y_C = \frac{y_A + (X_C - X_A) \tan(\theta_A + \eta_A)}{\tan(\theta_A + \eta_A)} \tag{3.19}
\]

A finite difference form of equation (3.19) enables us to calculate \(V_C\) and \(\Theta_C\).
\[
V_c = \frac{1}{\cot \mu_A + \cot \mu_B} \left\{ \cot \mu_A \left[ 1 + \mathcal{L}_A (x_c - x_A) \right] + \cot \mu_B \left[ 1 + \mathcal{M}_B (x_c - x_B) \right] + \theta_B - \theta_A \right\} \tag{3.20}
\]

where
\[
\mathcal{L}_A = \frac{\tan \mu_A \sin \mu_A \sin \theta_A}{\gamma_A \cos (\theta_A + \mu_A)} \quad \mathcal{M}_B = \frac{\tan \mu_B \sin \mu_B \sin \theta_B}{\gamma_B \cos (\theta_B - \mu_B)}
\]
\[
\Theta_c = \theta_A + \cot \mu_A \left[ \frac{V_c - V_A}{V_A} - \mathcal{L}_A (x_c - x_A) \right] \tag{3.21}
\]

The method (as described above) of plotting the flow field makes use of the same procedures which are explained in section 2-8. The only real difference is the form of the equations which are used to find the flow properties at an unknown point in the flow. The method of ref. 7 also requires an iteration procedure which involves rewriting equations (3.18) through (3.21) with the coefficients as average values between points A and C, and B and C. Once again, except for the form of the equations, this iteration procedure is the same as that described in section 1-9.

In suggesting a method for design of the transition region of the nozzle, refs. 7 and 8 are identical. Where Guentert and Neumann do not recommend two particular known data curves as starting points, Dillaway uses the Mach 1.15 line and the nozzle wall curve. The form of the characteristic equations used by Dillaway has been discussed
previously in section 3-3. The iteration procedure recommended in these two references is also identical.

By using a high speed computing machine (the ENIAC at the Ballistic Research Laboratories, Aberdeen Proving Ground), ten supersonic axially symmetric nozzle shapes were computed by the method of characteristics (ref. 6). Nozzle contours can be obtained accurately from them by interpolation for exit Mach numbers and for a range of ratios of nozzle length to throat diameter.

3.5 Summary

Axially symmetric nozzles are generally designed by utilizing the method of characteristics. This method is applied by starting from two known data curves and computing a characteristic net of Mach lines from which the flow quantities can be evaluated. The nozzle is usually divided into four regions of flow:

I - Throat,
II - Radial Flow,
III - Transition,
IV - Uniform Flow.

The assumption of radial source flow in the nozzle saves computation time and provides known data curves from which calculation can begin. Applying the method of characteristics in the design of a nozzle requires a numerical-
graphical procedure with iteration to provide accuracy. Although many forms of the characteristic equations are used, the design procedures are very similar.
CHAPTER IV

DESIGN OF THE MACH NUMBER EIGHT NOZZLE

4.1 General Design Procedure

Since the method of characteristics has been utilized with accuracy in the past, it was decided to incorporate this into the design of the present Mach number eight nozzle. With this method, it would certainly be necessary to iterate the calculations to some degree which meant that much time would be spent on numerical computations. To shorten this time as much as possible, uniform radial source flow was assumed. This also provides a data curve for the construction of the characteristic net in the transition region.

4.2 Graphical Construction of Radial Flow Region

Design of the Mach eight nozzle was begun by graphically locating the region of radial flow. Point D (fig. 1) was first located by taking the $r/r^*$ ratio associated with $M = 8.03$ in the Radial Flow tables (see Table I for sample values). By letting $r^* = 2$ centimeters, point D was located at $r = 27.82$ centimeters from the assumed origin of source flow. It was decided to work in centimeters for

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ease of computation and graphical construction.

In the Mach number 7 nozzle (ref. 11) the expansion angle was chosen as 13.5° to permit the flow to expand in such a manner as to reduce the chances of flow separation and also to be sufficiently small so as to not overexpand and necessitate the introduction of compression waves. For the Mach number 10 nozzle (ref. 5) the expansion angle was chosen as 16° for the same reason. Based on these and other designed nozzles, an angle of 15° was chosen for the present Mach number 8 nozzle. Radial lines were drawn from the apparent origin of source flow at increments of 1.5°. By the method of section 3-2, point C was located and Mach lines BC and CD were constructed. This construction proceeded as follows:

a) $\gamma_C = \gamma_D - 15^\circ = 47.875 - 15^\circ = 32.875^\circ$
b) from Table I, $\gamma_C = 3.27 \times 6.54 \text{ cm}$.
c) $\Theta_C = 15^\circ$
d) point C located along 15° radial line at $\gamma = 6.54 \text{ cm}$.

The same procedure was repeated for points B, and E through $\gamma$ (see fig. 1). Table I gives radial flow variables corresponding to the Prandtl-Meyer function for points B, C, D and E through $\gamma$. This table depicts only a portion of the table illustrated in ref. 5.
4.3 Design of the Transition Region

Once the radial flow region has been constructed, the Mach line CD is a known data line which can be utilized as a starting point for the characteristic net in region III. From point D, a characteristic line can be drawn at an angle of \( \theta_p + \phi_p \) which in this case is \( \phi_p \) since \( \phi_p = 0 \). Since the flow along this line is uniform and parallel, the line is straight (see section 3.2 and 3.4 for a more thorough explanation of this reasoning). The Mach line DE is a second known data line which can be used in the construction of the characteristic net.

At any point along Mach line DE \( \gamma = 47.875^\circ \), \( \psi = 7.15^\circ \) and \( \phi = 0^\circ \). Point 1 (fig. 1) was arbitrarily chosen and by the method described in section 2.8, point 2 was located. In this instance, point \( \gamma \) corresponds to the known point on a \( \gamma \) characteristic and point 1 to the known point on a \( \xi \) characteristic. Once point 2 was located, the iteration procedure as explained in section 2.9 was used to locate point 2'. Sample calculations are shown in fig. 2. Upon locating point 2', this point and point \( \gamma \) were then utilized in locating point 3, etc.

This procedure continued (with points being arbitrarily selected along DE) until a large enough net existed so that the nozzle boundary could be constructed.
4.4 Construction of the Nozzle Wall

The nozzle wall was drawn in graphically by interpolating between the values of flow direction at the upper and lower points of the appropriate flow field. For example, from diagram 4-1, assuming that the nozzle wall enters the flow field at e at an angle of 10° (the same angle it had in the previous flow field) and that the flow angles at a and c are 7° and 10° respectively. Measure the straight line distance a-c and also measure where the nozzle wall crosses this line. From the diagram, a-c = 30 mm., and the assumed nozzle wall (dotted line) crosses the line a-c 21 mm. from point a. The true nozzle wall should, therefore, cross this flow field at an angle of

\[
\frac{21}{30} (10^\circ - 7^\circ) + 7^\circ = 9.1^\circ.
\]

From point e, the wall is now drawn at an angle of 9.1°.

By using the above procedure from point C to point...
E, the nozzle wall was constructed. See Table III for computed flow angles along the nozzle wall.

4.5 Throat and Test Section Dimensions

To determine the area ratio between the test section and the throat, the one dimensional area relationship may be used.

\[
\frac{A_t}{A^*} = \frac{1}{M_t} \left[ \frac{2}{Y+1} \left( 1 + \frac{\gamma - 1}{2} M_t^2 \right) \right] \frac{\frac{Y+1}{2(r-1)}}
\]

For the present Mach number eight nozzle, this area ratio is equal to 193.5.

When radial flow is assumed, from continuity considerations, the mass flow through the throat radius \( y_r \) is equal to the mass flow through the sonic arc which in turn is the same mass flowing through the chord of the sonic arc (diagram 4-2).

\[
y_r = 2 r^* \sin \frac{1}{2} \omega \quad (4.2)
\]
From equation (4.1)

\[
\frac{A_i}{A^*} = \frac{\pi y_i^2}{\pi y_T^2} \quad \text{or} \quad \frac{y_i}{y_T} = \left( \frac{A_i}{A^*} \right)^{\frac{1}{2}}
\] (4.3)

The throat radius is therefore:

\[
y_T = 2 (2) \sin 7.5^\circ = 0.522 \text{ cm.}
\]

The test section radius is:

\[
y_i = y_T (13.91) = 7.3 \text{ cm.}
\]

This value of the test section radius compares favorably with that arrived at by graphically constructing the nozzle wall (fig. 1).

4.6 Design of the Throat Region

The throat region was designed last. Rather than assume a linear distribution in Mach number between points B and A (the axial position of the throat), an analytical method of design was adopted. Appropriate boundary conditions were imposed on the nozzle contour in region I, and these conditions are satisfied by the general cubic equation which describes the wall curvature.

The boundary conditions imposed are:

a) \( y = y^* \), \( x = x^* \)

b) \( \frac{dy}{dx} \bigg|_{x = x^*} = 0 \)
These conditions are satisfied by the cubic
\[ y = A(x-x_*)^3 + B(x-x_*)^2 + C(x-x_*) + D \]
Applying boundary condition (a) \[ D = y* \]
Applying boundary condition (b) \[ C = 0 \]
Applying boundary condition (c) \[ A = -\frac{\tan \omega}{3(x_c-x_*)^2} \]
Applying boundary condition (d) \[ B = \frac{\tan \omega}{(x_c-x_*)} \]
Hence the equation of the nozzle wall in region II is
\[ y = y* + \frac{\tan \omega}{(x_c-x_*)} \left[ \frac{(x-x_*)^2 - (x-x_*)^3}{3(x_c-x_*)} \right] \]  \hspace{1cm} (4.4)
By applying the condition \( y = y_1 \) at \( x = x_1 \), equation (4.4) can be solved for \( x_1 \).

\[ y_1 = y* + \tan \omega \left[ \frac{x_c(x_c-x_*)}{3} \right] \] \hspace{1cm} (4.5)

Since \( y* = 5.22 \text{ mm}, \ y_1 = r_2 \sin \omega = 17 \text{ mm}, \) and \( x_c = 63 \text{ mm}. \)
\[ x_1 = -3 \text{ mm}. \]

For selected values of \( x \) between \( x^* \) and \( x_c \) equation
(4.4) was evaluated for the corresponding \( y \) and the resulting
wall curve \((x, y)\) was plotted between the actual throat and
the inflection point \( C \) (fig. 1).
4.7 Summary and Conclusions

An axisymmetric, hypersonic nozzle was designed using the method of characteristics to convert the flow into uniform parallel stream-lines. The nozzle throat was designed analytically and uniform radial flow was assumed in a portion of the nozzle in order to reduce computation time and to provide a known data curve. The numerical-graphical construction of the characteristic net utilized simple, straightforward equations. Net computation proceeded from two known points to an unknown point in the flow field of the transition region. Iteration permitted a close approximation to the flow parameters at the unknown point.

In the present method of design of the Mach eight nozzle, only one iteration was performed. A second or third iteration would improve the accuracy of the nozzle wall contour. For ease of computation and graphics, the flow net in the transition region was arbitrarily chosen as coarse. The selection of a finer net would again have resulted in a more exact solution.

This method of design is time consuming and tedious but can easily be programmed on a digital computer.

A suggested area for further study would be the building and testing of the Mach number eight hypersonic nozzle. A second area for future study would be to program...
on an electronic computer the characteristic net computations required in the design of the nozzle transition region.
REFERENCES


APPENDIX

Symbols

c \quad \text{local speed of sound in the flow}

\( u, v \) \quad \text{components of flow velocity parallel and perpendicular to nozzle axis of symmetry, respectively}

\( u_x, v_y, \text{ etc.} \) \quad \text{designation for } \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \text{ etc.}

q, V \quad \text{flow velocity}

x \quad \text{distance along nozzle axis of symmetry}

y \quad \text{distance normal to nozzle axis of symmetry}

r \quad \text{distance of a point from the apparent origin in radial source flow}

r^* \quad \text{critical radius in source flow (on which } M=1 \text{)}

s, n \quad \text{stream-line direction and normal to stream-line direction, respectively}

u, \Theta \quad \text{magnitude of velocity expressed in normal coordinates}

A \quad \text{nozzle cross sectional area}

M \quad \text{Mach number}

R \quad \text{distance measured normal from nozzle axis of symmetry to point in the flow field}

\( \gamma \) \quad \text{ratio of specific heats of the gas (for air } \gamma = 1.4\text{)}

\( \eta, \xi \) \quad \text{designation for left and right running Mach (characteristic) lines in the flow}

\Theta \quad \text{local flow angle}
### Symbols

- $\gamma$: Mach angle
- $\nu$: Prandtl-Meyer expansion angle in three dimensional flow
- $\nu'$: Prandtl-Meyer expansion angle in two dimensional flow
- $\rho$: density of the fluid
- $\omega$: nozzle expansion angle

### Subscripts

- $p$: $(r_d, \theta_z, \text{etc.})$ values at a point $P$ in the flow
- $l$: $(A_l, M_l, \text{etc.})$ values in test section of the nozzle
- $*$: $(r^*, y^*, \text{etc.})$ values where $M=1$ (either in the assumed radial flow or the nozzle throat)
- $\tau$: $(y_T, \text{etc.})$ values at the designed throat of the nozzle
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Computations for the Characteristic Net

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<tr>
<td>9°a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>na</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10°</td>
<td>40.265</td>
<td>5.41</td>
<td>10.65</td>
<td>17.69</td>
<td>.185</td>
<td>.303</td>
<td>20</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>32.875</td>
<td>4.00</td>
<td>14.487</td>
<td>15.00</td>
<td>.250</td>
<td>.258</td>
<td>17</td>
<td></td>
<td></td>
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</tbody>
</table>
C1   | 39.21   |     |         |         |        |        |        |          |          |
| 10°a |         |     |         |         |        |        |        |          |          |
| Ca   |         |     |         |         |        |        |        |          |          |
C1   | 39.32   | 5.19| 11.11   | 20.44   |        |        |        |          |          |
### TABLE III

**Computations for Construction of the Nozzle Wall**

<table>
<thead>
<tr>
<th>Flow Field</th>
<th>Computation</th>
<th>Wall Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((n, 10', C', C'))</td>
<td>(\frac{1}{2}(20.44 - 13.5) + 13.5)</td>
<td>16.97</td>
</tr>
<tr>
<td>((9', 29', 30', 10'))</td>
<td>(\frac{8}{17}(13.84 - 15.61) + 15.61)</td>
<td>14.8</td>
</tr>
<tr>
<td>((8', 28', 29', 9'))</td>
<td>(\frac{11}{15}(12.75 - 13.18) + 13.18)</td>
<td>12.87</td>
</tr>
<tr>
<td>((28', 38', 39', 29'))</td>
<td></td>
<td>10.8</td>
</tr>
<tr>
<td>((27', 37', 38', 28'))</td>
<td>(\frac{9}{20}(9.13 - 9.0) + 9.0)</td>
<td>9.06</td>
</tr>
<tr>
<td>((26', 36', 37', 27'))</td>
<td>(\frac{14}{19}(8.07 - 7.13) + 7.13)</td>
<td>7.83</td>
</tr>
<tr>
<td>((36', 46', 47', 37'))</td>
<td>(\frac{2}{23}(7.6 - 6.5) + 6.5)</td>
<td>6.6</td>
</tr>
<tr>
<td>((35', 45', 46', 36'))</td>
<td>(\frac{12}{22}(5.7 - 5.0) + 5.0)</td>
<td>5.38</td>
</tr>
<tr>
<td>((34', 44', 45', 35'))</td>
<td>(\frac{22}{25}(4.36 - 3.47) + 3.47)</td>
<td>4.25</td>
</tr>
<tr>
<td>((44', 54', 55', 45'))</td>
<td>(\frac{6}{20}(4.02 - 2.97) + 2.97)</td>
<td>3.28</td>
</tr>
<tr>
<td>((43', 53', 54', 44'))</td>
<td>(\frac{15}{20}(2.72 - 1.86) + 1.86)</td>
<td>2.49</td>
</tr>
<tr>
<td>((53', 63', 64', 54'))</td>
<td>(\frac{3}{15}(2.54 - 1.67) + 1.67)</td>
<td>1.82</td>
</tr>
</tbody>
</table>
### TABLE III (Cont'd)

**Computations for Construction of the Nozzle Wall**

<table>
<thead>
<tr>
<th>Flow Field</th>
<th>Computation</th>
<th>Wall Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (52^\circ, 62^\circ, 63^\circ, 53^\circ) )</td>
<td>( \frac{13}{18}(1.52 - .75) + .75 )</td>
<td>1.31</td>
</tr>
<tr>
<td>( (62^\circ, 72^\circ, 73^\circ, 63^\circ) )</td>
<td>( \frac{5}{22}(1.4 - .67) + .67 )</td>
<td>0.83</td>
</tr>
<tr>
<td>( (61^\circ, 71^\circ, 72^\circ, 62^\circ) )</td>
<td>( \frac{16}{22}( .61 - 0.0) + . )</td>
<td>0.45</td>
</tr>
<tr>
<td>( (71^\circ, 81^\circ, 82^\circ, 72^\circ) )</td>
<td>( \frac{6}{24}( .58 - 0.0) )</td>
<td>0.15</td>
</tr>
</tbody>
</table>
FIGURE 2

Computations

\[ \nu_i' = \frac{1}{2}(47.875° + 46.375°) + \frac{1}{2}(0 - 1.5°) + \frac{1}{2} \left[ .1245(0)(\frac{58°}{6.5°}) + .136(.02620)(\frac{45°}{6.5°}) \right] = 47.075° \]

\[ \Theta_i = \frac{1}{2}(47.875 - 46.375) + \frac{1}{2}(0 + 1.5) + \frac{1}{2} \left[ .1245(0)(\frac{58°}{6.5°}) - .136(.02622)(\frac{45°}{6.5°}) \right] = 0.8° \]

\[ \nu_i' = \frac{1}{2}(47.875 + 46.375) + \frac{1}{2}(0 - 1.5) + \frac{1}{2} \left[ .128(.007)(\frac{58°}{10°}) + .133(.020)(\frac{45°}{10°}) \right] = 46.865° \]

\[ \Theta_i = \frac{1}{2}(47.875 - 46.375) + \frac{1}{2}(0 + 1.5) + \frac{1}{2} \left[ .128(.007)(\frac{58°}{10°}) - .133(.020)(\frac{45°}{10°}) \right] = 1.3° \]
# FIGURE 2 (Cont'd)

## Tabulated Values

<table>
<thead>
<tr>
<th>Pt</th>
<th>( V )</th>
<th>M</th>
<th>( \phi )</th>
<th>( \theta )</th>
<th>( \text{Sine} \phi )</th>
<th>( \text{Sine} \theta )</th>
<th>R</th>
<th>( \Delta \xi )</th>
<th>( \Delta \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>47.875</td>
<td>8.03</td>
<td>7.15</td>
<td>0.0</td>
<td>0.124</td>
<td>0.0</td>
<td>6</td>
<td>58</td>
<td></td>
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<tr>
<td>v</td>
<td>46.375</td>
<td>7.35</td>
<td>7.815</td>
<td>1.5</td>
<td>0.136</td>
<td>0.026</td>
<td>6</td>
<td>45</td>
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<tr>
<td>2</td>
<td>47.075</td>
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<td>7.50</td>
<td>0.8</td>
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<td>14</td>
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<tr>
<td>( v_{avg} )</td>
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<td></td>
<td>7.33</td>
<td>0.4</td>
<td>0.128</td>
<td>0.007</td>
<td>10</td>
<td>58</td>
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</tr>
<tr>
<td>( v_{avg} )</td>
<td></td>
<td></td>
<td>7.66</td>
<td>1.15</td>
<td>0.133</td>
<td>0.020</td>
<td>10</td>
<td>45</td>
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<tr>
<td>2'</td>
<td>46.865</td>
<td>7.53</td>
<td>7.60</td>
<td>1.3</td>
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