BLACKBODY SIMULATOR CAVITY RADIATION THEORY

by

Frederick Otis Bartell

A Dissertation Submitted to the Faculty of the

COMMITTEE ON OPTICAL SCIENCES (GRADUATE)

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1978

Copyright 1978 Frederick Otis Bartell
I hereby recommend that this dissertation prepared under my direction by Frederick Otis Bartell
entitled Blackbody Simulator Cavity Radiation Theory
be accepted as fulfilling the dissertation requirement for the degree of Doctor of Philosophy

Dissertation Director

As members of the Final Examination Committee, we certify that we have read this dissertation and agree that it may be presented for final defense.

Final approval and acceptance of this dissertation is contingent on the candidate's adequate performance and defense thereof at the final oral examination.
STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the copyright holder.

SIGNED: [Signature]

[Signature]
ACKNOWLEDGMENTS

Many people have contributed to the work of this dissertation, and it is not possible to name them all; however, the role of some of them should be acknowledged. I want to thank E. A. Tulus, who fostered scientific accomplishments in a production environment; E. R. Ban, M. Kamhi, J. J. Gardner, and I. L. Fischer, who pressed harder than I did for improved blackbody simulators; and D. N. Anderson, who provided valuable analytical criticisms. I want to express my greatest appreciation to my dissertation director, Professor W. L. Wolfe, for his continuing guidance; to my parents, who encouraged me in many ways; and to my wife, whose contribution was the most important of all.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>viii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>HISTORY OF BLACKBODY THEORY</td>
<td>6</td>
</tr>
<tr>
<td>The Concept of a Blackbody</td>
<td>6</td>
</tr>
<tr>
<td>Spectral Blackbody Theory</td>
<td>7</td>
</tr>
<tr>
<td>Blackbody Simulator Theory</td>
<td>8</td>
</tr>
<tr>
<td>The Preliminary Period: 1892 to 1927</td>
<td>9</td>
</tr>
<tr>
<td>The Precomputer Period: 1927 to 1960</td>
<td>11</td>
</tr>
<tr>
<td>The Computer Period: 1960 to Present</td>
<td>14</td>
</tr>
<tr>
<td>THEORY</td>
<td>18</td>
</tr>
<tr>
<td>Basic Analysis</td>
<td>18</td>
</tr>
<tr>
<td>Comments on Eqs. (4) through (6a)</td>
<td>25</td>
</tr>
<tr>
<td>Isothermal and Polythermal Cavities: More</td>
<td>28</td>
</tr>
<tr>
<td>Comments about Eq. (6)</td>
<td>28</td>
</tr>
<tr>
<td>Temperature Considerations</td>
<td>29</td>
</tr>
<tr>
<td>Wavelength Considerations</td>
<td>35</td>
</tr>
<tr>
<td>Summary</td>
<td>38</td>
</tr>
<tr>
<td>THEORETICAL AND EXPERIMENTAL STUDIES OF THE PROJECTED SOLID ANGLE EFFECT</td>
<td>40</td>
</tr>
<tr>
<td>Projected Solid Angle Effect</td>
<td>40</td>
</tr>
<tr>
<td>Approximate Values of the Projected Solid Angle</td>
<td>42</td>
</tr>
<tr>
<td>A Simplified Expression</td>
<td>45</td>
</tr>
<tr>
<td>Experimental Setup</td>
<td>51</td>
</tr>
<tr>
<td>Theoretical and Experimental Results</td>
<td>53</td>
</tr>
<tr>
<td>COMPARISONS AMONG THEORIES</td>
<td>55</td>
</tr>
<tr>
<td>Emissivity</td>
<td>56</td>
</tr>
<tr>
<td>Integral-Equation Analyses</td>
<td>58</td>
</tr>
<tr>
<td>Gouffé's Theory</td>
<td>59</td>
</tr>
<tr>
<td>De Vos' Theory</td>
<td>61</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS--Continued

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treuenfels' Theory</td>
<td>62</td>
</tr>
<tr>
<td>Kelly's Theory</td>
<td>63</td>
</tr>
<tr>
<td>Nicodemus' Theory</td>
<td>63</td>
</tr>
<tr>
<td>Comparison Summary</td>
<td>64</td>
</tr>
<tr>
<td>Previous Comparisons</td>
<td>66</td>
</tr>
<tr>
<td>An Ecumenical Theory</td>
<td>68</td>
</tr>
<tr>
<td><strong>FOUR SPECIAL CAVITY SHAPES</strong></td>
<td>71</td>
</tr>
<tr>
<td>Spheres</td>
<td>71</td>
</tr>
<tr>
<td>Cones</td>
<td>72</td>
</tr>
<tr>
<td>Reentrant Cones</td>
<td>72</td>
</tr>
<tr>
<td>Cavities with Convex-Inward Wall Elements</td>
<td>74</td>
</tr>
<tr>
<td><strong>THE QUANTITIES $T_m$ and $\int_{\text{ap}} d\Omega$</strong></td>
<td>76</td>
</tr>
<tr>
<td>Other Theories</td>
<td>76</td>
</tr>
<tr>
<td>$T_m$</td>
<td>78</td>
</tr>
<tr>
<td>$\int_{\text{ap}} d\Omega$</td>
<td>79</td>
</tr>
<tr>
<td>The Accuracy of $T_m$ and $\int_{\text{ap}} d\Omega$</td>
<td>80</td>
</tr>
<tr>
<td><strong>HOW TO CALCULATE EFFECTIVE CAVITY EMISSIVITY</strong></td>
<td>82</td>
</tr>
<tr>
<td>First Approximations</td>
<td>82</td>
</tr>
<tr>
<td>Second Approximations</td>
<td>83</td>
</tr>
<tr>
<td>Third Approximations</td>
<td>84</td>
</tr>
<tr>
<td>Computer Calculations</td>
<td>84</td>
</tr>
<tr>
<td><strong>CONCLUSIONS AND EXTENSIONS</strong></td>
<td>85</td>
</tr>
<tr>
<td><strong>APPENDIX A.</strong> Derivation of Equation (6) from Equation (4)**</td>
<td>88</td>
</tr>
<tr>
<td><strong>APPENDIX B.</strong> $\Delta T/T_t$ and the Coefficient $\xi$</td>
<td>98</td>
</tr>
<tr>
<td><strong>APPENDIX C.</strong> Integral Equation Analyses</td>
<td>105</td>
</tr>
<tr>
<td><strong>APPENDIX D.</strong> Gouffé's Theory</td>
<td>117</td>
</tr>
<tr>
<td><strong>APPENDIX E.</strong> De Vos' Theory</td>
<td>129</td>
</tr>
<tr>
<td><strong>REFERENCES</strong></td>
<td>142</td>
</tr>
</tbody>
</table>
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph of Cosine Distribution of Blackbody Simulator Radiation Source</td>
<td>2</td>
</tr>
<tr>
<td>2. Blackbody According to Kirchhoff (1960)</td>
<td>6</td>
</tr>
<tr>
<td>3. Cavity Arrangement</td>
<td>19</td>
</tr>
<tr>
<td>4. Temperature Details</td>
<td>31</td>
</tr>
<tr>
<td>5. Cavity Interior</td>
<td>42</td>
</tr>
<tr>
<td>6. Experiment</td>
<td>51</td>
</tr>
<tr>
<td>7. Experimental Verification of Theory</td>
<td>53</td>
</tr>
<tr>
<td>8. Reentrant Cone Variations, Normalized to $d=1.0$ and $\omega=0.5$</td>
<td>74</td>
</tr>
<tr>
<td>D-1. Gouffé's Reversal for Spherical Cavities</td>
<td>120</td>
</tr>
<tr>
<td>D-2. Geometry for a Spherical Cavity</td>
<td>121</td>
</tr>
<tr>
<td>D-3. Parameter $A$ for Conical Cavities</td>
<td>127</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
</tr>
<tr>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>1. Cavity Emissivity According to Gouffé for Several Typical Popular Isothermal Cavities</td>
<td>3</td>
</tr>
<tr>
<td>2. Summary of Eqs. (2) to (19)</td>
<td>38</td>
</tr>
<tr>
<td>3. ((L_\zeta/L_d)) and (\left[(\partial T/\partial T)(T/L)\right]<em>{T</em>\zeta})</td>
<td>39</td>
</tr>
<tr>
<td>4. Approximate Expressions for (\pi^{-1}\int_{ap} d\Omega)</td>
<td>44</td>
</tr>
<tr>
<td>5. Approximate Values for (\pi^{-1}\int_{ap} d\Omega)</td>
<td>45</td>
</tr>
<tr>
<td>6. Values of (G_5) when (x = (14388 \mu m K)/\lambda T)</td>
<td>48</td>
</tr>
<tr>
<td>7. Gouffé's Five Expressions for (\epsilon_0)</td>
<td>60</td>
</tr>
<tr>
<td>8. Values of (A) and (B) for Eq. (41)</td>
<td>65</td>
</tr>
<tr>
<td>9. Reentrant Cone Designs</td>
<td>73</td>
</tr>
<tr>
<td>D-1. Analytic Geometry for Spherical Cavity</td>
<td>122</td>
</tr>
<tr>
<td>D-2. Gouffé's (\epsilon_0) and My (\epsilon_\alpha(\vec{r})) for Conical Cavities</td>
<td>128</td>
</tr>
<tr>
<td>E-1. De Vos' Temperature Parameter (\kappa)</td>
<td>141</td>
</tr>
</tbody>
</table>
ABSTRACT

A multiple-reflection analysis has been performed with the assumptions of gray, Lambertian walls and a cold aperture, and the result is a compact, closed-form expression that emphasizes the geometrical effects of the projected solid angle as seen from various points on the back wall of the cavity. Variations of this expression give special forms with terms representing temperature variations parallel to and perpendicular to the cavity wall. Other variations provide for analysis according to the radiation laws of Stefan-Boltzmann, Planck, Wien, or Rayleigh-Jeans. This theory is characterized as being approximate, because the solutions contain a special mean temperature and a special mean value of the projected solid angle; and although these mean values can be approximated, they cannot, in general, be determined precisely.

A simplified version of the theory has been shown to be valid to an accuracy of about 2% for good-quality blackbody simulators used in typical practical applications. This simplified expression has been compared with experimental data for conical, cylindrical, and spherical cavities, and the basic characteristics have been confirmed concerning the angular distribution of radiation from blackbody simulator cavities with these three shapes.

The form of the approximate isothermal solution of this dissertation is such that it lends itself to convenient comparisons with the analyses of 21 other papers. Informative, critical comparisons are
stimulated by the close similarities that are brought out in this dissertation among these diverse papers.

Recommended procedures are given for determining the approximate value of the effective emissivity of blackbody simulator cavities.
INTRODUCTION

The motivation for this dissertation occurred in December 1956 when I received the graph that is reproduced here as Fig. 1. It shows the output signal as a function of angle for the best blackbody simulator that was commercially available at that time. Note that at 5° to the left (R in Fig. 1) and 10° to the right (L in Fig. 1), the signal falls off more than three times as much as it would if the source obeyed the Lambert cosine law.

At the time these data were obtained, there was a consensus of opinion within the infrared community that almost perfect blackbody simulators were easy to build. Therefore, the relatively large disagreement in Fig. 1 between the data and the cosine law was somewhat of a paradox. My explanation of the situation was that the blackbody simulator theories in vogue at that time solved the wrong problems, and the optimistic solutions to these wrong problems led to overly optimistic opinions about the quality of blackbody simulators and the ease with which they might be built. The most popular theory in 1956 for the design and evaluation of blackbody simulators was that of Gouffé (1945). His theory determines a single figure of merit for a given combination of cavity shape and wall emissivity; his figure of merit is the isothermal cavity emissivity as seen by a small, on-axis detector that views only paraxial rays. Blackbody simulators are commonly used with collimators having f/numbers in the vicinity of f/4, and in the format of Fig. 1 that involves data
Fig. 1. Graph of Cosine Distribution of Blackbody Simulator Radiation Source.

Circles show experimental data for best blackbody simulator radiation source that was commercially available in 1956; solid curve represents theoretical values.
points out to 7° R and 7° L. Figure 1 shows that paraxial rays alone do not properly describe that blackbody simulator when rays 7° to the right or 7° to the left are to be used.

The insufficiency of this single figure of merit is further shown by comparing typical examples with typical radiometric measurement accuracies. Table 1 gives the isothermal, on-axis cavity emissivities for several different blackbody-simulator cavities, calculated on the basis of Gouffé's theory. Incidental to this comparison I have found errors in Gouffé's work (described and corrected in this dissertation) that affect some of his tabulated results, and the corrected values are given in parentheses alongside the uncorrected values. Note that, according to this most popular blackbody simulator theory of 1956, typical blackbody simulator designs had emissivities very close to 1.000.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Wall emissivity</th>
<th>Cavity emissivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>28.07° Cone</td>
<td>4.0 0.9</td>
<td>0.992 (0.998)</td>
</tr>
<tr>
<td></td>
<td>4.0 0.8</td>
<td>0.979 (0.996)</td>
</tr>
<tr>
<td>14.25° Cone</td>
<td>8.0 0.9</td>
<td>0.997 (0.999 788)</td>
</tr>
<tr>
<td></td>
<td>8.0 0.8</td>
<td>0.992 (0.999 523)</td>
</tr>
<tr>
<td>Cylinder</td>
<td>4.0 0.9</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td>4.0 0.8</td>
<td>0.984</td>
</tr>
<tr>
<td></td>
<td>8.0 0.9</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>8.0 0.8</td>
<td>0.994</td>
</tr>
<tr>
<td>Sphere</td>
<td>4.0 0.9</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>4.0 0.8</td>
<td>0.986</td>
</tr>
<tr>
<td></td>
<td>8.0 0.9</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>8.0 0.8</td>
<td>0.996</td>
</tr>
</tbody>
</table>

\[ L/R = \text{(length)}/(\text{aperture radius}) \]
The entries in Table 1 should be viewed in the perspective of the kinds of radiometric measurement accuracies that were obtainable at that time. Eleven years later, Nicodemus (1967, p. 263) had this to say:

Radiometry enjoys the dubious distinction of relatively poor attainable precision and accuracy, especially when contrasted with the measurement of frequency or time, ... where one part in $10^{10}$ or even $10^{11}$ is achieved. In radiometry, accuracies of a few percent or more are often acceptable as being results of careful work, and extreme precautions are required to achieve a fraction of one percent, or even one percent.

Table 1 and Nicodemus' statement show how in 1956 the infrared community might have accepted the view that, compared to the accuracy of infrared measurements, the quality of blackbody simulators was very good.

Throughout 1956, I had been skeptical about that widely held belief, and when the information in Fig. 1 became available in December of that year, I was convinced that a different kind of theory was called for. The new theory, I felt, should not provide a single, on-axis figure of merit for each cavity; on the contrary, it should forecast the variations in cavity output as a function of angle as shown in Fig. 1. In 1958, I made the first step toward such a theory—a theory based on the projected solid angle (Nicodemus, 1968, p. 1361) of the cavity aperture as seen from different points on the back wall of the cavity. This approach was first published in 1972 (Bartell and Wolfe, 1972), but I have used it since 1958 to design and compare blackbody simulators.

Among the conclusions that could be drawn from that projected solid-angle analysis were that there are special advantages in spherical cavities and special disadvantages in conical cavities.

Those nascent ideas have been extended so that the more comprehensive blackbody simulator theory now includes analytical considerations
of the cavity effect as well as the original analytical considerations of tendencies toward certain types of temperature distributions. Results are presented in terms of effective cavity emissivity.

The theory has been extended (Bartell and Wolfe, 1976a) from its original dependence on the Stefan-Boltzmann law, so it can now be used equally well with other radiation laws such as those of Wien, Rayleigh-Jeans, and Planck.

The theory has also been extended (Bartell and Wolfe, 1976a) to cover temperature differences both parallel to and perpendicular to the cavity wall surface.

Comparisons have been made between this theory and 21 other papers on cavity theory. Previously unsuspected similarities have been found among these papers (Bartell and Wolfe, 1976b).

Polythermal cavity effects have been analyzed.

Experimental verification for the theory has also been made (Bartell and Wolfe, 1976a).

The rest of this dissertation describes the extended theory and the experimental findings, and relates them to the work of others.
The Concept of a Blackbody

The concept of a blackbody was originated by Kirchhoff (1860).

Figure 2 is a reproduction of his definition of a blackbody.

§ 1. Before the body C (fig. 1), imagine two screens, \( S_1 \) and \( S_2 \), to be placed, containing openings 1 and 2, whose dimensions must be regarded as infinitely small in comparison with their distance apart, and each of which has a middle point. Through these two openings a pencil proceeds from the body C. Of this pencil let that part be considered which consists of waves, the length of which lies between \( \lambda \) and \( \lambda + \lambda \), and let this be divided into two component parts polarized in the perpendicular planes \( a \) and \( b \) passing through the axis of the pencil. Let the intensity of the part polarized in \( a \) be \( E \lambda \). Let \( E \) be then the radiating power of the body.

Conversely, a pencil of rays polarized in plane \( a \) and having waves of the length \( \lambda \), falls on the body C through the openings 2 and 1. Of this, part is absorbed by the body, the rest being partly reflected and partly transmitted. Let the ratio of the absorbed to the incident rays be called \( A \); then \( A \) will represent the power of absorption of the body.

The magnitudes \( E \) and \( A \) depend on the nature and temperature of C, on the position and form of the openings 1 and 2, on the magnitude \( \lambda \), and on the position of the plane \( a \). It will be shown that the ratio of \( E \) to \( A \) is independent of the nature of the body; it will then necessarily follow that it cannot be affected by the position of the plane \( a \), and its independence of the position and form of the openings 1 and 2 will then be easily deduced, so that it only remains to be determined how far it depends on the temperature of C and the wave-length \( \lambda \).

The proof I am about to give of the law above stated, rests on the supposition that bodies can be imagined which, for infinitely small thicknesses, completely absorb all incident rays, and neither reflect nor transmit any. I shall call such bodies \textit{perfectly black}, or, more briefly, \textit{black} bodies. It is necessary in the first place to investigate the radiating power of bodies of this description.

§ 2. Let \( C \) be a black body. Let its radiating power (generally indicated by \( E \)) be called \( e \). It will be shown that \( e \) remains the same when \( C \) is replaced by any other black body of the same temperature.
Note that Kirchhoff defines a blackbody as an ideal or imagined body. I use the same definition. I shall extend the idea by using the term blackbody simulator for real hardware that is called on to function as a blackbody. Blackbody simulators are commonly called blackbodies, but that is poor usage. Hudson (1969, p. 67) presents an interesting discussion of that point. According to my definition, and Kirchhoff's, a real body can approximate or simulate a blackbody, but it cannot be a blackbody.

**Spectral Blackbody Theory**

During the last decade of the nineteenth century, there was a great surge of activity, both theoretical and experimental, concerning blackbodies and blackbody simulators. The theories gave the variation with wavelength of the energy emitted from a blackbody. I call these theories spectral blackbody theories. Examples are the Rayleigh-Jeans, Wien, and Planck laws (Forsythe, 1937). The activity at the end of the nineteenth century and on into the twentieth century had to do with finding the spectral blackbody theory that would most closely agree with experimental measurements. The climax of these investigations was Planck's pronouncement of his quantum theory of radiation (Planck, 1900a, 1900b, 1972); that theory is universally accepted today and is generally considered to be the cornerstone of modern physics.

For about 60 years after Planck's landmark work, there was little change in the status of spectral blackbody theory. Since about 1960 there has been an increasing amount of research into the coherence properties of the Planckian blackbody field, the effects of the cavity wall,
and the effects of finite-size and of small-size cavities. The earliest papers on coherence effects are those by Bourret (1960), Kano and Wolf (1962), Sarfatt (1963), and Mehta and Wolf (1964a, 1964b). Significant papers on surface corrections and the effects of small- or finite-sized cavities include those by Baltes and Kneubühl (1971, 1972), Caren (1972), and Steinele, Baltes, and Pabst (1975, 1976). These recent developments in spectral theory are reviewed by Baltes (1977). It should be noted that these papers complement and extend the Planck theory, and the recent theoretical work is in harmony with, not in conflict with, the ideas of 75 years ago.

**Blackbody Simulator Theory**

A second kind of theory associated with blackbodies deals with the closeness to perfection of real or ideal cavities in simulating blackbodies. This kind of theory can be called *blackbody simulator theory*, and it is the kind that is investigated in this dissertation.

The development of blackbody simulator theory has been less orderly than that of spectral blackbody theory. It has been marked by serious mistakes and misinterpretations, and there remain important areas where there are differences of opinion concerning correctness and importance.

The history of blackbody simulator theory can be divided into three periods:

(1) The preliminary period: 1892 to 1927

(2) The precomputer period: 1927 to 1960

(3) The computer period: 1960 to present.
The Preliminary Period: 1892 to 1927

The earliest bona fide example of blackbody simulator theory was by Buckley (1927), but between 1892 and 1927 there were several papers on cavity geometry and interreflections that are appropriate for inclusion in this history.

Sumpner (1892, 1893) showed that, for a surface that radiates according to a cosine law, radiation reflected from any part of a spherical cavity is uniformly distributed over the spherical surface (Walsh, 1919/1920; Williams, 1961; Williams and Becklund, 1972).

Nicodemus (1968, p. 1361) says the following about these two papers of Sumpner:

An interesting property of a spherical surface, first presented by Sumpner [1892, 1893], is that the projected solid angle subtended at any surface element $dA$, by any portion of the same spherical surface, is determined completely by its area, irrespective of the configuration (its shape and its position, with respect to $dA$, on the spherical surface).

Nicodemus' statement about the projected solid angle is true, and it is very important for blackbody simulator cavities. However, it would have been better if he had said: "An interesting property of a spherical surface, the foundation of which was developed by Sumpner, . . . ."

Walsh (1919/1920) found a closed-form expression for the flux received from a disc (radiating according to a cosine law) by a second parallel and coaxial disc. The reader is referred to Williams (1961) and Williams and Becklund (1972) for a brief discussion of this work including Walsh's expression.

Bartlett (1920) found an interesting expression for the radiation received from an open-ended isothermal cylinder by a coaxial disc.
that has a radius not larger than the cylinder radius and that is located outside the cylinder. He found that the amount of radiation would be equal to the difference between the two amounts that would be received if the cylinder were replaced, successively, by two discs, each with the same radius, temperature, and emissivity as the inner wall of the cylinder, with one disc located at one end of the cylinder and the other disc at the other end of the cylinder. (See Williams, 1961, and Williams and Becklunand, 1972, for a brief but comprehensive discussion of this paper.)

Yamauti (1926) performed a detailed analysis of "The Light Flux Distribution of a System of Interreflecting Surfaces," and his final expression of illumination resembles a solution of the Fredholm equation of the second kind (Pogorzelski, 1966, p. 41):

\[ \varepsilon(x) = E(x) + \int_S \varepsilon(x, y) \, \varepsilon(y) \, dS_y \]

where \( E(x) \) and \( \varepsilon(x) \) are the initial and final illumination at \( x \), and \( \varepsilon(x, y) \) is a function that gives the light flux per unit area received at \( x \) from unit area at \( y \) when the illumination at \( y \) is unity (see Buckley, 1928a, p. 890). Yamauti's result is:

\[ \varepsilon(x) = E(x) + \frac{1}{P} \int_S P(y, x) \, E(y) \, dS_y \]

where the \( P \) and \( P(y, x) \) resemble quantities that are defined by Fredholm's series. However, Yamauti (as well as Buckley, 1928a, discussing Yamauti) does not distinguish between one-dimensional and two-dimensional variables of integration, as is done in Appendix C; Yamauti does not mention
integral equations at all. Buckley (1928a, p. 890) describes Yamauti's paper as follows:

Yamauti's paper is an exceedingly able one and very elegant from the point of view of formal mathematics, but in giving the solution of the problem in the Fredholm form and not giving the integral equation itself the value of his paper is somewhat obscured. Indeed, the Fredholm solution . . . is so complicated that it can only be regarded as an existence theorem and it is doubtful whether it will prove of any assistance in solving even the artificially simplified problems to which we are reduced whenever we meet the difficulties of practical cases.

The Precomputer Period: 1927 to 1960

Between 1927 and 1960, blackbody simulator theory became clearly identified and a number of different methods of analysis were introduced.

Buckley (1927) showed in his Figure 4 the distribution of radiation near the end of a uniformly heated infinite cylinder, and the curves are plotted as fractions of $cT^n$. This is an expression of local effective hemispherical emissivity, and it is the earliest paper I have found on what I call "blackbody simulator theory"--the theory that deals with the closeness to perfection of real or ideal cavities in simulating blackbodies. Buckley sets the problem up as an integral equation of the steady-state conditions, and then expands the kernel as a series of exponentials.

In a later paper, Buckley (1928a) treats six different cavity shapes by integral equation techniques, and he uses several different mathematical methods for the solutions. He describes his integral equation formulation of the cavity radiation problem, and he reports that he has obtained the same equation from both a multiple reflection analysis and a steady-state model. He also states that the Fredholm solution to
the corresponding Fredholm integral equation of the second kind is the same expression that Yamauti gives as his final result, but he does not comment on differences between one-dimensional and two-dimensional variables of integration. In the appendix to his paper, Buckley synthesizes a Fredholm integral equation of the second kind from multiple-reflection considerations, and he notes that the Liouville-Neumann method of solution of that equation determines the successive reflected radiation terms. This is one of the very first bona fide papers on blackbody simulator theory, and yet it is remarkably comprehensive.

Moon (1940), in a paper entitled "On Interreflections," solves for the radiant exitance rather than for an emissivity. He formulates the problem as an integral equation resembling the Fredholm integral equation of the second kind and then converts to that standard form. He uses Hilbert-Schmidt theory for his solutions, and gives a table of kernels for nine different interreflection configurations.

Moon's paper has some valuable mathematical considerations for readers who are interested in blackbody simulator theory, but it presents only limited results in final form. Conversion of the integral equation to standard Fredholm form has been mentioned. Moon describes how a single integral equation can be used for a "surface that is composed of any number of flat or curved pieces with a different reflection factor on each." (This argument is called upon in Appendix C of this dissertation to replace a set of integral equations by a single integral equation.) Moon gives a number of mathematical theorems to aid in the Hilbert-Schmidt analysis.
Moon's results in terms of radiant exitance (he calls this "luminosity") are limited to a sphere, any complete enclosure with constant initial radiant exitance, and an infinitely long semicircular cylinder.

A final note of interest concerning Moon's paper is the fact that he gives short, two- and three-sentence reviews of nine previous papers related to his work (Moon, 1940, footnote 1).

The paper of Gouffé (1945) is perhaps the best known and most widely cited on blackbody simulator theory. It is important to this dissertation, and it is the subject of Appendix D.

The paper of De Vos (1954a) is also well known, and is also important to this dissertation. It is the subject of Appendix E.

Vollmer (1957) studies a circular cylinder. He formulates the problem in terms of a pair of integral equations, and his solution has the form of an exponential and a sum of exponentials. Vollmer's analysis follows a paper by Buckley (1934) that treats a cylinder closed on one end and that is an extension of an earlier study (Buckley, 1928b) of an infinite cylinder.

Vollmer also performed an experimental study of the radiometric output of a cylindrical cavity with one end closed. This is one of the first papers that compare blackbody simulator theory with corresponding measurements.

An interesting part of Vollmer's paper concerns a theoretical and experimental treatment of temperature gradients within the cavity walls. His measurements are made from thermocouples that are imbedded different distances in the walls. The only papers I have found that
follow up this important part of blackbody simulator theory, design, and evaluation are Geist (1973, p. 1327), Bartell and Wolfe (1974, p. 111), and Bartell and Wolfe (1976a, p. 16).

Williams (1961) presents an outstanding review article of most of the important papers on blackbody simulator theory published up to 1957. (His material is presented in a slightly modified form in Williams and Becklund, 1972.) He includes key analytic expressions, figures, and tables from the reviewed articles, plus a few analytical and comparative comments of his own. (A mistake I have found in this review is described in detail in the section entitled "Previous Comparisons," p. 66.) It is appropriate that this fine review article be the last in what I call the precomputer period. Although it was published in 1961, none of the reviewed articles is more recent than 1957, and computer solutions are mentioned only in the last paragraph, for possible future studies.

The Computer Period: 1960 to Present

The first paper I have found describing computer techniques to determine cavity emissivity is that by Sparrow and Albers (1960). Since 1960 there have been some important publications that do not involve computer solutions—such as Nicodemus (1968) on spherical cavities and the Bartell and Wolfe papers (1976a, 1976b) with their closed-form, approximate solutions—but the standards of accuracy for all papers on blackbody simulator theory have been significantly higher since Sparrow and his coworkers introduced computer computations into this field.

Sparrow and Albers (1960), Sparrow, Albers, and Eckert (1962), and Sparrow and Jonsson (1963) use computer techniques to solve integral
equation formulations of the cavity problem. These three papers deal with, respectively, the semi-infinite cylinder, the finite-length cylinder, and the cone.

Sparrow (1965) considers the nonisothermal cylinder and introduces the concept of effective emissivities that may be greater than 1.0. That idea is based on his definition of apparent emissivity (he calls it "apparent emittance"), which is the sum of emitted and reflected radiation divided by the blackbody radiation associated with a reference temperature. If the reference temperature is the local temperature, and if all the surrounding temperatures are higher, then the effective emissivity may be greater than 1.0.

Lin and Sparrow (1965) determine the emissivity of cones and cylinders having walls that emit diffusely but reflect specularly.

Kelly and Moore (1965) provide the most accurate comparison I have found between blackbody simulator theory and experiment. The theories of Buckley (1927, 1928b, 1934), Gouffé (1945), and Sparrow, Albers, and Eckert (1962) are compared against experimental data for shallow cylindrical cavities. Length-to-radius ($L/R$) ratios of 0.5, 1.0, 1.5, and 2.0 are studied. Buckley's theory shows the poorest agreement: the difference between theory and experiment amounts to several per cent at $L/R$'s of 2.0 and 1.5, and as much as 7% at $L/R$ of 0.5. For the on-axis Gouffé theory, Kelly and Moore derived an off-center correction, and for this modified Gouffé theory the agreement with experiment is better than 1%. For Gouffé's theory without modification the agreement is still better than 1% for $L/R$'s of 2.0 and 1.5, about 1% to 2% for $L/R$ of 1.0,
and about 2% to 3% for $L/R$ of 0.5. For the Sparrow, Albers, and Eckert theory the agreement with experiment is better than 1%.

Nicodemus (1968) derives an exact expression for the emissivity of an ideally diffuse, gray, isothermal spherical cavity. This paper has a number of very important features, summarized briefly here as follows:

1. For the initial assumptions of diffuse, gray, isothermal, spherical cavity, the analysis is exact.

2. Nicodemus' approach lends itself to extension beyond his initial assumptions. This dissertation follows his method and generalizes it to include polythermal, nonspherical cavities. (The analysis of this dissertation begins with nondiffuse, nonuniform surface assumptions, but rather early a uniform Lambertian wall surface is assumed.)

3. Cavity emissivity is expressed as a ratio of radiances, and that is the best way to describe the emissivity of a cavity.

4. Nicodemus says: "It appears that, even under ideal conditions, the spherical configuration is the only one that will have uniform isotropic emissivity over a wide angle, approaching a full hemisphere, across the entire aperture" (Nicodemus, 1968, p. 1359).

5. The appendix to Nicodemus' paper gives an informative discussion of projected solid angles and radiance. It contains the statement about projected solid angles and spherical cavities that has already been quoted in the discussion of Sumpner's papers (p. 9).

Sydnor (1969) expands on the point mentioned in the appendix to Buckley's paper (1928a) that the integral-equation formulation of the
cavity problem can be reduced using the Liouville-Neumann series solution to the Fredholm integral equation of the second kind.

Bedford (1970) provides an excellent review of the papers by Gouffé (1945), De Vos (1954a), Sparrow and Albers (1960), Sparrow, Albers, and Eckert (1962), Sparrow and Jonsson (1963), Sparrow (1965), and others. Among the features of his article are the following:

(1) A footnote about a discontinuity in mathematical logic by Gouffé. I call this step Gouffé's surprise, and explain it in detail in Appendix D.

(2) The third approximation in De Vos' series. De Vos (1954a) gives only the first and second approximations.

(3) A brief description of integral equation treatment of the cavity problem including some of the characteristics, advantages, and problems.

Bedford and Ma (1974, 1975, 1976) use integral equation techniques to solve the problems of cones, cylinders, cylinder-cone combinations, and double cones. Features of these papers include the following:

(1) Reasons that the diffuse model is a good approximation for blackbody simulator theory.

(2) Discussions of singular points and how they are treated.

(3) Analysis of both isothermal and nonisothermal cavities.

(4) Emissivities that can be greater than 1.0. They are given for nonisothermal cavities under conditions similar to those of Sparrow (1965).

Papers by Bartell and Wolfe (1976a, 1976b) have been described on page 5 of this dissertation.
TheorY

Basic Analysis

Blackbody simulator theory, as already noted, describes how closely the performance of a blackbody simulator approaches that of an ideal blackbody. In this dissertation this comparison is expressed as an emissivity based on radiance; other types of emissivity that are often used are discussed in the section entitled "Emissivity" (p. 56).

Blackbody simulators are traditionally made from cavities, but other configurations are possible and occasionally a simple blackened surface is used. The blackbody simulator theory discussed here is a theory of cavity radiation. It is based on multiple reflections within the cavity. The present analysis is similar to Nicodemus' (1968) study of an isothermal, spherical cavity; it is in fact an extension of Nicodemus' treatment, to include polythermal cavities and nonspherical shapes. We begin with non-Lambertian wall surfaces, but most of our study, like all of Nicodemus', deals with Lambertian walls. Our first equation, referring to the physical situation shown schematically in Fig. 3, gives \( L(\vec{r}) \), the radiance along the vector \( \vec{r} \); it is the same as Nicodemus' Eq. (3):

\[
L(\vec{r}) = L_{10} + L_{210} + L_{3210} + \cdots + L_{n(n-1)\cdots 210} + \cdots ,
\]

where

\[ L_{10} = \text{radiance along } \vec{r} \text{ due to emission at } \alpha_1, \]
$L_{210} = \text{radiance along } \mathbf{r} \text{ due to emission from the entire cavity followed by a single reflection at } \alpha_1,$

$L_{n(n-1)\ldots210} = \text{radiance along } \mathbf{r} \text{ due to emission from the entire cavity followed by } (n-2) \text{ reflections over the entire cavity followed by a single reflection at } \alpha_1.$

For gray wall surfaces:

$$L_{n(n-1)\ldots210} = \int \frac{f_{210}}{h} \int \frac{f_{321}}{h} \ldots \int \frac{\varepsilon_{n(n-1)}f_{n(n-1)(n-2)}}{h}$$

$$\cdot \pi^{-1} \sigma T_n^4 \, d\Omega_n(n-1) \ldots d\Omega_2,$$

where $\int_{h}$ represents integration over a hemisphere; $f_{n(n-1)(n-2)}$ is the bidirectional reflectance-distribution function (BRDF) (Nicodemus, 1970) at $\alpha_{n-1}$ for radiation arriving from the direction of $\alpha_n$ and departing in the direction of $\alpha_{n-2}$; $\varepsilon_{n(n-1)}$ is the directional emissivity of the wall surface from $\alpha_n$ in the direction of $\alpha_{n-1}$; $\sigma$ is the Stefan-Boltzmann constant; $T_n$ is the temperature at $\alpha_n$; and $d\Omega_n(n-1)$ is the element of
projected solid angle (Nicodemus, 1968) at \(a_{n-1}\) subtended by an element of area at \(a_n\).

If \(L^{bb}(T_i)\) is the radiance from a blackbody at temperature \(T_i\), where \(i\) is any integer, then

\[
L^{bb}(T_i) = \sigma T_i^4 \pi^{-1}.
\]

The first expression of cavity emissivity is thus

\[
\varepsilon_L(\hat{r}, T_1) \equiv \frac{L(\hat{r})}{L^{bb}(T_1)}
\]

\[
= \varepsilon_0 + \int_{\mathcal{h}} \varepsilon_{21} \int_{\mathcal{h}} \varepsilon_{32} \int_{\mathcal{h}} \varepsilon_{n} \frac{T_2}{T_1} \frac{T_3}{T_1} \frac{T_n}{T_1} \, d\Omega_{21} \, d\Omega_{32} \, d\Omega_{21}
\]

\[
+ \cdots + \int_{\mathcal{h}} \int_{\mathcal{h}} \int_{\mathcal{h}} \varepsilon_n (n-1) \frac{T_n}{T_1} \, d\Omega_{21} \cdots d\Omega_{21} \cdots (T_n/T_1)^4 d\Omega_{n-1} \cdots d\Omega_{21} + \cdots .
\]

The important assumptions used in arriving at Eq. (2) are that the walls are gray, and the small areas \(a_n\) on the back wall associated with elements of projected solid angle \(d\Omega_{(n+1)n}\) must not include any
points of discontinuity such as the apex of a cone or the seam of a cylinder. The integrals \( \int d\Omega_{(n+1)n} \) are not defined for such points. When points of discontinuity are encountered, the \( \alpha_n \) must be redefined as a number of smaller areas \( \alpha_{ni} \) that exclude the discontinuity points, and the original integral containing the discontinuity points in \( \alpha_n \) must be replaced by a sum of integrals that do not include the discontinuity points in their \( \alpha_{ni} \). Equation (2) then describes all of the cavity wall except the points of discontinuity.

The general nature of Eq. (2) makes it applicable to most cavity situations including those that involve walls with directional reflectances and emissivities. However, with the many terms, the different \( e_{ij} \)'s and \( f_{i,j,k} \)'s in each term, and the many different values of each \( e_{ij} \) and each \( f_{i,j,k} \), it is proper that we seek some type of simplification before proceeding. The most popular set of restrictions to Eq. (2) is the assumption that the cavity wall surfaces are uniform and Lambertian and the assumption that the radiation coming in through the aperture can be neglected. Then

\[
\begin{align*}
\epsilon_n(n-1) &= \epsilon = \text{constant}, \\
\bar{f}_n(n-1)(n-2) &= \bar{f} = (1-\epsilon)\pi^{-1} = \text{constant}, \\
\int_{\bar{h}} &= \int_{\text{wall}}.
\end{align*}
\]

Bedford and Ma (1974, p. 339) present the following justification for assuming a diffuse wall surface:
A completely general calculation of the emissivity of even a simple cavity is a problem of considerable mathematical complexity and, within most present-day budgets, of prohibitive cost. Consequently, the calculations must comprise a number of assumptions and a good deal of simplification. In this paper, we limit the discussion to cavities with walls that emit and reflect diffusely. We also assume that the wall emissivity is uniform (i.e., has the same value at different locations on the wall), but this is not particularly restrictive, and we do admit the possibility of different emissivities for different surfaces in the cavities.

It is difficult to estimate to what extent the effective emissivity of a diffuse cavity differs from that of a real one. A few calculations . . . have replaced purely diffuse reflection with purely specular reflection, or with a mixture of diffuse plus specular reflection, obtaining for corresponding cavities generally higher effective emissivities. However, experimental verification is lacking. In experiments to compare measured and predicted values of local irradiance, Schornhorst, Toor, and Viskanta . . . found it necessary to use highly reflecting surfaces with significant specular components arranged in open configurations, to distinguish between various theoretical models. Even then, the diffuse model is, by and large, a fair approximation. These are just the opposite of the conditions desired in a blackbody source, so it appears likely that in this case the diffuse model will be a good approximation.

We will follow the assumptions of Eqs. (3), but for different reasons. One reason is that the combination of Eq. (2), restricted by Eqs. (3), leads to the useful compact approximate solutions of pages 25 to 39. A second reason is the fact that more than one third of this dissertation and its appendixes is devoted to discussions of comparisons among this and other blackbody simulator theories, and nearly all of these other theories rely on Eqs. (3). In following this course, it is reassuring to know that Bedford and Ma commend it as a "good approximation."

De Vos (1954a) uses a different technique to arrive at an equation that retains the directional characteristics of emissivity and reflectance and yet is far less complicated than Eq. (2). He simply limits
the number of terms in what would otherwise be an infinite series. Note that he has a different series from mine (unity minus small correction terms), but we could follow his procedure to simplify Eq. (2) by using only a few terms. De Vos' cavity emissivity research (De Vos, 1954a) was carried on in conjunction with a study of the emissivity of tungsten ribbon (De Vos, 1954b), and he wanted to take into account the partial specularity of ordinary metal surfaces. Bedford (1970, p. 179) gives a valuable review of De Vos' paper (1954a) and of others who have followed De Vos' analysis, and he concludes that application of De Vos' equation "is, in practice, formidable."

Lin and Sparrow (1965) find the emissivity of conical and cylindrical cavities whose walls emit diffusely but reflect specularly. Their formulation of the problem is different from Eq. (2), but their assumptions suggest a third way that Eq. (2) might be simplified, and that is by assuming some functional forms for the $\varepsilon_n(n-1)$ and $f_n(n-1)(n-2)$ that would reduce the complexity of Eq. (2) sufficiently to allow it to be solved.

It seems that the best way to simplify Eq. (2) is to use Eq. (3), and we will proceed in this way.

Equation (2) then becomes:
\[\varepsilon_L(\mathbf{x}, T_1) = \varepsilon + \varepsilon(1-\varepsilon) \pi^{-1} \int_{\text{wall}} (T_2/T_1)^4 \, d\Omega_{21} \]
\[+ \varepsilon(1-\varepsilon)^2 \pi^{-2} \int_{\text{wall}} \int_{\text{wall}} (T_3/T_1)^4 \, d\Omega_{32} d\Omega_{21} \]
\[+ \ldots \]
\[+ \varepsilon(1-\varepsilon)^{n-1} \pi^{1-n} \int_{\text{wall}} \ldots \int_{\text{wall}} (T_n/T_1)^4 \, d\Omega_{n(n-1)} \ldots d\Omega_{21} \]
\[+ \ldots . \]  

(4)

Key steps in the solution of Eq. (4) are the application of mean value theorems of integral calculus (Courant, 1936) to replace \(T_2 \ldots T_n \ldots\) inside the integrals by \(T_m\) outside the integrals and to unfold the nested integrals thus:

\[
\int_{\text{wall}} \ldots \int_{\text{wall}} d\Omega_{n(n-1)} \ldots d\Omega_{21} = \left( \pi - \int_{\text{ap}} d\Omega_n \right)^{n-2} \left( \pi - \int_{\text{ap}} d\Omega_{21} \right),
\]

(5)

where \(T_m\) is a weighted average cavity temperature, \(^1\) \(\int_{\text{ap}}\) indicates

1. The weighted average cavity temperature \(T_m\) is determined for a given cavity evaluation point; and areas of the cavity wall are weighted, for the most part, according to their projected solid angle (Nicodemus, 1968, p. 1361) as seen from the cavity evaluation point.
integration over the aperture, and \( \int_{\text{ap}} \) is a weighted average value of \( \int_{\text{ap}} d\Omega \). The solution of Eq. (4) is

\[
\varepsilon_L(\mathbf{r}, T_1) = \varepsilon + (T_m/T_1)^4 (1-\varepsilon) - (T_m/T_1)^4 \frac{(1-\varepsilon)}{\pi} \int_{\text{ap}} d\Omega
\]

\[
= \frac{(T_m/T_1)^4}{\pi \varepsilon \pi} \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\text{ap}} d\Omega \left(1 - \frac{1}{\pi} \int_{\text{ap}} d\Omega\right)
\]

\[
= \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\text{ap}} d\Omega
\]

Equation (6) may also be written in the form

\[
\varepsilon_L(\mathbf{r}, T_1) = \varepsilon + \frac{(T_m/T_1)^4 (1-\varepsilon)}{\pi} \left(1 - \frac{1}{\pi} \int_{\text{ap}} d\Omega\right)
\]

\[
= \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\text{ap}} d\Omega
\]

Comments on Eqs. (4) through (6a)

The analytical steps from Eqs. (4) through (6a) are the mathematical heart of this dissertation. The mathematics are presented

1. The average integral is a weighted average of the projected solid angle of the aperture as seen from a point on the cavity wall. After a cavity evaluation point has been selected, the projected solid angle of the aperture is then considered for all points on the cavity wall. A weighted average of these projected solid angles is determined by weighting areas on the cavity wall, for the most part, according to their projected solid angle as seen from the cavity evaluation point.
separately in Appendix A; the physical interpretation and the applications are described here.

The form of Eq. (6) might raise the interesting criticism that Eq. (6) is not a significant development from Eq. (4) since $T_m$ and $\int_{ap} d\Omega$ might be defined just to make Eqs. (4) and (6) agree. This type of objection is strengthened when it is realized that instead of appealing to mean value theorems of integral calculus for the validity of Eqs. (A-2) and (A-5) in Appendix A, one might write them as defining equations for $\overline{T}$ and $\int_{ap} d\Omega_{n(n-1)}$, thus:

$$
\overline{T}_{2}^4 \equiv \int_{wall} \frac{T_{2}^4 d\Omega_{21}}{\int_{wall} d\Omega_{21}}
$$

$$
\int_{ap} d\Omega_{n(n-1)} \equiv \int_{wall} \frac{d\Omega_{n-1} d\Omega_{n-1}(n-2)}{\int_{wall} d\Omega(n-1)(n-2)}
$$

It might be said that an equation that cannot be solved [Eq. (4)] has been replaced by two quantities that cannot be evaluated ($T_m$ and $\int_{ap} d\Omega$)! But that point of view does not take into account the usefulness of Eq. (6) as developed in this dissertation and summarized as follows:

(1) The form of Eq. (4) is a clumsy infinite series, and it is almost impossible to use it to study real problems. Equation (6), on
the other hand, is compact, and except for \( T_m \) and \( \int_{\text{ap}} d\Omega \), all its elements can be found rather easily.

(2) For practical blackbody simulator cavities, \( T_m \) can nearly always be estimated within rather close limits; \( \int_{\text{ap}} d\Omega \) also can be estimated reasonably well.

(3) For some cavity situations, \( T_m \) and \( \int_{\text{ap}} d\Omega \) can be found quite accurately. (See the chapter entitled "Four Special Cavity Shapes," page 71.)

(4) The form of Eq. (6) leads to interesting comparisons with other theories, and by appealing to these other theories, we can gain knowledge about the troublesome \( T_m \) and \( \int_{\text{ap}} d\Omega \). (See the section entitled "Comparison Summary," page 64.)

(5) More accurate determinations of \( T_m \) and \( \int_{\text{ap}} d\Omega \) are described in the chapter on "The Quantities \( T_m \) and \( \int_{\text{ap}} d\Omega \)," page 76.

In conclusion, we should comment on the physical significance of the individual terms in Eqs. (6) and (6a). The first term in both equations corresponds to the radiation emitted from a surface at temperature \( T_1 \) with emissivity \( \varepsilon \). The second term in Eq. (6) corresponds to the radiation originating in the cavity at a weighted mean temperature \( T_m \) and then reflected from the surface with reflectance \( 1-\varepsilon \). The third term in Eq. (6) corresponds to the deficiency in reflected radiation due to the aperture. The fourth term is a higher-order multiple-reflection correction. Note that nearly all cavity geometrical effects are manifest in \( \int_{\text{ap}} d\Omega \) and \( \int_{\text{ap}} d\Omega \). The second term in Eq. (6a) is equivalent to the second, third, and fourth terms in Eq. (6), and this second term of Eq. (6a) may be thought of as an expression of the cavity effect.
Isothermal and Polythermal Cavities:
More Comments about Eq. (6)

The steps leading to Eq. (6) have been described as the mathematical heart of this dissertation, and correspondingly Eq. (6) is the most important mathematical development. The next few sections present specialized analyses based on Eq. (6) involving temperature variations and several radiation laws. Most of the rest of this dissertation compares the present analysis with the blackbody simulator papers of many other authors, including Williams (1961), Buckley (1928b), Gouffé (1945), Moon (1940), DeVos (1954a), Peavy (1966), Treuenfels (1963), Kelly (1966), Nicodemus (1968), Vollmer (1957), Sparrow and Jonsson (1963), Sparrow, Albers, and Eckert (1962), Sydnor (1969), Chandos and Chandos (1974), Bedford and Ma (1974), and Bedford (1970). Most of these authors (e.g., Treuenfels, Kelly, Williams, Buckley, Bedford, Vollmer, Nicodemus, Chandos and Chandos, Sydnor, and Sparrow, Albers, and Eckert) have considered only isothermal cavities, and others (e.g., Gouffé, De Vos, Peavy, and Sparrow and Jonsson) have made their polythermal analyses a somewhat separate part of their treatment. Only a few papers (e.g., Bedford and Ma), like this dissertation, solve, first, the polythermal cavity problem, and then proceed to the isothermal cavity as a special case. This dissertation emphasizes the importance of variations in blackbody simulator output as a function of azimuth angle as illustrated by Fig. 1; and a major factor in these variations is the presence of temperature variations within the cavity. Thus it is appropriate that this analysis feature a polythermal solution to the cavity problem in the form of Eq. (6). However, since the isothermal cavity problem has been treated so
extensively by other authors, and because the comparisons among theories in the last sections of this dissertation involve only isothermal equations, it is proper to present at this point the isothermal version of Eq. (6):

\[
\epsilon_L(\hat{r}, T_1) = \epsilon + (1-\epsilon) \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
- \frac{(1-\epsilon)^2}{\epsilon \pi} \int_{ap} d\Omega \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
1 + \frac{(1-\epsilon)}{\epsilon \rho} \int_{ap} d\Omega
\]

and this can be simplified to

\[
\epsilon_L(\hat{r}) = \frac{\epsilon \left[ 1 + (1-\epsilon)(B-A) \right]}{\epsilon (1-B) + B}
\]

(7)

where

\[
A = \pi^{-1} \int_{ap} d\Omega
\]

\[
B = \pi^{-1} \int_{ap} d\Omega, \text{ a weighted average of } A.
\]

Temperature Considerations

Greater understanding can be realized concerning surface temperature differences if \( T_m \) is expressed in terms of small deviations from \( T_1 \):

\[
T_m = T_1 + \Delta T_m.
\]

(8)
Then
\[
(T_m / T_1)^4 = (1 + \Delta T_m / T_1)^4 = 1 + 4\Delta T_m / T_1,
\]

and Eq. (6) becomes
\[
\varepsilon_L(\mathbf{r}, T_1) = \varepsilon + (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} \! d\Omega \right)
\]
\[
- \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\text{ap}} \! d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} \! d\Omega \right)
\]
\[
1 + \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\text{ap}} \! d\Omega
\]
\[
+ 4(\Delta T_m / T_1)(1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} \! d\Omega \right)
\]
\[
- 4(\Delta T_m / T_1) \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\text{ap}} \! d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} \! d\Omega \right)
\]
\[
1 + \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\text{ap}} \! d\Omega.
\]

This is the solution with tangential temperature difference terms.

It is very important for blackbody simulators to have the cavity wall temperature as uniform as possible. A corollary to the desirability of uniform temperature is the need for accurate information about the temperature distribution over the cavity wall surface. However, the accurate measurement of cavity wall surface temperature is a difficult problem. It is easier, and more reliable, to provide rear-located or side-located temperature-transducer wells as shown in Fig. 4. These temperature-transducer wells have advantages over temperature transducers inserted into the cavity aperture because the latter might damage the
cavity surface or disturb the cavity wall temperature, and they normally cannot be employed while the cavity is in use because they would interfere with the function of the cavity aperture. Radiometric techniques cannot be used to measure back-wall temperatures, because of multiple reflections within the cavity.

Most blackbody simulators are provided with one or more temperature transducers mounted in a well or wells at the rear or side. Therefore it is desirable, for blackbody simulatory theory, to use, for the working reference temperature, the more subtle (but normally available) transducer-well temperature $T_t$ instead of the more obvious (but less available) cavity wall surface temperature $T_1$. The relationship between $T_t$ and $T_1$ is shown in Fig. 4. This use of $T_t$ instead of $T_1$ has been suggested by several other investigators (Vollmer, 1957, p. 928; Geist, 1973, p. 1327; Bartell and Wolfe, 1974, p. 111).

---

Fig. 4. Temperature Details.
The solution for $\varepsilon_L(\vec{r}, T_\perp)$ instead of for $\varepsilon_L(\vec{r}, T_1)$ is obtained by multiplying Eq. (6) by $(T_1/T_\perp)^4$ and replacing the remaining $T_1$ by

$$T_1 = T_\perp + \Delta T_\perp$$

(10)

so that

$$(T_1/T_\perp)^4 = 1 + 4\Delta T_\perp/T_\perp.$$ 

Then

$$\varepsilon_L(\vec{r}, T_\perp) = \varepsilon(1 + 4\Delta T_\perp/T_\perp) + (1-\varepsilon)(T_m/T_\perp)^4$$

$$- (T_m/T_\perp)^4 (1-\varepsilon) \pi^{-1} \int_{ap} d\Omega$$

$$+ (T_m/T_\perp)^4 \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{ap} d\Omega \left(1 - \pi^{-1} \int_{ap} d\Omega \right) \frac{1 + (1-\varepsilon)/\varepsilon \pi}{1 + \int_{ap} d\Omega}.$$ 

There is a steady-state heat balance between energy conducted through the core and energy radiated out the aperture, and this can lead to an evaluation of $\Delta T_\perp/T_\perp$ in terms of the geometrical and thermal properties of the cavity. That derivation is quite lengthy, so we have separated it from the rest of the analysis, and we provide it as Appendix B. The following result is approximately correct for most cavities of interest:
\[ \frac{\Delta T_T}{T_T} = -\Xi \pi^{-1} \left[ \int_{a_p} d\Omega \left( 1 + 4\Xi \pi^{-1} \int_{a_p} d\Omega \right)^{-1} \right] \]

where

\[ \Xi = \frac{\varepsilon \sigma T_e^3 r_1^\alpha'}{\cos \beta \cos \gamma} \]

\[ \begin{cases} \frac{r_1^{1-\alpha'} - r_1^{1-\alpha'}}{k_{\text{coat}}(1-\alpha')} + \frac{r_T^{1-\alpha''} - r_1^{1-\alpha''}}{k_{\text{core}}(1-\alpha'')} & \text{when } \begin{cases} \alpha' \neq 1 \\ \alpha'' \neq 1 \end{cases} \\ \text{replace first term by: } \frac{\ln(r_{\text{coat}}/r_1)}{k_{\text{coat}}} & \text{when } \alpha' = 1 \\ \text{replace 2nd term by: } \frac{\ln(r_T/r_{\text{coat}})}{k_{\text{core}}} & \text{when } \alpha'' = 1. \end{cases} \]

The symbols \( \beta, \gamma, r_1, r_{\text{coat}}, \) and \( r_T \) are shown on Fig. 4; \( \alpha' \) and \( \alpha'' \) are shape parameters; \( k_{\text{coat}} \) and \( k_{\text{core}} \) are the thermal conductivities of the coating and core. Note that \( \alpha' \) and \( \alpha'' \) in Eq. (11) are nearly the same for most cavities of interest. However, their values depend on the shape of the isothermal surfaces within the cavity walls, so the \( \alpha' \) (associated with the coating) that appears in the first term of \( \Xi \) and also outside the \( \Xi \) expression may differ somewhat from the \( \alpha'' \) in the second term of \( \Xi \) (\( \alpha'' \) is associated with the core).

When the value of \( \Delta T_T/T_T \) from Eq. (11) is substituted into the expression for \( \varepsilon_L(\mathbf{x}, T_T) \), the result is:
\[ \varepsilon_L(\hat{r}, T_\tau) = \varepsilon + \left( \frac{T_m}{T_\tau} \right)^4 (1-\varepsilon) - \left( \frac{T_m}{T_\tau} \right)^4 \frac{(1-\varepsilon)}{\pi} \int_{\text{ap}} d\Omega \]

\[ = \left( \frac{T_m}{T_\tau} \right)^4 \frac{(1-\varepsilon)^2}{\varepsilon\pi} \int_{\text{ap}} d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ - \frac{1 + \frac{(1-\varepsilon)}{\pi\varepsilon} \int_{\text{ap}} d\Omega}{1 + 4\varepsilon \pi^{-1} \int_{\text{ap}} d\Omega} \]

\[ = \frac{4\varepsilon \pi^{-1} \int_{\text{ap}} d\Omega}{1 + 4\varepsilon \pi^{-1} \int_{\text{ap}} d\Omega} . \quad (12) \]

This is the solution with a term for temperature differences that are perpendicular to the wall surface. The tangential and perpendicular temperature variation expressions of Eqs. (9) and (12) may be combined to provide a single equation:

\[ \varepsilon_L(\hat{r}, T_\tau) = \varepsilon + (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ - \frac{(1-\varepsilon)^2}{\varepsilon\pi} \int_{\text{ap}} d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ + 4(\Delta T_m/T_\tau)(1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ - \frac{4(\Delta T_m/T_\tau)}{\varepsilon\pi} \frac{(1-\varepsilon)^2}{\pi\varepsilon} \int_{\text{ap}} d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ - \frac{4\varepsilon \pi^{-1} \int_{\text{ap}} d\Omega \left( 1 + 4\varepsilon \pi^{-1} \int_{\text{ap}} d\Omega \right)^{-1}}{} . \quad (13) \]
Wavelength Considerations

The analysis just presented was based on the Stefan-Boltzmann equation for radiation over all wavelengths. (Recall that \( L_{bb}^b(T) = \sigma T^4 \)). That analysis can be repeated for a monochromatic or band limited expression of the Planck law, or for the Wien or Rayleigh-Jeans law. In a substitute for the above expression,

\[
L_{bb}^b(T_i) = L_i^b,
\]

where \( i \) is any integer and where \( L \) can refer to any of the radiation laws or expressions just noted. In that case, Eqs. (2) and (4) are the same as before except that \( (T_2/T_1)^4 \), \( (T_3/T_1)^4 \), and \( (T_n/T_1)^4 \) are replaced by \( (L_2/L_1) \), \( (L_3/L_1) \), and \( (L_n/L_1) \). Equation (6), which is my basic solution to the cavity problem, becomes

\[
\varepsilon_L(\vec{T}, T_1) = \varepsilon + (L_m/L_1)(1 - \varepsilon) - \frac{(1 - \varepsilon)}{\pi} \int_{\Delta \Omega} d\Omega
\]

\[
= \frac{(L_m/L_1)}{\varepsilon \pi} \left[ \frac{1}{\varepsilon} \int_{\Delta \Omega} d\Omega \right] \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right)
\]

\[
+ \frac{(1 - \varepsilon)}{\varepsilon \pi} \int_{\Delta \Omega} d\Omega .
\]  \hspace{1cm} (14)

Equations (8), (10), and (11) were used to develop the specialized temperature relations (9), (12), and (13) from the Stefan-Boltzmann-based Eq. (6). We may now develop the same kind of specialized temperature relations based instead on the general radiance equation, Eq. (14):
\[
\frac{L_m}{L_1} = \frac{L(T_m)}{L(T_1)} = \frac{L(T_1 + \Delta T_m)}{L(T_1)} = 1 + \left( \frac{\partial L}{\partial T} \frac{T}{L} \right)_{T_1} \frac{\Delta T_m}{T_1} \quad (15)
\]

Equations (14) and (15) lead to

\[
\varepsilon_L(\hat{r}, T_1) = \varepsilon + (1 - \varepsilon) \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
\frac{(1 - \varepsilon)^2}{\pi} \int_{ap} d\Omega \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
1 + \frac{(1 - \varepsilon)}{\pi} \int_{ap} d\Omega
\]

\[
+ \left( \frac{\partial L}{\partial T} \frac{T}{L} \right)_{T_1} \frac{\Delta T_m}{T_1} (1 - \varepsilon) \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
\frac{(1 - \varepsilon)^2}{\pi} \int_{ap} d\Omega \left( 1 - \pi^{-1} \int_{ap} d\Omega \right)
\]

\[
1 + \frac{(1 - \varepsilon)}{\pi} \int_{ap} d\Omega
\]

\[
(16)
\]

This is the general radiance solution for gray Lambertian cavity radiation with tangential temperature difference terms. Multiplying Eq. (14) by \((L_1/L_t)\) gives the term \(\varepsilon(L_1/L_t)\); and, like Eq. (15), this expands to

\[
\varepsilon(L_1/L_t) = \varepsilon + \varepsilon \left( \frac{\partial L}{\partial T} \frac{T}{L} \right)_{T_t} \frac{\Delta T_t}{T_t} \quad (17)
\]

Then Eqs. (11), (14), and (17) lead to
\[ \varepsilon_L (\vec{r}, T_t) = \varepsilon + (L_m / \varepsilon \pi) (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ \frac{L_m}{L_t} \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\Delta \Omega} d\Omega \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ 1 + (1-\varepsilon) \int_{\Delta \Omega} d\Omega \]

\[ \varepsilon \left( \frac{\partial L}{\partial T} \right)_{T_t} \frac{\Delta T_m}{T_t} \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ \frac{\partial L}{\partial T} \frac{\Delta T_m}{T_t} \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\Delta \Omega} d\Omega \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ 1 + (1-\varepsilon) \int_{\Delta \Omega} d\Omega \]

\[ \varepsilon \left( \frac{\partial L}{\partial T} \right)_{T_t} \varepsilon \pi^{-1} \int_{\Delta \Omega} d\Omega \]

\[ 1 + 4\varepsilon \pi^{-1} \int_{\Delta \Omega} d\Omega \]

(18)

This is a variation of Eq. (6) that has a term for temperature differences perpendicular to the wall surface.

Finally:

\[ \varepsilon_L (\vec{r}, T_t) = \varepsilon + (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ \frac{(1-\varepsilon)^2}{\varepsilon \pi} \int_{\Delta \Omega} d\Omega \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ 1 + (1-\varepsilon) \int_{\Delta \Omega} d\Omega \]

\[ \varepsilon \left( \frac{\partial L}{\partial T} \right)_{T_t} \frac{\Delta T_m}{T_t} \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right) \]

\[ \frac{(\partial L / \partial T)_{T_t} \Delta T_m / T_t (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\Delta \Omega} d\Omega \right)}{1 + (1-\varepsilon) \int_{\Delta \Omega} d\Omega} \]

\[ \varepsilon \left( \frac{\partial L}{\partial T} \right)_{T_t} \varepsilon \pi^{-1} \int_{\Delta \Omega} d\Omega \]

\[ 1 + 4\varepsilon \pi^{-1} \int_{\Delta \Omega} d\Omega \]
This is a variation of Eq. (6) that has terms for temperature differences both parallel to (fourth and fifth terms) and perpendicular to (sixth term) the cavity wall surface.

Summary

Equations (2) to (19) are summarized in Table 2.

Table 2. Summary of Eqs. (2) to (19).\(^a\)

<table>
<thead>
<tr>
<th>Type of equation</th>
<th>Based on the radiance laws of</th>
<th>Stefan-Boltzmann</th>
<th>Planck, Wien, or Rayleigh-Jeans</th>
</tr>
</thead>
<tbody>
<tr>
<td>The general expression for blackbody simulator cavities with gray walls</td>
<td>Eq. (2)</td>
<td>Eq. (2)</td>
<td>Eq. (2) with ((L_e/L_j)) in place of ((a_e/a_j)^4)</td>
</tr>
<tr>
<td>Blackbody simulator cavity problem with uniform, gray, Lambertian walls and a cold aperture</td>
<td>Eq. (4)</td>
<td>Eq. (4)</td>
<td>Eq. (4) with ((L_e/L_j)) in place of ((a_e/a_j)^4)</td>
</tr>
<tr>
<td>Approximate solution to the blackbody simulator cavity problem for gray, Lambertian walls and a cold aperture</td>
<td>Eqs. (6) and (6a)</td>
<td>Eq. (14)</td>
<td></td>
</tr>
<tr>
<td>Isothermal version of Eq. (6)</td>
<td>Eq. (7)(^c)</td>
<td>Eq. (7)(^c)</td>
<td></td>
</tr>
<tr>
<td>Variation of Eq. (6) with special terms for (\Delta T) parallel to cavity walls</td>
<td>Eq. (9)</td>
<td>Eq. (16)</td>
<td></td>
</tr>
<tr>
<td>Variation of Eq. (6) with special term for (\Delta T) perpendicular to cavity walls</td>
<td>Eq. (12)</td>
<td>Eq. (18)</td>
<td></td>
</tr>
<tr>
<td>Variation of Eq. (6) with special terms for (\Delta T) parallel to (terms 4 and 5) and perpendicular to (term 6) cavity walls</td>
<td>Eq. (13)</td>
<td>Eq. (19)(^d)</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\)Coefficients \((L_e/L_j)\) and \([(\partial L/\partial T)(T/L)]_T\), which appear frequently in these equations, are presented in detail in Table 3. Coefficient \(\bar{Z}\) is described in Eq. (11).

\(^b\)Equations (6), (6a), (7), (9), (12), (13), (14), (16), (18), and (19) are all approximate solutions to the blackbody simulator cavity problem. I prefer Eq. (6).

\(^c\)Isothermal Eq. (7) is used in the last sections of this dissertation for comparisons among different theories.

\(^d\)In the next few sections, Eq. (19) is simplified by approximation, and then compared with experimental data.
Table 3. \((L_i/L_j)\) and \(\frac{\partial L}{\partial T} \frac{(T/L)}{T_i}\).

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Stefan-Boltzmann</th>
<th>Monochromatic</th>
<th>Band-limited</th>
<th>Wien</th>
<th>Rayleigh-Jeans</th>
</tr>
</thead>
<tbody>
<tr>
<td>((L_i/L_j))</td>
<td>(\left(\frac{T_i}{T_j}\right)^4)</td>
<td>(\frac{e^{x_i} - 1}{e^{x_i} - 1})</td>
<td>1 + (\frac{(T_i - T_j)}{T_j(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} \frac{x_i e^{x_i} d\lambda}{e^{x_i} - 1})</td>
<td>(\frac{e^{x_i}}{e^{x_i} - 1})</td>
<td>(\frac{T_i}{T_j})</td>
</tr>
<tr>
<td>(\frac{\partial L}{\partial T} \frac{L}{T_i})</td>
<td>4</td>
<td>(\frac{x_i e^{x_i}}{e^{x_i} - 1})</td>
<td>(\frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{x_i e^{x_i} d\lambda}{e^{x_i} - 1})</td>
<td>(x_i)</td>
<td>1</td>
</tr>
</tbody>
</table>

\(x_i = \frac{\sigma}{\lambda T_i}\)
THEORETICAL AND EXPERIMENTAL STUDIES
OF THE PROJECTED SOLID ANGLE EFFECT

Projected Solid Angle Effect

It was noted in the Introduction that this project had its beginning in my belief that blackbody simulator theory, design, and evaluation should take into account the variations in output with angle or geometry such as those shown in Fig. 1. A review of the theory up to this point, and the equations from (6) to (19) in particular, shows that one geometrical function dominates over all others in its impact on $\varepsilon_L$, and that function is $\int_{\text{ap}} d\Omega$, the projected solid angle of the aperture as seen from a point on the wall of the cavity. This finding is important for two reasons:

(1) It is desirable to find a simple geometrical function that might be used in the evaluation of blackbody simulator cavities.

(2) The projected solid angle is a reasonably simple and well known radiometric function, and it is related by a factor of $\pi$ to the configuration factor $F$, which is an even better known and more widely used function in heat transfer technology. Thus:

$$F_{\text{ap}} = \pi^{-1} \int_{\text{ap}} d\Omega.$$

There have been many publications on the configuration factor for different geometries, and some give methods for calculations and
tables of configuration factors. Some writers use different names for configuration factor, i.e., angle factor, shape factor, or view factor. Examples of references on configuration factors are Siegel and Howell (1972, Chapter 7, pp. 170-234, Appendix B, pp. 748-782, Appendix C, pp. 783-791); Sparrow and Cess (1966, Chapter 4, pp. 113-136, Appendix A, pp. 300-310), who call it "angle factor"; Wiebelt (1966, Appendix 4, pp. 240-261); and Kreith (1962, Appendix V, pp. 201-229), who calls it "shape factor."

Equations (11) and (19) show that:

1. Cavity geometrical effects are manifest in \( \int_{ap} d\Omega, \int_{ap} d\Omega, \Delta T_m, \) and \( \Sigma, \) where \( \Sigma \) depends on \( \alpha', \alpha'', \beta, \gamma, r_1, r_\alpha, r_t, k_{coat}, \) and \( k_{core}. \)

2. The most important shape-dependent factor is \( \int_{ap} d\Omega, \) followed by \( \Delta T_m, \int_{ap} d\Omega, \alpha', \alpha'', \beta, \gamma, r_1, r_\alpha, r_t, k_{core}, \) and \( k_{coat}, \) listed approximately in the order of their importance. The factors \( \Delta T_m \) and \( \int_{ap} d\Omega, \) both vary with \( \int_{ap} d\Omega, \) and for many cavity shapes the dependence is approximately a proportionality.

We could find out more about the relationship between \( \int_{ap} d\Omega \) and cavity geometry, and between \( \int_{ap} d\Omega \) and cavity output signals, by carefully reviewing the derivation of Eq. (19) and then analyzing different cavity shapes for the roles played by the quantities listed above. But it is easier and more informative to make a less detailed theoretical analysis based on approximations and combine it with an experimental study. The appropriate experiment should correspond to Fig. 3 (page 19), but the cavity or the detector is moved so that varying back wall locations correspond to \( \alpha_1. \) The detector and the system aperture are
both chosen to be small, so the signal at the detector is proportional to \( L(\mathbf{r}) \). Before discussing the simplified theory or the experiment, we will derive the approximate values of the projected solid angle for cavities of importance.

**Approximate Values of the Projected Solid Angle**

To investigate the projected solid angle, \( \int_{ap} d\Omega \), of customary cavity shapes, we begin with

\[
\int_{ap} d\Omega = \int_{ap} \cos \phi_1 \cos \phi_2 \, v^{-2} \, ds,
\]

where \( ds \) is an element of aperture surface area and \( \phi_1 \), \( \phi_2 \), and \( v \) are defined in Fig. 5.

---

**Fig. 5. Cavity Interior.**
We can obtain some simple and useful approximate results by assuming

\[
\begin{align*}
\phi_1 & = \text{constant over } d\Omega, \\
\phi_2 & = \text{constant over } \int_{ap} d\Omega.
\end{align*}
\]  

(21)

These assumptions are fairly accurate for small apertures and deep cavity locations, but they are less reliable for large apertures and for points near the aperture. When Eq. (21) is valid,

\[
\int_{ap} d\Omega = \cos\phi_1 \cos\phi_2 \alpha_{ap} \nu^{-2}.
\]  

(22)

Let us restrict ourselves to circular apertures of radius \(\omega\). Then, for a general cavity shape,

\[
\pi^{-1} \int_{ap} d\Omega = \begin{cases} \\
\frac{\nu^2 \omega^2}{\nu^4} (y \cos u - z \sin u) & \text{rectangular coordinates} \\
\frac{\nu^2}{\nu^2} \sin(\theta - u) \cos \theta & \text{or} \\
\frac{\nu^2}{\nu^2} (\sin \theta \cos u - \cos \theta \sin u) \cos \theta & \text{polar coordinates} \\
\end{cases}
\]  

(23)

Table 4 gives approximate expressions for \(\pi^{-1} \int_{ap} d\Omega\) for the customary shapes of sphere, cone, and cylinder and also for the less common shapes of a cone at the end of a cylinder and a reentrant cone.

Table 5 gives approximate values for \(\pi^{-1} \int_{ap} d\Omega\) for different values of \(\theta\) for various shapes. (Note that the cylinder values are all side wall values except for the five entries marked with an asterisk, which are back-wall values.)
<table>
<thead>
<tr>
<th>Cavity shape</th>
<th>Polar coordinates</th>
<th>Rectangular coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>General shape</td>
<td>$\frac{\omega^2}{v^2} \sin(\theta - \omega) \cos \theta$</td>
<td>$\frac{m \omega^2}{v^4} (\gamma \cos u - \gamma \sin u)$</td>
</tr>
<tr>
<td>Eqs. (23)</td>
<td>$\frac{\omega^2}{v^2} (\sin \theta \cos u - \cos \theta \sin u) \cos \theta$</td>
<td></td>
</tr>
<tr>
<td>Sphere</td>
<td>$\frac{\omega^2}{4R^2}$</td>
<td>$\frac{m \omega^3}{(s^2 + \omega^2)^2}$</td>
</tr>
<tr>
<td>Cone</td>
<td>$\frac{(h \sin \theta + \omega \cos \theta)^3 \cos \theta}{\sqrt{\omega^2 + h^2}}$</td>
<td>$\sqrt{1 + \frac{\omega^2}{h^2}} \left[ s^2 + \omega^2 \left(1 - \frac{\omega}{h} \right)^2 \right]^2$</td>
</tr>
<tr>
<td>Cylinder</td>
<td>$\sin^3 \theta \cos \theta$</td>
<td></td>
</tr>
<tr>
<td>Side</td>
<td>$\frac{\omega^2}{l^2} \cos^4 \theta$</td>
<td>$\frac{\omega}{(l + h)^2} \left( s^2 + \omega^2 \left(1 - \frac{\omega}{h} \right)^2 \right)^2$</td>
</tr>
<tr>
<td>Back</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cone at end of cylinder</td>
<td>Same as cylinder</td>
<td>Same as cylinder</td>
</tr>
<tr>
<td>Cylindrical part</td>
<td>(above)</td>
<td>(above)</td>
</tr>
<tr>
<td>Conical part</td>
<td>$\frac{(h \sin \theta + \omega \cos \theta)^3 \cos \theta}{\sqrt{\omega^2 + h^2}}$</td>
<td>$\frac{m \omega^3}{(h + l)^2}$</td>
</tr>
<tr>
<td>(above)</td>
<td>$\frac{(h \sin \theta + \omega \cos \theta)^3 \cos \theta}{\sqrt{\omega^2 + h^2}}$</td>
<td>$\frac{m \omega^3}{(h + l)^2}$</td>
</tr>
<tr>
<td>Reentrant cone</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Near aperture</td>
<td>$\frac{[g \sin \theta - (d - \omega) \cos \theta]^3 \cos \theta}{\sqrt{g^2 + (d - \omega)^2}}$</td>
<td>$\sqrt{(d - \omega)^2 + g^2} \left[ s^2 + \left( g + \frac{(d - \omega) \omega}{g} \right)^2 \right] \left[ s^2 + \left( d - \frac{(g - g) d}{h} \right)^2 \right] \left[ s^2 + \left( d - \frac{(g - g) d}{h} \right)^2 \right]$</td>
</tr>
<tr>
<td>Near apex</td>
<td>$\frac{\omega^2 (h \sin \theta + \omega \cos \theta)^3 \cos \theta}{d^2 \sqrt{d^2 + h^2}}$</td>
<td>$\frac{\omega^2 \omega^2}{(h + g)^2}$</td>
</tr>
<tr>
<td>Assumptions: $\phi_1$</td>
<td>$\frac{\omega}{v^2} = \text{constant over } \int_{\text{ap}} \frac{d\Omega}{\omega}$.</td>
<td>$\phi_2$</td>
</tr>
<tr>
<td>Symbols:</td>
<td>$R$ = sphere radius</td>
<td>$h$ = axial length of cone</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$b$ = distance off axis</td>
<td>$l$ = axial length of cylinder</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$g$ = axial length of cylinder</td>
<td>$d$ = radius of base of cone</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\omega$</td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\omega$</td>
<td></td>
</tr>
</tbody>
</table>
Table 5. Approximate Values for $\int_{ap} d\Omega$.

<table>
<thead>
<tr>
<th>Cavity shape</th>
<th>$\theta$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5.828</th>
<th>8</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>All $\theta$</td>
<td>0.1600</td>
<td>0.0900</td>
<td>0.0554</td>
<td>0.0278</td>
<td>0.0151</td>
<td>0.0098</td>
<td>0.0025</td>
</tr>
<tr>
<td>Cylinder$^a$</td>
<td>$10^\circ$</td>
<td>0.2352*</td>
<td>0.1045*</td>
<td>0.0588*</td>
<td>0.0277*</td>
<td>0.0052</td>
<td>0.0052</td>
<td>0.0052</td>
</tr>
<tr>
<td></td>
<td>$20^\circ$</td>
<td>0.1949*</td>
<td>0.0376*</td>
<td>0.0376</td>
<td>0.0376</td>
<td>0.0376</td>
<td>0.0376</td>
<td>0.0376</td>
</tr>
<tr>
<td></td>
<td>$30^\circ$</td>
<td>0.1083</td>
<td>0.1083</td>
<td>0.1083</td>
<td>0.1083</td>
<td>0.1083</td>
<td>0.1083</td>
<td>0.1083</td>
</tr>
<tr>
<td></td>
<td>$40^\circ$</td>
<td>0.2034</td>
<td>0.2034</td>
<td>0.2034</td>
<td>0.2034</td>
<td>0.2034</td>
<td>0.2034</td>
<td>0.2034</td>
</tr>
<tr>
<td></td>
<td>$50^\circ$</td>
<td>0.2890</td>
<td>0.2890</td>
<td>0.2894</td>
<td>0.2894</td>
<td>0.2894</td>
<td>0.2894</td>
<td>0.2894</td>
</tr>
<tr>
<td>Cone</td>
<td>$10^\circ$</td>
<td>0.2603</td>
<td>0.1181</td>
<td>0.0707</td>
<td>0.0390</td>
<td>0.0255</td>
<td>0.0197</td>
<td>0.0109</td>
</tr>
<tr>
<td></td>
<td>$20^\circ$</td>
<td>0.4498</td>
<td>0.2508</td>
<td>0.1751</td>
<td>0.1180</td>
<td>0.0905</td>
<td>0.0775</td>
<td>0.0552</td>
</tr>
<tr>
<td></td>
<td>$30^\circ$</td>
<td>0.6291</td>
<td>0.4030</td>
<td>0.3090</td>
<td>0.2329</td>
<td>0.1934</td>
<td>0.1739</td>
<td>0.1387</td>
</tr>
<tr>
<td></td>
<td>$40^\circ$</td>
<td>0.7396</td>
<td>0.5265</td>
<td>0.4316</td>
<td>0.3504</td>
<td>0.3062</td>
<td>0.2838</td>
<td>0.2417</td>
</tr>
<tr>
<td></td>
<td>$50^\circ$</td>
<td>0.7393</td>
<td>0.5745</td>
<td>0.4963</td>
<td>0.4264</td>
<td>0.3867</td>
<td>0.3661</td>
<td>0.3265</td>
</tr>
</tbody>
</table>

$^a$All side wall values except for those marked with an asterisk, which are back wall values.

$^b$This table includes a broad range of shape and $\theta$ values, but good-quality blackbody simulator operations are limited to the unshaded areas.

A Simplified Expression

The simplification of the theory begins with Eq. (19), which can be rewritten as

$$
\varepsilon_L(T_t,T_t) = \varepsilon + G_1G_2 - \frac{G_1G_2G_3G_4}{1 + G_3G_4} + G_1G_2G_5G_6 - \frac{G_1G_2G_3G_4G_5G_6}{1 + G_3G_4}
$$

$$
= \frac{\varepsilon G_5 \pi^{-1} \int_{ap} d\Omega}{1 + 4\pi^{-1} \int_{ap} d\Omega}
$$

where

$$
G_1 = 1 - \varepsilon
$$

$$
G_2 = 1 - \pi^{-1} \int_{ap} d\Omega
$$

$$
G_3 = \frac{(1 - \varepsilon)}{\varepsilon}
$$

$$
G_4 = \pi^{-1} \int_{ap} d\Omega
$$

$$
G_5 = \left(\frac{\partial L}{\partial T} \frac{T}{L}\right)_{T_t} = x_t \frac{e^x_t}{(e^x_t - 1)}
$$

where $x_t = c_2/\lambda T_t$

$$
G_6 = \Delta T_m/T_t.
$$
The simplified expression we are about to obtain resembles just the first two terms of Eq. (24), and that is a great simplification. Because of its simplicity, the new equation will be useful for preliminary considerations of most blackbody simulator cavity problems, and of course we are about to use it in comparing our theory with a set of experimental measurements. However, the simplification process reduces accuracy, and the conditions of the approximations are restrictive, so for tasks that require more accuracy—and this would include final calculations for most problems—the full Eq. (24) should be used.

For the simplified version of Eq. (24), only practical blackbody simulator cavities will be considered, and for back wall primary radiating surface locations (like \( \alpha_1 \) in Fig. 3), only areas that are directly viewed in practical applications will be used. Temperature variations over the back-wall primary radiating surface will be limited to those that allow only 1% variation in Stefan-Boltzmann radiation. Thus

\[
M = \sigma T^4
\]

\[
\frac{\Delta M}{M} = \frac{\Delta T}{T} = \frac{1}{100}
\]

so

\[
\frac{\Delta T}{T} = \frac{1}{400} \text{ (primary radiating surface).}
\]

Total cavity temperature differences 10 times that amount will be allowed:

\[
\frac{\Delta T}{T} = \frac{1}{40} \text{ (total cavity).}
\]
As a criterion for simplifying Eq. (24), terms that amount to less than 1% will be dropped.

We begin with the third term of Eq. (24), and we examine first $G_1G_3$ and then $G_2G_4$. For most practical blackbody simulator cavities, $\varepsilon$ will be more than $5/6$ (0.833), and then $G_1G_3$ will be less than $1/30$ (0.0333). To estimate the range of practical $G_2G_4$ values, we refer to the footnote on page 25, which describes $\int d\Omega$ as a weighted average of $\int d\Omega$. The weighting depends on projected solid angles in such a way that, for simple shapes like cones and cylinders, nearby points with similar values of $\int d\Omega$ are more heavily weighted, and thus an approximate value for $\int d\Omega$ at a given point is just the unaveraged quantity, $\int d\Omega$, itself. Since $\pi^{-1}\int d\Omega$ is always less than 1 for points on a cavity wall, $G_2G_4$ is approximately equal to $(1-x)x$, where $0 \leq x \leq 1$, and so the maximum value of $G_2G_4$ is 0.25. Therefore, the maximum value of $G_1G_2G_3G_4$ is 0.0083. Since we are using a 1% criterion for this approximate simplified expression, we may neglect the third term in Eq. (24).

The fourth term is $G_1G_2G_5G_6$. We have just commented that, according to the footnote on page 25, $\int d\Omega$ will normally assume values close to $\int d\Omega$. By a similar logic, according to the footnote on page 24, $T_m$ will normally be close to $T_1$, so $\Delta T_m = T_m - T_1$ will usually have small values. We have assumed that total temperature spreads, $\Delta T/T$, will be less than 1/40, and by virtue of this footnote, the main zone of temperatures that will contribute to $\Delta T_m$ will be limited to about 1/3 the total temperature spread. And finally, $\Delta T_m$ will probably be about 1/3 the spread in that zone. Therefore, for the conditions specified for
the development of this simplified expression, \( G_6 = \Delta T_m / T_t \) will be about 1/360 or smaller. This argument applies better to more open cavity configurations like cones and cylinders, but more closed cavity configurations like spheres and reentrant cones tend to have smaller total temperature differences, so the limitation of \( G_6 = \Delta T_m / T_t \) less than 1/360 is reasonable. We have seen that \( G_1 \) is less than 1/6 and \( G_2 \) is less than 1.0, so \( G_1 G_2 G_6 \) is less than 1/2160. Therefore, the fourth term, \( G_1 G_2 G_5 G_6 \), will be less than 1% whenever \( G_5 \) is less than 21.6. Table 6 shows that that condition is met for all but the few combinations of \( \lambda \) and \( T_t \) in the shaded part of the table. Furthermore, in the field of infrared technology, lower-temperature sources are usually associated with longer wavelength operations, so the shaded part of Table 6 represents a region of limited importance. Finally, for the unshaded combinations of \( \lambda \) and \( T_t \) shown in Table 6, we can ascribe 1% to the fourth term in arriving at the simplified version of Eq. (24).

<table>
<thead>
<tr>
<th>( \lambda ), ( \mu m )</th>
<th>( T_t ) in kelvins</th>
<th>100</th>
<th>300</th>
<th>500</th>
<th>800</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>143.88</td>
<td>47.96</td>
<td>28.776</td>
<td>17.985</td>
<td>14.388</td>
<td>7.199</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>71.94</td>
<td>23.98</td>
<td>14.388</td>
<td>8.994</td>
<td>7.199</td>
<td>3.698</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>47.96</td>
<td>15.987</td>
<td>9.593</td>
<td>6.010</td>
<td>4.836</td>
<td>2.638</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>28.776</td>
<td>9.593</td>
<td>5.773</td>
<td>3.698</td>
<td>3.049</td>
<td>1.886</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>17.985</td>
<td>6.010</td>
<td>3.698</td>
<td>2.514</td>
<td>2.155</td>
<td>1.516</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>9.593</td>
<td>3.334</td>
<td>2.249</td>
<td>1.717</td>
<td>1.555</td>
<td>1.259</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>4.836</td>
<td>2.638</td>
<td>1.555</td>
<td>1.330</td>
<td>1.259</td>
<td>1.125</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.886</td>
<td>1.259</td>
<td>1.151</td>
<td>1.093</td>
<td>1.074</td>
<td>1.036</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.074</td>
<td>1.024</td>
<td>1.014</td>
<td>1.009</td>
<td>1.007</td>
<td>1.004</td>
<td></td>
</tr>
</tbody>
</table>
The fifth term in Eq. (24) is just \( G_5 G_6 \) times the third term. But we have seen that the third term is less than 0.0083 and \( G_6 \) is less than 1/360. Therefore, whenever \( G_5 \) is less than 360 the fifth term will be less than 0.0083. Table 6 shows that for all \( \lambda \) and \( T_t \) considered, \( G_5 \) is less than 0.4 times 360, so for all those cases, the fifth term is less than 0.0033. Thus, to a root sum square (r.s.s.) accuracy of 1.34% we can use, in place of Eq. (24):

\[
\varepsilon_L(\vec{r}, T_t) = \varepsilon + (1 - \varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) - \frac{\varepsilon G_5 \Xi \pi^{-1} \int_{\text{ap}} d\Omega}{1 + 4 \Xi \pi^{-1} \int_{\text{ap}} d\Omega}. \tag{25}
\]

From Eq. (11), the last term equals \( \varepsilon G_5 \Delta T_t / T_t \), and since we are limiting surface temperature differences in the primary radiating surface area to \( \Delta T / T = 1 / 400 \), the value of this term will be \( \varepsilon G_5 / 400 \) or smaller, and to an accuracy of 1% of the term its denominator is 1.0. Furthermore, \( \varepsilon \) and \( G_5 \) are constant to 1/400, so the last term in Eq. (25) is a constant to about the same degree that \( \Xi \) is a constant. It can be shown from Appendix B that \( \Xi \) is approximately constant when the isothermal surfaces are parallel to the cavity walls; and since here \( \Delta T_t / T_t \leq 1 / 400 \), \( \Xi \) is a constant to about 1% or better. Thus Eq. (25) can be rearranged to yield

\[
\varepsilon_L(\vec{r}, T_t) = 1 - \left( 1 - \varepsilon + \frac{\varepsilon G_5 \Xi}{1 + 4 \Xi \pi^{-1} \int_{\text{ap}} d\Omega} \right) \pi^{-1} \int_{\text{ap}} d\Omega
\]

or

\[
\varepsilon_L(\vec{r}, T_t) = 1 - \left( \text{a constant} \right) \pi^{-1} \int_{\text{ap}} d\Omega. \tag{26}
\]
Equation (26) is valid to 1.7\% r.s.s. accuracy for good-quality blackbody simulator cavities.

For the present experiment we are measuring the variations of $\varepsilon_L$ with angle, and in order to emphasize the different characteristics of different cavity shapes, we extend the angular range to ±50°, which is beyond the domain of good-quality blackbody simulator performance for our cone and cylinder. If we reexamine the approximations leading to Eq. (26) in light of our experimental conditions, we find:

\[
\varepsilon \approx 0.9 \\
\lambda = 2.3 \, \mu m \\
T \approx 400^\circ C = 673.15 \, K
\]

and therefore

\[
\varepsilon = (14388 \, \mu m-K)/\lambda T = 9.3
\]

and $\Delta T/T$ in the primary radiating surface may be 10 times that used in the assumptions leading to Eq. (26). Reviewing the derivation of Eq. (26), we see that the third and fifth terms will still be less than 1\%, but the fourth term may be several per cent, and the assumption that the last term in Eq. (25) is a constant times $\pi^{-1}\int_{ap} d\Omega$ may also be in question to an accuracy of a few per cent. Therefore, in applying Eq. (26) to our experiment, we might expect differences between theory and experiment of several per cent near the seam of the cylinder and at larger angles for the cone and cylinder where larger $\Delta T/T$ occur.

We can rewrite Eq. (26) as follows:

\[
\varepsilon_L(\hat{r},T) = \frac{L(\hat{r})}{L^{\Delta \hat{r}}(T)} = 1 - (a \, \text{constant}) \pi^{-1}\int_{ap} d\Omega. \tag{26a}
\]
Each of the three parts of Eq. (26a) is approximately proportional to $S$, the signal from the detector in the experiment. Thus:

$$S = (1 - (a \text{ constant}) \pi^{-1} \int \frac{d\Omega}{ap} ) (a \text{ new constant}).$$

If we let $S_0$ be the signal when $\hat{r}$ is on axis, and let a third constant $D$ be the product of the first two constants, then

$$S = S_0 - D \left[ \pi^{-1} \int \frac{d\Omega}{ap} - \left( \pi^{-1} \int \frac{d\Omega}{ap} \right) \text{ on axis} \right].$$ (27)

We may now compare the theoretical expression Eq. (27) with experimental data.

**Experimental Setup**

Equation (27) was tested using blackbody simulators of three cavity types: a sphere, a cone, and a cylinder. The experiment is described in Fig. 6.
Each cavity was made from a 4-in. (10.16-cm) cube of aluminum, and each had an aperture radius of 0.5 in. (1.27 cm) and cavity depth of 2.91 in. (7.40 cm). This provided a sphere radius of 1.5 in. (3.81 cm). The cavity surfaces were coated with fine granulated quartz (sand) lightly bonded to the core with Aquadag (colloidal graphite in water) and Aerodag (an aerosol of micrometer-sized graphite in isopropyl alcohol); both materials are available from Acheson Colloids Company, Port Huron, Mich. The coating provided a diffuse surface and increased the effective wall emissivity. Because quartz is a relatively poor thermal conductor, the coating also allowed thermal gradients to develop and thus demonstrate fine structure variations in the signal as a function of back-wall location; a simple aluminum cavity with its higher conductivity might have masked detailed variations in the signal.

The simulators had cartridge-type heaters in each of four corners 13/16 in. (2.06 cm) in from the sides, to permit operation at 400°C. Radiation from the sources was detected by a 0.7-mm-square lead sulfide detector. Readout was performed by a Princeton Applied Research (PAR) model HR-8 lock-in amplifier connected to a Hewlett-Packard Moseley model 7000AM xy recorder. The tests were run by rotating the turntable, which was centered at the cavity aperture. The detector signal was connected to the y axis of the plotter, and a rotation-sensing transducer was connected to the x axis. The plotter gains for both vertical and horizontal axes were adjusted for convenient displays, and the horizontal and vertical scales were marked from turntable and amplifier readings. Alignment was performed with a small He-Ne laser and was checked radiometrically through the final readout system.
Theoretical and Experimental Results

The experimental results for the cone, cylinder, and sphere are shown in Fig. 7 along with corresponding theoretical curves based on Eq. (27). The horizontal zero position and horizontal scale factor for the theoretical curves were selected to agree with the experimental curves, and the value of $S_0$ in Eq. (27) was selected for each theoretical curve so as to make the maxima of the theoretical and experimental plots agree. A single value of $D$ in Eq. (27) was used for all three theoretical curves, and that value of $D$ was chosen to make the theoretical and experimental curves for the cylinder agree at $+50^\circ$.

![Diagram of theoretical and experimental results for cone, cylinder, and sphere.](image-url)

Fig. 7. Experimental Verification of Theory.
Figure 7 shows agreement between theory and experiment as follows:

(1) The curves for the sphere, both theoretical and experimental, are much more uniform over the range $-50^\circ$ to $+50^\circ$ than are the curves for the cylinder or cone.

(2) The decrease in signal at large angles is greatest for the cone, as shown on both the theoretical and the experimental curves.

(3) The maximum signal for the cone occurs at $0^\circ$, and a double maximum occurs for the cylinder near $-10^\circ$ and $+10^\circ$; these features are shown on both the theoretical and the experimental curves.

The beginning of this dissertation criticized the standard practice whereby blackbody simulator cavities have been described by a single, on-axis, effective cavity emissivity. Rather than a single figure of merit, what was needed was a theory and a method of describing blackbody simulator cavity quality that would properly account for the angular distribution of the emitted radiation. Figure 7 shows that the theory developed in this dissertation gives useful approximate values for that angular distribution.

Comparison measurements involving the absolute value of the radiometric output signals were not attempted because such measurements would require about an order of magnitude greater experimental accuracy than the few per cent involved in the data of Fig. 7. This dissertation and this figure give an approximate solution for the problem that was undertaken. The future holds the solution to the absolute value problem.
COMPARISONS AMONG THEORIES

We can compare this work with major analyses of the past--both multiple-reflection studies and integral-equation presentations. The theory is based on multiple reflections, and it can be compared with the multiple-reflection studies of Gouffé (1945), De Vos (1954a), Treuenfels (1963), Kelly (1966), and Nicodemus (1968). Furthermore, as shown in Appendix C, the Liouville-Neumann series solution of the Fredholm integral equation of the second kind acts as a bridge to connect this theory to most integral-equation formulations of the cavity problem. The papers that are compared are limited to those that specify isothermal, diffuse cavity walls. Although the best known papers on blackbody simulator theory can be placed in a form that fosters convenient comparisons, there are a number of important studies that do not fit into the present comparison scheme. Three types of papers are not compared here:

(1) Papers that feature directional emissivity or reflectance for the cavity walls. An example is Lin and Sparrow (1965), who specify diffuse emissivity and specular reflectance.

(2) Papers that feature polythermal cavity effects, such as that by Campanaro and Ricolfi (1966b).

(3) Papers that feature a detector-oriented cavity emissivity, such as that by Campanaro and Ricolfi (1967); this paper solves for what is called the "normal emissivity" of the cavity, as defined on page 58.
The applicable parts of several important papers that might have been eliminated by one or two of the three exclusion points just given are included in the present comparison. De Vos (1954a) introduced the concept of directional emissivity and reflectance into blackbody simulator theory, and his paper also provides for nonisothermal cavities, but his paper is an important part of the present comparison. He applied his theory to several shapes of cavities with isothermal diffuse walls, and Quinn (1967a) followed De Vos' method for cylinders with isothermal diffuse walls. Bedford and Ma (1974, 1975) treat the nonisothermal cavity and also solve for a detector-oriented cavity emissivity. However, they first solve for the effective local hemispherical emissivity for isothermal cavities, and their solutions are among the best integral equation analyses of blackbody simulator theory. Previous comparisons are also described, and the exclusions just mentioned do not apply to these.

Emissivity

Before proceeding with the comparison, we should describe the different cavity figures of merit that are involved. The two most important cavity emissivities are $\varepsilon_L(\hat{r}, T)$, which is based on radiance, and $\varepsilon_\alpha(\hat{r}, T)$, which is the local hemispherical emissivity. These quantities are called effective or apparent emissivities, and they are associated with the sum of emitted and reflected radiation. They are defined by Eqs. (28) and (29):

$$
\varepsilon_L(\hat{r}, T) = \frac{L_{\text{emit}}(\hat{r}, T) + L_{\text{refl}}(\hat{r})}{L(T)}
\quad \text{(28)}
$$
\[
\varepsilon_\alpha(\mathbf{\hat{r}}, T) = \frac{M_{\text{emit}}(\mathbf{\hat{r}}, T) + M_{\text{refl}}(\mathbf{\hat{r}})}{M(\mathbf{T})^{bb}}.
\]

where \( T \) is the temperature of the wall surface at the point identified by \( \mathbf{\hat{r}} \). When there is no question about the temperature that is involved (e.g., isothermal cavities), \( \varepsilon_L(\mathbf{\hat{r}}, T) \) and \( \varepsilon_\alpha(\mathbf{\hat{r}}, T) \) are often replaced by \( \varepsilon_L(\mathbf{\hat{r}}) \) and \( \varepsilon_\alpha(\mathbf{\hat{r}}) \). The symbols \( L \) and \( M \) are radiance and exitance, respectively, for either emitted (emit) or reflected (refl) radiation. The superscript \( bb \) signifies radiation from a blackbody. For a diffuse, gray, Lambertian surface, it is shown by Eqs. (3) and by Nicodemus [1965, Eqs. (5), (8), (18), (22); 1970] and Klein (1970, p. 124), that

\[
\begin{align*}
M_{\text{emit}}(\mathbf{\hat{r}}, T) &= \pi L_{\text{emit}}(\mathbf{\hat{r}}, T) \\
M(\mathbf{T})^{bb} &= \pi L^{bb} \\
M_{\text{refl}}(\mathbf{\hat{r}}) &= \rho E = (1-\varepsilon)E \\
L_{\text{refl}}(\mathbf{\hat{r}}) &= \int E = (1-\varepsilon)E/\pi
\end{align*}
\]

where \( E \) is irradiance and \( \rho \) is hemispherical reflectance. Substitutions into Eqs. (28) and (29) show

\[
\begin{align*}
\varepsilon_L(\mathbf{\hat{r}}) &= \varepsilon_\alpha(\mathbf{\hat{r}}) \\
\varepsilon_L(\mathbf{\hat{r}}, T) &= \varepsilon_\alpha(\mathbf{\hat{r}}, T).
\end{align*}
\]

Another relationship of interest in comparing different theories is Eq. (3) of Kelly (1966), which he establishes (Kelly, 1965) as

\[
\varepsilon_\alpha(\mathbf{\hat{r}}) = 1 - \rho_\alpha(\mathbf{\hat{r}}).
\]
He defines $\varepsilon_a(\mathbf{r})$ as done above, and he calls $\rho_a(\mathbf{r})$ the apparent local reflectance, defining it in a manner consistent with the cavity analyses of Gouffé (1945), Treuenfels (1963), and Kelly (1966).

The expressions appearing in Eqs. (30) and (31) are extensively used in the comparisons that follow. There are two other expressions of cavity emissivity that often appear in the literature, but they are not related to the three expressions in Eqs. (30) and (31) in any simple way and are not used in these comparisons. They are the apparent hemispherical emissivity of the cavity aperture (Sparrow and Jonsson, 1963) and a detector-oriented emissivity (Bedford and Ma, 1974, 1975) for which the radiation incident on a detector is determined and then that quantity is divided by an amount equal to what the incident radiation would be if the cavity were replaced by an ideal blackbody. A special case of detector oriented emissivity is called the normal cavity emissivity. This special case calls for a small detector to be located on axis, perpendicular to the axis, and at a great distance from the aperture (Sparrow and Heinisch, 1970).

**Integral-Equation Analyses**

The largest class of blackbody simulator cavity theories is that involving integral equations. Because the present study of integral-equation formulations of the blackbody simulator cavity problem is so extensive and would break the continuity if it were discussed here, it is presented separately as Appendix C. That appendix shows that most integral-equation expressions of the cavity problem can be reduced to a single Fredholm integral equation of the second kind; and furthermore
the solution to that integral equation is the right-hand side of Eq. (4). The solution to the right-hand side of Eq. (4) is the right-hand side of Eq. (6); or for the isothermal case it is the right-hand side of Eq. (7). Thus Appendix C shows that most isothermal integral equation analyses can be reduced to an equation that is much like Eq. (7):

\[
\varepsilon_{a}^{(1)} = \frac{\varepsilon[1 + (1 - \varepsilon)(B-A)]}{\varepsilon(1-B) + B} \quad (32)
\]

where

\[
A = \pi^{-1} \int_{ap} d\Omega
\]

\[
B = \pi^{-1} \int_{ap} d\Omega, \text{ a weighted average of } A.
\]

**Gouffé's Theory**

Probably the best known and most widely cited paper on blackbody simulator cavities is that by Gouffé (1945). As we did for integral equations, we now present a separate analysis of Gouffé's theory (Appendix D). It has not been recognized until now, but Gouffé presents not one, but five, different analytic expressions as his solution for the blackbody simulator cavity problem. They are presented here as Table 7. Quantities in this table are described on pages 118, 121, 122, and 124 in Appendix D. Note that:

\[
\varepsilon_{0} = \frac{\varepsilon[1 + (1-\varepsilon)(B-A)]}{\varepsilon(1-B) + B} \quad (33)
\]
Table 7. Gouffé's Five Expressions for $\varepsilon_0$.

<table>
<thead>
<tr>
<th>Gouffé's equation</th>
<th>My equation</th>
<th>$A$</th>
<th>$B$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) (D-2)(D-3)</td>
<td>(7)</td>
<td>$\pi^{-1} \int_{\Lambda_p} d\Omega$</td>
<td>$\pi^{-1} \int_{\Lambda_p} d\Omega$</td>
<td>Correct equation; general cavity</td>
</tr>
<tr>
<td>(3) (D-2)(D-6)</td>
<td></td>
<td>$\omega/\pi$</td>
<td>$s/S$</td>
<td>General cavity</td>
</tr>
<tr>
<td>(D-7)</td>
<td></td>
<td>$s/S$</td>
<td>$s/S$</td>
<td>Spherical cavity only</td>
</tr>
<tr>
<td>(2 bis)(4) (D-8)(D-9)</td>
<td></td>
<td>$s/S_0$</td>
<td>$s/S$</td>
<td>General cavity</td>
</tr>
<tr>
<td>(2 bis)(4) (1°)(2°)(3°) (D-8)(D-9) (D-11)(D-13)</td>
<td></td>
<td>$s_0/S_0$</td>
<td>$s/S$</td>
<td>General cavity when Eq. (3°) is substituted into Eq. (4)</td>
</tr>
<tr>
<td>(2 bis)(4) (3°) (D-7)(D-11) (D-13)</td>
<td></td>
<td>$s_0/S$</td>
<td>$s_0/S$</td>
<td>Spherical cavity only</td>
</tr>
</tbody>
</table>

Gouffé's analysis has several serious errors, and these have different results for different-shaped cavities. His recipe for spherical cavities (see the last row of data in Table 7) and his table of $\varepsilon_0$ values for spherical cavities provide correct results. His recipe for cylindrical cavities (see the next to last row of data in Table 7) and his table of $\varepsilon_0$ values for cylindrical cavities give good results for shallow cavities but poor results for elongated cavities. His recipe for conical cavities (see the next to last row of data in Table 7) and his table of $\varepsilon_0$ values for conical cavities give values much too low.

My Eq. (31) and Treuenfels (1963) and Kelly (1966) show that Gouffé's $\varepsilon_0$ in Eq. (33) is the same as

$$\varepsilon_0 = 1 - \rho_\alpha(\hat{r}) = \varepsilon_\alpha(\hat{r}).$$ (34)
De Vos' Theory

Another oft-cited analysis is by De Vos (1954a). When his Eqs. (6) and (9) are applied to cavities with diffuse walls and are written with my conventions, they become

\[ L(r) = L^{bb} \left( 1 - \frac{1 - \varepsilon}{\pi} \int_{\text{ap}} d\Omega_{21} \right) = L^{bb} \varepsilon_0 \]  
(35)

and

\[ L(r) = L^{bb} \left( 1 - \frac{1 - \varepsilon}{\pi} \int_{\text{ap}} d\Omega_{21} - \frac{(1 - \varepsilon)^2}{\pi^2} \int_{\text{wall}} \int_{\text{ap}} d\Omega_{32} d\Omega_{21} \right. 
- \frac{\varepsilon(1 - \varepsilon)}{\pi} \int_{\text{wall}} k d\Omega_{21} \), 
(36)

where

\[ k = \frac{\sigma_2 \Delta T}{\lambda T} \]

\( \sigma_2 \) is the second radiation constant, and \( \lambda \) is the wavelength. De Vos calls his Eq. (6) [my Eq. (35)] his first-order approximation. We might give a similar name to the isothermal version of the first three terms of Eq. (6), and then our first-order equation would agree exactly with De Vos' for a diffuse cavity. De Vos calls his Eq. (9) [my Eq. (36)] a second-order approximation, and in that equation the first two terms are simply a restatement of Eq. (35); the third term is a new isothermal second-order approximation term, and the last term is his lowest-order term to represent cavity temperature differences. There is no direct analog to this third term among Eqs. (2), (4), and (6) although it is tempting to compare it with the third term of Eqs. (2) and (4). These terms, however, have an extra factor of \( \varepsilon \) or \( \varepsilon_{32} \), and the signs are
positive where his are negative. The best way to make the desired comparison is to extend De Vos' isothermal, diffuse wall theory to an infinite number of terms. When this is done (see Appendix E), the results are identical in form to isothermal Eq. (7). Like Gouffé, De Vos announces his results as $\varepsilon_0$, a single figure of merit for each cavity, but in Eq. (35) he identifies $\varepsilon_0 = L(\mathbf{r})/L^b$, which is just $\varepsilon_L(\mathbf{r})$.

**Treuenfels' Theory**

Treuenfels (1963), according to his own discussion and discussions of Kelly (1965, 1966), solves for a quantity that is the same as those of Eq. (34), so we may write Treuenfels' result as follows, where we have substituted $F^*$ for his $f$ because we have previously used $f$ for BRDF:

$$
\varepsilon_a(\mathbf{r}) = 1 - \rho_a(\mathbf{r}) = \frac{1 - (1-F^*)\rho - \rho F^*}{1 - (1-F^*)\rho},
$$

which reduces to

$$
\varepsilon_a(\mathbf{r}) = \frac{E}{\varepsilon(1-F^*) + F^*}. \quad (37)
$$

Treuenfels calls $F^*$ a mean square view factor of the cavity opening seen by the cavity wall, and gives the expressions

$$
F^* = \frac{\int F^2 \, dA}{\int F \, dA} \quad (38a)
$$

$$
F = \pi^{-1} \int_{ap} d\Omega. \quad (38b)
$$
Note that $F$ is the view factor or configuration factor of the aperture as seen from a point on the cavity wall, and this quantity was described earlier in the section entitled "Projected Solid Angle Effect" (p. 40). Equation (37) is the same as Eq. (32) when $A = B = F^*$. 

**Kelly's Theory**

Kelly's (1966) analysis is similar to Gouffé's and Treuenfels' but more thorough. His Eq. (4) is

$$
\varepsilon_a(r) = \frac{\varepsilon[1 + (1-\varepsilon)(s/F)]}{\varepsilon(1 - s/S) + s/S}
$$

(39)

and this is similar to my Eq. (7) when $A = F$ and $B = s/S$.

**Nicodemus' Theory**

Nicodemus (1968) limits his analysis to the isothermal, Lambertian, spherical cavity, and since (see page 18) this analysis is a generalization of his, conversely, his is a special case of this. Thus it is not surprising to find that Nicodemus' solution [his Eq. (11)] is a special case of Eq. (7), where

$$
A = B = F = \pi^{-1} \int_{ap} d\Omega = s_0/S,
$$

(40)

where $s_0$ is the area of the spherical cap of the aperture, and $S$ is the total area of the sphere.
Comparison Summary

The study of the similarities among the theories can now be summarized with Table 8 and the following equation, which resembles isothermal Eq. (7):

\[ \epsilon_L(\hat{r}) = \epsilon_\alpha(\hat{r}) = 1 - \rho_\alpha(\hat{r}) = \frac{\epsilon[1 + (1-\epsilon)(B-A)]}{\epsilon(1-B) + B}. \quad (41) \]

The terms in Table 8 are as follows:

\[ F = \pi^{-1} \int_{ap} d\Omega = \text{the configuration factor (equals } \pi^{-1} \text{ times the projected solid angle of the aperture seen from a point on the cavity wall surface)} \]

\[ \overline{F} = \text{a weighted average of } F \]

\[ F^* = \frac{\int F^2 \, dA}{\int F \, dA} \]

\[ \omega = \text{the solid angle corresponding to } \overline{F} \]

\[ s = \text{aperture area} \]

\[ S = \text{total cavity surface area (equals aperture area plus wall area)} \]

\[ s_0 = \text{area of spherical cap of aperture} \]

\[ S_0 = \text{area of a fictitious spherical cavity that has the same depth as the cavity being considered (see Appendix D, p. 124).} \]

Note that the \( \overline{F} \) of Bartell and Wolfe and the \( \overline{F} \) of the integral equations are the same because they are developed from the same equation, an infinite series of positive terms. But these two \( \overline{F} \)'s may or may not be equal to the \( \overline{F} \) of De Vos because De Vos' quantity is derived from a
Table 8. Values of A and B for Eq. (41).

<table>
<thead>
<tr>
<th>Type of theory</th>
<th>References</th>
<th>Remarks</th>
<th>Quantity solved</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>This dissertation</td>
<td>--</td>
<td>Eq. (7)</td>
<td>$e_L(r)$</td>
<td>$F$</td>
<td>$\overline{F}$</td>
</tr>
<tr>
<td>Integral equation</td>
<td>Buckley (1928b), Sparrow and Albers (1960), Williams (1961), Sparrow and Jonsson (1963), Sydnor (1969), Bedford (1970), Chandos and Chandos (1974)</td>
<td>Eqs. (32), (C-5)</td>
<td>$e_\alpha(\aleph)$</td>
<td>$F$</td>
<td>$\overline{F}$</td>
</tr>
<tr>
<td>Integral equation</td>
<td>Moon (1940), Sparrow and Cess (1966, Eq. 3-40), Bedford (1970)</td>
<td>A constant times Eq. (32)</td>
<td>A constant times $e_\alpha(\aleph)$</td>
<td>$F$</td>
<td>$\overline{F}$</td>
</tr>
<tr>
<td>De Vos</td>
<td>De Vos (1954a)</td>
<td>Eq. (E-7); $\overline{F}_{DeV}$ may or may not equal $\overline{F}$</td>
<td>$e_L(r)$</td>
<td>$F$</td>
<td>$\overline{F}_{DeV}$</td>
</tr>
<tr>
<td>Kelly</td>
<td>Kelly (1966)</td>
<td>Eq. (39)</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$F$</td>
<td>$a/S$</td>
</tr>
<tr>
<td>Gouffé</td>
<td>Gouffé (1945)</td>
<td>His Eq. (2), my Eqs. (D-2),(D-3); general cavity</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$\omega/\pi$</td>
<td>$a/S$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>His Eq. (3), my Eqs. (D-2),(D-6),(D-7); spherical cavity only</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$a/S$</td>
<td>$a/S$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>His Eqs. (2 bis), (4), my Eqs. (D-8),(D-9); general cavity</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$s/S_0$</td>
<td>$a/S$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>His Eqs. (2 bis), (4),(1°),(2°), (3°), my Eqs. (D-8),(D-9),(D-11),(D-13); general cavity when Eq. (3°) is substituted into Eq. (4)</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$s_0/S_0$</td>
<td>$a_0/S$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>His Eqs. (2 bis), (4),(3°), my Eqs. (D-7),(D-11),(D-13); spherical cavity only</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$a_0/S_0$</td>
<td>$a_0/S$</td>
</tr>
<tr>
<td>Treuenfels</td>
<td>Treuenfels (1963)</td>
<td>Eqs. (37),(38a), (38b)</td>
<td>$1 - \rho_\alpha(\aleph)$</td>
<td>$F^*$</td>
<td>$F^*$</td>
</tr>
<tr>
<td>Nicodemus</td>
<td>Nicodemus (1968)</td>
<td>Eqs. (40),(43); spherical cavity only</td>
<td>$e_L(r)$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>
different equation, an infinite series of mostly negative terms. Most likely a uniqueness argument will establish that the $\overline{F}$ of De Vos is the same as the other $\overline{F}$, but we make no such claim at present.

**Previous Comparisons**

The agreement shown by Eq. (41) and Table 8 represents a satisfying state of affairs for past cavity theories. But the situation is not so satisfying for several previous comparisons among theories. Williams (1961, p. 569) says De Vos' first-order approximation [De Vos' Eq. (6)] "... agrees precisely with Gouffé's first approximation when $r^\alpha \xi = \rho/\pi." But this condition by Williams is almost never satisfied, as may be seen from De Vos' expression for a diffuse surface where there is a cos$\theta$ on the right-hand side [see De Vos' Eq. (3)]. Furthermore, as we have noted, Gouffé's first approximation should be

$$\varepsilon_0 = 1 - \frac{\tilde{\rho}}{\pi} \int_{ap} d\Omega$$

instead of

$$\varepsilon_0 = 1 - \frac{\rho \omega}{\pi}.$$  

Thus, comparing Eqs. (35) and (42), we can now improve on Williams' statement by saying: De Vos' first-order approximation, for a diffuse surface, agrees precisely with the corrected form of Gouffé's first approximation.

Sparrow and Jonsson (1963) in their Fig. 2 compare Gouffé's results against their own for conical cavities. The differences are
appreciable, and Gouffé's values are in error; however, Sparrow and Jonsson's presentation is equally in error. The Gouffé data are local effective hemispherical emissivities evaluated on axis. Gouffé calls his results $\varepsilon_{0}$, but Eq. (34) shows they are $1 - \rho_{\alpha}(\vec{r}) = \varepsilon_{\alpha}(\vec{r})$. Table D-2 shows these Gouffé values are much too low. The Sparrow and Jonsson data are the "apparent hemispherical emissivity of the cavity as a whole."

The Sparrow and Jonsson data thus tend to average local emissivity effects over the entire cone; and since most of the cone area is near the base where local emissivities are lower, rather than near the apex where local emissivities are higher, the Sparrow and Jonsson data are even lower than the Gouffé data. Thus Sparrow and Jonsson's Fig. 2 makes a faulty comparison, and as a result it shows that Gouffé's data are much higher than they should be, when in fact they are much lower than they should be.

Campanaro and Ricolfi (1967) make an error that is very similar to that of Sparrow and Jonsson's; and the Fig. 2 of each paper is quite similar: both pairs of authors compare their own results with Gouffé's for conical cavities. But where Sparrow and Jonsson use hemispherical emissivity for the cavity aperture, Campanaro and Ricolfi use a normal cavity emissivity. Campanaro and Ricolfi's values are close to those of Sparrow and Jonsson for large cone angles (when the half angle is more than 45°), but for small cone angles Campanaro and Ricolfi's values come closer to what Gouffé's values should be. Thus Campanaro and Ricolfi make a faulty comparison like Sparrow and Jonsson, but the results are not as misleading.
Kelly (1966) compares his results with Gouffé's in forms given by my Eq. (41) and Table 8. Kelly says Gouffé's $s_0/S_0$ (see the next to last Gouffé value of $A$ in Table 8) in place of his own correct $F$ is an approximation by Gouffé. But, as demonstrated in Appendix D, Gouffé's use of $s_0/S_0$ is an error, not an approximation for tilted back-wall shapes, and for cones the error is serious.

Both Nicodemus (1968) and Fecteau (1968) arrive at generally correct conclusions regarding the sameness in results among the theories of Gouffé (1945), De Vos (1954a), Sparrow and Jonsson (1962), and Nicodemus (1968) for the isothermal, diffuse, spherical cavity. However, both Nicodemus and Fecteau fail to distinguish between Gouffé's final correct expressions for the sphere (the last Gouffé $A$ and $B$ entries in Table 8) and his earlier incorrect expressions (the first two Gouffé entries for $A$ and also for $B$ in Table 8). Nicodemus says his Eq. (11) is the same as Gouffé's Eq. (3), but Nicodemus' $s$ is the area of the aperture's spherical cap whereas Gouffé's $s$ is just the aperture area. Similarly, Fecteau says his Eq. (6) is one form of the formula given by Gouffé, but again Fecteau's $s$ is the area of the aperture's spherical cap whereas Gouffé's $s$ is just the aperture area.

An Ecumenical Theory

There is an ecumenical character to this theory that is evidenced by the Bartell and Wolfe (1976b) paper, which gives part of the present analysis and is entitled "Cavity Radiators: An Ecumenical Theory."

One ecumenical feature of this dissertation is the combination of Eq. (41) and Table 8. It has been shown that the isothermal version
of the present theory, and also the work of 21 other papers, have results that can all be expressed in terms of Eq. (41) with different functional forms for $A$ and $B$ for the different theories. These different $A$'s and $B$'s are given in Table 8. The unifying nature of the single equation for the many theories (both integral-equation and multiple reflection) is in contrast to previous comparisons among theories where differences rather than similarities have most often been emphasized, e.g., Williams (1961, p. 571), Kelly and Moore (1965, pp. 37-38), Quinn (1967a, pp. 1108-1110), Hudson (1969, pp. 71-72), and Bedford (1970, pp. 177-182). Geist (1973, p. 1325) says that "the exact relation between the two schools [(1) integral equations and (2) multiple reflections] has not been clarified."

A second ecumenical feature of this dissertation is the detailed comparisons among the different theories that are stimulated by the form of Eq. (41) and Table 8. The different entries in the $A$ and $B$ columns of the table raise questions about the different assumptions and different analyses that have led to the different results. One example of this kind of comparison is Appendix D, which developed after it was noted that $A = F = \pi^{-1} \int_{\text{ap}} d\Omega$ according to my Eq. (41) whereas $A = \omega/\pi$ according to Gouffé's Eq. (2). A second example of this kind of comparison is based on Eqs. (16) and (36) rather than on Eq. (41) and Table 8. This second comparison involves the discrepant factor of $\varepsilon$ described in the last half of Appendix E.

A third ecumenical feature of this dissertation concerns the identity of the right-hand sides of Eqs. (4) and (C-4) (see Appendix C). Because of that identity, integral-equation formulations of the blackbody
simulator cavity problem can be handled with the solutions in Table 2 and Eq. (26); and furthermore, persons using tabulated or plotted outputs from computer calculations of integral-equation blackbody simulator problems can supplement those results with one of the approximate closed form equations of this dissertation.
FOUR SPECIAL CAVITY SHAPES

It is clear from the footnote on page 25 that evaluation of $\overline{F} = \pi^{-1} \int_{\text{ap}} d\Omega$ will in general be difficult. Table 8 and the discussion of other theories suggest that approximate values of $\overline{F}$ are given by $s/S$ or $F^*$. There are two geometries, however, that lend themselves to very accurate evaluation of $\overline{F}$, namely, points on a sphere and points very close to the apex of a nearly perfect cone. Therefore spheres and cones are the first two types of special cavity shapes. Table 4 for reentrant cones leads to some interesting design considerations, so reentrant cones are the third special group of cavity shapes. The fourth group of cavity shapes in this chapter is cavities with wall elements that are convex inward.

Spheres

According to Nicodemus (1968), both $F$ and $\overline{F}$ for a sphere are always equal to $s_0/S$, where $s_0$ is the area of the spherical cap associated with the aperture and $S$ is the total area of the sphere (wall plus aperture spherical cap). Then not only have we exactly evaluated $F$ and $\overline{F}$ in Eq. (41), but we also have $B = A$, so Eq. (41) simplifies to

$$\varepsilon_L(\hat{r}) = \varepsilon \left[ \frac{1}{\varepsilon (1 - s_0/S) + s_0/S} \right],$$

which is the same as Nicodemus' (1968) Eq. (11). I prefer:

$$\varepsilon_L(\hat{r}) = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{s_0}{S}}.$$

(43a)
Cones

According to Kelly (1966), in the infinitesimal area near the vertex of a nearly perfect cone,

\[ F = \bar{F} = \sin^3(\theta/2), \]

where \( \theta \) is the cone vertex angle. Then Eq. (41) simplifies to

\[ \epsilon_L(\hat{r}) = \epsilon \left[ \frac{1}{\epsilon + (1-\epsilon)\sin^3(\theta/2)} \right]. \] \hspace{1cm} (44)

which is the same as Kelly's Eq. (5). A better form is:

\[ \epsilon_L(\hat{r}) = \frac{1}{1 + \frac{1-\epsilon}{\epsilon} \sin^3(\frac{\theta}{2})}. \] \hspace{1cm} (44a)

Reentrant Cones

The entries in Table 4 suggest several interesting design variations for the reentrant cone. I am not sure whether these designs have any merit, but they are examples of how Table 4 can be used to study blackbody simulator designs. When \( \int_{\text{ap}} d\Omega \) at the apex of the rear cone is equal to \( \int_{\text{ap}} d\Omega \) on the rear cone near the junction of the front and rear cones, we have:

\[ z = v = g + h, \quad \text{at apex}, \]

\[ z = g \]

\[ v = (d^2 + g^2)^{1/2} \quad \text{at junction}. \]
This condition is an approach toward improving the uniformity of the output radiation of conical cavities, and it leads to:

\[
\frac{\pi^{-1}}{\Delta_{ap}} d\Omega = \frac{a^2 d(g + h)}{(g + h)^4} \frac{g + h}{(d^2 + h^2)^{3/4}}, \quad \text{at apex},
\]

\[
\frac{\pi^{-1}}{\Delta_{ap}} d\Omega = \frac{a^2 dg}{(d^2 + g^2)^2} \frac{g + h}{(d^2 + h^2)^{3/4}}, \quad \text{at junction}.
\]

So:

$$\frac{1}{(g + h)^3} = \frac{g}{(d^2 + g^2)^2}.$$ 

Let us solve this for \(g\) and \(h\) as a function of \(h/g\) for the normalized case where \(d = 1\):

\[
g = \frac{1}{\left[ (1 + h/g)^{3/2} - 1 \right]^{1/2}}.
\]

Equation (45) leads to the entries in Table 9.

<table>
<thead>
<tr>
<th>(h/g)</th>
<th>100</th>
<th>10</th>
<th>5</th>
<th>3</th>
<th>1.0</th>
<th>0.333</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g)</td>
<td>0.0314</td>
<td>0.168</td>
<td>0.270</td>
<td>0.378</td>
<td>0.740</td>
<td>1.36</td>
<td>1.78</td>
<td>2.55</td>
<td>8.15</td>
</tr>
<tr>
<td>(h)</td>
<td>3.14</td>
<td>1.68</td>
<td>1.35</td>
<td>1.13</td>
<td>0.740</td>
<td>0.454</td>
<td>0.357</td>
<td>0.255</td>
<td>0.0815</td>
</tr>
</tbody>
</table>

Figure 8 illustrates a number of designs from Table 9. Note the large cone angles. For \(g > h\) the cone angle is more than 90°. Even the
Fig. 8. Reentrant Cone Variations, Normalized to \(d = 1.0\) and \(\omega = 0.5\).

extreme case of \(h/g = 100\) has a cone angle of 2 arc cot(3.14) \(\approx 35^\circ\). This is in marked contrast to most analyses of conical cavities, where small cone angles are suggested for best design.

**Cavities with Convex-Inward Wall Elements**

Throughout this dissertation it has been assumed that radiant energy exchange can take place between any two elements on the cavity wall surface. That assumption is implied in nearly all theoretical studies of blackbody simulator cavities. However, Geist (1973, p. 1326) has used an interesting function \(K_{n,\omega}\), which is defined to be unity whenever the line between points \(n\) and \(\omega\) does not intersect the cavity wall.
surface, and zero otherwise. If the $d\Omega_{n}(n-1)$'s are redefined to be equal to Geist's $K_n(n-1)$ times their former values, and then these new $d\Omega_{n}(n-1)$'s are used throughout the analysis of this dissertation, the present theory can be extended to include cavity shapes for which not all surface elements are in a straight, unobscured line from all other surface elements, i.e., cavities with convex-inward wall elements. It should be noted that cavities with convex-inward wall elements are rather common, including for example cavities with internal baffles or grooved walls, e.g., Quinn (1967a, p. 1111). Another example of this type of cavity is a cylinder the back wall of which is an array of pyramids (Quinn, 1967b, p. 221).
THE QUANTITIES $T_m$ AND $\int_{ap} d\Omega$

The quantities $T_m$ and $\int_{ap} d\Omega$ are described in the footnotes on pages 24 and 25 and are developed further on pages 25 through 27. They represent the mathematical bridge in proceeding from the formidable problem that is Eq. (4) to the compact approximate solution that is Eq. (6). The difficulty of that mathematical problem is appreciated when it is realized that Eqs. (4) and (6) are general polythermal expressions and yet even isothermal, as well as polythermal, representations have elicited rather strong remarks from several prominent researchers about the severity of the problem. Their comments are summarized below.

Other Theories

Treuenfels (1963, p. 1165), describing isothermal conical and cylindrical holes, says, "These cavities are not easily adapted to calculation. A graphical method (Herman's method) was used to find the view factor."

Sparrow, Albers, and Eckert (1962, pp. 74-75) present their formulation of the isothermal, gray, diffuse cylindrical cavity in the form of their Eqs. (8) and (9), and comment: "The formidable mathematical problem embodied in equations (8) and (9) is not amenable to a closed-form solution, and it was necessary to use numerical means."

Sparrow and Jonsson (1963, pp. 817-819), describing the solutions to their diffuse isothermal cone problem [Eqs. (5) and (19)] and their diffuse, uniform wall heat flux, cone problem [Eqs. (11) and (19)],
says: "A detailed study of the governing integral equations (5) and (11) and the geometrical factor expression (19) reveals that closed-form, analytical solutions are not possible."

Bedford and Ma (1974, p. 340) describe their analysis of isothermal and polythermal diffuse cones and cylinders in this way:

Exact analytical solution of these integral equations is impossible. Sparrow and Jonsson used numerical integration (Simpson's rule) and successive iteration. This method has the disadvantage that it is inaccurate at and near those points at which the kernels in the integrals are singular (at the junctions of any two surfaces, at the apex of the cone); frequently, these regions are in the field of view of a detector. The resulting errors can be large; they accumulate from iteration to iteration.

We have eliminated this difficulty, thereby improving the accuracy, by using a refinement of the zonal approximation of heat-transfer analysis. The first integral in Eq. (2), for example, can be rewritten

$$
\int_{0}^{L} \varepsilon(x) \, d^2F_{x_0,x} \, = \, \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \varepsilon(x) \, d^2F_{x_0,x},
$$

where $x_i = 0$, $x_{n+1} = L$, and $n$ is an integer. Because $\varepsilon(x)$ is a slowly varying function of $x$, we may take $\varepsilon(x)$ to be a constant over the range of each of the subintegrals in Eq. (8) provided that we choose $n$ appropriately. This allows $\varepsilon(x)$ to be taken outside the integrals, to give

$$
\int_{0}^{L} \varepsilon(x) \, d^2F_{x_0,x} \, = \, \sum_{i=1}^{n} \frac{1}{2}[\varepsilon(x_{i+1}) + \varepsilon(x_i)] \times \int_{x_i}^{x_{i+1}} d^2F_{x_0,x},
$$

where, for the constant value of $\varepsilon(x)$ within each band, we have chosen the mean of its values at the end points. The integrals remaining in Eq. (4) can almost always be evaluated explicitly, even for rather complex cavity geometries.

Bedford and Ma (1975, pp. 566-568) formulate the diffuse isothermal and polythermal cylindro-cone problem with their integral Eqs.
We have solved Eqs. (3)-(5) by numerical iteration using an IBM 360/67 digital computer. Guidelines governing the necessary number of values of $\epsilon_a(x_i), \epsilon_a(y_j), \epsilon_a(a_k)$; the choice of starting values; the number of iterations; etc., are similar to those discussed previously [Bedford and Ma, 1974]. It is, of course, possible to solve this set of (typically 125) linear, simultaneous, algebraic equations by more straightforward methods: we have found matrix inversion to be prohibitively expensive and gaussian elimination, although useful in certain specific instances as a check on, for example, convergence, to be substantially more costly than numerical iteration.

These five references are, in a sense, discussing the problems of proceeding from Eq. (4) to Eq. (6). The compact simplicity of the approximate Eq. (6a) should be compared with the less accurate solution of Treuenfels [Eq. (37)] and the numerical or graphical (not analytical) solutions of the other four references. Thus the approximate quantities $T_m$ and $\int_{\Omega_p} d\Omega$ have been instrumental in finding a simple and compact solution to the blackbody simulator cavity problem. Furthermore, as the next two sections will show, $T_m$ and $\int_{\Omega_p} d\Omega$ are more than mathematical constructs whose approximate values must be guessed at; there are some definite procedures that can be followed to determine these quantities rather closely. Recall that approximate estimation of $T_m$ and $\int_{\Omega_p} d\Omega$ were discussed on page 27.

The value of $T_m$ should be selected as part of a study of the temperature distribution throughout the cavity. Traditionally the accurate measurement of cavity wall temperatures has been one of the most difficult problems in the use of blackbody simulators. The logic of pages 46 to 48 leads to the conclusion that good-quality blackbody
simulators in practical applications have \( \Delta T_m/T_m < 1/360 \). But in following the method of cavity evaluation developed in this dissertation, one can do better than just measure cavity temperatures conventionally and then estimate \( T_m \). This new method promises considerable improvement in the handling of temperature measurements. Instead of using Eq. (6) or (6a), we should work with Eqs. (19) and (11), and we note that all the variables can be rather easily found except perhaps \( \Delta T_m, \int_{ap} d\Omega, \alpha', \alpha'', \beta, \) and \( \gamma \). Note that the most important temperature involved is \( T_t \), a temperature transducer temperature, and that is much easier to work with than a cavity wall temperature. The \( \alpha ' s \) and \( \beta \) and \( \gamma \) can be estimated, or they can be measured reasonably accurately by building and testing dummy cavities with many temperature transducer wells. The quantity \( T_m \) can then be estimated much more accurately than the one part in 360 mentioned above if a careful study of the cavity interior temperature is made and the results are combined with a careful map of the projected solid angle of the aperture as seen from all points on the cavity wall.

The recipe for \( T_m \) given on page 24 employs a one-term approximation in going from Eq. (A-1) to Eq. (A-4) in Appendix A. For highest accuracy, the number of terms should be increased until there is no significant change in \( T_m \) when another term is added.

\[
\int_{ap} d\Omega
\]

Like \( T_m, \int_{ap} d\Omega \), is described in a footnote (page 25), and further discussion is presented on pages 26, 27, and 47. For a preliminary estimate of \( \int_{ap} d\Omega \), page 47 suggests the simple unaveraged quantity \( \int_{ap} d\Omega \) be used. For greater accuracy, a map is needed that gives the projected
solid angle of the aperture as seen from all points on the cavity wall. A second map is also needed, which is really a set of maps giving the projected solid angle of small constant areas at a number of locations as seen from all points on the cavity wall. The recipe on page 25 can be followed using these two maps (or map and set of maps). When the highest accuracy is desired, it is recommended (in a recommendation similar to that for $T_m$) that the method of calculating $\int_{ap} d\Omega$ be patterned after the way it is developed in Appendix A.

The Accuracy of $T_m$ and $\int_{ap} d\Omega$

We have characterized the theory of this dissertation as being approximate because Eqs. (6) to (19) include the quantities $T_m$ and $\int_{ap} d\Omega$ and, in general, these quantities cannot be precisely determined. However, the section entitled "$T_m$" suggests that the theory of this dissertation presents a very accurate method for incorporating temperature measurements into blackbody simulator cavity emissivity calculations.

The outlook for accurate determinations of $\int_{ap} d\Omega$ is not so favorable as for $T_m$. But the two maps of $\int_{ap} d\Omega$ (or one map and one set of maps) should be attainable to a reasonable degree of completeness and accuracy without excessive computer time. Then if these results are combined according to a plan determined by the derivation of $\int_{ap} d\Omega$ in Appendix A, the result should be a reasonably accurate finding for this quantity. Furthermore, Eq. (19) is a complete expression of my results; the discussion on pages 46 to 49 shows that the third and fifth terms of Eq. (19) are the least important, and these are the only terms that contain $\int_{ap} d\Omega$. 
Finally, it should be noted that, in the chapter on "Four Special Cavity Shapes," exact evaluation of $\int_{\text{ap}} d\Omega$ is given for points on a sphere and points in the vicinity of the apex of a cone.
HOW TO CALCULATE EFFECTIVE CAVITY EMISSIVITY

I recommend that the effective emissivity for blackbody simulator cavities be calculated according to three different levels of accuracy and complexity.

First Approximations

For the first approximation I recommend use of the first two terms of Eq. (19):

\[
\varepsilon_L(\hat{r}) = \varepsilon + (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right)
\]

\[
= 1 - \frac{1-\varepsilon}{\pi} \int_{\text{ap}} d\Omega.
\] (46)

A polythermal variation of the isothermal equation, Eq. (46), is given by Eqs. (25) and (26):

\[
\varepsilon_L(\hat{r}, T) = 1 - (\text{a constant}) \pi^{-1} \int_{\text{ap}} d\Omega,
\]

where the constant is

\[
1 - \varepsilon + \varepsilon \left( \frac{\partial L}{\partial T} \frac{T}{L} \right)_{T_t} \frac{\Delta T_t}{T_t}.
\]

The bracket is equal to the parameter \( G_5 \), and values of \( G_5 \) for typical \( \lambda \) and \( T \) are given in Table 6.
Second Approximation

For the second level of approximation, I recommend the use of isothermal Eq. (7). Equation (7) is an appropriate successor to Gouffé's theory, which it resembles. Note that Eq. (7) is equal to the first three terms of Eq. (19).

In using Eq. (7) at a second level of approximation, I recommend calculating the appropriate values of $A = \pi^{-1}\int \omega \, d\Omega$. For $B = \pi^{-1}\int \omega \, d\Omega$, I recommend estimating its value with the assistance of the footnote on page 25, the comments on page 47, and the discussion of $\int \omega \, d\Omega$ in the previous chapter.

The form of Eq. (7)--and from it, Eq. (41)--was chosen to resemble the work of other authors [Gouffé, 1945, Eq. (2); Treuenfels, 1963, p. 1163; Kelly, 1966, Eq. (4); and Nicodemus, 1968, Eq. (11)]. However, it is easier to perform calculations when this equation is rearranged:

$$
\varepsilon_{L}(\tau) = 1 + \frac{(1-\varepsilon)(B-A)}{1 + \frac{1-\varepsilon}{\varepsilon} B}.
$$

(48)

It is instructive to manipulate Eq. (48) further. Table 5 and the discussion on page 47 show that, for good quality blackbody simulators, the second term in the denominator of Eq. (48) will always be less than 0.02. Therefore, to an accuracy of better than 0.0004, Eq. (48) becomes

$$
\varepsilon_{L}(\tau) = 1 - (1-\varepsilon)A - \frac{(1-\varepsilon)^2}{\varepsilon} B.
$$

(49)
Equation (49) shows more clearly than Eq. (48) the relative importance of $A$ and $B$. However, I prefer Eq. (48) to Eq. (49) for calculations.

**Third Approximations**

For the third level of approximation I recommend use of the full Eq. (19). For most applications, parameters that cannot be readily calculated or measured should be estimated. For improved accuracy, the previous chapter should be consulted. That chapter describes $T_m$ and $\int d\Omega_{ap}$ in detail, and it also discusses $\alpha'$, $\alpha''$, $\beta$, and $\gamma$. The final chapter of this dissertation considers future studies that may result in very high accuracy determinations of effective cavity emissivity by using Eq. (19).

**Computer Calculations**

Computer calculations are generally associated with integral equation formulation of the blackbody simulator cavity problem, and they probably correspond to the third level of approximation just described. In Appendix C I recommend that persons using numerical and graphical computer evaluations for effective cavity emissivity also consider the parallel use of some of the approximate closed-form analytic expressions given in this dissertation.
CONCLUSIONS AND EXTENSIONS

This dissertation is the result of a search for a blackbody simulator theory that would properly take into account the variations with angle of the output of blackbody simulator radiation sources. A multiple reflection analysis has been performed with the assumptions of gray, Lambertian walls and a cold aperture, and the result has been a compact, closed-form expression that emphasizes the geometrical effects of the projected solid angle as seen from various points on the back wall of the cavity. Variations of this expression give special forms with terms representing temperature variations parallel to and perpendicular to the cavity wall. Other variations provide for analysis according to the radiation laws of Stefan-Boltzmann, Planck, Wien, or Rayleigh-Jeans. This theory is characterized as being approximate, because the solutions contain a special mean temperature and a special mean value of the projected solid angle; and although these mean values can be approximated, they cannot, in general, be determined precisely.

A simplified version of the theory has been shown to be valid to an accuracy of about 2% for good-quality blackbody simulators used in typical practical applications. This simplified expression has been compared with experimental data for conical, cylindrical, and spherical cavities, and the basic characteristics have been confirmed concerning the angular distribution of radiation from blackbody simulator cavities with these three shapes.
The form of the approximate isothermal solution of this dissertation is such that it lends itself to convenient comparisons with the analyses of 21 other papers. Informative, critical comparisons are stimulated by the close similarities that are brought out in this dissertation among these diverse papers.

Finally, recommended procedures are given for determining the approximate value of the effective emissivity of blackbody simulator cavities, and these recommendations are given at three different levels of approximation.

It is recommended that followup work be done in the areas of $T_m$, $\int_{\text{ap}} \hat{d} \Omega$, and $\varepsilon$. Experimental temperature studies of different cavity designs could lead to greatly improved accuracies for the quantities $\alpha'$, $\alpha''$, $\beta$, and $\gamma$, which would lead to better values for $\varepsilon$. Computer studies of different cavity designs could result in accurate maps of the projected solid angle of the aperture as seen from all points on the cavity wall, and such studies could also provide sets of maps of the projected solid angle of small constant areas at a number of locations as seen from all points on the cavity wall. By finding better values of $T_m$, $\int_{\text{ap}} \hat{d} \Omega$, and $\varepsilon$, these experimental and computer studies might lead to greatly improved evaluations of cavity radiation sources.

The next step should be the study of cavities with directional surface characteristics. The mathematics of the present dissertation are formidable enough, and the extension to general directional emissivity and reflectance represents such a very great increase in complexity, that some type of limited directional properties should probably be specified. One example might be that of Lin and Sparrow (1965), where
the emissivity is diffuse and the reflectivity is specular. Another possibility is the development of new simplified reflectance relationships that may arise from current studies of the bidirectional reflectance distribution functions (BRDFs) of materials. An example of this BRDF work is given in Bartell et al. (1976, pp. 3-7, 3-8, and 7-1 through 7-33).
APPENDIX A

DERIVATION OF EQUATION (6) FROM EQUATION (4)

We begin with Eq. (4):

$$\varepsilon_L(\hat{r},T_1) = \varepsilon + \varepsilon(1-\varepsilon)\pi^{-1} \int_{\text{wall}} (T_2/T_1)^4 d\Omega_{21}$$

$$+ \varepsilon(1-\varepsilon)^2\pi^{-2} \int_{\text{wall}} \int_{\text{wall}} (T_3/T_1)^4 d\Omega_{32} d\Omega_{21}$$

$$+ \cdots$$

$$+ \varepsilon(1-\varepsilon)^n\pi^{-1-n} \int_{\text{wall}} \cdots \int_{\text{wall}} (T_n/T_1)^4 d\Omega_{n(n-1)} \cdots d\Omega_{21}$$

$$+ \cdots.$$ 

Figure 3 showed that \(\alpha_1\) and hence temperature \(T_1\) are constant for the evaluation of \(\varepsilon_L(\hat{r},T_1)\), so \(T_1\) can be removed from the integrals. Therefore,

$$\varepsilon_L(\hat{r},T_1) T_1^4 = \varepsilon T_1^4 + \varepsilon(1-\varepsilon)\pi^{-1} \int_{\text{wall}} T_2^4 d\Omega_{21}$$

$$+ \varepsilon(1-\varepsilon)^2 \pi^{-2} \int_{\text{wall}} \int_{\text{wall}} T_3^4 d\Omega_{32} d\Omega_{21}$$

$$+ \cdots$$

$$+ \varepsilon(1-\varepsilon)^n\pi^{-1-n} \int_{\text{wall}} \cdots \int_{\text{wall}} T_n^4 d\Omega_{n(n-1)} \cdots d\Omega_{21}$$

$$+ \cdots.$$  

(A-1)
According to the mean value theorem of integral calculus (Courant, 1936), there are some values of temperature $\overline{T}_2$, $\overline{T}_3$, and $\overline{T}_{32}$ such that

$$
\int_{\text{wall}} T_2^4 \, d\Omega_{21} = \overline{T}_2^4 \int_{\text{wall}} d\Omega_{21}
$$

and

$$
\int_{\text{wall}} \int_{\text{wall}} T_3^4 \, d\Omega_{32} \, d\Omega_{21} = \int_{\text{wall}} \overline{T}_3 \int_{\text{wall}} d\Omega_{32} \, d\Omega_{21}
$$

and

$$
\int_{\text{wall}} \int_{\text{wall}} \overline{T}_3 \, d\Omega_{32} \, d\Omega_{21} = \overline{T}_{32} \int_{\text{wall}} \int_{\text{wall}} d\Omega_{32} \, d\Omega_{21}.
$$

Therefore, there is some $\overline{T}_{n(n-1)\cdots(3)(2)}$ such that

$$
\int_{\text{wall}} \cdots \int_{\text{wall}} T_n^4 \, d\Omega_{n(n-1)} \cdots d\Omega_{32} \, d\Omega_{21} = \overline{T}_{n(n-1)\cdots(3)(2)} \int_{\text{wall}} \cdots \int_{\text{wall}} d\Omega_{n(n-1)} \cdots d\Omega_{21}
$$

and therefore

$$
\varepsilon_{\varphi}(\overline{T}, \overline{T}_1) T_1^4 = \varepsilon T_1^4 + \varepsilon (1-\varepsilon) \overline{T}_2 \int_{\text{wall}} d\Omega_{21}
$$

$$+ \varepsilon (1-\varepsilon)^2 \overline{T}_3 \int_{\text{wall}} \int_{\text{wall}} d\Omega_{32} \, d\Omega_{21}
$$

$$+ \varepsilon (1-\varepsilon)^3 \overline{T}_{32} \int_{\text{wall}} \int_{\text{wall}} \int_{\text{wall}} d\Omega_{33} \, d\Omega_{32} \, d\Omega_{21}
$$

$$+ \cdots
$$

$$+ \varepsilon (1-\varepsilon)^n \overline{T}_{n(n-1)\cdots(2)} \int_{\text{wall}} \cdots \int_{\text{wall}} d\Omega_{n(n-1)} \cdots d\Omega_{21}
$$

$$+ \cdots
$$

(A-4)
To evaluate these terms, we first consider the inner integral

\[
\int_{\text{wall}} d\Omega_{n(n-1)} = \int_{2\pi \text{ ster}} d\Omega_{n(n-1)} - \int_{\text{aperture}} d\Omega_{n(n-1)}.
\]

Recall that \(d\Omega_{n(n-1)}\) is the projected solid angle at the point \(n-1\) subtended by an element of area \(d\alpha\) at point \(n\). But

\[
\int_{2\pi \text{ ster}} d\Omega_{n(n-1)} = \int_0^{2\pi} \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta \, d\phi
\]

in agreement with Nicodeumus' (1965) Eq. (7) and the equations that follow it. (Recall that \(d\Omega\) here corresponds to \(d\Omega'\) of Nicodemus.) Thus

\[
\int_{2\pi \text{ ster}} d\Omega_{n(n-1)} = \pi.
\]

Let "aperture" be designated by "ap," so that

\[
\int_{\text{wall}} d\Omega_{n(n-1)} = \pi - \int_{\text{ap}} d\Omega_{n(n-1)}.
\]

Let

\[
I_2 = \int_{\text{wall}} \int_{\text{wall}} d\Omega_{n(n-1)} \, d\Omega_{(n-1)(n-2)}.
\]

Then

\[
I_2 = \int_{\text{wall}} \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)} \right] \, d\Omega_{(n-1)(n-2)}
\]

\[
= \pi \int_{\text{wall}} d\Omega_{(n-1)(n-2)} - \int_{\text{wall}} \int \int_{\text{ap}} d\Omega_{n(n-1)} \, d\Omega_{(n-1)(n-2)}.
\]
The inner integral of the second term of $I_2$ is the projected solid angle of the aperture as seen from a general point $n-1$. This integral then becomes a weighting factor for the outer integral, which sums projected solid angles of all small area elements at points $n-1$ as seen from a general point $n-2$. The inner integral will have different values depending on where the point $n-1$ is located. However, as we perform the outer integration, the mean value theorem of integral calculus (Courant, 1936) provides that there is some average value of the inner integral such that

$$
\int_{\text{wall}} \int_{\text{ap}} d\Omega_{n(n-1)} d\Omega_{(n-1)(n-2)} = \int_{\text{ap}} d\Omega_{n(n-1)} \int_{\text{wall}} d\Omega_{(n-1)(n-2)}. 
$$

(A-5)

Thus $I_2$ becomes

$$
I_2 = \pi \int_{\text{wall}} d\Omega_{(n-1)(n-2)} - \int_{\text{ap}} d\Omega_{n(n-1)} \int_{\text{wall}} d\Omega_{(n-1)(n-2)} = \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)} \right] \int_{\text{wall}} d\Omega_{(n-1)(n-2)}.
$$

The last integral is evaluated like

$$
\int_{\text{wall}} d\Omega_{n(n-1)} = \pi - \int_{\text{ap}} d\Omega_{n(n-1)}.
$$

Therefore

$$
I_2 = \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)} \right] \left[ \pi - \int_{\text{ap}} d\Omega_{(n-1)(n-2)} \right].
$$
Let

\[ I_3 = \int_{\text{wall}} \int_{\text{wall}} \int_{\text{wall}} d\Omega_{n(n-1)} d\Omega_{n-1}(n-2) d\Omega_{n-2}(n-3) \]

\[ = \int_{\text{wall}} I_2 d\Omega_{n-2}(n-3) \]

\[ = \int_{\text{wall}} \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)} \right] \left[ \pi - \int_{\text{ap}} d\Omega_{n-1}(n-2) \right] d\Omega_{n-2}(n-3) \]

\[ = \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)(n-2)} \right] \int_{\text{wall}} \left[ \pi - \int_{\text{ap}} d\Omega_{n-1}(n-2) \right] d\Omega_{n-2}(n-3) . \]

The mean value theorem that was used to extract the first bracket from the integral is described by Courant (1936).

The aperture integral with three subscripts that is outside the wall integral is indeed different from that with two subscripts inside the wall integral. Physical reasoning (based on integration over the same surface) indicates that these two integrals will not be much different—especially for larger numbers of subscripts. Then

\[ I_3 = \left[ \pi - \int_{\text{ap}} d\Omega_{n(n-1)(n-2)} \right] \int_{\text{wall}} \int_{\text{wall}} d\Omega_{n-1}(n-2) d\Omega_{n-2}(n-3) . \]
The double integral can be solved the same way \( I_2 \) was solved. Then

\[
I_3 = \left( \pi - \int_{\text{wall}} d\Omega_{n(n-1)(n-2)} \right) \left( \pi - \int_{\text{wall}} d\Omega_{(n-1)(n-2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-2)(n-3)} \right).
\]

In general one can write

\[
I_n = \int_{\text{wall}} \cdots \int_{\text{wall}} d\Omega_{n(n-1)\cdots(n-2)} \\
= \left( \pi - \int_{\text{wall}} d\Omega_{n(n-1)(n-2)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-1)(n-2)(n-3)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-2)(n-3)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-3)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-4)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-5)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-6)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-7)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-8)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-9)\cdots(2)} \right) \\
\times \left( \pi - \int_{\text{wall}} d\Omega_{(n-10)\cdots(2)} \right).
\]

Note that there is some average integral

\[
\int_{\text{wall}} d\Omega_n
\]

such that
\[
\left[ \pi - \int_{ap} d\Omega_{n(n-1)(n-2)\cdots(3)(2)} \right] \left[ \pi - \int_{ap} d\Omega_{(n-1)(n-2)\cdots(3)(2)} \right] \cdots \\
\times \left[ \pi - \int_{ap} d\Omega_{32} \right] = \left[ \pi - \int_{ap} d\Omega_n \right]^{n-2}.
\]

Therefore

\[
I_n = \int_{wall} \cdots \int_{wall} d\Omega_{n(n-1)} d\Omega_{(n-1)(n-2)} \cdots d\Omega_{32} d\Omega_{21}
\]

\[
= \left[ \pi - \int_{ap} d\Omega_n \right]^{n-2} \left[ \pi - \int_{ap} d\Omega_{21} \right],
\]

where the first integral is an average integral evaluated over the aperture from some average point within the cavity and the second integral is the projected solid angle of the aperture as seen from \(a_1\) in Fig. 3. Thus, Eq. (A-4) becomes

\[
\varepsilon_L^{T_1} = \varepsilon_T^{T_1} + \varepsilon(1-\varepsilon) \pi^{-1} T_2^{\frac{1}{2}} \left[ \pi - \int_{ap} d\Omega_{21} \right]
\]

\[
+ \varepsilon(1-\varepsilon)^2 \pi^{-2} T_3^{\frac{1}{2}} \left[ \pi - \int_{ap} d\Omega_3 \right] \left[ \pi - \int_{ap} d\Omega_{21} \right]
\]

\[
+ \varepsilon(1-\varepsilon)^3 \pi^{-3} T_4^{\frac{1}{2}} \left[ \pi - \int_{ap} d\Omega_4 \right] \left[ \pi - \int_{ap} d\Omega_{21} \right]
\]

\[
+ \cdots
\]

\[
+ \varepsilon(1-\varepsilon)^{n-1} \pi^{-n} T_{n(n-1)\cdots432}^{\frac{1}{2}} \left[ \pi - \int_{ap} d\Omega_n \right]^{n-2} \left[ \pi - \int_{ap} d\Omega_{21} \right]
\]

\[
+ \cdots .
\]

(A-6)
The $\overline{T}_{n(n-1)\ldots32}$ are average temperatures selected so that Eq. (A-3) would be true. Therefore the $(n-1)$ different $\overline{T}_{n(n-1)\ldots32}$ of Eq. (A-6) will be about the same value, but slightly different. However, for any set of values for all the quantities in Eq. (A-6) there is some quantity $T_m$ such that all the $\overline{T}_{n(n-1)\ldots32}$ can be replaced by $T_m$ and Eq. (A-6) will still be valid.

Similarly, the different $\int_{ap} d\Omega_n$ can all be replaced by a single new average integral $\overline{d\Omega}$. At this point we will also drop the subscripts $21$ from $\int_{ap} d\Omega_{21}$ because we will no longer deal with multiple forms of the integral, and without the subscripts the simple $21$ operation is the one that would normally be performed. Thus:

$$\varepsilon_L(\vec{x}, T_1)T_1^4 = \varepsilon T_1^4 + \varepsilon (1-\varepsilon)T_m^4 \pi^{-1}(\pi - \int_{ap} d\Omega) \sum_{n=0}^{\infty} (1-\varepsilon)^n \pi^{-n}(\pi - \int_{ap} d\Omega)^n.$$

But

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = (1-x)^{-1}$$

when $-1 < x < 1$, and this is always true for the present case. Therefore,

$$\varepsilon_L(\vec{x}, T_1)T_1^4 = \varepsilon T_1^4 + \frac{\varepsilon (1-\varepsilon)T_m^4 \pi^{-1}(\pi - \int_{ap} d\Omega)}{1 - (1-\varepsilon) \pi^{-1}(\pi - \int_{ap} d\Omega)}.$$

The denominator in the last expression is equal to
\[ 1 - 1 + \varepsilon + (1-\varepsilon) \pi^{-1} \int_{\Omega} d\Omega = \varepsilon \left( 1 + \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\Omega} d\Omega \right). \]

Therefore

\[ \varepsilon_L(\mathbf{r}, T_1) = \varepsilon + \frac{(T_m/T_1)^4(1-\varepsilon) \left( 1 - \pi^{-1} \int_{\Omega} d\Omega \right)}{1 + \frac{(1-\varepsilon)}{\varepsilon \pi} \int_{\Omega} d\Omega}. \]  

(A-7)

And this is Eq. (6a).

The second term of Eq. (A-7) has the form

\[ \alpha(1-\varepsilon)(1-\beta)(1+\gamma)^{-1}, \]

but

\[ (1+\gamma)^{-1} = \frac{1+\gamma-\gamma}{1+\gamma} = 1 - \gamma(1+\gamma)^{-1}. \]

Therefore the second term becomes

\[ \alpha(1-\varepsilon)(1-\beta) - \alpha(1-\varepsilon)(1-\beta)\gamma(1+\gamma)^{-1} \]

\[ = \alpha(1-\varepsilon) - \alpha(1-\varepsilon)\beta - \alpha(1-\varepsilon)\gamma(1-\beta)(1+\gamma)^{-1}. \]

So Eq. (A-7) becomes
\[ \epsilon_L(\hat{r}, T_1) = \epsilon + (T_m/T_1)^4(1-\epsilon) - (T_m/T_1)^4 \frac{(1-\epsilon)}{\pi} \int_{ap} d\Omega \]

\[ - \frac{(T_m/T_1)^4 \frac{(1-\epsilon)^2}{\epsilon \pi} \int_{ap} d\omega \left(1 - \pi^{-1} \int_{ap} d\Omega\right)}{1 + \frac{(1-\epsilon)}{\epsilon \pi} \int_{ap} d\Omega}, \]

(A-8)

which is Eq. (6).
APPENDIX B

\[ \Delta T_t/T_t \text{ AND THE COEFFICIENT } \varepsilon \]

The general arrangement for the derivation of \( \Delta T_t/T_t \) is shown in Fig. 4. First, observe that for good blackbody simulator design the core material must have high thermal conductivity to minimize thermal gradients and the cavity wall surface must have high emissivity. In general, no single material has both these properties, so in general good cavity design calls for a metal core with an internal surface coating.

Consider a very small isothermal element of area (AREA) located somewhere between the temperature transducer at temperature \( T_t \) and the wall surface at temperature \( T_1 \). Then conductive heat transfer across that area will be in accordance with

\[
d\dot{Q}/dt = -k(\text{AREA})(d\dot{T}/dn), \quad (B-1)
\]

where

\[
\begin{align*}
    k &= \text{coefficient of thermal conductivity} \\
    \text{AREA} &= \text{element of area} \\
    T &= \text{temperature} \\
    \eta &= \text{displacement perpendicular to isothermal surfaces}.
\end{align*}
\]

Consistent with Fig. 4, we choose the positive directions outward for \( d\dot{Q} \), \( d\dot{T} \), and \( d\eta \). Then since, in general, the temperature decreases inward (in the direction from the heaters to the cavity wall surface), the heat
transfer will be inward; thus, in agreement with Eq. (B-1), $dT/\partial n$ is greater than zero and $dQ/dt$ is less than zero. If we let

$$P = \text{power} = -dQ/dt,$$  \hspace{1cm} (B-2)

then $P$ will be a positive quantity equal to the rate of conductive heat transfer across AREA.

Figure 4 shows that

$$dr = d\eta \cos \beta,$$  \hspace{1cm} (B-3)

where

$dr$ is measured normal to the axis,

$d\eta$ is measured normal to the isotherms,

$\beta$ is the angle between these two directions.

Now consider the mathematical construct, which we will call a "heat tube," and which we show in Fig. 4. This heat tube is defined by moving the edges of the very small area element perpendicular to isothermal surfaces so that the heat tube extends from the $T_1$ isotherm to the $T_\text{f}$ isotherm. We have specified that AREA be very small so that it is small compared to the coating thickness and the distance from the core-coating interface to the temperature transducer; we can then neglect the awkward geometrical situation at the place where the heat tube intersects the core-coating interface.

Note that the area of intersection between the heat tube and an isothermal surface will vary along the heat tube in a way that is
dependent on the cavity shape. If the isotherms are cylindrical or conical, the area will be proportional to $r$. If the isotherms are spherical, the area will vary according to $r^2$. Then if we consider Eq. (B-1) as applying to any of the small isothermal areas defined by the heat tube, we have

$$\text{AREA} = \Delta \delta r^\alpha,$$  \hfill (B-4)

where

$$\Delta \delta = \text{a constant chosen to make Eq. (B-4) true; physically it is the value AREA would have at } r=1 \text{ if the heat tube extended to that location}$$

$$r = \text{the distance off axis}$$

$$\alpha = \text{an exponent that depends on the cavity shape; for cylindrical or conical isotherms, } \alpha = 1, \text{ and for spherical isotherms, } \alpha = 2.$$

Cavity shapes that are exactly or approximately conical, cylindrical, or spherical can be represented approximately by Eq. (B-4) with $\alpha$ values that are at or near 1.0 or 2.0. Many other shapes can be represented by Eq. (B-4) when other values of $\alpha$ are considered. Shapes for which no simple expression, $\Delta \delta r^\alpha$, can be found can still be represented by a series of such terms with different values of $\alpha$ in each term; and when integration is required further along in this analysis, numerical techniques may be used. We are not so much interested in finding values of $\alpha$ to represent different cavity shapes as we are in applying the mathematical form of Eq. (B-4) to lead to an expression for $\Delta T_7$ in terms of the thermal structure of the cavity. More is said about $\alpha$ (and its successors $\alpha'$ and $\alpha''$) further on in this derivation, in connection with
Eqs. (B-6) and (B-7). The quantities $a'$ and $a''$ are also discussed on page 33 (where the results of this appendix are introduced into the main theory of this dissertation) and on pages 79 and 86 (where improved evaluations of $a'$ and $a''$ are discussed).

When Eq. (B-4) is applicable, Eqs. (B-1), (B-2), (B-3), and (B-4) give

$$dT = -\frac{Pdr}{k\Delta s r^\alpha \cos \beta}.$$  \hspace{1cm} (B-5)

Then

$$\Delta T_t = T_1 - T_t = \Delta T_{\text{coating}} + \Delta T_{\text{core}} = -\int_{T_1}^{T_t} dT.$$  

Recall that because of the way $\Delta T_t$ was defined in Eq. (10), it will normally be less than zero. Then

$$\Delta T_t = -P \left[ \int_{r_1}^{r_{cc}} \frac{dr}{k_{\text{coating}} \Delta s r^\alpha \cos \beta} + \int_{r_{cc}}^{r_t} \frac{dr}{k_{\text{core}} \Delta s r^\alpha \cos \beta} \right],$$

where

$r_1$ = the distance off axis where the heat tube reaches the cavity wall surface

$r_{cc}$ = the distance off axis where the heat tube crosses the core-coating interface

$r_t$ = the distance off axis of the temperature transducer.

Strictly speaking, in most cases $\Delta s$, $\alpha$, and $\beta$ will differ in the two integrals. However, for common simple cavity shapes the two $\Delta s$'s
and the two $\beta$'s will be nearly the same, so $(\Delta s \cos \beta)$ can be brought out of the bracket as a common factor with little loss of accuracy. Furthermore, there are some average values, $\Delta s$ and $\beta$, that will make that factoring operation exactly correct. There is no simple way to combine the two terms involving $r^\alpha$, so we use two different symbols, $\alpha'$ and $\alpha''$, for the $\alpha$ values in the two terms.

We can now rewrite the last equation in a more compact way, remembering that $\Delta s$ and $\beta$ are special average values and the two $\alpha$'s ($\alpha'$ and $\alpha''$) are probably about the same although they may be different:

$$\Delta T_t = -\frac{P}{\Delta s \cos \beta} \left[ \frac{1}{k_{\text{coat}}} \int_{r_1}^{r_{cc}} \frac{dr}{r^{\alpha'}} + \frac{1}{k_{\text{core}}} \int_{r_{oc}}^{r_t} \frac{dr}{r^{\alpha''}} \right].$$  \hspace{1cm} (B-6)

The integration of these terms will depend on whether $\alpha'$ and $\alpha''$ are equal to or unequal to 1.0. Therefore our next equation has several alternative forms:

$$\Delta T_t = -\frac{P}{\Delta s \cos \beta} \left[ \right],$$  \hspace{1cm} (B-7)

where

$$\left[ \right] = \begin{cases} \frac{r_{cc}^{1-\alpha'} - r_1^{1-\alpha'}}{k_{\text{coat}}(1-\alpha')} + \frac{r_t^{1-\alpha''} - r_{cc}^{1-\alpha''}}{k_{\text{core}}(1-\alpha'')} & \text{when } \{ \alpha' \neq 1, \alpha'' \neq 1 \} \\
\text{Replace 1st term by } \frac{\ln(r_{cc}/r_1)}{k_{\text{coat}}} & \text{when } \alpha' = 1 \\
\text{Replace 2nd term by } \frac{\ln(r_t/r_{cc})}{k_{\text{core}}} & \text{when } \alpha'' = 1.\end{cases}$$

The quantity $P$ in Eq. (B-7) must equal the net power radiated away from that part of the cavity wall surface that is intercepted by the heat
tube. To a first order of approximation, that will be the power that is radiated out the aperture. This calculation is similar to the recipe of least sophistication in Siegel and Howell (1972) and the definition of view factor in Özisik (1973). We neglect the power that is radiated back through the aperture, and we include a factor of \( \varepsilon \), which is the cavity wall emissivity. Then

\[
P = \varepsilon \sigma T_1^4 \left( \int \frac{da_1}{\text{heat tube}} \right) F_{1-ap}
\]

where

\[
\int \frac{da_1}{\text{heat tube}} = \frac{\Delta \theta r_1^{a'}}{\cos \gamma} = \text{area of cavity wall intercepted by the heat tube},
\]

\[
F_{1-ap} = \pi^{-1} \int \frac{d\Omega}{\text{ap}} = \text{view factor or configuration factor of the aperture as seen from } a_1,
\]

as may be seen from Eq. (B-4) and Fig. 4. Then the radiated power becomes

\[
P = \varepsilon \sigma T_1^4 \frac{\Delta \theta r_1^{a'}}{\cos \gamma} \pi^{-1} \int \frac{d\Omega}{\text{ap}}.
\]

This value of \( P \) must be equal to that in the expression for \( \Delta T \) in Eq. (B-7), so

\[
\Delta T_t = -\frac{\varepsilon \sigma T_1^4 r_1^{a'}}{\cos \beta \cos \gamma} \left[ \right] \pi^{-1} \int \frac{d\Omega}{\text{ap}}.
\]
But Eq. (10) leads to

\[ T_1^{\text{th}} = T_t^{\text{th}} (1 + 4\Delta T_t/T_t), \]

and when this is substituted into Eq. (B-8) we obtain the following result, which is approximately correct for most blackbody simulator cavities of practical value:

\[ \frac{\Delta T_t}{T_t} = -\frac{\Xi \pi^{-1} \int_{\text{ap}} d\Omega}{1 + 4\Xi \pi^{-1} \int_{\text{ap}} d\Omega} \]  

(B-9)

where

\[ \Xi = \frac{\varepsilon \sigma T_t^3 r_1^{4'}}{\cos \delta \cos \gamma} \]
APPENDIX C

INTEGRAL EQUATION ANALYSES

Since Sparrow and Albers (1960) introduced computer techniques for the solution of these problems, integral equation formulation has been the most popular type of blackbody simulator theory. An example is Eq. (15) of Chandos and Chandos (1974), which may be written in our nomenclature as follows:

\[ \varepsilon_{\alpha}(\vec{r}_1) = \varepsilon + \frac{1-\varepsilon}{\pi} \int_{\text{wall}} \varepsilon_{\alpha}(\vec{r}_2) \, d\Omega_{21}. \]  

(C-1)

Note that our \( d\Omega \) is equal to the \( d\Omega\cos\theta \) of Chandos and Chandos. Many authors use an equation like (C-1): Buckley [1928a, Eq. (3)]; Williams [1961, Eq. (4)]; and Sparrow and Cess [1966, Eq. (3-40)].

There is a simple derivation of Eq. (C-1) that amounts to little more than the statement that the effective local hemispherical emissivity is based on emitted plus reflected radiation [see Eq. (29)]. That seems to be Williams' (1961) approach. A more detailed development is given by Sparrow and Cess (1966, p. 94).)

Equation (C-1) is a simple, direct form for the integral equation expression of the blackbody simulator problem. However, there are some alternative forms of the equation that are very important. These variations involve: (1) a one-dimensional variable instead of a two-dimensional variable, (2) polythermal cavities, (3) an unknown
function that is a constant times $\varepsilon_a(r)$ instead of $\varepsilon_a(kr)$ alone, and (4) a family of integral equations instead of a single integral equation.

Alternative Forms of Integral Equations

One-Dimensional Integration Variables

The blackbody simulator cavity wall is a two-dimensional surface in three-dimensional space; thus the two-dimensional variable of integration, $d\Omega_2$, in Eq. (C-1) makes that equation an appropriate general expression. The four examples cited have two-dimensional variables of integration. On the other hand, a number of authors use an equation similar to Eq. (C-1) but they use a one-dimensional integration variable like

$$
\varepsilon_a(x) = \varepsilon + (1-\varepsilon) \int \varepsilon_a(y) K(x,y) \, dy,
$$

where $K(x,y)dy = dF_{x-y}$, the configuration factor (see pp. 40 and 41 for a discussion of configuration factor) of an area element at $y$ as seen from a point at $x$. Equation (C-2) is similar to Eq. (4) of Buckley (1928a), Eq. (1a) of Moon (1940), and Eq. (1) of Sydnor (1969). It is important to note that Eq. (C-2) is limited as follows: Buckley (1928a, p. 891) considers "... only those cases in which the points ... on the surface can each be expressed in terms of a single variable"; Moon (1940, p. 195) requires that all surfaces be "sufficiently symmetrical so that the position of a point can be specified as a function of a single variable"; and Sydnor (1969, p. 1288) assumes that "that position can be described by a variable $x$ ranging from 0 to 1." Sydnor goes on
to say that "the results to follow apply to any cavity"; and with cer-
tain qualifications this is true, as is shown below. However, Sydnor's
statement without these qualifications is too strong, as Sydnor, him-
self, agrees (Sydnor, 1977).

This appendix gives the connection between integral equation
analyses and the present development by showing the relationship between
Eqs. (6) and (C-1); the bridge is the Neumann series solution to the
Fredholm integral equation of the second kind. But Eqs. (6) and (C-1)
have two-dimensional variables, and most studies of the Fredholm integral
equation of the second kind are performed in terms of a one-dimensional
variable. Therefore it is important to reconcile the one-dimensional
Eq. (C-2) with the two-dimensional Eq. (C-1). Courant and Hilbert (1953;
p. 152) provide that reconciliation:

The developments of §§1 to 6 and of §8 can be significantly
generalized in two directions.
In the first place, all the arguments remain valid if we con-
sider integral equations for functions of several, say m, inde-
dependent variables. Thus, suppose \( f(s) \) and \( \phi(s) \) are continuous
functions of the variables \( s_1, s_2, \ldots, s_m \) in a fixed finite re-
region \( G \); suppose also that \( K(s,t) \) is a continuous function of the
variables \( s_1, s_2, \ldots, s_m \) and \( t_1, t_2, \ldots, t_m \), each set of vari-
ables ranging over the region \( G \); let \( ds \) denote the volume element
\( ds_1 ds_2 \ldots ds_m \) of \( G \) and \( dt \) the corresponding volume element
\( dt_1 dt_2 \ldots dt_m \) with the understanding that all integrals are to be
taken over the region \( G \). Then the integral equation

\[
f(s) = \phi(s) - \lambda \int K(s,t) \phi(t) \, dt
\]

is an integral equation for a function \( \phi(s) \) of \( m \) variables, and
its kernel \( K(s,t) \) is a function of \( 2m \) variables; our whole theory
remains valid word for word.

The §6 referred to here is Courant and Hilbert's Neumann series
solution of this equation. This equation becomes the one-dimensional
variable Eq. (C-2) when \( \phi(s) = \varepsilon_\alpha(x) \), \( f(s) = \varepsilon \), \( \lambda = 1 - \varepsilon \), \( K(s,t) = K(x,y) \), \( \phi(t) = \varepsilon_\alpha(y) \), and \( dt = dy \); and it becomes the two-dimensional variable Eq. (C-1) when \( \phi(s) = \varepsilon_\alpha(r_1) \), \( f(s) = \varepsilon \), \( \lambda = 1 - \varepsilon \), \( \phi(t) = \varepsilon_\alpha(r_2) \), and \( K(s,t) dt = \pi^{-1} d\Omega_{21} \).

The problems of the continuity of the functions and the convergence of the integrals and series can be handled in three ways: (1) In the paragraphs immediately following the above quotations, Courant and Hilbert discuss far less stringent requirements of continuity and convergence. (2) Moon (1940) provides a detailed argument of the continuity question for an equation like Eq. (C-2); his presentation is repeated later in this section. (3) Sydnor (1969) uses a physical argument, and we can adopt that point of view and say that if \( \varepsilon_\alpha(r_1) \) applies to a practical blackbody simulator cavity, then the integrals and series involved must converge to the corresponding physical values.

Polythermal Cavities

If Eq. (C-1) is modified by adding the factor \( (T_2/T_1)^4 \) inside the integral, then the new equation can be used for polythermal cavities:

\[
\varepsilon_\alpha(r_1, T_1) = \varepsilon + \frac{1-\varepsilon}{\pi} \int_{\text{wall}} (T_2/T_1)^4 \varepsilon_\alpha(r_2) d\Omega_{21}.
\]

Unknown Functions Other Than \( \varepsilon_\alpha(r) \)

Instead of solving for \( \varepsilon_\alpha(r) \), one group of authors solve for some constant times \( \varepsilon_\alpha(r) \). For example, Moon [1940, Eqs. (1) and (1a)] solves for

\[
L(s) = \sigma T^4 \varepsilon_\alpha(s),
\]
and he calls this quantity "luminosity" and uses the letter \( L \) for it—although today we prefer to call it "radiant exitance" using the letter \( M \). Another example of this case is Bedford (1970), where he reviews a number of papers by Sparrow and his coworkers. Bedford's Eq. (6.25) involves

\[
B(x_0) = \sigma T^h \varepsilon_\alpha(x_0)
\]

and although he uses the symbol \( B(x_0) \) instead of the modern symbol \( M \), he does use the modern name "radiant exitance."

Families of Integral Equations

Still another group of integral equation analyses has a family of integral equations where each equation is somewhat like Eq. (C-1). Bedford (1970, p. 181), following his Eq. (6.27), which is like Eq. (C-1), says: "If the cavity under consideration is composed of multiply connected surfaces (e.g., a closed-end cylinder), a set of simultaneous equations is obtained; each equation of the set is of the form of Eq. (6.27) and may contain several terms on the right-hand side."

A number of authors follow that recipe: Vollmer [1957, Eqs. (1a) and (1b)]; Sparrow, Albers, and Eckert [1962, Eqs. (4) and (7)]; Peavy [1966, Eqs. (1) and (2)]; Bedford and Ma [1974, Eq. (2) plus a similar equation]; Chandos and Chandos [1974, Eqs. (22) and (23)]; and Bedford and Ma [1975, Eq. (2) plus two similar equations]. Since we are about to demonstrate the connection between integral equation analyses and my analysis by showing how Eqs. (6) and (C-1) are related, it is important to reconcile the several-term, several-equation expressions of this
group of authors with the single-equation format of Eq. (C-1). Moon (1940, p. 196) provides that reconciliation:

One of the theoretical advantages of integral equations over differential equations is that no trouble is introduced by functions that are not analytic. In fact, the theorems of integral-equation theory apply equally well to a much broader class of functions--functions of class $L^2$. Thus both $L(t)$ and $K(s,t)$ may be discontinuous functions, provided they are of integrable square.

Because of the foregoing property of integral equations, Eq. (1a) is not restricted to reflecting surfaces consisting of a single smooth sheet. The surface may be composed of any number of flat or curved pieces with a different reflection factor on each; and Eq. (1a) will continue to apply, though the kernel will generally have a discontinuity at each break in the surface. For example, the kernel can be written very easily for the interreflections within a rectangular prism of infinite length, having a different reflection factor on each side, $dt$ being an infinitesimal length on the surface. Here $\rho(s)$ is a step-function which must be combined with $E_1(s,t)$ to form the discontinuous kernel. A solution of Eq. (1a) would then give a piecewise-continuous function of four parts, representing the actual luminosity distribution on the four interior faces. The problem is formulated easily, though it has not yet been solved.

Recall that Moon's Eqs. (1) and (1a) use

$$L(s) = \sigma T^4 \varepsilon_a(s)$$

where our Eq. (C-1) has $\varepsilon_a(t_1^+).$ The symbol $K(s,t)$ represents the kernel in Moon's Eq. (1a), and $t$ is the dummy variable. The symbol $\rho(s)$ is the reflectivity. The symbol $E_1(s,t)$ is related to $K(s,t)$, to $\rho(s)$, to the change from a surface element to a one-dimensional element, and to a constant, $\lambda$. Physically $E_1(s,t)$ is the irradiance at $s$ due to a unit source at $t$. 
Previous Solutions

Buckley (1927, 1928b) and Vollmer (1957) use Whittaker's (1918) method to solve their integral equations. They assume a solution in the form of a sum of exponentials and solve for the coefficients of the exponential terms.

Buckley (1928a, p. 910) shows that for an infinite cylinder a Liouville-Neumann series is the solution for both a multiple-reflection treatment and an integral equation formulation.

Moon (1940) uses the Hilbert-Schmidt theory of integral equations to obtain his solutions.

Sydnor (1969) obtains the Liouville-Neumann series solution to the Fredholm integral equation of the second kind by an iterative process of substituting successive results back into the integral equation. Sydnor's first guess for the effective cavity wall emissivity is just the emissivity of the wall material.

Since 1960 many authors have published papers with computer solutions to integral equation forms of different blackbody simulator problems. These include Sparrow and Albers (1960), Sparrow, Albers, and Eckert (1962), Sparrow and Jonsson (1963), Peavy (1966), Chandos and Chandos (1974), and Bedford and Ma (1974, 1975). The most popular general computer technique has been an iterative procedure with numerical integrations involving Simpson's rule. Several authors have used special techniques to reduce computer costs, improve accuracies, and overcome the problems of discontinuities of the integrands or their derivatives. (See for example page 77.)
It should be noted that all of these previous solutions (involving both analytical and computer techniques) have integral equations with one-dimensional variables of integration—like Eq. (C-2).

A New Solution

When the foregoing discussion of integral equations (with the alternative forms and the previous solutions) is combined with my analysis from Eqs. (1) to (6), the foundation is laid for a new solution to the integral equation formulation of the blackbody simulator cavity problem. The new solution comprises the following five steps:

1. Begin with any integral equation or set of integral equations that is like an equation or set of equations discussed earlier, i.e.: Buckley [1928b, Eq. (2)]; Williams [1961, Eq. (4)]; Sparrow and Jonsson [1963, Eq. (1)]; Sydnor [1969, Eq. (1)]; Bedford [1970, Eq. (6.27)]; Chandos and Chandos [1974, Eq. (15)]; Moon [1940, Eqs. (1) and (1a)]; Bedford [1970, Eq. (6.25)]; Vollmer [1957, Eqs. (1a) and (1b)]; Sparrow, Albers, and Eckert [1962, Eqs. (4) and (7)]; Peavy [1966, Eqs. (1) and (2)]; Bedford and Ma [1974, Eq. (2) plus a similar equation]; Chandos and Chandos [1974, Eqs. (22) and (23)]; or Bedford and Ma [1975, Eq. (2) plus two similar equations].

2. If the integral equation(s) involve a constant times $\varepsilon_{\alpha}(\vec{r})$ instead of $\varepsilon_{\alpha}(\vec{r})$, divide by that constant.

3. If a set of integral equations like those described by Bedford (1970, p. 181) is involved instead of a single integral equation like Eq. (C-1), appeal to Moon (1940, p. 196) to replace the set of integral equations by an appropriate single integral equation like (C-1).
4. Solve the integral equation by the iterative scheme used by many authors to obtain the Neumann series solution; for example, follow the simple presentation of Kanwal (1971, pp. 26-27). The more general two-dimensional-variable equation, Eq. (C-1), is preferred over the less general one-dimensional-variable equation, Eq. (C-2); but we will use the still more general polythermal equation, Eq. (C-3):

\[
\varepsilon_\alpha(x_1, T_1) = \varepsilon + \frac{1-\varepsilon}{\pi} \int_{\text{wall}} \left( \frac{T_2}{T_1} \right)^4 \varepsilon_\alpha(x_2) \, d\Omega_{21} \quad (C-3)
\]

\[
\varepsilon_\alpha(x_1, T_1)_0 = \varepsilon
\]

\[
\varepsilon_\alpha(x_1, T_1)_1 = \varepsilon + \frac{\varepsilon(1-\varepsilon)}{\pi} \int_{\text{wall}} \left( \frac{T_2}{T_1} \right)^4 \, d\Omega_{21}
\]

\[
\varepsilon_\alpha(x_1, T_1)_2 = \varepsilon + \frac{\varepsilon(1-\varepsilon)}{\pi} \int_{\text{wall}} \left( \frac{T_2}{T_1} \right)^4 \, d\Omega_{21}
\]

\[
+ \frac{\varepsilon(1-\varepsilon)^2}{\pi^2} \int_{\text{wall}} \left( \frac{T_2}{T_1} \right)^4 \int_{\text{wall}} \left( \frac{T_3}{T_2} \right)^4 \, d\Omega_{32} \, d\Omega_{21}
\]

\[
\varepsilon_\alpha(x_1, T_1)_3 = \varepsilon_\alpha(x_1, T_1)_2
\]

\[
+ \frac{\varepsilon(1-\varepsilon)^3}{\pi^3} \int_{\text{wall}} \left( \frac{T_2}{T_1} \right)^4 \int_{\text{wall}} \left( \frac{T_3}{T_2} \right)^4 \int_{\text{wall}} \left( \frac{T_4}{T_3} \right)^4 \, d\Omega_{43} \, d\Omega_{32} \, d\Omega_{21}
\]

Note that point 3 is fixed for the inner integral in the last term of the last equation, so the \((T_3/T_2)^4\) factor can be brought inside that
integral. A similar argument applies to the \((T_2/T_1)^4\) factor, so all the temperature ratios can be combined to give \((T_4/T_1)^4\) inside the inner integral. Therefore the solution becomes:

\[
e_{\alpha}(T_1,T_1) = \epsilon + \frac{\epsilon (1-\epsilon)}{\pi} \int_{\text{wall}} (T_2/T_1)^4 \, d\Omega_{21}
\]

\[
+ \frac{\epsilon (1-\epsilon)^2}{\pi^2} \int_{\text{wall}} \int_{\text{wall}} (T_3/T_1)^4 \, d\Omega_{32} \, d\Omega_{21}
\]

\[
+ \cdots
\]

\[
+ \frac{\epsilon (1-\epsilon)^{n-1}}{\pi^{n-1}} \int_{\text{wall}} \cdots \int_{\text{wall}} (T_n/T_1)^4 \, d\Omega_{n(n-1)} \cdots d\Omega_{21}
\]

\[
+ \cdots.
\]  

(C-4)

Note that the right-hand side of Eq. (C-4) is identical to the right-hand side of Eq. (4).

5. Apply Appendix A to obtain the right-hand side of Eq. (6) from the right-hand side of Eq. (C-4). Thus our new solution to the integral equation problem has resulted in the same approximate Eq. (6) that we obtained from the main analysis of this dissertation—and that analysis is a multiple-reflection rather than integral-equation study.

Features of the New Solution

This new method of handling the integral equation formulation of the blackbody simulator cavity problem has several features:

1. Nearly all integral equation treatments of this problem are covered.
2. The approximate solution, Eq. (6), includes polythermal cavities.

3. The isothermal version of Eq. (6) is Eq. (7), and this is included as a special case. Furthermore, all of the special solutions listed in Table 2 are also available from Eq. (6).

4. The simplified expression, Eq. (26), applies to these integral equation problems, and this equation provides a useful mathematical tool for the preliminary design and comparison of blackbody simulator cavities. The usefulness of Eq. (26) can be increased by tables of projected solid angles or configuration factors such as Table 4 of this dissertation or the references cited on page 41. Experimental verification of Eq. (26) is given by Fig. 7. This figure also illustrates how Eq. (26) can be used for the preliminary comparison of different cavity designs.

5. All of the approximate solutions [Table 2 and Eq. (26)] are closed-form analytic expressions whereas most other integral equation solutions are in the form of tabulated or plotted computer outputs. These expressions are useful even to those with operating computer programs because many design characteristics and tendencies (which might otherwise require a very large amount of computer output to prove or to even anticipate) can be recognized from the mathematical structure of the solution. The solutions of Table 2, and the simplified Eq. (26), and the published tables of projected solid angle and configuration factors such as Table 4 can all be drawn upon to supplement the computer. The important point when we compare these new solutions against the traditional tabulated or plotted computer outputs is that we are not
considering competing theories the way a number of authors have done in the past (e.g., Quinn, 1967a, Fig. 3, compares the theories of Buckley, Quinn, Sparrow, and Gouffé); on the contrary, we are considering multiple complementary forms of the same answer to the same problem.

6. My closed-form approximate solutions lend themselves to convenient and informative comparisons with other blackbody simulator cavity theories; these comparisons use the right-hand side of Eq. (7), the isothermal solution:

\[ \varepsilon_\alpha(\hat{r}) = \frac{\varepsilon[1 + (1-\varepsilon)(B-A)]}{\varepsilon(B) + B}, \]

(C-5)

where

\[ A = \pi^{-1} \int_{ap} d\Omega \]

\[ B = \pi^{-1} \int_{ap} d\Omega, \text{ a weighted average of } A. \]
APPENDIX D

GOUFFÉ'S THEORY

Gouffé's 1945 paper is the best known and most widely cited paper on the theory of blackbody simulators. According to Hudson (1969, p. 68), "Most of the papers describing the construction of highly precise blackbodies for use as calibration standards use the method of Gouffé to calculate their effective emissivity." In view of the prominent position of Gouffé's paper, it is surprising to find that it contains four non-rigorous steps: (1) Gouffé's error, (2) Gouffé's approximation, (3) Gouffé's reversal, and (4) Gouffé's surprise.

Gouffé's Error

Gouffé begins his analysis with a serious error in his first expression [his Eq. (1)]:

$$\varepsilon_0 = 1 - \rho \omega_{ap} \pi^{-1}, \quad (D-1)$$

where $\omega_{ap}$ is the solid angle of the aperture as seen from a back wall point.

The error in Eq. (D-1) is that $\int_{\omega_{ap}} d\Omega$, the corresponding projected solid angle, should appear in place of $\omega_{ap}$. This error is minor for on-axis back wall locations in the case of cavities with small apertures and back walls that are perpendicular to the axis, but the error is more serious for larger apertures, for off-axis back-wall
locations, and especially for cavities with tilted back-wall arrangements such as conical cavities.

**Gouffé's Approximation**

Gouffé's analysis determines the fraction of flux incident through a blackbody simulator cavity aperture that is absorbed after an infinite number of diffuse reflections. He makes the key assumption that, following two reflections, the flux is uniformly distributed over the cavity wall surface. This assumption leads to the quantity $s/S$ and to Gouffé's Eq. (2), which is:

$$
\varepsilon_0 = \frac{\varepsilon[1 + (1-\varepsilon)(B-A)]}{\varepsilon(1-B) + B},
$$

(D-2)

where

$$
A = \frac{\omega ap}{\pi},
$$

$$
B = \frac{s}{S},
$$

(D-3)

$s$ = area of aperture

$S$ = area of cavity wall plus aperture

$\omega ap$ = the solid angle of the aperture as seen from a point on the back wall of the cavity.

Gouffé's Eq. (2) [my Eq. (D-2)] is identical to my Eq. (41). However, my expressions for $A$ and $B$ are

$$
A = \pi^{-1} \int_{ap} d\Omega,
$$

$$
B = \pi^{-1} \int_{ap} d\Omega, \text{ a weighted average of } A.
$$
Gouffé's use of $\omega_\text{ap}/\pi$ in place of the more correct $\pi^{-1}\int_{\text{ap}} d\Omega$ is due to his error. Gouffé's use of $s/S$ instead of the more correct $\pi^{-1}\int_{\text{ap}} d\Omega$ is due to his simplifying approximation about uniform flux after two reflections; and his approximation gives good results for shallow cavities but not for elongated cavities (see Sanders and Stevens, 1954, pp. 179, 180; and Quinn, 1967a, p. 1110).

**Gouffé's Reversal (Spherical Cavity)**

Following his Eq. (2), Gouffé discusses simplifying this equation by noting that $1 + (1-\varepsilon)(B-A)"\text{demeure toujours très voisin de l'unité. Ce facteur est même rigoureusement égal à 1 lorsque le corps-noire est de forme sphérique, car on a alors: } s/S = \Omega/\pi."$ This quotation may be translated to read that $1 + (1-\varepsilon)(B-A)"\text{always remains very close to unity. This factor is even rigorously equal to 1 when the blackbody is of spherical shape, for then one has: } s/S = \Omega/\pi"$ or, in my notation,

$$s/S = \omega/\pi.$$  \hspace{1cm} (D-4)

Equation (D-4) is Gouffé's reversal for a spherical cavity. In place of the erroneous $\omega/\pi$, he substitutes $s/S$; and for the spherical cavity specified here, $s/S$ is in error in the opposite direction and to a greater degree than $\omega/\pi$. Figure D-1 shows both $\omega/\pi$ and $s/S$ and also the correct quantity $\pi^{-1}\int_{\text{ap}} d\Omega$. Note that the two errors are equal only when $\phi$ is 0 or 90°; $\phi = 0$ corresponds to no opening, and $\phi = 90°$ corresponds to no cavity. The angle $\phi$ is described in Fig. D-2, and the relationships between $\phi$ and the functions that are plotted in Fig. D-1 are given in Table D-1.
Before reversal
\( \omega_{ap}/\pi = [2 \sin(\phi/2)]^2 \)

Correct expression
\( \pi^{-1} \int_{\Omega} d\Omega = \sin^2 \phi \)

After reversal
\( s/S = (1/2 \sin 2\phi)^2 \)

Fig. D-1. Gouffé's Reversal for Spherical Cavities.
See Fig. D-2 for meaning of \( \phi \).
Fig. D-2. Geometry for a Spherical Cavity.
Table D-1. Analytic Geometry for Spherical Cavity.

Input quantities:  

\[ r_{ap} = \omega = \text{aperture radius} \]
\[ \lambda = \text{cavity depth} \]

Radii:

\[ r_1 = \left(\omega^2 + \lambda^2\right)^{\frac{1}{2}} \]
\[ r_0 = \frac{1}{2} r_1^2/\lambda \]
\[ = \frac{1}{2} \left(\omega^2 + \lambda^2\right)/\lambda \]

Segment heights:

\[ h_1 = r_1 - \lambda \]
\[ = \left(\omega^2 + \lambda^2\right)^{\frac{1}{2}} - \lambda \]
\[ h_0 = 2r_0 - \lambda \]
\[ = \omega^2/\lambda \]

Alternative values of \( A \):

\[ \frac{1}{4} \int_{ap} d\Omega = \frac{s_0}{S} = \frac{\omega^2}{\omega^2 + \lambda^2} = \sin^2 \phi \]

\[ \frac{\omega_{ap}/\pi}{\pi r_1^2} = \frac{s_1}{\omega^2 + \lambda^2} = \frac{2 \left[ \omega^2 + \lambda^2 - \lambda \left(\omega^2 + \lambda^2\right)^{\frac{1}{2}} \right]}{\omega^2 + \lambda^2} = \left[ 2 \sin(\phi/2) \right]^2 \]

\[ \frac{s/S}{4\pi r_0^2} = \frac{\omega^2 \lambda^2}{\left(\omega^2 + \lambda^2\right)^2} = \left(\frac{1}{2} \sin 2\phi \right)^2 \]

Note: This table is illustrated by Fig. D-2.
Gouffé has said that \[1 + (1-\varepsilon)(B-A)\] is equal to 1 for a spherical cavity, and that is true, because, for a sphere, the correct values of both \(A\) and \(B\) are given by

\[
A = B = F = \bar{F} = \pi^{-1}\int_{\text{ap}} d\Omega = \pi^{-1}\int_{\text{ap}} d\bar{\Omega} = s_0/S. \quad (D-5)
\]

(See page 71 and Table D-1 for these relations.) And with Gouffé's reversal, his value of that factor is also 1 because his values of both \(A\) and \(B\) are given by

\[
A = B = s/S. \quad (D-6)
\]

Therefore, for a spherical cavity, Gouffé's Eq. (3)

\[
\varepsilon_0 = \frac{\varepsilon}{\varepsilon(1-B) + B}, \quad (D-7)
\]

where \(B = s/S\), has the same form as the correct equation, Eq. (43), except that for Eq. (43) \(B = s_0/S\).

**Gouffé's Reversal (General Cavity)**

Gouffé's reversal for a general cavity is an extension of his reversal for a sphere. His Eqs. (2 bis) and (4) are repeated here as Eqs. (D-8) and (D-9):

\[
\varepsilon_0 = \frac{\varepsilon}{\varepsilon(1-B) + B} (1 + k) \quad (D-8)
\]

\[
k = (1-\varepsilon)(B-A), \quad (D-9)
\]
where $A = s/S_0$, $B = s/S$, and $S_0$ is the area of a fictitious spherical cavity that has the same depth as the cavity being analyzed with the depth measurement made in a direction normal to the aperture surface.

Note that Eqs. (D-8) and (D-9) together are the same as Eq. (D-2) except that $A = \omega_{ap}/\pi$ has been replaced by $A = s/S_0$. This replacement equation

$$\frac{s}{S_0} = \frac{\omega}{\pi},$$

(D-10)

which is not written out by Gouffé the way Eq. (D-4) is, but is implied by him in his Eqs. (2 bis) and (4) [my Eqs. (D-8) and (D-9)] is Gouffé's reversal for a general cavity.

**Gouffé's Surprise (Spherical Cavity)**

Following his Eq. (4), Gouffé presents analytic expressions for $s/S$ in terms of $l$ and $\omega$ (see Fig. D-2 and Table D-1 for the meaning of $l$ and $\omega$) for three important cavity shapes. His values for the cylinder $(2(1 + l/\omega))^{-1}$ and cone $(1 + (1 + l^2/\omega^2)^{3/2})^{-1}$ are correct (and that can be verified easily); but his value for the sphere

$$s/S = (1 + l^2/\omega^2)^{-1}$$

(D-11)

is Gouffé's surprise. Table D-1 shows that $s/S$ for a sphere is actually equal to

$$\frac{s}{S} = \frac{\pi \omega^2}{\pi (\omega^2 + l^2)^{2/3} l^2} = \left(\frac{l \omega}{\omega^2 + l^2}\right)^2$$

(D-12)

while it is $s_0/S$, not $s/S$, that is equal to:
Gouffé's surprise is clearly an invalid mathematical substitution. But the remarkable thing about Gouffé's surprise is that it follows in sequence from (1) Gouffé's error, (2) Gouffé's approximation, and (3) Gouffé's reversal; and then it leads to the correct expression for the effective emissivity of a spherical cavity blackbody simulator.

**Gouffé's Tables for Spheres**

Following Gouffé's surprise [Eq. (D-11)], he gives tables of $s/S$, $s/S - s/S_0$, $k$, and $\varepsilon_0$. In the entries for spheres for $s/S$, he uses Eq. (D-11); and in the entries for spheres for $\varepsilon_0$, he uses Eqs. (D-7) and (D-11). And so Gouffé's tabulated values of $\varepsilon_0$ for a spherical cavity blackbody simulator are correct. They are the same as one can obtain using Eq. (43) or Nicodemus' Eq. (11). (A minor exception is the case of $\ell/\omega = 5$, where my value is 0.963 and Gouffé's is 0.964.)

**Gouffé's Surprise (General Cavity)**

Although Gouffé's surprise [Eq. (D-11)] deals with a problem in spherical geometry, it has an impact on Gouffé's general cavity analysis as well. When he tabulates the quantities $s/S - s/S_0$, $k$, and $\varepsilon_0$ for cylindrical and conical cavities, Gouffé (in a manner consistent with his surprise substitution of $s_0$ for $s$ for the spherical cavity) uses $s_0/S_0$ according to Eqs. (D-11) and (D-13) instead of $s/S_0$ according to Eq. (D-12). This use of Eq. (D-11) for the cylinder and cone is Gouffé's surprise for the general cavity.
Gouffé's Five Expressions for $\varepsilon_0$

This appendix shows that Gouffé's 1945 paper presents five different expressions for $\varepsilon_0$. All five can be represented by Eq. (D-2), which is the same as Eq. (41), but each of the five has a different combination of functions for the parameters $A$ and $B$. Table 7 gave these values of $A$ and $B$ for these five expressions for $\varepsilon_0$.

Gouffé's Tables for Cylinders and Cones

Because of "Gouffé's surprise" for the general cavity, his determination of the parameter $A$ for on-axis cylindrical cavities is correct [see the entry in Table 7 identified with his equations (2 bis), (4), (1°), (2°), and (3°)]. Thus the only flaw in his tabulated calculations for cylinders is his approximation $s/S$ in place of the more correct, but less available, $\pi^{-1}\int_{\phi \Omega} d\Omega$. A weakness in my own theory is that I do not have a recipe for finding $\pi^{-1}\int_{\phi \Omega} d\Omega$ for cavity shapes like cylinders, so I appeal to the language of Sanders and Stevens (1954, pp. 179-180) and Quinn (1967a, p. 1110) and say that Gouffé's $\varepsilon_0$ values for shallow cylinders are fairly good, but for elongated cylinders they are not very good.

Gouffé's recipe for finding $\varepsilon_0$ for conical cavities, and his corresponding tabulations of values, represent his most serious mistakes. His surprise entry for $A$ in the bottom two rows of Table 7 gives correct $\varepsilon_0$ values for spheres and reasonable approximations for shallow cylinders; but it does not account for the tilted back wall of a cone, and as a result his values of $A$ are much too high for most cones of practical interest. Figure D-3 shows plots of $A$ for a cone according to both
Fig. D-3. Parameter $A$ for Conical Cavities.

See Fig. D-2 for meaning of $\phi$. 

Gouffé's Eqs. (2 bis)(4)(20)

$s_0/S_0 = \sin^2\phi$

$\pi^{-1} \int_{\Delta p} d\Omega = \sin^3\phi$
Gouffé's erroneous $s_0/S_0$ and my correct value of $\pi^{-1}\int_{\text{ap}} d\Omega$. And Table D-2 compares Gouffé's tabulated values of $\varepsilon_0$ for a cone against my correct values, which were calculated from Eq. (44). Wall emissivities for Table D-2 are 0.5. For both Fig. D-3 and Table D-2, we observe the logic of Kelly (1966), and find, for the vicinity of the apex for a cone:

$$F = \pi^{-1} \int_{\text{ap}} d\Omega = \pi^{-1} \int_{\text{ap}} d\Omega = A = B = \sin^3 \phi. \quad (D-14)$$

<table>
<thead>
<tr>
<th>Full cone angle</th>
<th>Half cone angle</th>
<th>$L/R$</th>
<th>$\varepsilon_0$ According to Gouffé</th>
<th>$\varepsilon_0(r)$ According to my Eq. (44)</th>
</tr>
</thead>
<tbody>
<tr>
<td>53.13°</td>
<td>26.57°</td>
<td>2</td>
<td>0.795 (0.806)$^a$</td>
<td>0.918</td>
</tr>
<tr>
<td>36.87°</td>
<td>18.43°</td>
<td>3</td>
<td>0.863</td>
<td>0.969</td>
</tr>
<tr>
<td>28.07°</td>
<td>14.04°</td>
<td>4</td>
<td>0.894</td>
<td>0.986</td>
</tr>
<tr>
<td>22.62°</td>
<td>11.31°</td>
<td>5</td>
<td>0.912 (0.913)</td>
<td>0.993</td>
</tr>
<tr>
<td>11.42°</td>
<td>5.71°</td>
<td>10</td>
<td>0.953 (0.954)</td>
<td>0.999016</td>
</tr>
<tr>
<td>5.72°</td>
<td>2.86°</td>
<td>20</td>
<td>0.976</td>
<td>0.999875</td>
</tr>
</tbody>
</table>

$^a$Values in parentheses are my corrections of arithmetic errors by Gouffé.

Note: $\varepsilon = 0.5$ for all entries.
APPENDIX E

DE VOS' THEORY

De Vos (1954a) introduced into blackbody simulator theory the idea of directional emissivity and reflectance. As has been pointed out (p. 23), De Vos was studying the emissivity of tungsten ribbon and wanted to take into account the specularity of ordinary metal surfaces. He then applied his theory to several cavity shapes with the special conditions of isothermal diffuse walls. Quinn (1967a) later used De Vos' theory to study cylinders with isothermal diffuse walls. For this special case of isothermal, diffuse wall cavities, we can compare De Vos' theory with this one and with that of 20 other papers (see Table 8).

Comparison of Theories

De Vos' first-order approximation is given by his Eq. (6). For a cavity with Lambertian wall surfaces and a single aperture, that equation can be written in the nomenclature of this dissertation as Eq. (35), repeated here as Eq. (E-1):

\[
L(\vec{T}) = L^{bb} \left(1 - \frac{1 - \epsilon}{\pi} \int_{ap} d\Omega_{21} \right) = L^{bb} \epsilon_0 . \tag{E-1}
\]

His second-order approximation equation is his Eq. (9), written as my Eq. (36) and repeated here as Eq. (E-2):

129
\[ L(\tau) = L^{bb} \left( 1 - \frac{1}{\pi} \int_{ap} d\Omega_{21} - \frac{(1-\epsilon)^2}{\pi^2} \int_{\text{wall}} \int_{ap} d\Omega_{32} d\Omega_{21} \right. \]
\[ \left. - \frac{\epsilon(1-\epsilon)}{\pi} \int_{\text{wall}} k d\Omega_{21} \right), \quad (E-2) \]

where

\[ k = \frac{L^{bb}(T_1) - L^{bb}(T_2)}{L^{bb}(T_1)} \]
\[ = \frac{\Delta L^{bb}}{L^{bb}(T_1)} = \left( \frac{\partial L^{bb}}{\partial T} \frac{T}{L^{bb}} \right)_{T_1} \frac{(T_1 - T_2)}{T_1} \]

De Vos uses Wien's law, so according to De Vos' Eq. (8) and Table 3 for Wien's law,

\[ k = \frac{\sigma_2}{\lambda T_1} \frac{(T_1 - T_2)}{T_1} \]

The first two terms in Eq. (E-2) are identical to all of Eq. (E-1), and both groups of two terms are like the first two terms in Eq. (19).

When this comparison is combined with the comments about the De Vos and Gouffé theories (p. 66), it is seen that the first-order approximations of this dissertation and of De Vos' theory agree with the corrected form of Gouffé's first-order approximation. The last term in Eq. (E-2) resembles the fourth term in Eq. (19). More is said later about that interesting comparison of temperature-effect terms; now we take up the issue that represents the main function of this appendix, that is, the
comparison of the first three terms of Eq. (E-2) with the theory of this dissertation. The third term of Eq. (E-2) is quite similar to the third terms of Eqs. (2) and (4); however, the terms in Eqs. (2) and (4) have an extra factor of $\varepsilon$ or $\varepsilon_{32}$, and the signs are positive whereas the sign of the third term in Eq. (E-2) is negative. Instead of making a term-by-term comparison, we will extend De Vos' isothermal, diffuse wall theory to an infinite number of terms. The infinite series that results will be found to have a compact sum that is comparable to Eqs. (6), (6a), and (7).

A key step in De Vos' theory is his Eq. (5), which gives the radiation from all parts of the cavity wall that reflects from a back wall element of area (like $a_1$ in Fig. 3) and then goes out through the aperture. De Vos' Eq. (5) contains the quantity $I_n^w$, which in the language of my pages 18 and 19 and Fig. 3 is $L_{21}$, the radiance from a general area increment $a_2$ toward $a_1$. De Vos obtains his output radiance in the form of two levels of approximations [Eqs. (E-1) and (E-2)] when he substitutes in two levels of approximation for $L_{21}$. We can continue that sequence and obtained De Vos' third level and $n$th level of approximation. My third approximation term for cavities with a single aperture and isothermal, Lambertian wall surfaces is

$$-\frac{(1-\varepsilon)^3}{\pi^3} \int_{\text{wall}} \int_{\text{wall}} \int_{\text{ap}} d\Omega_{31} d\Omega_{32} d\Omega_{21}.$$  

Apparently, only three other authors have given De Vos' third term. Bedford (1970) gives the general directional reflectance expression, and my expression for the general case agrees with his. Campanaro
and Ricolfi (1966a) give the general third-order term, but their apparent order of integration is suspicious. However, they immediately apply their general equation to a diffuse sphere, and for that special case the order of integration does not matter. Fecteau (1968) gives, in his Eq. (1), the third approximation term as a triple product rather than in the form of a triple integral. The topic of his paper is the diffuse spherical cavity, and for that special case his third approximation term is valid. However, his discussion suggests that his Eq. (1) applies to a general diffuse isothermal cavity, but it does not.

We can find that the following is the $n$th term in De Vos' development for the case of an isothermal diffuse cavity with a single aperture:

$$- \frac{(1-\varepsilon)^n}{\pi^n} \int_{\text{wall}} \cdots \left[ \int_{\text{wall}} \int_{\text{ap}} d\Omega_{(n+1)n} d\Omega_{32} d\Omega_{21} \right].$$

Therefore, the complete isothermal diffuse wall expression of De Vos' theory becomes

$$\varepsilon_{L}(\mathbf{r}) = \frac{L(\mathbf{r})}{L^{bb}} = 1 - \frac{1-\varepsilon}{\pi} \int_{\text{ap}} d\Omega_{21}$$

$$- \sum_{n=2}^{\infty} \frac{(1-\varepsilon)^n}{\pi^n} \int_{\text{wall}} \cdots \left[ \int_{\text{wall}} \int_{\text{ap}} d\Omega_{(n+1)n} d\Omega_{32} d\Omega_{21} \right].$$

(E-3)

Repeated applications of one of Courant's (1936) expressions of the mean value theorem of integral calculus show that there are average
integrals \( \int_{ap} d\Omega \frac{d\Omega}{n} \) such that the \((n+1)\)th term in Eq. (E-3) is equal to

\[
- \frac{(1-\varepsilon)^n}{n!} \int_{ap} d\Omega \frac{d\Omega}{(n+1)n} \int_{wall} \cdots \int_{wall} d\Omega_{n-1} \cdots d\Omega_{21} ,
\]

and by a simple averaging process there is some average integral \( \int_{ap} d\Omega \) such that the sum in Eq. (E-3) then becomes

\[
\int_{ap} d\Omega \sum_{n=2}^{\infty} \frac{(1-\varepsilon)^n}{n!} \int_{wall} \cdots \int_{wall} d\Omega_{n-1} \cdots d\Omega_{21} .
\]

This multiple integral is the same as the one labeled \( I_n \) in Appendix A, where it was found to be

\[
I_n = \left[ \pi - \int_{ap} d\Omega_n \right]^{n-2} \left[ \pi - \int_{ap} d\Omega_{21} \right] .
\]

Therefore we can write Eq. (E-3) as

\[
\varepsilon_{L}(\vec{x}) = 1 - \frac{1-\varepsilon}{\pi} \int_{ap} d\Omega_{21}
\]

\[
- \int_{ap} d\Omega \left( \pi - \int_{ap} d\Omega_{21} \right) \sum_{n=2}^{\infty} \frac{(1-\varepsilon)^n}{n!} \left( \pi - \int_{ap} d\Omega_n \right)^{n-2}
\]
or

\[ e_L(\vec{r}) = 1 - \frac{1 - \varepsilon}{\pi} \int_{\text{ap}} d\Omega_{21} \]

\[ - \frac{(1 - \varepsilon)^2}{\pi} \int_{\text{ap}} d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega_{21} \right) \]

\[ \times \sum_{n=0}^{\infty} \frac{(1 - \varepsilon)^n}{\pi^n} \left( \pi - \int_{\text{ap}} d\Omega \right)^n. \]  

(E-4)

By an averaging process, there is some single-valued average integral, \[ \int_{\text{ap}} d\Omega \] , that can replace all the \[ \int_{\text{ap}} d\Omega_n \] in the last equation. As was done in Appendix A, the subscripts 21 can now be dropped from \[ \int_{\text{ap}} d\Omega_{21} \] because we will no longer deal with multiple forms of the integral, and without the subscripts, the simple 21 operation is the one that would normally be performed. Thus

\[ e_L(\vec{r}) = 1 - \frac{1 - \varepsilon}{\pi} \int_{\text{ap}} d\Omega \]

\[ - \frac{(1 - \varepsilon)^2}{\pi} \int_{\text{ap}} d\Omega \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \]

\[ \times \sum_{n=0}^{\infty} \frac{(1 - \varepsilon)^n}{\pi^n} \left( \pi - \int_{\text{ap}} d\Omega \right)^n. \]

The summation is almost identical to one in Appendix A, and following that analysis we see that the sum is equal to:
Therefore:

\[ \varepsilon_{L}(\hat{r}) = 1 - \frac{\varepsilon - \pi}{\pi} \int_{\text{ap}} d\Omega - \frac{(1 - \varepsilon)^2}{\varepsilon \pi} \int_{\text{apDev}} \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \frac{1}{1 + \frac{\varepsilon - \pi}{\pi} \int_{\text{apDev}} d\Omega} \]

(E-5)

There is a similarity in the development of \( \int_{\text{ap}} d\Omega \) and \( \int_{\text{apDev}} d\Omega \), so their values will be similar for most cavity problems of interest. But regardless of their values, there is some single average integral, which we will call \( \int_{\text{ap Dev}} d\Omega \), that can be substituted for the two average integrals in Eq. (E-5) and still maintain the validity of the equation. Thus:

\[ \varepsilon_{L}(\hat{r}) = 1 - \frac{\varepsilon - \pi}{\pi} \int_{\text{ap}} d\Omega - \frac{(1 - \varepsilon)^2}{\varepsilon \pi} \int_{\text{ap Dev}} \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right) \frac{1}{1 + \frac{\varepsilon - \pi}{\pi} \int_{\text{ap Dev}} d\Omega} \]

(E-6)

Equation (E-6) is identical in form to the isothermal version of Eq. (6), and since the isothermal version of Eq. (6) is Eq. (7), we can rewrite Eq. (E-6):
\[
\varepsilon_L(t) = \frac{\varepsilon[1 + (1-\varepsilon)(B-A)]}{\varepsilon(1-B) + B}
\]  
(E-7)

where

\[
A = \pi^{-1}\int_{ap} \, d\Omega \\
B = \pi^{-1}\int_{ap \text{ DeV}} \, d\Omega
\]

Equations (7) and (E-7) are identical except that Eq. (E-7) has \(B = \pi^{-1}\int_{ap \text{ DeV}} \, d\Omega\) where Eq. (7) has \(B = \pi^{-1}\int_{ap} \, d\Omega\). It might be that a uniqueness argument will establish that the two average integrals are the same, but the two quantities were developed in similar, yet definitely different, ways, so we should await a future decision concerning their identity.

**Comparison of Temperature Terms**

We now return to the comparison of the last term in Eq. (E-2), which we refer to as LTE-2, and the fourth term in Eq. (19). In the earlier discussion of Eq. (E-2), it was said that LTE-2 is

\[
\text{LTE-2} = -\frac{\varepsilon(1-\varepsilon)}{\pi} \int_{\text{wall}} k \, d\Omega_{21}
\]

where

\[
k = \left( \frac{\partial_{L}b}{\partial T} \frac{T}{L^{bb}} \right)_{T_{1}} \frac{(T_{1} - T_{2})}{T_{1}}
\]  
(E-8)
so
\[
\text{LTE-2} = - \left( \frac{\partial L^b}{\partial T} \right) \frac{T}{L^b} T_1 \frac{\varepsilon (1-\varepsilon)}{\pi} \int_{\text{wall}} \frac{T_1 - T_2}{T_1} \, d\Omega_{21}.
\]

Temperature \(T_1\) is a constant during the integration, but \(T_2\) varies as point 2 assumes different locations. However, \(T_2\) inside the integral can be replaced by an average temperature \(\overline{T_{\text{DeV}}}\) outside the integral. If we follow the sign convention of Eq. (8), where

\[
T_m = T_1 + \Delta T_m,
\]

we have \(\overline{T_{\text{DeV}}} = T_1 + \Delta T_{\text{DeV}}\), so a sign reversal takes place and we have

\[
\text{LTE-2} = + \left( \frac{\partial L^b}{\partial T} \right) \frac{T}{L^b} T_1 \frac{\varepsilon (1-\varepsilon)}{\pi} \frac{\Delta T_{\text{DeV}}}{T_1} \int_{\text{wall}} \, d\Omega_{21}.
\]

Then we can apply one of the projected solid angle integral relations developed in Appendix A, on page 90:

\[
\int_{\text{wall}} d\Omega_n(n-1) = \pi - \int_{\text{ap}} d\Omega_n(n-1).
\]

And if we drop the subscripts 21 from \(d\Omega_{21}\) as we did earlier in this appendix, we have

\[
\text{LTE-2} = + \left( \frac{\partial L^b}{\partial T} \right) \frac{T}{L^b} T_1 \frac{\Delta T_{\text{DeV}}}{T_1} \varepsilon (1-\varepsilon) \left( 1 - \pi^{-1} \int_{\text{ap}} d\Omega \right).
\]
When we compare LTE-2 with the fourth term of Eq. (19), we see they are the same except for three things:

(1) The reference temperature in LTE-2 is \( T_1 \) and in Eq. (19) it is \( T_t \); however, this can be readily reconciled, as De Vos uses \( T_1 \) as his reference temperature and we do also in our Eq. (16). Therefore, if we compare LTE-2 with the fourth term of Eq. (16) instead of with the fourth term of Eq. (19), we have \( T_1 \) in the same places.

(2) Equation (16) has \( \Delta T_m \) where LTE-2 has \( \overline{\Delta T}_{DeV} \). These mean temperature differences are very similar, but they are not the same because \( \overline{\Delta T}_{DeV} \) is based on a single integration whereas \( \Delta T_m \) is based on many multiple integrals.

(3) The third difference between LTE-2 and the fourth term in Eq. (16) [or (19)] is more serious. De Vos' expression has an extra factor of \( \varepsilon \) compared to mine. Most probably De Vos' extra \( \varepsilon \) is an error on his part, and one can show that an alternative development can be made from De Vos' equations, using De Vos' type of analysis, and the result will be an alternative form of LTE-2 that does not have the extra \( \varepsilon \).

De Vos begins his development of the term LTE-2 with his Eq. (7), which gives \( L_{21} \), the radiance from \( a_2 \) toward \( a_1 \):

\[
L_{21} = L_{bb}^{bb}(T_1) \left( 1 - \frac{1-\varepsilon}{\pi} \int_{ap} d\Omega_{21} - k\varepsilon \right). \tag{E-9}
\]

Comparing this to De Vos' Eq. (6) or my Eq. (E-1), we see that \( L_{21} \) might alternatively be expressed as

\[
L_{21} = L_{bb}^{bb}(T_2) \left( 1 - \frac{1-\varepsilon}{\pi} \int_{ap} d\Omega_{21} \right). \tag{E-10}
\]
But from De Vos' Eqs. (7) and (8):

\[
\kappa = \frac{L_{bb}(T_1) - L_{bb}(T_2)}{L_{bb}(T_1)},
\]

(E-11)

so \( L_{bb}(T_2) \) in Eq. (E-10) can be replaced by

\[
L_{bb}(T_2) = L_{bb}(T_1)(1 - \kappa).
\]

Therefore Eq. (E-10) becomes

\[
L_{21} = L_{bb}(T_1) \left( 1 - \frac{1 - \varepsilon}{\pi} \int_{ap} d\Omega - \kappa + \kappa \frac{1 - \varepsilon}{\pi} \int_{ap} d\Omega \right)
\]

or, to first-order approximation,

\[
L_{21} = L_{bb}(T_1) \left( 1 - \frac{1 - \varepsilon}{\pi} \int_{ap} d\Omega - \kappa \right).
\]

(E-12)

Equation (E-12) is my alternative expression for Eq. (E-9), which is the same as De Vos' Eq. (7). Note that the last term in the parentheses in Eq. (E-12) is \( \kappa \) and for Eq. (E-9) it is \( \kappa \varepsilon \). If an equation like Eq. (E-12) instead of one like Eq. (E-9) is used for De Vos' Eq. (7) in the development of LTE-2, an alternative LTE-2 is found:

\[
\text{LTE-2} = \left( \frac{2L}{T_T} \right) T_T \frac{\Delta T_{\text{DeV}}}{T_1} (1 - \varepsilon) \left( 1 - \frac{1}{\pi} \int_{ap} d\Omega \right),
\]

and this expression for LTE-2, we have just shown, is (except for the \( \Delta T_m \) vs. \( \Delta T_{\text{DeV}} \) difference) the same as the fourth term in Eq. (16).
The above analysis shows why the extra $\varepsilon$ is probably an error on De Vos' part. As a final step, De Vos' Eqs. (7) and (8) should be corrected to read:

\[
I_n^w = I_B(1 - \sum_h r_n^{wh} d\Omega_n^h - k_n)
\]
and
\[
I_w^0 = I_B(1 - \sum_h r_w^{0h} d\Omega_w^h - \sum_h \int r_n^{wh} d\Omega_n^h r_w^{0n} d\Omega_w^n - \int k_n r_w^{0n} d\Omega_w^n)
\]
or, in my nomenclature:

\[
L_{21} = L^{bb}(T_1) \left( 1 - \frac{1-\varepsilon}{\pi} \int_{ap} d\Omega_{21} - k \right)
\]
and

\[
L(\vec{r}) = L^{bb} \left( 1 - \frac{1-\varepsilon}{\pi} \int_{ap} \Omega_{21} \right)
- \frac{(1-\varepsilon)^2}{\pi^2} \int_{wall} \int_{ap} d\Omega_{32} d\Omega_{21} - \frac{1-\varepsilon}{\pi} \int_{wall} k d\Omega_{21} \right).
\]

De Vos' Temperature Parameter $k$

Equation (E-8) gives:

\[
k = \left( \frac{9L^{bb}}{2T L^{bb}} \right) \frac{T_1}{T_2} \frac{(T_1 - T_2)}{T_1}
\]
and De Vos provides a functional value for Eq. (E-8) only in terms of the Wien law:
We can extend De Vos' \( k \) by using our Table 3 to give Table E-1.

\[
k = \frac{c_2 \Delta T}{\lambda T}.
\]

<table>
<thead>
<tr>
<th>Radiation law</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stefan-Boltzmann</td>
<td>( \frac{4\Delta T}{T} )</td>
</tr>
<tr>
<td>Monochromatic Planck</td>
<td>( \frac{x e^x}{e^x - 1} \frac{\Delta T}{T} )</td>
</tr>
<tr>
<td>Band-limited Planck</td>
<td>( \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{x e^x}{e^x - 1} \frac{d\lambda}{T} )</td>
</tr>
<tr>
<td>Wien</td>
<td>( \frac{x}{T} \frac{\Delta T}{T} )</td>
</tr>
<tr>
<td>Rayleigh-Jeans</td>
<td>( \frac{\Delta T}{T} )</td>
</tr>
</tbody>
</table>

Where \( x = c_2/\lambda T \).
REFERENCES


Sydnor, C. L., Jet Propulsion Laboratory, Pasadena, Calif. (1977), personal communication to the author.


