A Solution of Euler's Geometric Equations for the Motion of a Rigid Body with a Fixed Point Under No Forces

by

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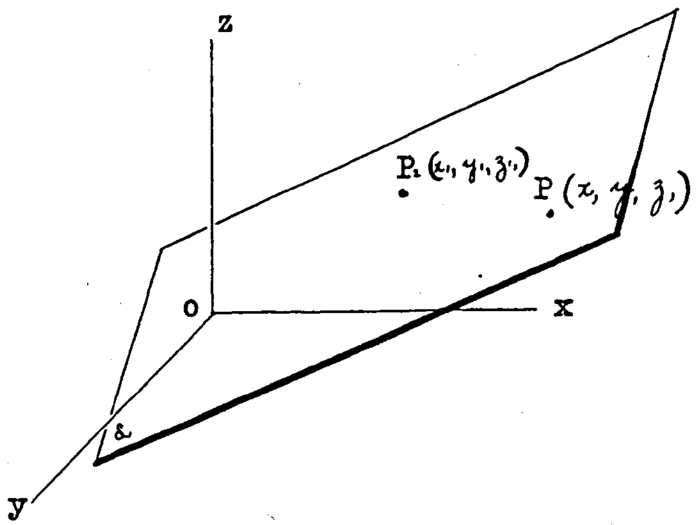
The instantaneous state of motion of a rigid body, one of whose points is fixed, if not a state of rest, is a rotation about a line through the point. To prove this it is necessary to show that if one point of the body has zero velocity there exists a line l through O all of whose points have zero velocity.

Consider a body of which one point O has zero velocity and let P_1 and P_2 be any two points of the body not in line with O. The velocities of P_1 and P_2 must be normal to OP_1 and OP_2 respectively; because O is fixed and OP_1 and OP_2 are rigid lines. If the velocity of either of these points were zero, the line joining this point to O would be the required axis of rotation. We assume, therefore, that these velocities are both different from zero. We can also assume that these velocities are not parallel; for if they happen to be so we could replace one of the two points by a point whose velocity is not parallel to those of P_1 and P_2; otherwise the motion would be a translation which is impossible for a body with a fixed point.

Let the plane through OP normal to the velocity of P_1 be called α; the plane through OP_2 normal to the velocity of P_2 be called β. These two planes must intersect in a line l which of course passes through O.

Let us now prove the following:
If a point in a plane, which passes through the origin has a velocity which is normal to the plane, then every point in the plane has a velocity normal to the plane.



Let the velocity of $P_1 (x_1, y_1, z_1)$ be normal to the plane α , which passes through O. Let $P (x, y, z)$ be any other point in the plane. We have the following relations;

(1) $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 =$ constant

(2) $x^2 + y^2 + z^2 =$ constant

(3) $x_1^2 + y_1^2 + z_1^2 -$ constant

Setting (2) and (3) in (1) we have

(4) $xx_1 + yy_1 + zz_1 =$ constant

Now rotate the axes until the xy plane coincides with the plane α . The quantities (1), (2), (3), (4), are invariant to this change of axes.

Differentiate (4) with respect to the time and we have

(5) $$x_1\frac{dx}{dt} + x\frac{dx_1}{dt} + y_1\frac{dy}{dt} + y\frac{dy_1}{dt} + z_1\frac{dz}{dt} + z\frac{dz_1}{dt} = 0$$

The velocity of P_1 being now normal to the xy plane is $\frac{dz_1}{dt}$, and we have that $z = 0$, $z_1 = 0$, $\frac{dx_1}{dt} = 0$, $\frac{dy_1}{dt} = 0$

The expression (5) now becomes

$$x_1\frac{dx}{dt} + y_1\frac{dy}{dt} = 0$$

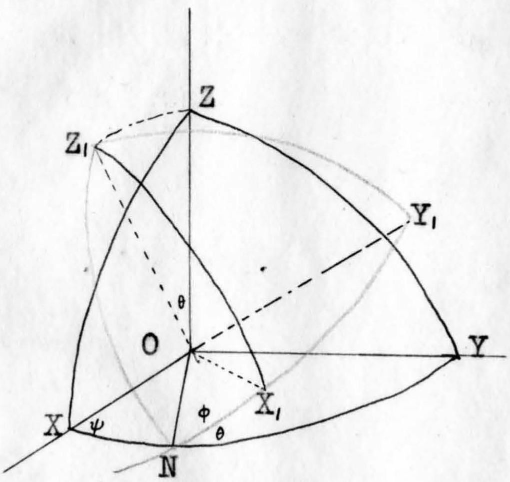
This is true for all values of x_1 and y_1 so that

$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 0$$

We have thus proved that the velocity of P must be $\frac{dz}{dt}$ or normal to the xy plane.

Now consider any point P' on the line ℓ . It being in the plane α must have a velocity normal to α because the velocity of P_1 is normal to α . But the point P' lies also in the plane β so its velocity must be normal to β . It must also have a velocity normal to the line OP' because O is fixed and OP' is a rigid line. This means that the velocity of P' is zero, and the line ℓ is the axis of rotation.

The position of axes fixed in the body and moving with it, with respect to axes fixed in space can be assigned by the three angles θ , ϕ , ψ , as given in the figure.



X, Y, Z, the space axes
X_1, Y_1, Z_1, the body axes.

These angles fully determine the relative position of one trihedral with respect to the other. The angular velocity can be resolved along the axes ON, OZ, and OZ_1 into the components $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, $\frac{d\phi}{dt}$. Hence, the sum of the projections of these components on OX_1 must be equal to p , the projection of Ω on OX_1. Similiarly the sum of the projections on OY_1 equals q, and the sum of the projections on OZ_1 equals r.

As the figure shows the direction cosines of ON, OZ, and OZ with respect to the moving trihedrial are,

	X_1	Y_1	Z_1	
N	$\cos\phi$	$-\sin\phi$	o	
Z_1	o	o	1	(6)
Z	$\sin\theta\sin\phi$	$\sin\theta\cos\phi$	$\cos\theta$	

So that

$$p = \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi$$

(7) $$q = -\frac{d\phi}{dt}\sin\phi + \frac{d\psi}{dt}\sin\theta\cos\phi$$

$$r = \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta$$

Solving for $\frac{d\theta}{dt}$ $\frac{d\phi}{dt}$ $\frac{d\psi}{dt}$ we have

$$\frac{d\theta}{dt} = p\cos\phi - q\sin\phi$$

(8) $$\frac{d\phi}{dt} = -p\sin\phi\cot\theta - q\cos\phi\cot\theta + r$$

$$\frac{d\psi}{dt} = p\sin\phi\csc\theta + q\cos\phi\csc\theta$$

When the axes in the body are the principal axes, the relation between the angular momentum H and the angular velocity Ω are given by,

$$(9) \qquad \begin{aligned} H_1 &= A\,p \\ H_2 &= B\,q \\ H_3 &= C\,r \end{aligned}$$

where H_1, H_2, H_3, are the projections of the angular momentum H on the body axes and A, B, C, are the moments of inertia with respect to these axes. For motion under no forces H is constant in magnitude and direction and we can use this fact to determine the values of θ, ϕ, ψ.

Let the invariable direction of H be taken as the axis OZ. The direction cosines of H taken from (6) are

$$(10) \qquad \begin{aligned} H_1 &= H \sin\theta \sin\phi \\ H_2 &= H \sin\theta \cos\phi \\ H_3 &= H \cos\theta \end{aligned}$$

Using (9) we have

$$(11) \qquad \begin{aligned} A\,p &= H \sin\theta \sin\phi \\ B\,q &= H \sin\theta \cos\phi \\ C\,r &= H \cos\theta \end{aligned}$$

For motion under no forces we have also the relations

$$(12) \qquad A^2 p^2 + B^2 q^2 + C^2 r^2 = H^2$$

This is the integral of angular momentum.

$$(13) \qquad A p^2 + B q^2 + C r^2 = T$$

This is the integral of kinetic energy.

Take the last equation of (8)

$$\sin\theta \frac{d\psi}{dt} = p \sin\phi + q \cos\phi$$

multiply both sides by $H^2 \sin\theta$

$$H^2 \sin^2\theta \frac{d\psi}{dt} = (H p \sin\phi \sin\theta + H q \cos\phi \sin\theta) H$$

using (11), we have

$$\frac{d\psi}{dt} = \frac{(A p^2 + B q^2) H}{H^2 - H^2 \cos\theta}$$

but $H \cos\theta = C r$, so that

$$(14) \quad \begin{cases} d\psi = \dfrac{(A p^2 + B q^2) H}{A^2 p^2 + B^2 q^2} dt & \text{and from (11) we have also} \\ \tan\phi = \dfrac{A p}{B q} \\ \cos\theta = \dfrac{C r}{H} \end{cases}$$

The value of $p, q,$ and r can be found as an elliptic function solution of Euler's dynamical equations. The values given below are taken from Greenhill's "The Elliptic Functions."

If $A > B > C$

$p = P \operatorname{cn}(nt) \qquad q = -Q \operatorname{sn}(nt) \qquad r = R \operatorname{dn}(nt)$

$$P^2 = \frac{H^2 - C\,T}{A(A - C)} \qquad Q^2 = \frac{H^2 - C\,T}{B(B - C)} \qquad R^2 = \frac{A\,T - H^2}{C(A - C)}$$

$$n^2 = \frac{(A\,T - H^2)(B - C)}{A\,B\,C} \qquad k^2 = \frac{H^2 - C\,T}{A\,T - H^2}\,\frac{A - B}{B - C}$$

Using these values of p , q , and r , then the values of θ, ϕ, ψ are given by equations (14).

Putting the values of p , q , and r from (11) into the equations (8) we have,

$$\text{(15)} \qquad \begin{aligned} \frac{d\theta}{dt} &= H \sin\theta \left(\frac{\cos\phi \sin\phi}{A} - \frac{\cos\phi \sin\phi}{B} \right) \\ \frac{d\phi}{dt} &= H \cos\theta \left(-\frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B} + \frac{1}{C} \right) \\ \frac{d\psi}{dt} &= H \left(\frac{\sin^2\phi}{A} \quad \frac{\cos^2\phi}{B} \right) \end{aligned}$$

From these we can get

$$\frac{d\theta}{H \sin\theta \left(\frac{\cos\phi \sin\phi}{A} - \frac{\cos\phi \sin\phi}{B} \right)} = \frac{d\phi}{H \cos\theta \left(-\frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B} + \frac{1}{C} \right)}$$

or

$$\cot\theta \, d\theta = \frac{\frac{\cos\phi \sin\phi}{A} - \frac{\cos\phi \sin\phi}{B}}{\frac{1}{C} - \frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B}} \, d\phi$$

which integrates into

$$\log \sin\theta = -\tfrac{1}{2} \log \left(\frac{1}{C} - \frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B}\right) + \log c_1$$

or

$$(16) \qquad \sin\theta \sqrt{\frac{1}{C} - \frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B}} = c_1$$

The constant, c_1, can be expressed in terms of T and H. Square (16) and set in it the values from (11) and we have

$$\frac{1}{C} - \frac{C^2 r^2}{H^2 C} - \frac{A^2 p^2}{H^2 A} - \frac{B^2 q^2}{H^2 B} = c_1^2$$

or

$$c_1^2 = \frac{1}{C} - \frac{T}{H^2}$$

Place the above value of c_1 in (16) and then solve it for $\cos\theta$. This is

$$(17) \qquad \cos\theta = \frac{\sqrt{T - H^2\left(\frac{\sin^2\phi}{A} + \frac{\cos^2\phi}{B}\right)}}{H\sqrt{\frac{1}{C} - \frac{\sin^2\phi}{A} - \frac{\cos^2\phi}{B}}}$$

Place this value of cos θ in the $\frac{d\phi}{dt}$ equation of (15) and we have an elliptic integral in ϕ which is

(18)
$$\int \frac{d\phi}{\sqrt{T-H^2\left(\frac{\sin^2\phi}{A}+\frac{\cos^2\phi}{B}\right)}\sqrt{\frac{1}{C}-\frac{\sin^2\phi}{A}-\frac{\cos^2\phi}{B}}} = t+\gamma$$

Equation (17) gives us

$$(19) \quad H^2\cos^2\theta\left(\frac{1}{C}-\frac{\sin^2\phi}{A}-\frac{\cos^2\phi}{B}\right)=T-H^2\left(\frac{\sin^2\phi}{A}+\frac{\cos^2\phi}{B}\right)$$

The last equation of (15) is

$$\frac{d\psi}{dt} = H\left(\frac{\sin^2\phi}{A}+\frac{\cos^2\phi}{B}\right)$$

Solve (19) for $\left(\frac{\sin^2\phi}{A}+\frac{\cos^2\phi}{B}\right)$ and we have

$$\frac{\sin^2\phi}{A}+\frac{\cos^2\phi}{B} = \frac{T-\frac{H^2}{C}\cos^2\theta}{H^2\sin^2\theta}$$

Place this value in the $\frac{d\psi}{dt}$ equation above and it becomes

$$(20) \qquad d\psi = \frac{T-\frac{H^2}{C}\cos^2\theta}{H\sin^2\theta}\,dt$$

By using equations (18), (19), and (20), ϕ, θ, ψ, can be found as functions of the time, and then p, q, r, can be obtained from (11)

In A = B, that is, if the momental ellipsoid is one of revolution, the results assume a simple form. Equation (18) becomes

$$\frac{\phi}{\sqrt{\left(T - \frac{H^2}{A}\right) \frac{A - C}{A C}}} = t + \gamma$$

or

$$\phi = \frac{1}{A} \sqrt{(A T - H^2) \frac{(A - C)}{C}} \, (t + \gamma)$$

Equation (19) becomes

$$\cos^2 \theta = \frac{(A T - H^2) C}{H^2 (A - C)}$$

Equation (20) becomes

$$\psi = \frac{H}{A} t + \tau$$

It can be seen from these three equations that $\theta, \frac{d\phi}{dt}, \frac{d\psi}{dt}$ are constants. The motion of the body consists, therefore, in the rotation of constant angular velocity $\frac{d\phi}{dt} = \frac{1}{A} \sqrt{(AT - H^2) \frac{(A - C)}{C}}$ about OZ_1, together with the turning of this axis OZ_1, with constant angular velocity $\frac{d\psi}{dt} = \frac{H}{A}$ about the axis OZ, the angle θ between these axes remaining constant. Such a motion is called a regular precession. The motion of the earth can be considered as that of a rigid body, symmetrical about an axis, acted on by forces all of which pass through the center of gravity. Although the earth is not perfectly rigid or perfectly symmetrical, the assumption is a close approximation. Suppose $B = C$. Euler's dynamical equations have a simply solution in this case. The results for p, q, and r, given below are taken from Jean's Theoretical Mechanics, page 309.

$$p = \Omega$$

$$q = E\cos(kt + \epsilon)$$

$$r = -E\sin(kt + \epsilon)$$

where $k = \frac{B-A}{B}$ and E, ϵ are constants of integration. Thus the components of the angular velocity at the instant t are:

$$\Omega, \qquad E\cos(kt + \epsilon), \quad E\sin(kt + \epsilon),$$

and the axis of rotation describes a cone with the period $\frac{2\pi}{k}$ or $\frac{2\pi}{\Omega}\frac{B}{B-A}$. If B is nearly equal to A, the period may be very great and the motion consequently very slow. This happens in the case of the earth. The motion of the axis of rotation gives rise to the phenomenon known as the variation of latitude, of which the period is about 428 days. As a period $\frac{2\pi}{\Omega}$ represents roughly one day, we conclude that for the earth $\frac{B-A}{B}$ is of the order $\frac{1}{428}$. The true value of this quantity is .00328, the discrepancy resulting from the imperfect rigidity of the earth.

If an attempt is made to solve Euler's Geometric Equations (8) when they are in that form, many and perhaps insurmountable difficulties are met with. ~~An interesting mathematical, but useless dynamical solution of them~~ A solution of them which is mathematically interesting but dynamically useless can be obtained considering p, q, and r as constant. This is as follows:

Let $\quad p \sin\phi + q \cos\phi = F$

then $\quad (p \cos\phi - q \sin\phi)\, d\phi = dF$

then from the first equation of (8)

$$\frac{d\theta}{dt} = \frac{dF}{d\phi} \quad \text{or} \quad \frac{d\phi}{dt} = \frac{dF}{d\theta}$$

Place this value of $\frac{d\phi}{dt}$ in the second equation of (8) and we have

$$\frac{dF}{d\theta} = -\cot\theta\ F + r$$

This is a linear differential equation of the first order of which an integrating factor is

$$e^{\int \cot\theta\, d\theta} = e^{\int \log \sin\theta} = \sin\theta$$

so that the solution is

$$F \sin\theta = \int r \sin\theta\, d\theta = -r\cos\theta + c_1$$

(21) $\quad F = p \sin\phi + q \cos\phi = \frac{-r\cos\theta + c_1}{\sin}$

From the first equation of (8), we have

$$\left(\frac{d\theta}{dt}\right)^2 = p^2\cos^2\phi - 2\,pq\,\sin\phi\cos\phi + q^2\sin^2\phi$$

also

$$F^2 = p^2\sin^2\phi + 2\,pq\,\sin\phi\cos\phi + q^2\cos^2\phi$$

so that

$$\left(\frac{d\theta}{dt}\right)^2 + F^2 = p^2 + q^2 = m^2 \quad \text{let us say.}$$

So that

$$\frac{d\theta}{dt} = \sqrt{m^2 - F^2} = \sqrt{m^2 - \left(\frac{-r\cos\theta + c_1}{\sin\theta}\right)^2}$$

so that finally

$$\frac{\sin\theta\, d\theta}{\sqrt{m^2 - c_1^2 - \cos^2\theta\,(m^2+r^2) + 2\,rc_1\cos\theta}} = dt$$

if we let

$$-\cos\theta = x \qquad 2\,rc_1 = b$$

$$m^2 - c_1^2 = a \qquad -(m^2+r^2) = c$$

the left side becomes

$$\frac{dx}{\sqrt{a + bx + cx^2}}$$

when $c < o$ this integrates into

$$\frac{1}{\sqrt{-c}}\,\sin^{-1}\left(\frac{-2\,cx - b}{\sqrt{b^2 - 4\,ac}}\right)$$

replacing for a, b, and c their values, we have

$$\frac{1}{\sqrt{m^2+r^2}}\,\sin^{-1}\left(\frac{-2(m^2+r^2)\cos\theta - 2\,rc_1}{\sqrt{4\,r^2c_1^2 + 4(m^2+r^2)(m^2-c_1^2)}}\right) = t + \alpha$$

or

$$\cos\theta = \frac{-r\,c_1}{m^2+r^2} - \frac{m\sqrt{m^2+r^2-c_1^2}}{m^2+r^2}\,\sin^{1}\left(\sqrt{m^2+r^2}\;t+\delta\right) \quad (22)$$

from (21), we get

$$(23) \qquad p\,\sin\phi + q\,\cos\phi = \frac{-r\,\cos + c_1}{\sin\theta}$$

and the last equation of (8) gives us

$$(24) \qquad \frac{d\psi}{dt} = \frac{F}{\sin\theta} = \frac{-r\,\cos\theta + c_1}{\sin^2\theta}$$

Equations (22), (23), and (24) would give the value of θ, ϕ, Ψ, considering p, q, and r, as constants. These results cannot be applied to an actual physical problem, because in no motion are p, q, and r, all constants.

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