

Discussion of Some of the Problems

"Arising from a Consideration of the Rotation in Space of an Ellipsoid of three unequal Axes about an Axis not the Axis of Figure."

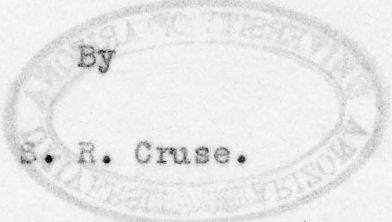
Being a Thesis offered to the Faculties of the University of Arizona in part fulfillment of the requirements for the

D e g r e e o f

M a s t e r o f S c i e n c e

By

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The seal of the University of Arizona is visible in the background, featuring the text "UNIVERSITY OF ARIZONA" and "FOUNDED 1891".

Discussion of Some of the Problems

Arising from a Consideration of the Rotation in
Space of an Ellipsoid of three unequal axes about
an axis not the axis of symmetry.

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Master of Science

Department of Mathematics



In some of his astronomical researches, particularly upon one of the moons of Jupiter and one of the variable stars, Dr. A.E. Douglass, Professor of Astronomy and Director of the Steward Observatory at the University of Arizona, found that a discussion of the motion of an ellipsoid of three unequal axes, rotating in space about an axis not the axis of figure, was necessary in order further to elucidate his work. Also, many of his observations tended to show that at least one of the variable stars has a variable period. It is the purpose of this paper to discuss the motion of this rotating ellipsoid, and to offer an explanation of the variable period by means of this discussion. It may be remarked at the outset that mathematical proof of this explanation of a possible variable period has not been established, owing to the fact that observations necessary in order to plot the requisite curves have not been available. It is hoped that at an early date these observations may be forthcoming, and that this problem may then be completed in such a way as to prove of some little practical value to Dr. Douglass in his work.

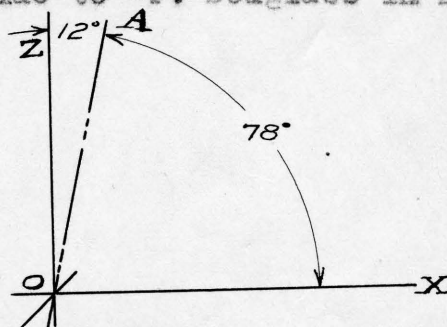


Fig. 1

The semi axes OX, OY, OZ of the ellipsoid under discussion are, respectively, $4\frac{1}{2}$, 5, 4. OA, the axis of rotation, lies in the plane of OX and OZ and makes with OZ an angle of 12° . OA passes through

O, the center of mass of the ellipsoid, which remains fixed during the rotation. This motion has been termed a "Poincot - motion" by Webster (vide "Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies", p. 256). In such motion, there is no translation, consequently there is neither linear momentum nor centrifugal resultant, and the only forces we have to deal with are the angular momentum and the centrifugal couple. This centrifugal couple produces in the body a tendency to right itself and to set itself in a stable position wherein the axis of rotation coincides with the axis of greatest inertia - that is, it tends to rotate about the axis of greatest radius of gyration. This is true of any rotating body and may be illustrated by a familiar example. If a ball be attached to the end of a cord whose other end is held in the hand and the ball be rotated on the end of the string, it will rise higher and higher as the velocity is increased, until it has assumed a position as far from the axis as possible, and will maintain this position despite further increases in the velocity. The axis of greatest inertia in our case is evidently the shortest, or Z axis. We shall first compute the value of this centrifugal couple for the ellipsoid under consideration.

Again referring to Webster's monumental work before cited, we find (p. 257) that the centrifugal couple S_c is equal to the resultant of the components of angular momentum of the ellipsoid along the axes of X and Z, multiplied by the angular velocity times the time - differential.

Stated in terms of a formula,

if H_x be the component of Angular Momentum along the axis of X,
 and H_z the component of Angular Momentum along the axis of Z,
 and H_2 the resultant of H_x and H_z ,
 and ω the angular velocity,
 then $S_c dt = H_2 \omega dt$ (1).

This is the Righting Power of the ellipsoid and measures the energy with which it tends to right itself. It is now necessary, before we go further, to calculate the values of H_x , H_y , H_z , the components of Angular Momentum along the axes of X, Y, and Z.

From Webster, p. 256,

$$\begin{aligned} H_x &= - E \omega, \\ H_y &= - D \omega, \\ H_z &= C \omega \end{aligned} \quad (2).$$

That is, H_x , the component of the Angular Momentum along the axis of X, is the product of the angular velocity ω into the Product of Inertia with respect to the X and Z axes, with a negative sign; H_y , the component of the Angular Momentum along the axis of Y, is the product of the angular velocity ω into the Product of Inertia with respect to the Y and Z axes, with a negative sign; and H_z , the component of the angular momentum along the Z axis, is the product of the angular velocity ω , into the Moment of Inertia of the Ellipsoid with respect to the Z axis.

Assuming that the ellipsoid, which is here being considered, represents a continuous distribution of mass - that is, has a uniform density of ρ , the infinitesimal element of volume is $d\tau$ and the infinitesimal element of mass is dm .

Therefore $dm = \rho d\tau = \rho dx dy dz$.

Now from Webster, Art. 53, p. 226, we have

$$\begin{aligned} H_x &= M V \bar{y} - \omega \sum m z x, \\ H_y &= M V \bar{x} - \omega \sum m z y, \\ H_z &= \omega \sum m r^2 \end{aligned} \quad (3).$$

Since we are considering here the motion of a body, one of whose points remains fixed, there is no translation. In the above formulae, \bar{x} and \bar{y} are the abscissa and ordinate, respectively, of the distance through which the center of mass moves. Inasmuch as in this case the center of mass remains fixed, \bar{x} and \bar{y} become zero, and consequently $MV\bar{y}$ and $MV\bar{x}$ vanish.

$$\begin{aligned} \text{Therefore } H_x &= -\omega \sum m z x, \\ H_y &= -\omega \sum m z y, \\ H_z &= \omega \sum m r^2 \end{aligned} \quad (4).$$

A comparison of the values of H_x , H_y and H_z in Equations (2) and (4) will show that

$$\begin{aligned} E &= \sum m z x = \int z x dm, \\ D &= \sum m z y = \int z y dm, \\ C &= \sum m r^2 = \int (x^2 + y^2) dm. \end{aligned}$$

But $dm = \rho d\tau = \rho dx dy dz$.

$$\begin{aligned} \text{Therefore } E &= \iiint \rho x dx dy z dz, \\ D &= \iiint \rho dx y dy z dz, \\ C &= \iiint \rho (x^2 + y^2) dx dy dz. \end{aligned}$$

We shall also need, in the course of this paper, the additional constants P , A , and B , P being the product of inertia with re-

spect to the axes of X and Y, A the moment of inertia about the X-axis, and B, the moment of inertia about the Y-axis. From the fact that the formulae for these constants are symmetrical, we may at once write

$$F = \iiint \rho \quad x \, dx \, y \, dy \, dz ,$$

$$A = \iiint \rho (y^2 + z^2) \, dx \, dy \, dz ,$$

$$B = \iiint \rho (x^2 + z^2) \, dx \, dy \, dz .$$

We may, however, simplify the formula for the three Moments of Inertia. From D'Alembert's "Analytical Mechanics", p. 219, we have

$$A = \rho \iiint (y^2 + z^2) \, dx \, dy \, dz = \rho V \frac{\iiint (y^2 + z^2) \, dx \, dy \, dz}{V}$$

$$B = \rho \iiint (x^2 + z^2) \, dx \, dy \, dz = \rho V \frac{\iiint (x^2 + z^2) \, dx \, dy \, dz}{V}$$

$$C = \rho \iiint (x^2 + y^2) \, dx \, dy \, dz = \rho V \frac{\iiint (x^2 + y^2) \, dx \, dy \, dz}{V}$$

We shall now employ the calculus to evaluate the six constants

E, D, F, A, B, C, taking them in that order. To obtain the limits between which we must integrate in order to secure the actual values of E, D, and F for this particular problem, consider the equation of our ellipsoid, which has semi-axial values of $\frac{10}{2}$, $\frac{9}{2}$ and $\frac{8}{2}$

for Y, X, and Z, respectively. Hence its equation is

$$\frac{x^2}{(\frac{10}{2})^2} + \frac{y^2}{(\frac{9}{2})^2} + \frac{z^2}{(\frac{8}{2})^2} = 1 . \quad (5)$$

Let us consider that part of the ellipsoid bounded by its surface and the three planes each containing two of the axes.

If we integrate with respect to Z, we build up a column composed of infinitesimal elements of mass $dm = \rho \, d\tau$, Z varying from zero to that value of Z

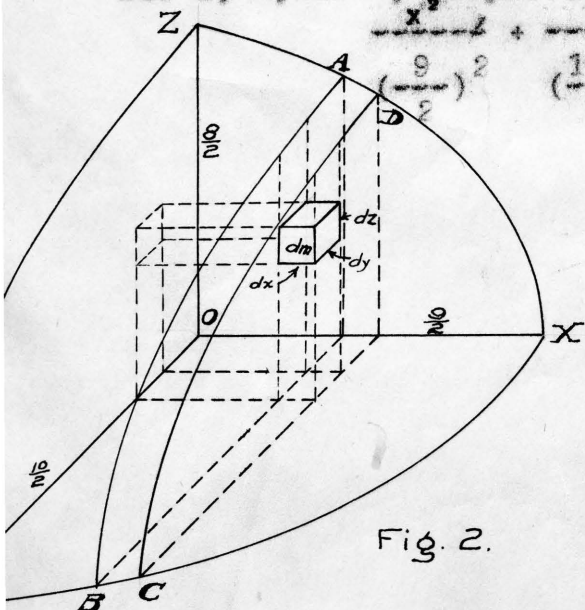


Fig. 2.

which will give a point on the surface of the ellipsoid, that is, from $z = 0$ to $z = 4 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}$, this last value

being obtained by solving Equation (5) for z .

Now if we integrate with respect to y , we collect all the columns in the slice A B C D, y varying from zero to the curve X Y, the equation of which is obtained from (5) by putting $z = 0$ and is $\frac{x^2}{(\frac{9}{2})^2} + \frac{y^2}{5^2} = 1$. Therefore y varies from $y = 0$ to

$$y = 5 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}$$

Finally, integrating with respect to x , we collect all the slices in the solid X Y Z, x varying from $x = 0$ to $x = \frac{9}{2}$. But the solid X Y Z is only $\frac{1}{8}$ of the ellipsoid. Therefore

$$E = 8 \int_0^{\frac{9}{2}} \int_0^{5 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \int_0^{4 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}} \rho x \, dz \, dy \, dx$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho x \, dx \, dy \int_0^{\frac{1}{2} z^2} dz$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho x \, dx \, dy \left[\frac{1}{2} z^2 \right]_0^{\frac{1}{2} z^2}$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5 \sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho x \, dx \, dy \left[1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2} \right]$$

$$\begin{aligned}
E &= 8 \int_0^{1/6} \rho x dx \int_0^{5\sqrt{1-\frac{4x^2}{81}}} \left(8 - \frac{32x^2}{81} - \frac{8y^2}{25}\right) dy \\
&= 8 \int_0^{1/6} \rho x dx \left\{ 8 \int_0^{5\sqrt{1-\frac{4x^2}{81}}} dy - \frac{32x^2}{81} \int_0^{5\sqrt{1-\frac{4x^2}{81}}} dy - \frac{8}{25} \int_0^{5\sqrt{1-\frac{4x^2}{81}}} y^2 dy \right\} \\
&= 8 \int_0^{1/6} \rho x dx \left\{ 8[y]_0^{5\sqrt{1-\frac{4x^2}{81}}} - \frac{32x^2}{81} [y]_0^{5\sqrt{1-\frac{4x^2}{81}}} - \frac{8}{75} [y^3]_0^{5\sqrt{1-\frac{4x^2}{81}}} \right\} \\
&= 8 \int_0^{1/6} \rho x dx \left\{ 40\sqrt{1-\frac{4x^2}{81}} - \frac{160x^2}{81}\sqrt{1-\frac{4x^2}{81}} - \frac{8}{75} (5\sqrt{1-\frac{4x^2}{81}}) \left(25 - \frac{100x^2}{81}\right) \right\} \\
&= 8 \int_0^{1/6} \rho x dx \left\{ (5\sqrt{1-\frac{4x^2}{81}}) \left(8 - \frac{32x^2}{81} - \frac{8}{3} + \frac{32x^2}{243}\right) \right\} \\
&= 8 \int_0^{1/6} \rho x dx \left\{ (5\sqrt{1-\frac{4x^2}{81}}) \left(\frac{16}{3} - \frac{64x^2}{243}\right) \right\} \\
&= 8 \int_0^{1/6} \rho x dx \left\{ \frac{80}{3} \sqrt{1-\frac{4x^2}{81}} - \frac{320x^2}{243} \sqrt{1-\frac{4x^2}{81}} \right\} \\
&= 8\rho \left\{ \int_0^{1/6} \frac{80}{3} x dx \sqrt{1-\frac{4x^2}{81}} - \int_0^{1/6} \frac{320x^3}{243} \sqrt{1-\frac{4x^2}{81}} \right\} \\
&= 8\rho \left\{ 3 \int_0^{1/6} 40 \cdot \frac{2x}{9} \sqrt{1-\frac{4x^2}{81}} dx - 3 \int_0^{1/6} 40 \cdot \frac{8x^3}{729} \sqrt{1-\frac{4x^2}{81}} dx \right\} \\
&= 8\rho \left\{ 120 \int_0^{1/6} \frac{2x}{9} \sqrt{1-\frac{4x^2}{81}} dx - 120 \int_0^{1/6} \frac{8x^3}{729} \sqrt{1-\frac{4x^2}{81}} dx \right\} \\
&= 8\rho \left\{ \frac{9}{2} \cdot 120 \int_0^{1/6} \frac{2x}{9} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} - \frac{9}{2} \cdot 120 \int_0^{1/6} \frac{8x^3}{729} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} \right\} \\
&= 8\rho \cdot 540 \left\{ \int_0^{1/6} \frac{2x}{9} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} - \int_0^{1/6} \frac{8x^3}{729} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} \right\} \quad (6)
\end{aligned}$$

Now if we let $\frac{2x}{9} = u$, then $\frac{4x^2}{81} = u^2$ and $\frac{2dx}{9} = du$

Substituting these values in Eq. (6) and temporarily omitting the limits, we have

$$E = 8\rho \cdot 540 \left\{ \left[\int u \sqrt{1-u^2} du \right] - \left[\int u^3 \sqrt{1-u^2} du \right] \right\} \quad (7)$$

(A)

(B)

Taking the brackets up separately:

$$\textcircled{A} \int u \sqrt{1-u^2} du = -\frac{(1-u^2)^{\frac{3}{2}}}{3}$$

(This is integrated directly by Formula 81, p. 455, Granville's "Calculus").

Replacing u by $\frac{2x}{9}$ and restoring the limits,

$$\begin{aligned} \int_0^{\frac{3}{2}} \frac{2x}{9} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} &= -\frac{(1-\frac{4x^2}{81})^{\frac{3}{2}}}{3} \Bigg|_0^{\frac{3}{2}} \\ &= -\frac{(1-\frac{4 \cdot 81}{81})^{\frac{3}{2}}}{3} + \frac{(1-\frac{4 \cdot 0}{81})^{\frac{3}{2}}}{3} = \frac{1}{3} \quad \textcircled{A} \end{aligned}$$

$$\textcircled{B} \int u^3 \sqrt{1-u^2} du = \int u^2 \sqrt{1-u^2} u du.$$

Let $1-u^2 = w^2$, then $\sqrt{1-u^2} = w$, $u^2 = 1-w^2$, and $u du = -w dw$

Substituting these values in \textcircled{B} ,

$$\begin{aligned} \int u^3 \sqrt{1-u^2} du &= \int (1-w^2) w (-w dw) \\ &= \int (-w^2 dw + w^4 dw) \\ &= \int w^4 dw - \int w^2 dw \\ &= \frac{w^5}{5} - \frac{w^3}{3} = w \left(\frac{w^4}{5} - \frac{w^2}{3} \right) \end{aligned}$$

Replacing w by $\sqrt{1-u^2}$,

$$\begin{aligned} \int u^3 \sqrt{1-u^2} du &= \sqrt{1-u^2} \left(\frac{1-2u^2+u^4}{5} - \frac{1-u^2}{3} \right) \\ &= \sqrt{1-u^2} \left(\frac{3-6u^2+3u^4-5+5u^2}{15} \right) \\ &= \frac{\sqrt{1-u^2}}{15} (3u^4 - u^2 - 2). \end{aligned}$$

Now replacing u by $\frac{2x}{9}$ and restoring the limits, we have

$$-\int_0^{\frac{3}{2}} \frac{8x^3}{729} \sqrt{1-\frac{4x^2}{81}} \cdot \frac{2dx}{9} = -\left[\frac{\sqrt{1-\frac{4x^2}{81}}}{15} \left(3 \cdot \frac{16x^4}{6561} - \frac{4x^2}{81} - 2 \right) \right]_0^{\frac{3}{2}}$$

$$= - \left[\left\{ \frac{\sqrt{1 - \frac{4 \cdot 81}{15}}}{15} \left(\frac{3 \cdot 16 \cdot (\frac{9}{2})^4}{6561} - \frac{4}{81} (\frac{9}{2})^2 - 2 \right) \right\} - \left\{ \frac{\sqrt{1 - \frac{4 \cdot 81}{15}}}{15} \left(\frac{3 \cdot 16 \cdot 0}{6561} - \frac{4}{81} \cdot 0 - 2 \right) \right\} \right]$$

$$= - \left[\{0\} - \left\{ \frac{1}{15} (-2) \right\} \right] = - \frac{2}{15} \quad \textcircled{B}$$

Replacing, in Eq. ⑦, the brackets ① and ② by their respective values, we have

$$E = 8\rho \cdot 540 \left[\frac{1}{3} - \frac{2}{15} \right] = 8\rho \cdot 540 \cdot \frac{108}{5} = \underline{\underline{864\rho}}$$

$$D = \Sigma mzy = \int \rho yz d\tau = \iiint \rho yz dx dy dz$$

Since we are dealing with the same ellipsoid, we have the same limits as in the calculation of E .

$$\therefore D = 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \int_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}} \rho dx dy dz$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho dx dy \int_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}} z dz$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho dx dy \left[\frac{1}{2} z^2 \right]_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}}$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho dx dy \left(8 - \frac{32x^2}{81} - \frac{8y^2}{25} \right)$$

$$= 8 \int_0^{\frac{9}{2}} \rho dx \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \left(8y dy - \frac{32x^2}{81} y dy - \frac{8}{25} y^3 dy \right)$$

$$= 8 \int_0^{\frac{9}{2}} \rho dx \left\{ \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} 8y dy - \frac{32x^2}{81} \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} y dy - \frac{8}{25} \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} y^3 dy \right\}$$

$$= 8 \int_0^{\frac{9}{2}} \rho dx \left\{ \left[4y^2 \right]_0^{5\sqrt{1 - \frac{4x^2}{81}}} - \frac{32x^2}{81} \left[\frac{1}{2} y^2 \right]_0^{5\sqrt{1 - \frac{4x^2}{81}}} - \frac{8}{25} \left[\frac{1}{4} y^4 \right]_0^{5\sqrt{1 - \frac{4x^2}{81}}} \right\}$$

$$= 8 \int_0^{\frac{9}{2}} \rho dx \left\{ 100 \left(1 - \frac{4x^2}{81} \right) - \frac{16x^2}{81} \left(25 - \frac{100x^2}{81} \right) - 50 \left(1 - \frac{8x^2}{81} + \frac{16x^4}{6561} \right) \right\}$$

$$= 8 \int_0^{\frac{9}{2}} \rho dx \left\{ 100 - \frac{400x^2}{81} - \frac{400x^2}{81} + \frac{1600x^4}{6561} - 50 + \frac{400x^2}{81} - \frac{800x^4}{6561} \right\}$$

$$\begin{aligned}
 D &= 8 \int_0^{\frac{9}{2}} \rho dx \left\{ 50 - \frac{400x^2}{81} + \frac{800x^4}{6561} \right\} \\
 &= 8\rho \left\{ 50 \int_0^{\frac{9}{2}} dx - \frac{400}{81} \int_0^{\frac{9}{2}} x^2 dx + \frac{800}{6561} \int_0^{\frac{9}{2}} x^4 dx \right\} \\
 &= 8\rho \left\{ 50 \left[x \right]_0^{\frac{9}{2}} - \frac{400}{81} \left[\frac{x^3}{3} \right]_0^{\frac{9}{2}} + \frac{800}{6561} \left[\frac{x^5}{5} \right]_0^{\frac{9}{2}} \right\} \\
 &= 8\rho \left\{ 50 \cdot \frac{25}{2} - \frac{400}{81} \cdot \frac{729}{8} + \frac{800}{6561} \cdot \frac{9 \cdot 6561}{160} \right\} \\
 &= 8\rho \{ 225 - 150 + 45 \} = 8\rho \{ 120 \} = \underline{\underline{960\rho}}
 \end{aligned}$$

$$F = \Sigma mxy = \int \rho xy d\tau = \iiint \rho xy dx dy dz$$

Again, in calculating F , we use the same limits as in calculating D and E .

$$\therefore F = 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \int_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}} \rho x dx dy dz$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2}}} \rho x dx dy \int_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}} dz$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} \rho x dx dy [z]_0^{4\sqrt{1 - \frac{x^2}{(\frac{9}{2})^2} - \frac{y^2}{5^2}}}$$

$$= 8 \int_0^{\frac{9}{2}} \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} \rho x dx dy (4\sqrt{1 - \frac{4x^2}{81} - \frac{y^2}{25}})$$

$$= 8 \int_0^{\frac{9}{2}} \rho x dx \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} 4y dy \sqrt{1 - \frac{4x^2}{81} - \frac{(y/5)^2}{1}}$$

$$= 8 \int_0^{\frac{9}{2}} \rho x dx \cdot 100 \int_0^{5\sqrt{1 - \frac{4x^2}{81}}} \frac{y}{5} \sqrt{1 - \frac{4x^2}{81} - \frac{(y/5)^2}{1}} \frac{dy}{5}$$

Integrating directly by Formula 81, p. 455, Granville's "Calculus".

$$F = 8 \int_0^{\frac{9}{2}} \rho x dx \cdot 100 \left\{ - \frac{[(1 - \frac{4x^2}{81}) - \frac{(y/5)^2]}{3} \right\}^{3/2} \Big|_0^{5\sqrt{1 - \frac{4x^2}{81}}}$$

$$= 8 \int_0^{\frac{9}{2}} \rho x dx \cdot \frac{100}{3} \left\{ - \frac{[(1 - \frac{4x^2}{81}) - \frac{25}{25}(1 - \frac{4x^2}{81})]}{1} + \frac{[(1 - \frac{4x^2}{81}) - \frac{0}{25}]^{3/2}}{1} \right\}$$

$$= 8 \int_0^{\frac{9}{2}} \rho x dx \cdot \frac{100}{3} \left(1 - \frac{4x^2}{81} \right)^{3/2} \Big|_0^{\frac{9}{2}}$$

$$F = \frac{800}{3} \rho \int_0^{\frac{9}{4}} x dx \left(1 - \frac{4x^2}{81}\right)^{\frac{3}{2}}$$

$$= \frac{81 \cdot 800}{4 \cdot 3} \rho \int_0^{\frac{9}{4}} \frac{2x}{9} \left(1 - \frac{4x^2}{81}\right)^{\frac{3}{2}} \frac{2dx}{9}$$

Again integrating directly by the same formula as before,

$$F = 5400 \rho \left\{ -\frac{\left(1 - \frac{4x^2}{81}\right)^{\frac{5}{2}}}{5} \right\}_0^{\frac{9}{4}}$$

$$= 5400 \rho \left\{ \left[-\frac{\left(1 - \frac{4}{81} \cdot \frac{81}{4}\right)^{\frac{5}{2}}}{5} \right] - \left[-\frac{\left(1 - \frac{4}{81} \cdot 0\right)^{\frac{5}{2}}}{5} \right] \right\}$$

$$= 5400 \rho \left\{ [0] + \left[\frac{1}{5}\right] \right\}$$

$$= \frac{1080}{5} \rho = \underline{\underline{1080 \rho}}$$

$$A = \rho V \frac{y^2 + z^2}{5}$$

$$= \rho \cdot \frac{4}{3} \pi \cdot \frac{3}{4} \cdot 5 \cdot 4 \cdot \frac{25+16}{5} \quad \left(V = \frac{4}{3} \pi xyz, x, y, z \text{ being the semi-axes of the ellipsoid.} \right)$$

$$= \frac{24}{5} \pi \rho \cdot \frac{25+16}{5}$$

$$= 24 \pi \rho \cdot 41 = \underline{\underline{984 \pi \rho}}$$

$$B = \rho V \frac{x^2 + z^2}{5}$$

$$= \rho \cdot \frac{4}{3} \pi \cdot \frac{3}{4} \cdot 5 \cdot 4 \cdot \frac{81+16}{5}$$

$$= 120 \pi \rho \cdot \frac{145}{5}$$

$$= \frac{6}{20} \pi \rho \cdot \frac{145}{20} = \underline{\underline{870 \pi \rho}}$$

$$C = \rho V \frac{x^2 + y^2}{5}$$

$$= \rho \cdot \frac{4}{3} \pi \cdot \frac{3}{4} \cdot 5 \cdot 4 \cdot \frac{81+25}{5}$$

$$= 120 \pi \rho \cdot \frac{181}{5}$$

$$= \frac{6}{20} \pi \rho \cdot \frac{181}{20}$$

$$= \underline{\underline{1086 \pi \rho}}$$

we shall now find H_2 , the resultant of H_x and H_z , and develop an expression for the value of the centrifugal couple, S_c .

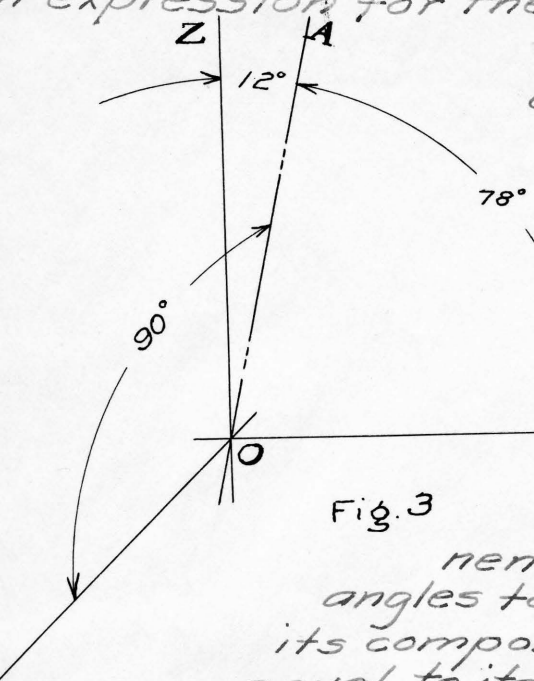


Fig. 3

As before noted, the axis of rotation, OA, lies in the plane of the X and Z axes, passing through O, the center of mass of the ellipsoid, and making with the X and Z axes, angles of 78° and 12° , respectively. Since OY is perpendicular to the plane XZ, the angle $AOY = 90^\circ$.

Now rotary motion, like linear motion, may be resolved into components along three rectilinear axes at right angles to one another; and like linear motion also, its component along each one of these axes is equal to its actual value, multiplied by the cosine of the angle its axis makes with each one of these axes. (See "Analysis of Rotary Motion as applied to the Gyroscope," by Gen. J. G. Barnard, U.S.A., p. 11).

If, therefore, the angular velocity of our ellipsoid about the axis OA be ω , and p, q, r , its components along the axes of X, Y and Z, respectively, then

$$p = \omega \cos 78^\circ$$

$$q = \omega \cos 90^\circ = 0$$

$$r = \omega \cos 12^\circ$$

Referring again to Webster, p. 258, we find that

$$H_x = Ap = 984\pi p\omega \cos 78^\circ$$

$$H_y = Bq = 0$$

$$H_z = Cr = 1086\pi p\omega \cos 12^\circ$$

Since H_2 is the resultant of H_x and H_z , we have

$$H_2 = \sqrt{H_x^2 + H_z^2}$$

$$= \sqrt{(984\pi p\omega \cos 78^\circ)^2 + (1086\pi p\omega \cos 12^\circ)^2}$$

$$= \pi p\omega \sqrt{(984 \cos 78^\circ)^2 + (1086 \cos 12^\circ)^2}$$

$$= 1081.79\pi p\omega$$

$$S_c dt = H_2 \omega dt = 1081\pi p\omega^2 dt \quad (8)$$

In the absence of observations necessary to give the value of w and to enable the differential equation of time to be formed, we cannot further develop the work along this particular line just at present, and must be content for the time being with leaving the value of Sc in the above form.

The tendency of the body to right itself as expressed by Eq. ⑧ will, of course, give the body a "wobble" during its rotation. If we can conceive of some source of power constantly increasing its speed of rotation, it would speedily assume its position of stability. However, we are here assuming that it is "motion under no forces", consequently, we do not have the body quickly swinging into its position of stability. Its motion takes the form of an oscillation, the axis of figure (Z axis) swinging up towards the axis of rotation, and describing smaller and smaller curves about it, then, due to the change in angular momentum developed by the centrifugal couple, swinging gradually away from it again. The curve described by this axis of figure about the axis of rotation (named by Poinset, the French mathematician who discovered it, the "herpolhode") is tangent to two concentric limiting circles. It has no points of inflection, but moves gradually inward from tangency to the outer limiting circle, in narrowing curves, to tangency to the inner limiting circle; then reversing the process and returning again to the point of tangency to the outer limiting circle. The cycle is then repeated, and so on indefinitely. The general shape of this curve is something like the accompanying sketch, modeled after the sketch in Webster, p. 264. It is understood, however, that it has been assumed and not

calculated, because, as hereinbefore explained, data necessary in order to plot this curve accurately for this particular problem have not been available.

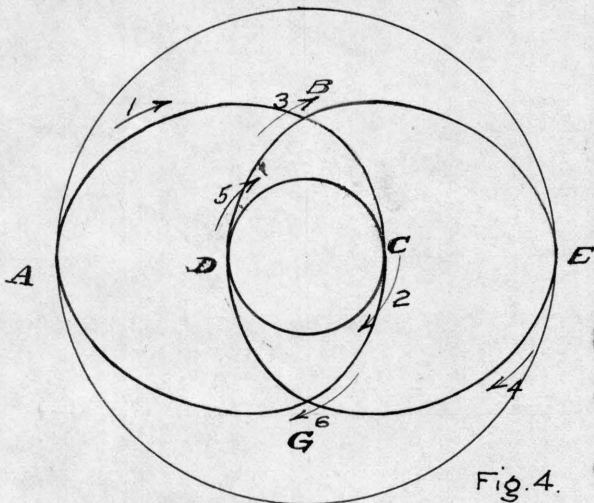


Fig. 4.

We shall now proceed to the attempt to explain by means of this curve, the variable period encountered by Dr. Douglas in his researches.

Suppose that we have an ellipsoid of the proportions of the ellipsoid under discussion, that is, with axes bearing the ratio of 10,9,8. Suppose that we thrust a long pin through this ellipsoid, the pin passing through the center of mass, lying in the plane of the axes whose lengths are 8 and 9 and making an angle of 12° with the axis whose length is 8. If we now give the ellipsoid a spin about the axis represented by the pin, and hold the pin rigid, the ellipsoid will rotate uniformly about the pin because it is constrained to do so. It will, however, possess a tendency to right itself, inasmuch as rotation about this axis will immediately develop the centrifugal couple expressed by Eq. (8). The centrifugal couple, though, is prevented from acting because of the rigidity of the pin. If now, the end of the axis whose length is 8 be made phosphorescent, say - or indeed, any point on the surface of the ellipsoid - be made phosphorescent, and the rotation viewed in a dark room, the phosphorescent spot will continue to describe uniform circles about the axis of rotation - that is, the pin - so long as the

rotation lasts and the rigidity of the pin continues.

If now, we could conceive of a dark void wherein gravitation does not act and we could suddenly release our ellipsoid from the pin while it is still spinning about the axis represented by the pin, and precipitate it into this void, we should see an immediate change in the motion of the phosphorescent spot. Instead of describing uniform circles about the axis of rotation, it would swing in toward the axis of rotation in narrowing curves till it reached a certain inner limiting position. Then swinging outward, reversing the process, until it reached an outer limiting position, then repeating the oscillatory cycle again, and so on. In other words, instead of describing circles, the phosphorescent spot would describe a curve something like the one represented by Fig. 4. That it actually will do this has been proven experimentally by the English physicists, Clerk-Maxwell and Searle, and by the American, Webster (See Webster, pp. 268 et seq.)

Let us now substitute for our small ellipsoid a heavenly body of similar ellipsoidal form, and rotating in a similar way. It would certainly seem that it would behave in a similar manner. If its rotation be marked by some point on its surface, its rotation might be observed when it is making one of its larger loops. It would then have a certain period apparently. If it be observed again when it is making a loop not the same as the first one and not similar to the first one, it would appear to have a different period. And a third time it might be observed in a still different phase; and the

period there found might differ from the other two. Of course, there is no way of knowing how many possible variations of period there might be until data sufficient to calculate the herpolhode are obtained; for upon the number of loops in the herpolhode, the diameters of its limiting tangent circles and the angular velocity of the body must the number of variable periods possible in any given case and their rate of variation depend. It is believed that enough has been said to make clear the generalization.

It is the very earnest hope of the writer, and his full intention, to develop these ideas and carry them to their logical conclusion. In order to do this, he has set himself four definite problems, as follows:

1. To determine the positions, diameters, and centers of the limiting circles.
2. To plot the herpolhode accurately for this particular ellipsoid.
3. To determine the time of one cycle.
4. To determine the number of possible variations of period and their rate of variation.

The solution of these problems will be earnestly sought after as soon as the necessary astronomical data are forthcoming, and as soon as the leisure essential to the consideration of matters like these is scarcely so hard-won a luxury as it is at present.

In conclusion the writer wishes to express his grateful appreciation to Drs. A.E.Douglass and H.B.Leonard for much kind assistance, and particularly does he wish to acknowledge his in-

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