APPLICATION OF CHANCE-CONSTRAINED PROGRAMMING TO THE MULTI-PERIOD CAPITAL BUDGETING PROBLEM UNDER RISK

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF FINANCE, INSURANCE AND REAL ESTATE

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE
WITH A MAJOR IN FINANCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

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STATEMENT BY AUTHOR

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ACKNOWLEDGMENTS

This thesis has benefited greatly from its director, Dr. John T. Emery, who tirelessly read its various drafts, helping to improve its ideas and style. For all his assistance and encouragement I am deeply grateful.

Thanks are also due to Dr. James E. Wert and Dr. Timothy L. Shaftel, second and third readers of this work, for their comments, questions and observations.

I have benefited greatly from my friendship with Ambrosio Goicoechea whose stimulation has led me to this subject and whose completed computer program has saved many hours of work for me.

Special thanks are due to Laurie, my wife, not only for her typing, editing and proof-reading, but for her patience, continuous encouragement and support that has made the completion of this work possible.

Although many people have helped with this thesis, I am fully responsible for it, especially any possible errors.

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ABSTRACT

This study will attempt to bring to bear chance-constrained programming to the multi-period capital budgeting problem under risk. The problem to be studied can be structured as a linear programming problem with stochastic coefficients; it can be expressed as follows:

$$\text{Maximize: } \sum_{j=1}^{n} E(b_j)x_j$$

$$\text{Subject to: } Pr\left(\sum_{j=1}^{n} c_{tj}x_j \leq C_t\right) \geq 1 - a_t$$

$$0 \leq x_j \leq 1 \quad (j = 1,2, \ldots, n)$$

$$0 \leq a_t \leq 1 \quad (t = 1,2, \ldots, T)$$

where $E(b_j)$ = the expected net present value of the j^{th} project,

Pr = probability,

x_j = the fraction of the j^{th} project undertaken,

c_{tj} = the present value of the outlay for the j^{th} project in the year t. It is a normally distributed random variable,

C_t = the present value of the budget in the year t, known with certainty,

$1-a_t$ = the prescribed confidence levels desired for each period, as stated, this linear problem has stochastic coefficients and cannot be solved directly with standard linear programming techniques. Chance-constrained programming can

be used to transform this stochastic linear problem to deterministic.

The resulting deterministic problem has non-linear constraints. Approximation will be used to change the non-linear constraints to linear. The resulting deterministic linear problem can be solved by applying the simplex method.

CHAPTER 1

INTRODUCTION

Although the origins of capital budgeting can be traced to the work of Irving Fisher,[1] Joel Dean was the first to articulate the capital budgeting problem in its current form.[2] He recommends approaching the capital budget through the application of the internal rate of return criterion. This procedure ranks projects in decreasing order of desirability. He argues that the standard for establishing a minimum acceptable rate of return should be based on the future average cost of capital.[3]

Dean's procedures were not subject to serious criticism until Lorie and Savage demonstrated three classes of capital budgeting problems where the internal rate of return criterion can lead to inappropriate decisions.

1. When the projects are not independent.
2. When there are limitations in the sum to be expended for more than one period.

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1. Irving Fisher, The Theory of Interest (New York: Macmillan and Company, 1930).

2. Joel Dean, Capital Budgeting (New York: Columbia University Press, 1951).

3. Ibid., pp. 62-81.

3. When the cash flows generated by the projects alter from positive to negative in different periods.[4]

Lorie and Savage, however, were able to drive alternatives to Dean's system for the first and third cases. No effective approach to the second case was devised until Weingartner applied linear programming to the problem of capital budgeting under credit rationing.[5]

Even though an extensive system of decision rules can be established with the procedures developed by these authors, all of the techniques require the assumption of deterministic variables. Since the capital budgeting problem involves future events and since clairvoyance is not a reliable forecasting procedure, the variables used in capital budgeting are stochastic by nature. One of the most promising techniques for handling capital budgeting under uncertainty is the application of chance-constrained programming. Naslund applied chance-constrained programming to the study of the multi-period capital budgeting with stochastic variables.[6]

Unfortunately chance-constrained programming has not received widespread acceptance as a feasible procedure for handling capital budgeting problems. The objective of this thesis is to develop the

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4. James H. Lorie and Leonard J. Savage, "Three Problems in Rationing Capital," Journal of Business, 28 (October, 1955), 229-239.

5. H. Martin Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems (Englewood Cliffs, N.J.: Prentice-Hall, 1963).

6. Bertil Naslund, "A Model of Capital Budgeting Under Risk," The Journal of Business, 39 (April, 1966), 257-271.

application of chance-constrained programming to the capital budgeting problem under risk. The material will be presented in a manner that only requires a basic knowledge of finance and linear programming.

To develop this model only a few primary assumptions are necessary. First both the investor and the decision-maker will be regarded as though they were the same. The distinction between risk and uncertainty is not relevant to the application considered here. For these reasons, investor will be used interchangeably with decision-maker and likewise risk and uncertainty. Also, it is assumed that the maximization of the net benefit is the only objective of the investor. The net benefit is defined as the difference between cash inflows (costs) and cash outflows (benefits or revenues). Only cases where the costs and revenues are measureable will be studied.

This paper is divided in three parts. The first examines the act of investing and addresses issues such as: What are the investor's motivations? What is the time value of the money? How should the cash flows generated by projects be measured and evaluated?

The second part is Chapters 3 and 4 in which the assumption that the outcomes (budgets, expenditures and the benefit of the projects) are known with certainty and can be expressed with deterministic variables is maintained. In Chapter 3 it is assumed that the investor has adequate capital to accept any project that meets the minimum acceptable standard for acceptance. To handle this problem, the net present value method is chosen. This method is also used to rank mutually exclusive projects. The capital rationing problem, in which the investor must operate within a fixed budget, is introduced in Chapter 4. The profitability index is selected as the way to study the single-period capital rationing. Linear programming is then used when the capital rationing is for more than one period and the projects are divisible. Integer programming is reserved for the study of multi-period capital rationing when the projects are indivisible.

In the third part, Chapters 5 and 6, the assumption that the outcome is known with certainty is relaxed. Chapter 5 will have a reference to stochastic linear programming and linear programming under uncertainty to study the multi-period capital budgeting when the multivariate probability of the outcome is known. In the same chapter, chance-constrained programming is applied to study the multi-period capital budgeting when only the budget or the cash outflows are normally distributed random variables.

Chapter 6 is an application of chance-constrained programming to the Lorie and Savage problem[7] when the expenditures or the budgets are normally distributed random variables.

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7. Lorie and Savage, "Three Problems in Rationing Capital," p. 234.

CHAPTER 2

PRELIMINARY ASPECTS OF CAPITAL BUDGETING

In this chapter, capital budgeting will be defined and its composition analyzed. The estimation and importance of cash flows in the capital budgeting decision is then examined. The concept of the time value of money is necessary for comparing cash flows at different points in time. Investor's motivations for investing also are brought forth in this chapter.

Capital budgeting refers to the allocation of capital for investment projects which are expected to be productive for a long period of time. Arbitrarially, a long period of time is defined as a time horizon with a duration greater than one year. When the allocation of capital extends less than one year, the typical procedure is to define these funds as working capital. The treatment of working capital is beyond the scope of this work.

The capital budgeting decision once made and implemented is often irreversible or only can be reversed at substantial cost. Although investments in productive equipment and capital may have a ready market, the price received from the sale of capital assets may represent only a fraction of their underlying value.

Since the consequences of the capital budgeting decision extends into the future, it is, by its nature, surrounded with uncertainty.

Motivation for Investing

A widely accepted model for investing originates in the economics of resource allocation over time. The assumed objective is to maximize expected utility over time. In the neoclassical model utility is viewed as resulting from consumption. An individual's utility function is thought of as being unique. The weights assigned to various consumption possibilities can vary and agreement among individuals need not be realized. For instance, a goal such as "keeping up with the Jones" may be as logical to one individual as it is illogical to another.

Although interpersonal utility comparisons are not allowed within the neoclassical framework, governmental agencies and various eleemosynary institutions frequently act to maximize a perceived utility function. Usually such social utility functions are meant to maximize the public benefit or the social good. For business firms, however, stockholders, when viewed as a group, receive the greatest benefit through the joint maximization of stock price appreciation and individual income. This is most easily achieved with profit maximization over time. The process of capital budgeting is meant to provide a logical and reproducible system for allocating capital in a manner which will allow the maximization of profit.

Van Horne provides the following definition stating: "Capital budgeting involves the generation of investment proposals; the estimate of cash flows for the proposals; the evaluation of proposals; the selection of projects based upon an acceptance criterion; and, finally, the continual reevaluation of investment projects after their acceptance."[1] The first and last aspects, the generation of investment proposal and the continual reevaluation of investment projects after their acceptance, are beyond the focus of this work. This chapter explores the estimate of cash flows. The evaluation of proposals and the selection of projects based upon an acceptance criterion are the subjects of Chapters 4, 5 and 6.

The expected benefit of a project should be expressed in terms of cash flows. As Peterson stresses, the investment evaluation should be based on cash flows since the firm cannot use the profits of one project for purposes of reinvestment, servicing of financial obligations, or payment of dividends if they are not converted to cash.[2] Along the same lines, Van Horne states that the firm invests cash hoping to receive cash returns in a greater amount in the future. He affirms this position with the statement, "Cash, not income, is what is important in capital budgeting."[3]

The Time Value of Money

To arrive at a systematic methodology for selecting among alternative use of scarce resources, it is necessary to devise standardizing procedures which allow the direct comparison of expenditures and

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1. James Van Horne, Financial Management and Policy (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1971), p. 47.

2. D. E. Peterson, A Quantitative Framework for Financial Management (Homewood, Illinois: Richard D. Irwin, Inc., 1969), p. 335.

3. Van Horne, Financial Management and Policy, p. 48.

benefits at different points in time. This methodology should be based on the opportunity costs of the resources and provide a method for equating cash flows at different points in time. The commonly accepted methodology for calculating the time value of money is presented here.

The time value of money (V) is a function of two variables $V = f(Q,t)$. Q represents quantity and t time. In this function $\frac{\delta f}{\delta Q} > 0$ and $\frac{\delta f}{\delta t} < 0$, in other words, for a given moment a bigger benefit is better than a smaller one and for a given quantity it is better to receive it sooner than later.[4] When two benefits are composed of different quantities and received in two different moments, it is necessary to have a financial law that would make them comparable. The interest rate will allow us the necessary comparison. The term "interest rate" will be used, assuming that it is only one, fixed, known and represented by (r).[5]

In the comparison of two quantities of capital which are expected at different moments of time (Q_1,t_1) and (Q_2,t_2), there are four variables (Q_1, Q_2, t_1, t_2). Using the interest rate as the equating mechanism it is only necessary to know three of these variables. From this base, it

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4. For a more extensive treatment of this, see: Lorenzo Gil Pelaez, Matematica de las Operaciones Financieras, Fasciculo I (Madrid: Copigraf, S.L., 1964), pp. 1-15.

5. There is, of course, a structure of interest rather than a unique rate. "There is no such thing as 'the' interest rate which keeps 'the' entrepreneurs from expanding, unless we assume very stable conditions in which there is no reason for any entrepreneur to think that the rates will change" F. A. Lutz, "The Structure of Interest Rates," Quarterly Journal of Economics, 55 (1940), p. 60.

is possible to develop a number of useful applications. Some applications of the interest rate are presented below.

1. What would the future value be of our present sum of money (S) after n years?

After one year there will be the same amount plus the interest accrued.

$$W_1 = S + rS = S(1+r)$$

In the second year the capital (W_1) will have accrued interest.

$$W_2 = W_1(1+r) = S(1+r)(1+r) = S(1+r)^2$$

In general, the value of the sum (S) after n years is

$$W_n = S(1+r)^n \qquad (2\text{-}1)$$

The process of finding the future value is normally known as capitalizing and the interest rate (r) employed for this process receives the name of rate of capitalization.

2. What is the present value (V_o) of an amount (Q) to be received in n periods from now?

From (2-1) it can be shown:

$$Q = V_o(1+r)^n$$

or

$$V_o = Q(1+r)^{-n} \qquad (2\text{-}2)$$

3. What is the present value (V_o) of an amount (Q) to be received during n periods, annuity, at the end of each period?

The problem can be stated:

$$V_o = Q(1+r)^{-1} + Q(1+r)^{-2} + \ldots + Q(1+r)^{-n};$$

factoring Q:

$$V_o = Q[(1+r)^{-1} + (1+r)^{-2} + \ldots + (1+r)^{-n}]$$

Since the expression between brackets [ ] is a geometric progression, its value is

$$\frac{1-(1+r)^{-n}}{r}$$

or

$$V_o = Q\frac{1-(1+r)^{-n}}{r} \qquad (2\text{-}3)$$

4. If the annuity is perpetual, $(1+r)^{-n}$ approaches zero as n approaches infinite. Integrating:

$$V_o = \frac{Q}{r} \qquad (2\text{-}4)$$

The process of finding the present value is normally known as <u>discounting</u>, and the interest rate (r) employed for this process receives the name of <u>discount rate</u>.[6]

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6. "We shall use the expression 'discount rate' instead of 'interest rate' whenever we shall be concerned with economic calculations rather than about financial transactions. A borrower and a lender agree on an interest rate, while decision makers use a discount rate." Pierre Masse, Optimal Investment Decisions (Englewood Cliffs: N.J.: Prentice-Hall, Inc., 1962), p. 14.

Estimating Cash Flows

To judge the attractiveness of projects, it is necessary to have estimates of expected cash flows. For the capital budgeting process to be effective, an efficient procedure must be developed to estimate the cash flows generated by the projects. A number of the important considerations for the analysis of cash flows are presented below. This particular list was originally given by Quirin.[7]

1. All direct and indirect benefits and costs generated by the project should be taken into account.
2. The cash flows should be considered on an incremental basis. For example, if the investor has to decide whether to accept or reject a project, the increment will represent the difference between cash flows if the project is accepted and if it is rejected.
3. The benefits and costs for the same project should be counted independently rather than taking their net difference. For example, Project A has an initial cost of $100 and an annual operating cost of $10. Its expected annual benefits are $60. Project B has an initial cost of $100 and an annual operating cost of $30. Its expected annual benefits are $80. The life for both projects is five years. The discount rate used is five percent. If the net is employed, the present value of the cash flows generated by each project is $116.48. The investor

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7. G. David Quirin, The Capital Expenditure Decision (Homewood, Illinois: Richard D. Irwin, Inc., 1967), pp. 59-94.

would be indifferent as to whether to accept Project A or B. But if the investor must operate within a fixed budget, the present value of future expenditures is $143.29 for Project A and $229.88 for Project B.[8] In this case, he would prefer Project A over B.

4. Double counting should be avoided. Probably the most common error in this respect is to charge the interest on the investment as an expense. The interest factor is taken into consideration by the use of the rate of discount, if interest on the investment also is charged as an expense. Double counting will result.

5. Opportunity cost is relevant. For example, the cash outflows for the students is not only their tuition but also the foregone income they would have earned if they were employed.

6. The appropriate period to be considered is the life of the project. When this life is very long (arbitrarily 50 years), it is frequently not practical to take into account the cash flows in the last years. For example, at a rate of discount of seven percent, a $10,000 benefit to be received in 50 years is worth today $339.48.

7. The changes in the price levels should be considered. The future cash flows should be units with purchasing power equivalent to the current value since in the investment, current units of money are going to be committed.

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8. The treatment of capital rationing is introduced in Chapter 4.

8. The effect of the tax rate on the investment should be taken into consideration. The income taxes do not affect all the investments in the same manner. Cash flows should be considered on an after tax basis.
9. Depreciation is a non-cash expense and, as such, should not be included in the cash flows. However, depreciation is a deductible expense in the computation of taxable income. It reduces the cash outflows originatedin the income tax payment and does influence the determination of cash flows.

CHAPTER 3

EVALUATION OF ALTERNATIVE PROPOSALS

This chapter examines various procedures for evaluating alternative investment proposals. Lorie and Savage defined a set of projects as mutually exclusive when, "acceptance of one proposal in such a set renders all others in the same set clearly unacceptable--or even unthinkable."[1] An example of mutually exclusive projects is the choice between two different sized dams that could be built in the same location. Obviously, the decision to build one automatically entails a decision not to build the other.

Methods for Evaluating Alternative Proposals

To develop methods for measuring the desirability of alternative proposals, it is assumed that the consequences of any project are known with certainty. Once the methods are established, this assumption can be relaxed. It is assumed that the life of the project is known, and the quantity and timing of cash flows is known and inalterable. It is easy to find examples where some or all of these situations are not known with certainty.[2] On the other hand, there are situations so close

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1. Lorie and Savage, "Three Problems in Rationing Capital," p. 229.

2. Uncertainty is introduced in Chapter 5.

to being in a world of certainty, that for practical reasons they are studied as such. One commonly cited example is buying United States Treasury Securities and holding them until maturity.[3] Furthermore, it is assumed that the investor has enough capital to accept any project which would meet the minimum criterion for acceptance. In other words, any investment that would meet the established criterion is acceptable.

The most widely accepted methods for measuring desirability are as follows.

Payback Period

The payback period is the time necessary to recover the initial investment. The definition can be clarified through the use of the following examples.

Example 3-1. If \$500 is invested in Project A, a perpetuity of \$100 every year will be received. The payback is five years (in five years the initial investment would be returned to the firm's treasury). This result was obtained by

$$\frac{\text{Initial investment}}{\text{Annual cash inflows}} = \frac{\$500}{\$100} = 5 \text{ years.}$$

Example 3-2. For a \$500 investment in Project B, the following series of cash flows would be received for eight years: \$10 in one year, \$20 in two years. \$40 in three years . . . and $10 \times 2^{n-1}$ in n years. The payback period is 5.594 years. This results from

3. Van Horne, Financial Management and Policy, p. 51.

	Cash flow (in dollars)	To be recovered (in dollars)
Initial expenditure	-500	500
First year	10	490
Second year	20	470
Third year	40	430
Fourth year	80	350
Fifth year	160	190
First 594/1,000 of Sixth year	190	0

Assuming that the $320 to be received during the sixth year is received in a uniform stream, $190 will have been received after 594/1,000 of the year is over. If Project A (Example 3-1) and Project B (Example 3-2) are mutually exclusive, the investor will accept Project A because its payback is shorter, 5 versus 5.594 years.

A major advantage of the payback method is that it is easy to calculate, but it has some important drawbacks.

1. It does not take into consideration the cash flows after the investment has been recovered. In our examples, the accepted project would continue producing $100 in the sixth period and successive ones and the project rejected would produce $360, $640 and $1280.
2. Payback does not take into consideration the time value of money. Dollars were added in different years as if they were equivalents.

The payback method is widely used but it can lead to undesirable decisions.

Average Rate of Return

This method does not deal directly with the cash flows. It calculates the ratio of the average annual net income to the average investment in the project, i.e., average book value after deducting depreciation.

Example 3-3. One project, with the initial cost of \$400, has a useful life of three years. Its salvage value at the end of that time is \$100. Using straight-line depreciation, the projected income from the project is shown in Table 3-1.

Table 3-1. Average rate of return problem.

	Year 1	Year 2	Year 3	Average
		(in dollars)		
Cash Income	100	200	300	200
Depreciation	100	100	100	100
Net Income	0	100	200	100
Book Value				
January 1	400	300	200	
December 31	300	200	100	
Average	350	250	150	250

Average rate of return is $\frac{\$100}{\$250}$ = .40 or 40%.

The average rate of return method is easy to calculate but has certain limitations. For one does not take into consideration the time value of the money. For this reason, it is an inefficient way of measuring the desirability of projects. This can result in sub-optimal decision making.

Net Present Value

Net present value is defined as the difference between the present value of the revenues (cash inflows) and the present value of the costs (cash outflows).

$$\text{Net present value} = \sum_t R_t(1+k)^{-t} - \sum_t C_t(1+k)^{-t} \qquad (3\text{-}1)$$

Where R_t = revenues in year t and C_t = costs in year t.

If the net present value is positive, then discounted revenues are larger than discounted costs and the project would be accepted. A negative net present value indicates a project which does not recover the opportunity cost of the invested funds. When the net present value is equal to zero, an investor would be indifferent between accepting or rejecting the project. This situation of indifference is caused because the return on the investment is equal to the investor's cost of capital.[4]

The cost of capital is defined as the rate of discount which makes the net present value equal to zero. In other words, the cost of

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4. Study of the cost of capital is beyond the scope of this paper. For information, see among others, James T. S. Porterfield, Investment Decisions and Capital Costs (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1965), pp. 42-63; G. David Quirin, The Capital Expenditure Decision, pp. 95-159; Ezra Solomon, "Measuring a Company's Cost of Capital," Journal of Business, 28 (October, 1955), pp. 240-252; James C. Van Horne, Financial Management and Policy, pp. 89-120.

capital is the rate of return on the project that would not change the market price of the stock of the firm if the project were undertaken. The cost of capital (k) is used as the discount rate in the net present value approach. It is assumed to be known and independent of the investment decisions. The use of net present value can be clarified with the following examples.

<u>Example 3-4</u>. Supposing the investor's cost capital is 20 percent in example 3-1, Project A, then the net present value would be equal to zero.

Calculations: Since example 3-1 is a perpetuity, then to calculate the present value of the inflows, (2-4) would apply.

$$\text{Present value of the inflows} = \frac{\$100}{.20} = \$500$$

Since the only outflow is the initial cost, the present value of the outflows is $500.

Net present value = 500 - 500 = 0.

In this case, an investor would be indifferent between accepting or rejecting project A.

<u>Example 3-5</u>. Supposing that the investor's cost of capital is 20 percent in example 3-2, Project B, then the net present value would be equal to $238.75.

Calculations: Present value of the outflows = $500.

$$\begin{aligned} PVI = {} & 10(1+k)^{-1} + 2 \times 10(1+k)^{-2} + 2^2 \times 10(1+k)^{-3} \\ & + \ldots + 2^7 \times 10(1+k)^{-8} \end{aligned}$$

$$PVI = 10(1+k)^{-1}[1 + 2(1+k)^{-1} + 2^2 \times (1+k)^{-2} + \ldots + 2^7 \times (1+k)^{-7}]$$

Where PVI is the present value of the inflows. The expression in the brackets is a geometric progression of the form $2(1+k)^{-1}$.

$$\text{Present value of the inflows} = 10(1.20)^{-1} \frac{1-2^8(1.20)^{-8}}{1-2(1.20)^{-1}} = \$738.85$$

Net present value = \$738.85 - \$500 = \$238.85.

Since the net present value is positive, the project should be accepted.

For the net present value method, projects with larger positive net present value are ranked as more desirable. In the above examples, (3-4 and 3-5), if Projects A and B are mutually exclusive the investor will accept Project B because its net present value is larger.

There are several advantages to the net present value technique. Besides being easy to calculate, it takes into consideration the whole life of the project and the time value of the money. Also, financial tables are published to facilitate computations.

Internal Rate of Return

The internal rate of return is defined as the rate (i) that equals the present value of the costs (cash outflows) with the present value of the revenues (cash inflows).

$$\sum_t C_t(1+i)^{-t} = \sum_t R_t(1+i)^{-t} \qquad (3\text{-}2)$$

Where C_t = costs in year t and R_t = revenues in year t.

If the internal rate of return is larger than the cost of capital, the investment would benefit stockholders and the project should be accepted. An internal rate of return smaller than the cost of capital indicates an undesirable project. When the internal rate of return is equal to the cost of capital, the investor is indifferent as to accepting or rejecting the project. This can be observed through the use of the examples employed previously.

Example 3-6. In the Example 3-1, the internal rate of return is 20 percent for this rate: The present value of the cash outflows is \$500 and the present value of the cash inflows is $\frac{\$100}{.20} = \$500$.

Example 3-7. In the Example 3-2 the internal rate of return is approximately 27.5 percent. For this rate the present value of the cash outflows is \$500, the present value of the cash inflows is approximately \$500.

For the internal rate of return method, projects with larger i, when i is larger than the cost of capital, are ranked as more desirable. In the examples studied above, 3-4 and 3-5, if Projects A and B are mutually exclusive the investor will accept Project B because its internal rate of return is larger.

The internal rate of return method takes into consideration the whole life of the project and the time value of the money. But, it is not easy to calculate without a computer and sometimes it can only be solved by a trial and error iterative process. Furthermore, certain cash flow patterns can generate more than one rate of return.[5]

Which is the Best Method for Ranking Alternative Proposals

Since the payback period and the average rate of return methods can lead to sub-optimal decisions, they have limited value as a criteria for ranking investments. The net present value and the internal rate of return methods, however, can produce different ranking to the same projects as will be seen in the next example.

Example 3-8. A firm with a cost of capital of ten percent has the opportunity to invest in one of the following mutually exclusive projects.

	Cash Flows	
Year	Project A (in dollars)	Project B (in dollars)
0	-2,225	-2,225
1	1,000	0
2	1,000	500
3	1,000	1,000
4	1,000	3,343
5	0	0

The net present value for Project A is \$910 and for Project B, \$1,116. Based on the net present value, Project B is more desirable.

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5. For an in-depth examination of the multiple rate of return problem, see Quirin, The Capital Expenditure Decision, pp. 49-56 and Ezra Solomon, "The Arithmetic of Capital Budgeting Decisions," Journal of Business, XXIX (April, 1956), pp. 240-252.

The internal rate of return for Project A is 25 percent and 22 percent for Project B. Based on the internal rate of return, Project A is more desirable. Both projects are ranked differently depending on the use of the net present value or the internal rate of return.[6]

The apparent conflict between the net present value and the internal rate of return is due to different assumptions with respect to the reinvestment rate of the cash outflows generated by the project. The net present value method assumes that these cash flows would be reinvested at the cost of capital (k) over the remaining life of the project. The internal rate of return method assumes that these cash flows would be reinvested at the internal rate of return (i) over the remaining life of the project.

The assumed reinvestment rate for the outflows in the net present value method is the same for all the projects, i.e., the cost of capital (k). This assumed reinvestment rate may be conservative since no future project with a rate of return less than the cost of capital is likely to be accepted. On the other hand, the assumed reinvestment rate for the outflows in the internal rate of return is going to be different for each internal rate of return. For projects with a high internal rate of return, a high reinvestment rate is assumed. For projects with a low internal rate of return, a low reinvestment rate is assumed. The assumed reinvestment rate in the internal rate of return would be an approximation to the real reinvestment rate on only a few occasions.

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6. This example was first presented by Ezra Solomon, The Theory of Financial Management (New York, London: Columbia University Press, 1963), pp. 134-135.

Van Horne states: "To the extent that we can regard the required rate of return, k, as an approximate measure of the opportunity rate for reinvestment, the present-value method is preferred over the internal-rate-of-return method."[7]

J. Hirshleifer suggests that the net present value is theoretically superior to the internal rate of return. Using isoquant analysis, he distinguishes between "production opportunities" and "market opportunities." The former are a sacrifice of present for future income through a production opportunity. The latter are a sacrifice of present for future income through a market or exchange opportunity. He uses the word "investment" to refer to both types of opportunities taken together. He shows that the net present value and the internal rate of return lead to identical answers in the accept/reject decision when the projects last two periods and there is a perfect capital market (the borrowing rate equals the lending rate and the capital is not rationed). If the projects are mutually exclusive and the capital market is perfect, the net present value ranks correctly what the internal rate of return fails to do. In conclusion, Hirshleifer states that the internal rate of return method for the multi-period[8] case is not generally correct and the net present value method is correct in a wide variety of cases to find the "productive" solution.[9]

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7. Van Horne, Financial Management and Policy, p. 65.

8. The study of the multi-period case of capital budgeting will start in Chapter 4.

9. J. Hirshleifer, "On the Theory of Optimal Investment Decision," Journal of Political Economy, 66 (August, 1958), pp. 329-352.

CHAPTER 4

CAPITAL RATIONING

The analysis in the preceding chapter was developed under the assumptions of a certainty and that the investor has adequate capital to accept any beneficial project. It was first assumed that the projects were independent and the best methods for accepting or rejecting them individually were examined. After developing several alternative approaches, the best criteria for ranking single mutually exclusive projects were considered.

This chapter will study a less rigid situation; the assumption that the investor has adequate capital to accept any beneficial project is relaxed and capital rationing is introduced. First, the single period case where the expenditure constraint only lasts for a period is studied. Afterwards, the multi-period case where the expenditure constraint if for more than one period is examined.

Capital rationing occurs when the investor must operate within a fixed budget. The investor, constrained to the budget, should accept the most beneficial set of projects instead of accepting all beneficial projects.

In the real world capital rationing is common. Frequently the investor imposes capital rationing on himself, i.e., the owners of a private corporation may not want to lose control over the company and consequently they might refuse to issue equity and limit themselves to the cash flows generated from operations and depreciation. Market conditions sometimes impose capital rationing because an investor is unable to secure enough funds in a short period of time to undertake all acceptable projects.

The investors need to rank the desirable projects, under capital rationing, in such a way that the set of projects which maximize benefits is selected.

The Single Period Case

In simple situation, the expenditure constraint lasts for only a single period. This case is of practical as well as theoretical interest because many investors have little insight into future investment opportunities and they must make decisions on available opportunities.[1]

Comparison Between the Net Present Value and the Profitability Index

Example 4-1. An investor has $875 available for investing and eight projects under consideration. Table 4-1 shows the initial outlay and the net present value of each project.[2]

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1. See Quirin, The Capital Expenditure Decision, pp. 177-184.

2. This was modified from one originally presented by Van Horne, Financial Management and Policy, p. 69.

Table 4-1. The net present value for the single period case.

Project	Initial Outlay (in dollars)	Net Present Value (in dollars)
A	150	- 5.5
B	175	28.
C	125	17.5
D	400	100.
E	100	5.
F	200	18.
G	100	19.
H	75	17.

If the investor ranks the projects based on their net present value, the accepted set of projects would be D, B, G, and F. There would be an initial expenditure of $875 and a net present value of $165. This result is not the best as will be demonstrated by ranking based on the profitability index.

The Profitability Index

The profitability index of a project is the ratio of future net cash flows over the initial cash outlay. The profitability index for the projects on Table 4-1 are shown on Table 4-2.

Table 4-2. The profitability index for the single period case.*

Projects	Initial Outlay (in dollars)	Net Present Value (in dollars)	Profitability Index
A	150	- 5.5	.97
B	175	28.	1.16
C	125	17.5	1.14
D	400	100.	1.25
E	100	5.	1.05
F	200	18.	1.09
G	100	19.	1.19
H	75	17.	1.22

* For the single period case $NPV = \Sigma R_t - C$ and $PI = \frac{\Sigma R_t}{C}$ or

$NPV = C(PI - 1)$ or $PI = \frac{NPV}{C} + 1$.

The investor would accept all the projects with a profitability index larger than one in a non-capital rationing situation. In the capital rationing single period case, the investor would rank the desirability of the projects by the one having the larger profitability index. The investor would accept desirable projects until the initial expenditure is exhausted. In this example, the investor would accept the set D, H, G, B and C. There would be an initial expenditure of \$875 (the same as when the ranking was based on the net present value), with a net present value of $181.50 versus $165.00.

The profitability index is better than the net present value in the single period case because it measures the relative profitability of every project versus the absolute contribution of the project measured by the net present value.

In the case that two or more of the projects are mutually exclusive, the one with the highest profitability index would be accepted. If in Table 4-2, Projects B and C are mutually exclusive, only B would be accepted. If Project D is contingent on Project A (the acceptance of Project D is dependent on the acceptance of Project A) both projects should be evaluated together to determine their joint profitability index as calculated below.

Net present value = ΣR_t - C, implies, ΣR_t = net present value + + C

where C = initial outlay and R_t = revenue in the year t.

For Project A, $\Sigma R_t = -5.5 + 150 = 144.5$.

For Project D, $\Sigma R_t = 100. + 400 = 500$.

The joint profitability index for A and D = $\frac{144.5 + 500.}{150 + 400} = 1.17$.

In this case, Projects A and D should be accepted before the ones with a lower profitability index.

The single period case limits the study to an expenditure constraint lasting for only one period. This case is not always realistic because large projects normally need several years to complete. They would be studied as the multi-period case.

So far, it has been assumed that the projects are indivisible; that they have been accepted or rejected as a whole not considering the possibility of only taking part of the project. For example, if an investor has capital to lend, it is reasonable to think that the quantity to be lended is divisible. Linear programming,the next method to be studied, is able to handle the multi-period capital rationing case and the divisible investments.

The Linear Programming Problem

The linear programming problem entails an optimizing process in which non-negative values for a set of decision variables x_1, x_2, ... x_n are selected so as to maximize (or minimize) an objective function in the form

$$\text{Maximize (minimize) } \sum_j b_j x_j$$

subject to resource constraints in the form

$$\sum_j c_{tj} x_j \leq C_t$$

where b_j, c_{tj} and C_t are given constants.

Depending upon the problem, the constraints may also be stated with equal-to signs (=) or greater-than-or-equal-to signs ($\geq$).

For linear programming to be applicable, the following conditions must exist.

1. The objective function and each constraint equation must be linear. This excludes exponents and cross products in the problem statement and implies proportionality.

2. The constants must be known and assumed to be deterministic. In other words, the probability associated with the occurrence of any b_j, c_{tj} and C_t value is presumed to be 1.0.
3. The decision variables must be divisible.[3]

The theory of linear programming is developed elsewhere.[4]

The Basic Model of Capital Rationing

The investor, limited by the budget in various years, seeks to maximize the net present value resulting from the adoption of a particular set of divisible projects. It is assumed that the annual budgets, the net present value and the annual cash outlay of the projects are known with certainty.

This situation is more general than the single period case. The capital rationing is for various periods and the projects do not have to be indivisible. The situation continues being in a world of certainty and has the structure to be solved by linear programming.

The basic model of capital rationing is

$$\text{Maximize (a)} \sum_{j=1}^{n} b_j x_j \qquad (4\text{-}1)$$

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3. Richard B. Chase and Nicholas J. Aquilano, Production and Operations Management: A Life Cycle Approach, 1st ed. (Homewood, Illinois: Richard D. Irwin, Inc., 1973), p. 215.

4. For reference, see A. Charnes, W. W. Cooper, and A. Henderson, An Introduction to Linear Programming (New York: John Wiley and Sons, 1953); George Bernard Dantzig, Linear Programming and Extensions (Princeton, New Jersey: Princeton University Press, 1963).

Subject to (b) $\sum_{j=1}^{n} c_{tj}x_j \leq C_t$

(c) $0 \leq x_j \leq 1$

where

b_j = the known net present value of the j^{th} project when discounted at the cost of capital (k).[5]

x_j = the fraction of the j^{th} project undertaken.

c_{tj} = the present value of the outlay for the j^{th} project in the year t.

C_t = the present value of the budget constraint in the year t.

The meaning of the equations in the basic model (4-1) is:

(a) The investor seeks to maximize the net present value resulting from the adoption of a particular set of projects.

(b) Each project involves fixed and known cash outlays in one or more of the periods (t) under consideration. The total cash outlays in any year cannot be larger than the fixed budget constraint for that year.

(c) It is not possible to take a negative part of the project $x_j \geq 0$. Furthermore, it is not possible to take the same project more than once $x_j \leq 1$.

The linear programming is applicable to the basic model of capital rationing because:

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5. Any further present value would be discounted at the cost of capital.

1. The objective function $\sum_j b_j x_j$ and the constraint equation $\sum_j x_j c_{tj} \leq C_t$ are linear.

2. The constants b_j, c_{tj} and C_t are known and assumed to be deterministic.

3. The decision variables x_j are divisible.

The Lorie-Savage Problem

Example 4-2. Nine investments have been proposed. The present value of the net outlays required in the first and second time periods and the present values of the investments are shown in Table 4-3. The finance committee has stated that \$50 and \$22 will be available for capital investments in the first and second periods, respectively. We shall consider these amounts to have present values of \$50 and \$20, respectively.[6]

Table 4-3. Table for the Lorie-Savage problem.

Investment Project	Present Value of Outlay, Period 1	Present Value of Outlay, Period 2	Present Value of Investment
1	\$12	\$ 3	\$14
2	54	7	17
3	6	6	17
4	6	2	15
5	30	35	40
6	6	6	12
7	48	4	14
8	36	3	10
9	18	3	12

6. Lorie and Savage, "Three Problems in Rationing Capital," p. 234.

The linear programming problem for Example 4-2 is

$$\text{Maximize } 14x_1 + 17x_2 + 17x_3 + 15x_4 + 40x_5 + 12x_6 + 14x_7 + 10x_8 + 12x_9$$

Subject to

$$12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 + 6x_6 + 48x_7 + 36x_8 + 18x_9 \leq 50$$

$$3x_1 + 7x_2 + 6x_3 + 2x_4 + 35x_5 + 6x_6 + 4x_7 + 3x_8 + 3x_9 \leq 20$$

$$0 \leq x_j \leq 1, \ (j = 1 \ldots 9)$$

its solution is shown in Table 4-4.

Table 4-4. Weingartner's solution of the Lorie-Savage problem.

$x_1 = 1.0$
$x_2 = 0$
$x_3 = 1.0$
$x_4 = 1.0$
$x_4 = 1.0$
$x_5 = 0$
$x_6 = 0.970$
$x_7 = 0.045$
$x_8 = 0$
$x_9 = 1.0$

Total net present value = \$70.27. The linear programming solution calls for the total adoption of projects 1, 3, 4 and 9, and partial acceptance of projects 6 to the extent of 97 percent and 7 to the extent of 4.5 percent. All the funds of the budgets are used.[7]

Mutually Exclusive and Contingent, Divisible Projects

The basic model had assumed that the projects are independent. In the case in which two or more projects are mutually exclusive it is only necessary to add the following equation to the basic model (4-1) for every set of mutually exclusive projects.

$$\sum_{j \in J} x_j \leq 1 \qquad (4\text{-}2)$$

where J is the set of mutually exclusive projects and j = any project in the set J.

Example 4-3. If projects 2 and 7 in Example 4-2 are mutually exclusive, the equation to add to the linear programming model is:

$$x_2 + x_7 \leq 1$$

The solution may suggest the adoption of fractions of both project 2 and 7. This solution is contrary to the definition of mutually exclusive projects. It should be interpreted as a suggestion of looking for an alternative project.

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7. Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems, pp. 18-19.

If the acceptance of project a is dependent on the acceptance of project b, contingent project, it is only necessary to add the following equation to the basic model (4-1)

$$x_a \leq x_b \tag{4-3}$$

The fraction of project b adopted must not be less than the fraction of project a adopted.

Shifting Funds Between Periods

The basic model assumes that funds cannot be carried over from one period to the next ones. This assumption is unrealistic in most of the cases. It can be easily disregarded by changing only the equation (b) in the basic model (4-1) to:

$$\sum_{t=1}^{T} \sum_{j=1}^{n} x_j c_{tj} \leq \sum_{t=1}^{T} C_t \tag{4-4}$$

In this model the meanings of x_j, c_{tj} and C_t are the same as in the basic model (4-1).

Example 4-4. If in Example 4-2 it is possible to shift funds between periods, the objective function will remain the same and the new resource constraint equations would be:

$$12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 + 6x_6 + 48x_7 + 36x_8 + 18x_9 \leq 50$$

$$15x_1 + 61x_2 + 12x_3 + 8x_4 + 65x_5 + 12x_6 + 52x_7 + 39x_8 + 21x_9 \leq 70$$

$$0 \leq x_j \leq 1, \ (j = 1,2, \ldots, 9)$$

The solution is shown in Table 4-5.

Table 4-5. Solution of the Lorie-Savage problem when shifting funds between periods is possible.

$x_1 = 1.0$
$x_2 = 0$
$x_3 = 1.0$
$x_4 = 1.0$
$x_5 = .354$
$x_6 = 1.0$
$x_7 = 0$
$x_8 = 0$
$x_9 = 0$

Total net present value = \$72.16, expended in the first period \$40.62, expended in the second period \$29.38. Comparing the solution on Table 4-5 with the one on Table 4-4, it is possible to see that: from the first period \$9.38 are shifted to the second, increasing the total net present value by \$1.89.

Dual Variables in
the Basic Model

Every linear programming problem, the primal problem, has an alternative formulation, the dual problem. The dual problem is useful in conducting certain types of sensitivity analysis in the primal, making changes in problem parameters and observing the effects of such changes on the optimal solution. The dual problem may also require less computational effort than its primal.[8] The dual problem of the basic model is:[9]

$$\text{Minimize } \sum_{t=1}^{T} p_t C_t + \sum_{j=1}^{n} v_j \qquad (4\text{-}5)$$

$$\text{Subject to } \sum_{t=1}^{T} p_t c_{tj} + v_j \geq b_j \quad (j = 1,2, \ldots, n)$$

$$p_t,\ v_j \geq 0 \qquad (t = 1,2, \ldots, T)$$

C_t, c_{tj} and b_j were defined in the basic model and here they have the same meaning. The variables p_t and v_t are new and are called dual variables.

The value of the dual variables for the optimal solution of Exercise 4-2 is shown in Table 4-6.

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8. Chase and Aquilano, Production and Operations Management: A Life Cycle Approach, p. 229. For further reference about the dual problem, see A. Charnes and W. W. Cooper, Management Models and Industrial Applications of Linear Programming, Volume I (New York: John Wiley and Sons, 1961), pp. 175-196.

9. Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems, p. 24.

Table 4-6. Dual variables for the Lorie-Savage problem.

$v_1 = 6.77$
$v_2 = 0$
$v_3 = 5.0$
$v_4 = 10.45$
$v_5 = 0$
$v_6 = 0$
$v_7 = 0$
$v_8 = 0$
$v_9 = 3.95$
$p_1 = 0.136$
$p_2 = 1.864$

Dual variables have an economic interpretation as marginal productivities, opportunity costs, or shadow prices.

The dual variables p_t are the opportunity costs associated with the budget constraints C_t. In Example 4-2, $p_1 = 0.136$ indicates that the net present value can be increased by \$0.136 if the budget in the first period is changed from \$50 to \$51. $p_2 = 1.864$ indicates that the net present value can be increased by \$1,864 if the budget in the second period is changed from \$20 to \$21. These dual variables p_t are valuable for the investor in deciding whether to shift funds from one budget to another or when and at what price the investor should borrow.

The dual variables v_j are a measure of the marginal productivity of each project associated with the constraints on the x_j, with the requirement that $x_j = 1$. The investor may wish to increase the scale of the projects with high v_j value.

The Integer Programming Problem

The integer programming problem solves for the integer values of the decision variables (x_j) which maximize or minimize the objective function in the form:

$$\text{Maximize (minimize) } \sum_j b_j x_j$$

The integer programming problem in which the answer has to consist entirely of integer numbers; no fractional or decimal parts are allowed in the solution.

The linear programming problem assumed that all the projects were divisible, but it is normal to find projects that are indivisible, i.e., machinery, plants, bridges, etc. If the linear programming problem is used for the study of indivisible projects, the solution may be integer or so close to a whole number that the use of integer programming will not contribute much. For example, for the solution in Example 4-2, $x_6 = 0.970$ and $x_7 = 0.045$, it is predictable that the integer programming solution will be $x_6 = 1$ and $x_7 = 0$. This does not always occur, i.e., $x_b = .5$ where x_b is the acceptance or rejection of the construction of a bridge (to build half a bridge is a patently absurdity). In this example the linear programming problem is not helping the investor to know if the project would be accepted or rejected. The integer programming method should be used in this case.[10]

The Integer Solution of the Basic Model

In this case, the assumption that the projects should be divisible is disregarded. The investor, limited by the budget in various years, seeks to maximize the net present value resulting from the adoption of a particular set of indivisible projects. The annual budgets, the net present value and the annual cash outlay of the projects are known with certainty.

This model is seen to be virtually the same as in the preceding basic model (4-1).[11]

$$\text{Maximize (a)} \quad \sum_{j=1}^{n} b_j x_j \qquad (4\text{-}6)$$

$$\text{Subject to (b)} \quad \sum_{j=1}^{n} c_{tj} x_j \leq C_t \quad (t = 1, 2, \ldots, T)$$

$$\text{(c)} \quad 0 \leq x_j \leq 1$$

$$\text{(d)} \quad x_j \text{ integer.}$$

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10. For further information about the integer programming method see among others, William J. Baumol, Economic Theory and Operations Analysis 2nd Ed. (Englewood Cliffs, N.J.: Prentice-Hall, 1965), pp. 148-166; Handy A. Taha, Operations Research: An Introduction (New York: Macmillan Publishing Co., Inc., 1971), pp. 304-350.

11. Weingartner, Mathematical Programming, p. 46.

The meaning of all of these equations but (f) was explained in this chapter under the section on, "The Basic Model of Capital Rationing."

Equation (f) says that x_j has to be integer and in conjunction with equation (c), restricting the values of x_j to $x_j = 1$ or $x_j = 0$. $x_j = 1$ means the acceptance and $x_j = 0$ means the rejection of the j^{th} project.[12]

The Lorie-Savage Problem with Integral Solutions

Example 4-4. If the projects in Example 4-2 are indivisible, the integer programming solution for it is shown in Table 4-7.

Table 4-7. Integer solution of the Lorie-Savage problem.

$x_1 = 1$
$x_2 = 0$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$
$x_6 = 1$
$x_7 = 0$
$x_8 = 0$
$x_9 = 1$

12. Balas has developed an algorithm for solving integer programming problems with variables that only can take the values zero or one. See Egon Balas "An Additive Algorithm for Solving Linear Programs with Zero-One Variables," Operations Research, 13 (July-August, 1965), pp. 517-546.

Total net present value = \$70. This solution calls for the acceptance of projects 1, 3, 4, 6 and 9.[13] This solution was anticipated based on the optimal values of x_j in Example 4-2.

Mutually Exclusive and Contingent, Indivisible Projects

In the case that two or more indivisible projects are mutually exclusive, it is only necessary to add the following equation to the model (4-6) for every set of mutually exclusive projects

$$\sum_{j \in J} x_j \leq 1 \qquad (4\text{-}7)$$

where J is the set of mutually exclusive projects and j = any project in the set J.

There will not be fractional solutions in this case and only one of the projects, at the most, will be accepted.

If project a is contingent on project b, the equation to add to model (4-6) is

$$x_a \leq x_b \qquad (4\text{-}8)$$

If project b is rejected $x_b = 0$, it is implied that project a wi will be rejected, $x_a = 0$. If project b is accepted, $x_b = 1$, it is not possible to assume that project a will also be accepted because, $x_a = 1$ or $x_a = 0$ are possible.

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13. Weingartner, Mathematical Programming, p. 46.

Shifting Funds Between Periods When the Projects are Indivisible

When it is possible to carry the funds from one period to the following ones, it is only necessary to change equation (b) in model (4-6) to:

$$\sum_{t=1}^{T} \sum_{j=1}^{n} x_j C_{tj} \leq \sum_{t=1}^{T} C_t. \qquad (4\text{-}9)$$

Integer Programming Drawbacks

To solve the integer programming problem, the problem is replaced by a linear programming problem whose optimal solution is the same as for the one replaced. Many of the added constraints in the process may result in being redundant. The integer programming problem does not always have an optimal solution. To find it, iterations are done in the model until they converge in the solution. It may happen that the number of necessary iterations is unmanageable or there is no convergency at all. In addition, to evaluate the dual variable of the integer programming problem is difficult and these variables are important for the sensitivity analysis over the integer programming problem.

Weingartner summarizes these ideas by saying that, "The procedures for obtaining integer solutions were found to have considerable drawbacks, both in terms of the computations required and in the interpretation of the solution."[14]

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14. Weingartner, Mathematical Programming, p. 108.

CHAPTER 5

CAPITAL BUDGETING UNDER RISK

In the preceding material both the future cash flows and the future annual budgets of the projects were assumed to have been known with certainty. The study of investment models under certainty or exact models has been justified by Haavelmo.[1] Although Haavelmo was concerned with macro-economic theories of investment, his comments are applicable to the models this paper has studied. The exact models may be used to gain insight into the way in which reality operates. He observes, "It may sometimes be intuitively obvious that if a certain proposition does not hold even under ideal conditions it would not hold in the real world."[2] In actuality, the exact models are an approximation to more realistic approaches. "That is, the possibility of reaching reasonably realistic, explanatory models by the mental process of working from the simple to the more complex is unquestionable."[3]

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1. Trygve Haavelmo, A Study in the Theory of Investment (Chicago: The University of Chicago Press, 1960), pp. 19-21.

2. Ibid., p. 20.

3. Ibid.

The investor, subject to an unknown future, is unable to make perfect forecasts. Various approaches to the resulting uncertainty are examined here.

Economic theorists have been distinguishing between risk and uncertainty, basing the distinction on whether the probability distribution of outcomes is known or unkonwn.[4] The problems examined here are limited to risk situations, although, the terms risk and uncertainty are used interchangeably.

In the remainder of this work, it is assumed that the investors base their decisions on a known probability distribution of the cash flows and the annual budgets. The cognizance of these probability distribution can be subjective or objective. An example of the first is when the investors, based on their judgment, specify the probability distribution. An example of the latter is when the investors, through computer simulation attempt to calculate a probability distribution of possible outcomes.[5]

Investors expose themselves to risk when they adopt a project where the outcomes are not known with certainty. The risk is the possibility of unforeseen adverse fluctuations in the cash flow generated by the project. Risk is associated with the dispersion of the

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4. R. Duncan Luce and Howard Raiffa, Games and Decisions (New York: John Wiley and Sons, Inc., 1957), p. 13.

5. For an example the application of computer simulation to the evaluation of risky projects, see David B. Hertz, "Risk Analysis in Capital Investment," Harvard Business Review, 42 (January-February, 1964), 95-106; David B. Hertz, "Investment Policies that Pay Off," Harvard Business Review, 46 (January-February, 1968), 96-108.

probability distribution of the possible outcomes: the greater the dispersion, the greater the risk. The conventional measure of dispersion of a probability distribution is the standard deviation.

The standard deviation for the probability distribution of cash flows in period t is defined as[6]

$$s = \left[\sum_{x=1}^{n} (A_{xt} - \bar{A}_t)^2 P_{xt}\right]^{\frac{1}{2}} \tag{5-1}$$

where

A_{xt} = a cash flow for the x^{th} possibility in period t.

$\bar{A}_t$ = the expected value of cash flows in period t. It is calculated by

$$\bar{A}_t = \sum_{x=1}^{n} A_{xt} P_{xt} \tag{5-2}$$

P_{xt} = the probability of occurrence of Z_{xt} in the period t.

Another measure of dispersion is the variance (s^2), which is the square of the standard deviation.

A measure of relative dispersion is the coefficient of variation (C of V).

$$\text{C of V} = \frac{s}{\bar{A}_t} \tag{5-3}$$

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6. Van Horne, Financial Management and Policy, p. 124.

The following situation will be studied:

$$\text{Maximize: } Z = \sum_{j=1}^{n} b_j x_j \tag{5-4}$$

$$\text{Subject to: } \sum_{j=1}^{n} c_{tj} x_j \leq C_t \qquad (t = 1,2, \ldots, T)$$

$$0 \leq x_j \leq 1 \qquad (j = 1,2, \ldots, n)$$

where

b_j = the net present value of the j^{th} project, known with uncertainty,

x_j = the fraction of the j^{th} project undertaken,

c_{tj} = the present value of the cash outflow for the j^{th} project in the year t, known with uncertainty,

C_t = the present value of the budget in the year t, known with uncertainty.

Model (5-4) can be expressed in matrix notations as

$$\text{Maximize: } z = bx \tag{5-5}$$

$$\text{Subject to: } Cx \leq d$$

$$0 \leq x \leq 1$$

where

b = a row vector with n elements,

x = a column vector with n elements,

C = a T by n matrix,

d = a column vector with T elements.

Risk is introduced in this model by allowing b_j, c_{tj}, C_t or any combination of them to be normally distributed random variables. The implicit information in Model (5-4) is that the investor, limited by the budget in various years, seeks to maximize the net present value resulting from the adoption of a particular set of divisable projects. The multivariate probability distribution for the elements b_j, c_{tj} and C_t is known. This probability is represented by

$$Pr(b_j, c_{tj}, C_t)$$

where Pr = the probability of simultaneous occurrence of specified values of b_j, c_{tj} and C_t.

The main approaches of dealing with this problem have been

1. Stochastic linear programming.
2. Linear programming under uncertainty.
3. Chance-constrained programming.

The remainder of this chapter examines the first two methods briefly and gives more extensive study of the last.[7]

Stochastic Linear Programming[8]

This method studies the statistical distribution of the objective (maximize Z). There are two approaches to the study of the

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7. The discussion of the two first methods is based on Bertil Naslund, "Mathematical Programming Under Risk," Swedish Journal of Economics (September, 1965), 240-255. The discussion of the last one is based on Handy A. Taha, Operations Research, 649-653, and Bertil Naslund, "A Model of Capital Budgeting Under Risk," 1966, 257-271.

8. For a presentation of this method, see Gerhard Tintner, "Stochastic Linear Programming with Applications to Agricultural Economics," in H. A. Antosiewics, ed., Second Symposium on Linear Programming, National Bureau of Standards, Washington, 1955.

stochastic linear programming, the "passive" and the "active" methods. The "passive" method is based on simulation. A Monte Carlo routine is used to generate values of b_j, c_{tj} and C_t from the multivariate probability distribution $Pr(b_j, c_{tj}, C_t)$. Multiple sets of these values are selected and a linear programming problem is solved for each of these sets. A distribution of the objective (maximize Z) is obtained with the solutions of these linear programming problems.[9]

The "active" method was formulated by Tintner.[10] This method applied to Model (5-5) will cause the following modification.

$$\text{Maximize: } z = bx \tag{5-6}$$

$$\text{Subject to: } CX \leq DU$$

$$0 \leq x \leq 1$$

where U = a T by n matrix with elements $u_{tj} \geq 0$ and $\sum_{j=1}^{n} u_{tj} = 1$.

X = a n by n diagonal matrix with components of the vector x in the diagonal.

D = a T by T diagonal matrix with components of vector d in the diagonal.

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9. Salazar and Sen used this stochastic linear programming to determine the expected return and risk of a combination of risky investments. See Rudolfo C. Salazar and Subrata K. Sen, "A Simulation Model of Capital Budgeting Under Uncertainty," Management Science, 15 (December 1968), B161-B179.

10. Gerhard Tintner, "A Note on Stochastic Linear Programming," Econometrica, 28 (April, 1960), 490-495.

Naslund explains, "The decision-maker first selects the matrix U then he observes the random parameters C_t, c_{tj} and(or) b_j and solves for X using regular linear programming methods. By repeating this, holding the U's fixed he will obtain a distribution of max Z for the matrix U selected."[11] A drawback with the stochastic linear programming is that it gives the statistical distribution of the objective, not the optimal solution that is desired in capital rationing.

Linear Programming Under Uncertainty[12]

In this approach, Model (5-5) will be divided in two stages. In the first stage an investment decision is made; the values of x are selected. Then, the random elements of C and/or d are observed; the budget might be violated due to randomness. In the second stage, if the budget has been violated the random variable y has to be added to the constraint $Cx \leq d$ and this constraint would change to:

$$Cx + Ay = d \tag{5-7}$$

where C, x and d have the same dimensions as in Model (5-5)

y = a column vector with n elements,

A = a T by n matrix. The addition of Cx and Ay is only possible if A has the same dimensions as matrix C.

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11. Naslund, "Mathematical Programming Under Risk," p. 243.

12. For a more extensive treatment of linear programming under uncertainty, see George B. Dantzig, "Linear Programming Under Uncertainty," Management Science, 1 (April-July, 1955), pp. 197-206 and George B. Dantzig, "Recent Advances in Linear Programming," Management Science, 2 (January, 1956), pp. 131-144.

Variables y have a cost, f, associated with them. Consequently, it is necessary to change the objective function Maximize z = bx to

$$\text{Maximize } E(bx + fy) \tag{5-8}$$

where

E = the expected value operator.

f = a row vector with n elements.

A drawback with linear programming under uncertainty is that it only can be used if the disadvantages of budgets violations can be stated numerically. It is difficult to measure the disutility associated with a budget larger than expected.

Naslund finds the following similarities between the "active" method of stochastic linear programming and the linear programming under uncertainty method.[13]

> In the former we would select the u:s first, then observe the random parameters and finally solve a regular linear programming problem. In the latter we solve for the first stage variables x (corresponding to the u:s) then we observe the random parameters and solve a regular linear programming problem for the second stage variable y. The difference between the methods lies in the ways in which the first stage variables are introduced in the formulation of the problem.

Chance-Constrained Programming[14]

Chance-constrained programming attempts to find the deterministic equivalent of some probabilistic problems. This paper examines two cases

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13. Naslund, "Mathematical Programming Under Risk," p. 247.

14. The original contribution in this area is by A. Charnes and W. W. Cooper, "Chance-Constrained Programming," Management Science (October, 1959), pp. 73-79.

from the general case in Model (5-4). The chance-constrained approach utilizes the expected values of the various random variables.

It is now assumed that c_{tj} are the only random variables in Model (5-4). They are normally distributed with mean $E(c_{tj})$, variance $Var(c_{tj})$ and standard deviation $s(c_{tj})$. The covariance between c_{ru} and c_{sv}, where $r, s \epsilon j$ and $u, v \epsilon t$, is represented by $Cov(c_{ru}, c_{sv})$. Since it is not possible to be certain that the constraints $\sum_j c_{tj}x_j \leq C_t$ are going to hold when c_{tj} are random variables, the constraints are changed to:

$$Pr(\sum_j c_{tj}x_j \leq C_t) \geq 1 - a_t \tag{5-9}$$

where

$1 - a_t$ = the prescribed confidence levels desired for each period $t = 1,2, \ldots, T$.

The new problem is:

$$\text{Maximize: } Z = \sum_j E(b_j)x_j \tag{5-10}$$

Subject to: (a) $Pr(\sum_j c_{tj}x_j \leq C_t) \geq 1 - a_t$

(b) $0 \leq x_j \leq 1$ $(j = 1,2, \ldots, n)$

(c) $0 \leq a_t \leq 1$ $(t = 1,2, \ldots, T)$

In Model (5-10) the investor wishes to maximize the expected net present value resulting from the adoption of a particular set of projects. He protects himself, by using a probabilistic constraint, against total outlays higher than the fixed budgets. The total outlay for each year should not be larger than the amount budgeted; except a_t of the time.

The constraints express that $100(1-a_t)$ percent of the years, the budget will be large enough to cover the outflow of money originated by the optimal set of projects.

Chance-constrained programming will be applied to Model (5-10) to find its deterministic equivalent.

The following equation will be defined,

$$h_t = \sum_j c_{tj} x_j \tag{5-11}$$

where h_t is normally distributed with mean

$$E(h_t) = \sum_j E(c_{tj}) x_j \tag{5-12}$$

and variance

$$Var(h_t) = x^t Dx \tag{5-13}$$

where

x = a column vector with n elements.

D = a n by n matrix. The elements of the diagonal are $Var(c_{tj})$ and the others $Cov(c_{ru}, c_{sv})$.

x^t = a row vector with n elements.

The standard deviation of h_t is

$$s(h_t) = [Var(h_t)]^{\frac{1}{2}} \tag{5-14}$$

Substituting the value of h_t in (5-9)

$$Pr(h_t \leq C_t) = Pr\left[\frac{h_t - E(h_t)}{s(h_t)} \leq \frac{C_t - E(h_t)}{s(h_t)}\right] \geq 1 - a_t \tag{5-15}$$

Since $\frac{h_t - E(h_t)}{s(h_t)}$ is normally distributed with zero mean and unit variance

$$Pr(h_t \leq C_t) = F(\frac{C_t - E(h_t)}{s(h_t)}) \geq 1 - a_t \tag{5-16}$$

where

F = the cumulative density function of the standard normal distribution.

It is always possible to find a value (K) of the standard normal distribution, such as

$$F(K) = 1 - a_t \tag{5-17}$$

Equation (5-17) can be moved to (5-16) and it would result in

$$Pr(h_t \leq C_t) = F(\frac{C_t - E(h_t)}{s(h_t)}) \geq F(K) \tag{5-18}$$

The comparison between (5-16) and (5-18) implies that $Pr(h_t \leq C_t) \geq 1 - a_t$ only if

$$\frac{C_t - E(h_t)}{s(h_t)} \geq K \tag{5-19}$$

Equation (5-19) yields the following deterministic non-linear constraint equivalent to the probabilistic constraint (a) in Model (5-10)

$$\sum_{j=1}^{n} E(c_{tj})x_j + K\, s(h_t) \leq C_t \tag{5-20}$$

If the outflows c_{tj} are independent, $Cov(c_{ru}, c_{sv}) = 0$,

$$Var(h_t) = \sum_j Var(c_{tj})x_j^2 \tag{5-21}$$

(5-20) would change to:

$$\sum_{j=1}^{n} E(c_{tj})x_j + K\left(\sum_{j=1}^{n} Var(c_{tj})x_j^2\right)^{\frac{1}{2}} \leq C_t \qquad (5\text{-}22)$$

The constraint (a) in Model (5-10)

$$Pr(\sum_j c_{tj}x_j \leq C_t) \geq 1 - a_t \qquad (5\text{-}23)$$

expresses in probabilistic terms, the safety margin required by the inventor. Its deterministic equivalent (5-22) expresses the quantity of monetary units that the investor must hold in order to obtain the desired safety margin.

In constraint (5-22) $\sum_j E(c_{tj})x_j$ represents the mean of the cash outflows and $K[\sum_j Var(c_{tj})x_j^2]^{\frac{1}{2}}$ is the safety margin required to be above the mean. This term can be interpreted as financial slack, i.e., money that on average will not be used.[15]

The last case is when only C_t are assumed to be normal random variables with mean $E(C_t)$, variance $Var(C_t)$ and standard deviation $s(C_t)$. Since it is not possible to be certain that the constraints $C_t \geq \sum_j c_{tj}x_j$ will hold when C_t are random variables, these constraints are changed to

$$Pr(C_t \geq \sum_j c_{tj}x_j) \geq a_t \qquad (5\text{-}24)$$

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15. Naslund, "A Model of Capital Budgeting Under Risk," p. 261.

where $1 - a_t$ = the prescribed confidence levels desired for each period $t = 1,2, \ldots T$.

The problem is now

$$\text{Maximize: } Z = \sum_j E(b_j)x_j \tag{5-25}$$

$$\text{Subject to: (a') } \Pr(C_t \geq \sum_j c_{tj}x_j) \geq a_t$$

$$\text{(b') } 0 \leq x_j \leq 1 \ (j = 1,2, \ldots, n)$$

$$\text{(c') } 0 \leq a_t \leq 1 \ (t = 1,2, \ldots, T)$$

In Model (5-25), the investor wishes to maximize the expected net present value resulting from the adoption of a particular set of projects. He protects himself against budgets lower than the fixed outlays by using a probabilistic constraint. The budget for each year should be large enough to cover the known outlays, except a_t percent of the time.

The analysis of this case is similar to the previous one.

$$\Pr(C_t \geq \sum_j c_{tj}x_j) = \Pr\left(\frac{C_t - E(C_t)}{s(C_t)} \geq \frac{\sum c_{tj}x_j - E(C_t)}{s(C_t)}\right) \geq a_t \tag{5-26}$$

Since $\frac{C_t - E(C_t)}{s(C_t)}$ is normally distributed with zero mean and unit variance

$$\Pr(C_t \geq \sum_j c_{tj}x_j) = F\left[\frac{\sum c_{tj}x_j - E(C_t)}{s(C_t)}\right] \geq a_t \tag{5-27}$$

It is always possible to find a value (K) of the standard normal distribution, such that:

$$F(K) = a_t \tag{5-28}$$

Equation (5-28) can be moved to (5-27) and it would result in

$$\Pr(C_t \geq \sum_j c_{tj}x_j) = F[\frac{\sum c_{tj}x_j - E(C_t)}{s(C_t)}] \geq F(K) \tag{5-29}$$

The comparison between (5-27) and (5-29) implies that

$$\Pr(C_t \geq \sum_j c_{tj}x_j) \geq a_t \tag{5-30}$$

only if[16]

$$\frac{\sum c_{tj}x_j - E(C_t)}{s(C_t)} \leq K \tag{5-31}$$

Equation (5-31) yields the following deterministic non-linear constraint equivalent to the probabilistic constraint (a') in Model (5-25)

$$\sum_j c_{tj}x_j \leq E(C_t) + K\, s(C_t) \tag{5-32}$$

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16. Taha, p. 652.

CHAPTER 6

CHANCE-CONSTRAINED PROGRAMMING SOLUTION TO THE LORIE-SAVAGE EXAMPLE

In the Lorie-Savage problem, Example 4-2 in Chapter 4, uncertainty is introduced by assuming that the present value of the costs are independent normally distributed random variables. Their mean and variance, the present value of investment and the budgets in both years are shown in Table 6-1.[1]

Table 6-1. Lorie-Savage problem under risk.*

Investment Project x_j	Expected PV of costs, Period 1 c_{1j}	Variance of costs, Period 1 $\mathrm{Var}(c_{1j})$	Expected PV of costs, Period 2 c_{2j}	Variance of costs, Period 2 $\mathrm{Var}(c_{2j})$	PV of Investment b_j
1	\$12	2	\$ 3	1	\$14
2	54	9	7	3	17
3	6	1	6	2	17
4	6	1	2	1	15
5	30	5	35	13	40
6	6	1	6	2	12
7	48	8	4	1	14
8	36	7	3	1	10
9	18	2	3	1	12

* Present Value of Budget Ceilings: C_1 = \$50; C_2 = \$20.

1. This table is a modification of the table in Lorie and Savage, p. 234.

Using confidence levels for both periods of 95%, this problem can be expressed as:

$$\text{Maximize: } Z = \sum_{j=1}^{9} b_j x_j \tag{6-1}$$

$$\text{Subject to: (a) } \Pr\left(\sum_{j=1}^{9} c_{ij} x_j \leq 50\right) \geq .95$$

$$\text{(b) } \Pr\left(\sum_{j=1}^{9} c_{2j} x_j \leq 20\right) \geq .95$$

where:

$$\sum_{j=1}^{9} b_j x_j = 14x_1 + 17x_2 + 17x_3 + 15x_4 + 40x_5 + 12x_6 + 14x_7 + + 10x_8 + 12x_9 \tag{6-2}$$

$$\sum_{j=1}^{9} c_{ij} x_j = 12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 + 6x_6 + 48x_7 + + 36x_8 + 18x_9 \tag{6-3}$$

$$\sum_{j=1}^{9} c_{2j} x_j = 3x_1 + 7x_2 + 6x_3 + 2x_4 + 35x_5 + 6x_6 + 4x_7 + 3x_8 + + 3x_9 \tag{6-4}$$

The deterministic constraints equivalent to (a) and (b) in Model (6-1):[2]

$$\sum_{j=1}^{9} c_{1j} x_j + K\left[\sum_{j=1}^{9} \mathrm{Var}(c_{ij}) x_j^{2}\right]^{\frac{1}{2}} \leq 50 \tag{6-5}$$

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2. See Equation (5-22) in Chapter 5.

$$\sum_{j=1}^{9} c_{2j}x_j + K[\sum_{j=1}^{9} Var(c_{2j})x_j^2]^{\frac{1}{2}} \leq 20 \tag{6-6}$$

where

$$\sum_j Var(c_{ij})x_j^2 = 2x_1^2 + 9x_2^2 + x_3^2 + x_4^2 + 5x_5^2 + x_6^2 + 8x_7^2 + + 7x_8^2 + 2x_9^2 \tag{6-7}$$

$$\sum_j Var(c_{2j})x_j^2 = x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2 + 13x_5^2 + 2x_6^2 + x_7^2 + + x_8^2 + x_9^2 \tag{6-8}$$

Although constraints (6-5) and (6-6) are non-linear, various methods for solving non-linear programming problems are available.[3]

Since this paper tries to solve the multi-period capital budgeting problem under uncertainty by using linear programming, it is necessary to find a linear approximation to the non-linear portions of the constraints. The approximation adopted by Naslund is:[4]

$$[\sum_{j=1}^{n} Var(c_{tj})x_j^2]^{\frac{1}{2}} \leq A - \sum_{j=1}^{n} (1-x_j) \left\{A - [A^2 - Var(c_{tj})]^{\frac{1}{2}}\right\} \tag{6-9}$$

where

$$A = [\sum_{j=1}^{n} Var(c_{tj})]^{\frac{1}{2}}$$

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3. Taha indicates the use of separable programming. See Taha, p. 683. Naslund refers to the gradient projection method and the cutting-plane method; Naslund, "A Model of Capital Budgeting Under Risk," p. 268.

4. Naslund, "A Model of Capital Budgeting Under Risk," p. 268.

Using (6-9):

$$\left[\sum_{j=1}^{9} \operatorname{Var}(c_{ij})x_j^2\right]^{\frac{1}{2}} \leq 6 - [(1-x_1)(6-\sqrt{34}) + (1-x_2)(6-\sqrt{25}) +$$

$$+ (1-x_3)(6-\sqrt{35}) + (1-x_4)(6-\sqrt{35}) +$$

$$+ (1-x_5)(6-\sqrt{31}) + (1-x_6)(6-\sqrt{35}) +$$

$$+ (1-x_7)(6-\sqrt{28}) + (1-x_8)(6-\sqrt{29}) +$$

$$+ (1-x_9)(6-\sqrt{34})] \tag{6-10}$$

or

$$\left[\sum_{j=1}^{9} \operatorname{Var}(c_{ij})x_j^2\right]^{\frac{1}{2}} \leq 0.17x_1 + x_2 + 0.08x_3 + 0.08x_4 + 0.43x_5 +$$

$$+ 0.08x_6 + 0.71x_7 + 0.61x_8 + 0.17x_9 +$$

$$+ 2.67 \tag{6-11}$$

(6-11) represents an equality rather than an approximation when all the projects are accepted. In the case that no project is accepted, i.e., $x_j = 0$ ($j = 1,2, \ldots, 9$), the value of (6-11) should be equal to zero and not 2.67.

One formula that will find the correct value when no project is accepted is:[5]

$$\left[\sum_{j=1}^{n} \operatorname{Var}(c_{tj})x_j^2\right]^{\frac{1}{2}} \leq \sum_{j=1}^{n} s(c_{tj})x_j \tag{6-12}$$

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5. *Ibid*., p. 270.

In the case under study:

$$\left[\sum_{j=1}^{9} \mathrm{Var}(c_{tj})x_j^2\right]^{\frac{1}{2}} \leq 1.41x_1 + 3x_2 + x_3 + x_4 + 2.24x_5 + x_6 + 2.83x_7 + 2.64x_8 + 1.41x_9 \qquad (6\text{-}13)$$

If all projects are accepted, the value of (6-13) will be 16.53 instead of 6.

Since a linear combination of (6-9) and (6-12) is a permissible linear approximation:

$$\left[\sum_{j=1}^{n} \mathrm{Var}(c_{tj})x_j^2\right]^{\frac{1}{2}} \leq p \sum_{j=1}^{n} s(c_{tj})x_j + (1-p)\left\{A - \sum_{j=1}^{n}(1-x_j)\left[A - \left[A^2 - \mathrm{Var}(c_{tj})\right]^{\frac{1}{2}}\right]\right\} \qquad (6\text{-}14)$$

The application of (6-14) to the problem under study when $p = .5$ results in:

$$\left[\sum_{j=1}^{9} \mathrm{Var}(c_{ij})x_j^2\right]^{\frac{1}{2}} \leq .79x_1 + 2x_2 + 0.54x_3 + 0.54x_4 + 1.34x_5 + 0.54x_6 + 1.77x_7 + 1.63x_8 + 0.79x_9 + 1.34 \qquad (6\text{-}15)$$

If all the projects are accepted, the value of (6-15) would be 11.28, which is an improvement over (6-13). If no project is accepted, the value of (6-15) is 1.34, which is an improvement over (6-11).

Since $1-a_t = .95$, the value of K, obtained from the standard normal tables would be:

$K = 1.645$

For the problem under study:

$$K\left[\sum_{j=1}^{9} Var(c_{1j})x_j^2\right]^{\frac{1}{2}} \leq 1.30x_1 + 3.29x_2 + 0.89x_3 + 0.89x_4 + 2.20x_5 + 0.89x_6 + 2.91x_7 + 2.68x_8 + 1.30x_9 + 2.20 \qquad (6\text{-}16)$$

Inserting (6-16) and (6-3) in (6-5) will result in:

$$13.30x_1 + 57.29x_2 + 6.89x_3 + 6.89x_4 + 32.20x_5 + 6.89x_6 + 50.91x_7 + 38.68x_8 + 19.30x_9 \leq 47.80. \qquad (6\text{-}17)$$

Where (6-17) is linear approximation to the non-linear constraint (6-5).

It is possible to find the linear approximation to the non-linear constraint (6-6) following the same methodology as was used for (6-5). The linear approximation tothe non-linear constraint (6-6) is:

$$3.90x_1 + 8.68x_2 + 7.32x_3 + 2.90x_4 + 39.23x_5 + 7.32x_6 + 4.90x_7 + 3.90x_8 + 3.90x_9 \leq 18.14 \qquad (6\text{-}18)$$

Now the proposed Lorie-Savage problem can be expressed as the following linear programming problem:

$$\text{Maximize: } 14x_1 + 17x_2 + 17x_3 + 15x_4 + 12x_6 + 14x_7 + 10x_8 + 12x_9 \qquad (6\text{-}19)$$

Subject to:

$$13.30x_1 + 57.29x_2 + 6.89x_3 + 6.89x_4 + 32.20x_5 + 6.89x_6 + 50.91x_7 + 38.68x_8 + 19.30x_9 \leq 47.80$$

$$3.90x_1 + 8.68x_2 + 7.32x_3 + 2.90x_4 + 39.23x_5 + 7.32x_6 + 4.90x_7 + 3.90x_8 + 3.90x_9 \leq 18.14$$

$$0 \leq x_j \leq 1 \quad (j = 1,2 \ \ldots, 9)$$

The solution is shown in the following table:[6]

Table 6-2. Solution of the Lorie-Savage problem under risk.*

x_1	= 1.0
x_2	= 0.0
x_3	= 0.997
x_4	= 1.0
x_5	= 0.0
x_6	= 0.0
x_7	= 0.028
x_8	= 0.0
x_9	= 1.0

* Total net present value is $58.34.

The linear programming solution calls for the total adoption of projects 1, 4 and 9, 99.7% of project 3 and 2.8% of project 7. Only $43.33 (86.66%) of the $50 budgeted for the first period (C_1), and only $14.10 (70.5%) of the $20 budgeted in the second period (C_2), are used.

6. The computer program, PROGRAM CHANCE, used to find this solution is listed in Appendix A.

Comparing the Lorie-Savage problem under certainty with this solution obtained under uncertainty, it is possible to see that with uncertainty the fraction of every project undertaken is not larger than the one accepted under certainty.[7]

With Certainty	With Uncertainty
$x_1 = 1.0$	$x_1 = 1.0$
$x_2 = 0.0$	$x_2 = 0.0$
$x_3 = 1.0$	$x_3 = 0.997$
$x_4 = 1.0$	$x_4 = 1.0$
$x_5 = 0.0$	$x_5 = 0.0$
$x_6 = 0.970$	$x_6 = 0.0$
$x_7 = 0.045$	$x_7 = 0.028$
$x_8 = 0.0$	$x_8 = 0.0$
$x_9 = 1.0$	$x_9 = 1.0$

The previous observation implies that the amount of money spent under uncertainty is less than under certainty, \$43.33 and \$14.10 versus \$50 and \$20 in this case. For the same reason the benefit obtained under uncertainty is less than the one obtained under certainty, \$58.34 versus \$70.25 in this case.

Another possible case to study is when only the budgets are random variables known with uncertainty. Its solution is straightforward because the new constraints do not contain non-linear elements.

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7. Naslund, "A Model of Capital Budgeting Under Risk," p. 264.

For example, in the Lorie-Savage problem, the budgets are the only random variables known with uncertainty, $s(C_1) = 3$ and $s(C_2) = 2$; the confidence levels desired for both periods is 95 percent.

The new constraints will be:[8]

$$\sum_{j=1}^{9} c_{1j}x_j \leq 50 - 1.645 \times 3 \qquad (6\text{-}20)$$

$$\sum_{j=1}^{9} c_{2j}x_j \leq 20 - 1.645 \times 2 \qquad (6\text{-}21)$$

$$0 \leq x_j \leq 1$$

Maximizing the objective function subject to these constraints expresses an ordinary linear programming problem similar to the one where all the variables are known with certainty.

Conclusions

Despite the apparent advantages of chance-constrained programming for capital budgeting applications, it has not been widely adopted. One of the greatest difficulties in applying this method is the necessity of establishing a meaningful confidence level. The nature of this decision requires management to make choices which reflect their perception of stockholder utility. Also, it may be difficult for the decision-maker to specify the probability a_t because he does not know *a priori* how different values of a_t are likely to affect the expected benefit.

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8. See Equation (5-32) in Chapter 5.

It is always possible, however, to solve the problem with different values of a_t and based on these solutions decide the value of a_t.[9] As stated by Naslund and Whinston:[10]

> By analyzing the dual variables an evaluation is obtained of changes in the parameters of the constraint. Some of the parameters denote the investor's willingness to take risks perhaps expressing verbally: "I am not willing to take a risk above 5% of losing more than $1000." The dual variables reflect how the expected value varies as a person alters his views on the degree of risk to which he is willing to expose himself. He might thus be willing to alter the constraints to obtain a more preferred balance between expected return and risk taking. The dual variables also are useful to the investor in deciding whether to seek additional capital for investment by providing a measure of the highest interest rate he should be willing to pay.

The use of (6-14) for finding a linear approximation to the nonlinear parts of the constraints (6-5) and (6-6) is not a drawback in that it is always possible to use more sophisticated techniques such as the cutting-plane method or separable programming which will arrive at an optimal solution more efficiently.

Summary and Direction for Future Research

This paper has endeavored to extend the technique of chance-constrained programming to the multiperiod capital budgeting problem under risk. To do this, maximization of the present value of net benefits was defined as the investor's objective.

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9. Naslund, "A Model of Capital Budgeting Under Risk," p. 260.

10. Naslund and Whinston, "A Model of Multi-period Investment Under Uncertainty," Management Science (January, 1962), p. 198.

Following the traditional capital budgeting framework, the net present value method was chosen for practical reasons as the optimum technique for maximizing investor utility when the investor has adequate capital to accept any project representing a net benefit or for purposes of ranking mutually exclusive projects. In the presence of single period capital rationing, the profitability index was selected as the method for identifying the set of integer projects that result in the maximization of the net present value. When the capital rationing is for more than one period, however, linear programming techniques become the only practical method of solution. For the multiperiod capital budgeting problem with uncertain outcomes, chance-constrained programming was selected as the appropriate procedure when only one of the variables (annual outlays or budgets) was a random variable. The procedures for applying this procedure were expanded here.

There are classes of capital budgeting problems not treated here which merit attention. For instance, chance-constrained programming was not employed when outlays and budgets for the same period were random variables. This case is more realistic than the ones studied in this paper. The development of a workable approach to this problem would be an important contribution to the capital budgeting process. Another direction to follow could be the use of chance-constrained programming and goal programming for analyzing capital budgeting decisions under uncertainty when the investor has more than one objective.

APPENDIX A

PROGRAM LISTING FOR THE SOLUTION OF THE LORIE-SAVAGE EXAMPLE UNDER RISK

```
PROGRAM CHANCE(INPUT, OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
COMMON SM(15,25), NBASIC(15), YVALUE(25)
NR=12
NC=22
SM(1,2)=-14
SM(1,3)=-17
SM(1,4)=-17
SM(1,5)=-15
SM(1,6)=-40
SM(1,7)=-12
SM(1,8)=-14
SM(1,9)=-10
SM(1,10)=-12
SM(2,2)=13.30
SM(2,3)=57.29
SM(2,4)=6.89
SM(2,5)=6.89
SM(2,6)=32.20
SM(2,7)=6.89
SM(2,8)=50.91
SM(2,9)=38.68
SM(2,10)=19.30
SM(2,11)=1
SM(2,22)=47.80
SM(3,2)=3.90
SM(3,3)=8.68
SM(3,4)=7.32
SM(3,5)=2.90
SM(3,6)=39.23
SM(3,7)=7.32
SM(3,8)=4.90
SM(3,9)=3.90
SM(3,10)=3.90
SM(3,12)=1
SM(3,22)=18.14
SM(4,2)=1
SM(4,13)=1
SM(4,22)=1
SM(5, 3)=1
SM(5,14)=1
SM(5,22)=1
SM(6, 4)=1
SM(6,15)=1
SM(6,22)=1
SM(7, 5)=1
SM(7,16)=1
SM(7,22)=1
SM(8, 6)=1
```

```
      SM(8,17)=1
      SM(8,22)=1
      SM(9, 7)=1
      SM(9,18)=1
      SM(9,22)=1
      SM(10, 8)=1
      SM(10,19)=1
      SM(10,22)=1
      SM(11, 9)=1
      SM(11,20)=1
      SM(11,22)=1
      SM(12,10)=1
      SM(12,21)=1
      SM(12,22)=1
  200 NTABLE=NTABLE+1
      CALL SIMPLEX (NR,NC,K)


      DO 202 I=1, NC
      WRITE (6,201) I, YVALUE(I)
  201 FORMAT (1H, 20X, I3, F 15.3)
  202 CONTINUE
      WRITE(6,7)NTABLE,LPP
    7 FORMAT (1H ,10X,14HLP TABLEAU NO.,I2,/
     1 11X,14HLP PROBLEM NO.,I2)
C
      NCC=NC- 1
      DO 68 I4=2,NR
      DO 67 J4=2,NCC
      IF(SM(I4,J4) .LT. .999 .OR. SM(I4,J4) .GT. 1.001)GO TO 67
      JJ4= J4
      SUM= 0.
      DO 66 I=1,NR
      SUM= SUM+ SM(I,JJ4)
   66 CONTINUE
      TOT=ABS(1.-SUM)
      IF(TOT .LT. .0001)NBASIC(I4-1)=JJ4-1
      IF(TOT .LT. .0001)GO TO 68
   67 CONTINUE
   68 CONTINUE
C
      NC1= NC-2
      DO 70 KK1=1,NC1
      YVALUE(KK1)= 0.
   70 CONTINUE
      NR1=NR-1
      DO 72 KK2=1,NR1
      N= NBASIC(KK2)
      IF(N .EQ. 0)GO TO 72
      YVALUE(N)= SM(KK2+1,NC)
```

```
   72 CONTINUE
C
C        *CHECK FOR OPTIMAL,FEASIBLE SOLUTION,
C             KFLAG2= 1, EITHER NOT OPTIMAL OR NOT FEASIBLE
C                   = 2, OPTIMAL AND FEASIBLE.
C
      KFLAG2= 2
      DO 60 K2=2,NR
      IF(SM(K2,NC) .LT. 0.)KFLAG2=1
   60 CONTINUE
      IF(KFLAG2 .EQ. 1)GO TO 200
      NCC= NC-1
      DO 65 K3= 2,NCC
      IF(SM(1,K3).LT. 0.)KFLAG2= 1
   65 CONTINUE
      IF(KFLAG2 .EQ. 1)GO TO 200
C
C
      DO 203 I=1, NC
      WRITE (6,201) I, YVALUE(I)
  203 CONTINUE
      STOP
      END
      SUBROUTINE SIMPLEX(NR,NC,K)
C        *SIMPLEX ALGORITHM. THE SIMPLEX ALGORITHM STARTS
C         WITH A SYSTEM IN BASIC, FEASIBLE FORM.
C
C                 NR= NO. OF ROWS IN SYSTEM MATRIX
C                 NC= NO. OF COLUMNS IN SYSTEM MATRIX
C                  K= BOUNDNESS INDICATOR,
C                   = 0, SOLUTION IS UNBOUNDED,
C                   = 1, BASIC, FEASIBLE SOLUTION IS AVAILABLE.
C
      COMMON SM(15,25), NBASIC(15), YVALUE(25)
C
      WRITE(6,5)
    5 FORMAT(1H1,20X,20H*SUBROUTINE SIMPLEX*,//)
C        *FIND CMOST, MOST NEGATIVE COST COEFFICIENT SM(1,J).
C
      J1=1
      K=0
      CMOST=0.
      NCC= NC-1
      DO 10 J=2,NCC
      IF(SM(1,J).GE.CMOST)GO TO 10
      J1= J
      CMOST= SM(1,J1)
   10 CONTINUE
      IF(CMOST.EQ.0.)RETURN
```

```
      I1=1
      R= 1.E+20
      DO 20 I2= 2,NR
      IF(SM(I2,NC) .LE. 0.)GO TO 20
      IF(SM(I2,J1).LE. 0.)GO TO 20
      RR= SM(I2,NC)/SM(I2,J1)
      IF(NCYCLE .LT. J8)
     1WRITE(6,15)RR
   15 FORMAT(1H ,25X, 4HR = ,E11.3)
      IF(RR .LT. R)I1= I2
      IF(RR .LT. R)R= RR
      K=1
   20 CONTINUE
C          *PIVOT ON ROW I1 AND COLUMN J1 OF SYSTEM MATRIX
C
      WRITE(6,25)I1,J1
   25 FORMAT(1H ,25X,14HPIVOT POINT  (,I2,1H,,I2,1H),/)
      ZTEST= 1.E-06
      IF(SM(I1,J1) .LT. ZTEST) WRITE(6,30)
   30 FORMAT(1H ,10X,24HNO PIVOT POINT AVAILABLE)
      IF(SM(I1,J1) .LT.ZTEST)K=0
      IF(SM(I1,J1) .LT. ZTEST)RETURN
      CALL PIVOT(NR,NC,I1,J1)
      RETURN
      END
      SUBROUTINE DUAL(NR,NC,K)
```

```
      COMMON/BLOCK5/NCYCLE,J8
      COMMON SM(15,25), NBASIC(15), YVALUE(25)
      WRITE(5,5)
    5 FORMAT(1H ,30X,17H*SUBROUTINE DUAL*,//)
```

```
      K= 0
      BMOST= 0.
      DO 10 I=2,NR
      IF(SM(I,NC) .GE. BMOST)GO TO 10
      I1= I
      BMOST= SM(I1,NC)
   10 CONTINUE
      IF(BMOST .EQ. 0.)RETURN
      L=NC-1


C
      J1=1
      R= -1.E+20
      DO 20 I2=2,L
      IF(SM(1,I2).LE. 0.)GO TO 20
      IF(SM(I1,I2).GE. 0.)GO TO 20
      RR= SM(1,I2)/SM(I1,I2)
      IF(RR .GT. R)J1= I2
      IF(RR .GT. R)R= RR
      K=1
      IF(NCYCLE .LT. J8)
     1WRITE(6,15)I2,RR
   15 FORMAT(1H ,35X,I2,5X,4HR = ,E11.3)
   20 CONTINUE
      WRITE(6,25)I1,J1
   25 FORMAT(1H ,35X,14HPIVOT POINT  (,I2,1H,,I2,1H),/)
      ZTEST= 1.E-06
      IF(ABS(SM(I1,J1)) .LT. ZTEST)WRITE(6,30)
   30 FORMAT(1H ,10X,24HNO PIVOT POINT AVAILABLE)
      IF(ABS(SM(I1,J1)) .LT. ZTEST)RETURN
      CALL PIVOT(NR,NC,I1,J1)
      RETURN
      END
      SUBROUTINE PIVOT(NR,NC,I,J)
C           *THIS SUBROUTINE PIVOTS ON ELEMENT (I,J) OF THE
C            SYSTEM MATRIX TO UPDATE IT.
C
C
C
      COMMON SM(15,25), NBASIC(15), YVALUE(25)
      IF(NCYCLE .LT. J8)
     1WRITE(6,5)
    5 FORMAT(1H ,40X,18H*SUBROUTINE PIVOT*,//)
      ZTEST= 1.E-06
      VALUE= 1./SM(I,J)
      DO 10 N= 1,NC
      SM(I,N)= VALUE*SM(I,N)
```

```
   10 CONTINUE
C
      DO 30 M=1,NR
      IF(M.EQ.I)GO TO 30
      IF(SM(M,J) .EQ. 0.)GO TO 30
      Q= SM(M,J)
      DO 20 NN=1,NC
      SS= SM(M,NN)-SM(I,NN)*Q
      IF(ABS(SS).LT.ZTEST)SS= 0.
      SM(M,NN)= SS
   20 CONTINUE
   30 CONTINUE
      RETURN
      END
```

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