THE VALIDITY OF MATHEMATICS TEXTBOOK SERIES IN GRADES 7-14 WITH STRUCTURE AS AN OBJECTIVE

by

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I hereby recommend that this dissertation prepared under my direction by Barbara Elpern Buchalter entitled "The Validity of Mathematics Textbook Series in Grades 7-14 With Structure as an Objective" be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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ABSTRACT

An investigation was made of the validity of selected mathematics textbook series for grades 7-14, which had as one of their objectives the teaching of mathematics through an understanding of its underlying structure. The nature of the structure of mathematics was explored with emphasis on unifying concepts and their interrelationships. The factors of mathematical structure were found to be concepts in the areas of sets, ordered sets, groups, fields, vector spaces, and logic and foundations.

The Taxonomy of Educational Objectives, Part 1, Cognitive Domain, edited by Benjamin Bloom, was examined for its implications in teaching mathematical structure, and taxonomic criteria were developed for evaluating the presentation of this structure at the six levels of cognitive learning--knowledge, comprehension, application, analysis, synthesis, and evaluation (lowest to highest).

Ratings for nineteen series comprising forty-five textbooks furnished data for a statistical analysis of the validity of these textbooks with respect to the objective of presenting the underlying structure of mathematics. Ratings by fifteen independent evaluators were used to establish the reliability of the rating procedure. The rating

instrument consisted of a Summary of Taxonomic Criteria, a Summary of Structure of Mathematics, and a Taxonomic Level Rating Data Sheet. The Spearman rank-correlation coefficient for paired rankings on each of the six taxonomic levels, showed significant correlation at the .01 level of significance between this investigator's ratings and each of the two independent evaluations and also between the ratings of the two other evaluators.

Analysis of variance between the differences in mean ratings on each taxonomic level of the textbook series, taken as a single group, indicated support at the .05 level of significance of the following hypotheses guiding the study: (1) The presentation of structure of mathematics at the two lowest levels of cognitive learning--knowledge and comprehension--rates higher than at the four highest levels--application, analysis, synthesis, and evaluation; (2) The presentation of structure of mathematics at the level of Application is no different from the presentation at the two lowest levels of cognitive learning; and (3) The presentation of structure of mathematics is no different at the two lowest levels of cognitive learning.

As a matter of interest, the nineteen series were analyzed singly to provide information on the presentation of structure of mathematics in each series.

Thus the majority of the textbook series evaluated failed to satisfy the criteria for testing their validity with respect to the objective of presenting mathematics through an understanding of its underlying structure. Since the textbooks evaluated comprise the majority of those now in use in mathematics classrooms at the secondary level, the implications are serious.

The procedure developed by this study for evaluation of textbooks was general and applicable to determining the validity of textbooks in any field with respect to their stated objectives.

CHAPTER 1

INTRODUCTION

Pioneers of the modern mathematics trend in the secondary schools stressed the study of mathematics through an understanding of its structures.[1] A commission of the National Council of Teachers of Mathematics has stated that attention should be devoted to the structure of mathematics and certain principles of logic.[2] For these reasons, presenting mathematics through an emphasis on its structures is now a stated objective of most mathematics textbooks series for secondary schools. The extent to which these textbooks contribute to the achievement of this objective has not been determined.

Statement of Problem

It is the purpose of this study to seek answers to the following questions: What is the validity of those mathematics textbook series for grades 7-14 which have the stated objective of presenting mathematics through an emphasis on its structures?

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1. Dorothy M. Fraser, Current Curriculum Studies in Academic Subjects (Washington, D. C.: National Education Association, Project on Instruction, 1962), pp. 37-38.

2. *Ibid.*, p. 29.

Importance of Study

Cronbach stated, "The sheer absence of trustworthy fact regarding the text-in-use is amazing."[3] This statement, made in 1955, is as true today. The teacher who wants to develop a modern mathematics program, has available a wide choice of textbooks. The validity of these textbooks, with respect to their objectives, has not been determined. Cronbach suggested the need for evaluation of textbooks to determine whether the stated objectives were achieved.[4]

Lumsdaine noted, "The usual textbook does not control the behavior of the learner in a way which makes it highly predictable as a vehicle of instruction or amenable to experimental research. It does not in itself generate a describable and predictable process of learner behavior, and this may be the reason there has been very little experimental research on the textbook."[5] He further pointed out that the paucity of experimental research on the characteristics of textbooks was noted by C. R. Carpenter at the Western Regional Conference on Title VII of the

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3. Lee J. Cronbach (ed.), Text Materials in Modern Education (Urbana: University of Illinois Press, 1955), p. 216.

4. Ibid., p. 165.

5. A. A. Lumsdaine, "Instruments and Media of Instruction," Handbook of Research on Teaching, N. L. Gage, (ed.), American Research Association (Chicago: Rand McNally Company, 1963), p. 585.

National Defense Education Act, held in Sacramento in 1960. Carpenter indicated that there were no known experimental comparisons of the effectiveness of alternate versions of text material.[6]

Bloom identified six levels of cognitive learning; knowledge, comprehension, application, analysis, synthesis, and evaluation (from lowest to highest). Throughout his detailed taxonomy Bloom emphasized the need and methods for a precise evaluation of objectives in the cognitive areas of learning.[7]

The significance of choosing the presentation of structure within the text to determine validity is evidenced by this partial definition of mathematics: "It is the study of abstract forms and structures and the relations among them."[8]

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6. Ibid., p. 586.

7. Benjamin Bloom, Taxonomy of Educational Objectives: Book 1, The Cognitive Domain (New York: Longmans, Green and Co., 1956).

8. L. H. Lange, "The Structure of Mathematics," The Structure of Knowledge and the Curriculum, G. W. Ford and Lawrence Pugno, editors (Rand McNally Curriculum Series, Chicago: Rand McNally Company, 1964), p. 52.

Bruner, a noted psychologist, wrote:

> Students, perforce, have a limited exposure to the materials they are to learn. How can this exposure be made to count in their thinking for the rest of their lives? The dominant view among men who have engaged in preparing and teaching the new curricula is that the answer to this question lies in giving students an understanding of the fundamental structure of whatever subjects we choose to teach.[9]

Hypotheses to be Tested

The following hypotheses, which will order and provide direction to the study, shall be tested: (1) The presentation of structure of mathematics at the two lowest levels of cognitive learning rates higher than at the four highest levels; (2) The presentation of structure of mathematics at the level of Application is not different from the presentation at the two lowest levels of cognitive learning; and (3) The presentation of the structure of mathematics is not different at the two lowest levels of cognitive learning. The decision level for rejection of these hypotheses was the .05 level of significance. If teaching the structure of mathematics at all levels of cognitive learning is the aim, the support of hypothesis (1) would indicate that the majority of textbooks do not accomplish this as well at the application, analysis, synthesis, and evaluation levels, as they do at the levels of knowledge and comprehension.

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9. Jerome Bruner, The Process of Education (Cambridge: Harvard University Press, 1960), p. 11.

Assumptions Underlying the Problem

The following will be assumed in conducting this study: (1) The nature of the structure of mathematics can be delineated by a classification scheme linking broad areas of mathematics by means of chains of abstract concepts: (2) Bloom's analysis of the objectives in the cognitive domain is applicable to the problem of determining criteria for evaluating the extent to which a textbook series achieves the objective of presenting mathematics through a study of structures;[10] and (3) It is possible to devise a quantified rating scale to evaluate the presentation of structure of mathematics textbooks, and from this to determine validity by statistical procedures.

Limitations of the Study

The following will limit the significance and scope of this study: (1) The results will be significant only to the extent the criteria and rating scale used are accepted as valid; and (2) Only mathematics textbook series which are written for grades 7-14, and which have the stated objective of presenting mathematics through a study of its structures will be evaluated. The book, Textbooks in Print, 1966[11] was used as the basic reference to determine whether

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10. B. Bloom, op. cit.

11. Textbooks in Print, 1966 (New York: R. R. Bowker Company), pp. 142-160.

all textbooks meeting the criteria of the study had been surveyed.

Definition of Terms

The following definitions will serve throughout this study:

(1) structure of mathematics. The abstract forms or patterns underlying the content of mathematics.[12]

(2) presentation of structure as an objective. Considered synonomous with objectives of presenting mathematics as an axiomatic system, as a postulational system, as an axiomatic-deductive system, or any words which denote study of mathematics through abstract patterns in a logical manner.

(3) basic generalizations. Those theorems and assumptions which provide a postulational system to characterize the structure under consideration.

(4) textbook series. A collection of textbooks from the same publisher which are intended for at least two consecutive years of mathematics study.

(5) validity. Construct validity, which is the relation of the textbook series to the concepts or theory which it is used to study, i. e. how well its presentation of structure relates to the concept of presentation of structure as defined by the selected criteria; it is a conditional validity.[13]

(6) level of cognitive learning. (from lowest to highest) Knowledge, comprehension, application, analysis, synthesis, and evaluation, as these are defined by Bloom.[14]

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12. Raymond L. Wilder, Introduction to the Foundations of Mathematics (second edition) (New York: John Wiley and Sons, Inc., 1965), p. 290.

13. Paul L. Dressel, et al., Evaluation in Higher Education (Boston: Houghton Mifflin Company, 1961), p. 448.

14. Bloom, op. cit., p. 18.

Resume of Related Literature

Literature on Structure

The current emphasis on the structure of mathematics emerged from the tenets of formalism, the foundational philosophy outlined by David Hilbert in his Grundlagen der Mathematik, and the logicist school of thought developed by Russell and Whitehead in their Principia Mathematica.[15] The result has been an emphasis on abstract symbolism, rigorous proofs, reliance on the axiomatic-deductive method and above all, a search for common comprehensive abstract structures.[16] These structures were identified by Delessart.[17]

Dienes gave three categories for situations pertaining to mathematical structure, and discussed principles needed to grasp structure.[18] Schwab stated that to study structures, the membership of the discipline must be

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15. Abraham S. Luchins and Edith H. Luchins, Logical Foundations of Mathematics for Behavioral Scientists (New York: Holt, Rinehart and Winston, Inc., 1965), p. 169.

16. Lange, op. cit., p. 50.

17. Andre Delessart, "What Does the Secondary-School Teacher of Mathematics Expect from the University," The Mathematics Teacher, 59: 280-281, March, 1966.

18. Z. P. Dienes, The Power of Mathematics (London: Hutchinson Educational Ltd., 1964), p. 39.

delineated and the conceptual structures identified.[19] Lange pointed out the necessary attributes for a good presentation of structure,[20] while Hartung and Fawcett stated these attributes in terms of behavioral goals.[21]

Literature on Establishing Validity

Astin distinguished between conceptual criterion and criterion performance. He maintained criterion measures cannot be validated, only logically analyzed for relevance to conceptual criterion.[22] Cronbach analyzed how well a textbook aim was realized by evaluating the amount and balance of material of the following types: description, narration, generalization and specialized vocabulary.[23]

Klein, in her definitive evaluation of the measurement of cognitive behavior as defined by Bloom's *Taxonomy*,

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19. Joseph J. Schwab, "Structure of the Disciplines: Meanings and Significances," *The Structure of Knowledge and Curriculum*, *op. cit.*, pp. 28-30.

20. Lange, *op. cit.*, p. 55.

21. Maurice Hartung and Harold Fawcett, "The Measurement of Understanding in Secondary School Mathematics," *The Measurement of Understanding*, Forty-fifth Yearbook of the National Society for the Study of Education, Part I, ed. Nelson B. Henry (Chicago: University of Chicago Press, 1946), p. 159.

22. Alexander W. Astin, "Criterion-Centered Research," *Educational and Psychological Measurement*, 24:819, 1964.

23. Cronbach, *op. cit.*, p. 56.

validated the levels of cognition given.[24] Cureton, in constructing a criterion, started with a trait-name and then constructed a corresponding criterion measure. He further emphasized that the criterion measure must be acceptable as an operational definition of the trait-name and if it is factorially complex, the factors must be assumed to correlate sufficiently with one another to make the single numerical index meaningful.[25] Bottrell, in a critical analysis of social studies textbooks, utilized a Data Work Sheet and Data Summary Sheet, with subsequent derivation of a quantity index and quality index. Her results were presented; however, in descriptive terms.[26]

Lawrence found that although the textbook is the major determining factor in the organization and content of the course of study, no procedures have been developed to determine the validity of the textbook.[27]

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24. Minnie Frances Klein, "Evaluation of Instruction: Measurement of Cognitive Behavior as Defined by the Taxonomy of Educational Objectives," (unpublished doctoral dissertation, University of California at Los Angeles, 1965), p. 10.

25. Edward E. Cureton, "Reliability and Validity: Basic Assumptions and Experimental Designs," Educational and Psychological Measurement, 25: 327-345, 1965.

26. Helen K. Bottrell, "A Critical Analysis of Human Relations Materials in State-Adopted Textbooks," (unpublished doctoral dissertation, University of Houston, 1954), p. 3.

27. John Dennis Lawrence, "The Application of Criteria to Textbooks in Secondary Schools of Los Angeles County," (unpublished doctoral dissertation, University of Southern California, 1961), p. 7.

Organization of Remaining Chapters

In Chapter 2 the evaluative procedures are outlined and the reliability of the rating procedure is established. This is done by developing a rating instrument, devising rating procedures, and analyzing statistically the correlation of ratings. This second chapter explains the selection of samples, the interpretation of rating scores, and the orientation of the evaluators. In this same chapter, the structure of mathematics is delineated, with particular emphasis on the tracing of the threads of common ideas throughout identified sub-categories of structure. This chapter also provides the background for the development of the criteria used in the evaluation of the textbooks. Included is a summary of Bloom's characterization of the levels of cognitive learning with particular applications to the area of this study. The final function of this chapter is to determine the reliability of the rating procedure.

Chapter 3 gives a complete analysis of the data from the textbook evaluation and presents conclusions drawn from this analysis.

Chapter 4 summarizes this study. This final chapter analyzes the findings of this study, records conclusions, and lists possible areas of further research to which this study may give impetus.

CHAPTER 2

EVALUATIVE PROCEDURES

An evaluation of the selected textbook series involved procedures for selection of textbooks, development of a rating instrument, selection and orientation of independent evaluators, rating each selected series by this investigator and other evaluators and statistical analyses of correlation between these ratings. These evaluative procedures were carried out as described in the rest of this chapter.

Selection of Textbook Series

Letters were sent to all publishers of mathematics textbooks for grades 7-14, listed in Textbooks in Print, 1966.[28] Sixty-two such letters were sent. The letter described the nature of this study and requested a reply giving names of textbooks the publishers felt met the scope of the study. The following excerpt from the letter indicated the type of books desired and information solicited.

> My research involves a study of the validity of those mathematics textbook series in grades 7-14, which have as a stated objective, "the study of mathematics through understanding of its underlying structure." The textbooks used for the study will be those which are in a series intended

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28. Textbooks in Print, 1966, op. cit.

> for at least two consecutive years of study as classroom texts in grades 7-14. Series which cover more than two years are especially desirable, since it is the over-all achievement of this objective that will be evaluated, not its achievement at just one level. Only textbooks which state, in the preface, teacher's edition, advertising materials, or letter from the author or publisher, that this series has as an objective the one under consideration, will be evaluated. The evaluation will be in terms of the level at which this objective may be expected to be realized as judged from the contents of the textbook series itself. These levels are the ones outlined in The Taxonomy of Educational Objectives: Part 1, Cognitive Domain, by Benjamin Bloom, et al.
>
> . . . I would appreciate a letter from you stating which textbook series you publish that you feel would satisfy the above criteria for this study. . . .

Responses were received to all the letters. From these responses, seventeen textbook series were recommended by the publishing company as suitable for the study. In addition, two other series were selected to be included. These latter selections were based on statements by the authors in the preface of the book that one of the principal objectives of the textbook was to present mathematics through a study of its underlying structure. All of the selected textbook series were listed in Textbooks in Print, 1966.

The complete list of selected textbook series is given in Appendix 1, page 79, and totals nineteen textbook series, encompassing a total of forty-five textbooks, published by thirteen different companies. An analysis of these series indicated one covered four years of college preparatory high school mathematics; eight covered material usually taught in seventh and eighth grades, with one of these series intended for non-college bound students; four covered material usually taught in two years in a high school college preparatory class; one two-year sequence was for high school general mathematics; one series was intended for grades seven, eight, and nine, with the latter book covering first year algebra; four encompassed three years of college preparatory mathematics; six included a geometry textbook. Based on observation and consensus of opinion of all those involved in the study, the selected series were the textbooks being used in the United States in the majority of mathematics classes in grades 7-12.

Development of Rating Instrument

The rating instrument employed for the evaluation of the selected textbook series consisted of three parts: (1) A Summary of the Structure of Mathematics; (2) A Summary of Taxonomic Criteria; and (3) A Taxonomic Level Rating Data Sheet. In order to develop these three instruments, it was necessary to characterize the structure of mathematics and to adopt a taxonomy of educational objectives related to the presentation of this structure. The final step was to devise a form for rating the presentation of each factor of mathematical structure, in terms of the extent to which this presentation fulfilled the taxonomic criteria.

Characterization of Structure of Mathematics

In order to delineate the underlying structure of mathematics, it was necessary to define broad areas of mathematics which would encompass this structure. The elements of these areas overlap in many cases, so that a description of the structure of mathematics was best understood by a diagrammatic approach relating these broad areas and the kind of quantities with which each is concerned. Delessart has employed a similar device, in trying to find a natural division which would permit assigning specific sections to secondary mathematics, and other sections to the university level.[29] This method makes use of what he called the natural ladder and the domain of structures, with a residue in which to place those topics in which mathematicians may take an interest, but which are difficult to place in the two principal charts.

Figure 1, page 15, consists of two charts and a list of residual areas, and is a modification of the one employed by Delessart in his presentation at the Conference on the Teaching of the Sciences and Economic Progress, held at Dakar, Benegal, January 14-22, 1965, organized jointly by the Inter-Union Commission on Science Instruction (ICSI) and the International Commission on Mathematics Instruction (ICMI).

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29. Delessart, op. cit.

Figure 1. Definition of Mathematical Structure

N:	set of natural whole numbers (0,1,2,3, . . .)
Z:	set of integers (0,1,-1,2,-2, . . .,n,-n . . .)
Z_n:	set of integers modulo rational whole numbers, i.e. classes of rational whole numbers.
Q:	set of rational numbers
R:	set of real numbers
E_2:	continuous Euclidean plane
E_3:	continuous "ordinary" Euclidean space
GE(2,R):	set of isometries of E_2
GE(3,R):	set of isometries of E_3
C:	set of complex numbers
R [x] :	set of polynominals in one variable x, with real coefficients
R [x,y] :	set of polynominals in two variables x and y, with real coefficients
R^n:	set of sequences of n real numbers,n=2,3, . . .
S^{n-1}:	real sphere of dimension n-1
P_{n-1}:	real projective space of dimension n-1
GL(n,R):	complete real linear group of degree n
E.Funct.:	class of sets of functions with real values, defined on non-empty subsets of R.

Chart A--The Natural Ladder

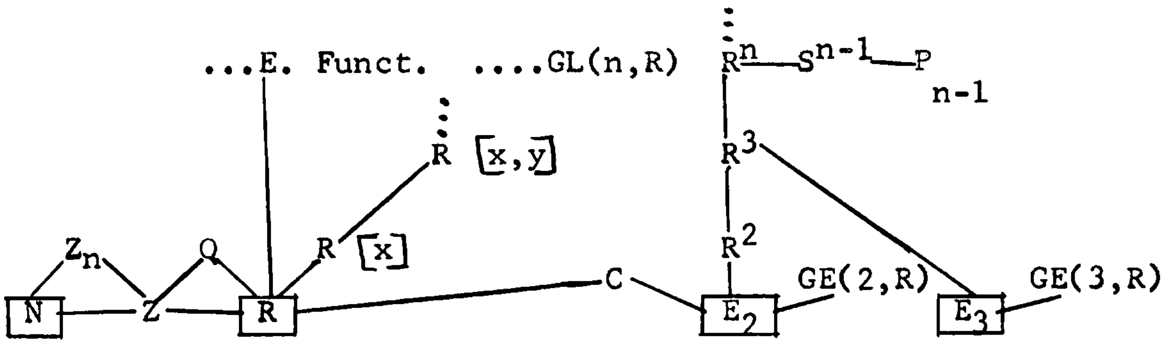


Chart B--Domain of Structures

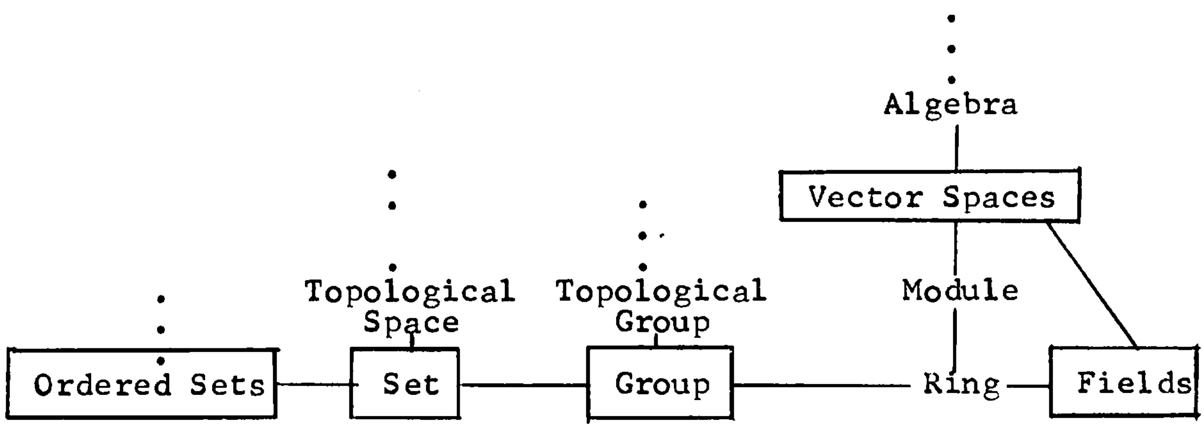


Chart C--Residue

(1) Logic

(2) Foundations of Mathematics

Figure 1. Definition of Mathematical Structure

Definition of Structure. It is well to emphasize that in Figure 1, diagrams A and B are merely sketched out. Even if one takes no account of the residual part, C, the union of A and B does not give a complete idea of mathematical structure. In addition, no attempt has been made to sketch the extensive branching possible throughout both charts. Diagrams A and B are not essentially independent. The links appearing in diagram A symbolize canonical procedures of construction elaborated in the domain of structures. Reciprocally, a structure appears in B only when it possesses a sufficient number of interesting models in the natural ladder. There are only a very small number of mathematical notions which are not prefigured in N, R, E_2 or E_3 and the objects which relate to them directly.

The ideas in frames (Figure 1) are fundamental. They are either in constant use in the building of mathematics, or they appear in all the practical applications, or else they provide vivid terminologies and prototypes in view of more advanced theories.

A vertical division does not appear naturally. Although there exist simple or fundamental notions (set, topological space, R, etc.) and compound or derived notions (vector space, topological group, etc.), many of these fundamental notions are not elementary, and many of the elementary notions are not simple. For the purpose of this study, it was not necessary to determine what constituted

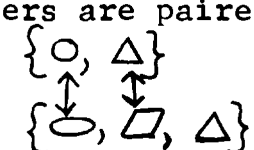
secondary school mathematics. This study assumed the following definition of mathematical structure, based on the diagrams presented in Figure 1, page 15:
Mathematical Structure (M) is a function which is the union of the product of charts A and B and basic concepts of logic and foundations of mathematics found in chart C.

Unifying Themes. The study of mathematics as a system of related ideas is basic to a program with the objective of presenting mathematics through an understanding of its underlying structure. At first, the approach is informal and intuitive with little verbalization of the principles involved. Gradually, the principles are identified and an awareness of mathematics as a system is developed. Major unifying themes provide the framework for this awareness. In this study the interrelationships of these unifying themes have been defined to come under the broad areas of sets, ordered sets, groups, vector spaces, fields, logic and foundations. The following is a brief examination of these interrelationships.

The concept of a set is the cornerstone of any study of mathematical systems, because of the clarity and precision it can provide. In general, the notion of a set arises with an awareness that objects, symbols, or ideas may share certain properties or characteristics. A set can be formed with objects having a common property. The objects are called members of the set. By investigating its properties, sets may be classified as finite or infinite, and useful notation introduced to designate sets. Once sets are designated, they are investigated to see how they relate to each other. In this way, ideas of subset, proper subsets, one-to-one correspondence and equivalence are clarified.

There is a close connection between sets and whole numbers. Each whole number is associated with a particular family of equivalent sets. For example, the family of sets equivalent to $\{\bigcirc, \triangle\}$ is associated with the number two. The whole number zero is associated with the empty set. This process is continued for all whole numbers.

Sets are also used to order the whole numbers. For example, a set of two and a set of three are displayed, and their members are paired as shown below:



This process leaves a member of the set of three unpaired, and so we state "2 is less than 3," which may be written $2 < 3$. Since there is only a single member left unpaired in the set of three, we are led to the idea that 3 is the number next after 2, or the successor to 2. This, in turn, leads to the idea that each whole number has a successor and hence, the set of whole numbers is infinite. This process also identifies zero as being the least whole number.

The concept of one-to-one correspondence, or pairing of members of two sets with exactly one member from each begins in kindergarten with matching clowns in one picture with clown hats in another. This idea is further developed in elementary school in determining how many are in a set. In junior high school, the concept of comparing the number of elements in the set of whole numbers, and the number of elements in the set of even numbers, further extends this concept of one-to-one correspondence as a means of determining the power of a set. In secondary school and undergraduate college work, a study of functions reveals one-to-one correspondence as a mapping and finally, in graduate studies the ideas of injections and isomorphism in abstract algebra place this concept of one-to-one correspondence in its proper perspective.

In the study of ordered sets, the properties of an equivalence relation are studied. These properties of reflexivity, symmetry, and transitivity, permeate much of mathematics. The trichotomy principle and properties of inequalities assure us that the whole numbers are totally ordered throughout. Operations can be defined in terms of Cartesian products, using ordered pairs. The integers, rational numbers, irrational numbers, real numbers, and complex number systems are all developed as extensions of the whole number system.

The five basic properties of positive integers, associativity, commutativity, existence of an additive identity, existence of a multiplicative identity, and the distributive property, constitute an algebra applicable to many systems.[30] These basic properties may be regarded as postulates for a particular type of algebraic structure, and any theorem formally implied by these postulates would be applicable to any other interpretation satisfying these five properties. The postulates of an ordered field form this underlying algebraic structure.

The study of the real number system as an ordered field and its relationship to the previous study of whole numbers, integers, and rational number systems, is basic in understanding mathematical structure. The principles of mathematical induction complete the properties of the real number system.

A group is one of the simplest algebraic structures of consequence. The importance of groups and semi-groups in algebra lies largely in the fact that many algebraic systems are actually groups or semi-groups with respect to one or more of the binary operations of the systems. Many algebraic structures contain the group structure of the semi-group structure within them as a substructure.

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30. Howard Eves and Carroll Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, (New York: Rinehart and Company, Inc., 1960), pp. 120-121.

Groups have been applied to geometry, too, in employing Felix Klein's definition of geometry: "A geometry is the study of those properties of a set S which remain invariant when the elements of set S are subjected to the transformations of some transformation group, T."[31]

Topology is perhaps the most general geometry in the sense its transformation group embraces many other transformation groups, and any theorem of topology holds within other geometries. Examples of these are Euclidean plane metric geometry, non-Euclidean plane geometry, line geometries, circle geometries, sphere geometries, plane equiform geometry, projective geometry and analytical geometry. Topology also has an algebraic structure.

The algebra of vectors is best recognized as an algebra by studying its development from the concepts of group, field, ring, module, and in general, the perception of those properties it has in common with the familiar algebraic structures previously studied. Often, vectors are studied only for their application and not through an understanding of their underlying structure.

Logic is the cement of mathematics. Through a study of the principles of logic, inductive and deductive reasoning may be utilized. In particular, the postulational

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31. Ibid., pp. 135-136.

approach to the study of mathematics requires an understanding of deductive reasoning and the underlying principles of logic.

From this brief summary it is obvious that the strands of mathematical structure evident throughout the study of mathematics are actually the properties of the six broad categories--sets, ordered sets, groups, vector spaces, fields and logic and foundations. The most careful description of these properties can not show all the interrelationships. Complete understanding can only come through a more than superficial study of each of the six basic areas in the domain of the structure of mathematics.

Application to Selected Series. The topics included in the six areas of structure of mathematics sometimes appear in high school textbooks under different headings. The study of sets may appear under that heading and usually includes set terminology and operations. More importantly, the study of sets will be found throughout all of the textbook wherever set theory is applied. Two areas where this usually occurs is in finding solution sets of equations and inequalities and in defining concepts in non-metric geometry. This latter area includes the study of lines, rays, segments, intervals, angles, and planes as sets of points.

Ordered sets are studied in units on functions and relations, graphing in the real number plane and order on the number line. Included also is the study of "less than" and "greater than" relations.

The study of groups does not always appear under that title in high school textbooks, although some recent texts use the word "group." Studies of closure, inverses, identity elements, and associative properties, all come under the study of groups. In geometry groups are sometimes studied in terms of translations and rotations.

The study of vector spaces is increasingly being included in high school mathematics textbooks under that title. The algebra of vectors is included in such books. However, even if vector spaces are never mentioned by that name, the algebra of number pairs introduces the same concepts. Matrices are often included in high school courses.

The study of fields underlies much of high school mathematics since the study of algebra takes up the largest proportion of time in a four-year sequence. The associative, commutative, and distributive properties of real numbers along with order relationships are now commonly found in high school and junior high school mathematics textbooks. Some textbooks go even further and specifically include a study of mathematical systems.

The study of foundations and logic may not be apparent as key units in the textbooks evaluated, but may be dispersed throughout other units. Topics in this category usually found in secondary mathematics textbooks are number theoretic problems, discussions of completeness and density of real numbers, distinction between numbers and numerals, application of rules of reasoning in proofs, and the study of deductive and inductive reasoning.

Some of the sub-areas under the six basic areas of structure of mathematics may appear in high school textbooks in disguised roles. Point-set topology is included in a study of sequences and in some senior level books in a study of limits. Some books include enrichment topics from topology in geometry textbooks.

These guidelines for finding the six areas of structure of mathematics in secondary school textbooks are not meant to be exhaustive. The first requirement for evaluating structure is a thorough knowledge of the mathematics in each of the six areas being evaluated. Without this background a competent and valid evaluation of the validity of mathematics textbooks with respect to their presentation of the structure is not possible. Guidelines, checklists, and summaries are only an aid to evaluation, not a substitute for knowledge of the nature of the thing being evaluated.

Development of Taxonomic Criteria

In order to evaluate the level at which structure of mathematics was presented in a textbook series, it was necessary to translate the basic concepts discussed in the preceding section into criteria written in behavioral terms.

For this purpose, Bloom's categorization of educational objectives in the cognitive domain was adopted. Corresponding to each level in the taxonomy--knowledge, comprehension, application, analysis, synthesis and evaluation--criteria were written to measure the relative presence or absence of material in the textbook series judged necessary for producing learning of structure at each level. These criteria were then judged valid by a panel of experts in the field. Finally, an evaluative model was designed to measure statistically the extent to which the given textbook series achieved the objective of presenting structure.

To understand the pertinence of the criteria developed, a short summary of the portions of Bloom's taxonomy appropriate to this study was included in this section.

Knowledge. The category of Knowledge may be divided into the following sub-categories:

1. Knowledge of Specifics--recall of specific and isolable bits of information.

(1) Knowledge of terminology--knowledge of the referents for specific verbal and nonverbal symbols.

(2) Knowledge of specific facts--distinguished from (1) since (1) usually represents conventions or agreements within a field, while facts represent findings testable by more than unanimity of the workers of the field. It includes knowledge about particular books, writings, and sources of information on specific topics and problems.

2. Knowledge of Ways and Means of Dealing with Specifics--knowledge of the ways of organizing, studying, judging and criticizing ideas and phenomena. Ways and means refers to processes rather than products. They indicate operations rather than results of operations. Involved are reflections on how field workers think and attack problems rather than the results of such thought or problem-solving.

(1) Knowledge of conventions--knowledge of characteristic ways of treating and presenting ideas and phenomena.

(2) Knowledge of trends and sequences--knowledge of the processes, directions, and movements of phenomena with respect to time.

(3) Knowledge of classifications and categories--knowledge of the classes, sets, division and arrangements which are regarded as fundamental or useful for a given subject field, purpose, argument, or problem. This includes only knowledge of these classifications, not their use.

(4) Knowledge of criteria--knowledge of the criteria by which facts, principles, opinions and conduct are tested or judged.

(5) Knowledge of methodology--knowledge of the methods of inquiry, techniques, and procedures employed in a particular subject field as well as those employed in investigating particular problems and phenomena.

3. Knowledge of the Universals and Abstractions in a Field--knowledge of the major ideas, schemes, and patterns by which

phenomena and ideas are organized.

(1) Knowledge of principles and generalizations--knowledge of particular abstractions which summarize observations of phenomena. All that is required is that the student be able to recognize or recall correct versions of principles and recall illustrations of them.

(2) Knowledge of theories and structures--knowledge of the body of principles and generalizations together with their interrelations which present a clear, rounded and systematic view of a complex phenomenon, problem or field. It differs from (1) in that emphasis here is on a <u>body</u> of principles and generalizations which are interrelated to form a theory or structure, while in (1) principles are treated as particulars which need not be related to each other.

<u>Comprehension</u>. This category includes those objectives, behaviors, or responses which represent an understanding of the literal message contained in a communication. The student may change the communication in his mind or in his overt responses to some parallel form more meaningful to him. Some responses may represent simple extensions beyond what is given in the communication itself.

1. Translation--means an individual can put a communication into another language, in other terms, or into another form of communication. It usually involves the giving of meaning to the various parts of a communication taken in isolation,

although such meaning may be partly determined by context. Individual competence in translation is usually dependent on possession of requisite or relevant knowledge. In order to engage in more complex thinking, an individual must conceptualize a given term in a communication as a general concept or aggregation of relevant ideas. An abstract idea may need to be transformed to concrete terms, or some part may need to be abstracted symbolically to facilitate thinking. Translation, then, may involve:

(1) Translation from one level of abstraction to another.

(2) Translation from one symbolic form to another form, or vice-versa.

(3) Translation from one verbal form to another.

2. Interpretation--to interpret a communication the reader must go beyond a part-by-part rendering of the communication to comprehend the relationships between its various parts. This involves a global view of what the communication contains. Interpretation also involves skill in recognizing essentials. The essential behavior in interpretation is that when given a communication, the student can identify and comprehend the major ideas in it as well as their interrelationships.

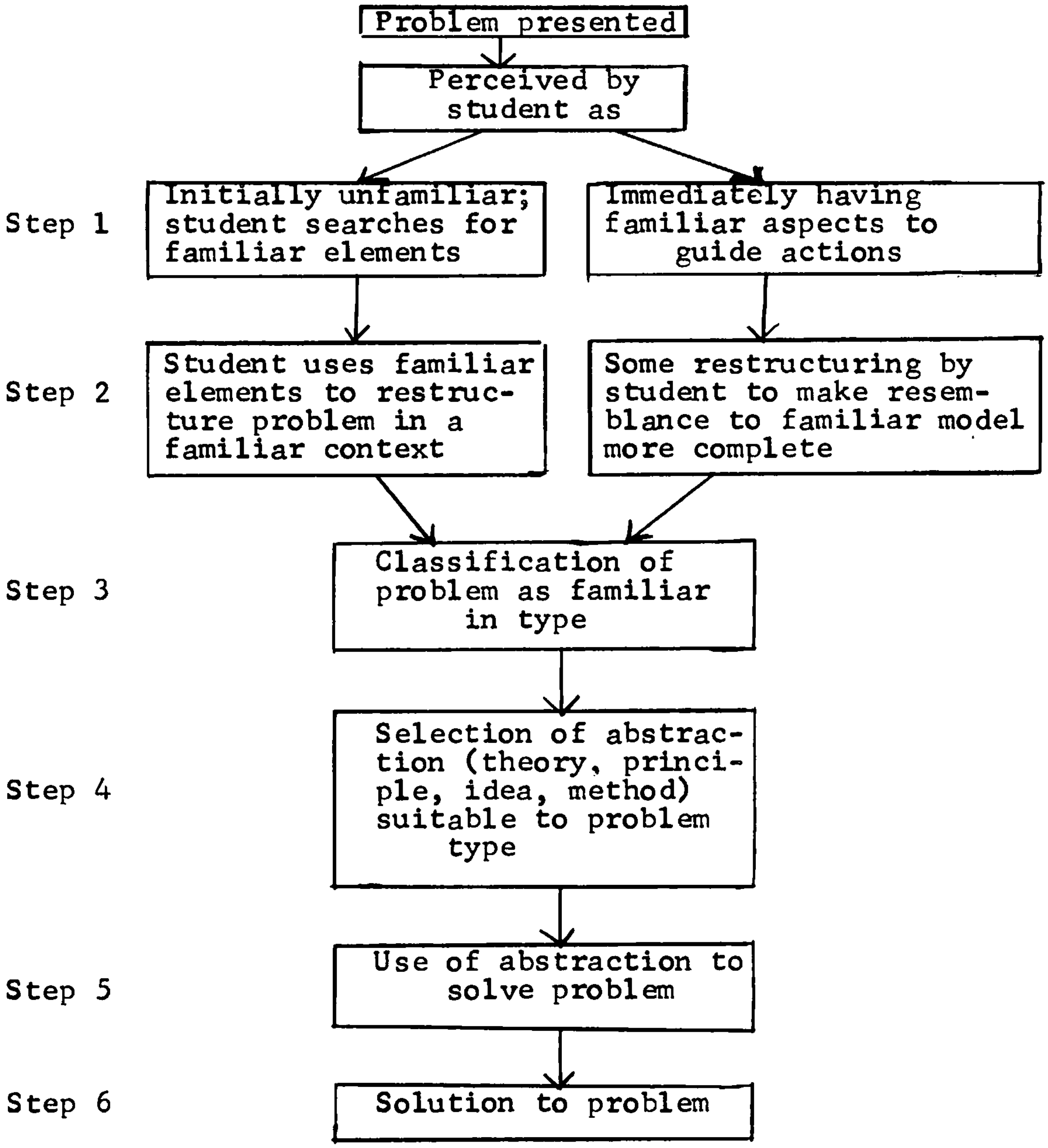
3. Extrapolation--although a writer tries to point out the consequences of a communication, he cannot exhaust these possibilities. The reader must, by extrapolation, perceive

the application of the ideas of the communication with the limits set by the author. Perception of these limits is also an important part of comprehension. In almost all cases the reader must recognize that an extrapolation can only be an inference which has some degree of probability. Extrapolation differs from application in that the thinking is characterized by the extension of what is given to other conditions. The thinking is less abstract than in the case of applications which make use of generalizations, rules of procedure, et cetera.

Application. The cognitive domain of the taxonomy is arranged like a nested sequence, i. e. each classification demands the skills and abilities which are lower in the classification order. Thus to apply something requires comprehension of the method, thoery, principle, or abstraction applied.

A problem of the comprehension category requires the student to know an abstraction well enough that he can correctly demonstrate its use when specifically asked to do so. Application goes beyond this. A demonstration of comprehension shows a student can use the abstraction when its use is specified. A demonstration of application shows he will use it correctly, given an appropriate situation in which no mode of solution is specified.

Figure 2 shows in diagrammatic form the problem-solving process of answering questions classified in the



Reproduced from Taxonomy of Educational Objectives, Part 1, Cognitive Domain, Benjamin Bloom (editor), et. al., op. cit., p. 121.

Figure 2. Application and Comprehension Processes

application category. Steps 1 through 4 are part of application but not of comprehension. Comprehension is best represented by a problem which starts with Step 5, Steps 1-4 being unnecessary because of the structuring of the problem situation. In writing application problems, it is necessary to observe the following:

1. Need for new yet real items
2. Need to identify general problem-solving ability versus application of specific principles
3. Need to diagnose failure on application problems
4. Need for adequate sampling of application

Analysis. Analysis emphasizes the breakdown of the material into its constituent parts and the detection of the relationships of the parts and the way they are organized. Although analysis may be an end in itself, it may also serve as a prelude to fuller comprehension or evaluation. Analysis deals with content and form. Analysis, as an objective, may be divided into three levels:

1. Analysis of elements--the student is expected to break down the material into its constituent parts and to identify or classify the elements of the communication.
2. Analysis of the relationships--the student is required to make explicit the relationships among the elements, to determine their connections and interactions.
3. Analysis of organizational principles--the student is

expected to recognize the organizational principles, the arrangement and structure, which hold together the communication as a whole.

Synthesis. Synthesis is defined here as the putting together of elements and parts so as to form a whole. The main difference between previously discussed categories and this one lies in the possibility that the former involve working with a given set of materials or elements which constitutes a whole in itself. On the other hand, the latter draws upon elements from many sources and puts these together into a structure or pattern not clearly there before.

Different kinds of synthesis may be differentiated primarily on the basis of the product. Three sub-categories of synthesis then emerge.

1. Production of a unique communication--this includes those objectives in which primary emphasis is upon communication, i.e., upon getting ideas, feelings and experiences across to others.

2. Production of a plan, or proposed set of operations--objectives in this sub-category aim at the production of a plan of operations. The production of the plan constitutes the act of synthesis. The product or plan of operations must satisfy the requirements of the task. These specifications, usually in the form of data, furnish the criteria to judge the soundness of the plan.

3. Derivation of a set of abstract relations--this sub-category includes objectives that require a student to produce, or derive, a set of abstract relations. There are two kinds of tasks involved:

(1) Those in which the student begins with concrete data or phenomena and which he must either classify or explain in some manner;

(2) Those in which the student begins with some basic propositions or other symbolic representations and from which he must deduce other propositions or relations.

Evaluation. Evaluation is defined as the making of judgments about the value, for some purpose, of ideas, works, solutions, methods, materials, et cetera. It involves the use of criteria as well as standards for appraising the extent to which particulars are accurate, effective, economical, or satisfying. The judgments may be either quantitative or qualitative and the criteria may be either those determined by the student or those which are given to him.

What is added in this category and not present in prior ones, are criteria including values. Evaluation is not necessarily the last step in thinking or problem-solving. It may be the preface to the acquisition of new knowledge, a new attempt at comprehension or application, or a new analysis and synthesis.

Judgments may be made in terms of internal or external evidence.

1. Judgments in terms of internal evidence are an evaluation of a communication from such evidence as logical accuracy, consistency, and other internal criteria.

2. Judgments in terms of external criteria are evaluations with reference to selected or remembered criteria.

General Criteria for Each Area of Structure of Mathematics

Utilizing the taxonomy described in the preceding section of this study, a list of criteria was developed for each level of the taxonomy. These criteria applied to each of the six areas of the structure of mathematics delineated in this chapter. These criteria then formed the basis for the rating scale which will be discussed in the next section.

Knowledge. The following criteria were used in this study at the lowest level of the taxonomy:

1. Student responses require precise formulation of basic definitions. Terminology may differ with different grade levels, but unambiguous definitions of all concepts presented by the textbook author is required at every level.

2. Student responses require recall of the referents for any symbols used. No judgment is made as to the degree of symbolism required, but this criteria applies to the evaluation of any symbolism introduced by the author of the textbooks.

3. Student responses require recall of other sources of information on the broad areas of structure of mathematics being evaluated.

4. Student responses require recall of the definitions of basic operations introduced by the author of the textbook. This may involve directly stating the definition of the operation or exhibiting a nonverbal awareness of these definitions by solving problems involving these operations.

5. Student responses require recognition of the same idea presented in various ways, which embody the various ways people in mathematics encounter this concept. This involves the non-standardization of vocabulary, symbolism, and problem presentation, and the recall of these varieties where applicable.

6. Student responses require recognition of the historical development or evolution of concepts in the area being evaluated.

7. Student responses require the recall of the categories and sub-categories of each area. This involves knowledge, not use, of classifications in each area.

8. Student responses require the recall of the methods of problem-solving. Here emphasis is on student responses showing they know such methods exist, not on students using these methods.

9. Student responses require the recall of basic theorems, principles, and generalizations presented by the author of the textbook.

10. Student responses require the recall of previously presented instances of generalizations.

Comprehension. The following criteria were used at this taxonomic level:

1. Student responses require demonstration of understanding of basic definitions by solving simple problems requiring interpretation of terminology.

2. Student responses require listing of instances of a given principle or generalization.

3. Student responses require stating generalizations or principles when given several instances of these.

4. Student responses require usage of symbols, presented by the author of the textbook, to translate words into mathematical sentences.

5. Student responses require translation of mathematical sentences into words.

6. Student responses require manipulations involving the basic operations presented in the textbook, when the choice of manipulation is not involved, i.e., the method of solution follows immediately from the presentation of the material.

7. Student responses require utilization of methods and procedures to solve problems which are direct applications of these procedures and follow imediately the textbook presentation of the procedures, or the procedure is specified.

8. Student responses require recognition of the reasoning involved in a proof presented to them in completed form.

9. Student responses require recognition of consequences of

generalizations when such consequences can be derived almost immediately from the given generalizations.

10. Student responses require proving statements in which the method of proof is exactly like that given in examples.

Application. It is well to repeat here that a demonstration of comprehension shows a student can use the abstraction when its use is specified. A demonstration of application shows he will use it correctly, given an appropriate solution in which no mode of solution is specified. Figure 2, page 30, was used to help distinguish between application and comprehension. The following criteria were used at this taxonomic level:

1. Student responses require the performance of manipulations when problems are varied and the student must choose method and operation. This differs from (6) and (7) under comprehension, in that here the methods and operations involved in the solution are not specified, nor do the problems follow directly a section which describes the method of solution of similar problems.

2. Student responses require solving problems embodying concepts and methods previously studied, but initially unfamiliar to them as examples of previous problems. Here emphasis is on recognizing that the problem is inherently like one already done, even though presented differently.

3. Student responses require solving problems from many areas of human endeavor, including areas outside of mathematics and science, where applicable.

4. Student responses require the ability to recognize what particulars are relevant to the validation of a judgment.

5. Student responses require the ability to detect logical fallacies in arguments.

6. Student responses require the ability to detect errors, by using techniques to verify answers.

7. Student responses require the recognition of the organizational scheme used by the author to present a broad area.

8. Student responses require the deciding of what premisses are needed in order to reach a given conclusion. In a sense, this is the process of planning a proof, by working backwards from the conclusion.

Synthesis. The following criteria were used at this taxonomic level:

1. Student responses require complete proofs of problems which differ significantly in their proof from the examples presented in the textbook.

2. Student responses require the devising of methods and techniques for attacking a problem, when such methods have not been presented previously. Many of the discovery approach type of lessons require this skill, but the degree of structuring given to the discovery determines whehter it is an example of synthesis or of comprehension.

3. Student responses require the utilization of elements from many sources to effect a structure or pattern not clearly there before, and should result in a product. This product can be a unique communication, a plan or proposed set of operations to be carried out, or a set of abstract relations.

Evaluation. The following criteria were used at this taxonomic level:

1. Student responses require the ability to assess, by internal standards, the general probability of accuracy in reporting facts from the care given to exactness of statement, documentation, proof, et cetera.

2. Student responses require the ability to apply given criteria, based on internal standards, to the judgment of the work.

3. Student responses require the ability to indicate logical fallacies in arguments. This criteria was included under analysis, but is also vital to evaluation.

4. Student responses require the comparison of major theories, generalizations, and facts about a broad area.

5. Student responses require the ability to compare, judging by external standards, a work with the highest known standards in the field--especially with other works of recognized excellence.

6. Student responses require the ability to distinguish between technical terminology which adds precision to a text

by permitting more appropriate definition of terms, and that which merely replaces a common name by an esoteric one.

7. Student responses require the ability to recognize the appropriateness of a given form of solution of a problem, to a given end.

8. Student responses require the ability to recognize the best of several means to a given end.

9. Student responses require the ability to determine the means which will serve best a given particular end.[32]

Other Applicable Criteria

Since the textbook series being evaluated ranged from those for seventh grade slow learners to those for accelerated honors students in the twelfth grade, not all six areas of structure defined in this chapter were found in each series evaluated. The primary reason for limiting this study to only series which covered at least two years of study, was to obtain more complete coverage of topics. Even then, in a two-year sequence, some areas designated basic in this study may not be considered at all. Yet mathematicians would not agree that the inclusion of these areas at that particular level was necessary or even desirable. Consequently, adjustment had to be made for rating these deleted areas, using criteria developed in this chapter. For this reason, ratings were based on the author's

32. Bloom, et. al., op. cit., et passim.

judgment of what was appropriate for the book. In those series which covered three or more years of study, a balanced presentation was necessary for a high rating in any area, so that merely adding more textbooks to the series did not give it any advantage over other series.

Another area of difficulty involved the question of how complete the presentation of generalizations, definitions, and operations of a given area should be at a given two-year level. For this reason, all ratings described so far were based on accepting the judgment of the textbook author as to which generalizations, definitions, and operations should be included at the grade level for which the series was written. Examples of these criteria were (2), (4), and (9) under Knowledge, as well as implicitly understood any time basic generalizations, definitions, and operations were mentioned in the criteria stated previously in this chapter. Thus internal criteria were used in these instances.

Additional external criteria were necessary to evaluate all the aspects of presenting the structure of mathematics in a two-year sequence of textbooks. These criteria represented a value judgment on the quality, completeness, and consistency of presentation. It was not enough to say a series rated high on presenting basic generalizations at the knowledge level, when this evaluation was based on only internal criteria.

External Criteria. For these reasons, the following criteria were needed as an adjunct to those already discussed, even though their evaluation was not statistically possible:

1. The inclusion of basic generalizations in the area presented by the authors of the textbook series is complete for the texts' grade levels. This includes consistency.
2. The basic generalizations included in the textbook series are correctly stated.
3. The inclusion of basic definitions in the areas presented by the authors of the textbook series is complete for the texts' grade levels. This includes consistency.
4. The basic definitions included in the textbook series are correctly stated.
5. The inclusion of basic operations in the areas presented by the authors of the textbook series is complete for the texts' grade levels. This includes the use of suitable symbolism.
6. The basic operations included in the textbook series are correctly defined.

Although the value judgment required to evaluate using these criteria was no different than that required to apply any of the previously stated criteria, no meaningful statistical analysis was possible. For this reason the external criteria were only indirectly involved in the study.

Rating Procedures

The criteria described in detail in the preceding sections and summarized in Appendix 2, pages 84-89, were used as the basis of the evaluation. These criteria were listed by number on the Taxonomic Level Rating Data Sheet, Appendix 3, page 90. This instrument was used to record all raw data. The procedure used to arrive at this data is discussed in the next section.

Scoring and Its Interpretation

Each of the six areas of mathematics underlying its structure--sets, ordered sets, groups, vector spaces, fields and logic and foundations--were evaluated on a ten point scale on each taxonomic criterion. A rating of ten represented the highest degree of satisfaction of a given criterion, with zero indicating the lowest.

Theoretically, the scores from zero to ten represented a continuum of values equally spaced. For example, a score of 9.6 theoretically existed. However, all scores were recorded in integral values from 0 to 10. The rating procedure, which is described in the next section, attempted to objectify the assignment of scores, so that a score of 7 for a single area of mathematics on a single criterion represented any score x such that $6.5 \leq x \leq 7.5$. All such scores on one taxonomic level under each of the six areas of mathematics were then summed to obtain an aggregate score. The six aggregate scores arising thus from the six areas evaluated were summed to obtain a score for a series on each taxonomic level. Since the number of criteria under each taxonomic level differed, these scores were not comparable in this form. Using arbitrarily the model of ten criteria under one taxonomic level and the ratings on these for the six areas of mathematics being evaluated, a total of six hundred points was the maximum for any series on a single taxonomic level. All aggregate scores obtained as described were scaled to this design of six hundred points maximum, so that the final data shown in Appendix 4, page 92 were comparable. Under this scoring scheme equal interval data were obtained. All of the nineteen selected series were rated in this manner by this investigator.

Ratings by Independent Evaluators

To establish the reliability of the ratings given each textbook series, each series was evaluated by at least two other mathematics educators. Fifteen different people served as evaluators. Some evaluated only one series, some evaluated only one taxonomic level for a series, but in the final tabulations, no evaluator contributed to more than two series in any one taxonomic level.

Orientation of Evaluators

The orientation of the evaluators was an integral part of the rating procedure. This briefing was threefold, and consisted of briefing on the structure of mathematics, briefing on the meaning of the taxonomic criteria, and briefing on the use of the Taxonomic Level Rating Data Sheet.

Although all of these people were familiar with Bloom's Taxonomy of Educational Objectives: Part 1, Cognitive Domain, and all of them had a good academic background in mathematics, it was necessary to review these areas. This was done not only as a review, but also to objectify the ratings as much as possible. The structure of mathematics has been defined in this chapter in terms of the diagrams shown in Figure 1, page 15. Each of the entries in these diagrams had to be clarified as well as the interrelationships between entries. In addition, guidelines had to be established for determining under what topic these areas might be presented in a secondary school mathematics textbook. The information presented on the pages immediately following Figure 1 of this study presented such guidelines.

Since the quantity being rated was presentation of structure of mathematics, each evaluator had to have the same frame of reference for the nature of this structure. Of prime importance was the fact that the characterization of structure of mathematics was made independent of the textbooks being evaluated. The question of how well structure was presented implied a definition of this structure prior to the evaluation. The fact that some of the concepts included in this independent determination of an external standard were not found later in the textbooks, does not invalidate the use of these concepts as part of the definition. Instead, this deficiency in the textbooks should be interpreted as their failure to fully present the underlying structure of mathematics. If instead one were to base the evaluation of the presentation of structure of mathematics on internal standards, i.e., on first listing what is contained in high school textbooks, then the validity question posed by the hypotheses guiding this study could not be determined. Criteria for validity studies must be externally determined.

With this rationale underlying the validity study of this investigation, it was essential the evaluators understand the diagrams on page 15, defining the structure of mathematics. Each evaluator was given, as part of the rating instrument, a copy of Appendix 5, pages 95-115, which is an outline of the basic concepts underlying the entries in Figure 1, page 15. Each evaluator also received a copy of Appendix 2, pages 84-89, "Criteria Summary Sheet," and a copy of the "Taxonomic Level Rating Data Sheet," Appendix 3, pages 90-91.

These tools for the rating were given in individual or group sessions and the material was discussed in detail. This involved also a discussion of each criterion and suggestions for what to look for in a secondary school mathematics textbook being evaluated. The evaluators were not told the hypotheses of this study.

In addition to aiding in determining the reliability of the rating procedure, the ratings of these evaluators along with those of this investigator, were used in the analysis of the individual textbook series.

Statistical Analysis of Correlation Between Ratings

To determine the reliability of the rating procedure, by a non-parametric technique, a Spearman Rank Correlation Coefficient r_s was calculated for each taxonomic level, using ratings for all nineteen series on that particular level. Three such r_s were calculated for each taxonomic level: (1) r_{AB}--correlation between this writer's ratings and those of the first evaluator; (2) r_{AC}--correlation between this writer's ratings and the second evaluator; and (3) r_{BC}--correlation between ratings of the two independent evaluators. The decision level for significant correlation was the .01 level.

Acceptance of significant correlation implies reliability of ratings within a single taxonomic level. The test of significance for $r_s \geqslant 0$, given by Bernstein,[33] was used. The hypothesis was H_o: $r_s' = 0$. If r_s was greater than the tabular value, H_o was rejected. Tabular values

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33. Allen L. Bernstein, A Handbook of Statistics Solutions for the Behavioral Sciences, (New York: Holt, Rinehart and Winston, Inc., 1964), pp. 69-70.

were those given by Bernstein.[34] Rejection at the .01 significance level of H_o implied the rankings were significantly correlated at this level of significance. Table 1 summarizes these results.

Summary and Conclusions

The development of a rating procedure was a vital part of this investigation, since a literature search did not show any prior statistical approaches to the determination of the validity of mathematics textbooks with respect to specified objectives. Criteria for selection of textbook series included recommendations of publishers and selections based on objectives stated by authors. Nineteen series were chosen and these included forty-five textbooks normally used in grades 7-12. In the estimation of those involved in the study, this sample of the total books used by students studying mathematics in grades 7-12 included those used by the majority of such students.

The rating instrument employed consisted of a Summary of the Structure of Mathematics, Appendix 5, page 95 a Summary of Taxonomic Criteria, Appendix 2, page 84, and a Taxonomic Level Rating Data Sheet, Appendix 3, page 90. Devising these evaluative tools involved a study of the nature of the structure of mathematics and a study of the

34. Ibid., p. 133.

Table 1

SPEARMAN RANK-CORRELATION COEFFICIENT r_s
FOR PAIRED RANKINGS ON EACH TAXONOMIC LEVEL

Taxonomic Level (1) Knowledge				
Textbook Series	Rankings by Evaluators* A	B	C	Results
A	2	3	2	Statistic:
B	18	14	17	
C	1	2	1	$r_s = 1 - \left\{\frac{6\sum d_i^2}{n^3 - n}\right\}$
D	5	5	5	
E	9	8	9	
F	16	15	15	Hypothesis: H_o
G	3	5	3	
H	11.5	11	9	$r_s{}' = 0$
I	4	1	4	
J	9	8	13	Decision Level:
K	11.5	11	12	
L	6	8	6	$r_{.01} = .523$
M	9	18	18	
N	13.5	13	9	$r < .523$ reject H_o
O	17	16	16	
P	13.5	11	9	Conclusions:
Q	15	17	14	$r_{AB} = .882$**
R	19	19	19	
S	7	5	9	$r_{AC} = .864$**
$\sum d_{AB}^2 = 134$	$\sum d_{AC}^2 = 155$	$\sum d_{BC}^2 = 104$		$r_{BC} = .909$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

Table 1--Continued

Taxonomic Level (2) Comprehension

Textbook Series	Rankings by Evaluators* A	B	C
A	9	11	7.5
B	11	11	11
C	7	4.5	7.5
D	2	4.5	1.5
E	4	4.5	12
F	15	17	15
G	2	4.5	5.5
H	6	9	4
I	5	4.5	3
J	2	4.5	1.5
K	10	4.5	10
L	12	16	13
M	16	14.5	16
N	18	19	18
O	17	14.5	17
P	8	11	9
Q	14	13	14
R	19	18	19
S	13	4.5	5.5

$\sum d^2_{AB} = 177.5$ $\sum d^2_{AC} = 145.5$ $\sum d^2_{BC} = 183.5$

Results

Statistic:

$r_s = 1 - \left\{\frac{6 \sum d_i^2}{n^3 - n}\right\}$

Hypothesis: H_o

$r_s' = 0$

Decision Level:

$r_{.01} = .523$

$r < .523$ reject H_o

Conclusions:

$r_{AB} = .895$**

$r_{AC} = .872$**

$r_{BC} = .840$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

Table 1--Continued

Taxonomic Level (3) Application				
Textbook Series	Rankings by Evaluators* A	B	C	Results
A	5	5	5	Statistic:
B	11	11	11	$r_s = 1 - \left\{\frac{6\sum d_i^2}{n^3 - n}\right\}$
C	8	9	8	
D	2	1.5	3	
E	3	4	1	Hypothesis: H_o
F	15	13.5	15	
G	1	3	2	$r_s' = 0$
H	13	15	13	
I	4	1.5	4	Decision Level:
J	7	6.5	7	
K	10	10	10	$r_{.01} = .523$
L	14	13.5	14	
M	18	18	18	$r < .523$ reject H_o
N	19	19	19	
O	17	16	17	Conclusions:
P	9	8	9	
Q	12	12	12	$r_{AB} = .905$**
R	16	17	16	
S	6	6.5	6	$r_{AC} = .995$**
$\sum d_{AB}^2 = 22.5$	$\sum d_{AC}^2 = 6$	$\sum d_{BC}^2 = 29.5$		$r_{BC} = .974$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

Table 1--Continued

Taxonomic Level (4) Analysis				
Textbook Series	Rankings by Evaluators* A	B	C	Results
A	5	4	5	Statistic:
B	10	10.5	9.5	$r_s = 1 - \left\{ \frac{6 \sum d_i^2}{n^3 - n} \right\}$
C	9	8	9.5	
D	2	1	2	
E	2	3	1	
F	15	12	14	Hypothesis: H_o
G	2	2	3.5	
H	11	10.5	11	$r_s' = 0$
I	6	6	6	
J	4	5	3.5	Decision Level:
K	12	13	12	
L	13	14.5	13	$r_{.01} = .523$
M	16	17	19	
N	17	16	16	$r < .523$ reject H_o
O	19	19	18	
P	8	9	8	Conclusions:
Q	14	14.5	15	
R	18	18	17	$r_{AB} = .982$**
S	7	7	7	
$\sum d_{AB}^2 = 21$	$\sum d_{AC}^2 = 18$	$\sum d_{BC}^2 = 28.5$		$r_{AC} = .984$** $r_{BC} = .975$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

Table 1--Continued

Taxonomic Level (5) Synthesis				
Textbook Series	Rankings by Evaluators* A	B	C	Results
A	2	2	2	Statistic:
B	10	10	8.5	$r_s = 1 - \left\{\frac{6 \sum d_i^2}{n^3 - n}\right\}$
C	11	7.5	10	
D	2	2	3	
E	2	2	1	
F	17	16.5	17	Hypothesis: H_o
G	6	6	6	
H	12	13	12.5	$r_s' = 0$
I	4	4	4	
J	7	7.5	7	Decision Level:
K	9	11	11	
L	14	16.5	14	$r_{.01} = .523$
M	17	16.5	17	
N	17	16.5	17	$r < .523$ reject H_o
O	17	16.5	17	
P	8	9	8.5	Conclusions:
Q	13	12	12.5	
R	17	16.5	17	$r_{AB} = .976$**
S	5	5	5	
$\sum d_{AB}^2 = 27.0$	$\sum d_{AC}^2 = 10.75$	$\sum d_{BC}^2 = 15.0$		$r_{AC} = .989$** $r_{BC} = .987$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

Table 1--Continued

Taxonomic Level (6) Evaluation				
Textbook Series	Rankings by Evaluators* A	B	C	Results
A	2	3	3	Statistic:
B	9	10	10	$r_s = 1 - \left\{\frac{6\sum d_i^2}{n^3 - n}\right\}$
C	8	9	11	
D	1	1	1	
E	3	2	2	
F	15	17	14	Hypothesis: H_o
G	5	5	5	
H	11	12	9	$r_s' = 0$
I	6	7.5	8	
J	7	6	6	Decision Level:
K	10	7.5	7	
L	17.5	17	17.5	$r_{.01} = .523$
M	17.5	17	17.5	
N	14	13	15	$r < .523$ reject H_o
O	17.5	17	17.5	
P	12	11	12	Conclusions:
Q	13	14	13	
R	17.5	17	17.5	$r_{AB} = .982$**
S	4	4	4	
$\sum d_{AB}^2 = 20.5$	$\sum d_{AC}^2 = 32.0$	$\sum d_{BC}^2 = 29.5$		$r_{AC} = .973$**
				$r_{BC} = .974$**

*Evaluator A was the investigator. Evaluators B and C were other independent raters.

**Significant correlation at the .01 level.

applications of Bloom's Taxonomy of Educational Objectives, to the presentation of the underlying structure of mathematics.

After a careful summarization of the bases for the criteria, specific criteria were listed under each of the six levels of cognitive learning. These criteria were in terms of behavioral responses by the student. Underlying these internal criteria were external criteria based on completeness, consistency and accuracy of the textbook series.

Each series was evaluated by this investigator. In order to establish the reliability of these ratings, selected persons rated the series after receiving briefing on the rating procedure. The orientation of the evaluators consisted of an attempt to make their ratings more objective by clarifying the definition of the structure of mathematics. Appendix 5, page 95, along with Figure 1, page 15, was used for this purpose. A similar clarification of the taxonomic criteria was made.

Each textbook series received a total of three scores on each entry in the Taxonomic Level Rating Data Sheet, Appendix 3, page 90. These ratings are given in Appendix 4, page 92. The scoring procedure furnished equal interval data based on the extent to which a given series met the established criteria.

Calculation of the Spearman Rank-Correlation Coefficient r_s for paired rankings on each of the six taxonomic levels showed significant correlation at the .01 level of significance between this investigator's ratings and each of the two independent evaluations and also between the ratings of the two other evaluators. This result established the significant reliability of the rating procedure for each taxonomic level. Correlation at the .01 level indicated there was a probability of 1 out of 100 of making an error in concluding these three ratings differed little because of the design of the experiment rather than because of chance. The high correlation between the ratings of the textbooks indicated the rating procedure was interpreted similarly by each person doing an evaluation. This was an important characteristic of a good rating procedure, since it was only valuable for repeated use if it was unambiguous.

CHAPTER 3

STATISTICAL ANALYSES OF EVALUATIONS

The statistical analyses of the evaluation described in the preceding chapter consisted of two facets; (1) To test the hypotheses of this study, an analysis of the differences in mean ratings of the entire group of selected series on the six levels of Bloom's taxonomy; and (2) Because of interest in the results, the analysis of each textbook series singly to characterize the differences in ratings on each taxonomic level.

Testing Hypotheses of Study

The next sections describe steps taken to test the following hypotheses of this study: (1) The presentation of the structure of mathematics, at the two lowest levels of cognitive learning is better than at the four highest levels, i.e., $\frac{\mu_1+\mu_2}{2} > \frac{\mu_3+\mu_4+\mu_5+\mu_6}{4}$, where the μ_i, $i = 1\text{-}6$ are the mean ratings of the correspondingly numbered taxonomic levels; (2) The presentation of the structure of mathematics at the level of Application is not different from the presentation at the two lowest levels of cognitive learning, i.e., $\frac{\mu_1+\mu_2}{2}=\mu_3$; and (3) The presentation of the structure of mathematics is not different at the two lowest levels of cognitive learning, i.e., $\mu_1=\mu_2$.

The decision level for rejection of the null hypothesis in each case was the .05 level of significance. Rejection of the null hypothesis in (1) implied support for the alternative hypothesis as stated in (1).

Nature of Data

A detailed description of the scoring procedure and characterization of the raw data was given in this study on page 43. One conclusion made then was that the data used for the evaluation was continuous and equal interval scoring was used. This indicated the use of parametric statistics was tenable if the other assumptions of the specific statistic were met.

In addition to the rating procedures already discussed, in applying the criteria to those series which included a textbook on geometry, some modifications were made in ratings. Due to the nature of the subject, all geometry textbooks require students to perform tasks under categories of application, analysis, and synthesis. Since a geometry text was evaluated along with other texts in a series for an aggregate rating, this total might have been unduly influenced by the geometry text and would not provide any information about the series in general. A comparison of these series with those not including geometry would have been impossible. For these reasons, when evaluating series containing a geometry textbook, high ratings were given only to those series which satisfied the criteria in more than the usual standard presentation of geometry. In most cases, this meant the algebra text had to show evidence of presenting mathematics at a given level before a high rating was given.

Assumptions Underlying Statistical Evaluations

Certain assumptions were made in employing the parametric technique of analysis of variance. According to Guenther, these assumptions are "the x_i are independently $N(\mu, \sigma^2)$."[35] This means $x_1, x_2, \ldots, x_n$ is a random sample from a population that is normal with mean μ and variance σ^2. Thus the technique of analysis of variance requires the following assumptions:

1. Randomness and independence of sampling
2. Normality of distribution of data
3. Homogeneity of Variance
4. Variable data with equal intervals, i.e., a continuum of possible scores.

These same assumptions underlie the F-Test and Scheffe's S-Method.[36] The design of this study was that of the fixed-effects model postulated by Guenther.[37] In this

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35. William C. Guenther, Analysis of Variance, (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1964), p. 27.

36. Henry E. Garrett, Statistics in Psychology and Education, (New York: David McKay Company, Inc., 1958), p.286.

37. Guenther, op. cit., p. 63.

case the only interest is in the r populations under investigation and any conclusions reached are valid only for the r populations. The selection of the nineteen series evaluated in this study was not random, but represented a nearly total sample of all the textbooks meeting the requirements of the study. Use of an exhaustive sample is an even more stringent fulfillment of the requirements for analysis of variance than use of a random sample. The independence criterion was easily met by examining the evaluative procedures described fully on page 43 of this study.

Although there was no information available for determining the exact nature of the distribution, both Guenther and Scheffe' concluded normality was not necessary. Guenther stated: "A further assumption we made in discussing analysis of variance technique was normality of the observations. Although the test is derived under this assumption, investigation has shown that failure to satisfy this condition has little effect upon the F-Test for equal means and the S-Method."[38] Scheffe' affirmed also: "The conclusion is that the effect of violation of the normality assumption is slight on inferences about means."[39]

The postulation of equal interval data was discussed in the preceding section of this paper, and the data used in

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38. Ibid., p. 63.

39. Henry Scheffe', The Analysis of Variance, (New York: John Wiley and Sons, Inc., 1959), p. 337.

this study met this criteria for using parametric statistics. Thus the only other assumption which had to be satisfied was that of homogeneity of variance.

Analysis of Homogeneity of Variance

The assumption of equal variances was tested by means of Bartlett's Test for Homogeneity of Variances. The data obtained when the textbook series were rated by this investigator were used to calculate the following statistic:

$$B = \frac{2.3026}{C}\left\{n\left(\log_{10} \sum_{i=1}^{k} n_i s_i^2 - \log_{10} n\right) - \sum_{i=1}^{k} n_i \log_{10} s_i^2\right\}$$

$$= \frac{B'}{C} \quad \text{where } C = 1 + \frac{1}{3(k-1)}\left\{\sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n}\right\} \quad \text{and where}$$

$s_1^2, s_2^2, \ldots, s_k^2$ are the variances of k independent samples having respectively $n_1, n_2, \ldots n_k$ degrees of freedom. This statistic is based on the hypothesis that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \ldots \sigma_k^2 = \sigma^2$, the variance, so the estimate of σ^2 can be obtained by pooling variances of the k samples, i.e.,

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2 + \ldots + n_k s_k^2}{n_1 + n_2 + \ldots + n_k} = \frac{\sum n_i s_i^2}{n} \quad \text{where}$$

$n = \sum n_i$. Even if some of the n_i are quite small, the statistic B has a chi-square distribution with $k - 1$ degrees of freedom.[40] In applying this test to the data of the study, $k = 19$ and $n_i = 5$ for each i. Since the pre-determined decision level was .05, the hypothesis of equal variances was tenable if $B < \chi^2_{.95}$. Hence if $B < \chi^2_{.95}$ the chance

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40. Helen M. Walker and Joseph Lev, Statistical Inference, (New York: Henry Holt and Company, 1953), p. 193.

was 5 out of 100 of making the error of assuming equal variances when in fact the variances were not equal. A summary of the results of this test are in Table 2. For this test, since there were six taxonomic levels, there were five degrees of freedom and $\chi^2_{.95}$ for five degrees of freedom is 28.9. The result obtained from the given data was B = 28.3 which is less than $\chi^2_{.95}$. Hence the conclusion followed that support of the hypothesis of equal variances was not in error in 95 cases out of 100. This implied the analysis of variance technique and subsequently the F-Test could be used to determine if there were any significant differences in the ratings of each taxonomic level when the nineteen textbook series were considered as one group.

Use of the F-Test

Rejection at the .05 level of the postulate of equal means implied the alternative hypothesis of differences in means of the taxonomic levels was tenable. The calculated value of F was 14.4 which was greater than $F_{5,108,.05} = 2.31$. Hence with a high probability of being correct, we can conclude at least two of the ratings differ enough to say the rating on one level was higher than on another. This meant the presentation of structure of mathematics was different on at least two of the levels of cognitive learning. The results of the F-Test are shown in Table 3, page 66.

Table 2

BARTLETT'S TEST FOR HOMOGENEITY OF VARIANCES OF MEAN RATINGS OF TEXTBOOK GROUP ON TAXONOMIC LEVELS

Group	k	$n_i = df$	$n = \sum n_i$	$\sum n_i s_i^2$	$\sum \log s_i^2$	$\sum \log n_i s_i^2$
Textbook Series	19	5	95	3,538,763	84.1572	420.786

$\log(\sum n_i s_i^2) = 6.5488$ $\log(\sum n_i) = 1.9777$

$$B = \frac{B'}{C} = \frac{2.3026}{C}\left\{\sum n_i(\log\sum n_i s_i^2 - \log\sum n_i) - \sum n_i \log s_i^2\right\}$$

where $C = 1 + \frac{1}{3(k-1)}\left\{\sum \frac{1}{n_i} - \frac{1}{n}\right\}$

$C = 1.0702$

$B' = 31.0125$

$B = 28.3$ $B < \chi^2_{.95} = 28.9$

Hypothesis: Variances are equal, i. e. $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = \sigma_6^2 = \sigma^2$

Conclusion: Hypothesis of equal variances tenable at the .05 level of significance.

Scheffe's Test

The F-Test tells us whether we may conclude the means are different. It does not locate these differences. The Scheffe' S-Test was then used to locate these significant differences when they existed at the .05 level.

Scheffe's procedure for handling multiple comparisons is called the S-method. Scheffe' has proved that the probability is 1 - a that all imaginable contrasts will be captured by the set of intervals given by

$$* \quad \hat{L} - S\hat{\sigma}_{\hat{L}} \leq L \leq \hat{L} + S\hat{\sigma}_{\hat{L}}$$

where

$$S^2 = (r - 1)\, F_{1-a,\ r-1,\ N-r}$$

r = number of groups
N = number of cases

and $\hat{\sigma}^2_{\hat{L}} = MS_W \sum_{j=1}^{t} \frac{c_j^2}{n_j}$ where MS_W is the mean square of the within group in an analysis of variance and the c_j are the coefficients in the linear combination of contrasts. Thus, the probability is 'a' that one or more false conclusions will be made.[41]

An interesting feature of the S-method is that one or more of the intervals given by * above will not cover zero whenever the F-Test rejects the hypothesis of equal means. Thus we are able to draw more conclusions than merely that all treatments are not the same. An additional advantage associated with this method is that it is known

41. Guenther, op. cit., p. 57.

to be affected very little if the assumptions of normality and equal variances are not satisfied.[42]

Using the data given in Table 2 for analysis of variance and the raw data given in Appendix 4, page 92, Scheffe's S-method was used to calculate the following contrasts to test each of the corresponding hypotheses:

1. Hypothesis: The presentation of structure of mathematics at the two lowest levels of cognitive learning is better than at the four highest levels. ("Better" defined by the inequality below.)

$$\text{Contrast: } \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5 + \mu_6}{4} > 0$$

For the data of this study, $r = 6$, $N = 114$, $a = .05$, $n_j = 19$, $F_{.95,5,108} = 2.31$, $MS_W = 31,396$. These figures were true for all the calculations. In computing the contrast postulated above, $c_1 = c_2 = \frac{1}{2}$, $c_3 = c_4 = c_5 = c_6 = -\frac{1}{4}$, $S = 3.40$, $\hat{\sigma}_{\hat{\psi}} = 35.2$, $S\hat{\sigma}_{\hat{\psi}} = 119.68$ and the value of the contrast was

$$271.3 - 119.68 \leq \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5 + \mu_6}{4} \leq 271.3 + 119.68$$

This conclusion is equivalent to supporting hypothesis (1) at the .05 level of significance.

2. Hypothesis: The presentation of the structure of mathematics at the level of application is not different from the presentation at the levels of knowledge and comprehension.

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42. Scheffe', *op. cit.*, pp. 66-72.

Table 3

ANALYSIS OF VARIANCES OF DIFFERENCES OF MEAN RATINGS ON TAXONOMIC LEVELS

Source of Variation	Df	Sum of Squares	Mean Squares
Among Groups	5	2,262,387	452,477
Within Groups	108	3,690,786	31,396
Total	113	5,953,173	

F-Test:

Hypothesis: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6$

Calculated F = 14.4 $F_{.95,5,108} = 2.31$

Conclusion: Reject the hypothesis at the .05 level.

Scheffe's S-Test:

S = 3.40 using

r = 6, N = 114, a = .05, n_j = 19 for j = 1,2,3,4,5,6

$F_{.95,5,108} = 2.31$, $MS_W = 31,396$

Hypothesis (1): $\frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5 + \mu_6}{4} > 0$

Conclusion:
$271.3 - 119.68 \leq \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5 + \mu_6}{4} \leq 271.3 + 119.68$

Rejection of null hypothesis and support of alternative hypothesis (1) at the .05 level.

Hypothesis (2): $\frac{\mu_1 + \mu_2}{2} - \mu_3 = 0$

Conclusions:
$168 - 169.32 \leq \frac{\mu_1 + \mu_2}{2} - \mu_3 \leq 168 + 169.32$

Support of hypothesis (2) at the .05 level

Hypothesis (3): $\mu_1 - \mu_2 = 0$

Conclusions:
$48 - 195.5 \leq \mu_2 - \mu_1 \leq 48 + 195.5$
Support of hypothesis (3) at the .05 level.

Contrast: $\frac{\mu_1 + \mu_2}{2} - \mu_3 = 0$

To compute this difference, $c_1 = c_2 = \frac{1}{2}$, $c_3 = -1$, $c_4 = c_5 = c_6 = 0$, $S = 3.40$, $\hat{\sigma}_{\hat{L}} = 49.8$, $S\hat{\sigma}_{\hat{L}} = 169.32$, and the value of the contrast was 168. Hence, the following was true:

$$168 - 169.32 \leq \frac{\mu_1 + \mu_2}{2} - \mu_3 \leq 168 + 169.32$$

Since this interval includes zero, $\frac{\mu_1 + \mu_2}{2}$ and μ_3 are not different. This conclusion is equivalent to supporting hypothesis (2) at the .05 level of significance.

3. Hypothesis: The presentation of the structure of mathematics is not different at the levels of knowledge and comprehension.

Contrast:

To compute this difference, $c_1 = 1$, $c_2 = -1$, $c_3 = c_4 = c_5 = c_6 = 0$, $S = 3.40$, $\hat{\sigma}_{\hat{L}} = 57.5$, $S\hat{\sigma}_{\hat{L}} = 195.5$, and the value of the difference was 48. Hence, the following was true:

$$48 - 195.5 \leq \mu_1 - \mu_2 \leq 48 + 195.5$$

Since this interval includes zero, μ_1 and μ_2 are not different. This conclusion is equivalent to supporting hypothesis (3) at the .05 level of significance.

Single Series Analyses

Although not pertinent to the hypotheses of this study, an analysis of each individual series was of interest enough to carry out the necessary calculations. An analysis of variance for each series was calculated with the following used for each one: Degrees of freedom (df) of among groups was 5, degrees of freedom of within groups was 12. The F-Test and subsequently Scheffe's S-Method with $F_{.95,5,12} = 3.94$ was then used to characterize the differences in means, when these differences existed. The three scores for the series given on each of the six taxonomic levels were used. These are tabulated in Appendix 4, page 92. Table 4 gives a tabulation of the results of the analysis of variance and the S-Test. It also shows how each entry should be interpreted. This table should be used if one is interested in examining how well a single textbook series presents the structure of mathematics when compared at the six taxonomic levels.

Conclusions of Statistical Analyses

The following pertinent conclusions were reached from the statistical analyses presented in the preceding sections: (1) The presentation of structure of mathematics, when considering the textbooks studied as a single group, rated higher when evaluated at the two lowest levels of cognitive learning than when evaluated at the four highest levels; (2) The presentation of structure of mathematics, considering the textbooks studied as a single group, was not different when comparing the two lowest levels of cognitive learning with the third level. This means at the .05 significance level, the textbooks do an equal job of presenting structure at the knowledge, comprehension, and application

Table 4

ANALYSIS OF VARIANCE OF DIFFERENCE OF
MEAN TAXONOMIC RATINGS OF
EACH TEXTBOOK SERIES

Textbook Series	Hypothesis (1)	Hypothesis (2)	Hypothesis (3)
	$\frac{\mu_1+\mu_2}{2} > \frac{\mu_3+\mu_4+\mu_5+\mu_6}{4}$	$\frac{\mu_1+\mu_2}{2} = \mu_3$	$\mu_1 = \mu_2$
A	=	Yes	Yes
B	Yes	Yes	Yes
C	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	Yes
D	=	Yes	$\mu_2 > \mu_1$
E	=	Yes	Yes
F	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	Yes
G	Yes	Yes	Yes
H	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	$\mu_2 > \mu_1$
I	Yes	Yes	Yes
J	Yes	Yes	$\mu_2 > \mu_1$
K	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	$\mu_2 > \mu_1$
L	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	Yes
M	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	Yes
N	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	$\mu_1 > \mu_2$
O	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	Yes
P	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	$\mu_2 > \mu_1$
Q	Yes	$\frac{\mu_1+\mu_2}{2} > \mu_3$	$\mu_2 > \mu_1$
R	Yes	$\frac{\mu_1+\mu_2}{2} \quad \mu_3$	Yes
S	Yes	Yes	$\mu_2 > \mu_1$

levels of cognitive learning; and (3) The presentation of structure of mathematics, considering the textbooks studied as a single group, was not different at the two lowest levels of cognitive learning. This means, at the .05 significance level, the textbooks do an equal job of presenting structure at the knowledge and comprehension levels.

The statistical tests used to arrive at these conclusions were the analysis of variance of differences in mean ratings on taxonomic levels of the textbook group, the F-Test, and Scheffe's S-Method for locating differences in multiple comparisons.

All of these results are shown in Table 3, page 66, and the conclusions reached supported the original hypotheses of this study.

As a matter of information, an analysis was made of each of the nineteen series studied. The results of this analysis are shown in Table 4, page 69.

CHAPTER 4

SUMMARY AND CONCLUSIONS

In order to study the validity of mathematics textbooks in grades 7-14, with respect to the objective of presenting the underlying structure of mathematics, the following hypotheses were tested:

1. The presentation of structure of mathematics at the two lowest levels of cognitive learning--knowledge and comprehension--rates higher than at the four highest levels--application, analysis, synthesis, and evaluation.
2. The presentation of the structure of mathematics at the level of Application is not different from the presentation at the two lowest levels of cognitive learning--knowledge and comprehension.
3. The presentation of the structure of mathematics is not different at the two lowest levels of cognitive learning--knowledge and comprehension.

The decision level for rejection of these hypotheses was the .05 level of significance.

Procedures

To develop criteria for the evaluation of selected textbook series, a study was made of the nature of the underlying structure of mathematics and the taxonomy of educational objectives in the cognitive domain. The first study led to a definition for "structure of mathematics," which divided the underlying structure into the six areas of sets, ordered sets, groups, vector spaces and logic and foundations. Unifying concepts in these areas were examined and their interrelationships explored. In order to clarify educational objectives, The Taxonomy of Educational Objectives, Part I, Cognitive Domain edited by Benjamin Bloom, was used as a primary source. The six taxonomic levels of knowledge, comprehension, application, analysis, synthesis and evaluation (lowest to highest levels) were defined. These two studies of mathematical structure and educational objectives culminated in a list of criteria for evaluating each textbook series on its presentation of mathematical structure in each of the six broad areas of mathematics.

Nineteen textbook series, each covering at least two years of study for grades 7-14, were selected for this investigation. Selections were made based on statements by publishers or authors that the series had as one of its objectives, the teaching of mathematics through an understanding of its underlying structure. A total of forty-five books were evaluated. These represented the majority of textbooks in use in mathematics classes in grades 7-12.

Fifteen persons in addition to this investigator aided in this study. These people were all involved in mathematics education in grades 7-12 and contributed by rating at least one series on at least one taxonomic level. Each series thus received two ratings on each criteria, in addition to the one given by this investigator, who rated all of the selected series on each of the six taxonomic levels. Extensive briefing on the rating procedure was given to all those involved in the evaluation. This orientation was an integral part of the rating procedure and consisted of discussions on the nature of the structure of mathematics, the meaning of the taxonomic criteria as defined for this study, and the use of the scoring sheet. Each evaluator was given a copy of the Summary of Structure of Mathematics, the Criteria Summary, and the Taxonomic Level Rating Data Sheet, to aid in making the evaluations as objective as possible. These aids are in Appendices 2, 3, 5 on pages 84, 90, and 95, respectively.

In order to establish the reliability of this rating procedure a Spearman rank-coefficient correlation was calculated to compare these three ratings received by each series. The correlation between this investigator's ratings and that of the first independent evaluator was significantly high, as well as the correlations between this investigator's ratings and the second evaluator, and the ratings of the two independent evaluators. These results indicated repeated independent ratings produced similar results. Hence the procedure itself can be considered reliable.

Conclusions

Statistical analyses established the support at the .05 level of all three hypotheses given at the beginning of this summary. To establish these results, the following tests were used: (1) Analysis of variance of differences of mean ratings on each taxonomic level, when considering all nineteen series as one group and using ratings given by this investigator; (2) The F-Test for differences in means; (3) The Scheffe' S-Test to locate the differences found by the F-Test. The Scheffe' S-method was used to make the following comparisons which were equivalent to the assertions made in the corespondingly numbered hypotheses: (The μ_i are the means on the coresponding taxonomic levels.)

1. $\frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4 + \mu_5 + \mu_6}{4} > 0$
2. $\frac{\mu_1 + \mu_2}{2} - \mu_3 = 0$
3. $\mu_1 - \mu_2 = 0$

These contrasts were chosen because they indicated the comparative quality of the presentation of structure of mathematics at the six levels of the taxonomy. If the aim was to present this structure at all levels of the taxonomy equally, then support of these statements leads to the conclusion that the textbook series studied did not meet the criteria for establishing validity.

Implication of This Study

In order to meet the established criteria for selection of textbook series, some excellent books were not included. These were chiefly because they were not part of at least a two-year series. This also accounted for why none of the selected textbooks were for grades 13 or 14. Texts for these levels usually occur singly and not as a part of a sequential series.

In evaluating a series as a whole, it was possible for the books within that series to differ greatly in how they met the criteria for the study. Yet in the over-all rating of the series, this deficiency might not have been noticeable. One example of this involved the postulational approach to the study of algebra. In some series the first year algebra text exhibited little of this approach, yet the second year algebra did an excellent job in this respect. This may have been deliberately done by some authors, who do not believe beginning students should have a formal approach. The quality of textbooks within a series varied greatly, with several having a much better book at the second year level than at the first year level. Very few geometry texts included topics of new interest in that area.

In assessing the conclusions reached by this study, it was possible for a series to rate highly on the single objective being evaluated and yet be deficient in other important aspects. Therefore, the validity of a textbook series on this one evaluated objective does not imply necessarily the series would be recommended for use by any of the evaluators. There are many other objectives which must be met before adopting a textbook. Significant differences in the presentation of structure at the knowledge and comprehension levels when compared to the four highest levels, is not necessarily an indictment of the series. In some cases, the author only intended to present structure at these lower levels. This could be true for texts intended for below-average or non-college bound students, or for students at the junior high school level. It may be true, too, for college preparatory courses where the author felt the students were not mathematically mature enough to attempt this material at the highest taxonomic levels. The essential thing is to understand clearly the objectives for the course, and then select the textbooks which best meet these needs.

The results of this study showed that the validity of the majority of textbook series with the objective being evaluated, was questionable. Reliance on the textbook is widespread among mathematics teachers. This confidence may be misplaced, judging by the results of this study.

Implications for Further Study

Although of use to those in mathematics teaching, the evaluation of the selected series was intended for a more general purpose. It should be possible, following the guidelines established by this study, to devise a rating procedure for each objective of a course in any subject field. After determining the objectives of the course to carry out the philosophy of the school, it should be possible using procedures of this study, to develop criteria to determine the validity of proposed textbooks with respect to those objectives. In this way, a text or probably a multiple selection of texts should be chosen to be the best aid to the teacher in achieving these objectives.

The teachers, themselves, with the help of consultants in in-service sessions, could determine these criteria and conduct the evaluative study. The most important step is to clarify the nature of the area being studied. Too often teachers use the textbooks themselves to list topics which should be included to fulfill a given objective. The fallacy in this procedure is that in doing this, they are accepting the validity of the textbook with respect to the given objective. The topics necessary to teach to fulfill an objective and the criteria for determining at what educational level these objectives will be met must be determined independently of any proposed textbooks. This is essential if the evaluation is to be meaningful.

One of the most important implication of this study for further studies is the need for textbook authors to validate their books with respect to the claims they make for it. Validation of tests is now a standard practice. It does not seem unreasonable that publishers of textbooks, which are still the major media of instruction and exert more influence than tests, should adopt a similar procedure.

APPENDIX 1

TEXTBOOK SERIES EVALUATED

Throughout this study the following letters were used to designate the textbook series listed below. Notations after the texts indicate the level for which the series is usually used.

A. Peters, Max and William Schaaf. *Algebra, A Modern Approach, Book 1*. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1963. 562 pp.

_______. *Algebra, A Modern Approach, Book 2*. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1963. 696 pp.

Level: High School College Preparatory Mathematics

B. Weeks, Arthur W. and Jackson B. Adkins. *First Course in Algebra*, Revised Edition. Boston: Ginn and Company, 1965. 550 pp.

_______. *A Course in Geometry*. Boston: Ginn and Company, 1961. 552 pp.

_______. *Second Course in Algebra*. Boston: Ginn and Company, 1962. 630 pp.

Level: High School College Preparatory Mathematics

C. Peters, Max and William Schaaf. *Mathematics, A Modern Approach, Book 1*. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1965. 555 pp.

_______. *Mathematics, A Modern Approach, Book 2*. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1965. 504 pp.

Level: 7-8 grades

D. Beberman, Max and Herbert E. Vaughan. High School Mathematics, Course 1. Boston: D. C. Heath and Company, 1965. 598 pp.

_______. High School Mathematics, Course 2. Boston: D. C. Heath and Company, 1966. 628 pp.

Level: High School College Preparatory Mathematics

E. Van Engen, Henry, Maurice L. Hartung, H. Trimble, et al. Seeing Through Mathematics, Book 1. Glenview, Illinois: Scott Foresman and Company, 1962. 508 pp.

_______. Seeing Through Mathematics, Book 2. Glenview Illinois: Scott Foresman and Company, 1963. 528 pp.

_______. Fundamental Mathematical Structures, Algebra. Glenview, Illinois: Scott Foresman and Company, 1966. 607 pp.

Level: 7, 8, 9 grades (includes algebra)

F. Rosskopf, Myron F., Robert Lee Morton, H. S. Moredock, et al. Modern Mathematics Through Discovery, Book I. Morristown, New Jersey: Silver Burdett Company, 1966. 424 pp.

_______. Modern Mathematics Through Discovery, Book II. Morristown, New Jersey: Silver Burdett Company, 1966. 424 pp.

Level: 7-8 grades

G. Rosskopf, Morton, Joseph Hooten, Harry Sitomer, et al. Modern Mathematics, Algebra One. Morristown, New Jersey: Silver Burdett Company, 1966. 437 pp.

Rosskopf, Morton, Stephen S. Willoughby, Bruce Vogeli. Modern Mathematics, Algebra Two and Trigonometry. Morristown, New Jersey: Silver Burdett Company, 1964. 501 pp.

Rosskopf, Morton, Harry Sitomer, George Lenchner. Modern Mathematics, Geometry. Morristown, New Jersey: Silver Burdett Company, 1966. 566 pp.

Level: High School College Preparatory Mathematics

H. Johnson, Richard, Lona Lee Lendsey, William E. Slesnick. *Modern Algebra, First Course*. Reading, Mass.: Addison-Wesley Company, 1961. 628 pp.

Johnson, Richard, William E. Slesnick, G. E. Bates. *Modern Algebra, Second Course*. Reading, Mass.: Addison-Wesley Company, 1962. 594 pp.

Level: High School College Preparatory Mathematics

I. Nichols, Eugene D. *Modern Elementary Algebra, Revised*. New York: Holt, Rinehart, Winston, 1965. 466 pp.

Nichols, Eugene D., Ralph T. Heimer, Henry E. Garland. *Modern Intermediate Algebra*. New York: Holt, Rinehart, Winston, 1965. 582 pp.

Level: High School College Preparatory Mathematics

J. Nichols, Eugene, Frances Flournoy, Robert Kalin, *et al*. *Elementary Mathematics, Grade 7*. New York: Holt, Rinehart, Winston, 1966. 458 pp.

_______. *Elementary Mathematics, Grade 8*. New York: Holt, Rinehart, Winston, 1966. 456 pp.

Level: 7-8 grades

K. Hayden, Dunstan and E. J. Finan. *Algebra One*. Boston: Allyn and Bacon, Inc., 1965. 455 pp.

Fischer, and Dunstan Hayden. *Geometry*. Boston: Allyn and Bacon, Inc., 1965. 582 pp.

Level: High School College Preparatory Mathematics

L. Brumfiel, Charles, Robert Eicholz, Merrill Shanks, P. G. O'Daffer. *Arithmetic Concepts and Skills*. Reading, Mass.: Addison-Wesley Company, 1963. 389 pp.

Brumfiel, Charles, Robert Eicholz, Merrill Shanks. *Introduction to Mathematics*. Reading, Mass.: Addison-Wesley Company, 1963. 367 pp.

Level: 7-8 grades

M. Eicholz, Robert E., Phares G. O'Daffer, Charles Brumfiel, Merrill Shanks. Basic Modern Mathematics, Second Course. Palo Alto: Addison-Wesley Company, 1965. 380 pp.

_______. Modern General Mathematics. Palo Alto: Addison-Wesley Company, 1965. 386 pp.

Level: 8-9 (sometimes 9-10) non-college bound students average or below in ability

N. Wilcox, Marie S. and John E. Yarnelle. Mathematics, A Modern Approach, First Course. Palo Alto: Addison-Wesley Company, 1963. 385 pp.

Wilcox, Marie. Mathematics, A Modern Approach, Second Course. Palo Alto: Addison-Wesley Company, 1966. 371 pp.

Level: 7-8 (sometimes 8-9 general mathematics)

O. Stein, Edwin I. First Course in Fundamentals of Mathematics. Boston: Allyn and Bacon, Inc., 1966. 290 pp.

_______. Second Course in Fundamentals of Mathematics. Boston: Allyn and Bacon, Inc., 1966. 355 pp.

Level: High School General Mathematics

P. Kenner, Morton R., Dwain E. Small, Grace N. Williams. Concepts of Modern Mathematics, Book 1. New York: American Book Company, 1963. 477 pp.

_______. Concepts of Modern Mathematics, Book 2. New York: American Book Company, 1963. 451 pp.

_______. Concepts of Modern Mathematics, Book 3. New York: American Book Company, 1965. 491 pp.

Level: High School College Preparatory Mathematics

Q. Deans, Edwina, Robert B. Kane, George McMeen, Robert Oesterle. Exploring Mathematics, Grade 7. New York: American Book Company, 1966. 424 pp.

_______. Exploring Mathematics, Grade 8. New York: American Book Company, 1966. 424 pp.

Level: 7-8 grades

R. Mayor, John R., John A. Brown, Bona Lunn Gordy, Dorothy Sward. Contemporary Mathematics, First Course. Englewood Cliffs, New Jersey: Prentice-Hall Inc., 1964. 388 pp.

________. Contemporary Mathematics, Second Course. Englewood Cliffs: Prentice-Hall, Inc., 1966. 371 pp.

Level: 7-8 grades

S. Dolciani, Mary P., Simon Berman, Julius Freilich. Modern Algebra, Book I. Boston: Houghton Mifflin Company, 1965. 588 pp.

Jurgensen, Ray C., Alfred J. Donnelly, Mary P. Dolciani. Modern Geometry. Boston: Houghton Mifflin Company, 1965. 592 pp.

Dolciani, Mary P., Simon Berman, William Wooton. Modern Algebra and Trigonometry, Book 2. Boston: Houghton Mifflin Company, 1965. 658 pp.

Dolciani, Mary P., Edwin F. Beckenbach, Alfred Donnelly, et. al. Modern Introductory Analysis. Boston: Houghton Mifflin Company, 1964. 626 pp.

Level: High School College Preparatory Mathematics

APPENDIX 2

CRITERIA SUMMARY SHEET

1. Knowledge

(1) Student responses require precise formulations of basic definitions.

(2) Student responses require recall of the referents for any symbols used.

(3) Student responses require recall of other sources of information on the broad areas of structure of mathematics being evaluated.

(4) Student responses require recall of the definitions of basic operations introduced by the author of the textbook.

(5) Student responses require recognition of the same idea presented in various ways, which embody the various ways people in mathematics encounter this concept.

(6) Student responses require recognition of the historical development or evolution of concepts in the area being evaluated.

(7) Student responses require the recall of the methods of problem-solving.

(8) Student responses require the recall of the categories and subcategories of each area.

(9) Student responses require the recall of basic theorems, principles and generalizations presented by the author of the textbook.

(10) Student responses require the recall of previously presented instances of generalizations.

2. <u>Comprehension</u>

(1) Student responses require demonstration of understanding of basic definitions by solving simple problems requiring interpretation of terminology.

(2) Student responses require listing of instances of a given principle or generalization.

(3) Student responses require stating generalizations or principles when given several instances of these.

(4) Student responses require usage of symbols, presented by the author of the textbook, to translate words into mathematical sentences.

(5) Student responses require translation of mathematical sentences into words.

(6) Student responses require manipulations involving the basic operations presented in the textbook, when the choice of manipulation is not involved, i. e. the method of solution follows immediately from the presentation of material.

(7) Student responses require utilization of methods and procedures to solve problems which are direct applications of these procedures and follow immediately the textbook presentation of the procedures, or the procedure is specified.

(8) Student responses require recognition of the reasoning involved in a proof presented to them in completed form.

(9) Student responses require recognition of the consequences of generalizations when such consequences can be derived almost immediately from the given generalization.

(10) Student responses require proving statements in which the method of proof is exactly like that given in examples.

3. Application

(1) Student responses require the performance of manipulations when problems are varied and the student must choose method and operation.

(2) Student responses require solving problems embodying concepts and methods previously studied, but initially unfamiliar to them as examples of previous problems.

(3) Student responses require solving problems from many areas of human endeavor, including areas outside of mathematics and science, where applicable.

(4) Student responses require a recording of the choice of correct principles and a demonstration of its use in the problem.

(5) Student responses require the solution of a problem and the recording of the processes of application.

(6) Student responses require the recording of the correct abstraction involved in the problem.

4. Analysis

(1) Student responses require the ability to distinguish between fact and hypothesis.

(2) Student responses require the ability to recognize unstated assumptions.

(3) Student responses require the ability to distinguish a conclusion from statements which support it.

(4) Student responses require the ability to recognize what particulars are relevant to the validation of a judgment.

(5) Student responses require the ability to detect logical fallacies in arguments.

(6) Student responses require the recognition of the organizational scheme used by the author to present a broad area.

(7) Student responses require the ability to detect errors, by using techniques to verify answers.

(8) Student responses require the deciding of what premisses are needed in order to reach a given conclusion.

5. Synthesis

(1) Student responses require complete proofs of problems which differ significantly in their proof from the examples presented in the textbook.

(2) Student responses require the devising of methods and techniques for attacking a problem, when such methods have not been presented previously.

(3) Student responses require the utilization of elements from many sources to effect a structure or pattern not clearly there before, and should result in a product.

6. <u>Evaluation</u>

(1) Student responses require the ability to assess, by internal standards, the general probability of accuracy in reporting facts from the care given to exactness of statement, documentation, proof, etc.

(2) Student responses require the ability to apply given criteria based on internal standards, to the judgment of the work.

(3) Student responses require the ability to indicate logical fallacies in arguments, and to indicate the nature of the fallacy and exact reasoning involved.

(4) Student responses require the comparison of major theories, generalizations and facts about a broad area.

(5) Student responses require the ability to compare, judging by external standards, a work with the highest known standards in the field--especially with other works of recognized excellence.

(6) Student responses require the ability to distinguish between technical terminology which adds precision to a text by permitting more appropriate definition of terms, and that which merely replaces a common name by an esoteric one.

(7) Student responses require the ability to recognize the appropriateness of a given form of solution of a problem, to a given end.

(8) Student responses require the ability to recognize the best of several means to a given end.

(9) Student responses require the ability to determine the means which will serve best a given particular end.

APPENDIX 3

TAXONOMIC LEVEL RATING DATA SHEET

________Textbook Series Evaluation Number

CRITERIA* (10 points each)	AREAS OF MATHEMATICAL STRUCTURE** 1	2	3	4	5	6
1. Knowledge						
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
Total						
			Scaled Score (600 pts.)			
2. Comprehension						
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
Total						
			Scaled Score (600 pts.)			

*Numbers in the subdivisions listed under criteria correspond to those numbered criteria in Appendix 2, pp. 84-89.

**Numbers under areas of mathematical structure refer to the following: (1) Sets: (2) Ordered Sets; (3) Groups; (4) Vector Spaces; (5) Fields; and (6) Logic and Foundations.

CRITERIA (10 points each)	AREAS OF MATHEMATICAL STRUCTURE					
	1	2	3	4	5	6
3. Application						
1						
2						
3						
4						
5						
6						
Total						
				Scaled Score (600 points)		
4. Analysis						
1						
2						
3						
4						
5						
6						
7						
8						
Total						
				Scaled Score (600 points)		
5. Synthesis						
1						
2						
3						
Total						
				Scaled Score (600 points)		
6. Evaluation						
1						
2						
3						
4						
5						
6						
7						
8						
9						
Total						
				Scaled Score (600 points)		

APPENDIX 4

AGGREGATE SCALED SCORES
FOR TEXTBOOK SERIES

TEXTBOOK SERIES	TAXONOMIC LEVELS 1	2	3	4	5	6
Series A						
Evaluation A	579	566	535	525	600	432
Evaluation B	580	560	525	538	600	417
Evaluation C	577	575	534	513	583	445
Series B						
Evaluation A	406	556	375	101	120	33
Evaluation B	450	560	375	100	100	28
Evaluation C	410	556	328	113	167	33
Series C						
Evaluation A	600	570	440	118	103	40
Evaluation B	590	600	400	138	250	100
Evaluation C	580	575	450	113	133	28
Series D						
Evaluation A	496	600	580	600	600	409
Evaluation B	550	600	600	600	600	578
Evaluation C	500	600	567	585	567	544
Series E						
Evaluation A	480	593	560	600	600	409
Evaluation B	500	600	550	563	600	444
Evaluation C	480	550	600	600	600	500
Series F						
Evaluation A	446	457	130	29	0	2
Evaluation B	435	445	167	94	0	0
Evaluation C	430	460	83	44	0	6
Series G						
Evaluation A	548	600	582	600	403	266
Evaluation B	550	600	567	588	333	256
Evaluation C	542	580	585	563	373	279

Total possible score on any Taxonomic Level is 600 points. Letters of Series designate textbook series identified by these letters in Appendix 1, pages 79-83.

TEXTBOOK SERIES	TAXONOMIC LEVELS 1	2	3	4	5	6
Series H						
Evaluation A	475	582	178	81	43	27
Evaluation B	475	570	165	100	17	17
Evaluation C	480	585	183	89	33	37
Series I						
Evaluation A	515	588	550	478	520	166
Evaluation B	600	600	600	500	497	167
Evaluation C	530	590	550	475	533	50
Series J						
Evaluation A	480	600	478	528	253	164
Evaluation B	500	600	500	525	250	222
Evaluation C	458	600	517	563	267	156
Series K						
Evaluation A	475	558	383	66	157	28
Evaluation B	475	600	383	50	167	3
Evaluation C	473	565	375	61	137	22
Series L						
Evaluation A	490	551	152	59	7	0
Evaluation B	500	495	167	31	0	0
Evaluation C	483	545	138	56	17	0
Series M						
Evaluation A	480	422	40	11	0	0
Evaluation B	350	500	25	11	0	0
Evaluation C	400	431	47	4	0	0
Series N						
Evaluation A	465	334	28	10	0	7
Evaluation B	455	297	17	15	0	11
Evaluation C	480	340	30	11	0	3
Series O						
Evaluation A	415	407	68	4	0	0
Evaluation B	420	500	75	0	0	0
Evaluation C	418	420	62	6	0	0
Series P						
Evaluation A	465	569	413	133	177	23
Evaluation B	475	560	420	129	183	22
Evaluation C	480	569	408	151	167	20

TEXTBOOK SERIES	TAXONOMIC LEVELS 1	2	3	4	5	6
Series Q						
Evaluation A	460	538	235	30	37	9
Evaluation B	410	525	275	31	23	6
Evaluation C	455	535	242	31	33	9
Series R						
Evaluation A	281	321	70	8	0	0
Evaluation B	310	331	60	4	0	0
Evaluation C	315	320	67	8	0	0
Series S						
Evaluation A	489	550	513	449	480	392
Evaluation B	550	600	500	464	467	406
Evaluation C	480	580	525	444	483	393

Evaluation A was made by this investigator, while Evaluation B and C were made by independent evaluators.

APPENDIX 5

SUMMARY OF STRUCTURE OF MATHEMATICS

The purpose of this brief overview of the basic ideas in each of the six areas was to trace the threads of basic concepts tying together all of these areas. Perception of these threads is a primary prerequisite to understanding the underlying structure of mathematics. The decision of which material to include in this brief summary was based on its pertinence directly to the entries in Figure 1, page 15, and its potential use in briefing evaluators of the textbook series.

Most of the material, unless otherwise indicated, in the following pages, was culled from lecture notes taken by this investigator in graduate courses in the mathematics department, University of Arizona. These have not been footnoted, since the summary was an original synthesis of this material.

<u>Sets</u>

Let X be a set (Set is undefined.).

1. $x \in X \Rightarrow x$ is a member of X.
2. $x \notin X \Rightarrow x$ is not a member of X.
3. $\{x_1, x_2, \ldots, x_n\}$ --denotes a finite set consisting of $x_1, x_2, \ldots, x_n$ elements.

4. $\{x\}$ = singleton x.

5. P(x) a proposition about x $\Rightarrow \{x: \ P(x)\}$ denotes a set of all x such that P(x) is true. $\{x\}_{P(x)}$ or $\{x \in A: P(x)\}$ denotes set of all elements x of set A such that P(x) is true.

6. $[X$ and Y sets and $x \in X \Rightarrow x \in Y] \Rightarrow X \subseteq Y$ or $Y \supseteq X$ (X is a subset of Y).

7. Empty set $\emptyset$ is $\{x: \ x \neq x\}$. $\emptyset \subseteq X$, for each X.

8. Ⓕ a collection of sets, then $\bigcap$Ⓕ $= \{x: \ x \in A$ for every $A \in$ Ⓕ$\}$. $\bigcup$Ⓕ $= \{x: \ x \in A$ for some $A \in$ Ⓕ$\}$.

9. I a set and A_α a set for all $\alpha \in I$, then Ⓕ $= \{A_\alpha: \alpha \in I\}$ is a collection of sets and I is an index set for Ⓕ .

10. $\bigcup$Ⓕ $= \bigcup \{A_\alpha: \alpha \in I\}$.

11. X and Y sets and disjoint $\Leftrightarrow X \cap Y = \emptyset$.

12. Ⓕ is disjoint $\Rightarrow$ X and Y disjoint for each pair of distinct sets X and Y in Ⓕ .

13. X and Y sets, <u>complement of X in Y</u> = $\{y \in Y: \ y \notin X\} = Y - X = \widetilde{X}$ (with respect to Y).

14. Seven basic principles concerning sets:

Let S be the universal set, X and Y variables whose domain consists of the members of S, i.e., the domain of "e" in S. Then for each X and Y in S, and each $e \in X$ or Y:

(1) $[e \in X \cup Y \Leftrightarrow (e \in X$ or $e \in Y)]$

(2) $[e \in X \cap Y \Leftrightarrow (e \in X \text{ and } e \in Y)]$

(3) $[e \in \tilde{X} \Leftrightarrow e \notin X]$

(4) $e \notin \emptyset$

(5) $e \in S$

(6) $[Y \subseteq X \Leftrightarrow (e \in Y \Rightarrow e \in X$, for each e in X or Y)]

(7) $Y \subseteq X$ and $X \subseteq Y \Rightarrow X = Y$

15. From these seven basic principles we can derive the following statements about sets:

Let S be the universal set, X and Y variables whose domain consists of the members of S, then for every X, for every Y, and for every Z:

(1) Commutativity: $X \cup Y = Y \cup X$

$X \cap Y = Y \cap X$

(2) Associativity: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

$(X \cap Y) \cap Z = X \cap (Y \cap Z)$

(3) Distributivity: $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$

$(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$

(4) $X \cup \emptyset = X$

(5) $X \cap S = X$

(6) $\tilde{X} \cup X = S$ ($\tilde{X}$ with respect to S)

(7) $\tilde{X} \cap X = \emptyset$ ($\tilde{X}$ with respect to S)

(8) $[Y \subseteq X \Leftrightarrow (X \cap Y = Y)]$

Topological Space

1. A topology $\mathcal{J}$ is a family of sets such that

(1) For each $A, B \in \mathcal{J}$, $A \cap B = C$, a member of $\mathcal{J}$.

(2) If $\{A_\alpha\}$ is any family of $\mathcal{J}$ then $\bigcup A_\alpha$ is in $\mathcal{J}$.

2. By induction, if $A \cap B \in \mathcal{J}$ for any $A, B \in \mathcal{J}$ then $\bigcap_{i=1}^{n} A_i \in \mathcal{J}$, whenever $(A_i)_{1 \leq i \leq n} \in \mathcal{J}$.

3. Set $X = \bigcup \{A: \ A \in \mathcal{J}\} \in \mathcal{J}$ since $\mathcal{J}$ is a subfamily of itself. Every member of $\mathcal{J} \subset X$. $\mathcal{J}$ is a topology for X and X is the space of the topology $\mathcal{J}$. The pair $(X, \mathcal{J})$ is a topological space.[43]

Example: Using the definition that an open set is any set A such that $A \in \mathcal{J}$, and letting X be the set of real numbers, define a subset A of reals as an open set $(\in \mathcal{J})$, if and only if for each $x \in A$, there exists a and b such that $a < x < b$ and the open interval $\{y: \ a < y < b\} \subset A$. This topology is called the usual topology for the reals and is the family of all those sets which contain an open interval about each of their points.

Ordered Sets

Relations

1. Ordered pair $(x,y) = \{\{x\}, \ \{\{x\} \cup \{y\}\}\}$
2. $(x,y) = (u,v) \Leftrightarrow x = u$ and $y = v$.
3. Relation is a set of ordered pairs.
4. $xRy \Rightarrow (x,y) \in R$, R is a relation.
5. domain $R = \{x: \ xRy \ \text{ for some } y\}$.

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43. John L. Kelley, General Topology, The University Series in Higher Mathematics (Princeton, New Jersey: Van Nostrand Company, Inc., 1955), cf. p. 37.

6. range R = $\{y:\ xRy \text{ for some } y\}$.

7. Cartesian product of X and Y (both sets) is $\{(x,y):\ x \in X \text{ and } y \in Y\}$ and denoted $X \times Y$.

8. R is a relation $\Longleftrightarrow R \subset (X \times Y)$ for some X and Y.

9. R is reflexive in case xRx, for each x in domain of R.

10. R is symmetric in case $xRy \Longrightarrow yRx$, for each $(x,y) \in R$.

11. R is transitive in case xRy and $yRz \Longrightarrow xRz$, for each $x,y,z \in R$.

12. R a relation and $A \subseteq$ domain R, then <u>image of A</u> under R is $A^R = R(A) = \{y:\ xRy \text{ for some } x \in A\}$.

(1) A = domain R $\Longrightarrow$ R(A) = range R.

(2) $x \in$ domain R, then $R(\{x\})$ and $\{x\}^R$ are abbreviated by R(x) and x^R, respectively.

Mappings (functions)

1. Definition of mapping: xfy and $xfz \Longrightarrow y = z$, where f is a relation.

2. f a mapping and $x \in$ domain f, then image f(x) of x under f is the <u>value</u> of f at x. $f(x) = x^f$, f_x, fx and xf.

3. A any set, then the <u>inverse image</u> (or <u>pre-image</u>) of A under f is a set. $f^{-1}(A) = \{x \in \text{domain } f:\ f(x) \in A\}$.

4. Let f and g be mappings: $f = g \Longleftrightarrow$ f and g contain exactly same elements, i. e.

$f = g \Longleftrightarrow$ domain f = domain g and f(x) = g(x), for each x in dmn f = dmn g.

5. Denote f by $\{f(x):\ x \in A\}$ or $(f(x))_{x \in A}$

X_α a set, for all $\alpha \in I$, we speak of mapping (X_α) as a family. (I is an index set as defined in (9), page 96.)

6. A one-to-one correspondence (mapping) of f is such that $f(x) = f(y) \Longrightarrow x = y$ ($x \neq y \Longrightarrow f(x) \neq f(y)$).

1-1 mapping is called an <u>injection</u> and said to be injective.

7. An injective mapping of X onto Y is called a <u>bijection</u> (of X onto Y) and said to be bijective.

8. f a bijection of X onto Y $\Longrightarrow$ for each $y \in Y$ there exists a unique $x \in X$ such that $f(x) = y$. Under such assumption, define $f^{-1}(y) = x$. Hence f^{-1} is a bijection of Y onto X and called the <u>inverse</u> of f. (Note: $(f^{-1})^{-1} = f$.

9. If $g: X \longrightarrow Y$ and $f: Y \longrightarrow Z$, then the <u>composite</u> of f and g is $f \circ g: X \longrightarrow Z$ defined by:

$$(f \circ g)(x) = f(g(x)),\ x \in X.$$

10. If $X \subseteq Y$, then the identity injection of X into Y (or identity map) is the injection $i_X: X \longrightarrow Y$ defined by: $i_X(x) = x$, $x \in X$.

11. $f: X \longrightarrow X$, then $f \circ i_X = i_X \circ f$.

12. If f is a bijection of X onto Y, then $f \circ f^{-1} = i_X$ and $f^{-1} \circ f = i_X$. The converse is also true.

13. <u>restriction of f to A</u> is such that if a mapping g has domain A and $A \subseteq$ domain f(mapping) and $g(x) = f(x)$ for every $x \in A$: $g = f|A$ or $f|A = \{(a,y): a \in A \text{ and } (a,y) \in f\}$.

14. ϕ is a mapping.

(1) domain ϕ = range $\phi = \phi$

(2) f a relation such that dmn $f = \phi$ or range $f = \phi \Rightarrow f = \phi$.

Quotient Sets

1. R is an <u>equivalence relation</u> if and only if R is reflexive, symmetric and transitive.

2. If X is a set, then R is an <u>equivalence relation on X</u> in case:

   (1) R is an equivalence relation, and

   (2) X is the domain (therefore, also range) of R.

3. R is an <u>equivalence realtion on X</u> and $x \in X \Longrightarrow x^R$ is an equivalence class of X (modulo R). Reminder: $x^R = \{y:$ xRy for some $x \in \{x\}\}$.

4. Properties of equivalence classes:

   (1) $y \in x^R \Longleftrightarrow y^R = x^R$

   (2) Since $x \in x^R$, we call each element of x^R a <u>representative</u> of the equivalence class.

5. Let $X \neq \emptyset$. A set Ⓠ is a <u>quotient</u> set (or a partition) of X in case

   (1) Ⓠ is a collection of non-empty subsets of X.

   (2) Ⓠ is disjoint.

   (3) $X = \bigcup Q$.

6. If R is an equivalence relation on X, we call X/R the quotient set of X modulo (relative to) R. The mapping $\pi: x \longrightarrow x^R$ is a mapping of X onto X/R. π is called the natural or canonical mapping of X onto X/R.

Products

Let I be a set and consider a family $(X_\alpha)_{\alpha \in I}$ of sets X_α.

1. Cartesian Product of the family $(X_\alpha)_{\alpha \in I}$ means the set $\prod_{\alpha \in I} X_\alpha$ of all families $(X_\alpha)_{\alpha \in I}$ such that $X_\alpha \in X_\alpha$, for all $\alpha \in I$.

2. $\prod_{\alpha \in I} X_\alpha = \{f:$ f is a function from I to $\bigcup_{\alpha \in I} X_\alpha$ where $f(\alpha) \in X_\alpha\}$.

3. If $I \neq \emptyset$ and if $x = (x_\alpha)_{\alpha \in I}$ is in $\prod_{\alpha \in I} X_\alpha$, then each x_α is called a coordinate of x; in particular, x_α is called the coordinate of x of index α (or simply the αth coordinate of x).

Order Relations

1. If R is any relation, then we set F(R) = dmn R $\cup$ range R.

2. If X is a set, then by restriction of R to X, we mean the relation on $R' = R \cap (X \times X)$. We also refer to R' as the relation induced in X by R.

3. If R is a relation and X is a set, we say R partially orders X in case for each $x, y, z \in X$:

    (1) R partially orders X.

    (2) For each $x, y \in X$, xRy or yRx.

5. R is a linear (total, simple) ordering in case R linearly orders F(R).

6. R well orders X in case

    (1) R linearly orders X

    (2) For each non-empty subset A of X there is a least element relative to R, i. e. there exists an $x \ni xRy$ for all $y \in A$.

7. R is a well-ordering in case R well orders F(R).

8. We will usually denote a partial ordering by some symbol like $\leq$. Then $x < y$ means $x \leq y$ and $x \neq y$.

9. By a partially ordered set we mean an ordered pair $(P, \leq)$ such that P is a non-empty set and $\leq$ is a partial ordering $F(\leq) = P$. Therefore, for each $x, y, z \in P$ we have:

(1) $x \leq x$

(2) $x \leq y$ and $y \leq x \Rightarrow x = y$

(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$

Usually denote $(P, \leq)$ by P only.

10. A partially ordered set $(P, \leq)$ is a linearly ordered set (or chain) in case $\leq$ is a linear ordering.

11. $(P, \leq)$ is a well-ordered set in case $\leq$ is a well-ordering. Examples:

(1) Let P be a collection of sets. For each $A, B \in P$, define $A \leq B$ in case $A \subseteq B$. Then $(P, \leq)$ is a partially ordered set, and usually referred to as partially ordered with respect to set inclusion.

(2) Let P be the set of all positive integers and for each $x, y \in P$ define $x \leq y \Leftrightarrow y - x$ is non-negative. Then $(P, \leq)$ is a chain and in fact, a well-ordered set.

12. If $x, y \in P$(a partially ordered set) then x is covered by $y \Leftrightarrow x < y$ and there does not exist $z \in P$ such that $x < z < y$. Sometimes $x \dot{<} y$ used to denote y covers x.

13. If P is finite then $x < y$ if and only if there exists a chain of the form: $x = x_1 \dot{<} x_2 \dot{<} \dots \dot{<} x_n = y$ in P.

14. Let P' be another partially ordered set. A mapping $f: P \longrightarrow P'$ is <u>order-preserving</u> or <u>isotone</u> $\Longleftrightarrow$ for each $x,y \in P$. $x \leq y \Longrightarrow f(x) \leq f(y)$, and f is bi-isotone in case f is isotone and $f(x) \leq f(y) \Longrightarrow x \leq y$.

15. If f is a bi-isotone mapping from P onto P' then f is an isomorphism (from P onto P').

16. Let $A \subseteq P$. An $x \in P$ is an upper (lower) bound of $A \Longleftrightarrow a \leq x$ $(x \leq a)$, for each $a \in A$.

17. $x \in P$ is a least upper bound or supremum or join (greatest lower bound or infimum or meet) of $A \Longleftrightarrow x$ is an upper (lower) bound of A and $x \leq y$, for each upper bound y of A ($y \leq x$, for each lower bound y of A).

18. If A has a sup., it is unique.

19. sup A $= \bigvee A$ = l.u.b. of A.

20. inf A $= \bigwedge A$ = g.l.b. of A.

21. $x \in P$ is maximal (minimal) $\Longleftrightarrow$ there does not exist $y \in P$ such that $x < y$ $(y < x)$.

21. We say a partially ordered set P is <u>inductive</u> in case every chain in P has a supremum in P.

22. <u>Axiom of Choice</u>: Any set can be well-ordered.

Cardinal Numbers

We say two sets "have the same cardinal number" in case there exists a bijection from one onto the other. If U is any family of sets and $A \in U$, then the cardinal number of A--denoted by $|A|$-- is the family of all elements of U having the same cardinal number as A. If A is finite, and contains n elements, then we write $|A| = n$, if A is infinite, write $|A| = \infty$.

Induction

1. Throughout, W will denote a linearly ordered set; if $\alpha \in W$ then $W/\alpha = \{x \in W: x < \alpha\}$ an initial segment of W.

2. Principle of finite induction: Let $N = \{1, 2, \ldots\}$ If $S \subset N$ and S satisfies (1) and (2) below then $S = N$:

(1) $1 \in S$

(2) $N/n \subset S \Rightarrow n \in S$.

3. A linearly-ordered set W satisfies the transfinite induction principle if $W_1 \subset W$ and W_1 satisfies (1) and (2) below $\Rightarrow W_1 = W$.

(1) $\wedge W$ exists and $\wedge W \in W$.

(2) $W/\alpha \subset W_1 \Rightarrow \alpha \in W_1$.

4. A linearly ordered set W satisfies P.T.I. $\Leftrightarrow$ the set is well-ordered.

Groups

Semi-Groups

1. A multiplicative system is a set X and a binary operation U on X ($X \times X \xrightarrow{\text{into}} X$). We say U is commutative $U(a,b) = U(b,a)$. Note: We usually write $U(a,b)$ as $a \cdot b$ or ab or $a \circ b$ or in special cases $a + b$.

Associativity $(ab)c = a(bc)$

Commutativity $ab = ba$

2. A <u>semi-group</u> is an associative multiplicative system.

3. The following are true:

(1) If $e \in S$, we say e is a <u>left-identity</u> in case $e \cdot x = x$, for all $x \in S$.

(2) We say e is a <u>right identity</u> in case $xe = x$, for all $x \in S$.

(3) We call e an identity $\Longleftrightarrow$ e is both a left and right identity. Examples of non-associative multiplicative systems: Vector Cross Product and Division.

(4) The identity is unique, if it exists.

4. S a semi-group, $x \in S$, we call x <u>right regular</u> if there is a $y \in S$ such that $xy = e$. x is <u>left regular</u> if there is a $y \in S$ such that $yx = e$. We call it <u>regular</u> if x is both left and right regular.

5. If x is regular, we call x <u>invertible</u> or a <u>unit</u>.

6. Let S be a semi-group and $x \in S$ a unit. Then there exits an element necessarily a unit of S, denoted by x^{-1} and called x-inverse which is unique with respect to $xx^{-1} = e$ and $x^{-1}x = e$.

7. A <u>group</u> is a semi-group with identity such that every element is a unit. This definition is equivalent to:

(1) $(G, \bullet)$ is a group $\Longleftrightarrow$ G is a multiplicative system and $(ab)c = a(bc)$.

(2) There is an $e \in G$ such that $e \cdot x = x \cdot e = x$, for all $x \in G$.

(3) $x \in G \Longrightarrow$ there is $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

Topological Groups

1. A set G of elements e, x, y, . . . is called a topological group if:

(1) G is a group

(2) G is a topological space

(3) The group operations in G are continuous, i.e., the transformations $G \times G \longrightarrow G$, $G \longrightarrow G$ given by $(x,y) \longrightarrow xy$ and $x \longrightarrow x^{-1}$ are continuous.

2. Examples:

(1) Reals with group structure of the additive group of reals and the topology of the real line R'.

(2) The set of complex numbers of unit modulus, with group structure of the multiplicative group of complex numbers and the topology of the circle S^1.

(3) Consider the general linear group GL(n,R) of non-singular linear transformations of R^n. If we fix a coordinate system in R^n, then GL(n,R) is represented as the multiplicative group of n x n non-singular real matrices. Writing such a matrix as $M = I_n + (m_{ij})$, we may take the number m_{ij} as coordinates of M and thus topologize GL(n,R) as a subspace of R^{n^2}. As topological subgroups we may take SL(n,R), the group of matrices of determinant +1, O(n) the orthogonal group and $R(n) = SL(n,R) \cap O(n)$, the rotation group which acts transitively and effectively on S^{n-1}. (Transitive in that for each $x,y \in S^{n-1}$, there is a $r \in R(n)$ such that $xr = y$ and effective in that only the identity

rotation leaves each point of S^{n-1} fixed.

(4) Consider the general linear group GL(n,C) in which we replace real numbers by complex numbers.[44]

Fields

Rings

1. A ring is an algebraic structure $(A, +, \cdot)$ where A is a set, $(A,+)$ is an Abelian group and $(A,\cdot)$ is an associative multiplicative system and $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

2. Examples:

(1) Set of integers (Z)

(2) Set of even integers

(3) Reals, rationals, or complex numbers under ordinary addition and multiplication.

(4) Set of all m x n matrices over a number field.

(5) Let X be a set. 2^X = all subsets of X; operations $A \cdot B = A \cap B$ and $A + B = (A-B) \cup (B-A)$ [all $x \in A$ or B but not in $A \cap B$].

(6) Any Abelian group, $a \cdot b = 0$, for all a,b in the group.

(7) Integers mod n and multiplication is "reduce mod n".

(8) Let X be any topological space, and R the reals, then C(X,R) under addition and multiplication pointwise.

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44. P. J. Hilton and S. Wylie, Homology Theory, (London: Cambridge University Press, 1960), pp.265-266.

3. A ring A is commutative, if commutative under multiplication, i. e. $ab = ba$ for all $a, b \in A$.

4. We say A has a unity element in case there exists $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$; and call 1 the unity element.

5. Any element x such that there exists a y such that $xy=1$ or $yx=1$ is such that x is called a unit and it is invertable. $n \cdot a = a^n$ where $a \in A$. If A has unity, this could mean $(n \cdot 1) \cdot (a)$.

6. A ring $\{a, b, c\} \subset A$ then

(1) $a \cdot 0 = 0 \cdot a = 0$

(2) $(-a) \cdot b = -(ab) = a(-b)$

(3) $(-a)(-b) = ab$

(4) $(b - c)a = ba - ca$

(5) $a(b - c) = ab - ac$

(6) Generalized Distributive Law

$$\sum_{i=1}^{n} a_i \sum_{i=j}^{m} b_j = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i b_j$$

(7) $ab = ba \Rightarrow (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

7. We say $a \in A$ is a divisor of 0 in case there is a $c \in A$ such that $c \neq 0$ and $a \cdot c = 0$ or $c \cdot a = 0$. Note: 0 is always a divisor of 0 except in a 0-ring. If $a \neq 0$ call 'a' a proper divisor of zero. Examples:

(1) Integers mod 6 have 2 and 3 as proper divisors of zero.

(2) $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = 0$ and neither is zero.

8. In a ring A the following are equivalent:

(1) "a" is not a divisor of 0.

(2) $ax = ay$ or $xa = ya \Longrightarrow x = y$.

9. An integral domain is a ring with no proper divisors of zero.

10. A division ring is a ring A such that $(A \sim \{0\}, \cdot)$ is a group. Example: 2 x 2 nonsingular matrices.

Properties of Fields

1. A field is a commutative division ring.

2. More specifically, a commutative ring F is called a field if the following field axioms are satisfied:

(1) F has at least two elements.

(2) F has an identity.

(3) Every element of F different from zero has an inverse.

(4) All three of the above axioms can be replaced by a single axiom: the elements of F which are different from zero form a group with respect to multiplication.[45]

3. An important consequence is that for all $a \neq 0$, $b \neq 0 \in F$, it is possible to form the quotient b/a. It is the unique solution of $ax = b$. Division by any element $a \neq 0$ is always permissable in a field, but division by zero is undefined.

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45. Oscar Zariski and Pierre Samuel, Commutative Algebra, Volume 1, (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1958), p. 10.

4. Examples: Set of all rational numbers, set of reals, and set of complex numbers.

5. Almost all mathematics studied at the secondary level encompasses examples given in (4) above, and hence field theory is the basis for the underlying structure of most mathematics taught in high school.

Vector Spaces

Groups with Operators

1. Let G be a group and M a set. Let Θ be such that Θ: $M \longrightarrow G^G$ = set of all maps from G to G. Let $m \in M$, then we say m operates on G via Θ in case $\Theta(m) \in E(G)$ = group of all endomorphisms on G. Call m an operator on G and we write m(x) for $\Theta(m(x))$. We say M operates on G via Θ in case m is an operator on G for all $m \in M$.

2. M-Group or group with operators is a group G along with a set of operators M on G.

Representation

1. Let G be an Abelian group, A a ring. By a representation of the ring A in G we mean a homomorphism φ such that φ: $A \longrightarrow E(G)$ = left ring of endomorphisms cf G.

2. A representation is faithful in case it is 1-1.

Module

1. Let M be an Abelian group, A a ring, then M is a left-A-module with respect to a map φ in case:

(1) M is an A-group with respect to φ, i. e. A is a set of operators on M with respect to φ.

(2) φ is a left representation of A in M.

2. A left-A-module M is faithful in case its representation is faithful.

3. Let M be an Abelian group, A a ring, $\varphi: A \longrightarrow M^M$. We write $\varphi(a)$ as $a(x)$, $a \in A$, $x \in M$. Then M is a left-A-module with respect to φ if and only if, for all $a, b \in A$ and $x, y \in M$:

(1) $a(x + y) = ax + ay$.

(2) $(a + b)(x) = a(x) + b(x)$

(3) $ab(x) = a(b(x))$

4. A submodule of a left-A-module is a subgroup of a group with operators A, and is a left-A-module.

5. Let M be a left-A-module, $\phi \neq B \subset A$, $\phi \neq S \angle A$(subgroup of M) then by $B \cdot S$ we mean set of all finite sums $\sum_{i=1}^{m} b_i s_i$ where $b_i \in B$ and $s_i \in S$. (Linear combination of elements in S with coefficients in B). We write $b \cdot S$ for $\{b\} \cdot S$ and $B \cdot s$ for $B \cdot \{s\}$.

6. Let M be a left-A-module with representation φ. We say M is unitary in case $M = A \cdot M$.

Properties of Vector Spaces

1. Let D be a division ring. By a left (resp. right) vector space over D we mean a unitary left-module V, i. e. $V = DV = \sum_{i=1}^{n} d_i v_i$.

2. V, V' left-V-vector spaces. By $L(V,V')$ we mean all linear transformations from V into V'.

3. Let $\alpha \in L(V,V')$. Then rank α--denoted $r(\alpha)$--means rank (dimension) of $\alpha[V]$. (This is a subspace since $V \xrightarrow{\alpha} V'$ and $\alpha[V] \subset V'$).

4. Nullspace of α --denoted $N(\alpha)$--means $K(\alpha)$ = kernel of α (A subspace, i. e. $\{x \in V: \alpha(x) = 0$ where 0 is the zero vector in V'$\}$.

5. The nullity of α --denoted by $n(\alpha)$--means the rank of $N(\alpha)$.

6. Let $\alpha \in (V,V')$, then Dim $V = r(\alpha) + n(\alpha)$

7. Let $\alpha,\beta \in L(V,V')$. Then $r(\alpha+\beta) \leq r(\alpha) + r(\beta)$ and $r(\alpha \cdot \beta) \leq \min \{r(\alpha)\ r(\beta)\}$. Note: Here V = V'.

8. Matrix--Let A be a ring. Let $|A| = n \neq 0$. Let $|I_n| = n$ be an index set, then the family $\{a_{ij}:(i,j) \in I_n \times I_n\}$--denoted (a_{ij})--where $a_{ij} \in A$, for all $(i,j) \in I_n \times I_n$ is called an $n \times n$ matrix.

9. Definition of operations on matrices:

$\oplus$ defined $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

We define (a_{ij}) to be row finite in case $a_{ij} = 0$ for almost all $j \in I_n$. So, we can define $(a_{ij}) \cdot (b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_{k \in I_n} a_{ik}b_{kj}$ if (a_{ij}) and (b_{ij}) are row finite.

10. The preceding definition of a vector space was developed as an extension of theory of groups, rings, and modules. The following alternative definition is the one usually

found in textbooks and is equivalent to the one given previously.

A vector space over a field F--denoted V(F)--is an algebraic system such that (V,+) is a group, F is a field, and $\cdot$ is a map from $F \times V \longrightarrow V$ such that

(1) $(\lambda_1 + \lambda_2) \cdot \alpha = \lambda_1 \alpha + \lambda_2 \alpha, \quad , \lambda_1, \lambda_2 \in F, \alpha \in V$

(2) $\lambda(\alpha + \beta) = \lambda \cdot \alpha + \lambda \cdot \beta$

(3) $1 \cdot \alpha = \alpha$; for all $\alpha \in V$

(4) $\lambda_1(\lambda_2 \alpha) = (\lambda_1 \lambda_2) \alpha$, for all $\alpha \in V$.

Dictionary of Terms

Group	Module	V-left-D-module	V-left-D-Vector Space
Subgroup	Submodule	An element $x \in V$	Vector $x \in V$
Homomorphism, etc.	A-homomorphism, etc.	Submodule	Subspace
Quotient Group	Quotient Module	Rank of module	Dimension
Product Group	Product Module	D-homomorphisms	Linear Transformation

Logic and Foundations of Mathematics

The principal areas of logic pertinent to this study are:

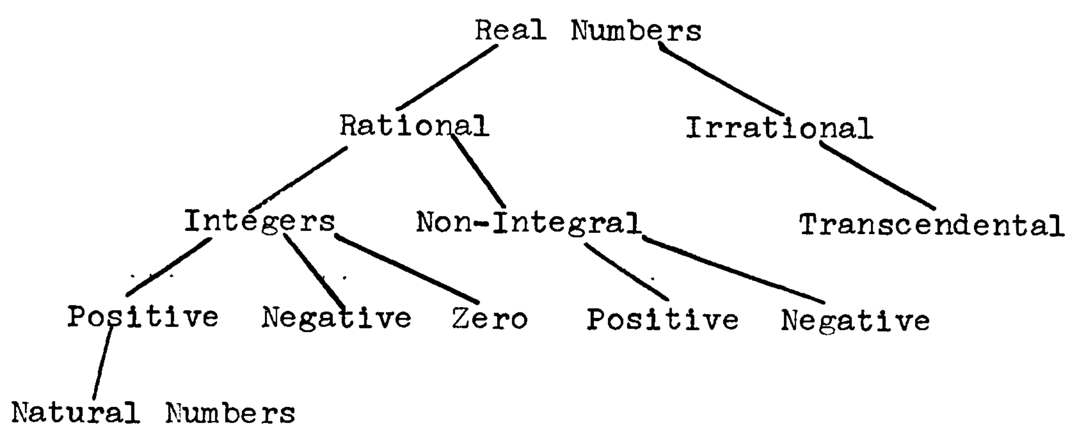
(1) Distinction between inductive and deductive reasoning.

(2) Understanding of the postulational system underlying the area of mathematics under study.

(3) The nature of proof in mathematics

(4) The use of two-valued logic system

(5) The systematic extension of mathematics in the development of the real numbers, starting with the natural numbers, until the real numbers emerge at the apex as depicted below. (Extended to complex numbers in more advanced secondary school mathematics.)



(6) Properties of Natural Numbers as the underlying assumptions of all mathematics.

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