Application of Dedekindian Schnitt to Definition of Logarithm

by

Vedder S. Hughay

Submitted in partial fulfillment of the
requirements for the degree of

Master of Arts

in the College of Letters, Arts, and Sciences, of the

University of Arizona

1932

Approved

A. A. Shaw

Heman Burr Leonard

May 10, 1932
INTRODUCTION

The following discussion is entirely concerned with a point of rigor. Important and involved as the establishing of such points admittedly is, it may seem that we have only succeeded in stating precisely and logically what we already knew to be so intuitively. That is, such discussions may seem from a creative point of view essentially sterile. This however is not the case.

Investigations as to the nature of irrational numbers and the nature of their defining sequences has frequently led to definite contributions to phases of mathematics previously assumed unrelated. To mention a very elementary example: the incommensurable nature of the ratio of the circumference to the diameter of a circle.

As a further illustration of the potentialities inherent in the subject consider the nature of the terms of sequences which define some number as the limiting sum to infinity of the terms of the sequence. For special cases criteria have been developed which allow the rational or irrational character of the limiting sum to be determined from the nature of the terms alone, that is, from the formula for the general term, without knowing what rational or
Introduction (Continued)

irrational value that limit may be. If we could develop criteria enabling us to always determine the rational or irrational character of the sum of a convergent infinite series from the formula for the general term we would be in a position to immediately establish Fermat's Last Theorem. For the statement that for any integral \( n > 2 \) there exists no set of rational numbers \( a, b, c \), such that

\[ a^n + b^n = c^n \]

implies that if \( a \) and \( b \) are rational \( c \), since some \( c \) exists satisfying the equation, must be irrational. But we can divide both members by the larger of \( a \) and \( b \) and take the \( n \)th root of both members and the transformed equation will be (say for \( a > b \))

\[
\left( 1 + \left( \frac{b}{a} \right)^n \right)^{\frac{1}{n}}
\]

where \( 0 < \frac{b}{a} < 1 \)

and \( \frac{c}{a} \) is irrational.

Expanding the right member above by the binomial formula we obtain an infinite series whose sum, according to the theorem, is irrational. Applying our criteria would immediately verify this and the theorem would be established.

---

(1) These remarkable theorems are discussed in Hobson's "Theory of Functions of Real Variable," Ch. I.
There are two well known methods of defining the important logarithmic function. Both arrive at identical conclusions regarding the properties of the function, the most fundamental of which is that given by the equation

$$\log fcn(x) + \log fcn(y) = \log fcn(xy)$$

It is the theoretical justification of these properties, and especially the inverse relation of the logarithmic to the exponential function, which the two methods treat differently.

In elementary algebra the exponential is defined by the equations

$$b^p = \sqrt[p]{b^p}, \quad b^{-\frac{p}{q}} = \frac{1}{b^{\frac{p}{q}}}$$

with $p, q$ positive integers and $b > 0$. And it follows that

$$b^m \cdot b^n = b^{m+n}$$

for $m, n$ any rational numbers.

The logarithm of $N$ to the base $b$ is then defined by the equations

$$L = \log_b N$$

if $b^L = N$, $N > 0$, $b > 0$, $b \neq 1$.

Hence we see at once that

$$\log_b (b^{L}) = L$$

and $b^{\log_b N} = N$

which are the conditions that the logarithmic and exponential functions be inverse functions.\(^{(1)}\)

This approach to the logarithm is open to the immediate objection that the exponential, and therefore its associated logarithm, are not defined for irrational values of

---

\(^{(1)}\) As to the precise meaning of inverse applied to operations see E. B. Wilson (p. 150, Advanced Calculus.) There can be, of course, no ambiguity here.
the exponent. Thus, if no two integers \( p \) and \( q \) exist such that \( b^{p/q} = N \), then we do not know what \( \log_b N \) is, and the logarithm will be discontinuous for such values of its base and argument. Since there exists an infinity of such pairs of values\(^{(1)}\) for \( b \) and \( N \) it becomes necessary, in establishing the continuity of the logarithmic function, to establish the continuity of the exponential function. That is, we must define the exponential for irrational values of the exponent and then identify \( L \), when it cannot be rational, with some irrational number.

It is customary to postpone the treatment of this problem to more advanced discussions. There it is attached indirectly in the following manner. The logarithmic function of \( N \) is defined by the equation

\[
\log \tan (N) = \int_{1}^{N} \frac{dt}{t}
\]

for \( N > 0 \),

where the right member exists for any rational or irrational positive \( N \). It is then shown that

\[
\log \tan (x) + \log \tan (y) = \log \tan (xy)
\]

Likewise, all the other characteristic properties of logarithms are deducible from the definition as an integral; and we find that for rational values this function is identical with the inverse of the exponential to the base \( e \), where \( e \) is given by the equation

\[
1 = \int_{1}^{e} \frac{dt}{t}
\]

Moreover, the function

\[
[\log \tan (N) / \log \tan (b)]
\]

is identical with the inverse of the exponential to the base \( b \).

\(\text{(1)}\)

Consult any elementary text on number theory.
for rational values. Finally, these functions are continuous\(^{1}\)
thru all values of \(b\) and \(\sqrt{b}\), whether the value of the function be rational or irrational.

Having thus rigorously developed a function, which is equivalent to our ordinary logarithm when it is rational and which is continuous thru irrational values, independently of the exponential, then the exponential is defined in terms of this function. By a series of simple operations with the equation

\[
\log \tan (b) = \int_{1}^{b} \frac{dt}{t}
\]

it is shown that for a rational \(L\)

\[
\log \tan (b^L) = L \log \tan (b)
\]

from which \(b^L = e^{L \log \tan (b)}\).

\(b^L\) is then defined for irrational values of \(L\) by this same equation, and the right member has a perfectly definite meaning which is equivalent to\(^{2}\)

\[
\lim_{n \to \infty} \left(1 + \frac{L}{n} \sum_{i=1}^{n} \frac{dt}{t}\right)^n
\]

In the above definition of \(b^L\) there is implied a fundamental postulate. Dedekind\(^{3}\) has demonstrated the logical necessity of conceiving of irrationals as the frontiers\(^{4}\) of sequences of rationals. Thus any irrational \(L = \lim_{L(L)}\) \(\rho, \beta\) integers.

---

\(^{1}\) See G. H. Hardy (Art. 204 Pure Mathematics).

\(^{2}\) Refer to G. H. Hardy (p. 368, Pure Mathematics). Our outline of the conventional treatment of logarithms is taken from his book.

\(^{3}\) For a complete exposition of Dedekind's method of introducing the irrational number, together with interesting historical information, see H. S. Carslaw (Ch. I, Fourier's Series and Integrals).
Therefore our definition stated explicitly involves
\[
\log \tan (b^{\frac{\lim t}{\sqrt{a}}}) = (\lim \frac{t}{\sqrt{a}}) \log \tan (b) = \lim \left[ \frac{\log \tan (b)}{\sqrt{a}} \right] = \lim \left[ \log \tan (b^{\frac{\lim t}{\sqrt{a}}}) \right]
\]
but this is simply (1)
\[
\int_{\frac{\lim t}{\sqrt{a}}}^{b^{\frac{\lim t}{\sqrt{a}}}} \frac{dt}{t} = \lim \left[ \int_{\frac{\lim t}{\sqrt{a}}}^{b^{\frac{\lim t}{\sqrt{a}}}} \frac{dt}{t} \right] = \int_{\frac{\lim t}{\sqrt{a}}}^{b^{\frac{\lim t}{\sqrt{a}}}} \frac{dt}{t} \quad \lim \frac{t}{\sqrt{a}} = L \text{ irrational.}
\]
So that we tacitly accept here the postulate that
\[
b^{\lim \frac{t}{\sqrt{a}}} = \lim \left[ b^{\frac{t}{\sqrt{a}}} \right] \quad \lim \frac{t}{\sqrt{a}} = L \text{ irrational.}
\]
This concept imposes itself upon our minds with more or less force. Dedekind's method of introducing the irrational involves the postulate, expressed symbolically, that
\[
\left[ \lim \left( \frac{t}{\sqrt{a}} \right) \right]^{n} = \lim \left[ \left( \frac{t}{\sqrt{a}} \right)^{n} \right] \quad a \text{ rational,}
\]
\[
\left[ \lim \left( \frac{t}{\sqrt{a}} \right) \right]^{n} = \lim \left[ \left( \frac{t}{\sqrt{a}} \right)^{n} \right] \quad a \text{ irrational.}
\]
Is it not consistent then to carry this viewpoint to its logical completion and assume further that
\[
b^{\lim \frac{t}{\sqrt{a}}} = \lim \left[ \left( \frac{t}{\sqrt{a}} \right) \right] \quad b \text{ rational,}
\]
\[
b^{\lim \frac{t}{\sqrt{a}}} = \lim \left[ \left( \frac{t}{\sqrt{a}} \right) \right] \quad L \text{ irrational.}
\]
Since the assumption that the exponential of the limit is uniquely defined by the limit of the exponential cannot be avoided even by the above indirect approach, let us accept it at the very start and see if by so doing we cannot at least avoid some of the indirection.

If the exponential of the limit equals the limit of the exponential, then the problem of evaluating numbers raised

(4)

This terminology is employed by E. B. Wilson (p. 36, et seq. Advanced Calculus) in an admirable summary of the problems considered in our thesis.

(1)

The step from the limit of the definite integral to the definite integral of the limit involves no doubtful analytical processes.
to irrational powers is at once reduced to familiar elementary concepts. For, we may express the irrational power as a frontier or limit of a sequence of rationals and determine the value of any number raised to that irrational power as the frontier or limit of the sequence of numbers raised to the rational powers which constitute the terms of the sequence defining the given irrational power. Needless to say, the result may be rational or irrational. Now that the exponential is defined for any real exponent we are in a position to establish the logarithm as that continuous function which is the inverse of the continuous exponential function. We must, however, identify \( \xi \), in the equation \( b^\xi = \gamma \), for those values of \( b \) and \( \gamma \) which admit of no rational \( \xi \) satisfying the equation, with some irrational member of the arithmetical continuum.

Consider two given\(^{(1)}\) values of \( b^\gamma \) and \( \gamma > 0 \) for which no rational \( \xi \) exists, negative, zero,\(^{(2)}\) or positive, such that \( b^\xi = \gamma \). Let us attempt to identify \( \xi \) with some irrational.

I. Divide all rational numbers into two classes such that all members of the upper class \( A \), upon exponentiating on \( b \), give numbers greater than \( \gamma \), while all members of the lower class \( B \), upon exponentiating on \( b \), give numbers less than \( \gamma \).

---

\(^{(1)}\) The case for \( 0 < b < 1 \) is established by an obvious extension of the identical arguments. Here we may let \( b = \frac{1}{b'} \), where \( b' > 1 \) and the position of the classes \( A \) and \( B \) is interchanged and we have, so to speak, a mirror image of the subsequent discussion and arrive at the same general picture.

\(^{(2)}\) The case where \( \gamma = 1 \) and hence \( \xi = 0 \) is trivial.
All rational numbers can be so divided since \( b^\frac{c}{d} \) is determinate and must be either less than, equal to, or greater than the given \( \gamma \); and by hypothesis it cannot be equal to \( \gamma \).

II. Every member of the upper class \( A \) is greater than every member of the lower class \( \beta \).

This follows from the fact that \( \exp^{\frac{c}{d}} \) is a monotonic function(1) of \( \frac{c}{d} \). That is, for rational values of the exponent, we know that an increase in the exponent results in an increase in the exponential and conversely, for any base \( ^\gamma \). Since the exponentials of the upper class are greater than those of the lower class, the same is true of the corresponding exponents.

We wish now to show that the upper class of rational exponents \( A \) can have no least member, and that the lower class \( \beta \) can have no greatest member. Having done this we will be able to exhibit \( L \), uniquely, as that member of the arithmetical continuum which divides the continuum into an upper and lower portion which contain, respectively, our upper and lower classes or rational members. For, the postulate that

\[
\lim_{n \to \infty} \left[ b^{\frac{c}{d}} \right] = \lim_{n \to \infty} \left( b^{\frac{c}{d}} \right)
\]

means that \( L \) does possess an exponential having the same

(1) A monotonic function is one whose values form a monotonic sequence, the indices of which constitute the set of values of its argument. In this case the indices range from \(-\infty\) to \(+\infty\) while the values of the sequence steadily increase from zero to \(+\infty\). We restrict ourselves in this discussion to the single valued real positive branch of the function.
monotonic property shown by exponentials with rational exponents. Since that exponential equals $N$ and lies between the exponentials of the upper and lower classes, therefore, its exponent $\lambda$ lies between the upper and lower classes of exponents. Hence $\lambda$ will then be identified with that irrational member of the arithmetical continuum which establishes the described irrational section.

III. (a). To show that the upper class $A$, whose exponentials of $b$ are greater than $N$, can have no least member, suppose there were such a number. Let it be $\beta$, then

$$b^\beta > N$$

and $b^\rho > N^\beta$ hence if

$$-\epsilon = \frac{N^\beta - b^\rho}{b^\rho}$$

then $b^\rho(-\epsilon) = N^\beta$ where $\epsilon$ is positive and rational (1) and $(\epsilon - \delta)$ is rational and less than unity by a definite fixed quantity. Consider a rational positive $\delta$ such that $\delta < \epsilon$, then we can find (2) a positive integer $m$ for which

$$(\delta - \epsilon)^m = 1 - \frac{m}{2} \delta^{1/2} + \frac{m(m-1)}{2!} \delta^2 - \ldots - (-1)^{m-1} \delta^m < \frac{1}{b}$$

hence $(\delta - \epsilon) < \delta^{1/2}$ and since $(\epsilon - \delta) < (\epsilon - \delta)$ therefore $(\delta - \epsilon) < b^{-\lambda}$ and $b^\rho(\delta - \epsilon) < b^\rho b^{-\lambda}$, then it follows that $N^\beta < b^{\rho - \lambda}$ and $N < b^{\rho - \lambda}$ where $\delta^{1/2}$ is rational and less than $\frac{\rho}{b}$ but is a member of the upper class $A$; hence the supposition is absurd.

(1)

The result of a finite number of combinations of the elementary operations of arithmetic upon a rational number is itself rational, see G. H. Hardy (Ch. I - Pure Mathematics).

(2)

We can always find a finite $m$ sufficiently large such that for a finite $\delta$, $(\delta - \epsilon)^m$ is as small as we choose.
III. (b) To show that the lower class $B$ whose exponentials of $b$ are less than $\sqrt[N]{S}$ can have no greatest number, suppose there were such a number. Let it be $\frac{R}{S}$, then

$$b^R > \sqrt[N]{S}$$

and $b^R > \sqrt[N]{S}$ hence if

$$e = \frac{\sqrt[N]{S}b^R}{b^R}$$

then $b^R(1+e) = \sqrt[N]{S}$ where $e$ is positive and rational and $(1+e)$ is rational and greater than unity by a definite fixed quantity. Consider a positive rational $S$ such that $S < e$ then we can find a positive integer $m$ such that

$$(1+e)^m = 1 + \frac{m}{2} e + \frac{m(m-1)}{2} e^2 + \cdots + e^m > b$$

hence $(1+e) > b^\frac{1}{m}$ and since $b^R(1+e) > (1+e)$

therefore $(1+e) > b^\frac{1}{m}$ then $b^R(1+e) > b^\frac{1}{m} b^\frac{1}{m}$

so that $\sqrt[N]{S} > b^R + \frac{1}{m}$ and $\sqrt[N]{S} > b^R + \frac{1}{m}$

where $\frac{R+1}{S}$ is rational and greater than $\frac{R}{S}$ but belongs to class $B$; hence our supposition is again absurd.

So we have demonstrated that neither the upper class of rational numbers whose exponentials of $b$ are greater than $\sqrt[N]{S}$ can have a least member, nor the lower class of rational numbers whose exponentials of $b$ are less than $\sqrt[N]{S}$ can have a greatest member. Hence $L$ is properly defined, according to the method of Dedekind, by these classes as that irrational number which brings about the section of the two classes.

We can now rigorously define the logarithm as the inverse of the exponential for any value which the exponent may assume. Since the exponential has the property that

$$b^m b^n = b^{m+n}$$

let $m = \log_b x$, $n = \log_b y$
substituting \(6^{\log_b x} \cdot 6^{\log_b y} = 6^{\log_b x + \log_b y}\)

but

\(6^{\log_b x} \cdot 6^{\log_b y} = x^y = 6^{\log_b x \cdot y}\)

hence

\(6^{\log_b x \cdot y} = 6^{\log_b x + \log_b y}\)

therefore

\(x^y = 6^{\log_b x + \log_b y}\)

for any real positive values of \(x\) and \(y\), and for any real positive value of \(b \neq 1\). And so the characteristic property of the logarithmic function is deduced from that of the exponential function.

Knopp\(^{(1)}\) has developed a method of establishing the fact that the above section or "schnitt" satisfies all the conditions, laid down by Dedekind and subsequent investigators, for sufficient definition of an irrational. His treatment is perhaps more analogous to that of Cantor than Dedekind. The particular process he employs is known as the theory of decimal sections. Since the methods of Cantor, and its extension to irrational exponentials by Knopp, parallels and is essentially equivalent to the method of Dedekind, and its extension here given, we will confine ourselves to mere mention of it. For an exhaustive consideration of the methods of Cantor and Dedekind and their equivalence the reader is advised to consult Hobson's "Theory of Functions of a Real Variable."

\(^{(1)}\) Refer to K. Knopp (p. 47 et seq. Theorie und Anwendugender Unendlichen Riehen).