THEORY OF A ZEEMAN RING LASER

by

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STATEMENT BY AUTHOR

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ABSTRACT

A theory of a ring laser subject to a uniform, axial dc magnetic field is given in extension of the non-Zeeman treatment by Lamb. The active medium consists of thermally moving atoms which have two electronic levels with arbitrary angular momenta. The electric field is treated classically for two circular polarizations of opposite sense in a cavity with any degree of cavity anisotropy. Losses due to back-scattering are also included. In addition, the results of a generalized treatment are given which includes arbitrarily oriented magnetic field, a general state of electric field polarization, varying isotopic abundance, and hyperfine structure. The self consistency requirement is used to obtain amplitude- and frequency-determining equations for multimode operation as functions of laser parameters. A general calculational technique, the "perturbation tree", is introduced in the calculation of the third-order component of the population matrix greatly simplifying the algebra involved by allowing it to be abstracted in tabular form. Two special cases are considered; four-wave, bidirectional operation and two-wave, unidirectional operation. Graphs showing coupling between various traveling waves in these two cases are given.
CHAPTER I

INTRODUCTION

In a recent paper by Sargent, Lamb and Fork (1967), the multimode theory of an optical maser, (henceforth referred to as the scalar theory), developed by Lamb (1964), was extended to include general states of electric field polarization and cavity frequency and loss anisotropy, a dc magnetic field at arbitrary angles to the maser axis, and an active medium consisting of two-level atoms (arbitrary number of isotopes) having two electronic states each of which could have arbitrary angular momentum and hyperfine structure. That theory was given for a standing wave, two mirror laser configuration. In the present treatment we consider a ring laser configuration such as is shown in Figure 1. The electric field is then represented by a set of traveling wave "modes". For readability, we consider a medium consisting of a single isotope where the atoms have no hyperfine structure, an axial magnetic field and an electric field with two circular polarizations. The results of a general treatment including effects neglected here are given in Appendix D. As in the previous work, pressure effects and spontaneous emission from the upper to lower low levels are neglected. The general state of loss anisotropy given by Sargent et al. (1967) is extended to include loss due to backscattering of one traveling wave mode into the traveling wave of the opposite direction. Frequency- and amplitude-determining equations are given for both
bidirectional and unidirectional Zeeman ring lasers. As a check on our calculations, the results are reduced to the standing wave, two mirror Zeeman laser and the scalar ring laser in Appendix E.

The calculation is similar to that in the Zeeman theory (Sargent et al. 1967) with the exception of the algebraic method. We introduce the "perturbation tree", a graphical representation of the third-order components of the population matrix. This technique greatly simplifies the calculation and allows repetitious algebra to be abstracted in tables. The "perturbation tree" is given in general form and may be applied to any laser problem utilizing perturbation theory. The coefficients appearing in amplitude-and frequency-determining equations calculated via the perturbation tree are evaluated exactly and are given in a form which is computer oriented.

To begin our discussion, we derive Maxwell's equations in a rotating frame to first order in \(|v/c|\), give the form of the electric field and induced polarization, and obtain the self-consistency equations in Chapter II. The equation of motion of the population matrix and the expression for the polarization of the medium are derived in Chapter III. In Chapter IV, the equation of motion for the population matrix is formally integrated to third-order in the electric field. The perturbation tree is also introduced. First and third-order contributions to the induced polarization are obtained in Chapters V and VI respectively. The amplitude-and frequency-determining equations for the bidirectional case are given in Chapter VII and for the unidirectional case in Chapter VIII. In Chapter IX, we reduce the general solution to
the case of two oppositely directed traveling waves each with two circular polarizations. Unlike the results of the standing wave theory, we encounter relative phase angles in the amplitude- and frequency-determining equations for this case. Competition between the various traveling waves is described in terms of a "coupling" parameter which is given explicitly for zero magnetic field. The reduction to a unidirectional Zeeman ring laser with one traveling wave consisting of two circular polarizations is accomplished in Chapter X.
CHAPTER II

ELECTROMAGNETIC FIELD EQUATIONS

In order to treat the ring laser we must obtain Maxwell's equations for a reference frame in uniform rotation (see Figure 1). Although the equations may be derived via the general theory of relativity (Menegozzi and Lamb n.d., Irvine 1964, Post 1967), we are only concerned with perimeter speeds \( v \), much less than the speed of light. Hence we derive Maxwell's equations to first order in \( |v/c| \). Following a treatment similar to Menegozzi and Lamb (n.d.) we assume the observer in the rotating frame may relate his field quantities \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \) to the fields \( \vec{E}^0(\vec{r}^0,t^0) \) and \( \vec{B}^0(\vec{r}^0,t^0) \) in the inertial frame by the equations of special relativity

\[
\vec{E}^0(\vec{r}^0,t^0) = \left[ \vec{E}(\vec{r},t) - \vec{v} \times \vec{B}(\vec{r},t) \right],
\]

\[
\vec{B}^0(\vec{r}^0,t^0) = \left[ \vec{B}(\vec{r},t) + \mu_0 \epsilon_0 \vec{v} \times \vec{E}(\vec{r},t) \right],
\]

where the physical measurements to determine \( \vec{E} \) and \( \vec{B} \) in the rotating frame are the same as those used to determine \( \vec{E}^0 \) and \( \vec{B}^0 \) in an inertial frame with any effects of inertial forces subtracted out (Irvine 1964). The details of the transformation are given in Appendix A. The macroscopic Maxwell's equations for a rotating frame to first order in \( |v/c| \)
This figure shows a possible ring laser geometry and the position of the vectors $\Omega$, $v$, $r$, and $s$ used in the derivation of Eqs. (2), (4), and (7). Here $\Omega$ is the rotation vector, $v$ is the instantaneous velocity at point $r$ on the laser axis and is equal to $\Omega \times r$, and $s$ is the unit vector along the laser axis. The unit vectors $i$ and $j$ are used to describe the polarization of the electric field. In the derivation of the electromagnetic field equations it is important to note we require the vector $\Omega$ to be perpendicular to the plane of the ring laser.
are found to be

\[ \text{div}[\mathbf{B} - \mu_0 \varepsilon_0 \nabla \times \mathbf{E}] = 0, \]

\[ \text{curl} \mathbf{E} + \left( \frac{\partial}{\partial t} \right) \left[ \mathbf{B} + \mu_0 \varepsilon_0 (\nabla \times \mathbf{E}) \right] = 0, \]

\[ \text{div}[\mathbf{E} - \nabla \times \mathbf{B}] = 0, \]

\[ \text{curl} \mathbf{B} - \mu_0 \varepsilon_0 \left( \frac{\partial}{\partial t} \right) \left[ \mathbf{E} - \nabla \times \mathbf{B} \right] = \mu_0 \mathbf{J}, \] (2)

where \( \mathbf{v} = \mathbf{v} \times \mathbf{r} \) is the instantaneous velocity at point \( \mathbf{r} \) on the laser axis and \( \mathbf{J} \) is the current density. Assuming the usual replacement for non-magnetic materials (VanVleck 1932)

\[ \mathbf{J} \rightarrow \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t}, \] (3)

and using Eq. (2), the wave equation to first order in \( |\mathbf{v}/c| \) becomes

\[ \text{curl} \text{curl} \mathbf{E} + \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

\[ + \mu_0 \varepsilon_0 \left( \frac{\partial}{\partial t} \right) \left[ \text{curl} \left( \mathbf{v} \times \mathbf{E} \right) + \mathbf{v} \times \text{curl} \mathbf{E} \right] \]

\[ = -\mu_0 \left( \frac{\partial}{\partial t} \right) [\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t}], \] (4)

where the induced polarization of the active medium \( \mathbf{P} \) will be described in the rest frame of the laser cavity. Following the treatment of
Sargent et al. (1967), we choose a current density to provide for differ- erent cavity resonances for linearly polarized radiation along unit vectors $\hat{i}$ and $\hat{j}$, circular birefringence, and different effective cavity lengths for different polarizations and traveling waves. These considerations lead to (Sargent et al. 1967)

$$\mathbf{J} = [\hat{\sigma}' - \nu \hat{\sigma}''(\partial/\partial t)] \cdot \mathbf{E},$$

where $\hat{\sigma}'$ and $\hat{\sigma}''$ are discussed by Sargent et al. (1967).

Since we are concerned with a traveling wave laser, the possibility exists for backscattering of one traveling wave mode into a mode of the opposite direction. This may be accounted for by including a loss term $\hat{\sigma}' \cdot \mathbf{E}$ in the current density (Aronowitz 1965, 1969, 1970). Using Eq. (5), the current density is

$$\mathbf{J} = [\hat{\sigma}' + \hat{\sigma}'' - \nu \hat{\sigma}''(\partial/\partial t)] \cdot \mathbf{E}.$$  \hspace{1cm} (6)

Neglecting the spatial variation in $\mathbf{E}$, curl $\mathbf{E} = \hat{s} \partial E/\partial s$, and curl curl $\mathbf{E}$ may be replaced by $-\partial^2 \mathbf{E}/\partial s^2$. Using these simplifications and Eq. (6), the wave equation, Eq. (4) reduces to

$$-\partial^2 \mathbf{E}/\partial s^2 + \mu_0 [\hat{\sigma}' + \hat{\sigma}'' - \nu \hat{\sigma}''(\partial/\partial t)] \cdot \mathbf{E}/\partial t$$

$$- 2\mu_0 \varepsilon_0 [(\hat{n} \times \hat{r}) \cdot \hat{s}] \partial^2 \mathbf{E}/\partial s \partial t = -\mu_0 \partial^2 \mathbf{E}/\partial t^2,$$  \hspace{1cm} (7)

where $s$ is the coordinate along the laser axis (as shown in Figure 1).
The electric field may be expanded in the form

$$E(s,t) = \frac{1}{2} \sum_{i=1}^{2} \sum_{n} \{ \hat{e}_i E_{ni}(t) \exp[-i(\nu_{ni} t + \phi_{ni}(t))] U_{ni}(s) + \text{c.c.} \},$$

(8)

where \( \hat{e}_i \) are any two orthogonal unit vectors in the transverse plane and may be complex. The amplitudes \( E_{ni}(t) \) and phases \( \phi_{ni}(t) \) are real, slowly varying functions of time. In this treatment we shall allow the subscript \( n \) to index both Fox and Li quasi-modes and traveling waves. Hence a "mode" will refer to a single Fox and Li quasi-mode and a single traveling wave. Then \( U_{ni}(s) \) are the complex eigenfunctions corresponding to the \( n \) th "mode" for \( \hat{e}_i \). We will use \( U_{ni}(s) = \exp(id_n K_{ni}s) \) in this treatment, where \( d_n = +1(-1) \) for the clockwise (counterclockwise) traveling wave. The polarization may similarly be written in the form

$$P(s,t) = \frac{1}{2} \sum_{i=1}^{2} \sum_{n} \{ \hat{e}_i P_{ni}(t) \exp[-i(\nu_{ni} t + \phi_{ni}(t))] U_{ni}(s) + \text{c.c.} \}. \quad (9)$$

where \( P_{ni}(s,t) \) is a slowly varying complex function of time. \( P_{ni}(t) \) will be referred to as the complex polarization.

In obtaining the self-consistency equations we drop the complex conjugates of the expressions for \( \hat{E} \) and \( \hat{P} \), which amounts to making the rotating wave approximation in the derivation of \( \hat{P} \) and equating the coefficients of the positive frequency term \( \exp(-i\nu t) \) to zero separately from that of the negative frequency term \( \exp(i\nu t) \). Then one substitutes Eq. (8) and (9) without complex conjugates into Eq. (7), projects the result onto the unit vectors \( \hat{e}_i \), and then onto the "mode" \( U_{ni}(s) \).
doing so, we neglect rapidly varying terms containing \( \ddot{E}_{ni}, \dot{\phi}_{ni}, \sigma_{ii}', \sigma_{sii}', \sigma_{ii}'\dot{E}_{ni}', \) and \( \ddot{\phi}_{ni} \).

Carrying out the indicated operations, noting that

\[
(v_{ni} + \dot{\phi}_{ni})^2 - \Omega_n \approx 2v(v_{ni} + \dot{\phi}_{ni} - \Omega_n),
\]

and dividing through by

\[-(2v) \exp[-i(v_{ni} t + \phi_{ni})],\]

the component equation is found to be

\[
(v_{ni} + \dot{\phi}_{ni} - \Omega_n) E_{ni} + iE_{ni} + L^{-1} d_n K_n \int_0^L ds (\Omega_T \times r) \cdot s
\]

\[+ i\frac{v}{2} \sum_{\mu i} \left\{ L^{-1} \int_0^L ds U^*_{ni}(s)(\sigma + \sigma_s)U_{ni'}(s)\right\} e_{ni'} E_{\mu i'} \]

\[\times \exp(i\psi_{ni, \mu i'}) = -\frac{i}{2} (\nu/\epsilon_0) P_{ni}, \quad (10)\]

where \( L \) is the round trip cavity length, \( \psi_{ni, \mu i'} = (v_{ni} - v_{\mu i'})t + \phi_{ni} - \phi_{\mu i'} \) are relative phase angles, and \( \sigma = \sigma' + i\sigma'' \).

Using the vector identity \((A \times B) \cdot C = A \cdot (B \times C)\), the first integral in Eq. (10) becomes

\[
\int_0^L ds (\Omega_T \times r) \cdot s = \Omega_T \cdot \int_0^L (r \times ds) = 2(\Omega_T \cdot A) = 2\Omega_T A, \quad (11)
\]
where \( \hat{A} \) is vector with magnitude equal to the area of the polygon formed by the laser and direction normal to the plane of this polygon (parallel to \( \hat{\Omega}_r \)). For the second integral it is convenient to define an "anisotropy matrix" which includes backscattering effects

\[
G = \left( \varepsilon_0 v L \right)^{-1} \int_0^L ds U_{n1}(s) \left( \bar{N} + \bar{S} \right) U_{\mu1}(s). \tag{12}
\]

If backscattering is not considered, and \( \bar{N} \) is a constant, \( n = \mu \) and this anisotropy matrix reduces to the form used by Sargent et al. (1967). In the present context, the real parts of the diagonal terms are the reciprocals of cavity Q's and the imaginary parts are frequency displacements of the \( i^{th} \) polarization from the nonbirefringent cavity frequency \( n \pi c/L \). The off-diagonal terms couple the "modes" via backscattering and polarization anisotropies. The summation over \( \mu \) will only include terms for which \( n \) and \( \mu \) are on the same Fox and Li quasi-mode, for the other terms lead to rapidly varying phase angles which the electric fields cannot follow. If the frequencies of the traveling waves "modes" (centered about a Fox and Li quasi-mode) are significantly different, several terms may be dropped due to rapidly varying phase angles. This corresponds to an inability of backscattered radiation to contribute to a "mode" if the frequencies of the "mode" and backscattered radiation are not approximately equal (Aronowitz 1965). For typographical simplicity we adopt the convention that vector subscripts index Fox and Li quasi-modes, traveling wave "modes" and polarizations. The component equation (Eq. 10) may then be written
\[(\psi_n^+ - \phi_n^- - \Omega_n^-) \tilde{E}_n^+ + i \left[ E_n^+ + \frac{1}{2} \nu \sum_{\mu} g_{n\mu} \exp(i \psi_{n\mu}^-) \right] = \frac{1}{2} (\nu/\epsilon_0) P_n^+, \quad (13)\]

where

\[\Omega_n^- = \Omega_n^- - 2d_x^P \Omega_n(A/L),\]

\[\psi_{n\mu}^- = (\nu_n^- - \nu_{\mu}^-) t + \phi_{n}^- - \phi_{\mu}^- . \quad (14)\]

Equating the real and imaginary parts separately to zero, we arrive at the self-consistency equations

\[E_n^+ + \frac{1}{2} \nu \sum_{\mu} \text{Im}[i g_{n\mu} \exp(i \psi_{n\mu}^-) E_{n\mu}^-] = - (\nu/\epsilon_0) \text{Im}(P_n^-), \quad (15a)\]

\[\nu_n^+ + \phi_n^+ = \Omega_n^- - \frac{1}{2} \nu \sum_{\mu} \text{Re}[i g_{n\mu} \exp(i \psi_{n\mu}^-)] E_{\mu}^+ E_{\mu}^- \]

\[- \frac{1}{2} (\nu/\epsilon_0) E_n^+ \text{Re}(P_n^-). \quad (15b)\]

If one neglects backscattering and the resulting \(G\) matrix is diagonal in the representation specified by \(\hat{e}_1\) and \(\hat{e}_2\), these equations reduce to

\[E_n^+ + \frac{1}{2} (\nu/Q_n^-) E_n^+ = - (\nu/\epsilon_0) \text{Im}(P_n^-), \quad (16a)\]

\[\nu_n^+ + \phi_n^+ = \Omega_n^- - \frac{1}{2} (\nu/\epsilon_0) E_n^+ \text{Re}(P_n^-), \quad (16b)\]
where

\[
\Omega_{n}^{+} = \Omega_{n}^{-} + \frac{\hbar}{2} \text{Im}(g_{n}^{+}) \cdot (17)
\]

If the losses are independent of polarization and traveling wave, Eq. (16) describes the amplitudes and frequencies for any set of orthogonal unit vectors.

Two useful bases for formulating the problem are the x-y basis for which

\[
\hat{e}_{1} = \hat{i}, \quad \hat{e}_{2} = \hat{j}, \quad (18)
\]

and the ± (circularly polarized) basis for which

\[
\hat{e}_{1} = \hat{e}_{-} = 2^{-\frac{1}{2}}(\hat{i} + \hat{j}),
\]

\[
\hat{e}_{2} = \hat{e}_{+} = 2^{-\frac{1}{2}}(\hat{i} - \hat{j}). \quad (19)
\]

Since we are concerned mainly with axial magnetic field we use the ± basis for the remainder of the calculation. As mentioned in Sargent et al. (1967), this is especially convenient for electric-dipole transitions in which the magnetic quantum number changes by +1(-1) contribute to \(E_{n+}(E_{n-})\). Properties of transverse magnetic fields and transformations of the G matrix are discussed in detail in Sargent et al. (1967).
CHAPTER III

POLARIZATION OF THE ATOMIC MEDIUM

The laser action takes place between two atomic levels a and b as depicted in Figure 2. The levels are connected by an electric-dipole transition and may be characterized by the total angular momentum \( J_a \) and \( J_b \) and other quantum numbers \( n_a \) and \( n_b \) respectively. The inclusion of isotopes with nuclear spin and hyperfine structure is straightforward but algebraically messy. Since they were included in Sargent et al. (1967), they will not be treated here. However, the completely general expressions in Appendix D contain both considerations.

A basis for a matrix representation consists of eigenvectors

\[ |nJm\rangle, \quad (20) \]

where \( n = n_a, n_b; J = J_a, J_b \); and \( m \) runs over the corresponding magnetic sublevels of \( a \) and \( b \). We will assume the decay operator \( \Gamma \) is diagonal in this basis with elements \( \Gamma_{mm} = \Gamma_m \). The Hamiltonian \( H \) has diagonal elements

\[ H' = H + \frac{\hbar^2}{2m} \mathbf{g}_{aa'} \mathbf{H}_{aa'}, \]

\[ H' = H + \frac{\hbar^2}{2m} \mathbf{g}_{bb'} \mathbf{H}_{bb'}, \quad (21) \]

where \( a' \) and \( b' \) are magnetic quantum numbers of the levels \( a \) and \( b \) respectively, \( \hbar \omega_a \) and \( \hbar \omega_b \) are the zero-field energies of \( a \) and \( b \), \( g_a \) and \( g_b \) are the Landé \( g \) factors for the levels, \( \mathbf{H} \) is the magnetic field.
This figure depicts a possible level scheme showing how levels a and b might be split into $2J_a + 1 = 3$ and $2J_b + 1 = 5$ sublevels respectively by an applied dc magnetic field. Here $J_a$ and $J_b$ are the total angular momenta of the levels a and b; $a'$ and $b'$ are the corresponding magnetic quantum numbers; $\omega_{ab}$ is the zero field optical frequency between levels a and b and $\omega_{a'b'}$ is a representative optical frequency in the presence of a magnetic field.
strength, and $\mu_B$ is the Bohr magneton. The off-diagonal elements of the Hamiltonian $V_{a'b'}$ are the matrix elements of the time dependent perturbation energy associated with the electric field

$$V_{a'b'} = - \langle n_a J_a' | e \vec{E} \cdot \vec{r} | n_b J_b' \rangle,$$

where $e$ is the charge of the electron and $\vec{r}$ is the position vector of the atom

$$\vec{r} = xi + yj + zs.$$

The position vector $\vec{r}$ may be written in spherical coordinates with complex combinations of $i$ and $j$ which make the calculation of matrix elements particularly easy:

$$\vec{r} = \frac{1}{2} \hat{r} \sin \phi [(i-ij) \exp(i\phi) + (i+ij) \exp(-i\phi)] + r \cos \phi \hat{s},$$

where $\theta$ and $\phi$ are the standard polar and azimuthal angles of $\vec{r}$ with respect to the atomic axis $(\hat{i}, \hat{j}, \hat{s})$. Writing Eq. (24) in terms of $\hat{e}_-$ and $\hat{e}_+$ given by Eq. (19) and substituting the result into Eq. (23), one has

$$V_{a'b'} = 2 \delta_{a'b'} [\hat{E}_- \hat{e}_+ \delta_{a'b'} + \hat{E}_+ \hat{e}_- \delta_{a'b'}].$$

where the dipole matrix elements $\delta_{a'b'}$ are given by
\[ \begin{align*}
\langle a'|b' \rangle & = \langle J_a | a' | J_b \rangle \exp(\pm i\phi) |b' \rangle, \quad a' = b' \pm 1, \quad J_a = J_b \pm 1, 0 \\
& = \langle J_a | a' | J_b \rangle, \quad a' = b', \quad J_a = J_b \pm 1, 0 \\
& = 0 \quad \text{otherwise} \\
\end{align*} \]

with explicit values (Condon and Shortley, 1967)

\[ \begin{align*}
\langle a'|b' \rangle & = \frac{1}{2} \Phi [\left( J_a \pm a' \right) \left( J_b \pm a' + 1 \right)]^{\frac{1}{2}} \\
& \quad \begin{cases} 
J_a = J_b - 1 \\
J_a = J_b \\
J_a = J_b + 1 
\end{cases} \\
= \Phi \left[ J_b^{2} - a'^{2} \right]^{\frac{1}{2}} \\
= \frac{1}{2} \Phi \left[ \left( J_a \pm a' \right) \left( J_a \pm a' + 1 \right) \right]^{\frac{1}{2}} \\
= \Phi a' \\
= \frac{1}{2} \Phi \left[ \left( J_a \pm b' \right) \left( J_a \pm b' + 1 \right) \right]^{\frac{1}{2}} \\
= \Phi \left[ J_a^{2} - b'^{2} \right]^{\frac{1}{2}} \\
\end{align*} \]
where $\rho$ is the reduced matrix element $(n_a J_a || er || n_b J_b)$. Substituting Eq. (8), expressed in the $\tilde{e}_\pm$ basis, into Eq. (25), one has

$$V_{a'b'} = -2^{-\frac{1}{2}} G_{a'b'} \sum_\mu \delta_{a',b'+\mu} E_\mu \exp[-i(v_\mu t + \phi_\mu)] U_\mu(s) + c.c.,$$

(28)

$$V_{b'a'} = V_{a'b'}^*,$$

where the polarizations $p_\mu^+ = +1$ (-1) for right (left) circularly polarized light.

The equation of motion of a density matrix

$$\rho(\alpha, s_0, t_0, v)$$

describing the pure state in which an atom is excited to the eigenstate $\alpha$ (i.e., $|nJm\rangle$), at place $s_0$, time $t_0$, and $s$ component of velocity $v$, is

$$\dot{\rho}(\alpha, s_0, t_0, v, t) = -i/\hbar \left[ H, \rho \right] - i/2 \left[ \Gamma \rho + \rho \Gamma \right],$$

(29)

where $H$ is the Hamiltonian and $\Gamma$ is the decay operator mentioned earlier. Using this density operator the average electric-dipole moment $\vec{p}$ is

$$\vec{p} = \text{Tr}(\rho \vec{e}).$$

(30)

All atoms arriving at $s$ at time $t$, regardless of $s_0$, $t_0$, $\alpha$, and $v$ contribute to the macroscopic polarization $\vec{P}(s, t)$ as follows (Sargent et al. 1967)
\[ \tilde{P}(s,t) = \sum_{\alpha} \int_{-\infty}^{t} dt_{0} \int_{s_{0}}^{L} ds_{0} \int_{0}^{\infty} dv \lambda_{\alpha}(s_{0},v,t_{0}) \text{Tr}(\rho_{\text{per}}) \]
\[ \times \delta[s-s_{0}-v(t-t_{0})], \]  
(31)

where we have assumed that the excitation mechanism always excites an atom to the eigenstate \( \alpha \), and \( \lambda_{\alpha}(s_{0},t_{0},v) \) is the number of atoms excited to the eigenstate \( \alpha \) per unit time per unit volume.

The effective Hamiltonian (Eq. 25) does not depend (Lamb 1964) on \( s_{0}, t_{0}, \) or \( \alpha \) but only on \( s, t, \) and \( v \). Proceeding as in Sargent et al. (1967), we use this fact to average the pure case over \( s_{0}, t_{0}, \) and \( \alpha \) before the integration of the equations of motion. We define a population matrix

\[ \rho(s,v,t) = \sum_{\alpha} \int_{-\infty}^{t} dt_{0} \int_{s_{0}}^{L} ds_{0} \lambda_{\alpha}(s_{0},t_{0},v) \rho(\alpha,s_{0},t_{0},v,t) \]
\[ \times \delta[s-s_{0}-v(t-t_{0})] \]  
(32)

where \( s, t, v, \) and \( t \) label an ensemble of atoms with velocity \( v \) which at time \( t \) arrive at place \( s \). This definition differs from Sargent et al. (1967) in the deletion of the superfluous time dependence of \( \rho \) on \( t \). In all calculations of \( P(s,t) \), \( t \) is set equal to \( t_{0} \), hence we may take derivatives with respect to \( t \) and arrive at the same result. We may now differentiate Eq. (32) with respect to \( t \) to find the equation of motion for \( \rho(s,v,t) \). Using the Leibnitz rule one has

\[ \frac{d\rho}{dt} = \lambda(s,v,t) + \sum_{\alpha} \int_{-\infty}^{t} dt_{0} \int_{s_{0}}^{L} ds_{0} \lambda_{\alpha}(s,v,t) \rho(\alpha,s_{0},t_{0},v,t) \]
\[ \times \delta[z-z_{0}-v(t-t_{0})]. \]  
(33)
Here we have assumed that the excitation operator $\lambda(s, v, t)$, with elements $\lambda_a(s_0, v, t_0)$ which are slowly varying in time and space, may be evaluated at $s, t, v$. Using Eq. (29) for $\dot{\rho}(a, s_0, t_0, v, t)$ in Eq. (33), interchanging the order of the Hamiltonian operator and the commutation operation with the integration, and invoking the definition of the population matrix, Eq. (32), the equation of motion for the population matrix becomes

$$\frac{d\rho(s, v, t)}{dt} = -(i/\hbar)[H, \rho(s, v, t)] - \frac{1}{2} [\Gamma \rho(s, v, t) + \rho \Gamma] + \lambda(s, v, t).$$

Combining the definition of $\rho(s, v, t)$, Eq. (32) with Eq. (31), $\dot{P}(s, t)$ becomes

$$\dot{P}(s, t) = \int_{-\infty}^{\infty} dv \text{ Tr } [\rho(s, v, t)e^r].$$

The equation for the complex polarization $P_n(t)$ may now be obtained by equating Eq. (9) and Eq. (35), projecting onto $U_n^*(s)$ and multiplying through by $2\exp[-i(v_n^* t + \phi_n^*)]$. We find

$$P_n(t) = 2 \exp[-i(v_n^* t + \phi_n^*)] L^{-1} \int_0^L ds U_n^*(s)$$

$$\cdot \int_{-\infty}^{\infty} dv \text{ Tr } [\rho(s, v, t) e^{(e(p_n^* \cdot r))}].$$

(36)
Noting that $e(p_n) \cdot r$ yields a matrix element given by $2^{-\frac{1}{2}}$ times the complex conjugate of Eq. (27) and performing the indicated trace operation, Eq. (36) yields

$$p_n(t) = 2\sqrt{2} \sum_{a} \sum_{b} \exp\left[-i(v_{a}^n + \phi_{a}^n)\right] \delta_{a',b'} \rho_{a',b'} + p_{n}^+$$

$$L^{-1} \int ds U_{n}^{*}(s) \int_{0}^{\infty} dv \rho_{a,b}(s,v,t).$$

(37)
CHAPTER IV

INTEGRATION OF THE EQUATIONS OF MOTION

Inserting the total Hamiltonian, Eqs. (21) and (22) into the equation of motion (34), one obtains the components

\[ \dot{\rho}_{a'b'} = -(i\omega_{a'b'} + \gamma_{a'b'})\rho_{a'b'} \]

\[ + \left( \frac{i}{\hbar} \right) \left[ \sum_{a''} V_{a''} \rho_{a''} - \sum_{b''} V_{a''} \rho_{b''} \right] \]  

\[ \rho_{a'a''} = -(i\omega_{a'a''} + \gamma_{a'a''})\rho_{a'a''} \]

\[ + \left( \frac{i}{\hbar} \right) \sum_{b''} \left( V_{b''} \rho_{a''} - V_{a''} \rho_{b''} \right) + \lambda_{a'a''}, \]  

\[ \rho_{b'b''} = -(i\omega_{b'b''} + \gamma_{b'b''})\rho_{b'b''} \]

\[ + \left( \frac{i}{\hbar} \right) \sum_{a''} \left( V_{a''} \rho_{b''} - V_{b''} \rho_{a''} \right) + \lambda_{b'b''}, \]  

\[ \dot{\rho}_{b'a'} = \rho_{a'b'}, \]  

where \( a'' \) and \( b'' \) are magnetic quantum numbers for the sublevels of \( a \) and \( b \) respectively,
\[ \omega_{a'a'} = \omega_a - \omega_{a'}, \]
\[ \gamma_{a'a'} = \frac{1}{2}(\gamma_a + \gamma_{a'}) + \gamma_{\text{phase}} \]
\[ \alpha, a' = a, b', a'', b''' , \] (42)

where \( \gamma_{\text{phase}} \) is the reciprocal of the phase diffusion time \( T_2 \) and the dot indicates differentiation with respect to \( t \). The iteration procedure to obtain Eqs. (38) - (41) to any desired order in the perturbation \( V_{a'b'} \) formally proceeds exactly as in the scalar theory (Lamb 1964) and the treatment in Sargent et al. (1967). Assuming the excitation rates have a separable form

\[ \lambda_\alpha(s, v, t) = W(v) \Lambda_\alpha(s, t), \] (43)

and defining the excitation rate density

\[ N_{a'b'}(s, t) = \Lambda_{a'}(s, t) \gamma_{a'}^{-1} - \Lambda_{b'}(s, t) \gamma_{b'}^{-1}, \] (44)

where \( \Lambda_{a'}(s, t) \) and \( \Lambda_{b'}(s, t) \) are the unsaturated population inversion densities for magnetic sublevels \( a' \) and \( b' \) respectively, the first-order contribution to \( \rho_{a'b'} \) becomes, as in Sargent et al. (1967),

\[ \rho_{a'b'}^{(1)}(s, v, t) = \left( i/\hbar \right) W(v) N_{a'b'}(s, t) \int_0^\infty d\tau' V_{a'b'}(s', t') \]
\[ \times \exp[-(\gamma_{a'b'} + i\omega_{a'b'})(\tau')]. \] (45)
Here $W(v)$ is the velocity distribution of the atoms, $t' = t - \tau'$, and $s' = s - v\tau'$. Substitution of Eq. (45) into Eqs. (39) and (40) yields the second order components of $\rho_{a'a''}$ and $\rho_{b'b''}$ (see Sargent et al. 1967). Using the second order results for $\rho_{a'a''}$ and $\rho_{b'b''}$ in Eq. (35), one has the third order components

\begin{equation}
\rho_{a'b'}^{(3)}(s,v,t) = -(i/\hbar^3) \int_0^\infty \int_0^\infty \int_0^\infty W(v) \frac{d\tau'}{\delta} \frac{d\tau''}{\delta} \frac{d\tau'''}{\delta} \times \exp[-(\gamma_{a'b'} + i\omega_{a'b'}\tau')]
\end{equation}

\begin{equation}
\times \sum_{a''b''} \{ V_{a''b''}(s',t') \exp[-(\gamma_{a''b''} + i\omega_{a''b''}\tau'')] V_{a'b'}(s'',t''') \}
\end{equation}

\begin{equation}
= \sum_{a''b''} \{ V_{b''a''}(s'',t'') N_{a''b''} \exp[-(\gamma_{b''a''} + i\omega_{b''a''}\tau'')] V_{a'b'}(s'',t''') \}
\end{equation}

\begin{equation}
= \sum_{a''b''} \{ V_{a'b''}(s',t') \exp[-(\gamma_{a'b''} + i\omega_{a'b''}\tau'')] \}
\end{equation}

\begin{equation}
\times \{ V_{a''b''}(s'',t'') N_{a''b''} \exp[-(\gamma_{a''b''} + i\omega_{a''b''})] V_{b''a''}(s'',t''') \}
\end{equation}

\begin{equation}
= \sum_{a''b''} \{ V_{a'b''}(s',t') \exp[-(\gamma_{a'b''} + i\omega_{a'b''}\tau'')] V_{a'b'}(s'',t''') \}
\end{equation}

where
Eq. (46) may conveniently be represented by a "perturbation tree" as shown in Figure 3, where we have defined

\[ e_{\alpha\alpha'} = \exp[-(\gamma_{\alpha\alpha'} + i\omega_{\alpha\alpha'})\tau'] \], \( \alpha, \alpha' = a', b', a'', b'' \),

In our tree, connecting lines and vertical ascent indicate multiplication and horizontal rows indicate addition. The earliest perturbation, contributing to \( \rho^{(0)} \) at time \( \tau'' \), connecting states \( |n_aJ_a\alpha\rangle \) and \( |n_bJ_b\alpha''\rangle \), \( |n_bJ_b\alpha''\rangle \) and \( |n_bJ_b\alpha''\rangle \), \( |n_aJ_a\alpha''\rangle \) and \( |n_bJ_b\alpha''\rangle \), is represented by the first (bottom) row of boxes. The first-order contribution connecting \( |n_bJ_b\alpha''\rangle \) and \( |n_aJ_a\alpha'\rangle \), \( |n_aJ_a\alpha'\rangle \) and \( |n_bJ_b\alpha''\rangle \), \( |n_aJ_a\alpha''\rangle \) and \( |n_bJ_b\alpha''\rangle \), at time \( \tau'' \) is represented by the second row. Finally, the third-order contribution connecting \( |n_aJ_a\alpha''\rangle \) and \( |n_bJ_b\alpha''\rangle \), \( |n_aJ_a\alpha''\rangle \) and \( |n_bJ_b\alpha''\rangle \), is represented by the third row. The \( e_{\alpha\alpha'} \) are the integrating factors which arise in the formal integration of Eqs. (38) - (41). The subscripts \( \nu, \rho, \) and \( \sigma \) will be used for the third-order contribution to the complex polarization \( P_\nu \) and the \( t = 1 \) which appear at the bottom are subscripts to integrals which appear in \( P_\nu \), (see Chapter VI). Up to this point we have not explicitly included the form of \( V_{a'b'} \). Thus our perturbation tree is perfectly general for any laser.
In this figure, the perturbation tree described in Chapter IV is shown. The $e_{\alpha\alpha'}$ are defined by Eq. (48). In the tree, connecting lines and vertical ascent indicate multiplication and horizontal rows indicate addition. The earliest perturbation is represented by the first (bottom) row of boxes. The first-order contribution is represented by the second row and finally, the third-order contribution is given by the third row. The $e_{\alpha\alpha'}$ are integrating factors which arise from the formal integration of the equations of motion for the population matrix. Since we have left the perturbation $V_{a'b'}$ in a general form, the tree may be used in any laser problem employing third-order perturbation theory by using the correct $V_{a'b'}$. Higher-order perturbation theory may be used if the tree is expanded appropriately.
calculation involving third-order perturbation theory (Lamb (1964)). In particular, by specifying the correct $U_n(s)$ one may do calculations for, 1) unidirectional ring laser by setting $U(s) = \exp(iKs)$ with only one traveling wave, 2) standing wave, two mirror laser by setting $U(z) = \sin(Kz)$, where $z$ is along the two mirror laser axis, and 3) the bi-directional ring laser by setting $U(s) = \exp(\pmiks)$ and considering oppositely directed traveling waves. We do 3), the most general case and specialize to 1) and 2) as a check on our calculations. The standing wave, two mirror laser was treated in Sargent et al. (1967). Although the unidirectional ring laser may be obtained by simplifying the standing wave analysis, to our knowledge this has not been done and will be presented here.
CHAPTER V

FIRST ORDER THEORY

The first order calculation formally proceeds in exactly the same manner as the standing wave, two mirror theory presented in Sargent et al. (1967), with the exception of the $U_n(s)$ and the convention that subscripts index "modes" as defined earlier. We shall assume a Maxwellian velocity distribution described by

$$W(v) = (\pi^3 u)^{-1} \exp[-(v/u)^2]$$

(49)

where $u$ is the most probable atomic speed. Combination of Eqs. (28), (45), and (37), and using the explicit form for $U_n(s)$, one has

$$P_n^{(1)}(t) = -2(i/\hbar) \sum_{a'b'} |\psi_{a'b'}|^2 \sum_{\mu} \delta_{a',b'+p,\mu} \delta_{a',b'+p,\mu}$$

$$\times L^{-1} \int_0^L \sum_{a'b'} N_{a'b'}(s,t) \exp[-i(d_sK_s-d_sK_s)s]$$

$$\times \exp[i[(v_{n,\mu}-v_{n,\mu}) t + \phi_{n,\mu} - \phi_{n,\mu}]] \int dv W(v) \int d\tau'$$

$$\exp{-i[y_{a'b'} + i(\omega_{a'b'} - v_{a'b'}) t + i(d_sK_s + i)]t'}$$

27
where we have freely interchanged the order of summation and integration. In order that the integrand of the integral over \( s \) be slowly varying, \( \delta_{n \mu} = \delta_{\mu \nu}. \) Since the velocity distribution is an even function in \( v, \) the \( \delta_{\mu \nu} \) does not affect the integral over \( v \) and may be dropped. Although the Kronecker deltas \( \delta_{a' b',+p_{\mu}} \) require that \( p_{\mu} = p_{\nu}, \) we choose not to make this simplification at this point.

Dropping the rapidly varying term and performing the time and velocity integrals, we find the first order contribution to the complex polarization

\[
P_{n \rightarrow} (t) = -2(\hbar K u)^{-1} E_{\mu} \exp(i \psi_{n \mu}) \sum_{n \mu} \sum_{a', b', +p_{\mu}} \delta_{a', b', +p_{\mu}} \delta_{a', b', +p_{\mu}} \delta_{a', b', +p_{\mu}} d_{a', \mu} d_{b', \mu} \]

\[
\times |\delta_{a' b', |^2 N_{a' b', +p_{\mu}} (n, +p_{\mu}) Z [\gamma_{a' b', +i(\omega_{a' b', -v_{\mu}}) \mu}],
\]

where the relative phase angle

\[
\psi_{n \mu} = (v - v_{\mu}) t + \phi - \phi_{n \mu},
\]

the plasma dispersion function (Hanson and Sargent, in prep; Fried and Conte 1961),
\[ Z(u) = iK\pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} dv \exp[-(v/u)^2] (u^2 + Kv)^{-1} \]  

and the Fourier components of the population inversion density

\[ N_{a'b'}(n-\mu) = L^{-1} \int_0^L ds \, N_{a'b'}(s,t) \exp[-i \frac{d_n(K_{n} - K_{n'}) s}{n - n'}]. \]  

We will frequently use the abbreviation

\[ \overline{N}_{a'b'}(t) = L^{-1} \int_0^L ds \, N_{a'b'}(s,t). \]  

Substitution of Eq. (51) into the self consistency equations (15a) and (15b), one has

\[ \dot{E} \equiv \sum_{\mu} \text{Im}[\hat{\alpha}_{\overline{n}\mu} \exp(i \psi_{\overline{n}\mu})] \frac{\dot{E}}{\mu}, \]  

\[ \nu \dot{\phi} + \phi = \Omega + \sum_{\mu} \text{Re}[\hat{\alpha}_{\overline{n}\mu} \exp(i \psi_{\overline{n}\mu})] \frac{E_{\mu} E_{\overline{n}}}{\nu t} \]  

where

\[ \hat{\alpha}_{\overline{n}\mu} = -\frac{1}{2} i \nu g_{\overline{n}\mu} + \nu (\hbar K u e_0)^{-1} \sum_{a'} \sum_{b'} \delta_{a',b',p_\mu} \delta_{a',b'} \delta_{p_\mu} \]

\[ \times \delta_{\overline{n},d_\mu} |(\rho_{a'b'})|^2 \]

\[ N_{a'b'}(n-\mu) Z \left[ Y_{a'b'} + i(\omega_{a'b'} - \nu_{\overline{n}\mu}) \right]. \]
If the frequency difference between opposite traveling waves on the same Fox and Li quasi-mode is large enough to make terms for which \( d_{nn} \neq d_{n} \) contain rapidly varying phase angles, Eqs. (56) and (57) reduce to equations similar to those given by Sargent et al. (1967);

\[
\begin{align*}
E_{\hat{n}} &= \sum_{i'} \Im[\tilde{\alpha}_{\hat{n}n'} \exp(i\psi_{\hat{n}n'})] E_{\hat{n}'} \\
\tilde{\Omega}_{\hat{n}} + \hat{\phi}_{\hat{n}} &= \Omega_{\hat{n}} + \sum_{i'} \Re[\alpha_{\hat{n}n'} \exp(i\psi_{\hat{n}n'})] E_{\hat{n}'} E_{\hat{n}}^{-1},
\end{align*}
\]

(59)

where \( \hat{n} = ni, \hat{n}' = ni' \), \( i \) and \( i' \) index only the polarizations, and

\[
\tilde{\alpha}_{\hat{n}n'} = -\frac{1}{2}ivg_{\hat{n}n'} + (\hbar k \varepsilon_{0})^{-1} \sum_{a',b'} \delta_{a',b'}+p_{\hat{n}}^{+} \delta_{a',b'}+p_{\hat{n}}^{-} \times |a'_{a'} b'_{b'}|^{2} N_{a_{a'}, b_{b'}} \cdot d_{\hat{n}}(\hat{n} \rightarrow \hat{n}') Z[\gamma_{a_{a'}, b_{b'}} + i(\omega_{a_{a'}, b_{b'}} - \nu_{n}^{+})].
\]

(60)

Equations (58) and (60) may be expressed in terms of the relative excitation as discussed in Sargent et al. (1967, Chapter V).
CHAPTER VI

THIRD ORDER THEORY

To obtain the third-order contribution to the complex polarization \( \mathbf{P}_{\mathbf{a}}(t) \) it is necessary to substitute the explicit form of \( V_{a'b'} \) into Eq. (46) and use that result in Eq. (37). The substitution of Eq. (28) for \( V_{a'b'} \) may be greatly simplified if it is noted that only positive frequency terms in \( V_{a'b'} \) lead to resonant contributions to the density matrix elements. In the same manner, only negative frequency terms of \( V_{b'a} \) lead to resonant contributions. The "perturbation tree" of Figure 3 simplifies the algebra of the indicated operation considerably. We associate subscripts \( \hat{\mu} \) with \( V_{a''b''} \), \( \hat{\rho} \) with \( V_{b''a''} \), and \( \hat{o} \) with \( V_{a'b''} \). Although the times of perturbation for \( \hat{\mu}, \hat{\rho} \) and \( \hat{o} \) change from branch to branch in Figure 3, the signs do not; \( \hat{\mu} \) and \( \hat{o} \) are associated with positive frequency terms; \( \hat{\rho} \) with the negative frequency terms. Each perturbation \( V_{a'b'} \), consists of a right, \( V_{a'b'}^{R} \), and left, \( V_{a'b'}^{L} \), traveling wave, e.g., \( V_{a'b'} = V_{a'b'}^{L} + V_{a'b'}^{R} \). Hence in the third order we have eight possible sub-branches for each branch in Figure 3. Because of the atomic motion, the perturbations are evaluated at different positions given by Eq. (47). In evaluating the branches of the tree, we encounter products of the form

\[
\exp[-s_1 iK\nu t'] \exp[-s_2 iK\nu(\tau'+\tau'')] \exp[-s_3 iK\nu(\tau'+\tau''+\tau'')] ,
\]

(61)
where the signs, \( s_n \), are associated with perturbations occurring at \( t', t'', \) and \( t''' \) for \( s_1, s_2, \) and \( s_3 \) respectively and are given by

\[
s_n = \begin{cases} 
\pm & \text{for } d_{\mu}, d_{\nu} = \pm 1 \\
\mp & \text{for } d_{\rho} = \pm 1 
\end{cases}
\]  

(62)

Table 1 summarizes the calculations for all 32 branches, where we have indicated only the sign on \( d_{\mu}, d_{\nu}, \) and \( d_{\rho} \). Eight of these are set to zero since they lead to rapidly varying terms in space (\( \exp(\pm i3Ks) \)). The tenth column indicates to which traveling wave the term contributes.

The times of perturbation (and hence the place) associated with \( \hat{\mu}, \hat{\nu}, \) and \( \hat{\sigma} \) are given for each group of eight in column one. An expansion of the \( t=1 \) branch in terms of \( V_{a'b'} \) and \( V_{a'b'} \) is given in Appendix B.

The \( T_{tw} \) column indicates which T integral is involved and will be discussed shortly.

Appropriately multiplying and summing the terms of the tree, as discussed in Chapter IV to form \( p_{a'b'}^{(3)} \), and substituting this result into Eq. (37), one has the third-order contribution to the complex polarization

\[
P_{n=1}^{(3)}(t) = (\hbar^3 Ku)^{-1} \sum_{\mu, \nu, \sigma} \sum \sum E_{\mu} E_{\nu} E_{\sigma} \exp(i\psi_{\mu\nu\sigma}) \sum_{a', b'} \sum_{a'' b''} \\
\times \delta_{a', b''; a''; b''} \delta_{b'', a''; b''; a''} \delta_{a', b''; a''; b''} \\
\times 4 \sum_{t=1} T_{t,w}(d_{\mu}, d_{\nu}, d_{\rho}, t)
\]  

(63)

where the relative phase angle
Table 1. Perturbation tree summary

This table summarizes the contributions of each branch of the perturbation tree of Fig. (3). The directions $d^+_{\mu}$, $d^+_{\rho}$, and $d^+_{\sigma}$ have been indicated by a plus (+1) and minus (-1) sign for simplicity. The signs $s_n$ are defined in Eqs. (61) and (62). The subscript $w$ for the $T_{tw}$ integrals may be calculated from the signs $s_n$ as given in Eq. (69). Each row contributes to the complex polarization of the traveling wave with direction specified by $d_n$. For each branch, $\hat{\mu}$, $\hat{\rho}$, and $\hat{\sigma}$ are associated with the perturbation as shown in the $t$ column.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$d^+_{\mu}$</th>
<th>$d^+_{\rho}$</th>
<th>$d^+_{\sigma}$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$w$</th>
<th>$T_{tw}$</th>
<th>$d^+_n$</th>
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<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
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<td>-</td>
<td>2</td>
<td>$T_{12}$</td>
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<td>3</td>
<td>$T_{13}$</td>
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<td>0</td>
<td>1</td>
<td>$T_{11}$</td>
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<td>$T_{12}$</td>
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<td>+</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{21}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>$T_{23}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>2</td>
<td>$T_{22}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>$T_{22}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>3</td>
<td>$T_{23}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{21}$</td>
<td>+</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{31}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>$T_{32}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>3</td>
<td>$T_{33}$</td>
<td>-</td>
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<td>+</td>
<td>-</td>
<td>3</td>
<td>$T_{33}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>2</td>
<td>$T_{32}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{31}$</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>$T_{42}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{41}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>$T_{43}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>3</td>
<td>$T_{43}$</td>
<td>+</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>1</td>
<td>$T_{41}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>2</td>
<td>$T_{42}$</td>
<td>+</td>
</tr>
</tbody>
</table>
The $T_{t,w}(d_{\mu}, d_{\rho}, d_{\sigma}, t)$ are integrals of the form

$$
T_{tw} = iK \frac{N_{tw}}{\pi^{1/2}} \int_{-\infty}^{\infty} dv \exp[-(v/u)^2] \int_{-\infty}^{\infty} d\tau' \int_{0}^{\infty} d\tau'' \int_{0}^{\infty} d\tau'''
$$

$$
\exp\{-[(v_{t1}+iC_{w1}Kv)\tau' + (v_{t2}+iC_{w2}Kv)\tau'' + (v_{t3}+iC_{w3}Kv)\tau''']\},
$$

(65)

where $N_{tw}$, $C_{wk}$ and $v_{tk}$ are defined in Table 2 and Table 3. The subscript $w$ is dependant on the directions $d_{\mu}, d_{\rho}, d_{\sigma}$ via signs $s_n$

$$
w = \begin{cases} 
1 & \text{for } s_1 = s_2 = s_3, \\
2 & \text{for } s_1 = s_2 = s_3, \\
3 & \text{for } s_1 = s_2 = -s_3.
\end{cases}
$$

(66)

The $T_{tw}$ integrals may be expressed in terms of the plasma dispersion function of Eq. (53), as discussed in Appendix C. For later convenience we define

$$
\begin{align*}
&v_1 \equiv v_{11} = v_{21} = v_{31} = v_{41}, \quad v_2 \equiv v_{12} = v_{22}, \quad v_3 \equiv v_{32} = v_{42}, \\
&v_4 \equiv v_{13}, \quad v_5 \equiv v_{23} = v_{33}, \quad v_6 \equiv v_{43}.
\end{align*}
$$

(67)
Table 2. Definition of the $N_{tw}$ integrals

This table defines the $N_{tw}$ which appear in the third order integrals of Eq. (65) in terms of the $N_{a',b'}$ which are defined in Eq. (54). The $C_{tk}$ in Eq. (65) are defined by the relationship $x_w = C_{w1} \tau' + C_{w2} \tau'' + C_{w3} \tau'''$.

<table>
<thead>
<tr>
<th>$N_{tw}$</th>
<th>$w = 1$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 1$</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu-\rho+\sigma})$</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu-\rho+\sigma})$</td>
</tr>
<tr>
<td>2</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu+\rho-\sigma})$</td>
</tr>
<tr>
<td>3</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{b''a''}, d_{n} (\overrightarrow{\mu-\rho+\sigma})$</td>
</tr>
<tr>
<td>4</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
<td>$N_{a''b''}, d_{n} (\overrightarrow{\mu+\rho+\sigma})$</td>
</tr>
</tbody>
</table>

$x_w \quad \tau''' - \tau' \quad \tau''' + \tau' \quad \tau''' + 2\tau'' + \tau'$
Table 3. Definition of the $u_{tk}$

This table defines the complex frequencies $u_{tk}$ used in the third order integrals of Eq. (65).

<table>
<thead>
<tr>
<th>$u_{tk}$</th>
<th>$k = 1$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 1$</td>
<td>$\gamma_{a''b''} + i(\omega_{a''b''} - \nu \sigma)$</td>
<td>$\gamma_{a'a''} + i(\omega_{a'a''} + \nu \sigma)$</td>
<td>$\gamma_{a'b'} + i(\omega_{a'b'} - \nu \sigma)$</td>
</tr>
<tr>
<td>2</td>
<td>&quot;</td>
<td>&quot;</td>
<td>$\gamma_{b''a''} + i(\omega_{b''a''} + \nu \sigma)$</td>
</tr>
<tr>
<td>3</td>
<td>&quot;</td>
<td>$\gamma_{b''b'} + i(\omega_{b''b'} + \nu \sigma)$</td>
<td>&quot;</td>
</tr>
<tr>
<td>4</td>
<td>&quot;</td>
<td>&quot;</td>
<td>$\gamma_{a''b'} + i(\omega_{a''b'} - \nu \sigma)$</td>
</tr>
</tbody>
</table>
The \( \sum_{t=1}^{4} T_{t,w}^{(d_{\mu}, d_{\rho}, d_{\sigma}, t)} \) may be expressed as three types of integrals. Using the definitions in Eq. (67), this is summarized in Table 4 along with "strong Doppler limit" (\( Ku >> \gamma \), see Appendix C) form of the integrals. With the help of Eq. (10) of Appendix C, the sum of \( T_{tw} \) for a homogeneously broadened medium where \( u \to 0 \), is given by

\[
T_{1i} + T_{2j} + T_{3k} + T_{4\ell} = i(Ku/u_1)
\]

\[
\times \left[ \frac{1}{u_2} \left( \frac{N_{1i}}{u_4} + \frac{N_{2j}}{u_5} \right) + \frac{1}{u_3} \left( \frac{N_{3k}}{u_5} + \frac{N_{4\ell}}{u_6} \right) \right].
\] (68)
CHAPTER VII

AMPLITUDE- AND FREQUENCY-DETERMINING EQUATIONS

Substitution of the real and imaginary parts of Eq. (63) for the third-order complex polarization into the self consistency equations (15) and adding this contribution to Eq. (56), one has the amplitude-and frequency-determining equations to third-order

\[
\mathbf{E}_n = \text{Im}(\sum \mathbf{a}_{n\mu} \mathbf{E}_\mu \exp(i\mathbf{\psi}_{n\mu}))
\]

\[
- \sum \sum \sum \mathbf{\mathcal{J}}_{n\mu\rho\sigma} \mathbf{E}_n \mathbf{E}_\mu \exp(i\mathbf{\psi}_{n\mu\rho\sigma})
\]

\[
\mathbf{v}_n + \mathbf{\phi}_n = \Omega_n \mathbf{E}_n^{-1} \text{Re}(\sum \mathbf{a}_{n\mu} \mathbf{E}_\mu \exp(i\mathbf{\psi}_{n\mu}))
\]

\[
- \sum \sum \sum \mathbf{\mathcal{J}}_{n\mu\rho\sigma} \mathbf{E}_n \mathbf{E}_\mu \exp(i\mathbf{\psi}_{n\mu\rho\sigma})
\]

where

\[
\mathbf{\mathcal{J}}_{n\mu\rho\sigma} = \frac{1}{2} \nu (\hbar^3 K u c_0)^{-1} \sum \sum \sum \delta_{a'b'a''b''} a'^{+p_\mu}_n b'^{+p_\mu}_n a''^{+p_\mu}_n b''^{+p_\mu}_n \delta_{a''b'} a', b' + p_\mu^\sigma b'', a'' - p_\mu^\sigma
\]

\[
\times \delta_{a', b'''+ p_\sigma^\mu}
\]

\[
\times \mathcal{P}_{b'a'b'} \mathcal{P}_{a''b''} \mathcal{P}_{b''a''} \mathcal{P}_{a'b'} \sum_{t=1}^4 \mathcal{C}_{t, w(d_\mu, d_\rho, d_\sigma, t)}
\]

38
The remaining symbols are defined by the equations indicated:
\[ \alpha_{\mu}, (58); \psi_{\mu}, (52); \psi_{\mu \rho \sigma}, (64); p_{\mu}, (28); \Omega_{n}, (14); \Omega_{a' b'}, (26) \]
and (27); \[ \tau_{t, w'}, \text{Tables 2 - 4} \text{and (65)}. \]

General amplitude- and frequency-determining equations taking into account arbitrarily oriented magnetic field, isotopes, general basis vectors, and hyperfine structure are given in Appendix D in extension of the theory of Sargent et al. (1967).
CHAPTER VIII

MULTIMODE UNIDIRECTIONAL THEORY

The reduction to a unidirectional ring laser is accomplished by requiring $d_n = d_{\mu} = d_{\rho} = d_{\sigma} = +1$ for traveling waves in the $+s$ direction. Since $\hat{n}, \hat{\mu}, \hat{\rho}, \hat{\sigma}$ no longer index traveling waves, there is no backscattering and the $G$ matrix (Eq. (12)) reduces to an anisotropy matrix for different polarizations as in Sargent et al. (1967), Section II. Specifically, the first-order $\hat{n} \rightarrow \hat{n} = n \hat{i}'$, where as in Chapter II, $i'$ indexes polarizations and now $n$ indexes only Fox and Li quasi-modes.

In third-order, only terms with $d_{\mu} = d_{\rho} = d_{\sigma}$ in Table 1, survive in the unidirectional limit. This results in considerable simplification since the only remaining integrals are those of Type 1 in Table 4.

The amplitude- and frequency-determining equations (69) and (70) become

$$\dot{E}_{nn} = \text{Im}\{\sum_{i} \hat{\alpha}_{nn} \cdot E_{nn} \exp(i\phi_{nn})\}$$

$$- \sum_{\mu} \sum_{\rho} \sum_{\sigma} \sum_{\eta} \sum_{\nu} \sum_{\gamma} \tilde{\sigma}_{\eta\nu\gamma\mu\rho\sigma} E_{\mu} E_{\rho} E_{\sigma} \exp(i\psi_{nn\mu\rho\sigma})$$

$$v_{nn} + \phi_{nn} = \Omega_{nn} + E_{nn}^{-1} \text{Re}\{\sum_{i} \hat{\alpha}_{nn} \cdot E_{nn} \cdot \exp(i\phi_{nn})\}$$

$$- \sum_{\mu} \sum_{\rho} \sum_{\sigma} \sum_{\eta} \sum_{\nu} \sum_{\gamma} \tilde{\sigma}_{\eta\nu\gamma\mu\rho\sigma} E_{\mu} E_{\rho} E_{\sigma} \exp(i\psi_{nn\mu\rho\sigma})$$

(72)

(73)
Table 4. The \( \sum \tau_{t, w(d_\mu^+, d_\rho^+, d_\sigma^+)} \) integrals

The three distinct types of the above integrals are given in this table as sums of the \( T_{tw} \) integrals. If \( K_{u} \gg \gamma \), the "strong Doppler limit" form may be used (see Appendix C). The \( u_i \) are defined in Eq. (67).

<table>
<thead>
<tr>
<th>( d_\mu^+ )</th>
<th>( d_\rho^+ )</th>
<th>( d_\sigma^+ )</th>
<th>( \sum \tau_{t, w(d_\mu^+, d_\rho^+, d_\sigma^+)} )</th>
<th>Strong Doppler Limit Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>( T_{12} + T_{21} + T_{31} + T_{42} )</td>
</tr>
<tr>
<td>Type 2</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>( T_{13} + T_{23} + T_{32} + T_{41} )</td>
</tr>
<tr>
<td>Type 3</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>( T_{11} + T_{22} + T_{33} + T_{43} )</td>
</tr>
</tbody>
</table>
where $\Omega_\mathbf{n}$ is the resonant frequency of the rotating cavity, $\mathbf{n}, \mathbf{\mu}, \mathbf{\rho}$ and $\mathbf{\sigma}$ index only Fox and Li quasi-modes and polarizations, $\mathbf{\sigma}_{nn'}$ is given by Eq. (60) without the $d_\mathbf{\sigma}$ on the $\mathbf{N}_{a'b'}$ and

$$\mathbf{\sigma}_{nn'} = \frac{1}{2}\nu (\hbar^2 K \nu \epsilon_0)^{-1} \sum_{a''b''} \sum_{a'b'} \sum_{a'b''} \delta_{a''b''} \delta_{a'b'} \delta_{a''b''} \delta_{a''b'} \delta_{a'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a'\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{a'\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a'\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}},$$

$$\left[ S_{b'M,a'M-p\mu} c_{b'M,a'M} g_{b'M,a'M} \sum_{t=1}^{4} T_{t,\mathbf{w}(t)} \right], \quad (74)$$

Equation (74) may be easily expressed in terms of the strong Doppler limit discussed in Appendix C. Using the form given in Table 4 for a Type 1 integral in this limit and noting from Table 2 that $N_{21} = N_{31}$, one has

$$\mathbf{\sigma}_{nn'} = \frac{i}{2}\nu (\hbar^2 K \nu \epsilon_0)^{-1} \sum_{a''b''} \sum_{a'b'} \sum_{a'b''} \delta_{a''b''} \delta_{a'b'} \delta_{a''b''} \delta_{a''b'} \delta_{a'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a'\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}} \delta_{a'\mathbf{n}} \delta_{b'\mathbf{n}} \delta_{a''\mathbf{n}} \delta_{b''\mathbf{n}},$$

$$\left[ S_{b'M,a'M-p\mu} c_{b'M,a'M} g_{b'M,a'M} \sum_{t=1}^{4} T_{t,\mathbf{w}(t)} \right], \quad (75)$$

where, for example

$$D(\omega_{a'a''} + \nu_\mathbf{n} - \nu_\mathbf{\sigma}) = [\gamma_{a'a''} + i(\omega_{a'a''} + \nu_\mathbf{n} - \nu_\mathbf{\sigma})]^{-1}, \quad (76)$$

where $\gamma_{a'a''}$ is the damping coefficient.
and \( N_b'' a''(t) \) is \( N_{21} \) (or \( N_{31} \)) of Table 2 and is given by

\[
N_b'' a''(t) = L^{-1} \int_0^L ds \, N_b'' a''(s,t) \exp[i(K_{\mu} - K_{\rho} + K_{\mu} - K_{\rho})s].
\]

(77)

Tables 2 and 3 may still be used to define the \( v_{tk} \) and \( N_{tw} \) involved in the \( T_{tw} \) integrals provided one remembers there is no summation over traveling waves. This correspondence is possible due to the generality of the perturbation tree of Figure 3. The perturbation tree for the unidirectional ring laser has no sub-branches similar to the previous case (see Appendix B) and is given directly by Figure 3 with the appropriate substitutions for the perturbations \( V_{a'b'} \). Single mode operation will be discussed in Chapter X.
CHAPTER IX

BIDIRECTIONAL, FOUR WAVE OPERATION

In this chapter we consider two oppositely directed traveling waves each with two circular polarizations. Mathematically this is a four "mode" problem. The electric field is then given by

\[ E(s,t) = \{ [E_1 e^{i(v_1 t + \phi_1)}] + E_2 e^{i(v_2 t + \phi_2)} \exp(iK_x s) \]

\[ + [E_3 e^{i(v_3 t + \phi_3)}] + E_4 e^{i(v_4 t + \phi_4)} \exp(-iK_z s) \]

\[ + \text{c.c.} \} \]

(78)

where \( e_- (e_+) \) is the basis vector for the left (right) circular polarization, the subscript notation is defined in Table 5, and subscript \( r(l) \) denotes right (left) traveling wave. In reducing the amplitude and frequency determining equations (69) and (70) for this case, we will make some simplifying assumptions: we take the strong Doppler limit form of the \( T_{tw} \) integrals of Table 4, assume \( \gamma_{a'b'} = \gamma_{a''b'} = \gamma_{b''a} = \gamma_{a'b''} = \gamma_{ab}; \gamma_{a'a''} = \gamma_{a} \gamma_{b}, \gamma_{b''b'} = \gamma_{b}, \) and take \( N_{tw} = N \). In these approximations, the integrals of Table 4 become

Type 1: \( (i \pi 2N) \mathcal{D} \left( \frac{i}{2} \omega_{a'a''} + \frac{i}{2} \omega_{b'b'} - \frac{i}{2} \nu_{\rho} + \nu_{\sigma} - \frac{i}{2} \nu_{\sigma} \right) \)

\[ [\mathcal{D}_a(\omega_{a'a''}, + \nu_{\rho} \nu_{\sigma}) + \mathcal{D}_b(\omega_{b''b'}, + \nu_{\rho} \nu_{\sigma})], \]  

(79)
Table 5. Traveling wave subscript definition

<table>
<thead>
<tr>
<th>( d_{\vec{n}} )</th>
<th>( p_{\vec{n}} )</th>
<th>&quot;Mode&quot; Subscript</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>4</td>
</tr>
</tbody>
</table>

The subscripts for the four traveling waves of Eq. (78) are defined in this table. As in previous tables we have dropped the 1 in the \( p_{\vec{n}} \).
Type 2: \((\pi^2 N) D \left( i \omega a'b' + \frac{i \omega}{2} a''b'' - \nu + \frac{i \nu}{2} - \frac{i \nu}{2} \right) D_b\)

\[
\times D_b \left( \omega b''b', + \nu - \nu \right), \tag{80}
\]

Type 3: \((\pi^2 N) D \left( i \omega a'b' + \frac{i \omega}{2} a''b'' - \frac{i \nu}{2} + \nu - \nu \right) D_a\)

\[
\times D_a \left( \omega a'a'' + \nu - \nu \right), \tag{81}
\]

where

\[
D(\Delta \omega) = (\gamma_a + i \Delta \omega)^{-1}, D_\alpha(\Delta \omega) = (\gamma_\alpha + i \Delta \omega)^{-1}, \alpha = a, b, \tag{82}
\]

and \(N = N_{a'b'}\), of Eq. (55).

We also assume there is no backscattering or cavity anisotropy for different polarizations and traveling waves. The G matrix is then diagonal with elements \(g_{\alpha n} = Q_{n}^{-1}\), where \(Q_{n}\) is the Q of the cavity for the nth "mode". In the reduction of \(\mathbf{\nu}_{nh \rho 0}\), Eq. (72), we encounter a total of thirty-six contributing terms. These are summarized in Tables 6 and 7 showing the magnetic sublevel transitions involved for each term, the type of integral encountered, and the coefficient of which the term is a part (see Eqs. (85) - (94)). The amplitude and frequency determining equations (70) and (71) become

\[
E_n^+ = E_n^+ \left[ \alpha_n + \sum_{\hat{n}m}^{4} \Theta_{\hat{n}m}^\rightarrow \frac{2}{m} \right] - \text{Im} \left\{ (\Theta_{\hat{n}ijk} + \Theta_{\hat{n}kji}) \exp(-ip_{\hat{n}d_{\psi}}) \right\} E_n^+ E E_n^+ E_{ij}, \tag{83}
\]
Table 6. Summary of the $\hat{\mathcal{O}}_{\mu\nu\rho\sigma}$ calculation

The calculation of the $\hat{\mathcal{O}}_{\mu\nu\rho\sigma}$ which yield the coefficients given in Eqs. (85) - (94) is summarized in this table. As in previous tables we have omitted the 1 on $p_\uparrow$ and $d_\uparrow$. The magnetic sublevel transitions are given by the dashes in column eight. A left (right) slanting dash, ($\backslash$) ($/$), indicates a change in the magnetic quantum number of -1 (+1). The ninth column indicates of which $\theta_{nm}$ the particular $\hat{\mathcal{O}}_{\mu\nu\rho\sigma}$ is a part.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\nu$</th>
<th>$\rho$</th>
<th>$\sigma$</th>
<th>Integral Type</th>
<th>$d_{n}^\uparrow$</th>
<th>$p_{n}^\uparrow$</th>
<th>Transitions</th>
<th>$\theta_{nm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1</td>
<td>+</td>
<td>-</td>
<td>|||</td>
<td>11</td>
</tr>
<tr>
<td>2222</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1</td>
<td>+</td>
<td>+</td>
<td>///</td>
<td>22</td>
</tr>
<tr>
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Table 7. Summary of the coefficients of the terms containing phase angles.

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<th>d_ρ^-</th>
<th>d_σ^-</th>
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This table summarizes the coefficients of the phase angle terms of Eqs. (83) and (84). The transitions are described in the same manner as in Table VI.
\[ \nu_n + \phi_n = \Omega_n + \sigma_n - \sum_{\tilde{m}=1}^{4} \tau_{nm} E_m \]

\[-\text{Re}\{ (\varphi_{nijkl} + \varphi_{nkjil}) \exp(-ip\cdot d\cdot \psi) \} E^{-1}_{ijk} \]

where \( i, j, \) and \( k \) are "modes" indices defined in Table 8. The coefficients are

\[ \alpha_n = F_1 \sum_{a',b'} \delta_{a',b'+p_n} |\varphi_{a'b'}|^2 Z_i [\gamma_{ab} + i(\omega_{a'b'} - \nu_n)]^{-\nu_n/Q_n}, \]  

\[ \sigma_n = F_1 \sum_{a',b'} \delta_{a',b'+p_n} |\varphi_{a'b'}|^2 Z_r [\gamma_{ab} + i(\omega_{a'b'} - \nu_n)], \]

\[ \theta_{nm} = 4F_3 \sum_{a',b'} \delta_{a',b'+p_n} |\varphi_{a'b'}|^4, \]

\[ \tau_{nm} = 0, \]

for \( nm = 12, 21, 34, 43; \)

\[ \theta_{nm} = (2F_3/\gamma) \sum_{a',b'} \delta_{a',b'+p_n} |\varphi_{a'b'}|^2 |\varphi_{a'-2p_n,b'}|^2 \gamma_a L(\delta_a) \]

\[ \{1 + (\gamma_b/\gamma_a)[1 - 2\delta_a^2/(\gamma_a \gamma_{ab})]L_a(2\delta_a)\} \]

+ same with \( \gamma_a \rightarrow \gamma_b, \delta_a \rightarrow \delta_b \) and \( \varphi_{a'-2p_n,b'} \rightarrow \varphi_{a',b'+2p_n}, \)
Table 8. The subscript definition for the phase angle term.

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This table defines the i, j, and k subscripts appearing in the phase angle term of Eqs. (83) and (84).
\[ \tau_{n\bar{m}} = \left(2F_{3}/\gamma\right) \sum_{a'} \sum_{b'} \delta_{a',b',+p_{n}} \left|\mathcal{O}_{a',b'}\right|^{2} \left|\mathcal{O}_{a'-2p_{n},b'}\right|^{2} \gamma_{a} \left(\delta_{a} \right) \left(p_{n}\delta_{a}/\gamma_{ab}\right) \]

\[ \times \left[1 + (1 + 2 \gamma_{ab}/\gamma_{a}) L_{a}(2\delta_{a})\right] \]

\[ + \text{same with } \gamma_{a}\leftrightarrow\gamma_{b}, \delta_{a}\leftrightarrow\delta_{b}, \text{ and } \mathcal{O}_{a'-2p_{n},b'} \rightarrow \mathcal{O}_{a',b'+2p_{n}} \]  

(89)

for \( n_{\bar{m}} = 13, 31, 24, 42; \)

\[ \theta_{n\bar{m}} = 4F_{3} \sum_{a'} \sum_{b'} \delta_{a',b',+p_{n}} \left|\mathcal{O}_{a',b'}\right|^{4} L \left[\omega_{a'b'} - (\nu_{n}^{+} + \nu_{m}^{+})\right] \]  

(90)

\[ \times \left[\omega_{a'b'} - (\nu_{n}^{+} + \nu_{m}^{+})\right] \gamma_{ab} \]

\[ \times [\omega_{a'b'} - (p_{n}\delta_{a}) - \nu_{n}^{+}] \]  

(91)

for \( nm = 23, 32, 14, 41; \)

\[ \theta_{n\bar{m}} = \left(2F_{3}/\gamma\right) \sum_{a'} \sum_{b'} \delta_{a',b',+p_{n}} \left|\mathcal{O}_{a',b'}\right|^{2} \left|\mathcal{O}_{a'-2p_{n},b'}\right|^{2} \gamma_{a} \]

\[ \times L \left[\omega_{a'b'} - (p_{n}\delta_{a}) - \nu_{n}^{+}\right] \]

\[ + \text{same with } \gamma_{a}\leftrightarrow\gamma_{b}, \delta_{a}\leftrightarrow\delta_{b}, \text{ and } \mathcal{O}_{a'-2p_{n},b'} \rightarrow \mathcal{O}_{a',b'+2p_{n}}, \]  

(92)
\[
\tau_{m} = (2F_{3}/\gamma) \sum_{a',b',+p_{n}} \delta_{a',b',+p_{n}} |\rho_{a',b',+p_{n}}|^2 |\rho_{a',-2p_{n},b',+p_{n}}|^2 \\
\times \gamma_{a}[\omega_{a',b',-}(p_{n} \delta_{a} - \nu_{n})]/\gamma_{ab} \\
\times L[\omega_{a',b',-}(p_{n} \delta_{a} - \nu_{n})] \\
+ \text{same with } \gamma_{a} \rightarrow \gamma_{b}, \delta_{a} \rightarrow \delta_{b}, \text{ and } \rho_{a',-2p_{n},b',+p_{n}} \rightarrow \rho_{a',b',+2p_{n}} 
\]

(93)

\[
\phi_{nijk} + \phi_{nki} = i(2F_{3}/\gamma) \sum_{a',b'} \delta_{a',b',+p_{n}} |\rho_{a',b',+p_{n}}|^2 \gamma_{a} \gamma_{b} \gamma_{ab} \\
\times D[\omega_{a',b',-}(\nu_{i} - \nu_{j} - \nu_{k})] \\
\times |\rho_{a',-2p_{n},b',+p_{n}}|^2 \rho_{a'} (2p_{n}) \\
+ \text{same with } \gamma_{a} \rightarrow \gamma_{b}, \delta_{a} \rightarrow \delta_{b}, \text{ and } \rho_{a',-2p_{n},b',+p_{n}} \rightarrow \rho_{a',b',+2p_{n}} 
\]

(94)

the relative phase angle

\[
\psi = (\nu_{1} - \nu_{2} - \nu_{3} + \nu_{4}) t + \phi_{1} - \phi_{2} - \phi_{3} + \phi_{4},
\]

(95)
\[ v = \frac{1}{4} \sum_{i=1}^{4} v_i, \]

\[ \gamma = \frac{1}{2} (\gamma_a + \gamma_b), \]

\[ F_1 = \nu N (\hbar K u o)^{-1}, \quad F_3 = \frac{1}{4} \pi \frac{1}{2} (\hbar^2 \gamma_a \gamma_b \gamma_{ab})^{-1} \gamma F_1, \quad (96) \]

\[ L_\alpha (\Delta \omega) = \gamma_\alpha^2 [\gamma_\alpha^2 + (\Delta \omega)^2]^{-1}, \quad \alpha = a, b, \]

\[ L (\Delta \omega) = \gamma_{ab}^2 [\gamma_{ab}^2 + (\Delta \omega)^2]^{-1} \quad (97) \]

\[ \omega_{a'b'} = \omega_o + (\mu_B/\hbar) H (g_a a' - g_b b') \quad (98) \]

\[ \delta_\alpha = (\mu_B/\hbar) H g_\alpha + \frac{1}{2} (\nu_+ - \nu_-), \quad \alpha = a, b, \quad (99) \]

where in the \( \delta_\alpha \) the \( \nu_+ (\nu_-) \) indicates the frequency which has \( p^+ = +1 \) (-1) e.q., in \( \theta_{12} \); \( \delta_\alpha = (\mu_B/\hbar) H g_\alpha + \frac{1}{2} (\nu_2 - \nu_1) \).

The frequency determining Eq. (84) may be recast in an equation of motion for \( \psi \)

\[ \dot{\psi} = a + b \sin \psi + c \cos \psi \quad (100) \]

where
The relative phase angle \( \psi \) vanishes in two special cases; if the traveling wave directions have the same frequency then \( \nu_1 = \nu_3 \) and \( \nu_2 = \nu_4 \); if the polarizations have the same frequency \( \nu_1 = \nu_2 \) and \( \nu_3 = \nu_4 \). Thus the phase angle may not be assumed to be rapidly varying allowing one to discard the term. In particular, the reduction to the standing wave, two mirror and the ring laser cases (Appendix E) would not succeed if the phase angle terms were dropped. In order to neglect the term, the frequency difference of both the two traveling waves and polarizations must be significant. In this decoupled approximation, Eq. (83) becomes

\[
I_n = 2I_n^\alpha_n - \sum_{m} \theta_{nm} I_m^\alpha_m,
\]

(104)
of Eq. (104) is given in Sargent et al. (1967), second paper, Chapter IV, and in O'Brien and Sargent (unpubl.). Of particular interest is the coupling between different "modes" described by a coupling parameter in Sargent et al. (1967)

\[ C_{nm} = (\theta_{nm} \theta_{nm}) / (\theta_{nn} \theta_{mm}) \]  

(105)

Examination of Eqs. (87) - (92) reveals that \( C_{nm} \) will decrease for increasing magnetic field strength due to terms such as \( L(\delta_a) \) and \( L(\omega_{ab} - \nu) \). Thus the maximum values of the coupling parameters may be obtained at zero magnetic field. In zero field we assume \( \delta_a \approx 0 \). Using Eq. (27) to calculate the matrix elements, we define
\[ M_1 = \sum_{a',b'} \sum_{\pm 1} \delta_{a',b',\pm 1} |\mathcal{G}_{a,b'}|^4 = \frac{1}{60} \mathcal{G}^4 \begin{cases} (J(J+1)(2J+1)(2J^2+2J+1) & \Delta J=0 \\ (J+1)(2J+1)(2J+3)(6J^2+12J+5) & \Delta J=\pm 1, \end{cases} \] (106)

\[ M_2 = 2 \sum_{a',b'} \sum_{\pm 1} \delta_{a',b',\pm 1} |\mathcal{G}_{a,b'}|^2 |\mathcal{G}_{a',b'}|^2 \begin{cases} J(J+1)(2J+1)(2J-1)(2J+3) & \Delta J=0 \\ (J+1)(J+2)(2J+1)(2J+3)(2J+5) & \Delta J=\pm 1, \end{cases} \] (107)

\[ M_3 = 2 \sum_{a',b'} \sum_{\pm 1} \delta_{a',b',\pm 1} |\mathcal{G}_{a,b'}|^2 |\mathcal{G}_{a',b'}|^2 = (1/60) \mathcal{G}^4 [J(J+1)(2J+1)(2J-1)(2J+3)] \Delta J=0, \pm 1, \] (108)

\[ M_4 = \sum_{a',b'} \sum_{\pm 1} \delta_{a',b',\pm 1} |\mathcal{G}_{a,b'}|^2 [|\mathcal{G}_{a',\pm 2,b'}|^2 + |\mathcal{G}_{a',b',\pm 2}|^2] = (1/60) \mathcal{G}^4 \begin{cases} J(J+1)(2J+1)(2J-1)(2J+3) & \Delta J=0 \\ (J+1)(2J+1)(2J+3)(2J^2+4J+5) & \Delta J=\pm 1, \end{cases} \] (109)
where $J = J_a$ and $J$ is defined in Chapter III. Substituting Eqs. (87) - (92) into Eq. (105), one has for the coupling between "modes" of the same direction and different polarization

$$C_{12} = C_{21} = C_{34} = C_{43} = \left(\frac{M_4}{M_1}\right)^2 = \begin{cases} \frac{(2J-1)(2J+3)}{(2J^2+2J+1)}^2 & \Delta J = 0 \\ \frac{(2J+4J+5)}{(6J+12J+5)}^2 & \Delta J = \pm 1, \end{cases} \quad (110)$$

same polarization but different direction

$$C_{13} = C_{31} = L^2[\omega_o - \frac{1}{2}(\nu_1 + \nu_3)],$$

$$C_{24} = C_{42} = L^2[\omega_o - \frac{1}{2}(\nu_2 + \nu_4)], \quad (111)$$

and different direction and polarization

$$C_{23} = C_{32} = [(\gamma_a M'' + \gamma_b M')/\tilde{\gamma}]^2 L(\omega_o - \nu_2) L(\omega_o - \nu_3),$$

$$C_{14} = C_{41} = [(\gamma_a M'' + \gamma_b M')/\tilde{\gamma}]^2 L(\omega_o - \nu_1) L(\omega_o - \nu_4), \quad (112)$$

where
These results agree with those of Fradkin and Khayutin (1970). The maximum coupling between "modes" with the same polarization but different direction, Eq. (111) is independent of J values and is always neutral (C=1) or weak (C<1). As shown in Figure 4, coupling between different polarizations of traveling waves of the same direction is always less than 1 for the \( \Delta J = \pm 1 \) transitions. However, the transitions for \( \Delta J = 0 \) lead to strong coupling for \( J > 1 \) in zero magnetic field. Figure 5 shows that the maximum coupling parameter, Eq. (112), for different directions and polarizations is always weak for both \( \Delta J = 0 \) and \( \Delta J = \pm 1 \) transitions.
Fig. 4. Coupling parameter versus J

The coupling parameter for waves traveling in the same direction but with opposite polarizations, Eq. (110), is plotted as a function of $J$ values where $J$ is the angular momentum of the upper level. The squares give values for $\Delta J = 0$ and the triangles for $\Delta J = \pm 1$. The values are taken at central tuning and for decay constants $\gamma_a = 18$ MHz and $\gamma_b = 40$ MHz.
Fig. 5. Coupling parameter vs. J for waves of opposite direction

In this figure, the coupling parameter for waves of opposite direction and polarization, Eq. (112), is plotted in the same manner as Fig. (4) for central tuning. Again, the decay constants are $\gamma_a = 18 \text{ MHz}$ and $\gamma_b = 40 \text{ MHz}$. 
CHAPTER X

UNIDIRECTIONAL, TWO WAVE OPERATION

We now consider two traveling waves in the same direction with opposite circular polarizations. The electric field takes the form

\[ E(s, t) = \frac{1}{2} (E_+ \hat{e}_+ \exp[-i(\nu_+ t + \phi_+)] + E_- \hat{e}_- \exp[-i(\nu_- t + \phi_-)]) \times \exp(iKs) + \text{c.c.} \]  

(115)

where we have dropped the \( \hat{n} \) subscript and use + (-) to indicate right (left) circularly polarized light. The relative phase angle, Eq. (95), encountered in Chapter IX vanishes in this case. The amplitude- and frequency-determining equations become

\[ \dot{E}_\pm = E_\pm (\alpha_\pm - \beta_\pm E_\pm^2 - \theta_+ E_+^2) \]  

(116)

\[ \nu_\pm + \dot{\phi}_\pm = \Omega_\pm + \sigma_\pm - \tau_\pm E_\pm^2 \]  

(117)

where

\[ \alpha_\pm = F_1 \sum_a \sum_b \delta_{a', b', \pm 1} |p_{a'b'}|^2 Z_i [\gamma_{ab} + i(\omega_{a'b'} - \nu_\pm)]^{-1/2} \nu/Q_\pm, \]  

(118)
\[ \beta \pm = 4F_3 \sum_{a', b'} \sum_{\delta_{a', b'} \pm 1} |\rho_{a', b'}|^{4}, \]  

(119)

\[ \sigma \pm = F_1 \sum_{a'} \sum_{b'} \delta_{a', b'} \pm 1 |\rho_{a', b'}|^{2} 2 \pi [\gamma_{a', b'}^{1} + i(\omega_{a', b'} - \nu_{\pm})], \]  

(120)

\[ \theta_{\pm} = (2F_3/\tau) \sum_{a'} \sum_{b'} \delta_{a', b'} \pm 1 |\rho_{a', b'}|^{2} |\rho_{a' \pm 2, b'}|^{2}, \]  

(121)

\[ \{\gamma_{a} L(\delta_{a}) + L_{a}(2\delta_{a}) [\gamma_{b}[1 - 2\delta_{a}^{2}/(\gamma_{a} \gamma_{ab})]L(\delta_{a}) \]  

\[ + [1 - \{(\pm 2\delta_{a})/(\gamma_{a} \gamma_{ab})\}][\omega_{a', b', -\nu_{\pm})]L(\omega_{a', b', -\nu_{\pm})] \} \]  

\[ + \text{same with } \gamma_{a} \leftrightarrow \gamma_{b}, \delta_{a} \leftrightarrow \delta_{b} \text{ and } \rho_{a' \pm 2, b'} \leftrightarrow \rho_{a', b' \pm 2}, \]  

\[ \tau_{\pm} = (2F_3/\gamma) \sum_{a'} \sum_{b'} \delta_{a', b'} \pm 1 |\rho_{a', b'}|^{2} |\rho_{a' \pm 2, b'}|^{2} \{ (\pm \delta_{a} \gamma_{b}/\gamma_{ab}) L(\delta_{a}) \]  

\[ + L \frac{(2\delta_{a})}{a} [(\pm \delta_{a})(2 + \gamma_{a} / \gamma_{ab}) L(\delta_{a}) \]  

\[ + \{(\pm 2\delta_{a}/\gamma_{ab}) + (\omega_{a', b', -\nu_{\pm}) / \gamma_{ab})]L(\omega_{a', b', -\nu_{\pm})] \} \]  

(122)

+ \text{same with } \gamma_{a} \leftrightarrow \gamma_{b}, \delta_{a} \leftrightarrow \delta_{b}, \text{ and } \rho_{a' \pm 2, b'} \leftrightarrow \rho_{a', b' \pm 2}, \]  

and the remaining constants are defined in Chapter IX. The equation similar to Eq. (104) may be solved in the same manner as a two-mode problem in the standing wave, two mirror laser, as in Sargent et al. (1967) and O'Bryan and Sargent (unpubl.). The coupling parameter
\[ C = \frac{\theta_{+-} \theta_{-+}}{\beta_{+-} \beta_{-+}} \]  

has a maximum value in zero field given by Eq. (11). As mentioned in Chapter IX, the coupling parameters exhibit a Lorentzian dependence on the magnetic field strength. This is illustrated in Figure 6 for central tuning and several J values. The dependence of the zero field value of C is given in Figure 4.
Fig. 6. Coupling parameter versus magnetic field strength

The Lorentzian dependence of the coupling parameter on the magnetic field is shown in this figure. In order of increasing maxima, the coupling parameter is plotted versus magnetic field strength for $J = 1\leftrightarrow 2$, $J = 0\leftrightarrow 1$, and $J = 2\leftrightarrow 2$, where $J$ is the angular momentum of the upper level. The dashed line represents neutral coupling and divides the graph into a strong-coupling region ($C>1$) and a weak-coupling region ($C<1$).
CHAPTER XI

DISCUSSION

A major portion of the work concerning the extension of the scalar theory to the ring laser configuration has been done by Aronowitz (1965, 1969, and 1970); and Aronowitz and Collins (1966, 1970). A more recent treatment has been given by Menegozzi and Lamb (n.d.) and O'Bryan and Sargent (unpubl.). The major difference between the treatments is the formulation of Maxwell's equations in a rotating frame. Aronowitz (1965) has made use of the electromagnetic field equations derived by Heer (1964) for a frame in uniform rotation. The present treatment and that of Menegozzi and Lamb (n.d.) employ a formulation similar to that of Irvine (1964) are incorrect, (Scully 1971). Former treatments have considered electric fields linearly polarized whereas the present work has considered arbitrary polarization and specifically two circular polarizations.

Extensive discussion of previous papers dealing with laser oscillators subject to dc magnetic fields is given by Sargent et al. (1967) in Chapter IX. The extension of the scalar theory to include a dc magnetic field, atoms with hyperfine structure and arbitrary angular momenta, and isotopic abundance given by Heer and Graft (1965) is general enough to describe the ring laser case given here. However,
they did not evaluate any of the coefficients in their amplitude and frequency determining equations (or in any special cases). They also assumed cavity losses to be isotropic, neglected backscattering, and used the field equations given by Heer (1964). Fradkin and Khayutin (1970) outlined a calculation similar to the presentation in Chapter IX. They obtained the coupling coefficients given in Eqs. (111) and (112). Their expressions differ in that they are of order $(\gamma_{ab}/Ku)^2$. This results from a higher order expansion of the plasma dispersion function in the "Doppler limit". The coupling coefficients derived here may be calculated exactly using the exact expressions for the $T_{tw}$ integrals given in Appendix B.

The present calculation gives the amplitude and frequency determining equations for the multimode operation of a ring laser subject to an axial dc magnetic field with possible anisotropic and backscattering losses and resonance and an active medium consisting of atoms which may have arbitrary angular momenta. Similar equations which include hyperfine structure, arbitrarily oriented magnetic field, and isotropic abundance have been given in an Appendix. The special case of bidirectional and unidirectional operation for a limited number of waves has also been given with graphs of coupling parameters versus $J$ value of the upper state and magnetic field strength.
APPENDIX A

ELECTROMAGNETIC FIELD EQUATIONS TO FIRST ORDER
IN |v/c| FOR A ROTATING FRAME

Maxwell's equations in a non-inertial, rotating frame, have been derived by Irvine (1964) using general relativity. Equation (2) may be obtained by reducing Eqs. (34) - (37) of Irvine (1964) to first order in |v/c|. It is illustrative, however, to derive Maxwell's equations from the special theory of relativity keeping only terms to order |v/c|. By using this approximation we may allow the rotation rate $\hat{\Omega}$ to be the same for each observer.

The electromagnetic field equations for an inertial observer (denoted by a superscript $^0$) are

$$\nabla^0 \cdot \vec{B} = 0, \tag{A1}$$

$$\nabla^0 \times \vec{E}^0 + \frac{\partial \vec{B}^0}{\partial t^0} = 0, \tag{A2}$$

$$\nabla^0 \cdot \vec{E}^0 = \rho^0/\varepsilon_0, \tag{A3}$$

$$\nabla^0 \times \vec{B}^0 - \mu_0 \varepsilon_0 \frac{\partial \vec{E}^0}{\partial t^0} = \mu_0 \vec{J}^0, \tag{A4}$$

where $(x^0, t^0)$ are the coordinates of the inertial observer. In the theory of Chapter II, we assume there are no free charges, but we
include the charge density $\rho$ here for generality. The coordinates are related by the four-vector transformation

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
  t
\end{pmatrix}
= S
\begin{pmatrix}
  x^0 \\
  y^0 \\
  z^0 \\
  t^0
\end{pmatrix}
\]

where the transformation matrix $S$ to first order in $|v/c|$ is given by

\[
S = \begin{pmatrix}
  \cos \Omega \ t^0 & \sin \Omega \ t^0 & 0 \\
  -\sin \Omega \ t^0 & \cos \Omega \ t^0 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]

Since $|v/c| < 1$, $dt = dt^0$ and difficulties in the definition of simultaneity do not arise. To first order in $|v/c|$ the electromagnetic fields obey the (special relativistic) transformations (Jackson 1962)

\[
\vec{E}^0 = (\vec{E} - \vec{v} \times \vec{B}), \quad \vec{B}^0 = (\vec{B} + \mu_0 \varepsilon_0 \vec{v} \times \vec{B}),
\]

where $\vec{v} = \vec{\Omega}_v \times \vec{r}$. Assuming the same transformation for the current and charge densities which govern the coordinates

\[
\rho^0 = \rho, \quad \vec{J}^0 = \vec{J} + \rho \vec{v}
\]
to first order in \(|v/c|\). Using the transformation matrix \(S\) of Eq. (A6) one may easily verify

\[
\varphi^0 = \varphi,
\]

(A9)

and from classical hydrodynamics

\[
(\partial/\partial t^0) = (\partial/\partial t) - \vec{V} \cdot \vec{V} + \vec{\Omega}_T \times \vec{r}.
\]

(A10)

In the substitution of Eqs. (A7) - (A10) into Eqs. (A1) - (A4) it is useful to note that since \(\vec{\Omega}_T\) is constant and perpendicular to the plane of the ring laser \((\vec{\Omega}_T = \Omega_T \hat{k} = \Omega_T \hat{k}^0)\) and \(\vec{V} = \vec{\Omega}_T \times \vec{r}\),

\[
\vec{\Omega}_T = \frac{1}{2} (\vec{V} \times \vec{V}),
\]

\[
(\vec{V} \cdot \vec{V}) = \vec{V} \cdot (\vec{\Omega}_T \times \vec{r}) = 0,
\]

\[
(\vec{\Omega}_T \times \vec{A}) = (\vec{A} \cdot \vec{V}) \vec{V},
\]

(A11)

where \(\vec{A}\) is any vector field. Performing the indicated substitutions and using identities (A11), one has to first order in \(|v/c|\)
\[ \nabla \cdot \left[ \mathbf{B} + \mu_0 \varepsilon_0 (\mathbf{\dot{v}} \times \mathbf{E}) \right] = 0, \]

\[ \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \left[ \mathbf{\dot{B}} + \mu_0 \varepsilon_0 (\mathbf{\dot{v}} \times \mathbf{E}) \right] = 0, \]

\[ \nabla \cdot \left[ \mathbf{E} - \mathbf{\dot{v}} \times \mathbf{B} \right] = \frac{\rho}{\varepsilon_0}, \]

\[ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 (\partial / \partial t) \left[ \mathbf{E} - \mathbf{\dot{v}} \times \mathbf{B} \right] = \mu_0 \mathbf{j}. \]  

\( (A12) \)
APPENDIX B

EXPANSION OF THE T = 1 BRANCH OF THE PERTURBATION TREE

Expressing the perturbation $V_{a'b'}$ as the sum of a right and left traveling wave, we write

$$V_{a'b'} = V_{a'b'}^r + V_{a'b'}^l.$$  \hfill (B1)

Using this form, each branch in Figure 3 yields eight sub-branches. Expanding the $t=1$ branch we obtain Figure B1. Employing the subscript associations of Chapter VI, using Eq. (46) and definition (48), the leftmost branch of Figure B1 yields

$$-(i/\hbar^3) W(v) \int \int \int \exp[-(\gamma_{a'b'} + i\omega_{a'b'})\tau'] \sum_{a''b''} \sum_{a'b'}^r$$

$$\times V_{a'b'}^r (s',t') \exp[-(\gamma_{a''a''} + i\omega_{a''a''})\tau''] V_{b''a''}(s'',t'')$$

$$\times \exp[-(\gamma_{a'b''} + i\omega_{a'b''})\tau''''] V_{a'b''}(s''',t''') N_{a'b''}(s,t).$$ \hfill (B2)

Noting the explicit form for $V_{a'b'}$ contains $\exp(iKs)$, using Eq. (47), interchanging the order of summation and integration, one has
Fig. B1. The perturbation tree expanded
\[-i(2\pi)^{-3} W(\nu) \sum_{\mu} \sum_{\rho} \sum_{\sigma} E \cdot E \cdot E \exp\left\{-i [\nu_{a^+} - \nu_{a^+} + \nu_{a^+}] t + \phi - \phi - \phi \right\}\]

\[\times \sum_{a''} \sum_{b''} \delta_{a'', b''} + p_{\mu} \delta_{b'', a''} - p_{\rho} \delta_{a'', b''} + p_{\sigma} \delta_{b'', a''} \frac{\mathcal{P}_{a''} b'' b''}{a''} \frac{\mathcal{P}_{a''} b'' b''}{a''} \frac{\mathcal{P}_{a''} b'' b''}{a''} \frac{\mathcal{P}_{a''} b'' b''}{a''} \]

\[\times \exp\left\{i (K_{\mu} - K_{\rho} + K_{\sigma}) s \right\} \int_{\tau'}^{\tau'''} \int_{\tau''}^{\tau''} \int_{\tau'''}^{\tau''} \exp\left\{-i K V (\tau'' + \tau''') \right\}\]

\[\times \exp\left\{-[\gamma_{a''} b'' + i (\omega_{a''} b'' - \nu_{a''} + \nu_{a''} - \nu_{a''})] (\tau') \right\}\]

\[\times \exp\left\{-[\gamma_{a''} a'' + i (\omega_{a''} a'' + \nu_{a''} - \nu_{a''})] (\tau'') \right\}\]

\[\times \exp\left\{-[\gamma_{a''} a'' + i (\omega_{a''} a'' - \nu_{a''})] (\tau''') \right\}, \quad (B3)\]

where for this branch \(d_{\mu} = d_{\rho} = d_{\sigma} = +1\) and \(s_1 = -s_2 = s_3 = + \) in agreement with Table 1. The remaining branches may be calculated in a similar manner.
APPENDIX C

INTEGRATION OF THIRD ORDER INTEGRALS

The third order integrals

\[
T_{tw} = iK^{-1}N_{tw} \int_{-\infty}^{\infty} dv \exp\left[-(v/u)^2\right] \int_{0}^{\infty} d\tau' \int_{0}^{\infty} d\tau'' \int_{0}^{\infty} d\tau^{'''}
\]

\[
\times \exp\left\{-[(v_1 + iC_1Kv)\tau']
\right.
\]

\[
+ (v_2 + iC_2Kv)\tau'' + (v_3 + iC_3Kv)\tau^{'''}\right\}
\]

(C1)

may be evaluated in terms of the plasma dispersion function of Eq. (53) using a simple partial fraction expansion. Here, for typographical simplicity, we use \(v_k\) for \(v_{tk}\) and \(C_k\) for \(C_{wk}\). The time integrations in Eq. (C1) may be performed immediately yielding

\[
T_{tw} = iK^{-1}N_{tw} \int_{-\infty}^{\infty} dv \exp\left[-(v/u)^2\right]\left[(v_1 + iC_1Kv)(v_2 + iC_2Kv)\right]
\]

\[
\times (v_3 + iC_3Kv)^{-1}.
\]

(C2)

Expanding the term in square brackets by partial fractions we find
\[ [ (v_1 + iC_1Kv) \ (v_2 + iC_2Kv) \ (v_3 + iC_3Kv) ]^{-1} \]

\[
= \left[ \frac{1}{C_1v_2 - C_2v_1} \right] \left\{ \begin{array}{c}
\left[ \frac{C_1}{C_1v_3 - C_3v_1} \left( v_1 + iC_1Kv \right) \right] \left[ \frac{-C_3}{v_3 + iC_3Kv} \right] \\
\left[ \frac{C_2}{C_2v_3 - C_3v_2} \left( v_2 + iC_2Kv \right) \right] \left[ \frac{-C_3}{v_3 + iC_3Kv} \right]
\end{array} \right\}
\]

\[ (C3) \]

Invoking definition (53) and using Eq. (C3), \( T_{tw} \) becomes

\[ T_{tw} = N_{tw} \left[ \frac{1}{C_1v_2 - C_2v_1} \right] \left\{ C \left[ \frac{Z(v_1/C_1) - Z(v_3/C_3)}{C_1v_3 - C_3v_1} \right] \\
- C_2 \left[ \frac{Z(v_2/C_2) - Z(v_3/C_3)}{C_2v_3 - C_3v_2} \right] \right\} (C4) \]

We may now substitute the definitions of \( C_{wk} \) from Table II into Eq. (C4):
\[ T_{t1} = (N_{t1}/u_2) \frac{[Z(u_1) + Z(u_3)]}{(u_1 + u_3)}, \tag{C5} \]

\[ T_{t2} = (N_{t2}/u_3) \frac{[Z(u_1) - Z(u_3)]}{(u_3 - u_1)}, \tag{C6} \]

and

\[ T_{t3} = \frac{1}{2} \left[ \frac{N_{t3}}{u_1 - i\omega_2} \right] \left[ \frac{Z(u_3) - Z(u_1)}{u_3 - u_1} - \frac{Z(u_3) - Z(i\omega_2)}{u_3 - i\omega_2} \right]. \tag{C7} \]

If \( v_1 = v_3 \), both the numerators and denominators of the terms involving \( (u_1 - u_3) \) in Eqs. (C6) and (C7) approach zero. In this limit, these terms become derivatives with respect to \( u_1 \) yielding

\[ T_{t2} = (N_{t2}/u_3) \frac{dZ(u_1)}{du_1} \tag{C8} \]

and

\[ T_{t3} = N_{t3}(u_1 - i\omega_2)^{-1} \left[ \frac{dZ(u_1)}{dZ_1} - \frac{Z(u_1) - Z(i\omega_2)}{u_3 - i\omega_2} \right]. \tag{C9} \]

As the atomic velocity \( u \to 0 \), \( Z(u) \to i(Ku/u) \), and (C5), (C6), and (C7) reduce to

\[ T_{tw} \to iKuN_{tw}/(u_1 u_2 u_3). \tag{C10} \]
In the case of large Doppler broadening where $K_\nu >> \text{decay rates and beat frequencies}$, $Z(\nu) \to i\pi^{1/2}$ and $T_{tw}$ reduces to the "strong Doppler limit"

\[ T_{t1} \to 2i\pi^{1/2}N_{t1}[u_2(u_1+u_3)]^{-1}, \quad T_{t2} \approx T_{t3} \approx 0. \]  

(C11)
APPENDIX D

GENERAL AMPLITUDE AND FREQUENCY EQUATIONS

To consider general amplitude-and frequency-determining equations involving an arbitrarily oriented magnetic field, isotopes, general basis vectors, and hyperfine structure, we follow the treatment of Sargent et al. (1967). As discussed in Chapter II of Sargent et al. (1967), we assume the jth isotope of the active medium has nuclear spin $I_j$, most probable $s$ component of velocity $u_j$ and a fractional abundance $a_j$ properly normalized such that

$$\sum_j a_j = 1 \quad \text{(D1)}$$

where $j$ indexes the isotopes present. We also assume the magnetic field does not break down the coupling between $I_j$ and $J$ ($F$ is a good quantum number). In the derivation of the perturbation energy $V_{a'b'}$, we introduce direction cosines $f_q(p_{\mu})$ given by

$$f_{\pm 1}(p_{\mu}) = \hat{e}(p_{\mu}) \cdot (\hat{i} \mp \hat{j})', \quad f_0(p_{\mu}) = \hat{e}(p_{\mu}) \cdot \hat{k}', \quad \text{(D2)}$$

where as in our treatment, $p_{\mu}$ indexes polarizations, $\hat{e}(p_{\mu})$ are the general basis and $j'$ and $k'$ are unit vectors along the $s$ and $y$ directions of the magnetic field (see Figure 1, Sargent et al. 1967). The matrix elements are given by Eq. (27) with $J_a$ and $J_b$ replaced by $F_a$ and
The general amplitude-and frequency-determining equations for a Zeeman ring laser corresponding to those given by Sar- 
gent et al. (1967) become

\[
\hat{E}_n = \text{Im}\left\{ \sum_{\mu} \hat{\alpha}_{n\mu} \hat{E}_\mu \exp(i\psi_{n\mu}) - \sum_{\mu} \sum_{\rho} \sum_{\sigma} \hat{\gamma}_{n\mu\rho\sigma} \hat{E}_\mu \hat{E}_\rho \exp(i\psi_{n\mu\rho\sigma}) \right\}
\]

\[
\nu_n + \mu_n = \Omega_n + E_n^{-1} \text{Re}\left\{ \sum_{\mu} \hat{\alpha}_{n\mu} \hat{E}_\mu \exp(i\psi_{n\mu}) \right\}
\]

\[
- \sum_{\mu} \sum_{\rho} \sum_{\sigma} \hat{\gamma}_{n\mu\rho\sigma} \hat{E}_\mu \hat{E}_\rho \exp(i\psi_{n\mu\rho\sigma}) \}
\]

where,

\[
\hat{\alpha}_{n\mu} = \frac{i}{\sqrt{2}} v_{g_{n\mu}} + \nu(nK_e)_{-1} \sum_{j} (a_j^* / u_j) \sum_{k=-1}^{1} f_k(p_{n\mu})^* f_k(p_{\mu})
\]

\[
\times \sum_{a} \sum_{b} \delta_{a',b'+k} \left| \omega_{a'b'} \right|^2 N_{a',b'} \sum_{k} (r_{a'b'}(\omega_{a'b'} \nu_{a'b'}))^* Z \left[ r_{a'b'}(\omega_{a'b'} \nu_{a'b'} \mu) \right],
\]

and
\[
\mathcal{J}_{\text{npq}} = v(n^2 K e_0)^{-1} \sum_j (a_j / u_j) \sum_{j=-1}^{1} f_{p_j}^* \sum_{k=-1}^{1} f_{p_k} \sum_{q=-1}^{1} f_{q_j} \sum_{r=-1}^{1} f_{q_r}^* \frac{1}{x} \sum_{s=-1}^{1} f_{r_s} (p_{\tilde{q}})^P
\]

\[
\times \sum_{n} \sum_{b' b'' a'a''} \delta_{a', b' + k} \delta_{a'', b' + q} \delta_{b'', a'' - r} \delta_{a', b'' + s}
\]

\[
\times \mathcal{P}_{b' a'} \mathcal{P}_{a'' b'} \mathcal{P}_{b'' a''} \mathcal{P}_{a' b''} \sum_{t=1}^{4} \mathcal{T}_{t,w} (d_{\mu}, d_{\rho}, d_{\delta}, t).
\]

(D5)

Tables II through IV may still be used to determine the \( \mathcal{T}_{t,w} \) provided one now subscripts the resonant frequencies \( \omega_{a'b'} \), \( K_{\tilde{n'}} \), and matrix elements \( \mathcal{P}_{a'b'} \) with the isotope index \( j \), which we have refrained from including for typographical simplicity. The procedure for expressing the coefficients in terms of the relative excitation may be found in Chapter V of Sargent et al. (1967).
APPENDIX E

REDUCTION TO STANDING WAVE, TWO MIRROR AND SCALAR RING LASERS

The coefficients for the standing wave, two mirror case may be obtained from those derived in Chapter IX by the proper reduction of the electric field Eq. (78) to the electric field for the two mirror case. If we allow

\[ K_1 = K_2 = K \]

\[ \nu_2 = \nu_4 = \nu_+; \nu_1 = \nu_3 = \nu_- \]

\[ E_2 = E_4 = \frac{i\nu}{2} E_+; E_1 = E_3 = \frac{i\nu}{2} E_- \]

\[ \phi_1 = \phi_2 = \phi_- - \frac{\pi}{2}; \phi_3 = \phi_4 = \phi_+ + \frac{\pi}{2} \]

where (+) indicates right (left) circularly polarized light, we obtain the electric field given by Sargent et al. (1967)

\[ \hat{E}(s,t) = i\frac{\nu}{2}[E_+ \hat{e}_+ \exp[-i(\nu_+ t + \varphi_+)] + E_- \hat{e}_- \exp[-i(\nu_- t + \varphi_-)] \]

\[ \times \sin(Ks) + \text{c.c.} \]

(E5)
The amplitude-and frequency-determining equations for this case become

\[
E_\pm = E_\pm (a_\pm - \beta_\pm E_\pm^2 - \theta_\pm E_\pm^2), \tag{E6}
\]

\[
\nu_\pm + i \phi_\pm = \Omega_\pm + \sigma_\pm + \rho_\pm E_\pm^2 + \tau_\pm E_\pm^2, \tag{E7}
\]

where

\[
a_\pm = \frac{1}{2} (\alpha_2 + \alpha_4) = \frac{1}{2} \sum_{a',b'} \sum_{\pm} \delta_{a',b',\pm 1} \left| \phi_{a',b'} \right|^2 \sum_{j=1}^{2} \frac{1}{2} \left[ \delta_{a,b} \pm i \left( \omega_{a',b'} - \nu_\pm \right) \right] - \frac{1}{2} \nu_\pm, \tag{E8}
\]

\[
\sigma_\pm = \frac{1}{2} (\sigma_2 + \sigma_4) = \frac{1}{2} \sum_{a',b'} \sum_{\pm} \delta_{a',b',\pm 1} \left| \phi_{a',b'} \right|^2 \frac{1}{2} \sum_{\pm} \left[ \gamma_{a,b} \pm i \left( \omega_{a,b} - \nu_\pm \right) \right], \tag{E9}
\]

\[
\beta_\pm = \frac{1}{8} \left( \theta_{22} + \theta_{44} + \theta_{24} + \theta_{42} \right) \frac{1}{33} \frac{1}{13} \frac{31}{11}
\]

\[
= \frac{1}{2} \sum_{a',b'} \sum_{\pm} \delta_{a',b',\pm 1} \left| \phi_{a',b'} \right|^4 \left[ 1 + L \left( \omega_{a',b'} - \nu_\pm \right) \right], \tag{E10}
\]
\[ \theta_{\pm} = \frac{1}{8} \left[ \theta_{21} + \theta_{23} + \theta_{41} + \theta_{43} + \text{Im}(\theta_{2134} + \theta_{2431} + \theta_{4213} + \theta_{4312}) \right] \]

\[ = \left( \frac{2F_3}{\gamma} \right) \sum \sum \delta_{a',b'} \pm 1 \left| \phi_{a'b'} \right|^2 \left| \phi_{a'2,b'} \right|^2. \]

\[ [\gamma_a L(\delta_a) + L(\omega_{a'b'} - (\pm \delta_a) - \nu_{\pm})] + \gamma_b L(a(2\delta_a) \times [(1 - 2\delta_{a'})/(\gamma_a \gamma_{ab})] L(\delta_a) + [1 - (\pm \delta_{a'})/(\gamma_a \gamma_{ab})] (\omega_{a'b'} - \nu_{\pm}) \]

\[ \times L(\omega_{a'b'}, -\nu_{\pm}) ] \]

+ same with \( \gamma_a \leftrightarrow \gamma_b, \delta_a \rightarrow \delta_b, \) and \( \phi_{a'2,b'} \rightarrow \phi_{a',b'} \pm 2, \) (E11)

\[ \rho_{\pm} = \frac{1}{8} (\tau_{24} + \tau_{42}) = \left( \frac{2F_3}{\gamma} \right) \sum \sum \delta_{a',b'} \pm 1 \left| \phi_{a'b'} \right|^2 [(\omega_{a'b'} - \nu_{\pm})/\gamma_{ab}] \]

\[ \times L(\omega_{a'b'}, -\nu_{\pm}), \] (E12)

\[ \tau_{-+} = \frac{1}{8} (\tau_{21} + \tau_{23} + \tau_{41} + \tau_{43} + \text{Re}(\theta_{2134} + \theta_{2431} + \theta_{4213} + \theta_{4312}) \]

\[ = \left( \frac{2F_3}{\gamma} \right) \sum \sum \delta_{a',b'} \pm 1 \left| \phi_{a'b'} \right|^2 \left| \phi_{a'2,b'} \right|^2. \]
\[
\begin{align*}
\left[ \gamma_a / \gamma_{ab} \right] \{ (\pm \delta_a) L(\delta_a) + (\omega_{a'b'} - (\pm \delta_a) - \nu_\pm) L(\omega_{a'b'} - (\pm \delta_a) - \nu_\pm) \} \\
+ L(2 \gamma_a) \{ (\pm \delta_a / \gamma_{ab}) (\gamma_a + 2 \gamma_{ab}) L(\delta_a) + \gamma_b \left[ (\pm 2 \delta_a / \gamma_a) + ((\omega_{a'b'} - \nu_\pm) / \gamma_{ab}) \right] \\
\times L(\omega_{a'b'} - \nu_\pm) \} + \text{same with } \gamma_a \leftrightarrow \gamma_b, \delta_a \rightarrow \delta_b, \text{ and } \phi_{a'+2,b'} \rightarrow \phi_{a',b'\pm2},
\end{align*}
\]

(E13)

and the remaining constants are defined in Chapter IX. Since the relative phase angle Eq. (95) vanishes in this reduction, the \(\exp(\pm i\psi)\) becomes unity and the term contributes to other coefficients. If these terms had been dropped, the reduction to the standing wave, two mirror case would have failed to agree with the results of Sargent et al. (1967) paper II.

The equations for the scalar ring laser (two waves traveling opposite directions with the same linear polarization) may also be obtained from Chapter IX. We allow

\[
\begin{align*}
v_1 = v_2 = v_r; \quad v_3 = v_4 = v_\ell, \\
E_1 = E_2 = \frac{1}{\sqrt{2}} E_r; \quad E_3 = E_4 = \frac{1}{\sqrt{2}} E_\ell, \\
\phi_1 = \phi_2 = \phi_r; \quad \phi_3 = \phi_4 = \phi_\ell,
\end{align*}
\]

(E14)  
(E15)  
(E16)
where \( r (\ell) \) denotes the right (left) traveling wave. The electric field is then polarized along the \( \hat{i} \) axis and is of the form

\[
E(s,t) = \frac{1}{\ell}[E_R \exp[-i(\nu_R t + \phi_R)] \exp(iK_R s) + E_L \exp[-i(\nu_L t + \phi_L)]\exp(-iK_L s)] + \text{c.c.}
\]

(E17)

The frequency and amplitude determining equations become

\[
\begin{align*}
\frac{\dot{E}_R}{E_R} & = \frac{\nu_R - \sigma_R}{\ell} E_R^2, & (E18) \\
\nu_R + \dot{\phi}_R & = \Omega_R + \sigma_R - \tau_R \frac{E_L^2}{E_R^2} & (E19)
\end{align*}
\]

\[
\begin{align*}
\alpha_R & = \frac{1}{\ell} (\alpha_1 + \alpha_2) = (F_1/\sqrt{2}) \frac{M_0}{Z_i} \gamma_{ab} + i(\omega_{ab} - \nu_R), & (E20) \\
\beta_R & = \frac{1}{2\sqrt{2}} (\theta_{11} + \theta_{22} + \theta_{12} + \theta_{21}) = 6\sqrt{2} F_3 M_1, & (E21)
\end{align*}
\]

\[
\begin{align*}
\frac{\dot{\theta}_R}{\ell} & = \frac{1}{\ell} \left[ \frac{\theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \text{Im} (\theta_{1243} + \theta_{1342} + \theta_{2134} + \theta_{2431})}{31413242 + 32413242 + 34213142 + 43124213} \right] + \beta_R L(\omega_{ab} - \nu_o), & (E22)
\end{align*}
\]

(E23)
\[
\tau_{\ell \ell} = \frac{1}{2\sqrt{2}} \left[ \tau_{13} + \tau_{14} + \tau_{23} + \tau_{24} + \text{Re} \left( \begin{array}{c} \rho_{123} + \rho_{134} + \rho_{2134} + \rho_{2413} \\ \rho_{3421} + \rho_{3124} + \rho_{4312} + \rho_{4213} \end{array} \right) \right]
\]

\[
= \frac{\beta_r}{\ell} \left[ (\omega_{ab} - \nu_0) / \gamma_{ab} \right] L(\omega_{ab} - \nu_0),
\]

(E24)

where

\[
\nu_0 = \frac{1}{2}(\nu_r + \nu_\ell),
\]

(E25)

\[
M_0 = \sum_{a'} \sum_{b'} \delta_{a',b',\pm 1} |\rho_{a'b'}|^2 = \frac{1}{6} \varphi^2 \left\{ \begin{array}{l} J(J+1)(2J+1) \Delta J = 0 \\ (J+1)(2J+1)(2J+3) \Delta J = \pm 1, \end{array} \right. \]

(E26)

\(\omega_{ab}\) is the atomic frequency, and \(M\) is defined by Eq. (106). The remaining constants are defined in Chapter IX.

As in the previous case the relative phase angle, Eq. (95) is zero and the phase angle terms contribute to the cross-saturation terms. The coupling between the two traveling waves is characterized by the coupling parameter

\[
C = (\theta_{\ell R} \theta_{\ell R}) / (\beta_r \beta_\ell) = L^2(\omega_{ab} - \nu_0),
\]

(E27)

which agrees with Eq. (111). These results agree with those obtained by O'Bryan and Sargent (unpubl.).
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