AVAILABILITY OF REPAIRABLE SYSTEMS WHEN REPAIR AND FAILURE RATES ARE NOT CONSTANT

by

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ABSTRACT

A modern trend in design is to develop a system capable of being returned to normal operation fast after a failure occurs. The availability function analytically expresses this capability. This thesis analyzes a possible analytical method for the determination of the availability. The evaluation of a systems availability in time is a stochastic process. In particular, the renewal process and the Markov process are applied. Associated with the stochastic processes is the distribution of time for each state. The exponential distribution gives no restriction for the availability calculation regardless of complexity of the system. But the aim of this study is to consider non-exponential distributions including the normal, lognormal, Weibull and gamma distributions, with emphasis on the gamma distribution.

For the availability derivation the Laplace transforms and the inverse Laplace transforms are used extensively.

CHAPTER 1

INTRODUCTION

During the past decade, the development of reliability and availability has been rapid, caused by the development of large and complex systems susceptible to failure and requiring restoration. The reliability is defined as the probability that a system has operated over the given interval without failure. To consider the benefit of repair capability, the availability function is introduced. Usually three aspects of availability are considered: (1) the availability $A(t)$, also known as an instantaneous availability, is the probability that the system is operating at time t, (2) the interval availability, defined as the expected fraction of a given interval of time (t_1,t_2) that the system will be able to operate, or

$$A(t_1,t_2) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} A(t)\,dt \qquad (1.1)$$

(3) the steady state availability, or the expected fraction of time in the long run that the system will be able to operate, or

$$A(\infty) = \lim_{t\to\infty} A(t) \qquad (1.2)$$

Obviously, the goal is to find the instantaneous availability, $A(t)$, because the interval availability and steady state availability are then easily determined using equations (1.1) and (1.2).

In general, the availability is a complex function which takes into account the configuration of the units in the system and the underlying distributions for the times to failure and the times to repair each unit.

Throughout the literature the availability function is derived assuming exponentiality. The scope of this study is to consider non-constant failure and repair rates. Chapter 2 is devoted to some commonly used distributions: normal, lognormal, gamma, Weibull, and the approximation of an arbitrary distribution. We focus our attention on the gamma distribution because it has two convenient features: the Laplace transform in a very simple analytical form, and it represents the distribution of summation of exponentially distributed random variables. The former will be obvious in Chapter 3 where we need the Laplace transform of the times-to-failure and of the times-to-repair distributions to find the availability function. The latter enables us to develop the Markov model for a given system even when the distributions for the times to failure and the times to repair are not exponential.

CHAPTER 2

FAILURE AND REPAIR RATES

2.1 General Remarks

When determining the availability function the stochastic process, Z(t), will be used, and the two-state, semi-Markov process will be applied. The two-state process is completely defined by the distributions of the times to failure X and the times to repair Y. The exponential is the most frequently used distribution. The reason for this is its very simple analytical form, which gives no difficulty in solving the systems of differential equations (Markov processes) or Laplace transforms and inverse Laplace transforms (semi-Markov processes and availability functions) in a general form. Regardless of its simplicity many authors find the exponential distribution a very convenient model to express physical behavior of complex systems. Epstein (1958), and Barlow, Proschan and Hunter (1965, p. 18) give thorough discussions of the exponential distribution, while Drenick (1960) gives a proof that the exponential is a good failure law for complex equipment. This discussion is repeated here briefly.

Consider a system consisting of many components. Each component has a failure distribution which is not necessarily exponential. A component failure causes a system failure, and after each failure occurs the component is immediately replaced. An unending sequence of failures and replacements constitutes a random process called a "renewal process." Drenick (1960) has proved that the time between system failure tends to an exponential distribution as the complexity of the system increases. However, the exponential time between system failures is reached only after a sufficiently long time: when the system consists of a mix of components of different ages, and when the system is in statistical equilibrium.

As is well known, the exponential distribution has a property that the component does not deteriorate with age. In other words, if a component has not failed up to a time T, the probability of success in its future for a mission t is the same as if the component were quite new and had just been placed in use at time T. This property can be expressed as:

$$P_s(t,T) = \frac{P(T+t)}{P(T)} = \frac{e^{-\lambda(T+t)}}{e^{-\lambda T}} = e^{-\lambda t} \tag{2.1}$$

or

$$P_s(t,T) = P_s(t). \tag{2.2}$$

This property makes the exponential distribution rather unsuitable for the repair time of the component, because it means that the time spent on a repair in the past has no influence on the time which will be spent for repair in the future. An exponential distribution, as a one-parameter distribution, has a variance equal to its mean. The repair times distribution is desired to have much more flexibility, with a variance independent of the mean. Boivard and Zagor (1961), Kline (1974), and many others, emphasize the lognormal distribution for repair time. The lognormal distribution has most of the probability mass concentrated close to the mean repair time which is in harmony with a physical behavior of repair time. But, some other unimodal distributions such as the gamma, normal and Weibull have a similar shape and might be chosen as an appropriate model. Before choosing any of these distributions, the best approach is to apply the K-S (or some other) goodness-of-fit test to see which of the competing distributions fit the data best. Sometimes a suitable analytical form and mathematical feature is a reason for favoring a certain distribution. This will be discussed later in this chapter.

The gamma, normal, lognormal and Weibull distributions are appropriate models for the probability law of the time to failure. Thus all of these distributions can be considered for both: probability law of time to failure and probability law of time to repair.

In contrast with the exponential distribution, which is one-parameter distribution, the lognormal, normal, gamma and Weibull are two or three parameter distributions. These parameters can always be related to the mean value m_X and the variance σ_X^2. For practical purposes the dimensionless parameter $\eta_X = \sigma_X/m_X$, called the coefficient of variation, is often used. For the exponential distribution the coefficient of variation is 1. Muth (1967, p. 28) has stated that aging of a component, as well as organization in a repair, would cause the distribution to be less dispersed about its mean value than the exponential. Thus we will be looking for distributions having coefficients of variation less than 1.

A probability law of random variable[1] X can be defined as the density function $f_X(x)$, as the distribution function $F_X(x)$, and as the hazard function $h_X(x)$. The hazard function is known under the names of "force of mortality," "Mills' ratio," "intensity function," or "hazard rate." Here we will refer to the hazard as the "failure rate" for time to failure and use "repair rate" for the time to repair. Cox (1962, pp. 3-6) calls the hazard function the "age-specific failure rate" $[\phi(x)]$, and defines it by equation

$$h_X(x) = \lim_{\Delta x \to 0+} \frac{\text{prob}(x < X \leq x + \Delta x \mid x < X)}{\Delta x} \tag{2.3}$$

Equation (2.3) can be rewritten as

$$h_X(x) = \lim_{\Delta x \to 0+} \frac{\text{prob}(x < X \leq x + \Delta x)}{\Delta x} \cdot \frac{1}{\text{prob}(x < X)} \tag{2.4}$$

The first part of equation (2.4) represents the probability density function $f_X(x)$ (Meyer, 1972, p. 73), and the denominator of the second part of equation (2.4) is the cumulative distribution function $F_X(x < X) = 1 - F_X(X < x)$. Thus equation (2.4) becomes

$$h_X(x) = \frac{f_X(x)}{1-F_X(x)} \tag{2.5}$$

Taking the integral and making conversions yields

$$F_X(x) = 1 - e^{-H_X(x)} \tag{2.6}$$

where

$$H_X(x) = \int_0^x h_X(u)du \tag{2.7}$$

---

1. In this section a positive and continuous random variable X is considered.

Equations (2.5) and (2.6) relate $f_X(x)$, $F_X(x)$ and $h_X(x)$. Equations (2.3) and (2.4) provide a physical interpretation of $h_X(x)$. It could be stated as follows: If E is an event, then $h_X(x)\Delta x$ represents the conditional probability that E will occur in interval $(x,x+\Delta x)$ given that E did not occur by x. Converting the random variable X to variable time t, then event E becomes a "failure" or a "repair completed." The hazard function is a function of time t and its increasing magnitude indicates aging of the component. In other words, for any component which deteriorates with age, the conditional probability that failure occurs in interval $(t,t+\Delta t)$ is higher than for any interval before t.

The purpose of this study is to consider the functions which have non-constant, increasing or decreasing hazard functions. A thorough research on this subject is given by Barlow, Proschan, and Hunter (1965, pp. 22-45) and Barlow, Marshall, and Proschan (1963) and in references therein. Through this study we will focus attention on the hazard functions of two-parameter distributions: gamma, normal, lognormal and Weibull. A general model will also be discussed later in this chapter.

As was mentioned before, the two-parameter distributions can always be related to the mean value, m_t, and the variance, σ_t^2. In Figs. 1 and 2 the behavior of the hazard functions and the density functions for the normal, Weibull, gamma and lognormal distribution with the same mean $m_t = 1$ and variance $\sigma_t^2 = 0.25$ is given. It is obvious from Fig. 1 that the hazard functions have increasing and fairly close values up to time 2.0. But, from Fig. 2, it follows

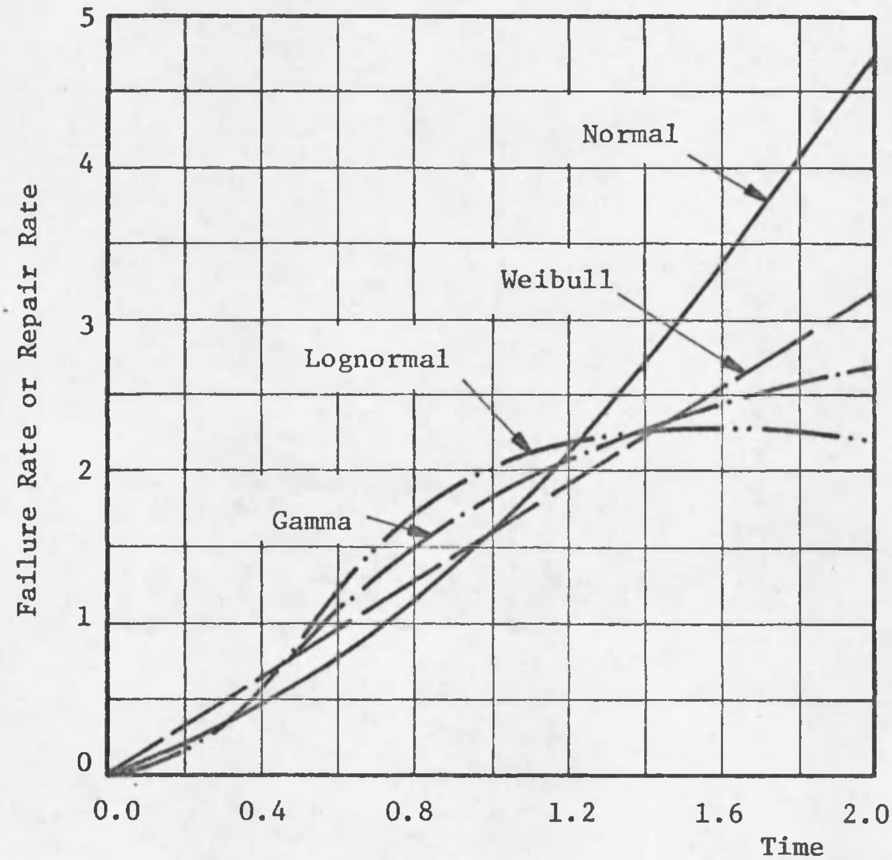


Fig. 1 Failure Rate and Repair Rate Functions Versus Time for Various Distributions of Times to Failure and Times to Repair with Mean Value = 1 and Standard Deviation = 0.5

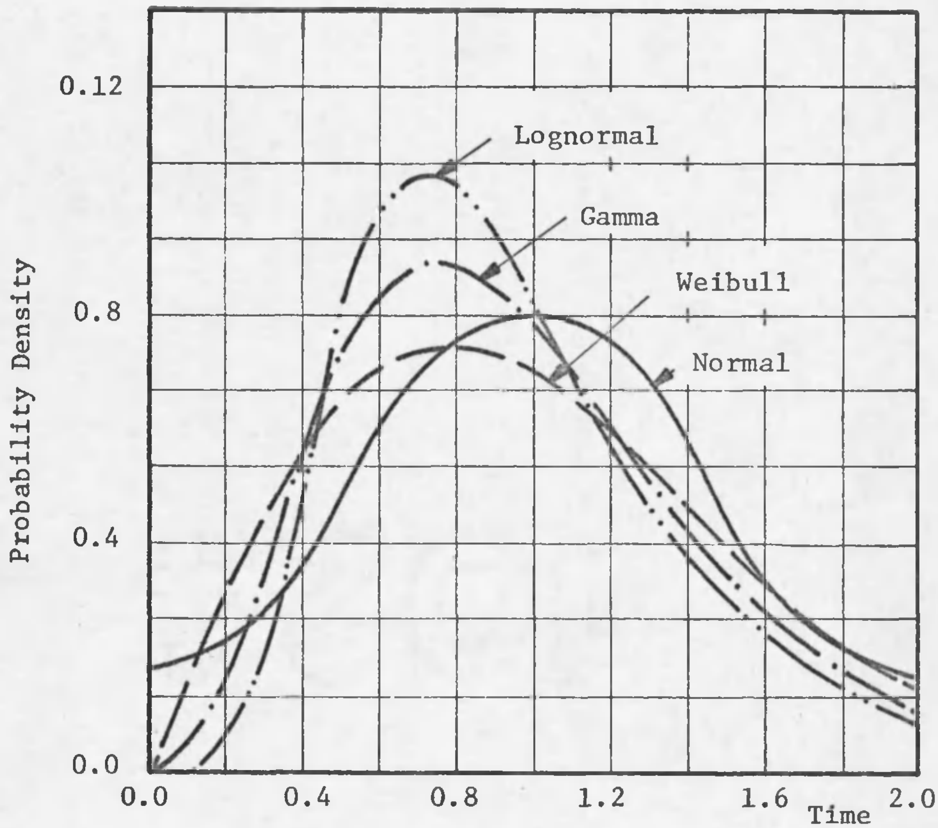


Fig. 2 Probability Density Functions with Mean Value = 1 and Standard Deviation = 0.5

that most of the probability mass is concentrated in the interval (0.2.0). The foregoing means that the large difference between the hazard functions is in the tails of the distributions.

Further properties of the hazard functions follow from Figs. 3, 4, 5 and 6. The gamma and Weibull distributions have an increasing $h_{G,W}(t)$ in the interval $(0,\infty)$ for parameter $\beta > 1$. The normal distribution has an increasing $h_N(t)$ in the interval $(0,\infty)$ for all values of its parameters μ and σ. The lognormal distribution has increasing and decreasing $h_L(t)$ with the property.

$$\lim_{t \to \infty} h_L(t) \to 0 \tag{2.8}$$

Muth (1967, p. 30) and Barlow, Proschan, and Hunter (1965, p. 13) consider the property (2.8) of $h_L(t)$ as a disadvantage, because it is not consistent with the component aging characteristics. However, they agree that the decreasing period is in the tail of the distribution where only about 0.05 of all of the probability mass lies, and that this could be neglected.

Now, additional properties of the normal, lognormal, Weibull, and gamma distributions will be discussed.

2.2 The Normal Distribution

The normal or Gaussian distribution is the best known statistical model. For a random variable N its probability density function is

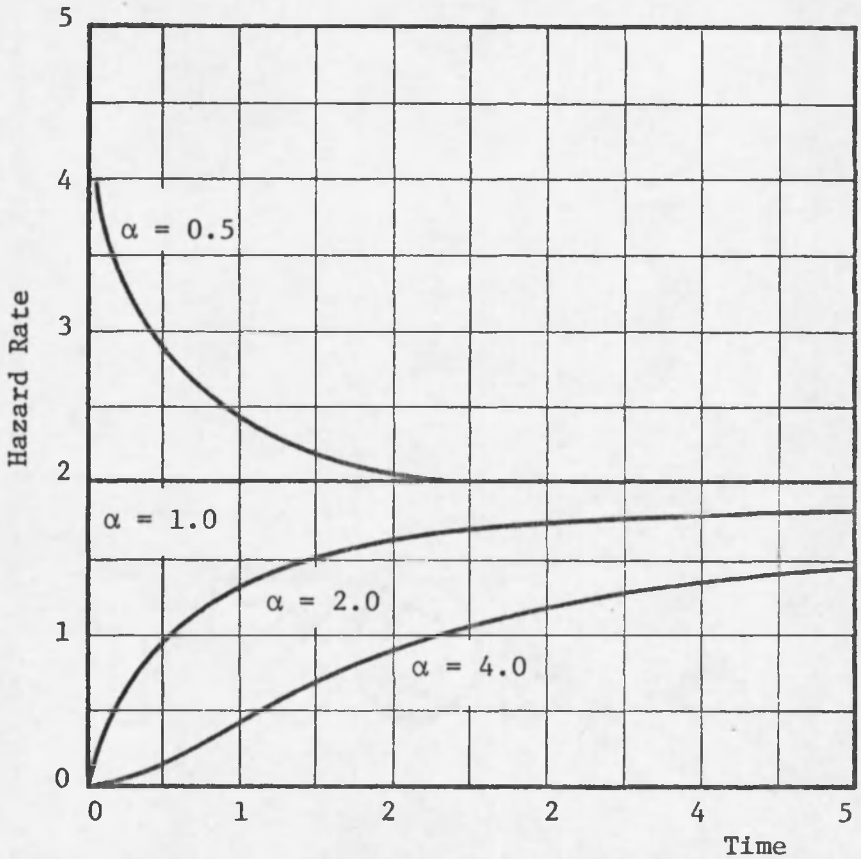


Fig. 3 Plot of Hazard Rate Functions of the Gamma Distribution for $\lambda = 2$

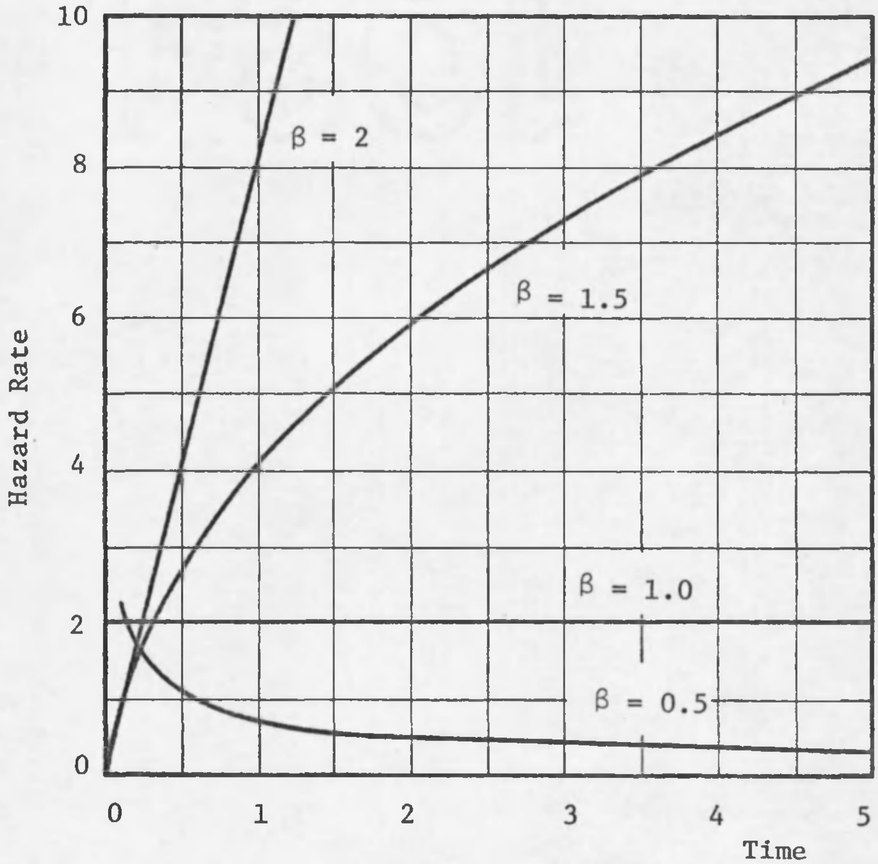


Fig. 4 Plot of Hazard Rate Functions of the Weibull Distribution for $1/\eta = 2$

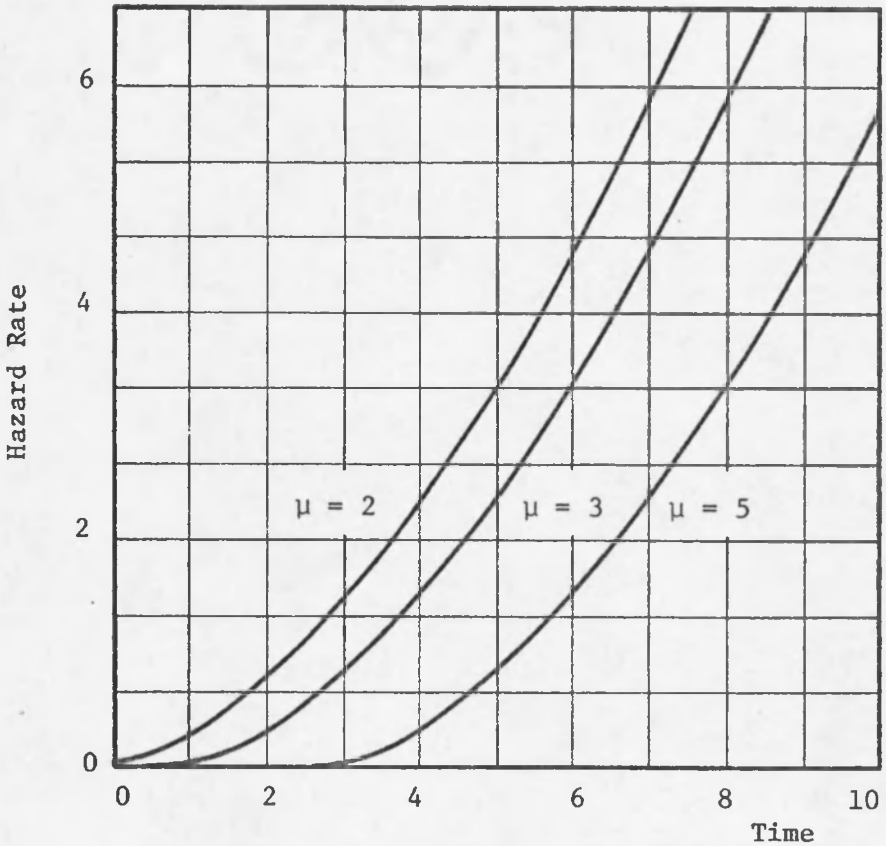


Fig. 5 Plot of Hazard Rate Functions of the Normal Distribution for $\sigma = 1$

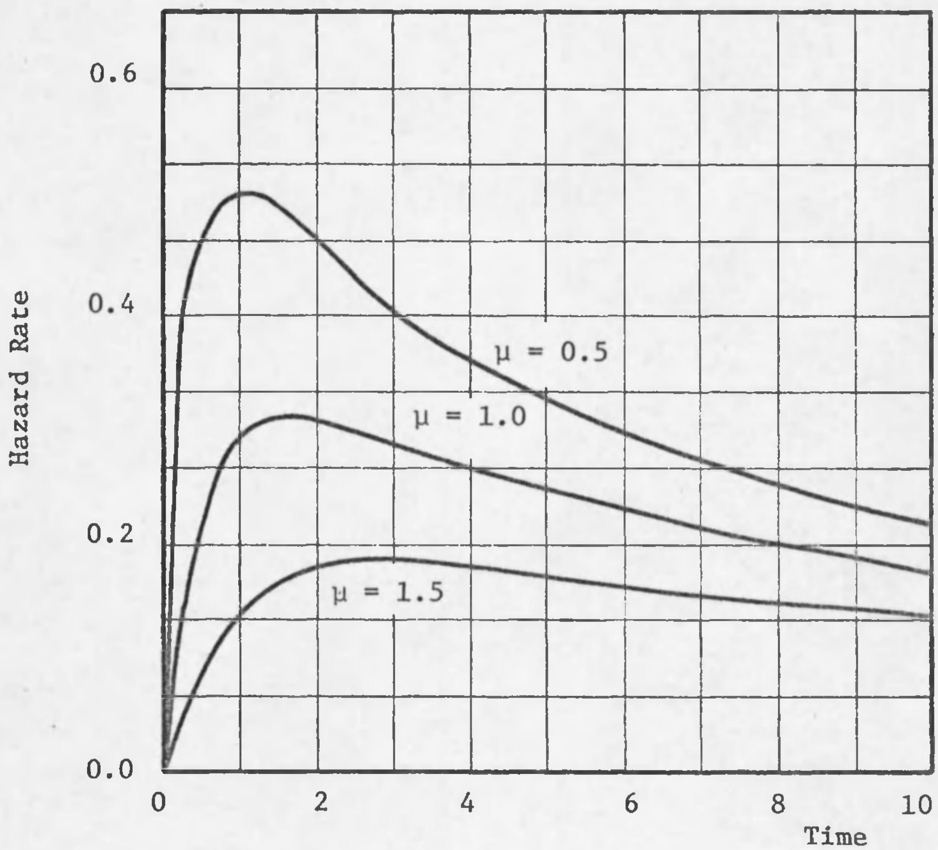


Fig. 6 Plot of Hazard Rate Functions of the Lognormal Distribution for σ = 1

$$f_N(t;\mu,\sigma) = \frac{1}{a\sigma\sqrt{2\pi}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right]$$
$$\sigma > 0, \; -\infty < \mu < \infty; \; 0 \leq t < \infty \tag{2.9}$$

where a is a normalizing constant, μ is mean value and σ is the standard deviation, or the location and the scale parameters, respectively. But, replacing the random variable N by the random variable $Z = (t-\mu)/\sigma$ gives the standardized normal density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{2.10}$$

and the standardized cumulative distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}x^2} dx \tag{2.11}$$

Equations (2.10) and (2.11) are tabulated and can be used to evaluate $f_N(t;\mu,\sigma)$ and $F_N(t;\mu,\sigma)$ for any parameters μ and σ.

A general form of the hazard function is shown by equation (2.5). It is obvious that both the numerator and the denominator tend to zero as $t \to \infty$. Thus, applying L'Hospital's rule, gives

$$\lim_{t\to\infty} h_X(t) = \lim_{t\to\infty} \frac{-f'_X(t)}{f_X(t)} \tag{2.12}$$

where the prime denotes differentiation with respect to t. From equation (2.9) it follows that

$$f'_N(t;\mu,\sigma) = \frac{-(t-\mu)}{a\sigma^3\sqrt{2\pi}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] \tag{2.13}$$

Substituting equations (2.13) and (2.9) into equation (2.12), we obtain the limit value of the hazard function as

$$\lim_{t\to\infty} h_N(t;\mu,\sigma) \to \infty \tag{2.14}$$

as shown in Fig. 5.

The moment-generating function (mgf) for the normal distribution (Meyer, 1972, p. 212) is defined as

$$M_N(z) = E(e^{zN}) = \exp(z\mu + \frac{1}{2}\sigma^2 z^2) \tag{2.15}$$

A well-known property of the mgf is

$$M^{(n)}(0) = E(X^n) \tag{2.16}$$

that is, the n^{th} derivative of the $M_X(t)$ evaluated at $z = 0$ yields the n^{th} moments of the random variable X, $E(X^n)$, about zero. When applying property (2.16) to equation (2.15), we have the familiar results

$$E(N) = M_N^{(1)}(0) = \mu \tag{2.17}$$

and

$$E(N^2) = M_N^{(2)}(0) = \sigma^2 + \mu^2 \tag{2.18}$$

In the same way, the higher moments of the variable N, $E(N^n)$, could be obtained.

The mgf may be used, also, to show the reproductive properties of the normal distribution (Meyer, 1972, p. 217). This property is formulated in the following way: Suppose that N_1 and N_2 are independent random variables with the normal distributions $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, respectively.[2] Let the random variable N be defined as $N = N_1 + N_2$, then N is also normally distributed with $E(N) = \mu_1 + \mu_2$

---

2. $N(\mu,\sigma^2)$ - is the denotation for the normal distribution with the parameters μ and σ.

and $V(N) = \sigma_1^2 + \sigma_2^2$, or

$$f_N(t;\mu,\sigma) = f_N(t;\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}) \tag{2.19}$$

We can complete the reproductive property as

$$f_N(t;\mu,\sigma) = f_N[t; \sum_{i=1}^{n} \mu_i, (\sum_{i=1}^{n} \sigma_i^2)^{1/2}] \tag{2.20}$$

for n independent random variables N_i.

One of the most important results of mathematical statistics is the central limit theorem, which is based on the normal distribution. This theorem may be stated as follows: Let $X_1, X_2, \ldots, X_n$ be a sequence of any n independent random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$, and let $X = X_1 + X_2 + \ldots + X_n$, then, for sufficiently large n, X is approximately normally distributed with $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

The estimation of normal distribution parameters is given in Hahn and Shapiro (1967, p. 74).

2.3 The Lognormal Distribution

For the lognormal distribution, L denotes a random variable and t is the same specific value of the random variable L. Then the lognormal density function is

$$f_L(t;\mu,\sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp \{- \frac{(\log t - \mu)^2}{2\sigma^2}\} , \ t > 0 \tag{2.21}$$

If N denotes a normal random variable, then the transformation $L = e^N$ produces the lognormal variable. From this transformation follows a a relationship between the lognormal and the normal, or the standardized normal, density functions

$$f_L(t;\mu,\sigma) = \frac{1}{t\sigma}\,\phi(\frac{\log t-\mu}{\sigma}) \tag{2.22}$$

and distribution functions

$$F_L(t;\mu,\sigma) = \Phi(\frac{\log t-\mu}{\sigma}) \tag{2.23}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standardized normal density and the cumulative distribution functions, respectively.

From equation (2.15) and transformation $L = e^N$ it follows that the k^{th} moment of L is

$$E(L^k) = E[e^{kN}] = \exp(k\mu + \frac{1}{2}\,k^2\,\sigma^2) \tag{2.24}$$

and the first and second moments are

$$E(L) = m_L = \exp(\mu + \frac{1}{2}\,\sigma^2)$$

and (2.25)

$$E(L^2) = \exp\{2(\mu + \sigma^2)\}$$

The variance is $V(L) = E(L^2) - \{E(L)\}^2$ which gives

$$V(L) = \sigma_L^2 = e^{2\mu}\,\{e^{2\sigma^2} - e^{\sigma^2}\} \tag{2.26}$$

The ratio of equation (2.25) and (2.26) gives

$$\eta_L = \{\exp(\sigma^2) - 1\}^{1/2} \tag{2.27}$$

where η_L denotes the coefficient of variation.

It was already stated that the limit value of the lognormal hazard function is zero. The proof follows from equations (2.21) and (2.12), whence

$$\frac{f_L'(t;\mu,\sigma)}{f_L(t;\mu,\sigma)} = -\frac{1}{t} - \frac{1}{t\sigma^2}(\log t - \mu) \tag{2.28}$$

It is clear that

$$\lim_{t\to\infty} h_L(t;\mu,\sigma) = 0 \tag{2.29}$$

The lognormal distribution has a reproductive property. The central limit theorem states that a product of independent and identically distributed random variables approaches a lognormal distribution. This is proven in Aitchison and Brown (1957, pp. 9-14).

2.4 The Weibull Distribution

Weibull proposed this distribution to represent the yield strength of steel, but it has been widely explored and applied in many other fields. Let W denote a Weibull random variable. Its density function is

$$f_W(t;\beta,\eta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp\{-(\frac{t}{\eta})^{\beta}\} \tag{2.30}$$

and its distribution and hazard functions are

$$F_W(t;\beta,\eta) = 1 - \exp\{-(\frac{t}{\eta})^{\beta}\} \tag{2.31}$$

and

$$h_W(t;\beta,\eta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \tag{2.32}$$

A graph of the hazard function is shown on Fig. 4. It may be seen that h_W is an increasing function in interval (0,) for a shape parameter $\beta > 1$.

The moment of W could be obtained using the exponential distribution with a random variable X

$$F_X(t) = 1 - e^{-t} \tag{2.33}$$

From equations (2.30) and (2.33)

$$W = (\frac{1}{\eta})^{-1} X^{1/\beta} \tag{2.34}$$

The k^{th} power of W is

$$W^k = (\frac{1}{\eta})^{-k} X^{k/\beta} \tag{2.35}$$

and

$$E(W^k) = (\frac{1}{\eta})^{-k} E(X^{k/\beta}) \tag{2.36}$$

Because X has an exponential distribution, $E(\cdot)$ is determined from

$$E(x^{k/\beta}) = \int_o^\infty t^{k/\beta} e^{-t} dt = \Gamma(1 + \frac{k}{\beta}) \tag{2.37}$$

Thus

$$E(W^k) = \frac{\Gamma(1 + \frac{k}{\beta})}{(\frac{1}{\eta})^k} \tag{2.38}$$

and

$$E(W) = m_W = \eta\Gamma(1 + \frac{1}{\beta})$$

$$Var(W) = \sigma_W^2 = \frac{\Gamma(1 + \frac{2}{\beta}) - \{\Gamma(1 + \frac{1}{\beta})\}^2}{(\frac{1}{\eta})^2} \tag{2.39}$$

or

$$\eta_W = [\frac{\Gamma(1 + \frac{2}{\beta})}{\{\Gamma(1 + \frac{1}{\beta})\}^2} - 1]^{1/2} \tag{2.40}$$

where η_W denotes the coefficient of variation.

It is well known that the Weibull distribution with parameter $\beta = 1$ represents the exponential distribution, and with $\beta = 2$ the Rayleigh distribution.

2.5 The Gamma Distribution

The family of gamma distributions is one of the best known and most thoroughly explored. It may be the appropriate model for the system time-to-failure and time-to-repair distributions. Further, it plays an important role in statistical queueing theory and in Bayesian analyses as a prior model to describe initial uncertainty concerning the rate of occurrence of some process.

The gamma function, denoted by Γ, is defined as

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx \text{ for } p > 0 \tag{2.41}$$

while the density function for the gamma random variable G is

$$f_G(t;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \text{ for } \lambda, \alpha > 0 \tag{2.42}$$

where α is the shape parameter and λ is the scale parameter. The effect of λ can be seen from the relation

$$f_G(t;\alpha,\lambda) = \lambda f_G(\lambda t;\alpha,1) \tag{2.43}$$

From equation (2.42) the cumulative distribution function is

$$F_G(t;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-\lambda x} dx \tag{2.44}$$

The integral of equation (2.44) is tabulated and known as the incomplete gamma function (Pearson, 1957). In most of our applications the parameter α will take an integer value n, where n = 1,2,....

Then the gamma distribution, equation (2.44), may be expressed in terms of the Poisson distribution (Meyer, 1972, p. 135) as

$$F_G(t;\alpha=n,\lambda) = 1 - \sum_{k=o}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

or (2.45)

$$F_G(t;\alpha=n,\lambda) = 1 - e^{-\lambda t}\{1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\}$$

It is seen from equation (2.42) that the exponential distribution is a special case at the gamma distribution when $\alpha = 1$. Furthermore, the sum of exponential variables is the gamma variable. This property will be used in Chapter 4, and therefore is derived here.

Suppose that T_1 and T_2 are independent, continuous random variables with density functions g and h, respectively. The joint density function is

$$f(t_1,t_2) = g(t_1)h(t_2) \tag{2.46}$$

Let $T = T_1 + T_2$, then $T_2 = T - T_1$ and equation (2.46) becomes

$$f(t,t_1) = g(t_1)h(t-t_1) \tag{2.47}$$

The density function at T is obtained by integrating $f(t,t_1)$ from $-\infty$ to ∞ with respect to t_1

$$f(t) = \int_{-\infty}^{\infty} g(t_1)h(t-t_1)dt_1 \tag{2.48}$$

The above integral is often called the convolution integral of g and h, and is usually written as g*h. Now taking T_1 and T_2 as exponentially distributed with parameters λ_1 and λ_2, respectively, equation (2.48) becomes

$$f(t) = \int_o^\infty \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 (t-t_1)} dt_1; \ t > 0 \tag{2.49}$$

whose solution is

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \tag{2.50}$$

When taking $\lambda = \lambda_1 = \lambda_2$, integral (2.48) becomes

$$f(t) = \int_o^\infty \lambda e^{-\lambda t_1} \lambda e^{-\lambda (t-t_1)} dt_1 \tag{2.51}$$

with the solution

$$f(t) = \lambda^2 t \ e^{-\lambda t}, \ t \geq 0 \tag{2.52}$$

This is a gamma distribution with parameters λ and $\alpha = 2$. The integral of equation (2.52) is the cumulative gamma distribution function

$$F_G(t;2;\lambda) = 1 - e^{-\lambda t} \{1+\lambda t\} \tag{2.53}$$

Now comparing equations (2.53) and (2.45) we conclude that the gamma distribution is the sum of n independent, exponentially distributed random variables. Further, applying the central limit theorem, it follows that the limit distribution of equation (2.45) is the normal distribution when $n \to \infty$, hence the gamma distribution (2.44) tends to normality as $\alpha \to \infty$. Papoulis (1957) states a central limit theorem for the sum of positive random variables, in which the limit distribution is a gamma distribution, or the distribution of arbitrary positive random variables has as an approximation the gamma distribution.

Now consider the gamma hazard function. A few curves of this function are shown in Fig. 3 for different values of α parameter.

It is seen that the gamma hazard function is increasing in interval $(0,\infty)$, and the limit value for $t \to \infty$ is λ. To prove this, we can apply equations (2.42) and (2.12) and obtain

$$\frac{-\ f'_G(t;\alpha;\lambda)}{f_G(t;\alpha;\lambda)} = \frac{-(\alpha-1)t^{\alpha-2}+\lambda t^{\alpha-1}}{t^{\alpha-1}} \tag{2.54}$$

and finally

$$\lim_{t\to\infty} h_G(t;\alpha,\lambda) = \lambda \tag{2.55}$$

In the same way, when applying equation (2.50), instead of (2.42), with equation (2.12) gives

$$\frac{-\ f'(t)}{f(t)} = \frac{\lambda_1-\lambda_2\ e^{-(\lambda_2-\lambda_1)t}}{1-e^{-(\lambda_2-\lambda_1)t}} \tag{2.56}$$

whose limit value depends on the value of $(\lambda_2-\lambda_1)$. If $(\lambda_2-\lambda_1) > 0$ then the exponential terms of equation (2.56) tend to zero and the result is λ_1. If $(\lambda_2-\lambda_1) < 0$, applying L'Hospital's rule yields a limit λ_2. So

$$\lim_{t\to\infty} \frac{-f'(t)}{f(t)} = \min(\lambda_1,\lambda_2) \tag{2.57}$$

This result may be extended to the more general case, i.e., for the gamma variable which is the sum of k exponentially distributed random variables, as

$$\lim_{t\to\infty} \frac{-f'(t)}{f(t)} = \min(\lambda_1,\lambda_2,\ldots,\lambda_k) \tag{2.58}$$

Meyer (1972, p. 213) has derived the mgf of the gamma distribution as

$$H_G(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \tag{2.59}$$

The mgf can be used to show that the sum of two independent, gamma distributed random variables with the same scale parameter λ is also gamma distributed, as follows: Let $f_{G_1}(t;\beta;\lambda)$ be the other gamma density function whose mgf is

$$M_{G_1}(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\beta} \tag{2.60}$$

then applying property (Meyer, 1972, p. 217)

$$M_{G_2}(t) = M_G(t)M_{G_1}(t) \tag{2.61}$$

gives

$$M_{G_2}(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha+\beta} \tag{2.62}$$

This is also the mgf of the gamma distribution with parameters $\alpha + \beta$ and λ. The foregoing statement is usually expressed in this form

$$f_G(t;\alpha;\lambda)* f_G(t;\beta;\lambda) = f_G(t;\alpha+\beta;\lambda) \tag{2.63}$$

Further, from the mgf, equation (2.69), we can derive the moments of G using equation (2.16), and we come up with the expression

$$E(G^k) = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)}{\lambda^k} \tag{2.64}$$

Thus, the mean value is

$$E(G) = m_G = \frac{\alpha}{\lambda} \tag{2.65}$$

the standard deviation is

$$Var(G) = \sigma_G^2 = \frac{\alpha}{\lambda^2} \qquad (\ .66)$$

and the coefficient of variation is

$$\eta_G = \frac{\sigma_G}{m_G} = \frac{1}{\sqrt{\alpha}} \tag{2.67}$$

The gamma function has a simple Laplace transform

$$f_G^*(s;\alpha,\lambda) = \left(\frac{\lambda}{s+\lambda}\right)^\alpha \tag{2.68}$$

It is this property that makes the gamma distribution a very convenient model for the availability functions. We will discuss this property later in Chapter 3.

A special case of the gamma distribution is a chi-square distribution with n degrees of freedom. It is obtained by substituting the parameters $\alpha = \frac{n}{2}$ and $\lambda = \frac{1}{2\sigma^2}$ where n is a positive integer and σ is the variance of the normal random variable X with zero mean. Usually this distribution is considered for $\sigma = 1$. The density function for the chi-square random variable, denoted by X_n^2, is given by

$$f_{X_n^2}(t) = \frac{1}{2^{n/2}\Gamma(n/2)} t^{n/2-1} e^{-t/2} \tag{2.69}$$

From the foregoing it follows that

$$P[X_n^2 \leq t] = F_G(t;\tfrac{n}{2},\tfrac{1}{2})$$

and, from equations (2.65) and (2.66), that

$$E(X_n^2)= n; \; Var(X_n^2) = 2n \tag{2.71}$$

The chi-square distribution is tabulated usually for n up to 45 because for larger n this distribution becomes very close to the normal distribution as is seen from the central limit theorem. It should be mentioned that this distribution has a large application in reliability testing, quality control, and in other fields.

The gamma distribution, given by equation (2.45), Cox (1962, p. 15), is called the special Erlangian distribution. Because n (or α) is an integer it has a very suitable analytical form. When dealing with the gamma distribution we will usually use α which has integer values. From equation (2.67) it is obvious that the coefficient of variation can take values < 1 and we will usually have

$$\frac{1}{\sqrt{n_1}} < \eta_X < \frac{1}{\sqrt{n_2}} \tag{2.72}$$

where n_1 is the greatest integer less than $1/\eta_G^2$ and $n_2 = n_1 + 1$. If we want to use the advantages of the Erlang distribution when η_X is given by equation (2.72) we can form the distribution of a random variable X by mixing two Erlang distributions with parameters n_1, $\frac{n_1}{m_X}$ and n_2, $\frac{n^2}{m_X}$, respectively. Thus

$$f_X(t) = a\ f_{G_1}(t;n_1, \frac{n_1}{m_X}) + (1-a)\ f_{G_2}(t;n_2, \frac{n_2}{m_X}) \tag{2.73}$$

where 0 < a < 1. The coefficient a may be found by matching the variance of both sides of equation (2.73). Applying equation (2.66) and the condition that G_1 and G_2 are independent gives the variance as

$$\sigma_X^2 = a\ \frac{m_X^2}{n_1} + (1-a)\ \frac{m_X^2}{n_2} \tag{2.74}$$

Dividing both sides of equation (2.74) by m_X^2 gives

$$\eta_X^2 = \frac{\sigma_X^2}{m_X^2} = \frac{a}{n_1} + \frac{(1-a)}{n_2} \tag{2.75}$$

or

$$a = \frac{n_1 n_2\ \eta_X^2 - n_1}{n_2 - n_1} \tag{2.76}$$

Thus, any gamma distribution can be expressed in the form of an Erlang distribution by using equation (2.73).

2.6 Approximation of an Arbitrary Distribution

Throughout the previous sections we were discussing the unimodal, two-parameter distributions for times to failure or times to repair with the emphasis on the gamma distribution. In practice, when dealing with large and complex systems, we will be faced with a more difficult problem. Only some arbitrary distribution for time to failure or time to repair will be available. The question arises how to find a suitable mathematical model to represent any distribution with the required accuracy. A very convenient approach to this problem is to express the arbitrary distribution in terms of Laguerre polynomials. The two main features of the Laguerre polynomials are: (1) possibility of matching successively higher moments of the distribution which is to be represented, and (2) the Laplace transform is simple. Muth (1967, pp. 41-47) has introduced this approach and it will be discussed here briefly.

Consider a continuous random variable X defined in $(0,\infty)$ with the density function $f(x)$ and with i^{th} moment of X given by

$$m_i = \int_0^\infty x^i f(x)dx \tag{2.77}$$

Next, denote $L_n^\alpha(x)$ as the generalized Laguerre polynomial (Szegö, 1939, pp. 96-97; and Abramowitz and Stegun, 1965, p. 774) defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} \{x^{\alpha+n} e^{-x}\} \tag{2.78}$$

or

$$L_n^{\alpha}(x) = \sum_{i=o}^{n} (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!} \tag{2.79}$$

Then the density function f(x) can be expressed in terms of the infinite Laguerre series as

$$f(x) = W(x;\alpha) \sum_{i=o}^{\infty} a_i L_i^{\alpha}(x) \tag{2.80}$$

where a_i are the coefficients to be found, and $W(x;\alpha)$ is the weighting function of form

$$W(x;\alpha) = x^{\alpha} e^{-x}, \ \alpha > -1 \tag{2.81}$$

The polynomials $L_n^{(\alpha)}(x)$ are orthogonal with respect to the weighting function and the orthogonal property is

$$(L_m^{\alpha}, L_n^{\alpha}) = \int_o^{\infty} L_m^{\alpha}(x) \ L_n^{\alpha}(x) \ W(x,\alpha) \ dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{m,n} \tag{2.82}$$

where $\delta_{m,n}$ is the Kronecker symbol defined as 1 if m = n and 0 otherwise (Spiegel, 1971, p. 184). Now, to determine the coefficients a_i we multiply equation (2.80) by $L_k^{\alpha}(x)$ and integrate, or

$$\int_o^{\infty} f(x)L_k^{\alpha}(x)dx = \int_o^{\infty} \sum_{i=o}^{\infty} a_i L_i^{\alpha}(x)L_k^{\alpha}(x)W(x;\alpha)dx$$

$$= \sum_{i=o}^{\infty} a_i \int_o^{\infty} L_i^{\alpha}(x)L_k^{\alpha}(x)W(x;\alpha)dx \tag{2.83}$$

The above relation with the condition (2.82) and equation (2.79) becomes

$$\sum_{i=o}^{k} (-1)^i \binom{k+\alpha}{k-i} \frac{1}{i!} \int_o^{\infty} x^i f(x)dx = \sum_{i=o}^{\infty} a_i \frac{\Gamma(\alpha+k+1)}{k!} \delta_{k,i} \tag{2.84}$$

Since $\delta_{k,i} = 0$ for $i \neq k$, and substituting equation (2.77) into equation (2.84) we determine the coefficient a_i in the form

$$a_k = \frac{k!}{\Gamma(\alpha+k+1)} \sum_{i=o}^{k} (-1)^i \binom{k+\alpha}{k-i} \frac{1}{i!} m_i \tag{2.85}$$

For a long number of situations we will apply the finite series instead of infinite series to approximate f(x). Then

$$g_n(x;\alpha) = W(x;\alpha) \sum_{i=o}^{n} a_i L_i^{\alpha}(x) \tag{2.86}$$

or,

$$g_n(\lambda x;\alpha) = W(\lambda x;\alpha) \sum_{i=o}^{n} b_i L_i^{\alpha}(\lambda x) \tag{2.87}$$

where g_n stands for f(x) expanded in n terms, λ is the scale parameter, and b_i are unknown coefficients. Introducing (λx) instead of (x) into equations (2.79) and (2.83) we have

$$L_n^{\alpha}(\lambda x) = \sum_{i=o}^{n} (-1)^i \binom{n+\alpha}{n-i} \frac{\lambda^i}{i!} x^i \tag{2.88}$$

and

$$\sum_{i=o}^{n} (-1)^i \binom{k+\alpha}{k-i} \frac{\lambda^{i+1}}{i!} \int_o^{\infty} x^i f(x)dx = \sum_{i=o}^{n} b_i \frac{\Gamma(\alpha+k+1)}{k!} \delta_{k,i} \tag{2.89}$$

From equation (2.89) it follows that the coefficients b_k are given by

$$b_k = \frac{k!}{\Gamma(\alpha+k+1)} \sum_{i=o}^{k} (-1)^i \binom{k+\alpha}{k-i} \frac{\lambda^{i+1}}{i!} m_i \tag{2.90}$$

It is seen that b_k depends on m_i, so looking back at the expression (2.87) we conclude that the moments, m_i, are the only relation between the functions f(x) and the terms of finite series g_n. Thus, when more terms n are taken, higher moments, m_i, are matched and the approximation becomes better.

In the terms for b_k [equation (2.90)], the two parameters α and λ are unknown. In order to determine α and λ one takes the first term of equation (2.87), $g_o(\lambda x;\alpha)$ to be the same known function, namely,

$$g_o(\lambda x;\alpha) = b_o \; W(\lambda x;\alpha) \tag{2.91}$$

where b_o is determined from equation (2.90) as

$$b_o = \frac{\lambda}{\Gamma(\alpha+1)} \tag{2.92}$$

and $W(\lambda x;\alpha)$ from equation (2.81) as

$$W(\lambda x;\alpha) = (\lambda x)^{\alpha} \; e^{-\lambda x} \tag{2.93}$$

So, equation (2.91) becomes

$$g_o(\lambda x;\alpha) = \frac{\lambda}{\Gamma(\alpha+1)} (\lambda x)^{\alpha} \; e^{-\lambda x} \tag{2.94}$$

which obviously is the gamma distribution with parameters $(\alpha+1)$ and (λ). When matching the first and second moments of f(x) and the gamma distribution, the parameters α and λ are given by

$$\alpha = \frac{m_1^2}{m_2-m_1^2} - 1 \tag{2.95}$$

$$\lambda = \frac{\alpha+1}{m_1} \tag{2.96}$$

and the coefficients b_1 and b_2 are zero. So, the approximation of some arbitrary distribution is the summation of the gamma distribution and the finite series of the form

$$g_n(\lambda x;\alpha) = f_G(x;\alpha+1,\lambda) + W(\lambda x;\alpha) \sum_{i=3}^{n} b_i L_i^\alpha(\lambda x)$$

$$= f_G(x;\alpha+1,\lambda) \{1 + \sum_{i=3}^{n} c_i L_i^\alpha(\lambda x)\} \tag{2.97}$$

where $c_k = b_k/b_o$. From the property of the gamma function

$$\Gamma(\alpha+k+1) = (\alpha+1)(\alpha+2)\ldots(\alpha+k)\Gamma(\alpha+1) \tag{2.98}$$

and denoting by M_i the ith moment of the underlying gamma distribution

$$M_i = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+i)}{\lambda^i} \tag{2.99}$$

it follows that

$$c_k = \sum_{i=o}^{k} (-1)^i \binom{k}{i} \frac{m_i}{M_i} \tag{2.100}$$

Because $c_o = 1$, $c_1 = 0$, $c_2 = 0$ and $L_o^\alpha(\lambda x) = 1$, we can rewrite equation (2.97) as

$$g_n(\lambda x;\alpha) = \sum_{i=o}^{n} c_i f_G(x;\alpha+1,\lambda) L_i^\alpha(\lambda x) \tag{2.101}$$

where

$$f_G(x;\alpha+1,\lambda) L_i^\alpha(\lambda x) = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x} L_i^\alpha(\lambda x) \tag{2.102}$$

From tables of Laplace transforms (Roberts and Kaufman, 1966, p. 55), we find[3]

$$L^*\{x^\alpha e^{-\lambda x} L_i^\alpha(\lambda x)\} = \frac{\Gamma(\alpha+i+1)}{i!} \frac{s^i}{(s+\lambda)^{\alpha+i+1}} \tag{2.103}$$

---

3. $L^*(\cdot)$ denotes Laplace transform.

Multiplying equation (2.103) by $\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)}$ gives the Laplace transform of equation (2.101) as

$$L^*\{g_n(\lambda x;\alpha)\} = \sum_{i=o}^{n} c_i \frac{\Gamma(\alpha+i+1)}{i!\Gamma(\alpha+1)} \frac{\lambda^{\alpha+1} \quad s^i}{(s+\lambda)^{\alpha+i+1}} \tag{2.104}$$

So, for any arbitrary density function f(x) we can find the Laplace transform of the form [equation (2.104)] using approximation by the finite series. It is obvious that equation (2.104) is a rational function when α is an integer. Therefore, the tendency will be to make α an integer, if it is not already. Two methods for doing this will be considered next.

In the first method we can take α rounded to the nearest integer when α, calculated from equation (2.95), is close to an integer. Then, to calculate the Laplace transform of $g_n(\lambda x;\alpha)$, we follow the previously described procedure except it should be noticed that the coefficients c_1 and c_2 will not be zero.

In the second method, the gamma distribution $f_G(x;\alpha+1,\lambda)$ from equation (2.97) is substituted by a mix of two Erlang distributions [see equation (2.73)], so

$$\begin{aligned} g_n(\lambda x;\alpha) = {} & (1-a)\ f_G(x;n_2+1,\lambda_2) \\ & + f_G(x;n_1+1,\lambda_1)\ \{a + \sum_{i=3}^{n} d_i L_i^{n_1}(\lambda_1 x)\} \end{aligned} \tag{2.105}$$

where: $a \geq 0.5$ calculated from equation (2.76), $\lambda_1 = \frac{n_1+1}{m_1}$; $\lambda_2 = \frac{n_2+1}{m_1}$; and n_1, n_2 are found from the relation

$$(n_1,n_2) < \frac{m_1^2}{m_2-m_1^2} - 1 < (n_1,n_2) + 1 \tag{2.106}$$

Now, looking back at equation (2.89) we will have

$$\sum_{i=o}^{n} (-1)^i \binom{k+\alpha}{k-i} \frac{\lambda^{i+1}}{i!} \int_o^\infty x^i\{f(x) - (1-a)\, f_G(x;n_2+1,\lambda_2)\}dx$$

$$= \sum_{i=o}^{n} d_i \frac{\Gamma(\alpha+k+1)}{k!} \delta_{k,i} \tag{2.107}$$

where the above integral is the ith moment of the difference of f(x) and gamma distributions, namely,

$$m_i^d = \int_o^\infty x^i\{f(x) - (1-a)\, f_G(x;n_2+1,\lambda_2)\}\, dx$$

$$= m_i - (1-a) \frac{(n_2+1)(n_2+2)\ldots(n_2+i)}{\lambda_2^i} \tag{2.108}$$

Similar to equation (2.100) the coefficient d_k will be

$$d_k = \sum_{i=o}^{k} (-1)^i \binom{k}{i} \frac{m_i^d}{M_i} \tag{2.109}$$

where M_i denotes the ith moment of the underlying gamma distribution $f_G(x;n_1+1,\lambda_1)$. From equation (2.105) it follows that the Laplace transform is of the form

$$L^*\{g_n(\lambda x;\alpha)\} = (1-a)\left(\frac{\lambda 2}{s+\lambda_2}\right)^{n_2+1} + a\left(\frac{\lambda_1}{s+\lambda_1}\right)^{n_1+1}$$

$$+ \sum_{i=3}^{n} d_i \binom{n_1+i}{n_1} \frac{\lambda_1^{n_1+s}\, s^i}{(s+\lambda_1)^{n_1+i+1}} \tag{2.110}$$

which is a rational function.

CHAPTER 3

AVAILABILITY OF A ONE-UNIT SYSTEM

3.1 Stochastic Processes

Stochastic processes are thoroughly explored and widely used. A general approach to the applied stochastic process is given by Bhat (1972), while the application of the stochastic, Markov and semi-Markov, processes for reliability and maintainability of complex systems are considered in the books by Barlow, Proschan and Hunter (1965, Chapter 5) and Shooman (1968). In this section, only some basic notions of the stochastic processes will be described which enable us to derive the availability function in the next section.

A stochastic process $\{X(t);t \geq 0\}$ is a family of random variables $X(t)$ indexed by the time parameter t. The random variable $X(t)$ can take some value, called the state, and the set of the possible values is called the state space. Both $X(t)$ and t may be either discrete or continuous, from which result the four possible models of the processes. Through this study $X(t)$ takes integer values in the finite state space $S\varepsilon\{0,1,...,n-1\}$ and t takes only positive values, so we will have discrete states and continuous time processes.

To be able to completely describe the process we have to know the joint distribution of the basic random variable $X(t)$. For the discrete set of points $t_1, t_2,...,t_n$ the joint distribution function is

$$P[X(t_1) \leq x_1; X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n] \tag{3.1}$$

The above equation is too complex and for practical application we need to make some simplifications. When taking the randon variables independent, then equation (3.1) reduces to the product of distributions of each random variable; furthermore, we are actually studying each state independently anyway. This is not the real situation, since between the states exists some dependence. The other assumption results in a Markov process which could be stated as follows: if the state $X(t_n)$ is known at the time t_n, then all information of the process before t_n does not influence to the future of the process, or for the discrete set of points t_1, $t_2, \ldots, t_n$, with $t_1 < t_2 < \ldots < t_n < \ldots$ and any real number $x_1, x_2, \ldots x_n$, we have condition

$$P\{X(t_n) \leq x_n | X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_{n-1}) \leq x_{n-1}\}$$

$$= P\{X(t_n) \leq x_n | X(t_{n-1}) \leq x_{n-1}\} \tag{3.2}$$

The condition of equation (3.2) applies the exponentially distributed time at any state $X(t)$. The extension of Markov processes results in semi-Markov processes which permit the use of arbitrarily distributed times at any state. The semi-Markov process is based on the concept of a regenerative stochastic process (Smith, 1955, and Feller, 1949) which could be described in the following way: consider the same event E, occurring at certain points in time t_i and connecting with the process $\{X(t), t \geq 0\}$ such that whenever E occurs the future behavior of the process is independent of its past. Then the process is regenerative and the time points t_i are points of regeneration. The differences $(t_i - t_{i-1})$ for $i = 1,2,\ldots$ are independently distributed random variables. A renewal process is a well--known regenerative process.

The semi-Markov process is completely determined by the pair $[P, F(t)]$ where $P = \{p_{ij}\}$ is the transition probability matrix and $F(t) = \{F_{ij}(t)\}$ is the matrix of the distribution functions. Each element of P, p_{ij}, represents the transition probability from state i to state j and it is given by

$$p_{ij}^{n,n+1} = P[X(n+1) = j \mid X(n) = i] \qquad (3.3)$$

or in words, p_{ij} is the conditional probability that the process will be in state j after n+1 steps given that it is in state i in n steps. For

$$p_{ij}^{n,n+1} = p_{ij}^{0,1} = p_{ij} \qquad (3.4)$$

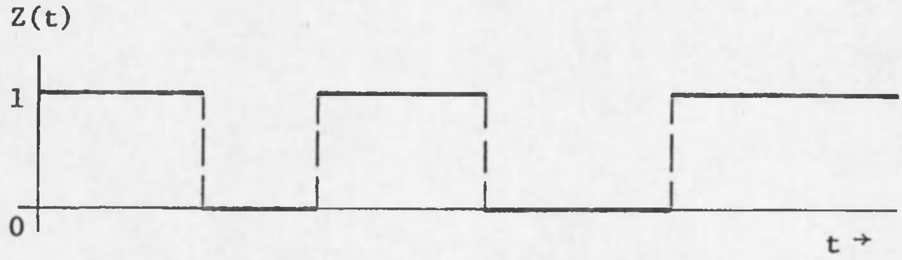
the process becomes stationary in time. To any element p_{ij} corresponds the time T_{ij} spent in the state i, given that the next state is j and the distribution function of the random variable T_{ij} is F_{ij}.

Associated with the stochastic process is the classification of states. So we consider an ergodic and transient set of states. For the ergodic set of states it is characteristic that the process enters each state infinitely and that all the states communicate. The non-ergodic set of states are called transient. The states of the transient set are absorbing states for which one has $p_{ii} = 1$. This means that once the process reaches the absorbing state it will remain in it forever.

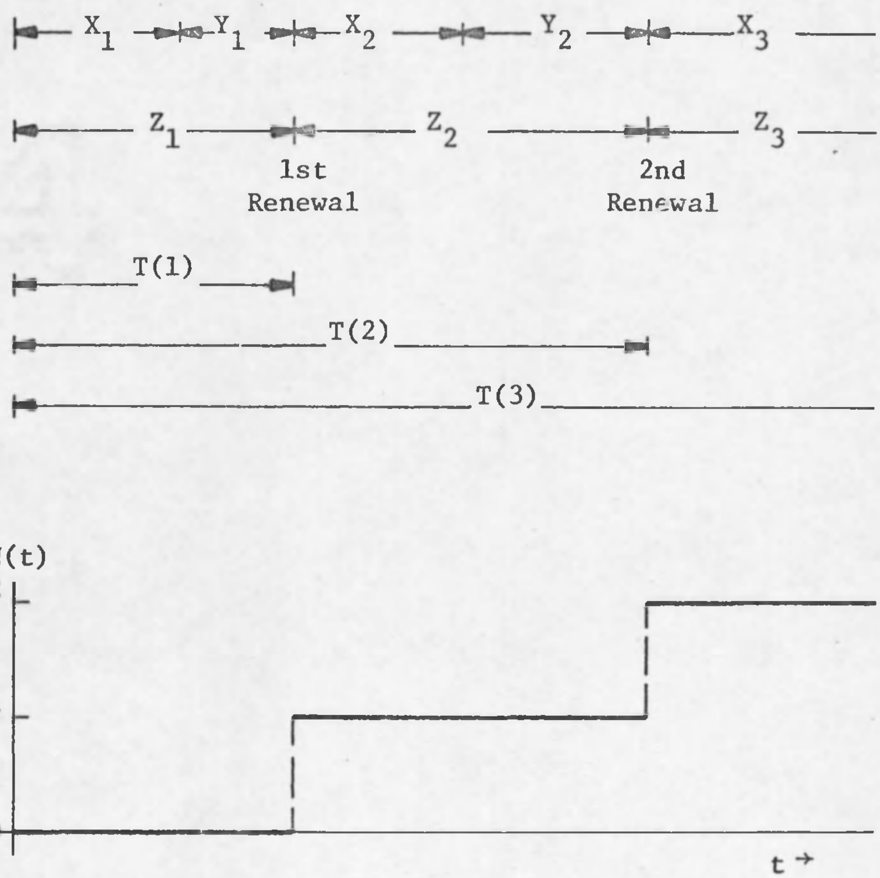
Now assume a process $\{X(t);t \geq 0\}$ where the random variable $X(t)$ takes the integer values in the finite state space $S\varepsilon\{0,1,2,\ldots n-1\}$. When looking at the system performance it is usually possible to divide the sample space S into two independent subsets S_1 and $S-S_1$, such that the subset S_1 takes all the possible operating states or the states for which the system is up, and the subset $S-S_1$ contains all non-operating states, or the states for which the system is down. This way one can reduce any process to a two-state process or any system to a one-unit system with the two occupying states up and down. For the two-state process the renewal theory (Cox, 1962) is developed and it will be used here to derive the availability function for a one-unit system.

3.2 Derivation of the Availability Function

Consider a two-state semi-Markov process $\{Z(t);t \geq 0\}$ such that $Z(t) = 1$ denotes operating (up) state and $Z(t) = 0$ denotes non-operating (down) state. Take a random variable X for the elapsed time in state 1, time to failure, and a random variable Y for the elapsed time in the state 0, time to repair. If the process initially starts at state 1, then X_1 is the time elapsed to the first failure, Y_1 is time elapsed to the first repair completed, and X_i, Y_i are the times elapsed to the ith failure occurs and the ith repair completed, respectively. The sum $Z_1 = X_1 + Y_1$ is the time for the first renewal and $Z_i = X_i + Y_i$ is the time for ith renewal. If all Z_i's are independent then the sequence of random variables $\{Z_1, Z_2, \ldots, Z_i, \ldots\}$ represents a renewal process. This process is illustrated in Fig. 7.



(a) State Probabilities in time t



(b) Number of Renewal Processes in Time

Fig. 7 Illustration of the Renewal Process

Usually we are considering two random variables associated with every renewal process: $T(n)$ denoting the time to the n^{th} renewal; and $N(t)$ denoting the number of renewals by time t. $T(n)$ is a continuous variable defined in interval $t(0,\infty)$ while $N(t)$ is a discrete variable and can take integer values $n = 0,1,2,\ldots$ It is seen from Fig. 7 that a relationship between $T(n)$ and $N(t)$ exists. For given t and $N(t) < n$, $T(n) > t$, or

$$P\{N(t) < n\} = P\{T(n) > t\} \tag{3.5}$$

In order to find the above probabilities we will determine first the distribution of $T(n)$. Let X_i, Y_i have the density functions $f_i(t)$, $g_i(t)$, respectively, and let $h_{Z_1}(t)$ denote the probability density function of Z_i, then, by the convolution theorem (2.48),

$$h_{(Z_i)}(t) = f_{(i)}(t)*g_{(i)}(t) \tag{3.6}$$

From Fig. 7

$$T(n) = Z_1 + Z_2 + \ldots + Z_n \tag{3.7}$$

Then applying again the convolution theorem we obtain the density function of $T(n)$, $h_{T(n)}(t)$, as

$$h_{T(n)}(t) = h_{Z_1}(t)*h_{Z_2}(t)* \ldots *h_{Z_n}(t) \tag{3.8}$$

We will be dealing only with all Z_i's which are identically distributed, so for that case, equation (3.8) reduces to

$$h_{T(n)}(t) = h^{(n)}(t) \tag{3.9}$$

where $h^{(n)}(t)$ is called the n-fold convolution of $h(t)$. Now the distribution function of $T(n)$ is

$$H_{T(n)}(t) = P\{T(n) \leq t\}$$

$$= \int_o^t h_{T(n)}(x)dx \tag{3.10}$$

and, from equation (3.5) we obtain

$$P\{N(t) < n\} = 1 - P\{T(n) \leq t\}$$

$$= 1 - H_{T(n)}(t) \tag{3.11}$$

For the discrete random variable N(t), any point of the density function could be found as

$$p_{N(t)}(n) = P\{N(t) = n\}$$

$$= P\{N(t) < n + 1\} - P\{N(t) < n\} \tag{3.12}$$

and substituting equation (2.11) gives

$$P_{N(t)}(n) = H_{T(n)}(t) - H_{T(n+1)}(t) \tag{3.13}$$

The expected number of renewals in interval (0,t) is the renewal function M(t). So, from the definition of the expected value and from equation (3.13) M(t) is

$$M(t) = E[N(t)]$$

$$= \sum_{n=o}^{\infty} n\, p_{N(t)}(n)$$

$$= \sum_{n=1}^{\infty} n\{H_{T(n)}(t) - H_{T(n+1)}(t)\}$$

$$= H_{T(1)}(t) - H_{T(2)}(t) + 2H_{T(2)}(t) - 2H_{T(3)}(t) + \ldots,$$

$$M(t) = \sum_{n=1}^{\infty} H_{T(n)}(t) \tag{3.14}$$

The renewal density function, $m(t)$, is the derivative of $M(t)$. Thus, from equations (3.14) and (3.9) we obtain

$$m(t) = \frac{d}{dt}\{M(t)\},$$

$$= \sum_{n=1}^{\infty} h_{T(n)}(t),$$

or

$$m(t) = \sum_{n=1}^{\infty} h^{(n)}(t) \tag{3.15}$$

Since $h^{(n)}(t)$ is the n-fold convolution of $h(t)$ with itself

$$h^{n}(t) = \int_{o}^{t} h^{(n-1)}(t-\tau)h(\tau)\, d\tau \tag{3.16}$$

and

$$m(t) = h(t) + \sum_{n=2}^{\infty} h^{(n)}(t)$$

$$= h(t) + \sum_{n=2}^{\infty} \int_{o}^{t} h^{(n-1)}(t-\tau)h(\tau)\, d\tau$$

$$= h(t) + \sum_{i=1}^{\infty} \int_{o}^{t} h^{(i)}(t-\tau)h(\tau)\, d\tau$$

$$= h(t) + \int_{o}^{t} \{\sum_{i=1}^{\infty} h^{(i)}(t-\tau)\}\, h(\tau)\, d\tau$$

or

$$m(t) = h(t) + \int_{o}^{t} m(t-\tau)h(\tau)\, d\tau \tag{3.17}$$

The renewal density function physically represents the expected number of renewals in interval $(t,t+\Delta t)$. It should be noticed that $m(t)$ is not necessarily the probability density function, namely,

$$\int_{o}^{\infty} m(t)\, dt \neq 1 \tag{3.18}$$

The Laplace transform of equation (3.17) is

$$m^*(s) = h^*(s) + m^*(s)h^*(s) \tag{3.19}$$

from which

$$m^*(s) = \frac{h^*(s)}{1-h^*(s)} \tag{3.20}$$

From equation (3.6) we obtain

$$h^*(s) = f^*(s)g^*(s) \tag{3.21}$$

so that

$$m^*(s) = \frac{f^*(s)g^*(s)}{1-f^*(s)g^*(s)} \tag{3.22}$$

Now to derive the availability function we start from the definition of availability. Availability is defined as the probability that the system will be operating at time t, or the process $\{Z(t);t \geq 0\}$ will be in state 1 at time t. This definition may be expressed analytically as

$$A(t) = P\{Z(t) = 1\} \tag{3.23}$$

For the renewal process $P\{Z(t) = 1\}$ is conditional, i.e., it is the probability that the system is operating given n renewals occurred. Thus

$$P\{Z(t) = 1\} = \sum_{n=o}^{\infty} P\{Z(t) = 1, N(t) = n\} \tag{3.24}$$

Assume that the nth renewal happened at time x, or $T(n) = x$. Then the probability that the system remains in state 1 in interval (x,t) is $\{1-F(t-x)\}$. Since the probability of nth renewal at x is given by $h_{T(n)}(x)$, we have

$$P\{Z(t) = 1,\ N(t) = n\} = \int_o^t \{1-F(t-x)\} h_{T(n)}(x)\ dx \tag{3.25}$$

and, from equations (3.23), (3.24) and (3.9) we have

$$A(t) = \sum_{n=o}^{\infty} \int_o^t \{1-F(t-x)\} h^{(n)}(x)\ dx \tag{3.26}$$

or

$$A(t) = 1-F(t) + \sum_{n=1}^{\infty} \int_o^t \{1-F(t-x)\} h^{(n)}(x)\ dx$$

$$= 1-F(t) + \int_o^t \{1-F(t-x)\}\{\sum_{n=1}^{\infty} h^{(n)}(x)\}\ dx \tag{3.27}$$

where substituting equation (3.15) into equation (3.27) gives

$$A(t) = 1-F(t) + \int_o^t \{1-F(t-x)\} m(x)\ dx \tag{3.28}$$

The Laplace transform of equation (3.28) is

$$A^*(s) = \frac{1}{s} - F^*(s) + \{\frac{1}{s} - F^*(s)\} m^*(s)$$

$$= \{\frac{1}{s} - F^*(s)\}\{1 + m^*(s)\}$$

$$= \{\frac{1}{s} - \frac{f^*(s)}{s}\}\{1 + \frac{f^*(s) g^*(s)}{1-f^*(s) g^*(s)}\} \tag{3.29}$$

and finally

$$A^*(s) = \frac{1-f^*(s)}{s\{1-f^*(s) g^*(s)\}} \tag{3.30}$$

The same result may be obtained by taking the Laplace transform of equation (3.26). Thus

$$A^*(s) = \sum_{n=o}^{\infty} \{\frac{1}{s} - F^*(s)\}\{h^*(s)\}^n$$

$$= \{\frac{1}{s} - F^*(s)\} \sum_{n=o}^{\infty} \{h^*(s)\}^n \tag{3.31}$$

Because $h^*(s) < 1$ the summation of equation (3.31) is a geometric series, so

$$A^*(s) = \frac{\frac{1}{s} - F^*(s)}{1-h^*(s)} \tag{3.32}$$

$$A^*(s) = \frac{1-f^*(s)}{s\{1-f^*(s)g^*(s)\}}$$

which is the same as equation (3.30).

Next, the principle of regenerative events will be applied to the derivation of $A^*(s)$. Consider a renewal process in a given internal $(0,t)$, and let E denote such events that whenever E occurs, the process enters state 1. It is assumed that, at the very beginning for $t = 0$, the system starts as new, or $X(0) = 1$. The probability that the process is in state 1 at time t is the summation of the probability that E did not happen in interval $(0,t)$, and the probability that the process is in state 1 given E occurred at time x where $0 \leq x \leq t$. Thus

$$P\{(Z(t) = 1\} = P\{Z(t) = 1, x > t\} + P\{Z(t) = 1, x < t\} \tag{3.33}$$

It has been already stated that

$$P\{Z(t) = 1, x > t\} = 1 - F(t) \tag{3.34}$$

The probability that E occurs at x is given by the probability density function $h(x)$, and the probability that the process is in state 1, given E occurs at x, may be considered as the availability of the process which starts at x, $A(t-x)$, so the second conditional probability of equation (3.33) is

$$P\{Z(t) = 1, x < t\} = \int_o^t A(t-x)h(x)\, dx \tag{3.35}$$

Finally

$$A(t) = 1-F(t) + \int_o^t A(t-x)h(x)\, dx \tag{3.36}$$

This is known as the availability integral equation. The Laplace transform of equation (3.36) is

$$\begin{aligned} A^*(s) &= \frac{1}{s} - F^*(s) + A^*(s)h^*(s) \\ &= \frac{1}{s} - \frac{f^*(s)}{s} + A^*(s)f^*(s)g^*(s) \end{aligned} \tag{3.37}$$

and the solution for $A^*(s)$ gives

$$A^*(s) = \frac{1-f^*(s)}{s\{1-f^*(s)g^*(s)\}}$$

a familiar result.

It is seen that the Laplace transform of availability is a function of the Laplace transforms of the time-to-failure and the time-to-repair density functions, so,

$$A^*(s) = A\{f^*(s), g^*(s)\} \tag{3.38}$$

Now it becomes obvious why the tendency in Chapter 2 was to emphasize the density functions which have Laplace transform in a convenient analytical form. Using functions described in the previous chapter we can find $A^*(s)$ and possibly $A(t) = L^{*-1}\{A^*(s)\}$ in the general form. This is the subject of the next section.

3.3 Solution of the Availability Function

Usually, when determining $A(t)$ from its Laplace transform, we apply either a numerical or an analytical method. Both methods have some disadvantages. Which one will be applied depends on the actual situation and the personal preference.

The numerical method may be used generally with no restrictions. For any A*(s) analytic in the half-plane and real $s \geq c$, the inverse Laplace transform is

$$A(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A^*(s)\, e^{st}\, ds \tag{3.39}$$

Thus, for any value of t in interval $(0,\infty)$ one can numerically calculate the integral of equation (3.39) and find A(t). This method may be preferred when a digital computer is available and when a partial fraction expansion becomes inpractical because of too many roots. This will be discussed later in this section.

The analytical approach takes either a partial fraction expansion or an approximation of A*(s) by a series whose inverse Laplace transform is known.

Using a partial function expansion, we can break down equation (3.30) into a sum of very simple terms where each term has an exponential inverse transform. So equation (3.30) can be rewritten as (Shooman, 1968, p. 108)

$$A^*(s) = \frac{N^*(s)}{D^*(s)} = \frac{N^*(s)}{\prod_{i=1}^{n} (s-s_i)} = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \ldots + \frac{A_n}{s-s_n} \tag{3.40}$$

where the s_i's are the roots of D*(s) and the coefficients A_i's are given by

$$A_i = \left\{\frac{N^*(s)}{D^*(s)} (s-s_i)\right\} \text{ evaluated for } s=s_i. \tag{3.41}$$

For function A*(s) the roots of the denominator may be obtained from

$$1 - f^*(s)g^*(s) = 0 \tag{3.42}$$

First, let's assume exponentially distributed time to failure and time to repair with parameters λ and μ, respectively. In this case $f^*(s) = \frac{\lambda}{\lambda+s}$ and $g^*(s) = \frac{\mu}{\mu+s}$ and equation (3.42) becomes

$$1 - (\frac{\lambda}{\lambda+s})(\frac{\mu}{\mu+s}) = 0 \tag{3.43}$$

or,

$$s(s+\lambda+\mu) = 0 \tag{3.44}$$

from which the roots are $s_1 = 0$ and $s_2 = -(\lambda+\mu)$, and the availability function is

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \tag{3.45}$$

Now, instead of exponential, let's take a gamma distributed time to repair with the density function $g_G(t;\alpha;\mu)$ and a gamma distributed time to failure with the density function $f_G(t;\beta,\lambda)$. According to equation (2.68) one has

$$g^*(s) = (\frac{\mu}{s+\mu})^{\alpha} \tag{3.46}$$

and

$$f^*(s) = (\frac{\lambda}{s+\lambda})^{\beta} \tag{3.47}$$

Substitution of the above equations into equation (3.42) gives

$$1 - (\frac{\mu}{s+\mu})^{\alpha}(\frac{\lambda}{s+\lambda})^{\beta} = 0 \tag{3.48}$$

or,

$$(s+\mu)^{\alpha}(s+\lambda)^{\beta} - \mu^{\alpha}\lambda^{\beta} = 0 \tag{3.49}$$

If α, β are integers, or the gamma distribution is an Erlang distribution, equation (3.49) becomes a polynomial of $(\alpha+\beta)$ degree. So, there are $(\alpha+\beta)$ roots to be calculated. Since α and β may take any integer values, the number of roots may be relatively large and unsuitable for practical applications. If the approximated distributions for time to failure and time to repair given by equation (2.105) are used instead of the gamma distribution, the number of roots may increase and also the difficulty to find $A(t)$. Thus, from the foregoing it is obvious that the partial fraction expansion method, though simple, has too many disadvantages: restriction to integer values α and β and the possibility of a large number of roots.

Approximating $A^*(s)$ by some known series may be the most general approach for inverting $A^*(s)$. Papoulis (1957) suggested three methods of inversion of the Laplace transform: (1) using the trigonometric set, (2) the Legendre set, and (3) the Laguerre polynomials. Again, we will focus our attention to the Laguerre polynomials, but with a slightly different approach than the one discussed by Papoulis (1957).

Consider a function $A_1(t)$ approximated by a finite series of the form

$$A_1(t) = e^{-\alpha t} \sum_{i=1}^{n} a_i L_i(\alpha t) \tag{3.50}$$

where $L_i(t)$ is the Laguerre polynomial given by

$$L_i(t) = e^{t} \frac{d^i}{dt^i} \{\frac{t^i e^{-t}}{i!}\} \tag{3.51}$$

or,

$$L_i(t) = \sum_{j=o}^{i} (-1)^j \binom{n}{n-j} \frac{t^j}{j!} \tag{3.52}$$

and

$$L_i(\alpha t) = \sum_{j=o}^{i} (-1)^j \binom{n}{n-j} \frac{\alpha^j t^j}{j!} \tag{3.53}$$

From the table of Laplace transforms by Roberts and Kaufman (1966, p. 55, equation 9.3.25) we obtain

$$L^*\{e^{-\alpha t} L_i(\alpha t)\} = \left(\frac{s}{s+\alpha}\right)^i \tag{3.54}$$

and from equation (3.50)

$$A_1(s) = \sum_{i=1}^{n} a_i \left(\frac{s}{s+\alpha}\right)^i \tag{3.55}$$

So, if we approximate the Laplace transform of $A_1(s)$ by a power series, such as that of equation (3.55), we will have the inverse Laplace transform given by the finite series of equation (3.50). It is obvious that the approximation of equation (3.50) is not convenient for the availability function. For $t = 0$ one has $A_1(0) = 0$ and for $t \to \infty$ gives $A(\infty) \to 0$, while we know that $A^*(0) = 1$ for $t = 0$ and for $t \to \infty$ the availability tends to some constant steady state value $A(\infty)$.[4] To satisfy the above conditions we introduce an availability function of the form

$$A(t) = e^{-\alpha t} \sum_{i=1}^{n} b_i L_i(\alpha t) + ce^{-dt} + a \tag{3.56}$$

---

4. Regardless of the underlying distributions for time to failure X and time to repair Y the steady state availability is $A(\infty) = m_X/(m_X+m_Y)$ where $m_X = E(X)$ and $m_Y = E(Y)$ (Shooman, 1968, p. 358).

where $d = \frac{1}{m_X} + \frac{1}{m_Y}$ (analogous to the exponential distribution). From the conditions $A(0) = 1$ and $\lim_{t\to\infty} A(t) = A(\infty)$ the coefficients a and c are

$$a = A(\infty) \tag{3.57}$$

and

$$c = 1 - A(\infty) - (b_1+b_2+ \ldots + b_n) \tag{3.58}$$

Analogous to equation (3.55) we may find the Laplace transform of equation (3.58) as

$$A^*(s) = \sum_{i=1}^{n} b_i \left(\frac{s}{s+\alpha}\right)^i + \frac{c}{s+d} + \frac{a}{s} \tag{3.59}$$

The parameter α and the coefficients b_i are unknown in equation (3.59). The parameter α can take any positive value. In order to determine the b_i's we choose a set of points $\{s_1,s_2,\ldots,s_n\}$ and calculate the values of $\{A^*(s_1), A^*(s_2),\ldots,A^*(s_n)\}$. Substituting these two sets into equation (3.59) gives n linear equations with n unknown coefficients b_i, whose solution represents no difficulty. So through n points, s_i, the function $A^*(s)$ is fitted. Taking n large enough we can satisfy any desired accuracy.

3.4 Example

Consider a one-unit system for which the availability function is to be found. Assume Weibull distributed time to failure and time to repair with parameters: $\beta_f = 3.0$ and $\frac{1}{\eta_f} = 0.05$ for time to failure; and $\beta_g = 2.635$ and $\frac{1}{\eta_g} = 0.1$ for time to repair.

To find the availability function we have to know the Laplace transform of the given distributions. Since the Weibull distribution does not have its Laplace transform in general form, the approximation up to n = 4, described in Section 2.6, will be used.

According to equation (2.38) the moments of the Weibull distribution, for time to failure, are

$$m_{f_1} = \frac{\Gamma(1 + \frac{1}{3})}{0.05} = 17.86150$$

$$m_{f_2} = \frac{\Gamma(1 + \frac{2}{3})}{(0.05)^2} = 360.82800$$

$$m_{f_3} = \frac{\Gamma(1 + \frac{3}{3})}{(0.05)^3} = 8000.00$$

and

$$m_{f_4} = \frac{\Gamma(1 + \frac{4}{3})}{(0.05)^4} = 190523.73$$

From equations (2.95) and (2.96)

$$\alpha_f = \frac{(17.8615)^2}{360.828-(17.8615)^2} - 1 = 6.63319$$

$$\lambda_f = \frac{6.63319 + 1}{17.86150} = 0.42736$$

and from equations (2.99) and (2.100)

$$M_{f_3} = \frac{(6.63319+1)(6.63319+2)(6.63319+3)}{(0.42736)^3} = 8133.28304$$

$$M_{f_4} = 8133.28304 \frac{(6.63319+4)}{0.42736} = 202365.08765$$

and

$$c_{f_3} = 1 - \frac{m_{f_3}}{M_{f_3}} = 1 - \frac{8000.00}{8133.28304} = 0.01639$$

$$c_{f_4} = 3 - 4\frac{m_{f_3}}{M_{f_3}} + \frac{m_{f_4}}{M_{f_4}}$$

or,

$$c_{f_4} = 3 - 4\frac{8000.00}{8133.28304} + \frac{190523.73}{202365.08765} = 0.00703$$

The Laguerre polynomials, $L_i^{\alpha_f}(\lambda_f t)$, are given by equation (2.88), so[5]

$$L_3^{\alpha_f}(\lambda_f t) = \binom{3+\alpha_f}{3} - \binom{3+\alpha_f}{2}\lambda_f t + \binom{3+\alpha_f}{1}\frac{\lambda_f^2 t^2}{2!} - \frac{\lambda_f^3 t^3}{3!}$$

$$= \frac{\Gamma(4+\alpha_f)}{\Gamma(4)\Gamma(\alpha_f+1)} - \frac{\Gamma(4+\alpha_f)}{\Gamma(3)\Gamma(\alpha_f+2)}\lambda_f t + \frac{\Gamma(4+\alpha_f)}{\Gamma(2)\Gamma(\alpha_f+3)}\frac{\lambda_f^2 t^2}{2!}$$

$$- \frac{\lambda_f^3 t^3}{3!}$$

$$= \frac{\Gamma(10.63319)}{3!\Gamma(7.63319)} - \frac{\Gamma(10.63319)}{2!\Gamma(8.63319)}(0.42736)t$$

$$+ \frac{\Gamma(10.63319}{1!\Gamma(9.63319)}\frac{(0.42736)^2}{2!}t^2 - \frac{(0.42736)^3}{3!}t^3$$

or,

$$L_3^{\alpha_f}(\lambda_f t) = 105.80258 - 17.77073t + 0.87969t^2 - 0.01301t^3$$

and

$$L_4^{\alpha}(\lambda_f t) = \binom{4+\alpha_f}{4} - \binom{4+\alpha_f}{3}\lambda_f t + \binom{4+\alpha_f}{2}\frac{\lambda_f^2 t^2}{2!} - \binom{4+\alpha_f}{1}\frac{\lambda_f^3 t^3}{3!}$$

$$+ \frac{\lambda_f^4 t^4}{4!}$$

---

5. $\binom{3+\alpha_f}{3} = \frac{\Gamma(3+\alpha_f+1)}{\Gamma(3+1)\Gamma(3+\alpha_f-3+1)}$. See Abramowitz and Stegun (1965, p. 256, equation 6.1.21).

$$= \frac{\Gamma(5+\alpha_f)}{\Gamma(5)\Gamma(\alpha_f+1)} - \frac{\Gamma(5+\alpha_f)}{\Gamma(4)\Gamma(\alpha_f+2)} \lambda_f t + \frac{\Gamma(5+\alpha_f)}{\Gamma(3)\Gamma(\alpha_f+3)} \frac{\lambda_f^2 t^2}{2!}$$

$$- \frac{\Gamma(5+\alpha_f)}{\Gamma(2)\Gamma(\alpha_f+4)} \frac{\lambda_f^3 t^3}{3!} + \frac{\lambda_f^4 t^4}{4!}$$

$$= \frac{\Gamma(11.63319)}{4!\Gamma(7.63319)} - \frac{\Gamma(11.63319)}{3!\Gamma(8.63319)} (0.42736)t$$

$$+ \frac{\Gamma(11.63319)}{2!\Gamma(9.63319)} \frac{(0.42736)^2}{2!} t^2$$

$$- \frac{\Gamma(11.63319)}{1!\Gamma(10.63319)} \frac{(0.42736)^3}{3!} t^3 + \frac{(0.42736)^4}{4!} t^4$$

or,

$$L_4^{\alpha}(\lambda_f t) = 281.25473-62.98652t + 4.67694t^2 - 0.13832t^3 + 0.00139t^4$$

Finally, from equation (2.97) the approximation of the given Weibull distribution for n = 2 is

$$g_2(\lambda_f t;\alpha_f) = f_G(t;\alpha_f+1,\lambda_f)$$

$$= \frac{(0.42736)^{7.63319}}{\Gamma(7.63319)} t^{6.63319} e^{-0.42736t}$$

$$= 6.22838\text{x}10^{-7} t^{6.63319} e^{-0.42736t} \tag{3.60}$$

and for n = 3 is

$$g_3(\lambda_f t,\alpha_f) = f_G(t;\alpha_f+1,\lambda_f) \{1+c_{f3}L_3^{\alpha_f}(\lambda_f t)\}$$

$$= 6.22838\text{x}10^{-7} t^{6.63319} e^{-0.42736t} \{2.73410$$

$$- 0.29126t + 0.01442t^2 - 0.000213t^3\} \tag{3.61}$$

and for n = 4 is

$$g_4(\lambda_f t;\alpha_f) = f_G(t;\alpha_f+1,\lambda_f)\{1+c_{f3}L_3^{\alpha_f}(\lambda+t) + c_{f4}L_4^{\alpha_f}(\lambda_f t)\}$$

$$= 6.22837\text{x}10^{-7}\ t^{6.63319}\ e^{-0.42736t}\ \{4.71132$$

$$- 0.73405t + 0.04730t^2 - 0.00118t^3 + 9.7705\text{x}10^{-6}t^4\} \quad (3.62)$$

From equation (2.104) the Laplace transforms of equations (3.60), (3.61) and (3.62) are

$$L^*\{g_2(\lambda_f t;\alpha_f)\} = \frac{c_o\lambda_f^{\alpha_f+1}}{(s+\lambda_f)^{\alpha_f+1}}$$

$$= \frac{(0.42736)^{6.63319+1}}{(s+0.42736)^{6.66319+1}} = \frac{0.00152}{(s+0.42736)^{7.63319}}$$

$$L^*\{g_3(\lambda_f t;\alpha_f)\} = \frac{c_o\lambda_f^{\alpha_f+1}}{(s+\lambda_f)^{\alpha_f+1}} + \frac{c_{f3}\Gamma(\alpha_f+3+1)}{3!\Gamma(\alpha_f+1)}\ \frac{\lambda_f^{\alpha_f+1}\ s^3}{(s+\lambda_f)^{\alpha_f+3+1}}$$

$$= \frac{(0.42736)^{6.63319+1}}{(s+0.42736)^{6.63319+1}} + \frac{0.01639\Gamma(6.63319+4)}{3!\Gamma(6.63319+1)}\ \frac{(0.42736)^{6.63319+1}\ s^3}{(s+0.42736)^{6.63319+4}}$$

$$= \frac{0.00152}{(s+0.42736)^{7.63319}}\ \{1 + 1.73410\ \frac{s^3}{(s+0.42736)^3}\}$$

and

$$L^*\{g_4(\lambda_f t;\alpha_f)\} = L^*\{g_3(\lambda_f t;\alpha_f)\} + \frac{c_{f4}\Gamma(\alpha_f+4+1)}{4!\Gamma(\alpha_f+1)}\ \frac{\lambda_f^{\alpha_f+1}\ s^4}{(s+\lambda_f)^{\alpha_f+4+1}}$$

$$= L^*\{g_3(\lambda_f t;\alpha_f)\} + \frac{0.00703\Gamma(6.63319+5)}{4!\Gamma(6.63319+1)}\ \frac{(0.42736)^{6.63319+1}\ s^4}{(s+0.42736)^{6.63313+5}}$$

$$= \frac{0.00152}{(s+0.42736)^{7.63319}}\ \{1 + \frac{1.73410\ s^3}{(s+0.42736)^3} + \frac{1.97722\ s^4}{(s+0.42736)^4}\}$$

For the time to repair distribution one has

$$m_{g1} = \frac{\Gamma(1 + \frac{1}{2.635})}{0.1} = 8.88570$$

$$m_{g2} = \frac{\Gamma(1 + \frac{2}{2.635})}{(0.1)^2} = 92.1136$$

$$m_{g3} = \frac{\Gamma(1 + \frac{3}{2.635})}{(0.1)^3} = 1067.0129$$

$$m_{g4} = \frac{\Gamma(1 + \frac{4}{2.635})}{(0.1)^4} = 13463.98$$

and

$$\alpha_g = \frac{(8.8857)^2}{92.1136-(8.8857)^2} - 1 = 5$$

$$\lambda_g = \frac{5+1}{8.8857} = 0.67524$$

Then

$$M_{g3} = \frac{(5+1)(5+2)(5+3)}{(0.67524)^3} = 1091.34$$

$$M_{g4} = 1091.34 \; \frac{(5+4)}{0.67524} = 14546.04$$

$$c_{g3} = 1 - \frac{1067.0129}{1091.34} = 0.02229$$

$$c_{g4} = 3 - 4\,\frac{1067.0129}{1091.34} + \frac{13463.98}{14546.04} = 0.01478$$

$$L_3^{\alpha_g}(\lambda_g t) = \binom{3+5}{3} - \binom{3+5}{2} (0.67524)t + \binom{3+5}{1} \frac{(0.67524)^2}{2!} t^2$$

$$- \frac{(0.67524)^3}{3!} t^3 = 56 - 18.90672t + 1.82379t^2 - 0.05131t^3$$

and

$$L_4^{\alpha_g}(\lambda_g t) = \binom{4+5}{4} - \binom{4+5}{3} (0.67524)t + \binom{4+5}{2} \frac{(0.67524)^2}{2!} t^t$$

$$- \binom{4+5}{1} \frac{(0.67524)^3}{3!} t^3 + \frac{(0.67524)^4}{4!} t^4$$

or

$$L_4^{\alpha_g}(\lambda_g t) = 126 - 56.72016t + 8.20708t^2 - 0.46181t^3 + 0.00866t^4$$

Thus the approximations are

$$g_2(\lambda_g t;\alpha_g) = \frac{(0.76524)^6}{\Gamma(6)} t^5 e^{-0.67524t}$$

$$= 7.89892 \times 10^{-4} t^5 e^{-0.67524t} \qquad (3.63)$$

$$g_3(\lambda_g t;\alpha_g) = 7.89892 \times 10^{-4} t^5 e^{-0.67524t} \{2.24824$$

$$- 0.42143t + 0.04065t^2 - 0.00114t^3\} \qquad (3.64)$$

$$g_4(\lambda_g t;\alpha_g) = 7.89892\text{x}10^{-4}\ t^5\ e^{-0.67524t}\ \{4.11052-1.25975t$$

$$+ 0.16195t^2 - 0.00797t^3 + 0.00013t^4\} \tag{3.65}$$

The Laplace transforms are

$$L^*\{g_2(\lambda_g t;\alpha_g)\} = \frac{0.09479}{(s+0.067524)^6}$$

$$L^*\{g_3(\lambda_g t;\alpha_g)\} = \frac{0.09479}{(s+0.67524)^6}\ \{1 + \frac{1.87229\ s^3}{(s+0.67524)^6}\}$$

and

$$L^*\{g_4(\lambda_g t;\alpha_g)\} = \frac{0.09479}{(s+0.67524)^6}\ \{1 + \frac{1.87229\ s^3}{(s+0.67524)^3} + \frac{1.86229\ s^4}{(s+0.67524)^4}\}$$

The Weibull distributions for time to failure and time to repair, with their approximations, equations (3.60), (3.61), (3.62) and (3.63), (3.64), (3.65) are shown in Figs. 8 and 9. It is obvious that the approximation with n = 4 is very close to the underlying Weibull distribution, and will be used in the calculation of the availability.

From equation (3.30) it follows that the Laplace transform of the availability function is

$$A^*(s) = \frac{1 - L^*\{g_4(\lambda_f t;\alpha_f)\}}{s[1-L^*\{g_4(\lambda_f t;\alpha_f)\}L^*\{g_4(\lambda_g t;\alpha_g)\}]} \tag{3.66}$$

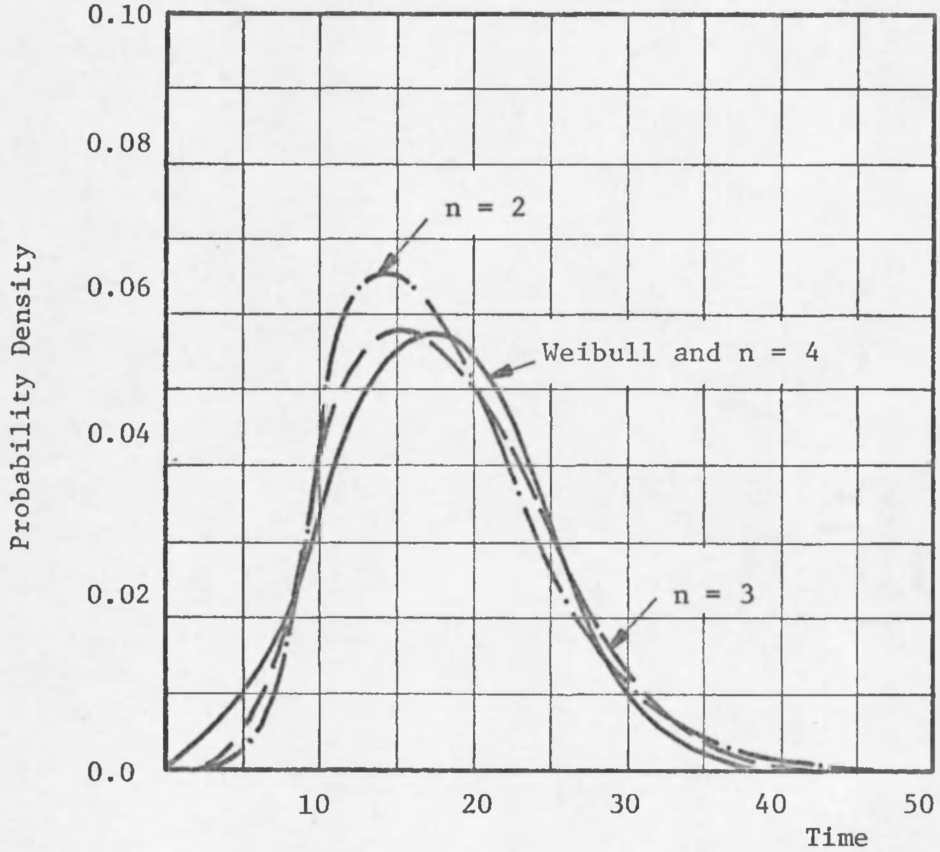


Fig. 8 Time-to-failure Density Functions: Weibull and its Approximations for n = 2, n = 3 and n = 4

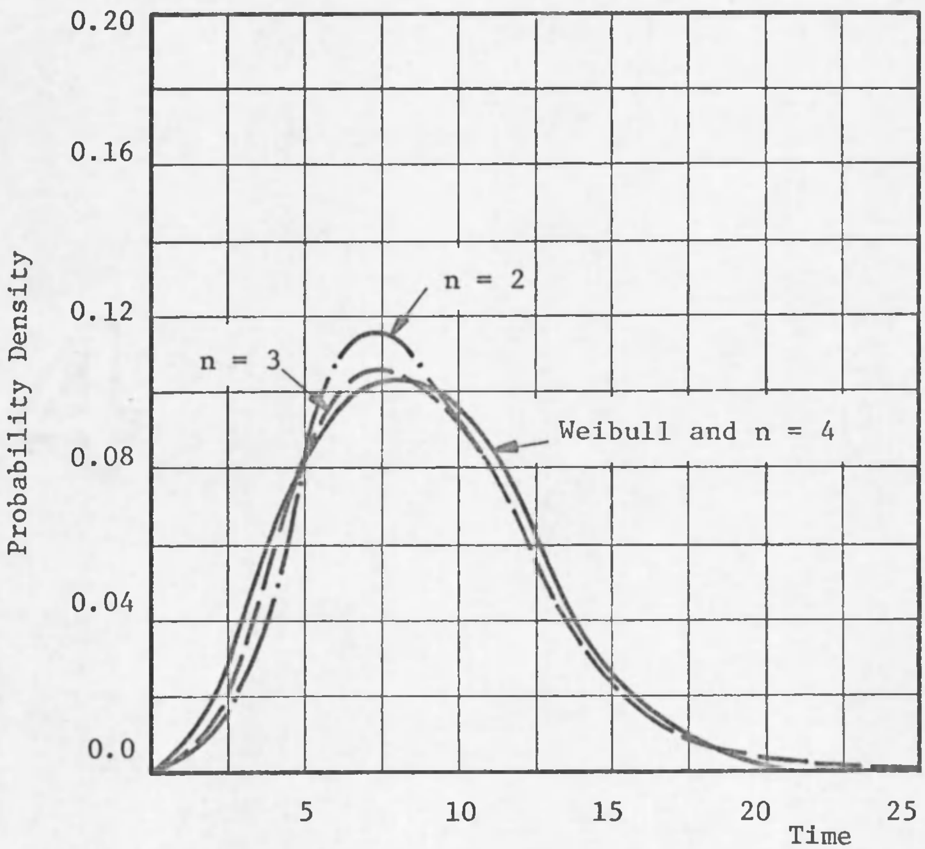


Fig. 9 Time-to-repair Density Functions: Weibull and its Approximations for n = 2, n = 3 and n = 4

Since $L^*\{g_4(\lambda_f t;\alpha_f)\}$ is not a rational function, the approximation for converting equation (3.66) will be applied.

The steady state availability is

$$A(\infty) = \frac{m_{fI}}{m_{f1} + m_{g1}} = \frac{17.86150}{17.86150 + 8.88570} = 0.66779$$

Hence from equations (3.57) and (3.58)

$$a = 0.66779, \qquad c = 0.33221 - (b_1 + b_2 + \ldots + b_n)$$

and

$$d = \frac{1}{17.8615} + \frac{1}{8.88570} = 0.16853$$

Taking $\alpha = 1$, $n = 4$ and for $s_1 = 2.0$, $s_2 = 5$, $s_3 = 20.0$, $s_4 = 100.0$ the following four linear equations result from equation (3.59):

$$0.50 = 0.66667b_1 + 0.44444b_2 + 0.29630b_3 + 0.19753b_4 + 0.15320$$
$$- 0.46114\ (b_1+b_2+b_3+b_4) + 0.33390$$

$$0.20 = 0.83333b_1 + 0.69444b_2 + 0.57870b_3 + 0.48225b_4 + 0.06428$$
$$- 0.19348\ (b_1+b_2+b_3+b_4) + 0.13356$$

$$0.05 = 0.95238b_1 + 0.90703b_2 + 0.86384b_3 + 0.82270b_4 + 0.01647$$
$$- 0.04958\ (b_1+b_2+b_3+b_4) + 0.03339$$

and

$$0.01 = 0.9901b_1 + 0.9803b_2 + 0.97059b_3 + 0.96098b_4 + 0.00332$$
$$- 0.00998\ (b_1+b_2+b_3+b_4) + 0.00668$$

From these, the coefficients k_i' are

$$b_1 = 0.50699$$

$$b_2 = -1.51577$$

$$b_3 = 1.51165$$

and

$$b_4 = -0.50287.$$

From equation (2.88) it follows that

$$L_1^o(\alpha t) = \binom{1}{1} - \alpha t$$

$$= 1 - t$$

$$L_2^o(\alpha t) = \binom{2}{2} - \binom{2}{1} \alpha t + \frac{(\alpha t)^2}{2!}$$

$$= 1 - 2t + 0.5t^2$$

$$L_3^o(\alpha t) = \binom{3}{3} - \binom{3}{2}\alpha t + \binom{3}{1} \frac{(\alpha t)^2}{2!} - \frac{(\alpha t)^3}{3!}$$

$$= 1 - 3t + 1.5t^2 - 0.16667t^3$$

and

$$L_4^o(\alpha t) = \binom{4}{4} - \binom{4}{3}\alpha t + \binom{4}{2} \frac{(\alpha t)^2}{2!} - \binom{4}{1} \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!}$$

$$= 1 - 4t + 3t^2 - 0.66667t^3 + 0.04167t^4$$

Now, from equation (3.56) the availability function is

$$A(t) = e^{-\alpha t} \{b_1L_1^o(\alpha t) + b_2L_2^o(\alpha t) + b_3L_3^o(\alpha t) + b_4L_4^o(\alpha t)\}$$

$$+ c\, e^{-dt} + a$$

or, numerically

$$A(t) = e^{-t}\{0.00109t + 0.00097t^2 + 0.0833t^3 - 0.02095t^4\}$$

$$+ 0.33221\, e^{-0.16853t} + 0.66779 \qquad (3.67)$$

The approximation with n = 3 for the same $s_2 = 5.0$, $s_3 = 20.0$, $s_4 =$ 100.0 gives the following three linear equations

$$0.20 = 0.83333b_1 + 0.69444b_2 + 0.57870b_3 + 0.06428$$
$$- 0.19348(b_1+b_2+b_3) + 0.13356$$

$$0.05 = 0.95238b_1 + 0.90703b_2 + 0.86384b_3 + 0.01647$$
$$- 0.04958(b_1+b_2+b_3) + 0.03339$$

and

$$0.01 = 0.9901b_1 + 0.9803b_2 + 0.97059b_3 + 0.00332$$
$$- 0.00998(b_1+b_2+b_3) + 0.00668$$

From these, the b_i's are

$$b_1 = 0.10343$$
$$b_2 = - 0.20890$$

and $$b_3 = 0.10549$$

The availability function then becomes

$$A(t) = e^{-\alpha t}\{b_1L_1^o(\alpha t) + b_2L_2^o(\alpha t) + b_3L_3^o(\alpha t)\} + c\, e^{-dt} + a$$

or

$$A(t) = e^{-t}\{0.00001-0.00280t+0.05378t^2-0.01758t^3\}$$
$$+ 0.3322\, e^{-0.16853t} + 0.66779 \qquad (3.68)$$

Similarly, the approximation with $n = 2$ for $s_3 = 20$ and $s_4 = 100$ gives the following two equations:

$$0.05 = 0.95238b_1 + 0.90703b_2 + 0.01647 - 0.04958\ (b_1+b_2)$$
$$+ 0.03339$$

and

$$0.01 = 0.9901b_1 + 0.9803b_2 + 0.00332 - 0.00998\ (b_1+b_2)$$
$$+ 0.00668$$

From these $\quad b_1 = 0.00394$

and $\quad b_2 = -\ 0.00398$

The availability function then becomes

$$A(t) = e^{-\alpha t}\ \{b_1L_1^o(\alpha t) + b_2L_2^o(\alpha t)\}\ + c\ e^{-dt} + a$$

or

$$A(t) = e^{-t}\ \{-0.00004 + 0.00401t - 0.00199t^2\}$$
$$+ 0.33225\ e^{-0.16853t} + 0.66779 \qquad (3.69)$$

The availability functions of equations (3.67), (3.68) and (3.69) are plotted on Fig. 10.

It should be noticed that selection of points "s," through which the A*(s) is fitted, is very important. Since, we are interested in the transient availability, when $t \to \infty$, then from the relation

$$\lim_{t\to\infty} A(t) = \lim_{s\to\infty} sA(s)$$

it follows that s should take large values.

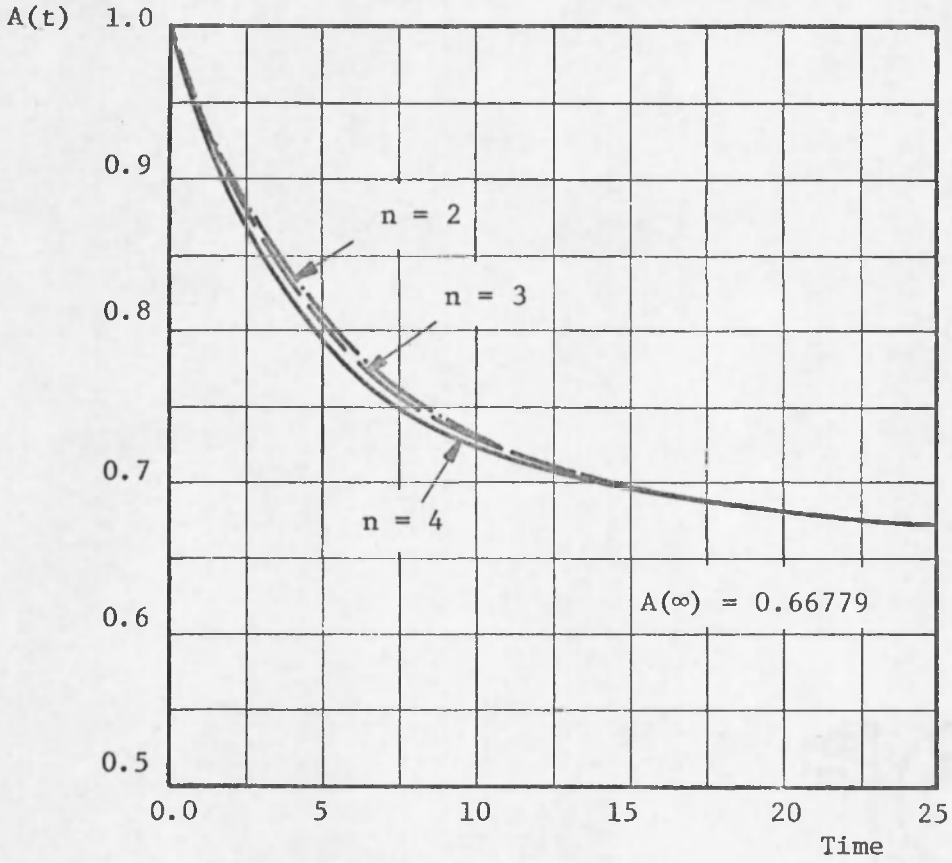


Fig. 10 Availability Function: The Approximations for n = 2, n = 3 and n = 4

CHAPTER 4

AVAILABILITY OF TWO OR MORE UNIT SYSTEMS

4.1 General Remarks

In recent years, there has been a tendency to construct complex systems consisting of two or more separate units. The availability function for complex systems is very difficult mathematically. In the literature there are only suggestions of procedures for a few very special cases.

In this chapter we shall discuss two approaches to the calculation of the availability of complex systems. The first technique is the reduction of a complex system to a one-unit (or a two-state) system which is thoroughly discussed in Chapter 3. Theoretically this reduction is possible, but practically it is very difficult as will be seen in Section 4.2. The second method is the use of a modified Markov model with added dummy states. Actually this method restricts the times-to-failure and the times-to-repair distributions to the special Erlang distribution.

4.2 Reduction to a One-unit System

This approach may be seen through the following example. Consider a system of two units in standby with single repair. For each unit we know the times-to-failure and the times-to-repair distributions. This system has only three possible states:

State 1: Unit 1 is operating, and Unit 2 is operable and in standby.

State 2: Unit 2 is operating, and Unit 1 has failed and is under repair.

State 3: Units 1 and 2 have failed, but only Unit 2 is under repair.

The corresponding state transition diagram is shown in Fig. 11.

Obviously, from States 1 and 3, the system can go only into State 2. But, from State 2, system can enter either State 1 or State 2. So, the transition probability matrix is

$$P = \begin{vmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ 0 & 1 & 0 \end{vmatrix}$$

where α is the probability of going from State 2 to State 1. The system will go to State 1 from State 2 only if the time to repair of Unit 1, Y_1, is less than the time to failure of unit 2, X_2. The probability $p_{21} = \alpha$ is then

$$\alpha = P\{Y_1 - X_2 < 0\}$$

or,

$$\alpha = \int_{x_2=o}^{\infty} \int_{y_1=o}^{x_2} g_{Y_1}(y_1) f_{X_2}(x_2) dx_2 \, dy_1 \tag{4.1}$$

Now, to reduce the above system to a two-state system we divide the state space $S = \{1,2,3\}$ into two disjoint subsets. From Fig. 11, the

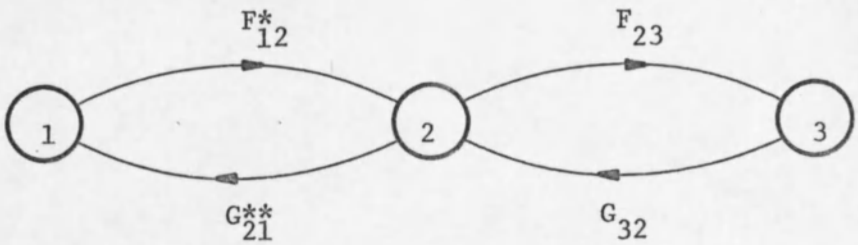


Fig. 11 State Transition Diagram for Two Units in Standby with Single Repair

F* denotes the time-to-failure distribution
G** denotes the time-to-repair distribution

the system is down when it is in State 3, $S_1 = \{3\}$. From State 3 it can enter only State 2 so the times-to-repair distribution for a two-state system is

$$G(t) = G_{32}(t)\ p_{32} \tag{4.2}$$

The system is up in States 1 and 2, $S_2 = \{1,2\}$. State 3 can be reached only from State 2. But before entering State 3 it may make n transitions between States 1 and 2. Thus, the times-to-failure distribution is

$$F(t) = p_{23}F_{23}(t) + p_{23}F_{23}(t) \sum_{n=1}^{\infty} p_{21}^{n}\ G_{21}^{(n)} * F_{12}^{(n)} \tag{4.3}$$

The times-to-failure distribution, equation (4.3), and the times-to-repair distribution, equation (4.2), completely determine the two-state system discussed in Chapter 3.

In terms of analytical form, equation (4.3) is very complex. For some other example it may have a different form, but still it is very complex. This feature restricts the practical application of this method.

4.3 Modified Markov Model

The Markov process applies only to exponentially distributed random variables. In Section 2.5 it is shown that the summation of exponentially distributed random variables is a gamma distributed random variable. This fact is used to apply Markov processes for two or more unit systems when times to failure and times to repair are gamma distributed. This may be illustrated by the following example.

Consider a two-unit standby system with single repair. Each unit has gamma distributed times to failure and times to repair. Physically this system has only three states:

State 1: Unit 1 is operating, and unit 2 is operable and in standby.

State 2: Unit 2 is operating, and unit 1 has failed and is under repair.

State 3: Units 1 and 2 have failed, but only unit 2 is under repair.

To set up the Markov graph for this example we insert a certain number of dummy states. Thus, the modified Markov graph looks like that in Fig. 12. It is seen that the time to failure of unit 1, t_{12}, consists of two exponentially distributed times, t_{14} and t_{42}. For $\lambda_{14} = \lambda_{42} = \lambda_1$, and from equation (2.52), the distribution of t_{12} is $f_{G1}(t;2,\lambda_1)$. The times-to-repair distribution is $f_{G1}(t;3,\mu_1)$, where $\mu_1 = \mu_{51} = \mu_{65} = \mu_{26}$. For unit 2 the times-to-failure distribution is $f_{G2}(t;2,\lambda_2)$ for $\lambda_2 = \lambda_{27} = \lambda_{73}$, and the times to repair is exponentially distributed with parameter $\mu_2 = \mu_{32}$. From Fig. 12 the transition probability matrix is

$$P = \begin{vmatrix} 1-\lambda_1 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 1-(\lambda_2+\mu_1) & 0 & 0 & 0 & \mu_1 & \lambda_2 \\ 0 & \mu_2 & 1-\mu_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 1-\lambda_1 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & 1-\mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & 1-\mu_1 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 1-\lambda_2 \end{vmatrix}$$

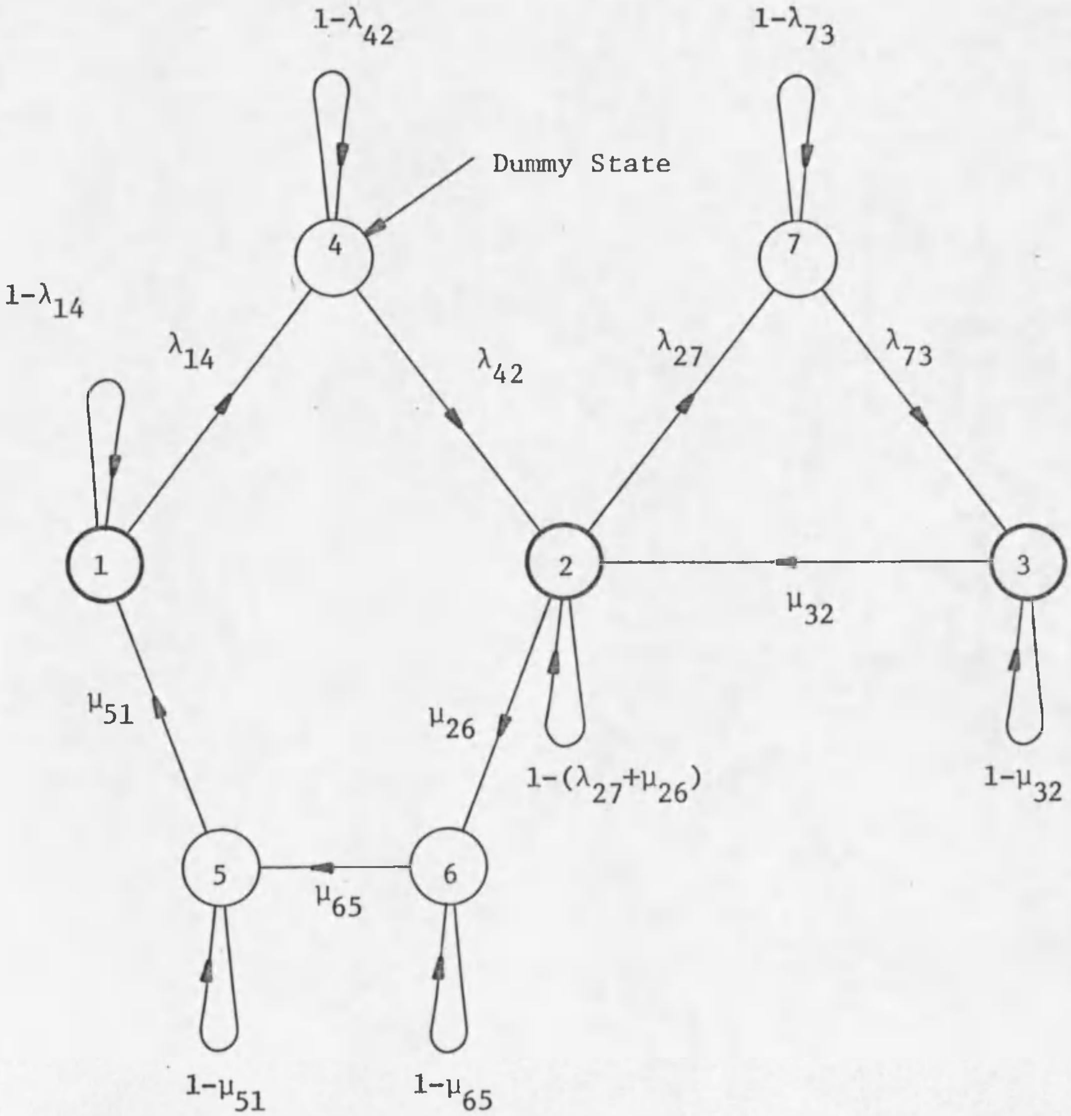


Fig. 12 Modified Markov Graph with Added Dummy States for a Two-unit Standby System

and the differential equations (Bhat, 1972, p. 140, equation 7.6.18) are

$$P_1'(t) = -\lambda_1 P_1(t) + \mu_1 P_5(t)$$

$$P_2'(t) = -(\lambda_2+\mu_1)P_2(t) + \mu_2 P_3(t) + \lambda_1 P_4(t)$$

$$P_3'(t) = -\mu_2 P_3(t) + \lambda_2 P_7(t)$$

$$P_4'(t) = \lambda_1 P_1(t) - \lambda_1 P_4(t) \tag{4.4}$$

$$P_5'(t) = -\mu_1 P_5(t) + \mu_1 P_6(t)$$

$$P_6'(t) = \mu_1 P_2(t) - \mu_1 P_6(t)$$

and

$$P_7'(t) = \lambda_2 P_2(t) - \lambda_2 P_7(t)$$

If $P_1(0) = 1$ and $P_2(0) = \ldots = P_7(0) = 0$, then the Laplace transform of equation (4.4) is

$$sP_1(s) - 1 = -\lambda_1 P_1(s) + \mu_1 P_5(s)$$

$$sP_2(s) = -(\lambda_2+\mu_1)P_2(s) + \mu_2 P_3(s) + \lambda_1 P_4(s)$$

$$sP_3(s) = -\mu_2 P_3(s) + \lambda_2 P_7(s)$$

$$sP_4(s) = \lambda_1 P_1(s) - \lambda_1 P_4(s) \tag{4.5}$$

$$sP_5(s) = -\mu_1 P_5(s) + \mu_1 P_6(s)$$

$$sP_6(s) = \mu_1 P_2(s) - \mu_1 P_6(s)$$

and $$sP_7(s) = \lambda_2 P_2(s) - \lambda_2 P_7(s)$$

For numerical values $\lambda_1 = 1.0$, $\lambda_2 = 0.8$, $\mu_1 = 0.5$, $\mu_2 = 0.4$, the solution of equation (4.5) gives the steady state availability (see Appendix A) as

$$A(\infty) = 1 - \frac{\lambda_1^2 \lambda_2^2 \mu_1^2}{f}$$

$$A(\infty) = 0.71428 \tag{4.6}$$

The availability function is

$$A(t) = 0.71429 - 0.04317\ e^{0.39t} - 0.65037\ e^{-0.93t}$$

$$+ e^{-0.70152t}\{0.20635 \cos (0.48228t) + 0.74395 \sin (0.48228t)\}$$

$$+ e^{-1.38848t}\{0.77290 \cos (0.44823t) - 0.06839 \sin (0.44823t)\}$$

This function represents a decaying sinusoid, which may also be seen from Fig. 13.

Using a corresponding number of states we may form a modified Markov model for any Erlang distribution for times to failure and times to repair, and for any number of units in the system. As the number of states increases the number of differential equations and the number of roots increase, too, but still this approach is applicable, especially when applying computer techniques.

The modified Markov model method may also be used for the reliability of a given system. For the above example, Fig. 13, just setting $\mu_{32} = 0$, the State 3 represents an absorbing state and the availability becomes the reliability.

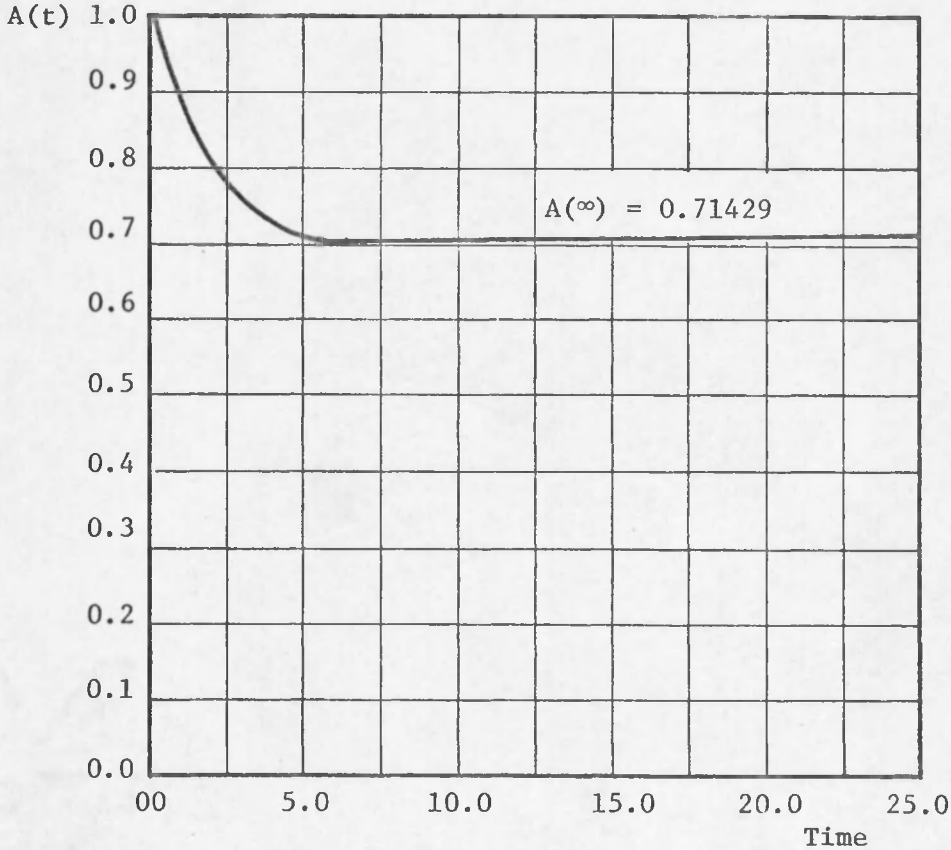


Fig. 13 The Availability Function of a Two-unit Standby System

CHAPTER 5

CONCLUSIONS

In order to get the availability function we first analyse the underlying times-to-failure and times-to-repair distributions. For constant failure and repair rates we can always set up the Markov model, regardless of system complexity, and come up with the availability function. For non-constant failure and repair rates, only special cases can be solved in a general form.

It was shown, in Chapter 2, that aging of the system causes increasing repair and failure rates. This fact puts the lognormal distribution in disfavor because it has a decreasing hazard function in the long-life region. The Weibull family is very flexible and may cover a large range of physical behaviors. In terms of analytical form the gamma family is very suitable. Any distribution can be successfully approximated by a finite series of Laguerre polynomials. The reason for this approximation is a convenient Laplace transform and desired accuracy can be reached by matching the higher moments of the distribution.

To derive the availability equation, stochastic processes were applied. In Chapter 3 a two-state system was discussed. Applying a renewal theorem the Laplace transform of the availability function was found. The inverse Laplace transform is possible in general form for the Erlang distribution and for the approximations given by equation (2.110). Obviously, the feasibility of this method depends on the number of roots of equation (3.42) which have to be found. The computer technique as a tool enormously increases the capability of this approach and makes it practical. The alternate approach is the approximation of A(s) by the series of Laguerre polynomials. Although this method does not have restrictions, in my personal opinion, it is very subjective because the selection of points through which the A(s) is fitted, is left to engineering judgment.

The modified Markov model is the best approach for determining the availability of complex systems. This method gives exact solutions for Erlang distributed times to failure and times to repair. The number of states in the model corresponds to the α parameter of the gamma distribution. So, if α increases, the number of states also increases. This makes the procedure longer but the method is still applicable. An example, given in Section 4.3, illustrates this method.

CHAPTER 6

RECOMMENDATIONS

It was shown that we have enough tools to solve any availability problem when a two-state system is involved. If a solution is not possible in a general form, then the approximation of the given distributions by a finite series of the Laguerre polynomial and the inverse Laplace transform by partial fraction expansion may be a very good approach.

For more complex systems the only possible technique is the modified Markov model. This method restricts the consideration to the Erlang distributed times to failure and times to repair. All other cases remain as a problem for further research.

For the numerical calculation of availability the Monte Carlo method may be applied. In general it has no restriction on the complexity of the system and its feasibility is known in advance. This method should be the subject of another study, and is not covered in this thesis.

APPENDIX A

SOLUTION OF EQUATION (4.5)

Rearranging equation (4.5) gives

1. $P_1(s)\{s+\lambda_1\} = \mu_1 P_5(s) + 1$

2. $P_2(s)\{s+\lambda_2+\mu_1\} = \mu_2 P_3(s) + \lambda_1 P_4(s)$

3. $P_3(s)\{s+\mu_2\} = \lambda_2 P_7(s)$

4. $P_4(s)\{s+\lambda_1\} = \lambda_1 P_1(s)$

5. $P_5(s)\{s+\mu_1\} = \mu_1 P_6(s)$

6. $P_6(s)\{s+\mu_1\} = \mu_1 P_2(s)$

7. $P_7(s)\{s+\lambda_2\} = \lambda_2 P_2(s)$

From equations (4), (5), (6), and (7) it follows that

$$P_4(s) = \frac{\lambda_1}{s+\lambda_1} P_1(s)$$

$$P_5(s) = \left(\frac{\mu_1}{s+\mu_1}\right)^2 P_2(s)$$

$$P_6(s) = \frac{\mu_1}{s+\mu_1} P_2(s)$$

and

$$P_7(s) = \frac{\lambda_2}{s+\lambda_2} P_2(s)$$

Thus equations (1), (2) and (3) become

$$1.\quad P_1(s)\{s+\lambda_1\} = \mu_1\left(\frac{\mu_1}{s+\mu_1}\right)^2 P_2(s) + 1$$

$$2.\quad P_2(s)\{s+\lambda_2+\mu_1\} = \mu_2 P_3(s) + \frac{\lambda_1^2}{s+\lambda_1} P_1(s)$$

$$3.\quad P_3(s)\{s+\mu_2\} = \frac{\lambda_2^2}{s+\lambda_2} P_2(s)$$

From equation (3)

$$P_2(s) = \frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2} P_3(s)$$

Substituting $P_2(s)$ into equations (1) and (2) gives

$$1.\quad P_1(s)\{s+\lambda_1\} = \mu_1\left(\frac{\mu_1}{s+\mu_1}\right)^2 \frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2} P_3(s) + 1$$

$$2.\quad (s+\lambda_2+\mu_1)\frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2} P_3(s) = \mu_2 P_3(s) + \frac{\lambda_1^2}{s+\lambda_1} P_1(s)$$

From equation (1)

$$P_1(s) = \frac{\mu_1^3}{(s+\mu_1)^2}\frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2(s+\lambda_1)} P_3(s) + \frac{1}{s+\lambda_1}$$

Finally equation (2) becomes

$$(s+\lambda_2+\mu_1)\frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2} P_3(s) = \mu_2 P_3(s) + \frac{\lambda_1^2\mu_1^3}{(s+\mu_1)^2}\frac{(s+\mu_2)(s+\lambda_2)}{\lambda_2^2(s+\lambda_1)^2} P_3(s)$$

$$+ \frac{\lambda_1^2}{(s+\lambda_1)^2}$$

from which

$$P_3(s)\left\{\frac{(s+\lambda_2+\mu_1)(s+\mu_2)s+\lambda_2)}{\lambda_2^2} - \mu_2 - \frac{\lambda_1^2\mu_1^3(s+\mu_2)(s+\lambda_2)}{(s+\mu_1)^2\lambda_2^2(s+\lambda_1)^2}\right\} = \frac{\lambda_1^2}{(s+\lambda_1)^2}$$

or,

$$P_3(s) = \{\lambda_1^2\lambda_2^2(s+\mu_1)^2\} \;/\; \{(s+\mu_1)^2(s+\lambda_2+\mu_1)(s+\mu_2)(s+\lambda_2)(s+\lambda_1)^2$$

$$- \mu_2\lambda_2^2(s+\mu_1)^2(s+\lambda_1)^2 - \lambda_1^2\mu_1^3(s+\mu_2)(s+\lambda_2)\}$$

Denoting

$$A = \mu_1^2 + 4\lambda_1\mu_1 + \lambda_1^2 + (\lambda_2+\mu_1)(2\mu_1+2\lambda_1) + \mu_2(3\mu_1+2\lambda_1+\lambda_2)$$

$$B = 2\lambda_1\mu_1^2 + 2\mu_1\lambda_1^2 + (\lambda_2+\mu_1)(\mu_1^2+4\lambda_1\mu_1+\lambda_1^2) + \mu_2(\mu_1^2+4\lambda_1\mu_1+\lambda_1^2)$$

$$+\mu_2(\lambda_2+\mu_1)(2\mu_1+2\lambda_1)$$

$$C = \mu_1^2\lambda_1^2 + (\lambda_2+\mu_1)(2\lambda_1\mu_1^2 + 2\mu_1\lambda_1^2) + \mu_2(2\lambda_1\mu_1^2+2\mu_1\lambda_1^2)$$

$$+\mu_2(\lambda_2+\mu_1)(\mu_1^2+4\lambda_1\mu_1 + \lambda_1^2)$$

$$D = \mu_1^2\lambda_1^2(\lambda_2+\mu_1) + \mu_2\mu_1^2\lambda_1^2 + \mu_2(\lambda_2+\mu_1)(2\lambda_1\mu_1^2+2\mu_1\lambda_1^2)$$

$$a = 3\mu_1 + 2\lambda_1 + 2\lambda_2 + \mu_2$$

$$b = A + \lambda_2(3\mu_1+2\lambda_1+\lambda_2+\mu_2)$$

$$c = B + \lambda_2 A - \mu_2\lambda_2^2$$

$$d = C + B\lambda_2 - \mu_2\lambda_2^2\,(2\mu_1+2\lambda_1)$$

$$e = D + C\lambda_2 - \mu_2\lambda_2^2(\mu_1^2+4\lambda_1\mu_1+\lambda_1^2) - \lambda_1^2\mu_1^3$$

and

$$f = \mu_2\mu_1^2\lambda_1^2(\lambda_2+\mu_1) + D\lambda_2 - \mu_2\lambda_2^2(2\lambda_1\mu_1^2+2\mu_1\lambda_1^2) - \lambda_1^2\mu_1^3(\mu_2+\lambda_2)$$

$P_3(s)$ becomes

$$P_3(s) = \{\lambda_1^2\lambda_2^2(s+\mu_1)^2\} \;/\; \{s(s^6+as^5+bs^4+cs^3 + ds^2 + es + f)\}$$

The availability of this system is

$$A(t) = 1 = P_3(t)$$

Taking the Laplace transform yields

$$A(s) = \frac{1}{s} - P_3(s)$$

from which the steady state availability is

$$\begin{aligned} A(\infty) &= \lim_{t\to\infty} A(t) \\ &= \lim_{s\to\infty} sA(s) \\ &= 1 - \frac{\lambda_1^2\lambda_2^2\mu_1^2}{f} \end{aligned}$$

Or, numerically

$D = 1.205, \qquad f = 0.56$

and

$$A(\infty) = 1 - \frac{(1)^2(0.8)^2(0.5)^2}{0.56} = 0.71428$$

When substituting the numerical values of λ_1, λ_2, μ_1, μ_2, $P_3(s)$ becomes

$$P_3(s) = \frac{0.64s^2+0.64s+0.16}{s(s^6+5.5s^5+12.63s^4+15.425s^2+10.59s^2+3.84s+0.56)}$$

The roots of the denominator are

$$s_1 = 0,\ s_2 = -\ 0.39,\ s_3 = -\ 0.93$$

$$s_4 = -\ 0.70152 + i\ 0.48228$$

$$s_5 = -\ 0.70152 - i\ 0.48228$$

$$s_6 = -\ 1.38848 + i\ 0.44823$$

$$s_7 = -\ 1.38848 - i\ 0.44823$$

Now, we may apply partial fraction expansion to $P_3(s)$, and obtain

$$P_3(s) = \frac{A}{s-s_1} + \frac{B}{s-s_2} + \frac{C}{s-s_3} + \frac{Ds+E}{(s-s_4)(s-s_5)}$$

$$+ \frac{Fs+G}{(s-s_6)(s-s_7)}$$

where

$$A = 0.28571$$

$$B = 0.04317$$

$$C = 0.65037$$

$$D = -\ 0.20635$$

$$E = -\ 0.8835$$

$$F = -\ 0.7729$$

$$G = -\ 1.00476$$

and

$$P_3(s) = \frac{0.28571}{s} + \frac{0.04317}{s+0.39} + \frac{0.65037}{s+0.93} + \frac{-0.20635s - 0.88835}{(s+0.70152)^2+(0.48228)^2}$$

$$+ \frac{-0.7729s - 1.00476}{(s+1.38868)^2+(0.44873)^2}$$

The inverse Laplace transform of $P_3(s)$ is

$$P_3(t) = 0.28571 + 0.04317\, e^{-0.39t} + 0.65037\, e^{-0.93t}$$

$$- e^{-0.70152t}\{0.20635 \cos (0.48228t) + 0.74395 \sin (0.48228t)\}$$

$$- e^{-1.38848t}\{0.77290 \cos (0.44823t) - 0.06839 \sin (0.44323t)\}$$

Finally, the availability of this system is

$$A(t) = 0.71429 - 0.04317\, e^{-0.39t} - 0.65037\, e^{-0.93t}$$

$$+ e^{-0.70152t}\{0.20635 \cos (0.48228t) + 0.74359 \sin (0.48228t)\}$$

$$+ e^{-1.38848t}\{0.77290 \cos (0.44823t) - 0.06839 \sin (0.44923t)\}.$$

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