HIERARCHICAL PROGRAMMING
AND APPLICATIONS TO ECONOMIC POLICY

by

Fidel Abraham Parraga

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF SYSTEMS AND INDUSTRIAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1 9 8 1

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Fidel Abraham Parraga

entitled

Hierarchical Programming and Applications to Economic Policy

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

A. Wayne Wymore	26 JAN 81 Date
Donald B. Schultz	26 Jan 81 Date
John L. Lemmel	Jan 26, 1981 Date
Lucien Duckstein	26 Jan 1981 Date
Mohindar S. Cheema	Jan 26, 1981 Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Robert L. Bulfin Dissertation Director	Jan 26 1981 Date

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED Fidel Barriaz-

DEDICATION

This dissertation is dedicated to all my professors.

ACKNOWLEDGMENTS

I gratefully acknowledge the opportunity Wilfred Candler gave me to work with him in his Multi-level Programming Research project in the Development Research Center of World Bank. I have thus benefitted not only by obtaining a dissertation topic but in great measure also from his ideas and his previous algorithmic research. Professor Robert Bulfin provided the perspective and the critical eye that upgraded the quality of the thesis both in content and form. Professors L. Duckstein, W. Wymore, J. Leonard and M. Cheema served on the dissertation committee and their encouragement and comments are appreciated.

I am indebted to Gary Kutcher for sharing with me an early version of his "El Fayoum" agricultural model; to Manuel Carrillo and Professor T. H. Mattheiss for their vertex enumeration algorithms; to Emmitt Summers, Jr. for his invaluable assistance in getting me acquainted with the Burroughs system, and to J. Kreusser for introducing me to the TEMPO Linear Programming System.

I also thank Ellen Alderman for putting her valuable organizational abilities to work in the presentation of my dissertation, and to Linda Begley for typing the manuscript.

TABLE OF CONTENTS

TABLE OF CONTENTS--Continued

TABLE OF CONTENTS--*Continued*

LIST OF TABLES

LIST OF ILLUSTRATIONS

HIERARCHICAL PROGRAMMING AND
APPLICATION TO ECONOMIC POLICY

Fidel Abraham Parraga, Ph.D.
The University of Arizona, 1981
Director: Professor Robert Bulfin

Outside the field of Mathematical Programming, conceptual models aimed at the hierarchical interactions of conflictive decision entities have appeared occasionally in the literature of Mathematical Economics and Game Theory. Thus, we have the monopolistic trading schemes authored by Shapley and Shubik, the leader-follower game proposed by Simaan and Cruz and the moral hazard problem in the Principal to Agent relationship. The novelty and value of the Mathematical Programming formulation of the hierarchical model lies in the fact that it is appropriate to carry out numerical experiments. Hierarchical Programming models can take on many forms, the objective functions and technical constraints can be linear or non-linear and the decision-making entities can have control over resource activities only, prices only or control over both. This dissertation focuses on the solution and economic policy applications of the two-level hierarchical model in which the objectives and technical constraints are linear and in which the decision

making entities have control over resource activities only. The (linear-resource) two-level problem is a non-convex problem and can have many local optima. Existing solution methodologies rely heavily on branch and bound techniques and other less orthodox enumeration procedures. The "Algorithm of Interceptions" developed here is based on findings regarding the topological and geometric structure of the feasible domain of the problem. This structure is first established by a body of theorems that assert that the feasible domain is a connected collection of faces of the polyhedron formed by the upper and lower level technical constraints. Furthermore, if a local optimum is not the global optimum then the polyhedron formed by the technical constraints and the hyperplane given by setting the upper objective to a level slightly above the local optima have at least one vertex that belongs to the two-level feasible domain. It is also demonstrated that if a Candler-Townsley high point is not a local optima, then it is possible to identify an alternative optimal basis associated with the same high point but having a higher high point. Once the algorithm is completely developed it is applied to numerical examples of the literature and economic models developed in the chapter of applications. The solutions to these problems and the solutions obtained using the other methodologies are then used for a comparison exercise of the methodology of interceptions against the other methodologies.

In the chapter of applications two existing linear programming models are recast as hierarchical models. In each case, policy instruments and a government concern are introduced in the upper level and the linear programming model is used as the corresponding lower level structure. The first model deals with the monetary policy exerted by the Federal Reserve System on the banking industry and the second with an agricultural policy problem.

CHAPTER 1

INTRODUCTION

1. The Hierarchical Model

There are two areas of research in Mathematical Programming, the construction of mathematical programming models, and the design of solution algorithms. The general programming model is an algebraic formulation of the decision variables, the conditions constraining them and a function to measure the quality of potential decisions. Building models is in general a difficult task and the case of Mathematical Programming is no exception. It involves a tradeoff between the conflicting objectives of constructing a model that describes accurately the configuration and dynamics of the system of interest and constructing a model that is useful to carry out computational experiments.

Nevertheless there has been extensive application of mathematical programming models to a wide spectrum of problems. Linear programming is probably the most widely used technique, since many applications exhibit the necessary characteristics of linear programming models. This fact, coupled with the well developed theory of linear programming, has led to the development of a number of commercial computer codes which can easily solve problems with thousands of variables.

Most importantly for the advancement of the art, the prolific application has brought the diversification of the model in response to particular demands of new situations. Such is the case of systems in which it is evident that there is more than one significant objective function, and of systems involving more than one decision making entity with the capability of affecting the system to promote their own interests. This class of problems have been classically dealt by the theory of games and utility functions. More recently the problem has been readdressed from a mathematical programming perspective, and a new body of models, solution concepts and methodologies have emerged under the name of Multi-objective Programming. See for example,[11,12,43].

However, there is a situation which does not fall within the domain of Multi-objective Programming, and has been treated by the theory of games [30,40] in an abstract manner only. It is the case of systems where the decision making entities have their own objective functions, and their actions have to conform to a sequential order constraint arising from a hierarchical arrangement of the decision entities. Models with this structure have been recently introduced in the field of Mathematical Programming by the independent work of two authors. The first author worked with systems of this nature in the context of agricultural policy problems [7] , and the second in the setting of strategic defense planning [6]. The problem in its simplest formulation (i.e., using linear relations) has a non-convex feasible domain, and to date there are no efficient solution procedures. This dissertation is in line with the research interest that the model has generated. It proposes a new solution algorithm based on structural findings of the feasible domain, and it also explores new areas of application. An example will best introduce the model in question, referred hereafter as the hierarchical model, or multi-level programming model.

The example chosen is related to the problem of national monetary policy. It has to do with the behavior of commercial banks, the aggregate of bank depositors and borrowers and the operations of the government regulatory agency, the Federal Reserve Board in the case of the United States.

The three entities involved have their own independent objectives and they will be in conflict most of the time. The government regulatory agency is concerned with the protection of the national financial structure and with the implementation of the government monetary policy. To achieve these goals it will attempt to influence the availability and cost of money and credit through its ability to impact segments of the bank portfolios. The banks and their clients will be assumed to be profit maximizers.

As conflicting as the objectives of the three entities may be, the decision making is hierarchical. The banks have to comply with the Federal Reserve Board regulations, and the clients of the banks will decide according to the opportunities the banks are able to offer them. In addition the decision making occurs by stages. The Federal Reserve Board decides first on the variables it controls, then the banks respond by adjusting their asset portfolios and finally consumers and investors reformulate expectations and restructure plans.

The variables controlled by the Federal Reserve Board are the discount rate and a number of parameters that impose limits on the risk and liquidity of the bank asset portfolio. Among these controllable parameters we have:

- The demand-deposit and savings-deposit reserve requirement parameters
- The capital adequacy ratio parameter
- The risk asset ratio parameter
- The pledged assets parameter

The banks decide on the levels of the assets in their portfolios, while consumer and investors control the level of their loans and deposits. A detailed model of the upper two levels of this system is developed in Chapter V.

2. The Algebraic Formulation of the Model

Hierarchical models can take on many forms, they can be linear or non-linear and can have decision making agents with control over resource activities only, prices only, or control over both. The following is the general three-level model in which we will indicate the variables controlled by each level, the technical and sequential conditions constraining them, and the inter-level relationship.

$$
\text{Third level}\left\{\begin{array}{l}
\max f_3 (x_1, x_2, x_3)\\
\quad 3.1 \quad g_3 (x_3) \geqq 0, \quad x_3 \in X_3\\
\quad 3.2 \quad (x_1, x_2) \text{ solves}\\
\text{Second level}\left\{\begin{array}{l}
\max f_2 (x_1, x_2; x_3)\\
\quad 2.1 \quad g_2 (x_2; x_3) \geqq 0, \quad x_2 \in X_2\\
\quad 2.2 \quad x_1 \text{ solves}\\
\text{First level}\left\{\begin{array}{l}
\max f_1 (x_1; x_2, x_3)\\
\quad 1.1 \quad g_1 (x_1; x_2, x_3) \geqq 0 \quad x_1 \in X_1\\
\quad 1.2 \quad \text{No sequential constraint}
\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

The objectives, constraints, and variables have been indexed to identify their level affiliation. Thus, constraints 3.1, 2.1, and 1.1 are the technical constraints for levels 3, 2, and 1. Constraints 3.2, 2.2, and 1.2 are the sequential or behavioral constraints for levels 3, 2, and 1.

Decision makers at level 3 manipulate variable x_3, a vector of resources or prices, within the technical possibilities given by 3.1, and can affect directly the objective functions f_2 and f_1, as well as the technical constraints 2.1 and 1.1. They can also indirectly affect the behavioral constraints 3.2, 2.2, and 1.2.

Decision makers at level 2 have to take variable x_3 as a given parameter and can manipulate variable x_2 within the possibilities allowed by the technical constraint 2.1, and thus they can affect directly the objective function f_1 and constraint 1.1; they can also affect indirectly the behavioral constraints 2.2 and 1.2.

Decision makers at level 1 have to take variables x_3 and x_2 as given parameters and optimize their objective function within the feasible choices offered by constraint 1.1. Constraint 1.2 is a dummy constraint that was added for purposes of symmetry only. Given a third level decision x_3, decision makers at level 2, optimize their objective function $f_2(x_1,x_2;x_3)$ within their technical feasible choices x_2 and the behavioral responses x_1. Decision makers at level 3, optimize their objective function $f_3(x_1,x_2;x_3)$ within their technical feasible choices x_3 and the behavioral responses (x_1,x_2) from the two-level sub-hierarchy.

The general n-level hierarchical programming model (G), can be defined in terms of a (n-1) level model, S(y) that depends parametrically on the n-level decision variable. Let y be the n-level variable and $x = (x_{n-1}, \ldots, x_1)$ the variable of the (n-1) subhierarchy, then G can be written as:

$$\begin{aligned} G:\quad & \max\ f(x,y) \\ \text{s.t.}\quad & (1)\ \ g(y) \geq 0 \quad y \in Y \\ & (2)\ \ x \text{ solves } S(y) \end{aligned}$$

Constraint (1) is the technical constraint for the n-level, and constraint (2) its behavioral constraint.

3. General Assumptions about the Model

In this section we shall make some general assumptions about the model and the system it represents. The first assumption regards the flow of information among the entities of the system. It will be assumed that each level has the capacity of obtaining accurate information not only about the technical and behavioral constraints in the lower levels but also of their objective functions. Only in this way will each level be able to predict the behavioral responses to each of its decisions, and select its most profitable decision.

Secondly, it will be assumed that the decision making entities are goal oriented and that each subhierarchy has reached a level of managerial sophistication to be able to learn and implement their most profitable program alternative. Thus, in order to apply n-level hierarchical programming to a particular system, it will be necessary that the decision makers in the (n-1) hierarchical subsystem to have mastered (n-1) hierarchical programming. Except for the special case of hierarchical two-level problems, this assumption will not often hold.

The third assumption is to exclude problems with ill conditioned solutions caused by alternative optimal solutions for the hierarchical model. We shall use here the formulation of the two-level linear model, since this assumption will be needed later for this particular model.

The two-level linear model can be written as:

$$
\begin{array}{ll}
(T): & \max \; px + qy \\
(1) & Ey \geqq e \\
(2) & \max \; cx + dy \\
 & Hx = b - Gy \\
 & \quad x \geqq 0
\end{array}
$$

Suppose $(\bar{x}, \bar{y})$ is a feasible solution to problem (T), giving the highest value for the two-level objective px + qy, among all feasible solutions to (T). However, given the upper level decision $\bar{y}$, it can be the case that the lower level has an alternative optimal solution response $\bar{\bar{x}}$, and thus an infinite number of them, such that $p\bar{\bar{x}} + q\bar{y} < p\bar{x} + q\bar{y}$. Therefore $\bar{y}$ is an ill conditioned decision in the sense that it would not guarantee the upper level entity the highest possible return px + qy.

One way to redefine the hierarchical optimality would be to record a function m(y), <u>exacting the minimum value</u> for px + qy among all alternative lower level optimal responses to y, and then maximize m(y). This is a minimax approach. Another approach would be to record the function m(y) exacting the maximum value for the two-level objective px + qy among all alternative lower level optimal responses to y and select y such that m(y) = max m(y). If y has alternative lower level optimal responses we can have a small change in the technical coefficients of constraint (2) above just to obtain a unique optimal lower level response to y. Since uniqueness can always be achieved in linear programming by small changes in the technical coefficients, the latter approach will be adopted.

Therefore, without loss of generality we can assume that we shall only deal with hierarchical models that are not ill conditioned.

Finally, it should be pointed out that in some particular systems non-dominated solutions may exist resulting in better objective values for each level than the hierarchical solution. These solutions may or may not be attainable according as to whether or not the decision entities in the system are willing to negotiate in a non-hierarchical fashion. (See Karwan and Bialas [4]).

4. Objectives of the Research

The principal objective of this dissertation is to establish a new solution algorithm for the two-level problem with constant prices, linear objectives, and linear technical constraints. The algebraic statement of this problem is

$$
\begin{aligned}
&(T) \quad \max\ px + qy \\
&\quad (1) \quad Ey \geqq e \\
&\quad (2) \quad \max\ cx + dy \\
&\qquad\qquad Hx \geqq b - Gy \\
&\qquad\qquad x \geqq 0
\end{aligned}
$$

The "Algorithm of Interceptions" proposed here is based on findings regarding the topological and geometric structure of the feasible domain of the problem. This structure is first established by a body of theorems that assert that the feasible domain is a connected collection of faces of a polyhedron, with points that satisfy a criterion of "behavioral optimality". Existing algorithms rely heavily on branch-and-bound techniques and other less orthodox procedures of enumeration, as reported in the literature review. The "Algorithm of Interceptions" in contrast rely on two types of intercepting elements. One type of interception is used for the local optimum procedure and another type for the passage from a local optimum to a better point in the feasible domain, if a better local optimum exists. If the current local optimum is the global optimum, the intercepting element (second type) is also used in establishing this fact. Once the algorithm is completely developed, it will be evaluated against the algorithms reported in the literature review. An attempt will also be made to generalize the algorithm to solve two-level problems in which the upper level controls lower level prices instead of resource activities.

Another important objective of this work is to explore new areas of application of the two-level hierarchical problem. Thus, two-level models will be developed for the regulation of commercial banking and for an agricultural production system.

5. Organization of the work

Chapter 2 reviews the research on hierarchical programming and related topics. In the first section, classical optimization models are reviewed and similarities and differences between them and the hierarchical model are established. In the second section a survey of existing solution algorithms for hierarchical programming is presented.

In Chapter 3 the framework and preliminary results for the new algorithm are developed. The algorithm itself is presented in Chapter 4, where it is computationally tested and compared to existing algorithms.

Two hierarchical models and their solutions are presented in Chapter 5. General conclusions and directions for further research and generalizations follow in Chapter 6.

CHAPTER 2

LITERATURE SURVEY

1. Origins

The two-level hierarchical programming model was first introduced in a linear programming context by Candler and Norton [7] . They sought to develop a normative-descriptive model to optimize government agricultural policy strategies using a behavioral model that replicates market equilibrium [18]. In this situation the government leads the decision making, but does not have control over all the economic decisions. Their formulation is in line with the economic policy ideas of Theil [42]. He observed that there are two kinds of variables policy makers have to consider: those which they can manipulate directly as "instruments" of policy planning, and those which fall beyond their influence and depend on chance or private decision entities.

Thus, the two-level programming formulation is associated with, and serves Theil's position, in at least three ways. First, it captures the hierarchical structure of policy making by introducing an explicit sequential constraint on the actions taken by the policy makers and the actions taken by the private entities. Second, it introduces the hypothesis that the private entities are rational optimizing agents. Finally, the linear programming association lends the model a computational adequacy that permits extensive numerical experimentation.

The idea of sequential interaction among decision making entities is not new in Mathematical Economics. In the next section we shall see that the microeconomic problem of market imperfections, specifically the oligopoly and monopoly in product or factor markets, leads to the formulation of symmetric game models. Subsequent work explored asymmetric game models, with structure similar to the two-level hierarchical model. However, these models are conceptual tools and do not amount to Mathematical Programming Models adequate to carry out numerical experiments. In section 2, we shall discuss other mathematical models which share some characteristics of the hierarchical model, but differ in one or more characteristic aspects of the hierarchical programming model as defined in this thesis.

2. Related Models

2.1 Stackelberg Games

Simaan and Cruz[40] introduced a solution concept, which they called "a stackelberg strategy", that applies to a non-zero sum game played by a leader player and a follower player. This solution concept corresponds to a two-level hierarchical problem, and in fact can be defined in the format introduced in section 3 of chapter 1.

A stackelberg strategy with player 2 as a leader is nothing else but a solution to the following two-level problem:

$$\begin{aligned} &\max \; J_2\,(u_1, u_2) \\ &\text{s.t.} \quad (1) \;\; u_2 \varepsilon\, U_2 \\ &\qquad\quad (2) \;\; \max_{u_1 \varepsilon\, U_1} J_1\,(u_1, u_2) \end{aligned}$$

Simaan and Cruz proved that there exists a stackelberg strategy for games in which

1) The admissible strategy sets U_1 and U_2 for players 1 and 2 respectively, are compact sets, and

2) The payoff functions J_1 and J_2 are continuous functions.

Although the leader-follower game was inspired by the study of duopoly models, the latter are not hierarchical models. The conflict in a duopoly can be resolved only when the production outputs of the two firms are at an equilibrium. This is in contrast to the hierarchical model in which the hierarchy arbitrates the conflict. To explain this point, we briefly review the basic facts of the Cournot and Stackelberg duopoly models.

In a duopoly, there are only two sellers of a commodity. The decision variables are the amounts each firm decide to produce (and sell). Their profits depend on the market price of the commodity and their production costs. The market price decreases with total (industry) output, according to a demand function, while each firm's production cost is an increasing function of the firm's output. The Cournot equilibrium occurs when each firm assumes that a change in its own output will not elicit a change in the other firm's output. With this additional assumption it can be shown [28] that the optimal output level of each firm is a decreasing function of the output of the other firm, as shown in Figure 1. Furthermore, the Cournot equilibrium is given by the intersection of the curves representing these functions.

If one of the firms, say firm 1, learns the way the other one reacts to changes in q_1, i.e. estimates or computes dq_2/dq_1, while firm 2 still assumes $dq_1/dq_2 = 0$, the stackelberg equilibrium occurs. This new equilibrium can be computed by correcting the optimal output function for firm 1.

The non-hierarchical character of the Cournot and Stackelberg game models becomes evident if we note that any industry output situation other than the equilibrium, such as the production vector $Q^1 = (q_1{}^1, q_2{}^1)$ in figure 1 will set off a chain of adjustments $Q^t = (q_1{}^t, q_2{}^t)$ over future periods of time t = 2, 3 ... convergent to the equilibrium point. Therefore, it is clear that neither of the firms has a privilege over the other; they are, rather, engaged in a symmetric game.

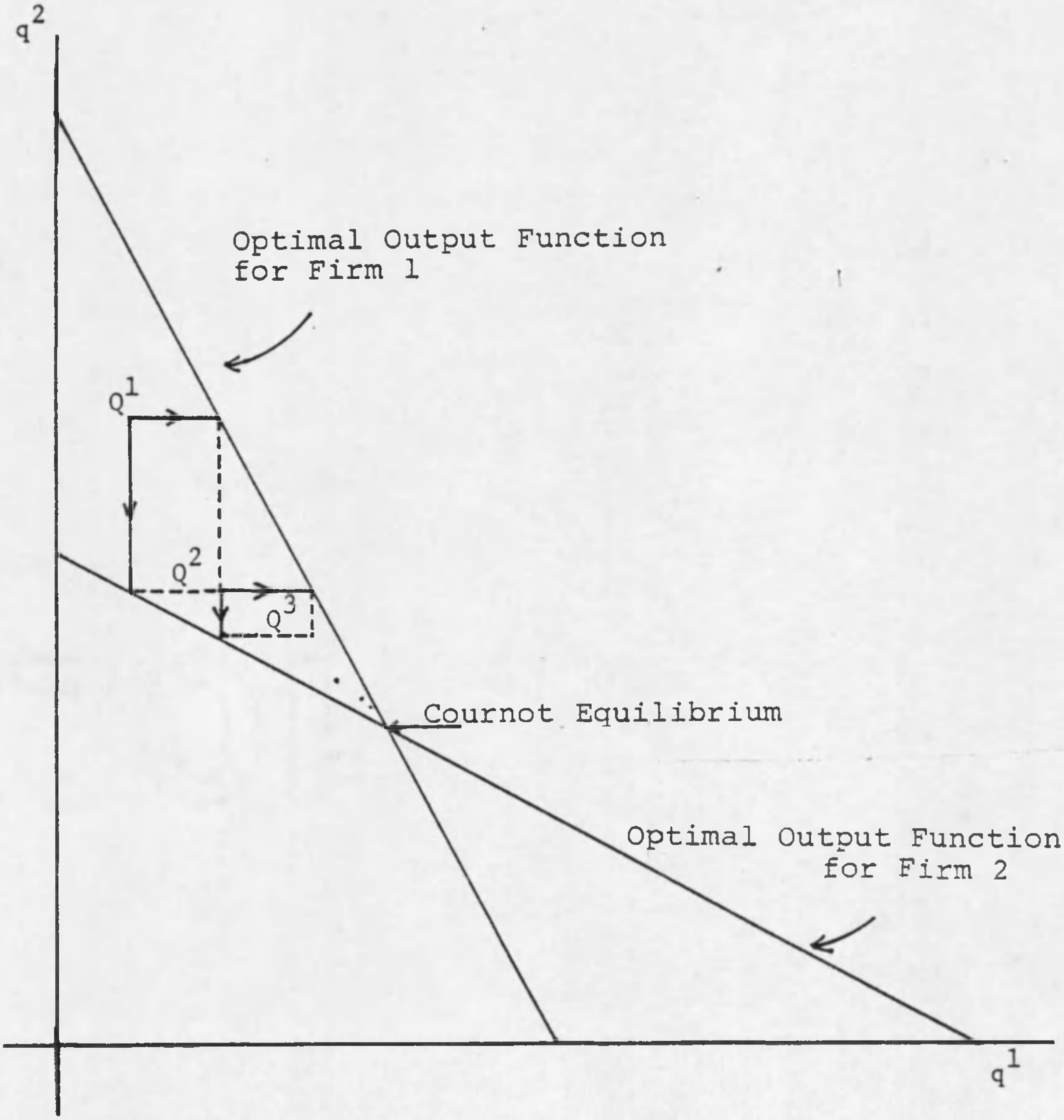

Figure 1. Equilibrium in a Duopoly

2.2 Asymmetric Monopolistic Games

2.2.1 An Edgeworth Box Model

In interpreting economic theory regarding the formation of prices and the operations of markets from a game theoretical point of view, Shapley and Shubik [39] produced three monopolistic trading schemes. One of these models, which we present below, possesses a hierarchical two-level structure.

The scheme in question regards the trading operations of two entities. The first one, a firm or group of firms, is capable of selecting a price schedule to exchange the (two commodities) of interest, while the second decides the amounts to be traded at the given price. The scheme is given in the framework of an "Edgeworth Box" which we next introduce.

The preferences of the two traders are represented in Figure 2 by two families of continuous differentiable <u>convex</u> indifference curves, denoted by $\psi(x',y')$ and $\phi(x,y)$, respectively, and where x' and y' (x and y respectively) are the amounts of the first and second commodities held by the first trader (second trader respectively).

Note that the traders use coordinates oppositely oriented. The coordinates of a point in the box relate by the transformation

$$x' = x - a$$
$$y' = y - b$$

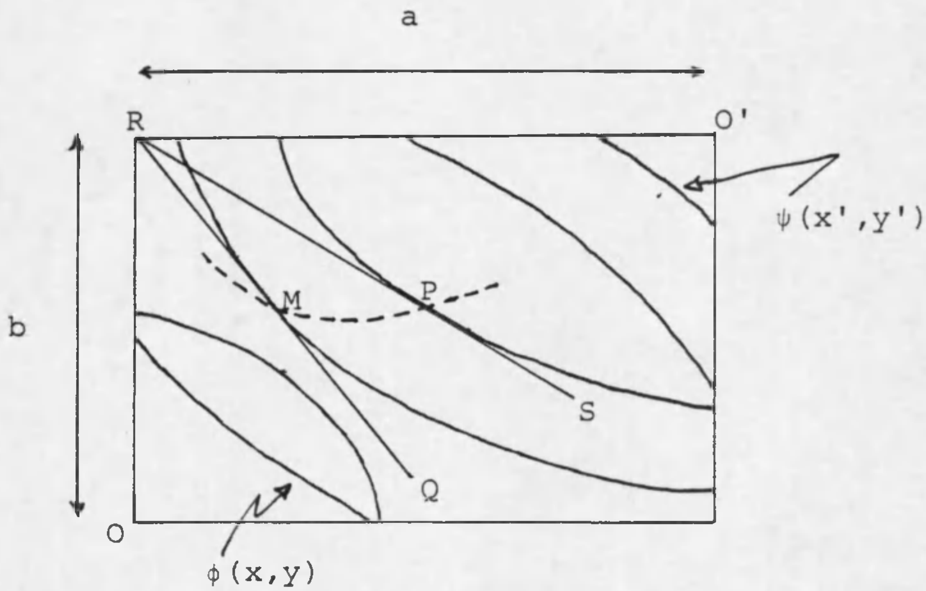

Figure 2. Traders in an Edgeworth Box

The origin of the first trader's coordinates is at O' and his initial holdings - a units of the first commodity and o of the second - are represented by the vector O'R. His indifference curves indicate that he wishes to trade in a manner that the final outcome is as "southwest" as possible, away from O'. The second trader's origin is at O, and OR represents his initial endowment of 0 units of the first commodity and b units of the second. According to his indifference curves, he would like to trade in such a way that the final outcome is as "northeast" as possible, away from O.

A price schedule between the commodities is given by a line with negative slope with respect to the coordinates of the second trader. In particular, a ray departing from point R into the box, such as ray RP, determines a price schedule. Furthermore, the points on the ray give the outcomes jointly attained by the two traders when their initial endowments are both represented by R.

We can now formally describe the trading scheme that exhibits a two-level hierarchical structure. The first trader selects any ray departing from R into the box, such as ray RS. The second trader responds by trading along the price line RS until his preference function is maximized as indicated by the point P, where the price line is tangent to his family of indifference curves. In fact, the second trader response curve - curve MP in Figure 2 - is given by the locus of tangency points between the price rays and the indifference curves of the second trader. The first trader is therefore faced with the problem of maximizing his preference function over the response curve of the second trader. The solution occurs at point M where the response curve is tangent to the first trader indifference curves. Of course the first trader does not directly choose trade M, he will rather choose the price ray RMQ, and then the second trader will trade up to the point M.

2.2.2 A Two-Level Problem with Disaggregated Lower Level

There has been work to extend monopoly and duopoly models, such as the one discussed in section 2.2.1, to models involving n entities. See for example references [15,29,30,39]. In reference [30] Kats produces an abstract monopolistic model for many entities. His model evolves naturally from the general game model in its <u>normal formulation</u>. Thus, he considers a collection of n+1 players, $N = \{0, 1, \ldots, n\}$ each one with a strategy set S_i, subset of some topological space. The collection of all possible outcomes denoted by S is defined as the cartesian product of the individual strategy sets, i.e. $S = S_0 \times \ldots S_1 \times S_n$. However, player 0 is assumed to have the capability to limit the choices of all other players. Accordingly, n power functions are defined to express the dominance of player 0 over each player i = 1, ... n.

$$D_i : S_0 \rightarrow \text{Subsets of } S_i$$

Thus, if player 0 chooses s_0 in S_0, player i sees his strategy set reduced from S_i to $D_i(s_0)$. Conforming to the "normal formulation" of games, a payoff function for each player is defined.

$$\pi_i : S \rightarrow R \qquad i = 0, 1, \ldots n$$

The function $\pi_i\ (s_0, \ldots, s_i, \ldots s_n)$ indicates that the payoff player i experiences does not depend solely on his own strategy s_i, but on everybody elses. Observe now that once player 0 chooses a strategy s_0 in S_0, the rest of the players $N_1 = \{1, 2, \ldots, n\}$ find themselves participating in a normal (symmetric) game, $\Gamma(s_0)$, with strategy sets $D_i(s_0)$ and payoff functions π_i. Therefore, this monopolistic model is a two-level problem with a disaggregated lower level and with player 0 in the upper level.

A solution concept for this problem can be formulated using two solution concepts commonly used in game theory, namely the Nash-equilibrium for the aggregate of players in the lower level and the max-min concept for the game played between the lower and upper levels. Thus, the following sets are defined:

$$\bar{A}\ (s_0) = \{(s_1, \ldots, s_n) \in S_1 x \ldots x S_n : (s_1, \ldots s_n) \text{ is a Nash equilibrium for } \Gamma(s_0)\}$$

Also, let

$\bar{S}_0 = \{s_0 \varepsilon S_0 : \bar{A}(s_0) \neq \phi\}$ and

$$\tilde{\pi}_0(s_0) = \min_{(s_1, \ldots, s_n) \,\varepsilon\, \bar{A}(s_0)} \pi_0(s_0, s_1, \ldots, s_n)$$

Finally, we will say that a strategy $s^* = (s_0^*, s_1^*, \ldots, s_n^*) \varepsilon S$ is a solution to the monopolistic game if and only if

1. $(s_1^*, s_2^*, \ldots, s_n^*) \varepsilon \bar{A} (s_0^*)$,

   $\tilde{\pi}_0(s_0^*) = \pi_0(s_0^*, s_1^*, \ldots, s_n^*)$ and

2. $\tilde{\pi}_0(s_0^*) \geqq \tilde{\pi}_0(s_0)$ for all $s_0 \varepsilon \bar{S}_0$.

Such an optimal strategy is proved to exist whenever

1. D_i is upper-semi-continuous for all $i = 1, 2, \ldots, n$.
2. $D_i (s_0)$ is non-empty, compact and convex for all $i = 1, 2, \ldots, n$ and for all $s_0 \varepsilon S_0$.
3. π_i is continuous for all $i \varepsilon N$ and quasiconcave in S_i for all $i = 1, 2, \ldots, n$.
4. $\tilde{\pi}_0$ is an upper-semi-continuous function.
5. S_0 is compact.

2.3 Optimal Control Models

These are dynamic models that have emerged primarily in physics and engineering to model "controllable motion".

There are, however, interpretations in other fields, such as, optimal capital growth in economics [17].

In optimal control theory the variables are functions of time, and they are partitioned into two classes as is the case in two-level hierarchical programming. The phase variable describes the state of the system at any point in time and the control variables make possible a continuous although limited and indirect control of the evolution of the phase variables. The evolution of the phase variables and their relationship to the control variables are described by a differential equation. The motivation for control is the optimization of a performance criterion expressed as a functional defined on the state and control variables. Mathematically the problem can be formulated as follows:

$$\max \int_{t_0}^{t_1} g(x(t),\ u(t))\ dt$$

s.t. (1) $u \varepsilon U$
(2) x solves $dx/dt = f(x,u)$

Where x is the phase variable and U is the collection of allowable control variables, most often, a collection of piece-wise continuous functions over the interval $[0, 1]$.

The similarity to hierarchical two-level programming is apparent if we notice that the phase variable is indirectly determined by the chosen control u and more directly determined by the differential equation (2). Thus, we might say that the phase variable x corresponds to the behavior of a lower level entity, which in spite of not being an optimizer, does have a predictable behavior of its own, and is sensitive to the decisions u of the upper level, who endeavors to optimize his performance criterion.

It should be indicated that the decision making in the optimal control model is of a continuous nature, as opposed to the "one-shot" decisions made by the agents of the two-level hierarchical model. The discretization of the continuous process of Optimal Control Theory leads to the problem of Dynamic Programming which shows a sharper relationship to Hierarchical Programming as will be seen next.

2.4 Dynamic Programming

Dynamic Programming is a model and a solution methodology for serial multistage decision systems [37]. In Dynamic Programming, as in Optimal Control Theory, the purpose is to control the evolution of a dynamic process in order to optimize an objective function. Although the dynamic process can have an infinite number of stages, Dynamic Programming emphasizes processes with a finite number of stages. A general serial multistage model can be described using Figure 3.

Each square represents a stage of the process, where the state variable x undergoes a transformation t, which is affected by a decision variable d. For example at stage n, the state variable is transformed from x_{n-1} to x_n according to the equation

$$x_n = t_n\,(x_{n-1}, d_n) \qquad d_n \in D_n(x_{n-1})$$

It is assumed that such a transformation results in an immediate return r_n given by

$$r_n = r_n(x_n, d_n)$$

and the optimization of the multi-stage decision problem can be written as:

$$\begin{aligned}
&\max\ r_1\,(x_1, d_1) \circ r_2\,(x_2, d_2) \circ \ldots \circ r_N\,(x_N, d_N) \\
&\text{s.t.}\quad x_n = t_n\,(x_{n-1}, d_n) \\
&d_n \in D_n(x_n-1) \text{ for } n=1, \ldots, N.
\end{aligned}$$

However, if the operation o, fulfills the sufficient conditions for decomposability, this optimization problem for N = 3 is equivalent to:

$$\max_{\substack{x_1=t_1(x_0,d_1)\\ d_1 \in D_1(x_0)}} \left\{ r_1(x_1,d_1) \circ \max_{\substack{x_2=t_2(x_1,d_2)\\ d_2 \in D_2(x_1)}} \left\{ r_2(x_2,d_2) \circ \max_{\substack{x_3=t_3(x_2,d_3)\\ d_3 \in D_3(x_2)}} r_3(x_3,d_3) \right\} \right\}$$

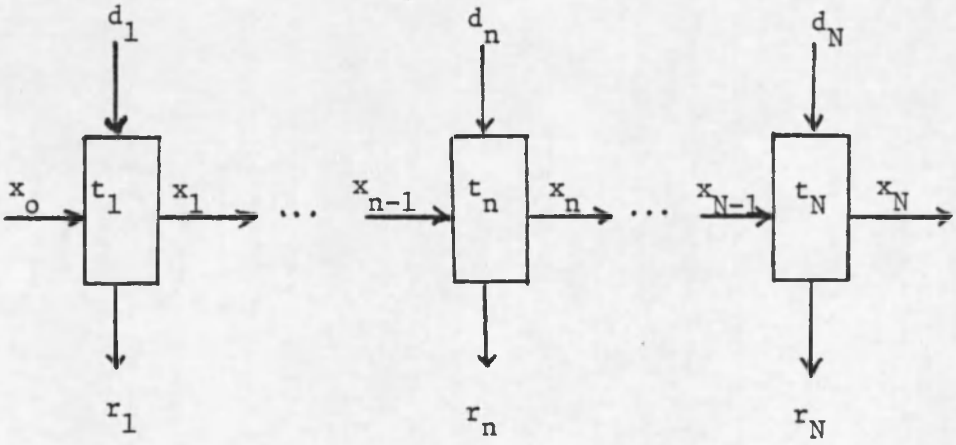

Figure 3. A Serial Multi-stage Model

The latter is the standard problem decomposition that makes Dynamic Programming a valuable solution methodology. Its relationship to Hierarchical Programming becomes manifest when rewritten as follows:

$$
\text{Third Level}\left\{
\begin{array}{l}
\max\ r_1(x_1,d_1) \circ r_2(x_2,d_2) \circ r_3(x_3,d_3) \\
\quad 3.1 \quad x_1=t_1(x_o,d_1)\ ,\ d_1 \in D_1(x_o) \\
\quad 3.2 \quad (d_2,d_3) \text{ solve} \\
\quad \text{Second Level}\left\{
\begin{array}{l}
\max\ r_2(x_2,d_2) \circ r_3(x_3,d_3) \\
\quad 2.1 \quad x_2=t_2(x_1,d_2)\ ,\ d_2 \in D_2(x_1) \\
\quad 2.2 \quad d_3 \text{ solves} \\
\quad \text{First Level}\left\{
\begin{array}{l}
\max\ r_3(x_3,d_3) \\
\quad 1.1 \quad x_3=t_3(x_2,d_3)\ ,\ d_3 \in D_3(x_2) \\
\quad 1.2 \quad \text{No constraint.}
\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

It can be seen that the decisions in the upper levels (earlier stages) condition the choices of the lower levels (subsequent stages). We can go so far as saying that a serial multi-stage decision problem decomposable according to Dynamic Programming is a particular case of the Hierarchical Programming Model. However, all decisions are made by a single agent which differentiates this model from the more general Hierarchical Programming Model.

2.5 Dantzig-Wolfe Decomposition Principle

The Dantzig-Wolfe decomposition principle for Linear Programming [16, 25, 31] has been interpreted as a two-level planning tool to optimize the objective function of a large system represented by a linear programming model of the form below.

$$
\begin{array}{lrcl}
(P) & \max & & c_1x + c_2y \\
& A_1x + A_2y & = & b \\
& D_1x & = & b_1 \\
& D_2y & = & b_2 \\
& x\ ,\ y & \geqq & 0
\end{array}
$$

Where $D_1x = b_1$ and $D_2y = b_2$ are independent constraints for the two sectors of the system, but $A_1x + A_2y = b$ and the objective function describe the inter-relationship between the sectors of the system.

The decomposition technique developed by Dantzig and Wolfe to solve problem (P) consists in the iterative solution of a restricted master and the sector problems. The restricted master is solved to obtain relative prices for the objective functions of the sector problems, and the sector problems are solved to recommend inclusion of new activities to the restricted master. When the sectors find they cannot recommend new activities to the restricted master, the previous restricted master provides a solution to problem (P).

Both in the two-level hierarchical programming model and the Dantzig-Wolfe decomposition model there are two levels of decision making but in the latter the entities in the second level collaborate unconditionally with the first level.

3. Major Methodologies of Solution[1]

Since the introduction of the two-level model in a mathematical programming context, there has been a constant research endeavor to establish efficient methodologies for its solution. Previous research has been done for the problem with linear and non-linear objectives. The following constitute the major achievements to date.

3.1 The T-set Algorithm

3.1.1 T-sets of the First Kind and Other Results

Candler and Townsley [8] set out to solve the following two-level hierarchical programming problem:

$$
\begin{aligned}
P1:\quad & \max\ px + qy \\
& (1)\quad y \geqq 0 \\
& (2)\quad \max\ cx + dy \\
& \qquad\qquad Hx = b - Gy \\
& \qquad\qquad\ \ x \geqq 0
\end{aligned}
$$

The objective $px + qy$ is referred to as the policy objective or upper objective function, while the objective $cx + dy$ is called the behavioral or lower objective function. When necessary, vector (p,q) will be denoted by f, and (c,d) by g. Likewise, y is called the policy variable,

---

1. Throughout this section we shall use standard Linear Programming terminology.

and x the behavioral variable. We shall also make the standard assumption that matrix [H,G] is of full rank, of dimension m x n; and that H is m x n_1, and G is m x n_2. For this and other algorithms, constraint (1) can be alternatively replaced by the slightly more general $Ey \geqq e$.

The T-set algorithm is based on four important results. The first one is a consequence of post-optimality analysis of the following problem which Candler and Townsley call Behavioral Problem.

$$\begin{aligned} P2(\bar{y}): \quad & \max\ cx + d\bar{y} \\ & Hx = b - G\bar{y} \\ & x \geqq 0 \end{aligned}$$

They exploit the fact that the optimal basis for $P2(\bar{y})$ - called a behavioral optimal basis, and denoted as BOB - remains optimal with changes of $\bar{y}$ as long as primal feasibility is not affected. That is, if B denotes the optimal basis of $P2(\bar{y})$, B remains optimal as long as $\bar{y}$ is changed within the set

$$\{y: Bx_b + Gy = b\ ,\ \text{for some}\ \ x_b \geqslant 0\}$$

This is so because the relative prices with respect to B, $c_j - c_B B^{-1}H_j$, do not depend on $\bar{y}$. In this manner they establish a problem which they refer to as the Policy Problem:

$$\begin{aligned} P3(B): \quad & \max\ px + qy \\ & Nx_N + Bx_B + Gy = b \\ & x_N = 0 \\ & x_B\ ,\ y \geqq 0, \end{aligned}$$

Where H has been partitioned as [N,B].

The policy problem gives the highest policy objective value within all the feasible policy settings y that keep basis B behavioral optimal basis (BOB). The optimal solution for P3(B), is called the high point for B, and the optimal value the high point value in B.

At the optimal solution of problem P3(B), all the activities in B and G have non-positive relative prices. However, this is not necessarily true for the activities in N. Those activities in N with positive relative prices constitute a set, which is called the T-set of the first kind associated with B, and is denoted as $T^1(B)$. Formally we have:

$$T^1(B) = \{H_j \in N : p_j - f_w W^{-1} H_j > 0\}$$

where W is the optimal basis for P3(B), and where f_w denotes the component from f=(p,q) corresponding to prices of activities in the basis W.

The second crucial element for the T-set algorithm is the following theorem.

Theorem 2.1 Candler and Townsley [8]

If B and B_k are optimal bases for the behavioral problems $P_2(\bar{y})$, $P_2(\bar{\bar{y}})$; and if the high point value for B is superior to the high point value for B_k, then B includes among its activities elements of the T-set $T^1(B_k)$.

Finally, the third and fourth results upon which the T-set algorithm is built are the following theorem and corollary.

Theorem 2.2 Candler and Townsley [8]

If the two-level problem P1, has an optimal solution then there exists a basis B, optimal for some behavioral problem $P2(\bar{y})$, such that the high point for B is an optimal solution for the two-level problem.

Corollary:

There exists an optimal solution to the two-level problem P1 that is a vertex of the technical polyhedron

$$\begin{aligned} Hx + Gy &= b \\ x,\ y &\geqq 0 \end{aligned}$$

A behavioral optimal basis B has to satisfy the relations

$$\begin{aligned} Bx_B + Gy &= b \\ x_B &\geqq 0 \end{aligned}$$

for some $y \geqq 0$.

It must also satisfy the relations

$$c_j - c_B\, B^{-1}\, H_j \leqq 0$$

for all the behavioral activities H_j.

A class of bases that includes the class of BOB bases is obtained if we only require the second condition,

$$c_j - c_B\, B^{-1} H_j \leqq 0$$

to be satisfied. Bases from this larger class will be used in the T-set algorithm and each one of them will be called a Behavioral Dual Feasible basis (BDF).

The algorithm is essentially a procedure to generate and enumerate bases from H, with special emphasis on BOB's. Although enumeration could be terminated before all bases have been exhausted, it is likely that this will occur very late in the process, or that all the bases get enumerated. The reason, as we shall see, is that the stopping rule is a sufficient condition for optimality and as such, it might not be effective as soon as the global optimum is attained.

3.1.2 T-sets of the Second and Third Kinds.

Many times in the course of the algorithm, given a collection Z of m_1 linearly independent vectors from H, $m_1 \leqq m$, it will be necessary to extend Z to a BDF basis B, by adding $m-m_1$ new vectors from H to Z. If the extension is not possible, one or more vectors from Z should be dropped and the extension attempted again for the new Z. The procedure is carried out in terms of linear programming as follows. First partition matrix H, as H = [Z,D] and solve the phase 1 problem below, to obtain a basis B including Z.

$$\text{Ph 1 :} \quad \min \; 1\, x_a$$

$$I\, x_a + Zx_z + Dx_d + Gy = b$$

$$x_a \qquad x_d \quad y \geq 0$$

$$x_z \text{ unrestricted}$$

If the optimal value for Ph1 is positive, then the resulting basis, although including Z is not a basis from H, as it includes artificial activities, a vector from Z should be dropped, and the corresponding phase 1 problem solved again. If on the other hand the optimal value of Ph1 is zero we proceed to the second phase problem

$$\text{P4:} \quad \max \; cx + dy$$

$$Zx_z + Dx_d + Gy = b$$

$$x_d \,, \quad y \geq 0$$

$$x_z \text{ unrestricted}$$

If the optimal value for P4 is finite then the resulting basis, B, includes Z and is BDF. This assertion follows by observing that the dual to P4 is

$$\text{DP4 :} \quad \min \lambda\, b$$

$$\lambda\, Z = -c_z$$

$$\lambda\, D \geq -c_d$$

$$\lambda\, G \geq -d$$

and that basic activities in the primal correspond to equality constraints in the dual.

If on the other hand, the optimal value for P4 is infinity, then DP4 is not feasible, and Z can be part of no BDF basis. A T-set of the second kind is defined in the latter case.

$$T^2 (B) = \{H_j \varepsilon Z{:}x_j \text{ can be increased arbitrarily in } P4\}$$

That is, T^2 is the collection of behavioral vectors that would be desirable to remove from Z - one at a time - to eventually obtain behavioral dual feasibility.

A second task performed repeatedly in the algorithm consists of checking whether or not a given BDF basis B is primal feasible as well, i.e., a BOB. If not, changes to remedy the situation are determined as follows:

Solve the phase 1 problem

$$\begin{aligned} P5: \min\ & 1\, x_a \\ & I\, x_a + B\, x_b + G_y = b \\ & x_a\ ,\ x_b\ ,\ y \geqslant 0 \end{aligned}$$

If at optimality P5 has a positive value indicating that B is not a BOB, then the optimal basis, say W, contains some artificial activities. Denote the index set of these activities by F(B). Note also that the dual optimal solution, π , should verify

$$\pi W = h$$

where h is a vector of ones for indices in F(B), and zero elsewhere. Therefore, the relative prices with respect to W, for activities in H can be written as

$$d - h W^{-1} H_j = - \sum_{i \in F(B)} W_i^{-1} H_j$$

where W_i^{-1} is the i-th row of the matrix W^{-1}. The relative prices for activities in B are nonnegative; however, activities in H-B may have negative relative prices. The latter are vectors which would lower the sum of infeasibilities if brought into the basis, and they constitute a T-set of the third kind. Formally we define

$$T^3(B) = \{H_j : \sum_{i \in F(B)} W_i^{-1} H_j > 0\}$$

3.1.3 The Stopping Rule

Suppose the BOB's: B_1, B_2, ... , B_k have been identified; and that consequently we also know the composition of the corresponding T^1-sets. It follows as a direct consequence of theorem 2.1 that if B is a BOB with a better high point than any of the high points for B_1, ... , B_k then B should include at least one element from each of the T^1 sets $T^1(B_1)$ ... , $T^1(B_k)$. This consequence arranged in contraposition produces the stopping rule for the T-set algorithm: If we could prove by enumeration or other means that there is no BOB basis B_{k+1} including at least one element from each of the T^1 sets T^1_1, ..., T^1_k, then we would conclude that the global optimum for the two-level problem is the highest high point among the high points produced by B_1, B_2, ... , B_k. It is important to note that there is no guarantee that the converse of the stopping rule will be true. This means that the stopping rule may not be satisfied as soon as the global optimum is attained. If the global optimum is among B_1, ... , B_k we could still produce a BOB: B_{k+1} satisfying the T-sets for B_1, ... , B_k, and get a high point for B_{k+1} inferior to the best known high point.

3.1.4 The Algorithm

Suppose we have obtained the BOB's: B_1, ... , B_k and the corresponding T^1-sets T^1_1, T^1_2, ---, T^1_k. Now it is necessary to find a new BOB: B_{k+1} including at least one element from each known T^1 set or else prove that such a BOB does not exist. The general plan will be to first find a collection Z of m_1, vectors ($m_1 \leq m$) satisfying each one of the T^1 sets, i.e., a set Z that includes at least one element from each known T^1-set. Subsequently, this collection

Z will be extended to a BDF basis and eventually modified to a BOB basis that in turn will produce a new T^1-set. Once a basis is visited, BOB or otherwise, it will be kept explicitly out of the search. This procedure will now be discussed in detail.

As a first step, the collection Z is obtained by solving the integer program

$$\text{IP:} \quad \sum_{j=1}^{n_1} \delta_{ij}\, y_j \geq 1 \qquad i = 1, \ldots, k$$

$$\sum_{j=1}^{n_1} y_j \leq m$$

$$y_j - y_j^2 = 0$$

where

$$\delta_{ij} = 1 \quad \text{if} \quad H_j \in T^1(B_i)$$
$$= 0 \quad \text{otherwise}$$

and where Z, which we now call Z_k, is obtained using a solution y to the problem above as follows

$$Z_k = \{H_j / y_j = 1\}$$

The second step in this procedure is the solution of problem P4 to find a basis $\bar{B}$ from H including Z_k. If the basis $\bar{B}$ is a BDF basis, we proceed to step 3, otherwise we obtain a T^2-set. In step 3 we solve problem P5 to check the primal feasibility for basis $\bar{B}$. If it is feasible we solve the policy problem for $B_{k+1} = \bar{B}$, add the corresponding $T^1{}_{k+1}$ set to the existing collection of T^1 sets and go to step 1. If basis $\bar{B}$ is not primal feasible, i.e. not a BOB, this means that problem P5 produces a T^3-set.

When a T^2-set results, we revise the integer program generating the collection of vectors Z, by adding T^2-set restrictions as follows:

IP:
$$\sum_{j=1}^{n_1} \delta_{ij}\, y_j \geq 1 \qquad i = 1 , \ldots, k$$
$$\sum_{j=1}^{n_1} y_j \leq m$$
$$\sum_{j=1}^{n_1}{}' \, \delta^2_{\bar{B}j}\, y_j \leq p - 1$$
$$y_j - y_j^2 = 0$$

where

$$\delta^2_{\bar{B}j} = 1 \text{ if } H_j \quad T^2(\bar{B})$$
$$= 0 \text{ otherwise}$$

and where p is the number of elements of $T_2(\bar{B})$. Once a T^2-set constraint is incorporated in the problem IP, it will remain in it, to prevent the previous basis $\bar{B}$ from recurring in the future.

When a T^3-set results, the problem IP is revised as follows.

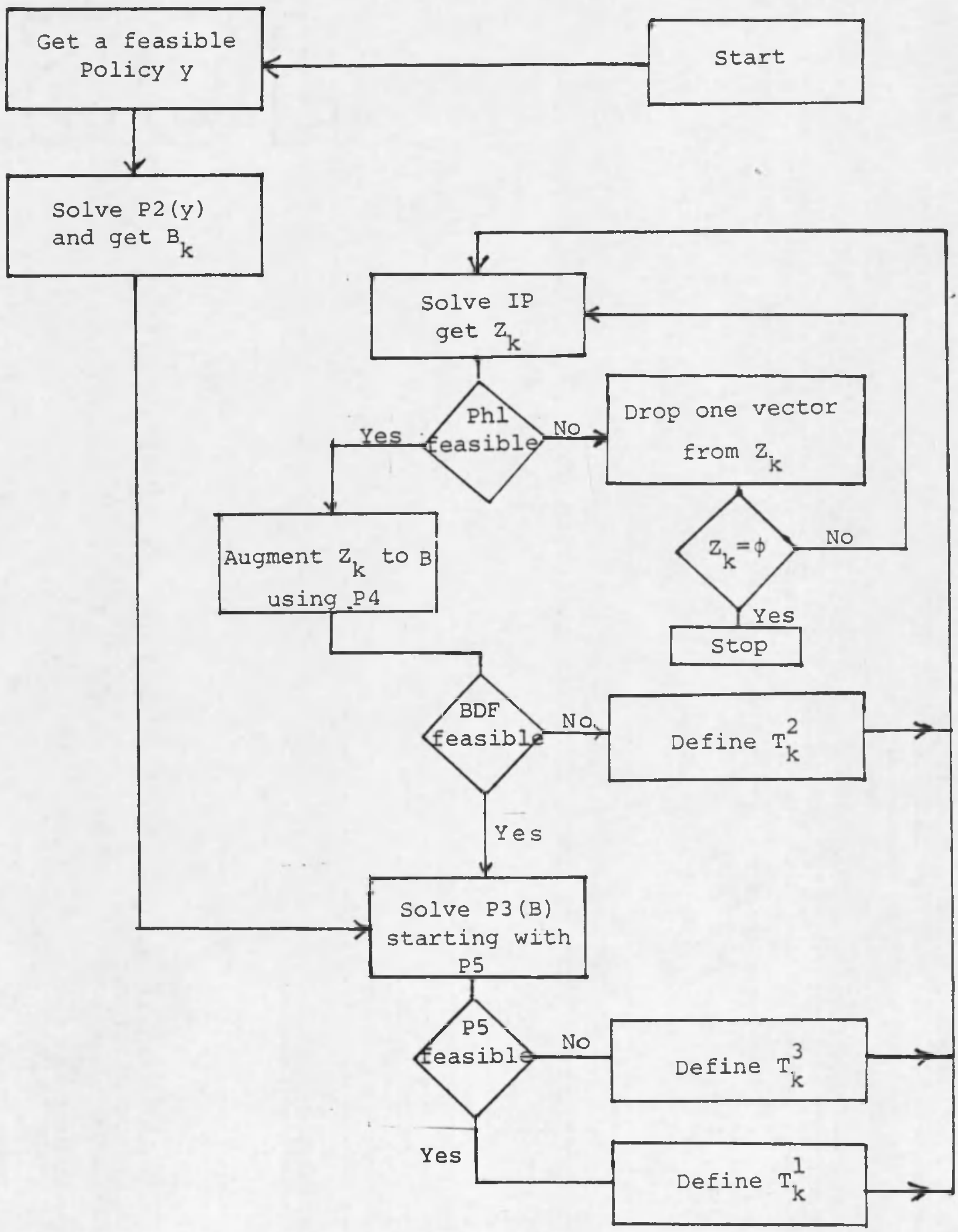

Figure 4. Flow Chart for the T-Set Algorithm

IP:

$$\sum_{j=1}^{n_1} \delta_{ij}\, y_j \geq 1 \qquad i = 1, \ldots, k$$

$$\sum_{j=1}^{n_1} y_j \leq m$$

$$\sum_{j=1}^{n_1} \delta^3_{\bar{B}j}\, y_j \geq 1$$

$$y_j - y_j^2 = 0$$

where

$$\delta^3_{\bar{B}j} = 1 \quad \text{if } H_j \in T^3(\bar{B})$$

$$= 0 \quad \text{otherwise}$$

A T^3-constraint has a double purpose, first it indicates which behavioral activities would potentially restore feasibility by lowering the sum of infeasibilities of problem P5, and second it prevents the previous basis B, from recurring in the future. A schematic diagram is given in Figure 4 which summarizes the T-set algorithm.

3.2 Max-Min Approach (Falk[22])

3.2.1 Preliminaries

The max-min problem is equivalent to a particular case of the two-level linear problem. Problem 1 below is the max-min problem, and problem 2 its reformulation to display the implicit two-level structure.

Problem 1

$$\max_{y} \min_{x} \{px + qy : Hx + Gy = b , x \geq 0, y \geq 0\}$$

Problem 2

$$\begin{aligned} &\max \quad px + qy \\ &(1) \quad y \geq 0 \\ &(2) \quad \max \quad -px - qy \\ &\qquad\qquad Hx + Gy = b \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}$$

The methodology presented by Falk to solve the max-min problem is based on the fact that an optimal solution occurs at a vertex of the technical polyhedron

$$\text{(3.2.a)} \qquad \begin{aligned} Hx + Gy &= b \\ x , y &\geq 0 \end{aligned}$$

The same property is true for the two-level problem [4, 8] . Furthermore, it is clear that the structure of the two-level feasible domain is determined only by the technical constraints and the lower objective function, thus the max-min problem and the two-level problem have the same feasible domain structure. In fact, in the next section we will extend Falk's branch and bound algorithm to the general two-level linear problem, in which the lower objective $- px - qy$ is replaced by $cx + dy$.

The vector (x,y) will be denoted by the single letter Z when convenient. Also since the basic variables of basic solutions to the technical polyhedron (3.2.a) will be scrutinized extensively the definition of the following set of indices will be found very useful:

$$B(p) = \{i : Z_i^p \text{ is a basic variable of } Z^p\}$$

where Z^p is the pth basic solution of the technical polyhedron (3.2.a) and $Z^p{}_i$ is the i-th basic variable of Z^p.

Finally, denote the optimal basic solution to the two-level problem by Z^* and by v^* the corresponding optimal value.

3.2.2 The Algorithm

The algorithm will proceed by stages. In the first stage, two problems are solved. The first problem Q^1 below will provide the first upper bound.

$$\begin{aligned} Q^1 : \max \quad & px + qy \\ & Hx + Gy = b \\ & x\ ,\ y \geq 0 \end{aligned}$$

The optimal value of Q^1, denoted u^1 is the first upper bound on v^*, and the optimal solution $Z^1 = (x^1, y^1)$ serves to define the second problem of the first stage:

$$\begin{aligned} R^1 : \max \ & cx + dy^1 \\ & Hx + Gy^1 = b \\ & x \geq 0 \end{aligned}$$

Problem R^1 is feasible (x^1 is a feasible point) and its optimal solution $\bar{Z}^1 = (\bar{x}^1, \bar{y}^1)$ provides the first lower bound on v^* given by

$$\ell^1 = p\bar{x}^1 + q\bar{y}^1 \qquad \text{(note that } \bar{y}^1 = y^1\text{)}$$

For the sake of a consistent notation along all stages of the algorithm, make $U^1 = u^1$ and $L^1 = \ell^1$.

If $L^1 = U^1$ then $\bar{Z}^1$, as well as Z^1 are optimal solutions to the two-level problem.[1]

If Z^1 is not a solution of the two-level problem we proceed to the second stage of the algorithm. Since Z^1 is not a solution, at least one of the basic variables of Z^1 must correspond to a non-basic variable of Z^*, i.e. Z_k^* must be non-basic for at least one $k \in B(1)$. The second step of the algorithm is initiated by setting up a problem

$$\begin{aligned} Q^k :\ & \max\ px + qy \\ & Hx + Gy = b \\ & x\,,\ y \geq 0 \\ & Z_{j_k} \ \text{non-basic} \end{aligned}$$

for each index j_n in $B(1) = \{j_2, \ldots, j_{m+1}\}$.

Many problems with side constraints of the form "Z_{j_k} non basic" like in Q^k above will be encountered while

---

1 If the two level problem satisfies the assumptions of uniqueness of the solution of the two-level problem, then $Z^1 = \bar{Z}^1$.

applying the algorithm. They will be solved starting with a phase 1 problem that includes among its artificial variables the z_{j_k} variables subject to the constraint of being non-basic. If at optimality the phase 1 problem has a positive value or is zero but degenerate in the sense that it has z_{j_k} (at zero level) which cannot be removed from the basis, then problem Q^k will be termed infeasible.

Let $z^k=(x^k,y^k)$ denote a solution of problem Q^k (if one exists), with u^k denoting its optimal value. Note that solution z^k is easily obtained from solution z^1 by driving the variable z_{j_k} out of the basis and thereafter holding $z_{j_k} = 0$. Note also that the number $U^2=\max\{u^2, \ldots, u^{m+1}\}$, is an upper bound for v^*, and a sharper bound than U^1 (i.e. $U^2 \leq U^1$) since U^1 is the optimal value of Q^1, which can be obtained from any Q^k $1 < k \leq m+1$, by just relaxing the constraint "z_{j_k} non-basic".

Associated with the optimal solution $z_k=(x^k,y^k)$ of Q^k we have the problem

$$R^k : \max cx + dy^k$$
$$Hx = b - Gy^k$$
$$x \geqslant 0$$

which is a feasible problem, since for instance x^k is a feasible point. The optimal solution $\bar{z}^k=(\bar{x}^k,\bar{y}^k)$ to problem R^k is feasible for the two-level problem and consequently

$$\ell^k = c\bar{x}^k + d\bar{y}^k$$

is a lower bound on v^*. The best lower bound up to the second stage is given by $L^2 = \max\{\ell^1, \ldots, \ell^{m+1}\}$. If $L^2=U^2$, we should have $L^2=v^*$, since $L^2 \leq V^* \leq U^2$. A solution of the two-level problem is given by that point $\bar{z}^k$ for which $\ell^k=L^2$. If $L^2 < U^2$, then we have no guarantee that $v^*=L^2$, and thus we go into the third stage.

In order to describe the continuation of the algorithm, it is convenient to introduce some additional notation. A problem Q^j is the <u>parent problem</u> of problem $Q^{j'}$, if problem $Q^{j'}$ is derived from problem Q^j by adding a constraint of the form "Z_i non basic", where index i belongs to B(j). Any problem Q^j may be the parent of m problems since B(j) contains m elements. Some of these m problems may be infeasible. If a problem Q^j is not a parent problem (i.e. if the problems corresponding to the set B(j) have not been set up), then problem Q^j is termed an <u>offspring problem</u>.

The branch and bound algorithm proceeds by a sequence of stages. The first stage consists of the problem Q^1 alone which is then an offspring problem. The second stage consists of those problems $Q^2, \ldots, Q^{m+1}$ corresponding to the elements of the set B(1). Thus Q^1 becomes a parent and $Q^2, \ldots, Q^{m+1}$ are offspring. The third stage of the method is obtained by defining offspring problems $Q^{m+2}, \ldots, Q^{2m+1}$ with one of the problems $Q^2, \ldots, Q^{m+1}$ becoming a parent. In general, the kth stage is initiated by selecting one of the current offspring problems Q^t and defining new offspring problems corresponding to the set B(t). Problem Q^t then becomes a parent. The selection of this new parent is made according to the branching rule described below. The succession of stages is depicted in Fig. 5 through the first 5 stages of a fictitious problem.

In general, the (k+1)st stage is defined from the kth stage as follows. Having completed the kth stage, problems $Q^1, \ldots, Q^{(k-1)m+1}$ have been set up and solved (or determined to be infeasible). Let $Z^j=(x^j,y^j)$ represent the solution of problem Q^j (if one exists), with u^j representing the optimal value (set $u^j= -\infty$ if Q^j is infeasible). The problem R^j is defined from Z^j as before and $\bar{Z}^j=(\bar{x}^j,\bar{y}^j)$ is the solution of problem R^j with ℓ^j denoting $c\bar{x}^j + d\bar{y}^j$ (set $\ell^j= -\infty$ if Q^j is infeasible).

The branching procedure of the algorithm assures that the optimal solution of the two-level problem is feasible for at least one of the current offspring problems Q^j (and hence feasible for R^j). Thus, with

$$L^k = \max \{\ell^1, \ldots, \ell^{(k-1)m+1}\} \text{ , and}$$

$$U^k = \max \{u^j : 1 \leq j \leq (k-1)m+1 \text{ and } Q^j \text{ is an offspring}\}$$

we have

$$L^k \leq v^* \leq U^k$$

The inequalities above follow since L^k represents the current best lower bound obtained after k stages while

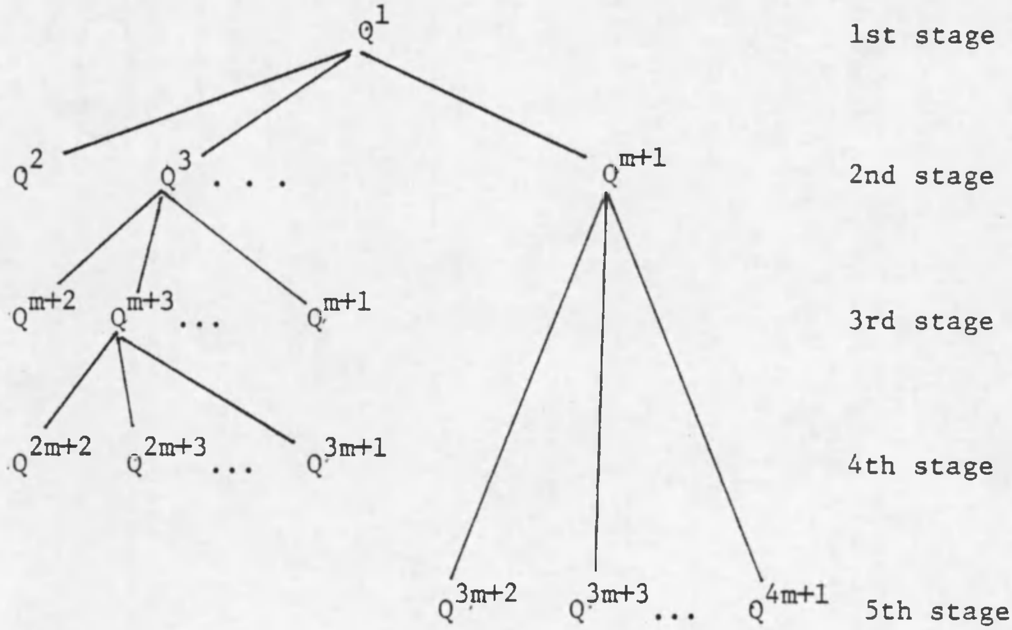

Figure 5. Branch and Bound Tree

U^k is an upper bound on all the current offspring, and the optimal solution for the two-level problem is feasible for at least one of these offspring. Note that since at each stage the best available lower bound is selected, we have

$$L^k \leq L^{k+1}$$

While $U^k \geq U^{k+1}$

results from the fact that any parent problem can be trivially restated in terms of any of its offspring by relaxing the latest added "non basic" constraint.

If $L^k = U^k$ at the kth stage, the two-level problem has been solved with $v^* = L^k$, and the solution is given by $z^{j(k)}$ whose superscript j(k) satisfies

$$\ell^{j(k)} = L^k.$$

If $L^k < U^k$, then a new parent problem should be selected. In principle this selection could be arbitrary and the algorithm still works, but the best bound first rule of integer programming would choose that offspring problem Q^j whose optimal solution u^j equals the upper bound U^k.

3.3 The Kuhn-Tucker Conditions Approach (Fortuny-Amat and McCarl [23])

3.3.1 General Approach

The behavioral constraint, or inner problem, can be replaced by the corresponding Kuhn-Tucker conditions. In general, the Kuhn-Tucker equations are only necessary optimality conditions, however, they are also sufficient for a class of two-level non-linear problems of which the linear problem is a particular case. Below we reproduce, save for minor changes and notation, the two-level problem dealt with by Fortuny-Amat and McCarl [23], and the equivalent using the Kuhn-Tucker conditions.

The two-level problem is:

$$
\begin{aligned}
T : \max\ & px + qy \\
\text{s.t.}\ & (1)\quad y \geq 0 \\
& (2)\quad \max\ cx + dy + (x,y)\, Q(x,y)^t \\
& \qquad\qquad Hx + Gy = b \\
& \qquad\qquad\qquad x \geq 0
\end{aligned}
$$

The Kuhn-Tucker equivalent is

$$
\begin{aligned}
K :\ & \max\ px + qy \\
& (1)\quad c + 2x^t Q_{11} + 2y^t Q_{21} + \lambda H + \mu = 0 \\
& (2)\quad \lambda \ \text{ unrestricted}, \ \mu \geq 0 \\
& (3)\quad Hx + Gy = b \\
& \qquad\ x\ ,\ y \geq 0 \\
& (4)\quad \mu x = 0
\end{aligned}
$$

Where Q is a symmetric matrix, and Q_{11}, Q_{12}, Q_{21} and Q_{22} are the components of a partition of Q that accomodates the expression of $(x,y)^t Q(x,y)$ in terms of x and y. The quadratic behavioral objective functions for which the Kuhn-Tucker conditions are equivalent to Behavioral Optimality are those in which Q_{11} is negative semi-definite.

The new formulation has removed the behavioral constraint - or inner problem - but it has introduced new constraints, among them the complementarity conditions which are hard to handle even with quadratic codes. Below we present some of the alternatives researchers have developed to tackle the computational problem.

3.3.2 Mixed Integer Programming

Fortuny-Amat and McCarl replaced each complementarity equation

$$v_j \, x_j = 0 \quad (v_j \geqq 0 \, , \, x_j \geqq 0)$$

by two equivalent inequalities

$$v_j \leqq M \, n_j \quad (v_j \geqq 0 \, , \, x_j \geqq 0)$$
$$x_j \leqq M(1-n_j)$$

Where n_j is a zero-one variable and M a sufficiently large upper bound on the variables v_j, and x_j. The resulting problem is a mixed integer problem, solvable by branch and bound techniques.

3.3.3 Special Ordered Sets

Bisshop and Meeraus [5] have solved two-level problems using the Fortuny-Amat formulation, with a commercial code for integer programming.

The computer programming code (APEX-III mixed integer programming option) is a branch and bound procedure utilizing a special ordered sets search [2]. Computational results, though limited, indicate that a large number of linear programs must be solved.

3.3.4 Separable Programming

Bard and Falk [1] also proposed and experimented with an alternative treatment of the complementary conditions of the Fortuny-Amat formulation.

The complementary equation

$$\sum_{1}^{n_1} v_j\, x_j = 0$$

is replaced by the equivalent set of equations:

$$\sum_{1}^{n_1} \{\min\,(0, w_j) + v_j\} = 0$$

$$w_j - x_j + v_j = 0 \qquad j = 1, \ldots, n_1$$

This transformation leaves the problem in the separable form which is amenable to their branch and bound algorithm for separable problems [21]. This procedure works by enclosing the feasible region of a separable non-convex program in a linear polyhedron which is then divided into disjoint subsets. A lower bound on the optimal value of the problem is found by minimizing the objective function over each of these subsets and selecting the smallest value obtained. An optimality check is made which, if successful, the algorithm terminates with a global solution of the piecewise linear approximation of the separable non-convex program. If the check fails, the subset corresponding to the smallest lower bound is further subdivided into either two or three linear polyhedra and the process continues as before with new and sharper bounds being determined. the process is finite and terminates with a global solution of the problem approximation. The experience with this method indicates that a large number of subproblems must be solved.

3.4 A Local Optimum Algorithm

3.4.1 Preliminaries

Karwan and Bialas [4] , explored the structure of the two-level feasible domain, and also the problem of controllability as a function of the number of policy variables. The main result of their paper is the following theorem.

3.4.a Theorem (Karwan and Bialas [4]):

If a point z in the two-level feasible set can be expressed as a non-trivial convex combination of points z_i in the one-level (or technical) feasible set, then the points z_i also belong to the two-level feasible set.

The two-level feasible set refers to the set

$$S^2 = \{(x, y): \ y \geq 0 \ \text{ and } \ x \ \text{ solves the behavioral problem } \ P2(y)\}$$

for the definition of problem P2(y) see section 3.1.1, and the one-level feasible set or technical feasible set refers to

$$S^1 = \{(x, y): \ Hx + Gy = b \ , \ x \geq 0 \ , \ y \geq 0\}$$

The above theorem permits them to obtain a high point and then search for adjacent extreme points to the current high point that are behavioral optimal and have a higher value than the current high point. Their procedure is reproduced below and it guarantees the attainment of a local optimum.

3.4.2 The Algorithm

Step 1: Solve the following problem via the simplex method

$$\max \ px + qy$$
$$Hx = b - Gy$$
$$x \geq 0$$

and obtain the optimal solution $\hat{z} = (\hat{x}, \hat{y})$ and optimal tableau $\hat{T}$.

Step 2: Set $y = \hat{y}$ and solve the following problem via bounded simplex, beginning with tableau $\hat{T}$.

$$\max \quad cx + dy$$

$$Hx + Gy = b$$

$$y = \hat{y}$$

$$x \geq 0$$

Let the optimal solution be $\bar{Z}$. If $\bar{Z} = \hat{Z}$, stop; $\hat{Z}$ is a global optimum solution. Otherwise, go to step 3a with current tableau $\bar{T}$ and relax the constraint $y=\hat{y}$.

Step 3a: If all non basic variables are equal to zero, go to step 4 with current tableau $\bar{T}$. Otherwise go to step 3b.

Step 3b: If $\bar{b}_i > 0$ for all i, go to step 3c. Otherwise, without loss of generality, choose ℓ such that $\bar{b}_\ell=0$ and choose $y_{\ell j}\neq 0$ in the tableau $\bar{T} = (t_{ij})$, with j corresponding to a non basic policy variable at non zero level. Bring x_j into the basic via a degenerate pivot. Go to step 3a.

Step 3c: Consider any non basic variable which is at a strictly positive value, say x_j. If the relative price with respect to the policy objective is non-negative, increase x_j until it enters the basis. If it is strictly less than zero, decrease x_j until either it reaches zero or it enters the basis. Go to step 3a.

<u>Step 4</u>: Beginning with the current tableau T solve the following problem via a modified simplex procedure:

$$\max\ px + qy$$
$$Hx + Gy = b$$
$$x\ ,\ y\ \geq 0$$

The modification is as follows. Given a candidate to enter the basis (one for which px + qy will increase) allow it enter only if the resulting basic solution, $\bar{Z}$ is contained in the two-level feasible set.

Existing numerical examples for the algorithms reviewed in this chapter will be presented in chapter 4 to compare the above algorithms to the algorithm proposed in this dissertation.

CHAPTER 3

FOUNDATIONS FOR AN ALGORITHM OF INTERCEPTIONS

1. Overview

The basic results for an algorithm of interceptions are developed in this chapter. For the sake of a clear geometric exposition, the two-level problem is formulated using only weak inequality relations:

$$
\begin{aligned}
T: \quad & \max\ p\,x + q\,y \\
& (1) \quad E\,y \geq e \\
& (2) \quad \max\ cx + dy \\
& \qquad\qquad Hx + Gy \geq b
\end{aligned}
$$

Matrices H and G are assumed to be of dimensions mxn_1 and mxn_2, and matrix E of $k \times n_2$.

The results of sections 2, 3, and 4, are in relation to the geometric and topological structure of the feasible domain for the two-level problem stated above. It will be shown that the feasible domain of problem T is a connected collection of behavioral optimal faces of the technical polyhedron

$$
\begin{aligned}
Hx + Gy &\geq b \\
Ey &\geq e
\end{aligned}
$$

A behavioral optimal face is a natural concept that follows from defining behavioral and policy problems for the two-level problem in its formulation above.

In sections 5 and 6 necessary and sufficient conditions for local optima, and global optima will be derived from the results regarding the two-level feasible domain structure.

2. Behavioral Optimality

Given a policy setting $y=\bar{y}$, satisfying the upper technical constraints

$$E\,y \geqslant e$$

the corresponding behavioral problem is defined as follows:

$$B\,P\,(\bar{y}):\quad \max\quad c\,x + d\,\bar{y}$$
$$H\,x + G\,\bar{y} \geqslant b$$

For a linear problem the Lagrange optimality conditions are the same as the Kuhn Tucker conditions, and as such they are necessary and sufficient for optimality.[1]

The Lagrange function for $BP(\bar{y})$ is

$$L_{\bar{y}} = (c + \lambda H)\,x + d\,\bar{y} + \lambda(G\bar{y} - b)$$

Note that the Lagrange function depends parametrically on $\bar{y}$. A pair (x,λ) is a saddle point for $L\bar{y}$ - and thus

1. The Kuhn Tucker conditions are necessary conditions for optimality and the Lagrange sufficient.

x is optimal for $BP(\bar{y})$, and λ optimal for the dual of $BP(\bar{y})$ - if and only if it satisfies the Lagrange optimality conditions:

$$2.a \quad c + \lambda H = 0$$

$$\lambda \geq 0$$

$$2.b \quad Hx + G\bar{y} \geq b$$

$$2.c \quad a_i^H x + a_i^G \bar{y} > b_i \text{ implies } \lambda_i = 0$$

(where $(a_i^H \ a_i^G)$ is the i^{th} row of matrix $[H,G]$)

The dual problem of $BP(\bar{y})$ is given by

$$\min_{\lambda \geq 0} \max_{x} L_{\bar{y}}(x,\lambda)$$

or more explicitly by

$$\text{DBP:} \quad \min \ \lambda(G\bar{y} - b)$$

$$\lambda H = -c$$

$$\lambda \geq 0$$

A basis for DBP or $BP(\bar{y})$ is given by an index subset J, of size n_1, from $\{1,2,\ldots,m\}$ such that the rows from H given by J, form a square nonsingular matrix H_J. The dual solution can be obtained by solving

$$(2.a)' \quad \lambda_N H_N + \lambda_J H_J = -c$$

$$\lambda_N = 0$$

(where N is the complement of J in $\{1, \ldots, m\}$) and it is feasible if

$$(2.a)'' \qquad \lambda_J = -c(H_J)^{-1} \geq 0$$

The primal basic solution is obtained by solving

$$(2.b)' \qquad H_J\, x = b_J - G_J\, \bar{y}$$

and it is feasible if $x = (H_J)^{-1}(b_J - G_J\, \bar{y})$, satisfies the remaining constraints of the primal problem:

$$(2.b)'' \qquad \text{i.e.} \quad a_i^H\, x \geq b_i - a_i^G\, \bar{y} \quad \text{for all } i \in N\, .$$

If the primal and dual solutions $(x, \lambda_N, \lambda_J)$ associated with the basis H_J (or basis J for short) are primal-feasible and dual-feasible, then $(x, \lambda_N, \lambda_J)$ satisfy 2.a, 2.b, and 2.c; and as such $(x, \lambda_N, \lambda_J)$ is a saddle point for $L_{\bar{y}}$. In particular, we can say that H_J determines an optimal solution for $BP(\bar{y})$.

3. Behavioral Optimal Faces

The optimality conditions (2.a)', (2.a)" do not depend on $\bar{y}$. Thus, if we change $\bar{y}$ to $\hat{y}$, and also change x to $\hat{x}$ so as to preserve equations (2.b)' and (2.b)", then the new pair $(\hat{x}, \lambda_N, \lambda_J)$ satisfy the conditions of saddle point for the Lagrange function $L_{\hat{y}}$, and therefore $\hat{x}$ solves $BP(\hat{y})$.

The collection of all possible variations (without changing the multiplier (λ_N, λ_J)) is then given by

$$(3.a) \quad a_i^H x + a_i^G y = b_i \qquad i \in J$$

$$(3.b) \quad a_i^H x + a_i^G y \geq b_i \qquad i \in N$$

Therefore, each solution (x,y) of (3.a) and (3.b) is such that x solves BP(y). If we add the upper level constraint

$$(3.c) \qquad E\,y \geq e$$

to the conditions (3.a) and (3.b) above we obtain a collection of points called a Behavioral Optimal Face. The Behavioral Optimal Face corresponding to a basis J will be denoted by BOF(J), and simply by BOF if the basis is not specified. Further, the highest value attained by the policy objective px+qy within a BOF will be called a high point.

Figure 6 illustrates graphically the solution of the behavioral problem BP($\bar{y}$), and the behavioral optimal face containing it.

The polyhedron with faces 1, 2, 3, 4 and 5 is the technical polyhedron. Line $\bar{z}$ - z_1, is the feasible domain for the behavioral problem BP($\bar{y}$). The optimal basis for BP($\bar{y}$) is given by lines $y=\bar{y}$ and face 1 and the behavioral optimal face that includes $\bar{z}$ is face 1. The two-level feasible domain is given by faces 1, 2 and 3.

The high point within the Behavioral Optimal Face 1 is point Q.

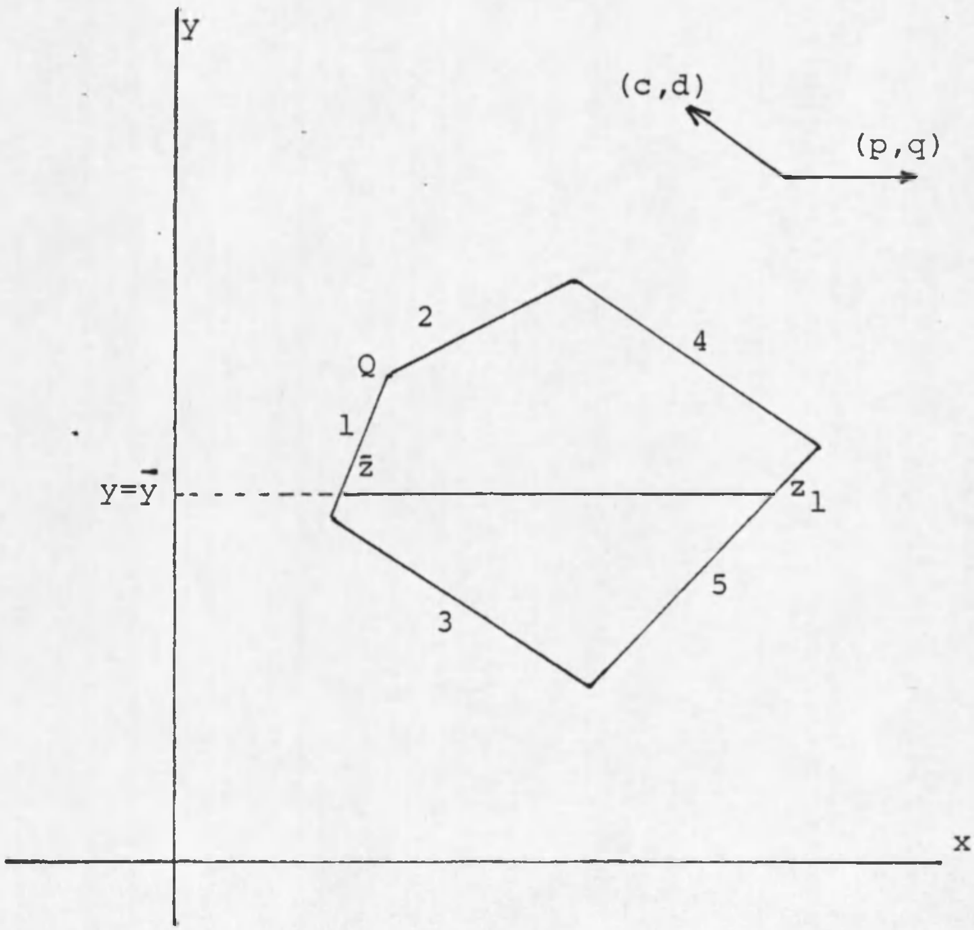

Figure 6. Behavioral
Solution and Behavioral Optimal Faces

4. The Two-Level Feasible Set is Connected

Consider the behavioral problem

$$\begin{aligned} BP(\bar{y}): \quad & \max\ cx + dy \\ & Hx + Gy - Iu = b \\ & y = \bar{y} \\ & u \geq 0 \end{aligned}$$

The dual to the above problem is

$$\begin{aligned} DBP: \quad & \min\ -\lambda b - \pi \bar{y} \\ & \lambda H = -c \\ & \lambda G + \pi = -d \\ & \lambda \geq 0 \quad \pi \text{ unrestricted} \end{aligned}$$

If DBP is feasible, the problem $BP(\bar{y})$ has an optimal solution for any $(b,\bar{y})$ that makes $BP(\bar{y})$ feasible. On the other hand, if DBP is not feasible, then $BP(\bar{y})$ is either unbounded or not feasible. Thus, the discussion below will be restricted to the case in which DBP is feasible, and the collection of <u>all</u> feasible policy settings will be denoted Y, that is

$$\begin{aligned} Y = \{\bar{y} /\text{there exists } x \in R^{n_1},\ y \in R^{n_2} \text{ and } u \geq 0 \\ \text{such that} \quad & Hx + Gy - Iu = b \\ & Ey \geq e \\ & y = \bar{y} \} \end{aligned}$$

The feasible set for the two-level problem will be denoted FS, that is

$$FS = \{ \binom{x}{y} \;/\; y \in Y \text{ and } x \text{ solves } BP(y) \}$$

The set Y is a convex polyhedron. This can be proved in terms of the following phase 1 problem.

$$
\begin{array}{llcl}
\text{Ph I:} & \min \; 1\,v^{+} + 1\,v^{-} & & \\
\partial : & Hx + Gy - u + v^{+} - v^{-} & = & b \\
k : & Ey & \geq & e \\
\theta : & y & = & \bar{y} \\
\rho : & u & \geq & 0 \\
\tau : & v^{+} & \geq & 0 \\
\sigma : & v^{-} & \geq & 0
\end{array}
$$

Where v^{+} and v^{-} are the artificial phase 1 variables, and $\pi = (\partial, k, \theta, \rho, \tau, \sigma)$ is the vector of simplex multipliers.

If $\pi_i = (\partial_i, k_i, \theta_i, \rho_i, \tau_i, \sigma_i)$ $i = 1, \ldots, k$ is the collection of <u>all</u> the optimal simplex multipliers then, it can be shown [41] that

$$Y = \{ y \in R^{n_2} \;/\; \theta_i y \geq -\partial_i b - k_i e \}$$

which is a convex polyhedron. As a trivial corollary of the above result we can state, for future reference, that Y is a closed and connected set.

In order to prove that FS is connected, we need to prove that FS is closed, and for the latter, we need the following result.

Lemma 1. The function $\sigma : Y \to R$, defined below is continuous.

$$\begin{aligned} \sigma(\bar{y}) = \max \; & cx + dy \\ & Hx + Gy - u = b \\ & y = \bar{y} \\ & u \geq 0 \end{aligned}$$

This is a standard result in the literature of linear programming. In fact $\sigma(\bar{y})$ is a polyhedral concave function given by

$$\sigma(\bar{y}) = \min_{j = 1, \ldots, R} \pi_j \binom{b}{\bar{y}}$$

where $\pi_1, \ldots, \pi_R$ is the collection of dual optimal simplex multipliers to problem BP($\bar{y}$) as $\bar{y}$ varies in Y.

Lemma 2. FS is closed.

Proof:

Let (x_n, y_n) be an arbitrary convergent sequence in FS. In order to prove that FS is closed, we have to show that the limit $(\bar{x}, \bar{y})$ of the sequence (x_n, y_n) belongs to FS. Since each element (x_n, y_n) of the sequence belongs to FS, we have that $y_n \in Y$ and x_n solves BP(y_n). Therefore, (y_n) is a sequence in Y, and since Y is closed, its limit $\bar{y}$, belongs to Y. This is the first condition for $(\bar{x},\bar{y})$ to be a member of FS. The second condition requires $\bar{x}$ to be equal to the solution of BP($\bar{y}$). We prove this next.

Note that $\sigma(y_n)$ converges to $\sigma(\bar{y})$, because σ is a continuous function. Also, since $\sigma(y_n)$ is equal to $cx_n + dy_n$, we see that $\sigma(y_n)$ converges to $c\bar{x} + d\bar{y}$, because the function $cx + dy$ is continuous. Therefore, by the uniqueness of a sequence limit we conclude that $\sigma(\bar{y}) = c\bar{x} + d\bar{y}$. To prove that $\bar{x}$ solves BP($\bar{y}$) it only remains to show that $\bar{x}$ is feasible for BP($\bar{y}$), that is, we should show that $x = \bar{x}$ satisfies

$$Hx + Gy \geq b$$
$$y = \bar{y}$$

However, this follows if we take limits as n goes to infinity in the following relations

$$H x_n + G y_n \geq b$$
$$y_n = y_n$$

which hold because x_n is feasible for BP(y_n).

Theorem 1. FS is connected

Proof:

Suppose FS is not connected. Let $\{FS_j\}_{j \in J}$ be the collection of its connected components. Then we have

(1) $FS = \bigcup_{j \in J} FS_j$

(2) Each FS_j is a nonempty closed in FS

(3) $FS_j \cap FS_i = \phi$ for $i \neq j$

Also since FS is closed in R^n, each FS_j is closed in R^n.

The sets FS and Y are related in a useful manner by the projection function

$$P: R^n \rightarrow R^{n_2}$$

defined by $P(x,y) = y$. This function is a well-known open topologic mapping. That is, it maps closed sets onto closed sets. Therefore, $P(FS_j)$ is a non-empty closed in R^{n_2}, and thus closed with respect to Y, since Y is closed. Besides

$$Y = P\ (FS) = \bigcup_{j \varepsilon J} P\ (FS_j)$$

Observe also that if $\bar{z} = (\bar{x},\bar{y})$ belongs to FS_k and if $z_1, \ldots, z_\ell$ is the collection of <u>all</u> the optimal basic solutions to:

$$\begin{aligned} BP(\bar{y}): \quad \max \quad & cx + dy \\ & Hx + Gy \geq b \\ & \qquad\quad y = \bar{y} \end{aligned}$$

then $\bar{z}$ belongs to the convex hull generated by $z_1, \ldots, z_\ell$, which being connected, must be entirely included in FS_k.[1]

---

1. If $BP(\bar{y})$ has an unbounded collection of optimal solutions, it would be necessary to replace the convex hull $z_1, \ldots, z_\ell$ by the convex hull $z_1, \ldots, z_\ell$ plus a cone generated by a finite number of vectors. However, it can be seen that the argument remains valid.

Now, we are ready for the final argument. Since Y is connected, there must exist $j \neq k$ such that

$$P(FS_j) \cap P(FS_k)$$

has at least one element, say $\hat{y}$. Therefore, there exists $\bar{x}$ and $\bar{\bar{x}}$ such that $\bar{z} = (\bar{x},\hat{y})$ belong to FS_j and $\bar{\bar{z}} = (\bar{\bar{x}},\hat{y})$ belong to FS_k. Let $z_1, z_2, \ldots, z_\ell$ be the collection of all the optimal basic solutions to $BP(\hat{y})$, where the number of solutions ℓ can be as small as one. By the observation made above, we infer that the convex hull $[z_1, \ldots, z_\ell]$ is included in FS_j since

$$\bar{z} \in [z_1, \ldots, z_\ell] \cap FS_j$$

We also infer that $[z_1, \ldots, z_\ell]$ is included in FS_k since

$$\bar{\bar{z}} \in [z_1, \ldots, z_\ell] \cap FS_k$$

This contradicts the condition "$FS_k \cap FS_j = \phi$ for $k \neq j$". We must therefore conclude that FS has only one connected component. That is we conclude that FS is connected.

5. Global Optimality

In this section we show how the connectedness of FS and theorem 2.4.a render a procedure that allows the transition from a local optimum to a better solution point within FS or else proves that the local optimum in question is the global optimum. The idea behind the procedure can be explained using Figure 7, which is similar to Figure 6.

If points $\bar{z} = (\bar{x},\bar{y})$ and $\bar{\bar{z}} = (\bar{\bar{x}},\bar{\bar{y}})$ are points in FS such that

$$\partial = p\bar{x} + q\bar{y} < p\bar{\bar{x}} + q\bar{\bar{y}} = \delta$$

as is the case in Figure 7, then FS and the plane $px + qy = \partial + \varepsilon$ have points in common for ε such that $\partial < \partial + \varepsilon < \delta$.

In order to carry the arguments of this section in the most general way we need to generalize theorem 2.4.a for the case in which the (technical) polyhedron

$$S^1 = \{ (x,y)/\ \begin{aligned} Hx + Gy &\geq b \\ Ey &\geq e \\ px + qy &= \gamma \} \end{aligned}$$

is unbounded. In this extension we need to use the standard linear programming result [41] that the polyhedron S^1 can be expressed as

$$S^1 = \{x,y)/\ (x,y) = \sum_{i=1}^{r} \lambda_i z_i + \sum_{j=1}^{s} \mu_j u_j, \quad \lambda_i \geq 0,\ u_j \geq 0,\ \sum_{i=1}^{r} \lambda_j = 1\}$$

where $z_i = (x_i,y_i)$ i=1, ...,r is the collection of all the extreme points of S^1 and $u_j = (v_j,w_j)$ j=1, ..., s is the collection of extreme rays of S^1.

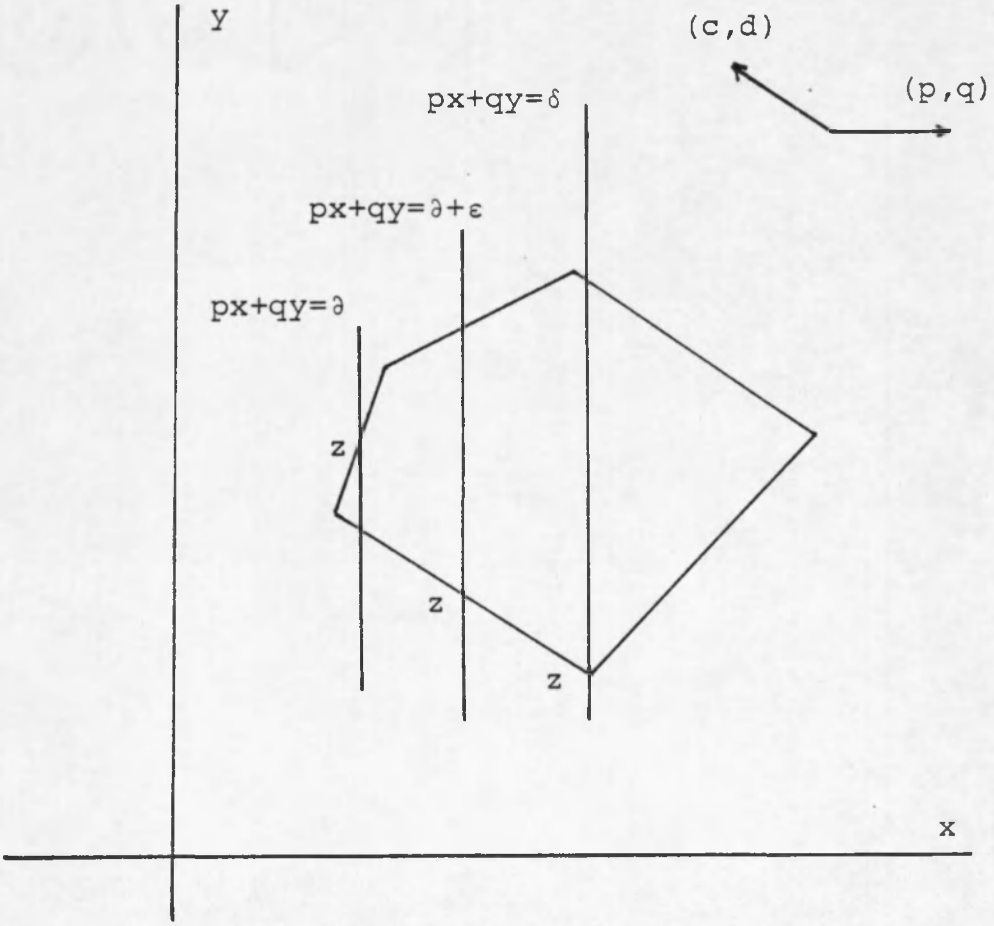

Figure 7. Interception
Using the Policy Objective Function

Theorem 5.a

If $z = \sum_{i=1}^{r} \bar{\lambda}_i z_i + \sum_{j=1}^{s} \bar{\mu}_j u_j$ belong to FS (where $\sum_{i=1}^{r} \bar{\lambda}_i = 1$

$\bar{\lambda}_i \geq 0$ and $\bar{\mu}_j \geq 0$) then $\bar{\lambda}_i > 0$ implies $z_i \in FS$.

Proof

Suppose $\bar{\lambda}_1 > 0$ and $z_1 = (x_1, y_1) \notin FS$, then solving the behavioral problem $BP(y_1)$ we get $\hat{x}_1$ such that $\hat{z}_1 = (\hat{x}_1, y_1) \in FS$, moreover $c\hat{x}_1 > cx_1$.

Next we shall see that

$$\hat{z} = \bar{\lambda}_1 \hat{z}_1 + \sum_{2}^{n} \bar{\lambda}_i z_i + \sum_{j=1}^{s} \bar{\mu}_j u_j \tag{5.a.1}$$

belongs to S^1. Since $\hat{z}_1$ belongs to S^1, it can be written as

$$\hat{z}_1 = \sum_{1}^{r} \lambda_i' \hat{z}_i + \sum_{1}^{s} \mu_j' u_j, \text{ with } \lambda_i' \geq 0 \; \mu_j' \geq 0 \text{ and } \sum_{1}^{r} \lambda_i' = 1 \tag{5.a.2}$$

Then replacing 5.a.2 in 5.a.1 we get

$$\hat{z} = \bar{\lambda}_1 \lambda_1' z_1 + \sum_{2}^{r} (\bar{\lambda}_i + \bar{\lambda}_1 \lambda_i') z_i + \sum_{1}^{s} (\bar{\mu}_j + \bar{\lambda}_1 \mu_j') u_j \tag{5.a.3}$$

where the coefficients of the z_i's add up to one, and the coefficients of the u_j's are non-negative. Therefore $\hat{z}$ belongs to S.

From the definitions of z and equation 5.a.1 above we see that z and $\hat{z}$ have the same y-component (policy setting) namely $y = \sum_1^r \bar{\lambda}_i y_i + \sum_1^s \bar{\mu}_j w_j$. However, it can be checked, using 5.a.1, that $(c, d)\,\hat{z} > (c,d)\,z$, which contradicts the fact that x should solve BP(y). Therefore we conclude that $z_1 \notin FS$ is not consistent with $\bar{\lambda}_1 > 0$, that in general $\bar{\lambda}_i > 0$ implies $z_i \in FS$ ||.

Now we are in a position to state and discuss the global optimality criterion.

Suppose we are at a high point $\bar{z} = (\bar{x},\bar{y})$ which is a local optimum with value say $p\bar{x} + q\bar{y} = \partial$, but there exists a high point $\bar{\bar{z}} = (\bar{\bar{x}},\bar{\bar{y}})$, in a BOF nonadjacent to $\bar{z}$, with value $p\bar{\bar{x}} + q\bar{\bar{y}} = \alpha > \partial$. Then, since the two-level feasible set is connected, there exists a <u>continuous</u> path $z(t) = (x(t), y(t))$, $0 \leq t \leq 1$ completely included in FS, such that $z(0) = \bar{z}$ and $\bar{\bar{z}}(1) = \bar{\bar{z}}$. Thus, if $\partial < \partial + \varepsilon < \alpha$, the value $\partial + \varepsilon$ is attained by the real valued continuous function $px(t) + qy(t)$ for some $0 < t_1 < 1$. The point $z(t_1) = (x(t_1), y(t_1))$ belongs to FS and satisfies

$$\begin{aligned} Hx + Gy &\geq b \\ Ey &\geq e \\ px + qy &= \partial + \varepsilon \end{aligned} \qquad (5.a.4)$$

If $z(t_1)$ is not an extreme point of the polyhedron 5.a.4, it can be expressed as

$$z(t_1) = \sum_{i=1}^{r} \lambda_i z_i + \sum_{j=1}^{s} \mu_j u_j$$

where z_i $i = 1, \ldots, r$ is the collection of all the extreme points of 5.a.4 and u_j is the collection of extreme rays to 5.a.4. By theorem 5.a.4, we conclude that at least one extreme point of the polyhedron 5.a.4 belongs to the two-level feasible set FS.

Consequently, an exhaustive search of the extreme points of polyhedron 5.a.4 will lead us to at least one point in FS[1] at the level $px + qy = \partial + \varepsilon$. If none of the extreme points for 5.a.4 are in the two level feasible set, then we can infer that there are no points in the two-level feasible domain FS with values equal to or higher than $\partial + \varepsilon$. Thus, high point $\bar{z}$ is an ε-global optimum. Theoretically, we could repeat the search process as ε converges to zero. If no points in FS are detected the high point $\bar{z}$ will be the global optimum with a value of $p\bar{x} + q\bar{y} = \partial$. In fact, we could enumerate all the alternative behavioral optimal basis in the polyhedron 5.a.4 with ε set equal to zero. If no alternative basis lead to a high point better than $\bar{z}$, then $\bar{z}$ is the global optimum. In practice it will suffice to choose ε sufficiently small.

6. Local Optimality

The procedure outlined in section 5 could be used to pass from a high point that is not a local optimum to

1. An extreme point (x,y) from 5.b is tested for membership in FS by solving BP(y).

another behavioral optimal face (adjacent or not). However, the procedure may require excessive computing resources, because it involves the enumeration of the vertices of a section of the technical feasible polyhedron. In this section we will develop a second intercepting procedure, less onerous than the first one, to pass from a high point that is not a local optimum to a point in an adjacent BOF. We will also show that if the procedure does not produce such a passage then the current high point is a local optimum.

The intercepting element in section 5 was the plane $px + qy = \partial + \varepsilon$. In this section the intercepting element will be a parallely displaced behavioral optimal face. Thus, we shall deal frequently with perturbations of the technical constraints of the lower level. To accommodate an appropriate algebraic presentation of the local optimum procedure, we will denote the rows of the matrix [H,G] by a^i, $i = 1, \ldots, m$. When necessary, we shall also use the partitions $a_i = (a^H{}_i, a^G{}_i)$, and $z = (x,y)$. To begin, we introduce the perturbed two-level problem.

$$
\begin{array}{lll}
T(J+\varepsilon): & \max\ px + qy & \\
 & (1)\ \ Ey \geqslant e & \\
 & (2)\ \ \max\ cx + dy & \\
 & \qquad a_i z \geqslant b_i + \varepsilon_i & i \in J \\
 & \qquad a_i z \geqslant b_i & i \in N
\end{array}
$$

where J is a subset from $\{1,2,\ldots,m\}$, and where vector ε has ε_i as its i-th component.

6.1 Perturbations of a Behavioral Optimal Face

In section 3 we saw that the points of the BOF(J):

$$
\begin{array}{lll}
3.a & a_i z = b_i & i \in J \\
3.b & a_i z \geq b_i & i \in N \\
3.c & E\,y \geq e &
\end{array}
$$

conform to saddle points with the same dual multiplier given by

$$
(2.a)' \quad \begin{array}{l} \lambda_N H_N + \lambda_J H_J = -c \\ \lambda_N \qquad\qquad\quad = 0 \end{array}
$$

Note that a change of the vector b does not affect equations (2.a)'. Therefore, if the equations below

$$
\begin{array}{lll}
6.1.a & a_i z = b_i + \varepsilon_i & i \in J \\
6.1.b & a_i z \geq b_i & i \in N \\
6.1.c & E\,y \geq e &
\end{array}
$$

have a non-empty solution set[1] BOF(J+ε), we can see that the points z in BOF(J+ε) verify the Lagrange conditions 2.a, 2.b, and 2.c with the same dual multiplier given by (2.a)' but with b changed to $b + \varepsilon$. Therefore, we conclude that BOF(J+ε) is a behavioral optimal face for T(J+ε).

---

1. A condition for BOF (J+ε) to be nonempty is to require BOF(J) to include a point z such that $a_i z > b_i$ for $i \in N$.

We can summarize the results above in the following proposition.

Proposition If BOF(J) is nontrivial (it has at least one interior point z, i.e. $a_i z > b_i$ for all $i \varepsilon N$) then BOF(J+ε) is a behavioral optimal face for T(J+ε). Furthermore, BOF(J) and BOF(J+ε) are associated with the same index set J, and also have the same adjacent faces.

6.2 Adjacent Behavioral Optimal Faces

Suppose the index set J determines a BOF for problem T, with a high point $\bar{z} = (\bar{x},\bar{y})$ which is not a local optimum. That is, suppose there exists an adjacent BOF (adjacent to $\bar{z}$) given by an index set K, with higher values than $p\bar{x} + q\bar{y}$.

The behavioral problem BP($\bar{y}$) is degenerate since basis J and K are alternative optimal bases which determine the same solution $\bar{z}$. The crucial problem here is to identify the index set K.

Let us consider the perturbed BOF

BOF $(J + \bar{\varepsilon}_1)$:

$$
\begin{array}{ll}
a_i\ z = b_i & i\ \varepsilon\ J \cap K \\
a_i\ z = bi + \varepsilon & i\ \varepsilon\ J - K \\
a_i\ z \geq bi & i\ \varepsilon\ N \\
E\ y \geq e &
\end{array}
$$

Note that the $\bar{\varepsilon}_1$ components are zero for $i \varepsilon J \cap K$ and ε for $i \varepsilon J-K$.

The perturbed BOF $(J+\bar{\varepsilon}_1)$ has the same adjacent face as BOF(J) for ε sufficiently small, but at least one point strictly inside BOF(K). The latter can be seen by noting that the points satisfying

$$
\begin{array}{ll}
a_i z = b_i & i \varepsilon K \\
a_i z = b_i + \varepsilon & i \varepsilon J-K \\
a_i z \geq b_i & i \varepsilon (J \cup K)^c \\
Ey \geq e &
\end{array}
$$

are included both in BOF$(J+\bar{\varepsilon}_1)$ and in the interior of BOF(K). Thus, the high point $\bar{\bar{z}} = (\bar{\bar{x}},\bar{\bar{y}})$ within BOF$(J+\bar{\varepsilon}_1)$ (for the problem $T(J+\bar{\varepsilon}_1)$) will be greater than $p\bar{x} + q\bar{y}$.

More importantly the problem

$$
\begin{array}{lll}
\text{6.2.a} \quad \max & cx + d\bar{\bar{y}} & \\
& a_i^H x + a_i^G \bar{\bar{y}} \geq b_i + \varepsilon & i \varepsilon J-K \\
& a_i^H x + a_i^G \bar{\bar{y}} \geq b_i & i \varepsilon (J-K)^c
\end{array}
$$

has alternative optimal bases J and K, the degenerate solution being $\bar{\bar{z}} = (\bar{\bar{x}},\bar{\bar{y}})$.

However, if we delete the ε's in problem 6.2.a, that is, if we consider

6.2.b $\max cx + d\bar{\bar{y}}$

$$a_i^H x + a_i^G \bar{\bar{y}} \geq b_i \qquad i = 1, \ldots, m$$

then J disqualifies as an optimal basis, since

$$a_i^H \bar{\bar{x}} + a_i^G \bar{\bar{y}} = b_i + \varepsilon > b_i \quad \text{for } i \in J\text{-}K$$ [1]

Nevertheless, we can check that K does not disqualify as an optimal basis. We should also note that $\bar{\bar{x}}$ solves 6.2.b, and therefore $\bar{\bar{z}}$ belongs to the two-level feasible set FS.

The above procedure in effect provides a way to pass from basis J to basis K. However, obtaining $\bar{\bar{y}}$ presupposes the knowledge of K itself. Nevertheless, we will show that the passage is possible even if we only know one element in J-K. The procedure and arguments are the same as before.

Let i_0 be an element in J-K. Then the high point $\bar{\bar{z}} = (\bar{\bar{x}}, \bar{\bar{y}})$ obtained within BOF(J+ $\bar{\varepsilon}_{i_o}$) (where $\bar{\varepsilon}_{i_o}$ is a vector of zero entries except for the i_0-th which is ε) should give a higher hight point than $p\bar{x} + q\bar{y}$, because as before BOF(J+ $\bar{\varepsilon}_{i_o}$) includes points interior to BOF(K). More importantly, the policy setting $\bar{\bar{y}}$ is associated with the BOF's BOF(J+ $\bar{\varepsilon}_{i_o}$) and BOF(K) for problem T(J+ $\bar{\varepsilon}_{i_o}$) but only with BOF(K) for problem T.

---

1. Note here that the Lagrange conditions are being used as necessary conditions for (behavioral) optimality.

Thus basis K can be identified by solving

$$BP\ (\bar{\bar{y}}) :\quad \max\ \ cx + d\,\bar{\bar{y}}$$

$$a_i\, z \geq bi \qquad\qquad i = 1, \ldots, m$$

As before (in 6.2.b) $\bar{\bar{x}}$ solves BP($\bar{\bar{y}}$), and therefore $\bar{\bar{z}}$ belongs to FS.

6.3 The passage from J to K

We have seen that if i_0 belongs to J-K then the high point within BOF(J+ $\bar{\varepsilon}_{i_o}$) is greater than the high point within BOF(J). Therefore, the optimal dual multiplier for $i_0 \varepsilon$ J-K in

$$PP(J):\quad \max\ \ p\,x + q\,y$$

$$a_i\, z = b_i \qquad\qquad i\ \varepsilon\ J$$

$$a_i\, z \geq b_i \qquad\qquad i\ \varepsilon\ N$$

has to be negative[1].

Thus, if we call L the set of indices in J with negative optimal multiplier in PP(J), we have

$$J\text{-}K \subset L \subsetneq J$$

Therefore, instead of searching for one element i_0 in J-K within J, we can reduce the search within L.

---

1. The rate of change of the optimal value of PP(J) with respect to b_i is the negative of the corresponding optimal dual multiplier.

Given an index i_0 in L, we will check for its membership in the set J-K in the following manner. First, we compute the high point $\bar{\bar{z}}$ within BOF(J+ $\bar{\varepsilon}_{i_0}$), then we solve the behavioral problem BP($\bar{\bar{y}}$) and see if $\bar{\bar{z}}$ belongs to the two-level feasible set FS.

If $\bar{\bar{z}}$ belongs to FS, we not only infer that i_0 belongs to J-K, but also identify the set K itself as the optimal basis of BP($\bar{\bar{y}}$). On the other hand, if all the optimal dual multipliers in J are non-negative; or if all the negative ones (those in L) produce a high point, that does not belong to FS, then it is clear that there does not exist any better adjacent BOF, and as such the current high point $\bar{z}$ is a local optimum.

CHAPTER 4

AN ALGORITHM OF INTERCEPTIONS

1. Overview

The algorithm presented here is based on results obtained in Sections 5 and 6 of Chapter 3. It consists of two procedures. The first one is designed to move along high points of adjacent BOF's, increasing the policy objective values at each iteration until a local optimum is attained. The intercepting element in this procedure is the current BOF parallely translated towards the interior of the technical polyhedron to permit the passage from a current high point to a point interior to an adjacent BOF, with higher values than the high point value of the current BOF. The passage is, of course only possible if a better adjacent BOF exists. Otherwise the algorithm will indicate that the current high point is a local optimum.

The second procedure is designed to pass from one high point to a point in a better[1] - adjacent or not - BOF. This procedure involves more computation than the previous one and therefore its use is only recommended to pass

---

1. BOF_2 is better than BOF_1 if BOF_2 includes points with higher values than the high point of BOF_1 (with respect to the policy objective).

from a local optimum to a better non-adjacent BOF. If a better BOF exists, it will find it, otherwise it will guarantee that no better BOF exists, and hence that the current local optimum is the global optimum. The intercepting element in the second procedure is a plane defined by setting the policy objective equal to a value slightly above the current level optimum.

2. Geometric Illustration

The algorithm is illustrated in Figure 8. The behavioral and policy variables are x and y, respectively, Arrow (p,q) gives the direction of increase for the policy objective, and (c,d) the direction of increase for the behavioral objective. Faces 1, 2, and 3 are the behavioral optimal faces, z is a local optimum and z the global optimum. Sections 2.1 and 2.2 below will refer to Figure 8.

2.1 Local Optimum Procedure.

S ppose we are currently in a point of face 2. The solution of the policy problem for this face produces the high point z = (x,y). If subsequently we solve the Behavioral Problem BP(y), we obtain a degenerate solution, since either face 2 or 3 will be a nonbasic binding restriction. If instead of solving the policy problem along face 2, we solve it along the element I - I', parallel to face 2 and interior to the technical polyhedron

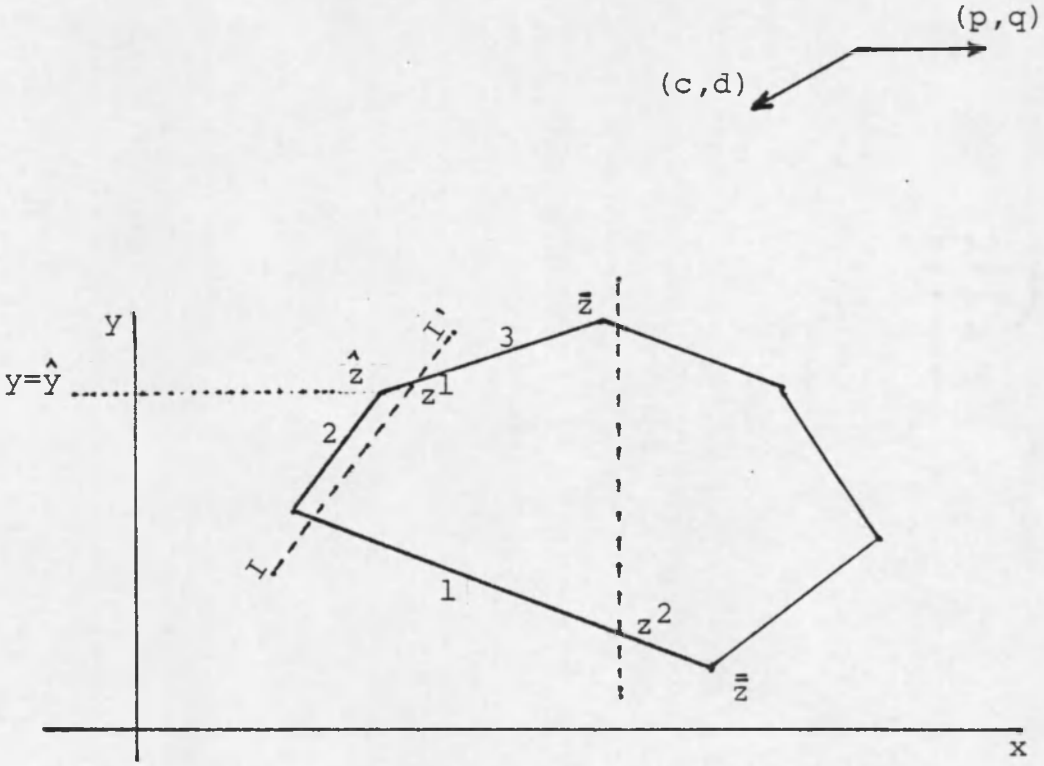

Figure 8. The Algorithm in Geometric Pictures

$$Hx + Gy \geq b, \quad Ey \geq e$$

that is if we solve

$$\max px + qy$$
$$a_i z = b_i + \varepsilon_i \qquad i \in J \qquad \text{where } J = \{2\}$$
$$a_i z \geq b_i \qquad i \in N$$
$$E\,y \geq e$$

then we obtain a new "high point" $z^1 = (x^1, y^1)$, which falls within face 3. If we now solve $BP(y^1)$, the alternative basis including face 2 disqualifies. The operation above allows us to pass from face 2 to face 3. This procedure can be repeated until a local optimum is attained. A rigorous discussion of the procedure is presented in Section 6 of Chapter 3.

2.2 Global Optimum Procedure

If we are at a high point $\bar{z}$ (Figure 8) which is a local optimum with value say $p\bar{x} + q\bar{y} = \gamma$, but with a better <u>nonadjacent</u> high point $\bar{\bar{z}}$ with value $p\bar{\bar{x}} + q\bar{\bar{y}} = \alpha$, ($\alpha > \gamma$), then as we show in Section 5 of Chapter 3, if $\gamma + \varepsilon$ is such that $\gamma < \gamma + \varepsilon < \alpha$ then at least one of the extreme points of

$$\text{2.2.a} \qquad Hx + Gy \geq b$$
$$Ey \geq e$$
$$px + qy = \gamma + \varepsilon$$

belongs to FS. In Figure 8 we see that z^2 is one such point. After detecting the extreme point z^2, the local optimum search can be restarted.

We also saw in Chapter 3 (Section 5) that if no extreme point of polyhedron 2.2.a belongs to FS then the current local optimum, ($\bar{\bar{z}}$ in Figure 8) is an ε-global optimum. It was also pointed out that we could enumerate all the alternative behavioral optimal basis in the polyhedron 2.2.a with ε set equal to zero. If no alternative basis leads to a high point better than $\bar{\bar{z}}$ then $\bar{\bar{z}}$ is the global optimum. A formal statement of the algorithm follows.

3. The Algorithm

1. Find a starting feasible policy setting $y = y^1$ by solving

$$\begin{aligned} \max\ & px + qy \\ & Hx + Gy \geq b \\ & Ey \geq e \end{aligned}$$

2. Solve the behavioral problem

$$BP(y^1):\ \max\ cx + dy^1$$
$$a^H{}_i x + a^G{}_i y^1 \geq b_i \qquad i = 1, \ldots, m$$

and record the index set J corresponding to the binding constraints at optimality.

3. Solve the policy problem

PP(J):

$$\max px + qy$$
$$a^H{}_i x + a^G{}_i y = b_i \qquad i \varepsilon J$$
$$a^H{}_i x + a^G{}_i y \geqslant b_i \qquad i \varepsilon N$$

and record the high point ∂ and the set L of indices in J with negative dual optimal multipliers in PP(J). Go to step 4.

4.0 If L is empty the current high point is a local optimum so go to step 5, otherwise go to 4.1.

4.1 Take $i_o \varepsilon$ L and solve

$$PP(J + \bar{\varepsilon}_{io}): \quad \max px + qy$$
$$a^H{}_{io} x + a^G{}_{io} y = b_{io} + \varepsilon$$
$$a^H{}_i x + a^G{}_i y = b_i \qquad i \varepsilon J - \{io\}$$
$$a^H{}_i x + a^G{}_i y \geqq b_i \qquad i \varepsilon N$$

and record the optimal solution $\bar{z} = (\bar{x},\bar{y})$ and go to 4.2

4.2 Solve $BP(\bar{y})$: $\max cx + d\bar{y}$

$$a^H{}_i x + a^G{}_i \bar{y} \geqslant b_i \qquad i = 1, \ldots, m$$

If $\bar{x}$ is a solution to $BP(\bar{y})$, $\bar{z}$ belongs to FS, so record new optimal basis J and go to step 3.

If $\bar{x}$ is not a solution to $BP(\bar{y})$, set $\bar{L}$ equal to $L - \{i_0\}$ and go to 4.0.

5. Enumerate the vertices of the polyhedron

$$I(\varepsilon): \quad \begin{aligned} a_i z &\geq b_i \\ Ey &\geq e \\ px + qy &= \gamma + \varepsilon \end{aligned}$$

using one of the algorithms designed for this purpose[1], with additional provision to terminate the enumeration as soon as a vertex $\bar{z} = (\bar{x},\bar{y})$ is found such that $\bar{x}$ solves $BP(\bar{y})$, i.e., a vertex $\bar{z}$ that belongs to FS. If such a vertex is found set the index set associated to the optimal basis equal to J and go to step 3.

If no vertex of $I(\varepsilon)$ belongs to FS then the current local optimum is an ε-global optimum. Go to step 6 only if improvements within ε are desired.

6. Enumerate the vertices of the polyhedron

$I(o)$:

$$\begin{aligned} a_i z &\geq b_i \\ Ey &\geq e \\ px + qy &= \gamma \end{aligned}$$

in the following manner:

---

1. In this work, we have used an algorithm and the corresponding computer program due to Matheiss [34].

6.0 Get a (new) vertex $\bar{z} = (\bar{x},\bar{y})$ of the polyhedron I(o). If $\bar{x}$ solves BP($\bar{y}$), $\bar{z}$ belongs to FS. Get the high point α of the basis detected in BP($\bar{y}$) and go to 6.1. If $\bar{x}$ does not solve BP($\bar{y}$) go to 6.0.

6.1 If $\alpha > \gamma$, set $\gamma = \alpha$, record the optimal basis as J and the set of indices of J with negative optimal dual multipliers in the policy problem PP(J) and L and go to 4.

If $\alpha = \gamma$ go to 6.0.

If no vertex of I(o) produce an alternative basis with a high point higher than the current ε-global optimum the current ε-global optimum is the global optimum.

4. A Numerical Example

In this section we present the detailed application of the algorithm of interceptions to the problem below:

$$\max\ x_1 + 40x_2 + 4x_3 + 8y_1 + 4y_2 \qquad \text{(UPOB)}$$

$$\text{Subject to:} \quad y_1 \geqq 0 \qquad \text{(P1)}$$

$$y_2 \geqq 0 \qquad \text{(P2)}$$

$$\max\ - x_1 - x_2 - 2x_3 - y_1 - 2y_2 \qquad \text{(LWOB)}$$

$$x_1 - x_2 - x_3 \geqq -1 \qquad \text{(R1)}$$

$$x_1 - 2x_2 + .5x_3 - 2y_1 \geqq -1 \qquad \text{(R2)}$$

$$-2x_1 + x_2 + .5x_3 - 2y_2 \geqq -1 \qquad \text{(R3)}$$

$$x_1 \geqq 0 \qquad \text{(R4)}$$

$$x_2 \geqq 0 \qquad \text{(R5)}$$

$$x_3 \geqq 0 \qquad \text{(R6)}$$

Constraints P1 and P2 are the technical constraints on the policy variables and constraints R1 through R6 the technical constraints of the lower level problem. The upper level objective is denoted UPOB and the objective of the lower level is denoted LWOB.

The algorithm proceeds as follows:

Step 1. The first step of the algorithm indicates to solve the problem.

max UPOB

Subject to the constraints

R1, R2, R3, R4, R5, R6, P1, P2

The solution values for y_1 and y_2 constitute the initial feasible policy setting. These values are $y_1 = 0$ $y_2 = 0$. The values for the other variables and the objective functions are:

UPOB = 65.5 LWOB = -5.00

$x_1 = 1.5$

$x_2 = 1.5$

$x_3 = 1.0$

Step 2. Using the policy setting obtained in Step 1, we solve the behavioral problem:

BP(y): max LWOB

Subject to the constraints

R_1, R_2, R_3, R_4, R_5, R_6

and $y_1 = 0$, $y_2 = 0$

The solution provides a feasible point for the two-level problem, and the behavioral optimal face BOF(J) that includes it. The solution and the corresponding objective functions values are:

UPOB	= 0	LWOB	= 0
x_1	= 0	y_1	= 0
x_2	= 0	y_2	= 0
x_3	= 0		

and the index set J is

$$J = \{ R4, R5, R6 \}$$

Step 3. Using the index set J obtained in the step above we construct the policy problem:

max UPOB

Subject to the constraints

(i) R4, R5, and R6 holding as equalities

(ii) R1, R2, R3, P1, P2

Its solution produces the high point within the BOF(J) and information of potential directions of local improvement.

The solution is:

$$\mathrm{UPOB} = 6.0 \qquad \mathrm{LWOB} = -1.5$$

$$x_1 = 0 \qquad y_1 = 0.5$$

$$x_2 = 0 \qquad y_2 = 0.5$$

$$x_3 = 0$$

$$L = \{ R4, R5, R6 \}$$

The set L indicates the constraints from J that have a negative optimal dual multiplier. They indicate directions that may lead to a better behavioral optimal face. The values for the multipliers in L are -1, -34, and -7 respectively.

Step 4.0 The set L is non-empty so we go to Step 4.1

Step 4.1 We take the first element from L, namely R4 and construct the perturbed policy problem

max UPOB

Subject to the constraints

(i) R4, R5, R6 holding as equalities

(ii) R1, R2, R3, P1, P2

where R4 has been perturbed by adding $\varepsilon = 0.1$ to its right hand side. The solution of this problem is:

$$
\begin{array}{ll}
UPOB = 6.1 & LWOB = -1.45 \\
x_1 = 0.1 & y_1 = 0.55 \\
x_2 = 0 & y_2 = 0.40 \\
x_3 = 0 &
\end{array}
$$

Step 4.2. Using the policy setting obtained in the step above, we set up the behavioral problem:

BP(y): max LWOB

subject to

R1, R2, R3, R4, R5, R6

and $y_1 = 0.55$ $y_2 = 0.40$

Solving this problem we obtain

$$
\begin{array}{ll}
UPOB = 6.1 & LWOB = -1.45 \\
x_1 = 0.1 & y_1 = 0.55 \\
x_2 = 0 & y_2 = 0.40 \\
x_3 = 0 &
\end{array}
$$

which is exactly equal to the solution obtained in Step 4.1. Therefore it belongs to the feasible domain of the two-level problem. Thus we record the index set of bindings constraints at the optimal solution of BP(y):

$$J = \{R2, R5, R6\}$$

and perform Step 3 again.

Step 3. Using the index set J obtained in the step above we set up the policy problem:

max UPOB

Subject to the constraints

(i) R2, R5 and R6 holding as equalities

(ii) R1, R3, R4, P1, P2

Its solution gives

UPOB = 6.5 LWOB = -1.25

$x_1 = 0.5$ $y_1 = 0.75$

$x_2 = 0$ $y_2 = 0$

$x_3 = 0$

and the directions of potential local improvement are given by L = { R5, R6 }. The corresponding values for the dual multipliers for the constraints in L are -34.5, -7.25.

Step 4.0. The index set L is non-empty, thus we go to Step 4.1.

Step 4.1. Take the first element of L, R5, and set up the corresponding perturbed policy problem:

max UPOB

Subject to the constraints

(i) R2, R5, R6 holding as equalities

(ii) R1, R3, R4, P1, P2

where R5 has been perturbed by adding $\varepsilon = 0.01$ to its right hand side. The solution gives

$$UPOB = 6.845 \qquad LWOB = -1.257$$
$$x_1 = 0.505 \qquad y_1 = 0.742$$
$$x_2 = 0.010 \qquad y_2 = 0$$
$$x_3 = 0.0$$

Step 4.2. The behavioral problem corresponding to the policy setting obtained in the step above has the following optimal solution

$$UPOB = 6.42 \qquad LWOB = -1.226$$
$$x_1 = 0.484 \qquad y_1 = 0.742$$
$$x_2 = 0 \qquad y_2 = 0$$
$$x_3 = 0$$

which is not identical to the solution in Step 4.1. Thus we set $L = L - \{R5\}$, that is $L =$ R6 and go to Step 4.0.

Step 4.0. The index set L is non-empty, proceed to Step 4.1.

Step 4.1. We set up the perturbed policy problem:

max UPOB

Subject to the constraints

(i) R2, R5, and R6 holding as equalities

(ii) R1, R3, R4, P1, P2

where constraint R6 has been perturbed by the addition of $\varepsilon = 0.07$ to its right hand side. The optimal solution gives:

$$\begin{array}{ll} \text{UPOB} = 7.008 & \text{LWOB} = -1.434 \\ x_1 = 0.518 & y_1 = 0.776 \\ x_2 = 0 & y_2 = 0 \\ x_3 = 0.07 & \end{array}$$

Step 4.2. The behavioral problem corresponding to $y_1 = 0.776$, $y_2 = 0$, gives the following solution:

$$\begin{array}{ll} \text{UPOB} = 7.003 & \text{LWOB} = -1.432 \\ x_1 = 0.517 & y_1 = 0.776 \\ x_2 = 0 & y_2 = 0 \\ x_3 = 0.069 & \end{array}$$

which is identical to the solution obtained in Step 4.1, therefore it belongs to the feasible domain of the two-level problem, and it is above the high point 6.5. Thus, we record the index of binding constraints

$$J = \{ R2, R3, R5 \}$$

and go to Step 3.

Step 3. Using the index set J obtained in the previous step, we set up the policy problem:

max UPOB

Subject to the constraints

(i) R2, R3 and R5 holding as equalities

(ii) R1, R4, R6, P1, P2

The solution is:

UPOB = 21.0 LWOB = -6.0

$x_1 = 1.0$ $y_1 = 1.5$

$x_2 = 0$ $y_2 = 0$

$x_3 = 2$

$L = \{R5\}$

The optimal dual multiplier for R5 is -29.669.

Step 4.0. L is non-empty, thus we go to Step 4.1.

Step 4.1. We set up the perturbed policy problem:

max UPOB

Subject to the constraints

(i) R2, R3, and R5 holding as equalities

(ii) R1, R4, R6, P1, P2

with constraint R5 modified by adding $\varepsilon = 0.05$. The solution gives:

UPOB = 22.483 LWOB = -6.45

$x_1 = 1.017$ $y_1 = 1.45$

$x_2 = 0.05$ $y_2 = 0$

$x_3 = 1.967$

Step 4.2. We solve the behavioral problem:

max LWOB

Subject to the constraints

R1, R2, R3, R4, R5, R6, P1, P2

and $y_1 = 1.45 \quad y_2 = 0$

The solution is

$$\begin{array}{ll} UPOB = 20.03 & LWOB = -6.15 \\ x_1 = 0.967 & y_1 = 1.45 \\ x_2 = 0 & y_2 = 0 \\ x_3 = 1.867 & \end{array}$$

which is different from the solution of Step 4.1, and below the high point 21. Thus, we make L = L − { R5 }, which then results empty. Therefore, the high point 21 is a <u>local optimum</u>. We go to Step 5.

Step 5. This step calls for the enumeration of the vertices of the polyhedron formed by the upper level and lower level constraints (R1, R2, R3, R4, R5, R6, P1, P2) and the intercepting hyperplane UPOB = 21 + ε. Theoretically, the enumeration should be stopped as soon as the behavioral optimality condition is satisfied by one vertex of the polyhedron. However, this would entail modifying existing vertex enumeration computer codes. In this exercise we use a computer code with no modifications, and thus we obtain all the vertices of the intercepting polyhedron.

Vertices of the Intercepting Polyhedron
at the Level UPOB = 22

Number	x_1	x_2	x_3	y_1	y_2
1	.0952	.5476	0	0	0
2	0	.5	0	0	.5
3	0	.4705	0	.0294	.7352
4	.7246	.4492	0	.4130	0
5	0	.5357	.1428	0	0
6	0	.4615	.5384	.1730	0
7	1.0112	.0337	1.9775	1.4662	0
8	0	.3333	.6666	.3333	.8333

The third vertex turns out to be behavioral optimal as shown by the solution of the following behavioral problem

max LWOB

Subject to the constraints

R1, R2, R3, R4, R5, R6

and $y_1 = 0.0294 \quad y_2 = .7352$

UPOB = 22 LWOB = -1.971

$x_1 = 0 \qquad y_1 = 0.029$

$x_2 = 0.471 \qquad y_2 = 0.735$

$x_3 = 0$

Observe that vertex 3 is the solution to the problem above. Thus, we record the set of binding constraints

$$J = \{ R3, R4, R6 \}$$

and go to Step 3.

Step 3. Solve the policy problem:

max UPOB

Subject to the constraints

(i) R3, R4, R6 holding as equalities

(ii) R1, R2, R5, P1, P2

the solution of which is

$$\begin{array}{ll} \text{UPOB} = 23 & \text{LWOB} = -2 \\ x_1 = 0 & y_1 = 0 \\ x_2 = 0.5 & y_2 = 0.75 \\ x_3 = 0 & \end{array}$$

The index set L is

$$L = \{ R4, R6 \}$$

and the corresponding dual multipliers are equal to -18.0, and -15.5 respectively.

Step 4.0. L is non-empty, so we go to Step 4.1.

Step 4.1. Solve the perturbed policy problem:

max UPOB

Subject to the constraints

(i) R3, R4, R6 holding as equalities

(ii) R1, R2, R5, P1, P2

with constraint R6 perturbed by the addition of $\varepsilon = 0.1$ to its right hand side. The solution is:

$$UPOB = 24.55 \qquad LWOB = -2.3$$
$$x_1 = 0 \qquad y_1 = 0$$
$$x_2 = 0.525 \qquad y_2 = 0.788$$
$$x_3 = 0.1$$

Step 4.2. The solution of the behavioral problem:

max LWOB

Subject to the constraints

R1, R2, R3, R4, R5, R6

and $y_1 = 0$, $y_2 = 0.788$

is identical to the solution of Step 4.1 except for minor numerical deviations:

$$UPOB = 24.571 \qquad LWOB = -2.304$$
$$x_1 = 0 \qquad y_1 = 0$$
$$x_2 = 0.525 \qquad y_2 = 0.788$$
$$x_3 = 0.101$$

Thus, we record the binding constraints in J

$$J = \{ R2, R3, R4 \}$$

and go to Step 3.

Step 3. We solve:

max UPOB

Subject to the constraints

(i) R2, R3, R4 holding as equalities

(ii) R1, R5, R6, P1, P2

The solution gives

UPOB = 29.2 LWOB = -3.2

$x_1 = 0$ $y_1 = 0$

$x_2 = 0.6$ $y_2 = 0.9$

$x_3 = 0.4$

The index set L is

$$L = \{ R4 \}$$

and the corresponding dual multiplier is -24.2.

Step 4.0 The set L is non-empty, thus we proceed to Step 4.1

Step 4.1. We set up the perturbed policy problem

max UPOB

Subject to the constraints

(i) R2, R3, R4 holding as equalities

(ii) R1, R5, R6, P1, P2

where constraint R4 has been perturbed by the addition of $\varepsilon=$ 0.02 to its right hand side. The solution is:

$$\begin{array}{ll} UPOB = 29.684 & LWOB = -3.224 \\ x_1 = 0.020 & y_1 = 0 \\ x_2 = 0.612 & y_2 = 0.888 \\ x_3 = 0.408 & \end{array}$$

Step 4.2. A behavioral problem is solved for the policy setting obtained above. The solution,

$$\begin{array}{ll} UPOB = 28.704 & LWOB = -3.104 \\ x_1 = 0 & y_1 = 0 \\ x_2 = 0.592 & y_2 = 0.888 \\ x_3 = 0.368 & \end{array}$$

is different from the solution obtained in Step 4.1, we make $L = L - \{R4\}$, what leaves L empty and therefore the high point 29.2 is a <u>local optimum</u>.

Step 5. We enumerate the intercepting polyhedron at the level $UPOB = 29.2 + \varepsilon$.

Vertices of the Intercepting Polyhedron

at the Level UPOB = 30

Number	x_1	x_2	x_3	y_1	y_2
1	.4761	.7380	0	0	0
2	.3888	.6944	0	0	.4583
3	.8405	.6811	0	.2391	0
4	.1654	.6992	.4661	0	0
5	.0330	.6198	.4132	0	.8801
6	1.1011	.3033	1.7977	1.1966	0

None of the vertices of the intercepting polyhedron satisfy the behavioral optimality condition, as can be shown by solving a behavioral problem for each of the policy settings in the vertices above. Thus, the high point 29.2 is the global optimum.

5. Comparison to Other Algorithms

An objective and definitive assessment of the algorithms studied in this dissertation would require a team effort that would provide the implementation of computer codes for each of the proposed algorithms, the criteria of comparison, and a wide sample of test problems.

In the present comparison we will use the numerical examples found in the literature surveyed in Chapter 2 and two problems of the chapter of applications. These test problems appear in the appendix, four of them in an explicit algebraic formulation and the other two (medium and large scale problems) in the standard MPS format used by most commercial computer codes, e.g. APEX III[1]. The sources and dimensions of the test problems are given in Table 1. The solutions used in this evaluation are those reported in the literature, except for the solutions given by the Special Ordered Sets method (SOS), which were obtained using the commercial computer code APEX III. Table 2 displays information on the availability of solutions for each problem and each methodology. As for the criteria of comparison we chose the following:

- Accuracy
- Speed of Convergence
- Data Storage Requirement

5.1 Accuracy

Each one of the algorithms, except Karwan's local optimum algorithm generate a finite sequence of subproblems the solutions of which theoretically converge to the global optimum. Every methodology consistently gave the same solution for each one of the test problems, except for problems 3 and 4 in which the separable programming method gave different (inferior) solutions than the solutions obtained using the SOS method and the mehod of interceptions. Table

1. APEX III is a commercial computer code produced and supported by the Control Data Corporation.

TABLE 1

TEST PROBLEMS, THEIR SOURCES AND DIMENSIONS

PROBLEM NUMBER	SOURCE	NUMBER OF POLICY VARIABLES	NUMBER OF BEHAVIORAL VARIABLES	NUMBER[1] OF POLICY CONSTRAINTS	NUMBER[2] OF BEHAVIORAL CONSTRAINTS
1	Candler [8]	2	3	2	6
2	Falk [22]	2	1	3	4
3	Bard and Falk [1] Example 2	2	2	3	4
4	Bard and Falk [1] Example 4	2	2	5	5
5	Banking Problem Chapter 5	3	20	6	34
6	Agricultural Policy Problem Chapter 5	2	47	4	86

[1] Include non-negative constraints on the policy variables

[2] Include non-negative constraints on the behavioral variables

TABLE 2

AVAILABLE SOLUTIONS FOR THE TEST PROBLEMS

PROBLEM NUMBER	ADAPTED MAX-MIN	T-SET	SEPARABLE PROGRAMMING	SOS	INTERCEPTIONS
1	NA	A	A	A	A
2	A	NA	A	A	A
3	NA	NA	A	A	A
4	NA	NA	A	A	A
5	NA	NA	NA	A	A
6	NA	A	NA	A	A

LEGEND A = Available
NA = Not Available

3 displays all the solutions obtained for each problem. The solutions are given in terms of the policy variables and the value of the upper level objective. The coincidence of all the solutions for the same problem validates the accuracy of each methodology; in particular the accuracy of the methodology of interceptions.

5.2 Speed of Convergence

Since integrated computer codes without manual interface either do not exist or are not available for all of the algorithms, we estimate the speed of convergence by the number of subproblems required to obtain the global optimum. This information is shown in Table 4. However, an important remark should be made here. Since the separable programming and SOS methodologies use the Kuhn-Tucker formulation of the two-level problem which include one additional constraint for each behavioral variable and one dual variable for each primal constraint, these methodologies have much larger subproblems. Consequently, it should take substantially more time to solve a subproblem for the SOS and separable programming methodologies than a subproblem for the other algorithms.

5.3 Data storage requirements.

A good estimate of the data storage requirements for a given algorithm is the size of the subproblems to be

TABLE 3

SOLUTIONS FOR THE TEST PROBLEMS

PROBLEM NUMBER	ADAPTED MAX-MIN	T-SET	SEPARABLE PROGRAMMING	SOS	INTERCEPTIONS
1	NA	UPOB = 29.2 $y_1 = 0$ $y_2 = 0.9$	UPOB = 29.2 $y_1 = 0$ $y_2 = 0.9$	UPOB = 29.2 $y_1 = 0$ $y_2 = 0.9$	UPOB = 29.392 $y_1 = 0$ $y_2 = 0.894$
2	UPOB = -7 $y_1 = 1$ $y_2 = 1$	UPOB = -7 $y_1 = 1$ $y_2 = 1$	UPOB = -7 $y_1 = 1$ $y_2 = 1$	UPOB = -7 $y_1 = 1$ $y_2 = 1$	UPOB = -7 $y_1 = 1$ $y_2 = 1$
3	NA	NA	UPOB = 1.75 $y_1 = 1$ $y_1 = 0$	UPOB = 3.25 $y_1 = 2$ $y_2 = 0$	UPOB = 3.25 $y_1 = 2$ $y_2 = 0$
4	NA	NA	UPOB = -2.55 $y_1 = 5$ $y_2 = 2$	UPOB = 0 $y_1 = 0$ $y_2 = 0.5$	UPOB = 0 $y_1 = 0$ $y_2 = 0.5$
5	NA	NA	NA	UPOB = 21.72 $G_1 = 0.6$ $G_2 = 0.03$ R4 = 3	UPOB = 21.72 $G_1 = 0.6$ $G_2 = 0.3$ R4 = 3
6	NA	TLAB = 17857.804 KCO = 0 KWAT = 1640.174	NA	Hit Time Limit with no feasible solution	TLAB = 17857.804 KCO = 0 KWAT = 1640.174

TABLE 4

NUMBER OF SUBPROBLEMS SOLVED TO ATTAIN GLOBAL OPTIMALITY

PROBLEM NUMBER	ADAPTED MAX-MIN	T-SET	SEPARABLE PROGRAMMING	SOS	INTERCEPTIONS
1		17	103	5	14
2	50	8	1	4	14
3			15	2	13
4			45	5	11
5				139	31 (3)
6		20 (1)		2000 (2)	24 (3)

1 Advanced Solution

2 Hit time limit with no feasible solution

3 Local Optimum guaranteed

TABLE 5

SUBPROBLEM DIMENSIONS BY PROBLEM AND METHODOLOGY

PROBLEM NUMBER	ADAPTED MAX-MIN	T-SET	SEPARABLE PROGRAMMING	SOS	INTERCEPTIONS
1	n = 8 m = 8	n = 8 m = 8	n = 14 m = 17	n = 14 m = 17	n = 5 m = 8
2	n = 6 m = 7	n = 6 m = 7	n = 10 m = 14	n = 10 m = 14	n = 3 m = 7
3	n = 6 m = 7	n = 6 m = 7	n = 10 m = 13	n = 10 m = 13	n = 4 m = 7
4	n = 7 m = 10	n = 7 m = 10	n = 12 m = 18	n = 12 m = 18	n = 4 m = 10
5	n = 34 m = 40	n = 34 m = 40	n = 68 m = 91	n = 68 m = 91	n = 23 m = 40
6	n = 91 m = 90	n = 91 m = 90	n = 177 m = 226	n = 177 m = 226	n = 49 m = 90

solved. This information is shown in Table 5. Additional data requirements arise from the bookeeping procedure involved in each method. An assessment of the latter requirements is not attempted here because it would involve the comparison of specialized branch and bound methods (separable programming, SOS, Max-Min) with vertex-polyhedron enumerations (algorithm of interceptions) with the special enumeration procedure of the T-set algorithm. The most effective way to compare them would be experimentally and for that we should have the algorithms implemented as integrated computer codes.

CHAPTER 5

APPLICATIONS

In this chapter two existing linear programming models are recast as two-level hierarchical models. In each case, a government concern is introduced in the upper level, and the chosen linear programming model is used as a behavioral model of the corresponding lower structure. The resulting hierarchical models are then solved using the methodology of interceptions. The numerical results should be interpreted bearing in mind the limitations of the behavioral models and the hypothetical policy manipulation.

1. A Two-Level Banking Model.

1.1 Linear Programming in Bank Portfolio Optimization.

Starting with Chambers and Charnes [10], a number of linear programming models have been developed to assist and describe the management of bank portfolios. To a certain extent the problem is one of multiple objectives, as bank portfolio managers are concerned not only with the returns on the portfolio but also with its liquidity and risk. However, if government regulations are effective in maintaining a minimum of risk and a minimum of illiquidity, it is reasonable to assume that bank portfolio managers will behave as exclusive profit maximizers. In fact, Linear Programming seems a natural formulation of the problem, if we observe that the Federal Reserve directives to impose acceptable limits on illiquidity and risk, such as the capital adequacy and risk ratio rules, are expressed as linear constraints on the assets banks hold. Thus, it is just a matter of expressing the return objective of the bank plus other relevant constraints in a linear algebraic form to obtain a linear programming formulation. Cohen and Hammer [13] experimented with three objective functions:

(1) The value of stockholder equity at the end of the final period.

(2) The present value of the net income stream over the planning period, and

(3) The sum of the above two objectives.

Although there is no conclusive judgment about the value of the model, Cohen and Hammer ([14] p. 413) report that at least one bank has used their model for more than five years, but they provide no information on the increase in profitability achieved or any other measure of the value banks attach to it. Fried [24] experimented with a modified expected return - standard deviation model, similar in principle to the linear programming models. His model involves maximizing expected return subject to probabilistic constraints on acceptable levels of risk and illiquidity.

He concludes that management policies of banks do not conform to the model and that they choose their portfolios inefficiently. An indication of banks' management aversion to analytical models is suggested by the studies on the adjustment process to deposit inflows. Hester and Pierce [27] conclude that the length of time for the adjustment process to be completed varied from eight weeks for cash to eight months for mortgages. Similar results have been substantiated by Melnik [36] and Russell [38].

As for the descriptive value of the model, Beazer [3] conducted a study involving fourteen banks from Chicago over a time span that varied from bank to bank but which extended to several quarterly periods. He used a linear programming model to generate representative quarterly portfolios for a period of 6 years and collected corresponding actual data. A statistical comparison of the 310 generated portfolios and the corresponding actual counterparts gives encouraging results. A regression study carried out on aggregate segments of the portfolios suggest that the model has the power to predict the loan portfolio segment, but it has no power to predict other important segments of the portfolio. However, he offers plausible explanations for the limitations of his model. First, he points out the bank-depositor relationship factor. Banks do not always prefer loan assets with higher rates of return to assets that serve their best depositors. A constraint reflecting this factor could be included, once the relevant data were available. Second, his study presents evidence and arguments to show that the regression might improve dramatically if actual portfolios would adjust to the new conditions with no time lags at each quarterly observation.

Very important evidence about the model is also given by a statistical regression run between shadow prices of the required reserve constraint and the federal funds interest rate. The closeness of the fit suggests that the regulatory restrictions do tend to determine the degree of risk the bank assumes ([3] p. 91).

1.2 The Banking Two-Level Model

In this section Beazer's linear programming model [3] will be recast as a linear two-level problem. Since the problem is one related to national monetary policy, in an actual application it would be necessary to select the type of bank the policy is designed for, and to test how it will affect other banks.

The Upper Level

In the upper level, the government strives to impact determined segments of the bank asset portfolio. For concreteness we shall suppose that the government objective is to affect the loan portfolio segment, to minimize it or to attain minimum deviations from a target level. This choice is also made because the lower level represented by Beazer's model can predict the loan portfolio best among all segments of the portfolio.

The government influence is effective through the manipulation of a collection of policy variables chosen from the following:

R_1 : The capital adequacy ratio.

G_1 : The demand-deposit requirement.

G_2 : The saving-deposit requirement.

R_2 : The real estate loans to time deposit ratio.

R_3 : The pledge assets to government holdings ratio.

R_4 : The risk asset ratio.

Not every combination of values for the policy variables is conducive to a feasible policy scenario. The domain of acceptable policy scenarios is given by the upper level constraints. In this example, we will experiment with two alternative sets of upper level constraints, each corresponding to different combinations of policy variables.

The first alternative set is given by

$$.1 \leq G_1 \leq .6 \qquad \text{(POL 1 and POL 2)}$$ [1]

$$.03 \leq G_2 \leq .3 \qquad \text{(POL 3 and POL 4)}$$

$$.5 \leq R_1 \leq .8 \qquad \text{(POL 5 and POL 6)}$$

The second set is given by

$$.1 \leq G_1 \leq .6 \qquad \text{(POL 1 and POL 2)}$$

$$.03 \leq G_2 \leq .3 \qquad \text{(POL 3 and POL 4)}$$

$$3.0 \leq R_4 \leq 12.0 \qquad \text{(POL 5 and POL 6)}$$

Given a new government policy scenario (in terms of specific values of the policy variables) the bank will adjust the levels of the asset holdings in its portfolio. It is assumed that the adjustment occurs according to Beazer's linear programming model, which we briefly describe next. Note the parametric dependence of some of the constraints on the policy variables.

The Lower Level

The variables of the lower level are the levels of the asset holdings expressed as a fraction of the total asset portfolio.[2] A list of the lower level variables and corresponding rates of return are listed in Table 6. Note

---

1. The equations of the model are referenced in the computer program by names. These names appear in parentheses to the side of each equation throughout this chapter. A complete list of the equations of the model appear in Fig. 9.

2. The total asset portfolio is equal to 100.

Table 6

Assets According To Liquidity Class

Liquidity Class	Variable	Asset	Rate of Return
C	X_1	Cash in process of collection	$c_1 = 0$
	X_2	Currency and coin	$c_2 = 0$
	X_3	Reserves with Federal Reserve	$c_3 = 0$
	X_4	Balances with banks	$c_4 = 0$
A_1	X_5	Loans to banks (fed. funds)	$c_5 > 0$
	X_6	Bills, certificates, govts. under one year	$c_6 > 0$
	X_7	Loans to brokers	$c_7 > 0$
A_2	X_8	Governments, 1-5 years	$c_8 > 0$
	X_9	Loans to finance companies	$c_9 > 0$
A_3	X_{10}	Governments 5-10 years	$c_{10} > 0$
	X_{11}	Loans for purchase of securities	$c_{11} > 0$
A_4	X_{12}	Governments over 10 years	$c_{12} > 0$
A_5	X_{13}	Real estate loans	$c_{13} > 0$
	X_{14}	Municipals and other securities	$c_{14} > 0$
	X_{15}	Agricultural, commercial, and individual loans	$c_{15} > 0$
	X_{16}	Consumer loans	$c_{16} > 0$
A_6	X_{17}	Other assets	$c_{17} > 0$
L_1	X_{18}	Capital adequacy vectors	$c_{18} = 0$
L_2	X_{19}	Capital adequacy vectors	$c_{19} = 0$
L_3	X_{20}	Capital adequacy vectors	$c_{20} = 0$

that the variables (assets) are grouped in categories of decreasing liquidity:

Table 7

Categories of the Variables in Table 6

C	Cash assets
A_1	Very short term assets
A_2	Government bonds 1-5 or equivalent
A_3	Minimum risk-assets
A_4	Intermediate assets
A_5	Portfolio assets

Variables x_{18}, x_{19}, x_{20} do not correspond to asset holdings, they are just auxiliary variables to express the capital adequacy constraint.

The lower level objective function will be to maximize the average rate of return of the portfolio, i.e.

$$\max \sum_{j=1}^{20} c_j X_j \qquad \text{(LWOB)}$$

The Constraints

The Capital Adequacy Ratio.

This constraint establishes a limit for the illiquidity of the bank asset portfolio and the bank liabilities as a function of the capital of the bank. It is, in effect, a tradeoff formula by means of which the Federal Reserve prescribes safe combinations of illiquidity and liabilities. The bank's actual capital K must be as large as the required capital Q, although in practice the requirement reduces to a capital adequacy ratio

$$Q/K \leq R_1$$

The range of the capital adequacy ratio R_1, that is considered acceptable by federal examiners is a rather broad one. But once a ratio is agreed upon, examiners pay very close attention to any sudden declines in the level. However, we consider R_1 a policy variable in one of our examples.

The required capital Q is broken down in two segments

$$Q = Q_1 + Q_2$$

The first segment Q_1 is a function of the assets (Table 6) plus the fixed amount, $40,000:[1]

$$Q_1 = 0.005A_1 + 0.04A_2 + 0.04A_3 + 0.06A_4 + 0.10A_5 + \$40,000.$$

(where A_1 stands for $X_5 + X_6 + X_7$ etc.)

---

1. The actual requirement is 15% of the first $100,000 of the portfolio + 10% of the next 100,000 + 5% of the next $300,000. In the example above we are assuming a bank with a portfolio larger than $500,000. The amount of $40,000 should be translated to an equivalent fraction, keeping in mind that $K + D_d + D_s = 100$.

The second segment Q_2 is a function of the liquidity needed D_L, the savings deposit D_S, and the demand deposits D_d.

$$
\begin{aligned}
Q_2 &= 0 \quad \text{if } D_L \leq M_1 \\
&= .065\ (D_L - M_1) \text{ if } M_1 < D_L \leq M_2 \\
&= .065\ (D_L - M_1) + .040(D_L - M_2) \text{ if } M_2 < D_L \leq M_3 \\
&= .065\ (D_L - M_1) + .040(D_L - M_2) + .095(D_L - M_3) \text{ if } M_3 < D_L
\end{aligned}
$$

Where:

D_L = (liquidity needed) = 0.47 D_d + 0.36 D_s + (deposits of banks and government) + (borrowings)

$M_1 = C + 0.995\ A_1 + 0.96\ A_2$

$M_2 = M_1 + 0.90\ A_3$

$M_3 = M_2 + 0.85\ A_4$

In the linear programming context the capital adequacy ratio constraint $Q_1 + Q_2 \leqq R_1K$ can be implemented by the following relations.

$$0.005A_1 + 0.04A_2 + 0.04A_3 + 0.06A_4 + 0.10A_5 + 0.065L_1 + 0.040L_2 + 0.095L_3 \leq R_1 K \quad \text{(CAR)}$$

$$C + .995A_1 + .96A_2 + L_1 \geq D_L \quad \text{(CAR 1)}$$

$$C + .995A_1 + .96A_2 + .9A_3 + L_2 \geq D_L \quad \text{(CAR 2)}$$

$$C + .995A_1 + .96A_2 + .9A_3 + .85A_4 + L_3 \geq D_L \quad \text{(CAR 3)}$$

Required Federal Reserves

Required reserves with the Federal Reserve System should be equal to a certain percent of demand deposits plus a certain percent of time deposits. Thus,

$$X_3 \geq G_1 \quad D_d + G_2 \; D_s$$

The required percent G_1 and G_2 will be policy variables in the two-level model, thus we will write:

$$X_3 - D_d \; G_1 - D_s \; G_2 \geq 0 \qquad \text{(RES)}$$

Since federal funds and the shadow price of this constraint have a high correlation in Beazer's model, we expect G_1 and G_2 to be effective policy instruments.

Real Estate Loans

Real estate loans must be less than the greater of either capital stock or a fraction of total time deposits. The fraction is 0.6 prior to September 1962 and 0.7 thereafter, however, it can be considered a potential variable.

The constraint is

$$X_{13} \leq \max \{K_1, R_2D_s\} \qquad \text{(REST)}$$

Pledged Assets

Both the federal government and most states require that banks pledge securities against their holdings of government-owned deposits. Since it is necessary that the securities be on deposit with the Federal Reserve before the government deposits can be accepted all banks observe a safety factor. We shall assume that the actual safety factor the bank uses is given and fixed.

The constraint is:

$$X_6 + X_8 + X_{10} + X_{12} \geq R_3 \, (D_{us} + D_{state}) \qquad \text{(PLDA)}$$

where

R_3 = the individual bank's safety factor

D_{us} = United States Government deposits

D_{state} = state government deposits

Transactions Balances and Balances with Banks

Transaction balances are holdings of vault cash required to take care of the daily business of providing cash to customers. In the model it is primarily a bookkeeping item inserted to maintain equality between assets and liabilities. The minimum required level is the one actually used by the bank. Thus,

$$X_2 \geq \text{Required currency holdings} \qquad \text{(TBAL)}$$

The balances held with other banks are usually maintained just at the level required to accommodate any necessary business and to compensate the depositee bank for services rendered. This constraint too, is primarily a bookkeeping item, although the dual variable gives a means of evaluating the cost of maintaining such balances. The minimum required balance in the model will be that used in the bank. Thus,

$$X_4 = \text{Required balances} \qquad \text{(BBAL)}$$

Risk Asset Ratio

The risk asset ratio requirement calls for the bank to hold capital greater than or equal to a fraction of its risk asset holdings. Typically, this fraction is 1/6 but there is considerable leeway in the figure acceptable to bank examiners. The constraint is

$$X_5 + X_7 + X_9 + X_{11} + X_{13} + X_{14} + X_{15} + X_{16} \leq R_4 K \qquad \text{(RAR)}$$

where K = Bank's capital

R_4 = The risk-asset capital ratio.

Other Assets and Cash in Process of Collection

These two items are strictly balancing items. The right hand side values should be actual ones taken from bank data. The constraints are:

		X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13	X14	X15	X16	X17	L1	L2	L3	G1	G2	R4	B
UPOB	N													1	1	1	1								
LWOB	N					◆	◆	◆	◆	◆	◆	◆	◆	◆	◆	◆	◆								
CAR	L					◆	◆	◆	◆	◆	◆	◆	◆	◆	◆	◆	◆		◆	◆	◆				◆
CAR1	G	1	1	1	1	◆	◆	◆	◆	◆									1						◆
CAR2	G	1	1	1	1	◆	◆	◆	◆	◆	◆	◆								1					◆
CAR3	G	1	1	1	1	◆	◆	◆	◆	◆	◆	◆	◆								1				◆
RES	G			1																		◆	◆		
REST	L													1											◆
PLDA	G						1		1		1		1												◆
TBAL	G		1																						◆
RBAL	G				1																				◆
RAR	L					1		1		1		1		1	1	1	1							◆	
LDDC	G					1	1																		◆
DAS	E																	1							◆
CIP	E	1																							◆
TOT	E	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1							◆
POL1	G																					1			◆
POL2	L																					1			◆
POL3	G																						1		◆
POL4	L																						1		◆
POL5	G																							1	◆
POL6	L																							1	◆

Figure 9. Tableau for the Two-Level Banking Model

X_{17} = Other assets (OAS)

X_1 = Cash in process of collection (CIP)

Total Assets

Total assets must equal total liabilities and both must equal unity since the values for all stipulations and activity levels are fractions of total assets. The constraint is:

$$\Sigma X_j = D_d + D_s + K = 100 \qquad \text{(TOT)}$$

A tableau for the model is shown in Figure 9. Coefficients other than ones or zeros are represented by asterisks. The complete list of coefficients is given in the appendix in the standard MPS format, used by most linear programming computer codes.

1.3 Computational Experience.

In this section we present the solutions for two configurations of the model, one in which the upper level has the power to decide on the reserve requirement parameters and the risk asset ratio and the second in which the upper level controls the reserve requirement parameters and the capital adequacy ratio. The solutions of these configurations are then compared to the current state of affairs configuration given by the following policy values:

$G1 = 0.2$ $R1 = 1.016$

$G2 = 0.05$ $R4 = 8.033$

and the following behavioral response:

$UPOB = 57.843$ $LWOB = 437.055$

$x_1 = 7.72$ $x_{10} = 0$

$x_2 = 0.32$ $x_{11} = 0$

$x_3 = 14.57$ $x_{12} = 0$

$x_4 = 1.30$ $x_{13} = 0$

$x_5 = 0$ $x_{14} = 0$

$x_6 = 17.461$ $x_{15} = 0$

$x_7 = 0.317$ $x_{16} = 57.843$

$x_8 = 0$ $x_{17} = 0.47$

$x_9 = 0$

The average rate of return for this portfolio is 4.37% and 57.84% of the portfolio is invested in the loans segment.

First Problem

The upper level objective is to minimize the loan portfolio segment, for this they count with the power to control the reserve requirement parameters G1, G2 and the risk asset ratio R4. The application of the algorithm of interceptions produce the following solution:

UPOB = 21.72	LWOB = 211.607
G1 = 0.6	R4 = 3.
G2 = 0.3	
x_1 = 7.72	x_{10} = 0
x_2 = 0.32	x_{11} = 0
x_3 = 47.691	x_{12} = 0
x_4 = 1.3	x_{13} = 0
x_5 = 0	x_{14} = 0
x_6 = 8	x_{15} = 0
x_7 = 0	x_{16} = 21.72
x_8 = 12.779	x_{17} = 0.47
x_9 = 0	

We can observe that the loan segment portfolio has been lowered from 57.8% of the total asset portfolio to just 21.7%. Most of this decrease has been transferred to the federal reserves (x_3) and government bonds (x_8). The objective of the lower level, the portfolio average rate of return experienced a decrease of 2.26 percentage points with the new policy.

Second Problem

In the second problem the objective of the upper level is again to minimize the loan portfolio segment. However, a new set of policy tools is considered this time: the capital adequacy ratio R1, and the reserve requirement parameters G1, G2. The application of the algorithm of interceptions produce the following solution:

UPOB = 34.62	LWOB = 358.459
G1 = 0.243	R1 = 0.5
G2 = 0.3	
x_1 = 7.72	x_{10} = 0
x_2 = 0.32	x_{11} = 0
x_3 = 24.03	x_{12} = 0
x_4 = 1.30	x_{13} = 0
x_5 = 0	x_{14} = 0
x_6 = 8.00	x_{15} = 0
x_7 = 23.537	x_{16} = 34.623
x_8 = 0	x_{17} = 0.470
x_9 = 0	

In this case the loan segment portfolio has been lowered from 57.8% of the total portfolio (in the current state of affairs) to 34.62%. Most of this decrease has been transferred to loans to brokers (x_7) and the federal reserves (x_3). The objective of the lower level, the portfolio average rate of return experienced a decrease of 8 tenths of a percentage point with the new policy.

In this numerical example, we find that the second combination of policy variables is more effective than the first combination, since the loss in rate of return for each point of decrease in the loan assets segment is half when the upper level uses the second combination of variables than when they use the first.

2. An Agricultural Policy Problem

2.1 Linear Programming for Agricultural Production Systems

Linear Programming has been proposed and used as a technical tool to improve the management of farms. It has also been used in the regional and national context to study alternative agricultural sector policies. The emphasis in the farm models is on the prescriptive aspect, while in the large scale sectoral models the emphasis is on the descriptive aspect. Within the first group of models we have the Purdue Top Form Model [35], which has been designed for large grain farms. The model determines the optimal acreage for planting each of several grains to maximize revenue in view of limited factors of production and scheduling constraints. The computer package to operate the model consists of an INPUT FORM, a MATRIX GENERATOR, an LP SOLVER, and a REPORT WRITER. The INPUT FORM is filled by the farmer, and from there the other elements of the package generate the model, solve the linear programming problem and produce a set of reports to the farmer.

At the agricultural sector level, linear programming models have been developed for Mexico [18], Brazil [32], Pakistan [20] and Egypt [33]. These models are built on the basis of smaller independent modular models, representing specific agricultural subareas which can be used individually or assembled in the sector framework. In the Mexican Linear Programming model Duloy and Norton [18,19] use the sum of consumers' and producers' surplus as the objective function and introduce a modeling element that incorporates demand functions and endogenous prices into the model. The result is a behavioral model that replicates market equilibrium. Subsequently, Hazell and Scandizzo [26] extended the model to include the impact of farmers risk attitudes on the production (supply) activities of the model.

2.2 A Two-Level Agricultural Model

In this two-level programming experiment we use the "El Fayoum" model, one of the fifteen components of the Agricultural Linear Programming Model for the Nile valley in Egypt [33].

The Upper Level

The long range Egyptian policy objectives in the agricultural sector include a distributional goal as stated by President Nasser's "National Charter of Arab Socialism" of 1962, and growth and development goals formally stated in the "First Five Year Plan 1960-1965". In this modeling exercise we will experiment with two intermediate policy objectives. The first one relates to closing the gap between national producer prices and the corresponding international prices[1] (Table 8). For this, the government will attempt to foster patterns of production so as to maximize net production value added at international prices. The second policy objective with which we experiment consists of implementing maximum employment. Each policy objective function leads to a different two-level programming exercise.

Although Egyptian policy instruments include direct price controls, investment projects, and production quotas, we propose to use only the latter (production quotas for cotton) plus a hypothetical government control of water deliveries to the region (Table 9). This choice of policy instruments is motivated by the notorious popular discontent with pricing policies that were perceived as extremely onerous in 1975.

The policy scenarios are limited by the upper level constraints which are an upper level limit on the area to plant cotton (60,000 feddans) in order to accomodate planting rotations, and an upper bound on the annual water delivery to the region (3,300 million cubic meters) due to

1. Prices are given in Egyptian pounds per ton (LE/ton).

Table 8

World and Domestic Prices for Products in the El Fayoum Model in LE/Ton (1978)

Crop	Symbol	Price	
		World	Egyptian
Short Berseem	SB	--	--
Wheat	WH	119	52
Barley	BA	95	56
Horsebeans	HB	250	112
Fenugreek	FG	245	110
Winter Tomatoes	WT	89	89
Cotton	CO	390	200
Rice	RI	125	50
Sorghum	SO	87	50
Nili Maize	MN	89	48
Summer Maize	MS	89	48
Nili Tomatoes	NT	63	63
Fruits	FR	62	62
Medical Crops	MC	85	85

Table 9

Policy Variables of the El Fayoum Model

Variable	Symbol	Unit
Annual cotton quota	KCO	thousands of feddans
Annual delivery to the region	KWAT	millions of cubic meters

maximum flow capacity of the irrigation system. These constraints can be written using the policy variables (Table 9) as follows:

$$KCO \leq 60$$

$$KWAT \leq 3300$$

The Lower Level

The objective of the lower level model is the maximization of agricultural net production value added at national prices. The behavioral variables are listed in Table 10. A variable with a subindex ending in small case letter u or v represents a crop grown with alternative technologies. these technologies are:

A: Actual technology with production of fodder.

B: Actual technology without production of fodder.

M: Mechanized with production of fodder.

N: Mechanized without production of fodder.

Subindex v can assume the values A, B, M, and N, while subindex u can only assume the values A and M.

The choice of values for the behavioral variables are constrained by the production factor constraints (land, water and labor), animal fodder requirements, upper bound requirements on the production of tomatoes and medical crops, and a set of pseudo-constraints that are used to

Table 10

Behavioral Variables of the El Fayoum Model

Product	Variable	Unit
Long Berseem	X_{LBu}	thousands of feddans
Short B rseem	X_{SBu}	"
Wheat	X_{WHu}	"
B rley	X_{BAu}	"
Horsebeans	X_{HBu}	"
Fenugreek	X_{FGu}	"
Winter Tomatoes	X_{WTu}	"
Rice	X_{RIu}	"
Shorgum	X_{SOv}	"
Nili Maize	X_{MNv}	"
Summer Maize	X_{MSv}	"
Nili Tomatoes	X_{NT}	"
Fruits	X_{FR}	"
Medical Crops	X_{MC}	"
Artificial Protein	APROT	tons
Artificial Starch	ASTAR	tons

NOTE: As explained in the text, subindices u and v indicate cropping technologies. Index v can take on the values A, B, M, and N, and index u can take on the values of A and M.

transform production from feddans to tons in order to compute revenues. Below we give the algebraic statement of the behavioral constraints as well as the behavioral and policy objectives.

Land Constraints

$$\Sigma a_{ij} X_j \leqq L_i \qquad i = O, N, D, 1, \ldots, 9$$ [1]

Where a_{ij} is the land input in month i for one unit of crop activity j and X_j the area (in thousands of feddans) of crop activity j. The total arable land in the El Fayoum is constant throughout the year and amounts to 375,000 feddans, i.e. $L_i = 375$.

Water Constraints

$$(1) \quad \Sigma w_{ij} X_j \leqq H_i \qquad i = O, N, D, 1, 2, \ldots, 9$$

$$(2) \quad \Sigma W_j \; X_j - KWAT \leqq O$$

In equation (1), w_{ij} is the water input in month i for one unit of crop activity j, and H_i is the monthly flow capacity of the irrigation system. Equation (2) implements the hypothetical government water delivery control. The coefficient W_j is the use of water by crop j over the year, i.e.,

---

1. The sequence i = O, N, D, 1, ..., 9 stands for the months of the year: October, November, etc.

$$W_j = \Sigma w_{ij}$$

Thus, equation (2) above states that the annual water demand must not exceed the government chosen value of KWAT.

Labor Constraints

$$\Sigma \; m_{ij} \; X_j \leq M_i \qquad i = O, N, D, 1, \ldots, 9$$

Where coefficient m_{ij} is the labor input in month i for one unit of crop activity j, M_i is the monthly labor supply, equal to 5,200 thousands man days.

Annual Fodder Requirements

The large animal population (cattle and buffalo) which is maintained for draught power purposes and for meat and dairy products, induces competition and interactions among the crops because of its fodder requirements. This can be expressed in terms of minimum required starch and protein derived from the two berseem crops in the winter and straws from wheat, barley, and carried-over berseem in the summer.

Starch Constraint

$$\text{ASTAR} + \Sigma \; s_j X_j \geq 217.602$$

Where s_j is the starch content (in tons/feddan) of 1 feddan of crop j, and ASTAR is an artificial starch activity.

Protein Constraint

$$APROT + \Sigma p_j X_j \geq 34.678$$

In this constraint p_j is the protein content (in tons/feddan) of 1 feddan of crop j, and APROT is an artificial protein activity.

Upper Bound Constraint

Marketing and rotation planting requirements impose upper bounds on the production of tomatoes and medical crops. Thus, we have that:

$$X_{WT} \leq 10$$
$$X_{NT} \leq 16$$
$$X_{MC} \leq 9$$

Transformation Pseudo Constraints

Corresponding to each crop, there exists a pseudo constraint to compute the total yield. For example, the total summer maize TOTMS (in thousands of tons) is given by the equation

$$-TOTMS + 1.58X_{MSA} + 1.98X_{MSB} + 1.58X_{MSM} + 1.98X_{MSN} = 0$$

Where the coefficients of the crop activities are the yields in tons per feddans. Note that the yields vary according to the technologies. In general, we have that the total production of crop C is given by

$$-TOTC + \Sigma y_{Cw} X_{Cw} = 0$$

where w is the techology subindex, and C the crop subindex.

Behavioral and Policy Objectives

In the first numerical experiment the policy and behavioral objectives are given by a linear function of the form

$$(1) \qquad \Sigma P_c \text{ (TOTC)} - \Sigma c_j X_j \qquad .$$

where p_c is the price of crop c in LE/tons, TOTC the total production in thousands of tons, and c_j the total unit cost of crop activity X_j. If national prices are used the linear function (1) gives the behavioral objective, denoted VAEP. If international prices are used, function (1) gives the policy objective, denoted VAIP. Both VAEP and VAIP are expressed in thousands of LE.

In the second numerical experiment we use the same behavioral objective VAEP, but use the maximization of total labor (in thousands of man days) TLAB, which is given by

$$TLAB = \Sigma t_j X_j$$

where t_j is annual labor required by one unit of activity X_j. A detailed coefficient tableau which shows the model in its entirety is given in Figure 10. The list of coefficients appears in the appendix in the standard MPS format.

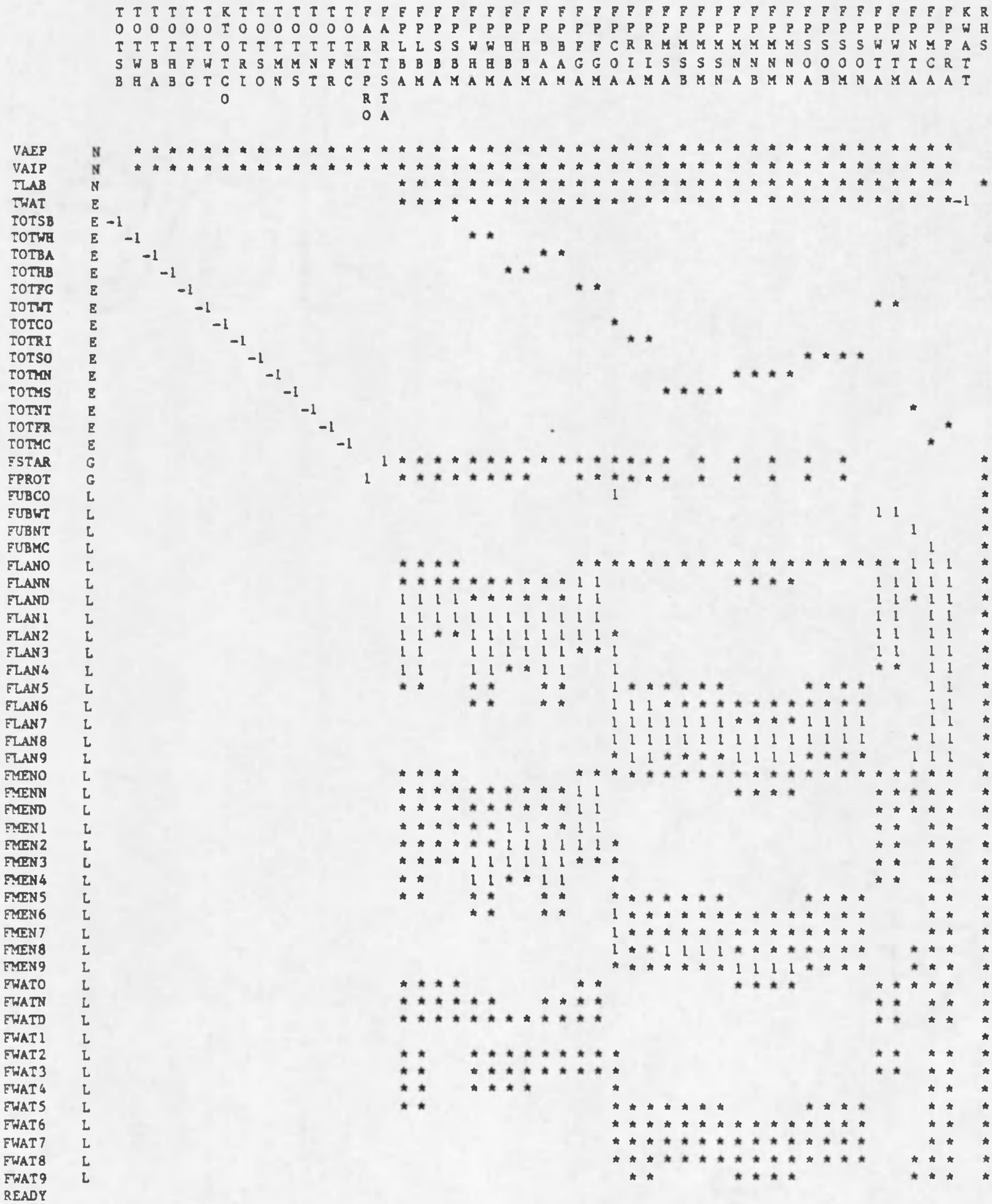

Figure 10. Tableau for the Agricultural Policy Problem

2.3 Computational Experience

The model has been solved for two cases, with a different objective function in each case. These solutions are reported below. No policy implications are drawn here as the data and the model require further refinement.

First Problem

This problem corresponds to the case in which the government seeks to maximize employment. A local optimum is obtained with

$$KCO = 0$$

$$KWAT = 1640.174$$

and the corresponding objective function values are

$$TLAB = 17857.804$$

$$VAEP = 60620.506$$

Second Problem

This problem corresponds to the case in which the government seeks to maximize net production value added at international prices. The following local optimum was attained:

$$KCO = 85.44$$

$$KWAT = 1688.516$$

and the corresponding objective function values are

$$VAIP = 118526.219$$

$$VAEP = 66385.997$$

CHAPTER 6

SUMMARY OF RESULTS AND CONCLUSIONS

1. Summary of Results

In Chapter 2, it was demonstrated that the branch and bound technique proposed by Falk to solve max-min problems can be specialized to solve the general two-level problem with linear objectives. In this chapter it was also shown that the Optimal Control Theory and Dynamic Programming were structurally related to the hierarchical two-level programming problem. In Chapter 3, the feasible domain of the two-level problem with linear objectives was studied. Two basic results for the test of global optimality were proven here. First it was established that the two-level feasible domain is connected. Secondly, it was shown that the intercepting plane $px + qy = \gamma+\varepsilon$ has a point in common with the two-level feasible set if $\gamma+\varepsilon$ falls within two high points. Further, it was shown that there exists at least one vertex from the polyhedron below, that belongs to the two-level feasible set:

$$I\ (\varepsilon):\quad \begin{aligned} Hx + Gy &\geq b \\ Ey &\geq e \\ px + qy &= \gamma + \varepsilon \end{aligned}$$

The elements for a local optimum algorithm were also established in this chapter. In Chapter 4, the algorithm of interceptions was formally stated and then it was applied to problems that had already been solved with other methodologies and to problems from the chapter of applications. Based on these solutions, the algorithm of interceptions was compared to the other algorithms with respect to accuracy, speed of convergence and data storage requirements.

In Chapter 5 two existing linear programming models were recast as hierarchical models. Computational results were then reported and policy implications discussed.

2. Conclusions

A body of knowledge regarding the two-level problem feasible domain has been established upon which the algorithm of interceptions was developed. This theoretical framework can be the basis for alternative strategies of solution. The algorithm of interceptions guarantees global optimal solutions for any problem. However in practice computational burden prohibits proving optimality for problems with more than 20 variables. The computational experience indicates that for large problems the algorithm does provide a solution that is likely to be the global optimum.

In the applications part of the thesis it has been shown that hierarchical programming is a valuable tool in the design of economic policy. The procedure to develop hierarchical programming models from existing linear programming models was illustrated with two examples. The numerical experiments for the Banking Problem suggest that the model is effective in selecting policy variables as well as their optimal values.

APPENDIX

TEST AND APPLICATION PROBLEMS

Problem 1

$$\max \quad -4x_1 + 40x_2 + 4x_3 + 8y_1 + 4y_2$$

(1) $\quad y_1,\ y_2 \geqq 0$

(2) $$\begin{aligned} \max\ - x_1 - x_2 - 2x_3 - y_1 - 2y_2 & \\ x_1 - x_2 - x_3 &\geqq -1 \\ x_1 - 2x_2 + 0.5x_3 - 2y_1 &\geqq -1 \\ -2x_1 + x_2 + 0.5x_3 - 2y_2 &\geqq -1 \\ x_1,\ x_2,\ x_3 &\geqq 0 \end{aligned}$$

Problem 2

$$\max \quad -8x + 2y_1 - y_2$$

(1) $$\begin{aligned} y_1 + y_2 &\leqq 2 \\ y_1,\ y_2 &\geqq 0 \end{aligned}$$

(2) $$\begin{aligned} \max\ 8x - 2y_1 + y_2 & \\ x + y_1 + y_2 &\leqq 3 \\ -x + y_1 &\leqq 0 \\ x - y_1 - y_2 &\leqq 1 \\ x &\geqq 0 \end{aligned}$$

PROBLEM 3

$$\max \quad -0.5x_1 + 2y_1 - y_2$$

$$(1) \qquad -y_1 - y_2 \geq -2$$

$$y_1,\ y_2 \geq 0$$

$$(2) \quad \max\ 4x_1 - x_2 - y_1 - y_2$$

$$-x_1 + x_2 + 2y_1 \geq 2.5$$

$$-x_2 - y_1 + 3y_2 \geq -2$$

$$x_1,\ x_2 \geq 0$$

PROBLEM 4

$$\max \quad 0.1x_2 - y_1$$

$$(1) \quad -2y_1 + 2y_2 \leq 1$$

$$y_1 + 4y_2 \leq 13$$

$$y_1 - 1.5y_2 \leq 2$$

$$y_1,\ y_2 \geq 0$$

$$(2) \quad \max \quad -0.1x_2 + y_1$$

$$x_1 - x_2 + y_2 \leq 4$$

$$2x_1 + 2x_2 + y_1 - 2y_2 \leq 8$$

$$11x_1 + 2x_2 \leq 44$$

$$x_1,\ x_2 \geq 0$$

PROBLEM 5. Coefficients for the Two-Level Banking Model

(For the constraint names and types see Figure 9)

COLUMNS

X1	CAR1	1.00000	CAR2	1.00000
X1	CAR3	1.00000	CIP	1.00000
X1	TOT	1.00000		
X2	CAR1	1.00000	CAR2	1.00000
X2	CAR3	1.00000	TBAL	1.00000
X2	TOT	1.00000		
X3	CAR1	1.00000	CAR2	1.00000
X3	CAR3	1.00000	RES	1.00000
X3	TOT	1.00000		
X4	CAR1	1.00000	CAR2	1.00000
X4	CAR3	1.00000	BBAL	1.00000
X4	TOT	1.00000		
X5	LWOB	2.96000	CAR	0.00500
X5	CAR1	0.99500	CAR2	0.99500
X5	CAR3	0.99500	RAR	1.00000
X5	LDDC	1.00000	TOT	1.00000
X6	LWOB	3.35000	CAR	0.00500
X6	CAR1	0.99500	CAR2	0.99500
X6	CAR3	0.99500	PLDA	1.00000
X6	LDDC	1.00000	TOT	1.00000
X7	LWOB	4.50000	CAR	0.00500
X7	CAR1	0.99500	CAR2	0.99500
X7	CAR3	0.99500	RAR	1.00000
X7	TOT	1.00000		
X8	LWOB	3.38000	CAR	0.04000
X8	CAR1	0.96000	CAR2	0.96000
X8	CAR3	0.96000	PLDA	1.00000
X8	TOT	1.00000		
X9	LWOB	3.38000	CAR	0.04000
X9	CAR1	0.96000	CAR2	0.96000
X9	CAR3	0.96000	RAR	1.00000
X9	TOT	1.00000		
X10	LWOB	3.36000	CAR	0.04000
X10	CAR2	0.90000	CAR3	0.90000
X10	PLDA	1.00000	TOT	1.00000
X11	LWOB	4.50000	CAR	0.04000
X11	CAR2	0.90000	CAR3	0.90000
X11	RAR	1.00000	TOT	1.00000

X12	LWOB	3.26000	CAR	0.06000
X12	CAR3	0.85000	PLDA	1.00000
X12	TOT	1.00000		
X13	UPOB	1.00000	LWOB	5.47000
X13	CAR	0.10000	RAR	1.00000
X13	TOT	1.00000	REST	1.00000
X14	UPOB	1.00000	LWOB	6.46000
X14	CAR	0.10000	RAR	1.00000
X14	TOT	1.00000		
X15	UPOB	1.00000	LWOB	4.00000
X15	CAR	0.10000	RAR	1.00000
X15	TOT	1.00000		
X16	UPOB	1.00000	LWOB	6.52000
X16	CAR	0.10000	RAR	1.00000
X16	TOT	1.00000		
X17	DAS	1.00000	TOT	1.00000
L1	CAR	0.06500	CAR1	1.00000
L2	CAR	0.04000	CAR2	1.00000
L3	CAR	0.09500	CAR3	1.00000
G1	RES	-66.21000	POL1	1.00000
G1	POL2	1.00000		
G2	RES	-26.55000	POL3	1.00000
G2	POL4	1.00000		
R4	RAR	-7.24000	POL5	1.00000
R4	POL6	1.00000		
RHS				
B	CAR1	49.03000	CAR2	49.03000
B	CAR3	49.03000	PLDA	6.36000
B	TBAL	0.32000	BBAL	1.30000
B	LDDC	8.00000	DAS	0.47000
B	CIP	7.72000	TOT	100.00000
B	REST	7.48000	CAR	7.36000
B	POL1	0.10000	POL2	0.60000
B	POL3	0.03000	POL4	0.30000
B	POL5	3.00000	POL6	15.00000

PROBLEM 6. Coefficients for the Two-Level Agricultural Model (For the constraint names and types see Figure 10)

COLUMNS

TOTSB	TOTSB	-1.00000		
TOTWH	VAIP	119.00000	VAEP	52.00000
TOTWH	TOTWH	-1.00000		
TOTBA	VAIP	95.00000	VAEP	56.00000
TOTBA	TOTBA	-1.00000		
TOTHB	VAIP	250.00000	VAEP	112.00000
TOTHB	TOTHB	-1.00000		
TOTFG	VAIP	245.00000	VAEP	110.00000
TOTFG	TOTFG	-1.00000		
TOTWT	VAIP	89.00000	VAEP	89.00000
TOTWT	TOTWT	-1.00000		
KTOTCO	VAIP	390.00000	VAEP	200.00000
KTOTCO	TOTCO	-1.00000		
TOTRI	VAIP	125.00000	VAEP	50.00000
TOTRI	TOTRI	-1.00000		
TOTSO	VAIP	87.00000	VAEP	50.00000
TOTSO	TOTSO	-1.00000		
TOTMN	VAIP	89.00000	VAEP	48.00000
TOTMN	TOTMN	-1.00000		
TOTMS	VAIP	89.00000	VAEP	48.00000
TOTMS	TOTMS	-1.00000		
TOTNT	VAIP	63.00000	VAEP	63.00000
TOTNT	TOTNT	-1.00000		
TOTFR	VAIP	62.00000	VAEP	62.00000
TOTFR	TOTFR	-1.00000		
TOTMC	VAIP	85.00000	VAEP	85.00000
TOTMC	TOTMC	-1.00000		
FARTPRO	FPROT	1.00000	VAIP	-1000.00000
FARTPRO	VAEP	-1000.00000		
FARTSTA	FSTAR	1.00000	VAEP	-1000.00000
FARTSTA	VAIP	-1000.00000		
FPLBA	TLAB	36.00000	VAEP	-1.65000
FPLBA	VAIP	-2.70000	TWAT	4.28400
FPLBA	FLANO	0.25000	FWATO	0.07800
FPLBA	FMENO	1.50000	FLANN	0.75000
FPLBA	FWATN	0.42000	FMENN	4.50000
FPLBA	FLAND	1.00000	FWATD	0.54000
FPLBA	FMEND	5.00000	FLAN1	1.00000
FPLBA	FMEN1	5.00000	FLAN2	1.00000
FPLBA	FWAT2	0.96000	FMEN2	5.00000
FPLBA	FLAN3	1.00000	FWAT3	0.95400
FPLBA	FMEN3	5.00000	FLAN4	1.00000
FPLBA	FWAT4	1.04400	FMEN4	5.00000

FPLBA	FLAN5	0.50000	FWAT5	0.28800
FPLBA	FMEN5	5.00000	FSTAR	2.16000
FPLBA	FPROT	0.48000		
FPLBM	TLAB	33.00000	VAEP	-5.40000
FPLBM	VAIP	-10.20000	TWAT	4.28400
FPLBM	FLANO	0.25000	FWATO	0.07800
FPLBM	FMENO	0.75000	FLANN	0.75000
FPLBM	FWATN	0.42000	FMENN	2.25000
FPLBM	FLAND	1.00000	FWATD	0.54000
FPLBM	FMEND	5.00000	FLAN1	1.00000
FPLBM	FMEN1	5.00000	FLAN2	1.00000
FPLBM	FWAT2	0.96000	FMEN2	5.00000
FPLBM	FLAN3	1.00000	FWAT3	0.95400
FPLBM	FMEN3	5.00000	FLAN4	1.00000
FPLBM	FWAT4	1.04400	FMEN4	5.00000
FPLBM	FLAN5	0.50000	FWAT5	0.28800
FPLBM	FMEN5	5.00000	FSTAR	2.16000
FPLBM	FPROT	0.48000		
FPSBA	TLAB	12.00000	VAEP	-1.65000
FPSBA	VAIP	-2.70000	TWAT	2.70000
FPSBA	FLANO	0.33000	FWATO	0.45600
FPSBA	FMENO	1.98000	FLANN	0.67000
FPSBA	FWATN	0.97800	FMENN	4.02000
FPSBA	FLAND	1.00000	FWATD	1.26600
FPSBA	FMEND	2.00000	FLAN1	1.00000
FPSBA	FMEN1	2.00000	FLAN2	0.50000
FPSBA	FMEN2	1.66000	FMEN3	0.34000
FPSBA	FSTAR	0.63000	FPROT	0.14000
FPSBM	TLAB	9.00000	VAEP	-5.40000
FPSBM	VAIP	-10.20000	TWAT	2.70000
FPSBM	TOTSB	7.00000	FLANO	0.33000
FPSBM	FWATO	0.45600	FMENO	0.99000
FPSBM	FLANN	0.67000	FWATN	0.97800
FPSBM	FMENN	2.01000	FLAND	1.00000
FPSBM	FWATD	1.26600	FMEND	2.00000
FPSBM	FLAN1	1.00000	FMEN1	2.00000
FPSBM	FLAN2	0.50000	FMEN2	1.66000
FPSBM	FMEN3	0.34000	FSTAR	0.63000
FPSBM	FPROT	0.14000		
FPWHA	TLAB	30.00000	VAEP	-10.05000
FPWHA	VAIP	-14.70000	TWAT	2.19600
FPWHA	TOTWH	1.12000	FLANN	0.12000
FPWHA	FWATN	0.24600	FMENN	0.96000
FPWHA	FLAND	0.88000	FWATD	0.55100
FPWHA	FMEND	7.04000	FLAN1	1.00000
FPWHA	FMEN1	2.00000	FLAN2	1.00000
FPWHA	FWAT2	0.47400	FMEN2	2.00000
FPWHA	FLAN3	1.00000	FWAT3	0.47600
FPWHA	FMEN3	1.00000	FLAN4	1.00000
FPWHA	FWAT4	0.44900	FMEN4	1.00000

FPWHA	FLAN5	0.88000	FMEN5	14.08000
FPWHA	FLAN6	0.12000	FMEN6	1.92000
FPWHA	FSTAR	0.35000	FPROT	0.00200
FPWHM	TLAB	21.00000	VAEP	-19.80000
FPWHM	VAIP	-34.20000	TWAT	2.19600
FPWHM	TOTWH	1.12000	FLANN	0.12000
FPWHM	FWATN	0.24600	FMENN	0.48000
FPWHM	FLAND	0.88000	FWATD	0.55100
FPWHM	FMEND	3.52000	FLAN1	1.00000
FPWHM	FMEN1	2.00000	FLAN2	1.00000
FPWHM	FWAT2	0.47400	FMEN2	2.00000
FPWHM	FLAN3	1.00000	FWAT3	0.47600
FPWHM	FMEN3	1.00000	FLAN4	1.00000
FPWHM	FWAT4	0.44900	FMEN4	1.00000
FPWHM	FLAN5	0.88000	FMEN5	9.68000
FPWHM	FLAN6	0.12000	FMEN6	1.32000
FPWHM	FSTAR	0.35000	FPROT	0.00200
FPHBA	TLAB	24.00000	VAEP	-2.63000
FPHBA	VAIP	-4.10000	TWAT	1.75200
FPHBA	TOTHB	1.06000	FLANN	0.12000
FPHBA	FMENN	0.96000	FLAND	0.88000
FPHBA	FWATD	0.65600	FMEND	7.04000
FPHBA	FLAN1	1.00000	FMEN1	1.00000
FPHBA	FLAN2	1.00000	FWAT2	0.38800
FPHBA	FMEN2	1.00000	FLAN3	1.00000
FPHBA	FWAT3	0.58400	FMEN3	1.00000
FPHBA	FLAN4	0.50000	FWAT4	0.12400
FPHBA	FMEN4	13.00000	FSTAR	0.30300
FPHBA	FPROT	0.02600		
FPHBM	TLAB	20.00000	VAEP	-8.63000
FPHBM	VAIP	-16.10000	TWAT	1.75200
FPHBM	TOTHB	1.06000	FLANN	0.12000
FPHBM	FMENN	0.48000	FLAND	0.88000
FPHBM	FWATD	0.65600	FMEND	3.52000
FPHBM	FLAN1	1.00000	FMEN1	1.00000
FPHBM	FLAN2	1.00000	FWAT2	0.38800
FPHBM	FMEN2	1.00000	FLAN3	1.00000
FPHBM	FWAT3	0.58400	FMEN3	1.00000
FPHBM	FLAN4	0.50000	FWAT4	0.12400
FPHBM	FMEN4	13.00000	FSTAR	0.30300
FPHBM	FPROT	0.02600		
FPBAA	TLAB	30.00000	VAEP	-4.34000
FPBAA	VAIP	-6.20000	TWAT	1.80000
FPBAA	TOTBA	1.12000	FLANN	0.17000
FPBAA	FWATN	0.49600	FMENN	1.96000
FPBAA	FLAND	0.83000	FWATD	0.51200
FPBAA	FMEND	9.54000	FLAN1	1.00000
FPBAA	FMEN1	1.50000	FLAN2	1.00000
FPBAA	FWAT2	0.43200	FMEN2	1.00000
FPBAA	FLAN3	1.00000	FWAT3	0.36000

FPBAA	FMEN3	1.00000	FLAN4	1.00000
FPBAA	FMEN4	1.00000	FLAN5	0.83000
FPBAA	FMEN5	11.62000	FLAN6	0.17000
FPBAA	FMEN6	2.38000	FSTAR	0.40300
FPBAM	TLAB	24.00000	VAEP	-14.09000
FPBAM	VAIP	-25.70000	TWAT	1.80000
FPBAM	TOTBA	1.12000	FLANN	0.17000
FPBAM	FWATN	0.49600	FMENN	1.26000
FPBAM	FLAND	0.83000	FWATD	0.51200
FPBAM	FMEND	6.22000	FLAN1	1.00000
FPBAM	FMEN1	1.50000	FLAN2	1.00000
FPBAM	FWAT2	0.43200	FMEN2	1.00000
FPBAM	FLAN3	1.00000	FWAT3	0.36000
FPBAM	FMEN3	1.00000	FLAN4	1.00000
FPBAM	FMEN4	1.00000	FLAN5	0.83000
FPBAM	FMEN5	9.98000	FLAN6	0.17000
FPBAM	FMEN6	2.04000	FSTAR	0.40300
FPFGA	TLAB	24.00000	VAEP	-2.63000
FPFGA	VAIP	-4.10000	TWAT	1.34800
FPFGA	TOTFG	1.07000	FLANO	0.50000
FPFGA	FWATO	0.15000	FMENO	8.00000
FPFGA	FLANN	1.00000	FWATN	0.34200
FPFGA	FMENN	1.00000	FLAND	1.00000
FPFGA	FWATD	0.36700	FMEND	1.00000
FPFGA	FLAN1	1.00000	FMEN1	1.00000
FPFGA	FLAN2	1.00000	FWAT2	0.37600
FPFGA	FMEN2	1.00000	FLAN3	0.50000
FPFGA	FWAT3	0.11300	FMEN3	12.00000
FPFGA	FSTAR	0.05600	FPROT	0.01300
FPFGM	TLAB	20.00000	VAEP	-9.38000
FPFGM	VAIP	-17.60000	TWAT	1.34800
FPFGM	TOTFG	1.07000	FLANO	0.50000
FPFGM	FWATO	0.15000	FMENO	4.00000
FPFGM	FLANN	1.00000	FWATN	0.34200
FPFGM	FMENN	1.00000	FLAND	1.00000
FPFGM	FWATD	0.36700	FMEND	1.00000
FPFGM	FLAN1	1.00000	FMEN1	1.00000
FPFGM	FLAN2	1.00000	FWAT2	0.37600
FPFGM	FMEN2	1.00000	FLAN3	0.50000
FPFGM	FWAT3	0.11300	FMEN3	12.00000
FPFGM	FSTAR	0.05600	FPROT	0.01300
FPCOA	TLAB	39.00000	VAEP	-26.20000
FPCOA	VAIP	-39.00000	TWAT	5.44800
FPCOA	FUBCO	1.00000	TOTCO	1.42400
FPCOA	FLANO	0.17000	FMENO	0.68000
FPCOA	FLAN2	0.50000	FWAT2	0.60000
FPCOA	FMEN2	12.00000	FLAN3	1.00000
FPCOA	FWAT3	0.55200	FMEN3	5.00000
FPCOA	FLAN4	1.00000	FWAT4	0.51600
FPCOA	FMEN4	8.00000	FLAN5	1.00000

FPCOA	FWAT5	0.72000	FMEN5	7.00000
FPCOA	FLAN6	1.00000	FWAT6	1.14000
FPCOA	FMEN6	1.00000	FLAN7	1.00000
FPCOA	FWAT7	1.28400	FMEN7	1.00000
FPCOA	FLAN8	1.00000	FWAT8	0.63600
FPCOA	FMEN8	1.00000	FLAN9	0.83000
FPCOA	FMEN9	3.32000	FSTAR	0.16100
FPCOA	FPROT	0.13000		
FPRIA	TLAB	45.00000	VAEP	-8.09000
FPRIA	VAIP	-11.90000	TWAT	11.85600
FPRIA	TOTRI	1.67000	FLANO	0.75000
FPRIA	FMENO	5.00000	FLAN5	0.50000
FPRIA	FWAT5	0.24600	FMEN5	4.00000
FPRIA	FLAN6	1.00000	FWAT6	3.00000
FPRIA	FMEN6	11.00000	FLAN7	1.00000
FPRIA	FWAT7	2.40000	FMEN7	11.00000
FPRIA	FLAN8	1.00000	FWAT8	3.57000
FPRIA	FMEN8	4.00000	FLAN9	1.00000
FPRIA	FWAT9	2.64000	FMEN9	10.00000
FPRIA	FSTAR	0.07700	FPROT	0.01000
FPRIM	TLAB	41.00000	VAEP	-16.34000
FPRIM	VAIP	-28.40000	TWAT	11.85600
FPRIM	TOTRI	1.67000	FLANO	0.75000
FPRIM	FMENO	3.00000	FLAN5	0.50000
FPRIM	FWAT5	0.24600	FMEN5	3.00000
FPRIM	FLAN6	1.00000	FWAT6	3.00000
FPRIM	FMEN6	10.00000	FLAN7	1.00000
FPRIM	FWAT7	2.40000	FMEN7	11.00000
FPRIM	FLAN8	1.00000	FWAT8	3.57000
FPRIM	FMEN8	4.00000	FLAN9	1.00000
FPRIM	FWAT9	2.64000	FMEN9	10.00000
FPRIM	FSTAR	0.07700	FPROT	0.01000
FPMSA	TLAB	33.00000	VAEP	-8.68000
FPMSA	VAIP	-12.40000	TWAT	3.72000
FPMSA	TOTMS	1.58000	FLANO	0.12000
FPMSA	FMENO	1.56000	FWAT5	0.76200
FPMSA	FMEN5	2.04000	FLAN5	0.12000
FPMSA	FLAN6	0.88000	FWAT6	1.02000
FPMSA	FMEN6	14.96000	FLAN7	1.00000
FPMSA	FWAT7	1.30200	FMEN7	2.00000
FPMSA	FLAN8	1.00000	FWAT8	0.63600
FPMSA	FMEN8	1.00000	FLAN9	0.88000
FPMSA	FMEN9	11.44000	FSTAR	0.17000
FPMSA	FPROT	0.02400		
FPMSB	TLAB	33.00000	VAEP	-8.68000
FPMSB	VAIP	-12.40000	TWAT	3.72000
FPMSB	TOTMS	1.98000	FLANO	0.12000
FPMSB	FMENO	1.56000	FLAN5	0.12000
FPMSB	FWAT5	0.76200	FMEN5	2.04000
FPMSB	FLAN6	0.88000	FWAT6	1.02000

FPMSB	FMEN6	14.96000	FLAN7	1.00000
FPMSB	FWAT7	1.30200	FLAN8	1.00000
FPMSB	FMEN7	2.00000	FWAT8	0.63600
FPMSB	FMEN8	1.00000	FLAN9	0.88000
FPMSB	FMEN9	11.44000		
FPMSM	TLAB	28.00000	VAEP	-13.93000
FPMSM	VAIP	-22.90000	TWAT	3.72000
FPMSM	TOTMS	1.58000	FLANO	0.12000
FPMSM	FMENO	1.56000	FLAN5	0.12000
FPMSM	FWAT5	0.76200	FMEN5	1.44000
FPMSM	FLAN6	0.88000	FWAT6	1.02000
FPMSM	FMEN6	10.56000	FLAN7	1.00000
FPMSM	FWAT7	1.30200	FMEN7	2.00000
FPMSM	FLAN8	1.00000	FWAT8	0.63600
FPMSM	FMEN8	1.00000	FLAN9	0.88000
FPMSM	FMEN9	11.44000	FSTAR	0.17000
FPMSM	FPROT	0.02400		
FPMSN	TLAB	28.00000	VAEP	-13.93000
FPMSN	VAIP	-22.90000	TWAT	3.72000
FPMSN	TOTMS	1.98000	FLANO	0.12000
FPMSN	FMENO	1.56000	FLAN5	0.12000
FPMSN	FWAT5	0.76200	FMEN5	1.44000
FPMSN	FLAN6	0.88000	FWAT6	1.02000
FPMSN	FMEN6	10.56000	FLAN7	1.00000
FPMSN	FWAT7	1.30200	FMEN7	2.00000
FPMSN	FLAN8	1.00000	FWAT8	0.63600
FPMSN	FMEN8	1.00000	FLAN9	0.88000
FPMSN	FMEN9	11.44000		
FPMNA	TLAB	33.00000	VAEP	-8.68000
FPMNA	VAIP	-12.40000	TWAT	3.72000
FPMNA	TOTMN	1.03000	FLANO	0.88000
FPMNA	FWATO	0.52800	FMENO	11.44000
FPMNA	FLANN	0.12000	FMENN	1.56000
FPMNA	FLAN6	0.12000	FWAT6	0.46800
FPMNA	FMEN6	2.04000	FLAN7	0.88000
FPMNA	FWAT7	0.90000	FMEN7	14.96000
FPMNA	FLAN8	1.00000	FWAT8	0.85200
FPMNA	FMEN8	2.00000	FLAN9	1.00000
FPMNA	FWAT9	0.97200	FMEN9	1.00000
FPMNA	FSTAR	0.17000	FPROT	0.02400
FPMNB	TLAB	33.00000	VAEP	-8.68000
FPMNB	VAIP	-12.40000	TWAT	3.72000
FPMNB	TOTMN	1.34000	FLANO	0.88000
FPMNB	FWATO	0.52800	FMENO	11.44000
FPMNB	FLANN	0.12000	FMENN	1.56000
FPMNB	FLAN6	0.12000	FWAT6	0.46800
FPMNB	FMEN6	2.04000	FLAN7	0.88000
FPMNB	FWAT7	0.90000	FMEN7	14.96000
FPMNB	FLAN8	1.00000	FWAT8	0.85200
FPMNB	FMEN8	2.00000	FLAN9	1.00000
FPMNB	FWAT9	0.97200	FMEN9	1.00000

FPMNM	TLAB	28.00000	VAEP	-13.93000
FPMNM	VAIP	-22.90000	TWAT	3.72000
FPMNM	TOTMN	1.03000	FLANO	0.88000
FPMNM	FWATO	0.52800	FMENO	11.44000
FPMNM	FLANN	0.12000	FMENN	1.56000
FPMNM	FLAN6	0.12000	FWAT6	0.46800
FPMNM	FMEN6	1.44000	FLAN7	0.88000
FPMNM	FWAT7	0.90000	FMEN7	10.56000
FPMNM	FLAN8	1.00000	FWAT8	0.85200
FPMNM	FMEN8	2.00000	FLAN9	1.00000
FPMNM	FWAT9	0.97200	FMEN9	1.00000
FPMNM	FSTAR	0.17000	FPROT	0.02400
FPMNN	TLAB	28.00000	VAEP	-13.93000
FPMNN	VAIP	-22.90000	TWAT	3.72000
FPMNN	TOTMN	1.34000	FLANO	0.88000
FPMNN	FWATO	0.52800	FMENO	11.44000
FPMNN	FLANN	0.12000	FMENN	1.56000
FPMNN	FLAN6	0.12000	FWAT6	0.46800
FPMNN	FMEN6	1.44000	FLAN7	0.88000
FPMNN	FWAT7	0.90000	FMEN7	10.56000
FPMNN	FLAN8	1.00000	FWAT8	0.85200
FPMNN	FMEN8	2.00000	FLAN9	1.00000
FPMNN	FWAT9	0.97200	FMEN9	1.00000
FPSOA	TLAB	34.00000	VAEP	-8.68000
FPSOA	VAIP	-12.40000	TWAT	3.60000
FPSOA	TOTSO	1.46000	FLANO	0.12000
FPSOA	FMENO	1.56000	FLAN5	0.12000
FPSOA	FWAT5	0.73800	FMEN5	2.04000
FPSOA	FLAN6	0.88000	FWAT6	0.99000
FPSOA	FMEN6	14.96000	FLAN7	1.00000
FPSOA	FWAT7	1.25400	FMEN7	2.00000
FPSOA	FLAN8	1.00000	FWAT8	0.61800
FPSOA	FMEN8	2.00000	FLAN9	0.88000
FPSOA	FMEN9	11.44000	FSTAR	0.17000
FPSOA	FPROT	0.02400		
FPSOB	TLAB	34.00000	VAEP	-8.68000
FPSOB	VAIP	-12.40000	TWAT	3.60000
FPSOB	TOTSO	1.90000	FLANO	0.12000
FPSOB	FMENO	1.56000	FLAN5	0.12000
FPSOB	FWAT5	0.73800	FMEN5	2.04000
FPSOB	FLAN6	0.88000	FWAT6	0.99000
FPSOB	FMEN6	14.96000	FLAN7	1.00000
FPSOB	FWAT7	1.25400	FMEN7	2.00000
FPSOB	FLAN8	1.00000	FWAT8	0.61800
FPSOB	FMEN8	2.00000	FLAN9	0.88000
FPSOB	FMEN9	11.44000		
FPSOM	TLAB	31.01000	VAEP	-13.93000
FPSOM	VAIP	-22.90000	TWAT	3.60000
FPSOM	TOTSO	1.46000	FLANO	0.12000
FPSOM	FMENO	1.56000	FLAN5	0.12000

FPSOM	FWAT5	0.73800	FMEN5	1.68000
FPSOM	FLAN6	0.88000	FWAT6	0.99000
FPSOM	FMEN6	12.33000	FLAN7	1.00000
FPSOM	FWAT7	1.25400	FMEN7	2.00000
FPSOM	FLAN8	1.00000	FWAT8	0.61800
FPSOM	FMEN8	2.00000	FLAN9	0.88000
FPSOM	FMEN9	11.44000	FSTAR	0.17000
FPSOM	FPROT	0.02400		
FPSON	TLAB	31.01000	VAEP	-13.93000
FPSON	VAIP	-22.90000	TWAT	3.60000
FPSON	TOTSO	1.90000	FLANO	0.12000
FPSON	FMENO	1.56000	FLAN5	0.12000
FPSON	FWAT5	0.73800	FMEN5	1.68000
FPSON	FLAN6	0.88000	FWAT6	0.99000
FPSON	FMEN6	12.33000	FLAN7	1.00000
FPSON	FWAT7	1.25400	FMEN7	2.00000
FPSON	FLAN8	1.00000	FWAT8	0.61800
FPSON	FMEN8	2.00000	FLAN9	0.88000
FPSON	FMEN9	11.44000		
FPWTA	TLAB	74.00000	VAEP	-40.35000
FPWTA	VAIP	-47.50000	TWAT	3.67200
FPWTA	FUBWT	1.00000	TOTWT	5.90000
FPWTA	FLANO	0.25000	FWATO	0.72000
FPWTA	FMENO	11.00000	FLANN	1.00000
FPWTA	FWATN	0.96000	FMENN	9.60000
FPWTA	FLAND	1.00000	FWATD	1.08000
FPWTA	FMEND	9.60000	FLAN1	1.00000
FPWTA	FMEN1	9.60000	FLAN2	1.00000
FPWTA	FWAT2	0.79800	FMEN2	9.60000
FPWTA	FLAN3	1.00000	FWAT3	0.11400
FPWTA	FMEN3	9.60000	FLAN4	0.75000
FPWTA	FMEN4	15.00000		
FPWTM	TLAB	71.00000	VAEP	-45.60000
FPWTM	VAIP	-58.00000	TWAT	3.67200
FPWTM	FUBWT	1.00000	TOTWT	5.90000
FPWTM	FLANO	0.25000	FWATO	0.72000
FPWTM	FMENO	8.00000	FLANN	1.00000
FPWTM	FWATN	0.96000	FMENN	9.60000
FPWTM	FLAND	1.00000	FWATD	1.08000
FPWTM	FMEND	9.60000	FLAN1	1.00000
FPWTM	FMEN1	9.60000	FLAN2	1.00000
FPWTM	FWAT2	0.79800	FMEN2	9.60000
FPWTM	FLAN3	1.00000	FWAT3	0.11400
FPWTM	FMEN3	9.60000	FLAN4	0.75000
FPWTM	FMEN4	15.00000		
FPNTA	TLAB	81.00000	VAEP	-54.85000
FPNTA	VAIP	-66.50000	TWAT	4.03200
FPNTA	FUBNT	1.00000	TOTNT	8.10000
FPNTA	FLANO	1.00000	FWATO	1.39200
FPNTA	FMENO	5.00000	FLANN	1.00000

FPNTA	FMENN	21.00000	FLAND	0.50000
FPNTA	FMEND	19.00000	FLAN8	0.25000
FPNTA	FWAT8	1.17000	FMEN8	17.00000
FPNTA	FLAN9	1.00000	FWAT9	1.47000
FPNTA	FMEN9	19.00000		
FPMCA	TLAB	89.50000	VAEP	-28.25000
FPMCA	VAIP	-38.90000	TWAT	10.63200
FPMCA	FUBMC	1.00000	TOTMC	5.88000
FPMCA	FLANO	1.00000	FWATO	0.99000
FPMCA	FMENO	24.00000	FLANN	1.00000
FPMCA	FWATN	0.42600	FMENN	4.00000
FPMCA	FLAND	1.00000	FWATD	0.53400
FPMCA	FMEND	4.00000	FLAN1	1.00000
FPMCA	FMEN1	4.50000	FLAN2	1.00000
FPMCA	FWAT2	0.96600	FMEN2	1.50000
FPMCA	FLAN3	1.00000	FWAT3	0.94800
FPMCA	FMEN3	3.50000	FLAN4	1.00000
FPMCA	FWAT4	1.05600	FMEN4	4.50000
FPMCA	FLAN5	1.00000	FWAT5	0.90600
FPMCA	FMEN5	25.50000	FLAN6	1.00000
FPMCA	FWAT6	1.20600	FMEN6	9.00000
FPMCA	FLAN7	1.00000	FWAT7	1.51800
FPMCA	FMEN7	3.00000	FLAN8	1.00000
FPMCA	FWAT8	1.08600	FMEN8	3.00000
FPMCA	FLAN9	1.00000	FWAT9	0.99600
FPMCA	FMEN9	3.00000		
FPFRA	TLAB	57.00000	VAEP	-27.65000
FPFRA	VAIP	-46.70000	TWAT	10.63200
FPFRA	TOTFR	4.00000	FLANO	1.00000
FPFRA	FWATO	0.99000	FMENO	0.50000
FPFRA	FLANN	1.00000	FWATN	0.42600
FPFRA	FMENN	0.50000	FLAND	1.00000
FPFRA	FWATD	0.53400	FMEND	2.00000
FPFRA	FLAN1	1.00000	FMEN1	31.00000
FPFRA	FLAN2	1.00000	FWAT2	0.96600
FPFRA	FMEN2	1.50000	FLAN3	1.00000
FPFRA	FWAT3	0.94800	FMEN3	1.50000
FPFRA	FLAN4	1.00000	FWAT4	1.05600
FPFRA	FMEN4	0.50000	FLAN5	1.00000
FPFRA	FWAT5	0.90600	FMEN5	9.50000
FPFRA	FLAN6	1.00000	FWAT6	1.20600
FPFRA	FMEN6	0.50000	FLAN7	1.00000
FPFRA	FWAT7	1.51800	FMEN7	8.50000
FPFRA	FLAN8	1.00000	FWAT8	1.08600
FPFRA	FMEN8	0.50000	FLAN9	1.00000
FPFRA	FWAT9	0.99600	FMEN9	0.50000
KWATT	TWAT	-1.00000	AUX	1.00000

RHS

RHS	FUBCO	60.00000	FUBWT	10.00000
RHS	FUBNT	16.00000	FUBMC	9.00000

RHS	FLANO	315.00000	FLANN	315.00000
RHS	FLAND	315.00000	FLAN1	315.00000
RHS	FLAN2	315.00000	FLAN3	315.00000
RHS	FLAN4	315.00000	FLAN5	315.00000
RHS	FLAN6	315.00000	FLAN7	315.00000
RHS	FLAN8	315.00000	FLAN9	315.00000
RHS	FMENO	5200.00000	FMENN	5200.00000
RHS	FMEND	5200.00000	FMEN1	5200.00000
RHS	FMEN2	5200.00000	FMEN3	5200.00000
RHS	FMEN4	5200.00000	FMEN5	5200.00000
RHS	FMEN6	5200.00000	FMEN7	5200.00000
RHS	FMEN8	5200.00000	FMEN9	5200.00000
RHS	FWATO	275.00000	FWATN	275.00000
RHS	FWATD	275.00000	FWAT1	275.00000
RHS	FWAT2	275.00000	FWAT3	275.00000
RHS	FWAT4	275.00000	FWAT5	275.00000
RHS	FWAT6	275.00000	FWAT7	275.00000
RHS	FWAT8	275.00000	FWAT9	275.00000
RHS	FPROT	34.67800	FSTAR	217.60200
RHS	TLAB	17291.00000		

LIST OF REFERENCES

1 Bard, J. and Falk, J. "An Explicit Solution to the Multilevel Programming Problem", Serial T-388, Program in Logistics, School of Engineering and Applied Science, The George Washington University, Washington, D.C., March 1979.

2 Beale, E.M.L. and Tomlin, J.A. "Special Facilities in a General Mathematical Programming System for Non-Convex Problems Using Ordered Sets of Variables", Proceedings of the Fifth International Conference on Operations Research, Travistock Publications, London, 1970.

3 Beazer, W.F., Optimization of Bank Portfolios, Lexington Books, D.C. Heath and Company, Massachusetts, Toronto, London, 1975.

4 Bialas, W. and Karwan, M. "Multilevel Linear Programming", Research Report No. 78-1, Department of Industrial Engineering, State University of New York at Buffalo, 1978.

5 Bisschop, J. and Meeraus, A. "The Solvability of Two-Level Programming Problems Using Commercial Software", World Bank Research Memorandum, Washington, D.C., 1979.

6 Bracken, J. and McGill, J. "Defense Applications of Mathematical Programs with Optimization Problems in the Constraints", Operations Research No. 21, June 1973.

7 Candler, W. and Norton, R. "Multilevel Programming and Development Policy", World Bank Staff Working Paper No. 258, Washington, D.C., May 1977.

8 Candler, W. and Townsley, R. "A Linear Two-Level Programming Problem", Development Research Center Discussion Paper No. 28, The World Bank, Washington, D.C., June 1978.

9 Candler, W. and Townsley, R. "A Linear Programming under Control of a Two-Level Hierarchy", Computers and Operations Research (forthcoming)

10 Chambers, D. and Charnes, A. "Intertemporal Analysis and Optimization of Bank Portfolios", Management Science VII, July 1961.

11 Charnes, A. and Cooper, W. "Goal Programming and Multiple Objective Optimization", European Journal of Operational Research, Vol. 1, May 1977.

12 Cochrane, J.L. and Zeleny, eds. Multiple Criteria Decision Making, University of South Carolina Press, S.C., 1973.

13 Cohen, K. and Hammer, F. "Linear Programming and Optimal Bank Asset Management Decisions", Journal of Finance 22, No. 2, May 1967.

14 Cohen, K. and Hammer F. Linear Programming Models for Optimal Bank Dynamic Balance Sheet Management, Eds. G.P. Szego and Karl Shell, Amsterdam, North Holland, 1972.

15 Cornwall, R. "The Concept of General Equilibrium in a Market Economy with Imperfectly Competitive Producers", Working Paper No. 20, Department of Economics, University of California at Davis, October 1972.

16 Dantzig, G.B. Linear Programming and Extensions, Princeton University Press, New Jersey, 1963.

17 Dorfman, R. "An Economic Interpretation of Control Theory", American Economic Review, Vol. 59, July 1969.

18 Duloy, J. and Norton, R. "CHAC, A Programming Model of Mexican Agriculture" in Multilevel Planning: Case Studies in Mexico, Eds. L. Goreaux and A. Manne, Amsterdam, North Holland, 1973.

19 Duloy, J. and Norton, R. "Prices and Incomes in Linear Programming Models", American Journal of Agricultural Economics, Vol. 57, Nov. 1975.

20 Duloy, J. and O'Mara, G. "Programming and Designing Investment: Indus Basin". World Bank Research Program, Abstracts of Current Studies, October 1979.

21 Falk, J. and Soland, R. "An Algorithm for Separable Nonconvex Programming Problems. Management Science 15, 1969.

22 Falk, J. "A Max-Min Problem", Mathematical Programming, Vol. 5, May 1973.

23 Fortuny-Amat, J. and McCarl, B. "Multilevel Programming", unpublished paper, Purdue University, W. Lafayette, Indiana, April 1979.

24 Fried, J. "Bank Portfolio Selection", Journal of Finance and Quantitative Analysis 5, No. 2, June 1970.

25 Haimes, Y. Hierarchical Analyses of Water Resource Systems, McGraw-Hill Inc., 1977.

26 Hazell, P. And Scandizzo, P. "Competitive Demand Structures under Risk in Agricultural Linear Programming Models", American Journal of Agricultural Economics, Vol. 57, 1974.

27 Hester, D. and Pierce, J. "Cross-Section Analysis and Bank Dynamics", Journal of Political Economy 76, No. 4, August, 1978.

28 Intriligator, M. Mathematical Optimization and Economic Theory, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1971.

29 Kats, A. "Monopolistic Trading Economics: A Case of Government Control", Public Choice, January, 1974.

30 Kats, A. "Non-Cooperative Monopolistic Games and Monopolistic Market Games". International Journal of Game Theory, Vol.3, Issue 4, Physica Verlag, Vienna, 1974.

31 Kornai, J. and Liptak, T. "Two Level Planning", Econometrics, Vol. 33, No. 1, 1965.

32 Kutcher, G. and Scandizzo, P. "Rural Development in Northeast Brazil". World Bank Research Program, Abstracts in Current Studies, October 1979.

33 Kutcher, G. "Master Plan for Water Resources Development and Use", Technical Report 16, UNDP-EGY/73/024. The World Bank, Washington, D.C., May 1980.

34 Matheiss, T. and Rubin, D. "A Survey and Comparison of Methods for Finding all Vertices of Convex Polyhedral Sets", Mathematics of Operations Research, Vol. 5, No. 2, May 1980.

35 McCarl, B. et al. "Experience with Mass Audience Linear Programming for Farm Planning". Mathematical Programming Study 9, North Holland Publishing Co., 1978.

36 Melnik, A. "Short Run Determinants of Commercial Bank Investment Portfolios: An Empirical Analysis", Journal of Finance 25, No. 3, June, 1970.

37 Nemhauser, G. Introduction to Dynamic Programming, John Wiley & Sons, 1966.

38 Russell, W. "An Investigation of Commercial Banks Aggregate Portfolio Adjustments", International Economic Review 10, No. 3, October 1969.

39 Shapley, J. and Shubik, M. "Concepts and Theorems of Pure Competition", Essays in Mathematical Economics in Honor of Oskar Morgenstern, ed M. Shubik, Princeton, 1967.

40 Simaan, M. and Cruz, J. "On the Stackelberg Strategy in Non-Zero Sum Games". Journal of Optimization Theory and its Applications, Vol. 11, No. 5, 1973.

41 Simmonard, M. Linear Programming, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1966.

42 Theil, H. Economic Forecasts and Policy, North-Holland Publishing Co., Amsterdam, 1961.

43 Zeleny, M. Linear Multi-objective Programming, Springer-Verlag, Berlin, 1974.